Advances in Differential Equations Volume 7, Number 7, July 2002, Pages 799–818

CLASSICAL SOLUTIONS OF THE TIMOSHENKO SYSTEM

R. Grimmer
Department of Mathematics, Southern Illinois University, Carbondale, IL 62901

E. Sinestrari
Dipartimento di Matematica, Università di Roma "La Sapienza"
P. Aldo Moro 7, 00185 Roma, Italy

(Submitted by: J.A. Goldstein)

Abstract. We prove the existence and uniqueness of a C^2–solution of the Timoshenko system for the motion of an elastic beam. In addition, we give pointwise estimates for the displacement, rotation, shear angle and their derivatives with constants explicitly calculated. The method of proof is based on the Hille–Yosida operator theory.

1. Introduction

In 1921, S.P. Timoshenko [8] proposed a model for the study of the transverse vibrations of a beam which takes into account the effect of the shear, its mathematical formulation is a system of partial differential equations for the unknowns $w(t,x)$ and $\varphi(t,x)$

$$\begin{aligned} &\varrho\Omega w_{tt}(t,x) = \lambda C\Omega(w_x(t,x)-\varphi(t,x))_x, \\ &\varrho I\varphi_{tt}(t,x) = EI\varphi_{xx}(t,x)+\lambda C\Omega(w_x(t,x)-\varphi(t,x)). \end{aligned} \tag{1.1}$$

Here $w(t,x)$ and $\varphi(t,x)$ are the displacement and the angular rotation at time t in the vibration plane of the element of the bar determined by its position x. ϱ is the density of the material, EI and c are the flexural and the modulus of rigidity and λ is a constant which depends upon the shape of the cross–section of area Ω.

In [8] equations in (1.1) are differentiated twice to eliminate φ and to obtain

$$\frac{\varrho^2 I}{\lambda C}\, w_{tttt}(t,x)+\varrho\Omega w_{tt}(t,x)-\varrho I\big(1+\frac{E}{\lambda C}\big)w_{xxtt}(t,x) = -EIw_{xxxx}(t,x) \tag{1.2}$$

Accepted for publication: June 2001.
AMS Subject Classifications: 74K10, 47D06.

(and a similar equation for φ). This equation must be satisfied by a C^4–solution of (1.1) and has been extensively studied under the name of Timoshenko beam equation.

For a treatment of the system we refer for instance to [5] where (1.1) is supplemented by certain boundary and initial conditions. The exponential decay of energy is proved by transforming the problem into a system of first order in time and then by applying the theory of semigroups of operators in a suitable L^2–space. More precisely, it is proved that the differential operator matrix which appears in the system is the generator of a semigroup of negative type. With this scheme one obtains solutions which verify system (1.1) in the L^2–sense with respect to the space variable.

In this paper, we want to find classical solutions of (1.1), i.e., C^2–solutions which verify (1.1) for each t and x. We suppose that an external force is exerted on the beam and we consider here the case of fixed ends (more general boundary conditions will be treated in a subsequent paper). Here we prove the existence and uniqueness of a C^2–solution under conditions which are necessary and sufficient (except for the regularity in time of the external force which is usual in problems of hyperbolic type, see Theorem 4.1 and Remark 4.2). The main consequence of this result is the possibility of getting pointwise estimates for w, φ and their derivatives. It must be noted that the constants which appear in these estimates can be explicitly calculated from the coefficients and the data. In this way (in addition to the energy estimate given by the L^2–approach) one obtains at each t an estimate of the maximum displacement and angular rotation of every element of the beam in terms of the initial data and the structural characteristics of the beam through constants exactly determined.

To prove these results we transform system (1.1) supplemented by forcing terms, boundary and initial conditions into a Cauchy problem for an ordinary differential equation in a suitable Banach space X of functions of the position x. As we look for classical solutions, X is given the sup–norm. This prevents us from using the classical semigroup theory because the homogeneous boundary conditions (a consequence of the fixed ends assumption) imply the non density of the domain of the operator appearing in the Cauchy problem in X. So we will use the theory of Hille–Yosida operators ([1]) and perturbation arguments, after proving the equivalence of our boundary–initial value problem to a first order differential problem. This approach has been previously introduced by the authors in the study of one–dimensional hyperbolic differential problems ([4]).

In the last part of the paper we want to find L^2–estimates for the X solution found above by using a suitable weighted L^2–norm. The main motivation for this is to find pointwise estimates for w and φ (and their derivatives) by constants independent of time when no external force is exerted on the beam. This proves (in this situation) the boundedness of w and φ for all time.

In addition to the usual notation we will denote by $W^{1,2}$ the Sobolev space of functions with first order partial derivatives in L^2 and if X is a Banach space, $\mathcal{L}(X)$ is the Banach space of the linear and continuous operators in X with the uniform norm.

We must thank Simon Brendle (University of Tübingen) for useful suggestions about the L^2–setting of Section 6.

2. The Timoshenko system

We will consider in this paper the motion of a beam of length l subject to external force (depending on time and space) and with fixed ends. As the initial position and velocity of each element of the beam are given, we are led to consider the following initial–boundary value problem in the unknown $w(t,x)$ and $\varphi(t,x)$:

$$\begin{cases} w_{tt}(t,x) = a^2(w_x(t,x) - \varphi(t,x))_x + f(t,x) \\ \varphi_{tt}(t,x) = c^2\varphi_{xx}(t,x) + e(w_x(t,x) - \varphi(t,x)) + g(t,x) \\ w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x) \\ \varphi(0,x) = \varphi_0(x), \quad \varphi_t(0,x) = \varphi_1(x) \\ w(t,x') = 0, \quad \varphi(t,x') = 0 \end{cases} \tag{P}$$

for $t \geq 0$, $x \in [0,l]$ and $x' = 0, l$. The positive constants a, c, e are defined by

$$a^2 = \sqrt{\tfrac{\lambda C}{\varrho}} \quad , \quad c = \sqrt{\tfrac{E}{\varrho}} \quad , \quad e = \tfrac{\lambda C \Omega}{I\varrho} \tag{2.1}$$

We will also consider and estimate the shear angle

$$s(t,x) = w_x(t,x) - \varphi(t,x). \tag{2.2}$$

Let us set

$$J = [0,l] \quad \text{and} \quad Q = \{(t,x) \; ; \; t \geq 0 \; , \; x \in J\}. \tag{2.3}$$

To reduce (P) to a first order system we introduce new unknowns as follows

$$\begin{cases} u_1 = \frac{1}{2}\ (a^{-1}w_t + w_x - \varphi), \quad u_2 = \frac{1}{2}\ (a^{-1}w_t - w_x + \varphi) \\ u_3 = \frac{1}{2}\ (c^{-1}\varphi_t + \varphi_x), \quad u_4 = \frac{1}{2}\ (c^{-1}\varphi_t - \varphi_x). \end{cases} \tag{2.4}$$

Substituting in (P) we obtain the following first order problem

$$\begin{cases} u_{1t}(t,x) = au_{1x}(t,x) - \frac{c}{2}[u_3(t,x) + u_4(t,x)] + \frac{1}{2a}f(t,x) \\ u_{2t}(t,x) = -au_{2x}(t,x) + \frac{c}{2}[u_3(t,x) + u_4(t,x)] + \frac{1}{2a}f(t,x) \\ u_{3t}(t,x) = cu_{3x}(t,x) + \frac{e}{2c}[u_1(t,x) - u_2(t,x)] + \frac{1}{2c}g(t,x) \\ u_{4t}(t,x) = -cu_{4x}(t,x) + \frac{e}{2c}[u_1(t,x) - u_2(t,x)] + \frac{1}{2c}g(t,x) \\ u_1(0,x) = \frac{1}{2}[a^{-1}w_1(x) + w_0'(x) - \varphi_0(x)] \\ u_2(0,x) = \frac{1}{2}[a^{-1}w_1(x) - w_0'(x) + \varphi_0(x)] \\ u_3(0,x) = \frac{1}{2}[c^{-1}\varphi_1(x) + \varphi_0'(x)], \quad u_4(0,x) = \frac{1}{2}[c^{-1}\varphi_1(x) - \varphi_0'(x)] \\ u_1(t,x') + u_2(t,x') = 0, \quad u_3(t,x') + u_4(t,x') = 0. \end{cases} \tag{P'}$$

Definition 2.1. A solution of problem (P) is a pair $(w, \varphi) \in [C^2(Q)]^2$ satisfying (P) for each $(t,x) \in Q$ and $x' = 0, l$. A solution of problem (P′) is a 4–tuple $(u_1, u_2, u_3, u_4) \in [C^1(Q)]^4$ satisfying (P′) for each $(t,x) \in Q$ and $x' = 0, l$.

Theorem 2.2. *If* (P) *has a solution, then the following compatibility and regularity conditions must by satisfied by the data:*

$$\begin{cases} w_0(x') = \varphi_0(x') = 0, \quad w_1(x') = \varphi_1(x') = 0 \\ a^2[w_0''(x') - \varphi_0'(x')] + f(0,x') = 0 \\ C^2\varphi_0''(x') + e[w_0'(x') - \varphi_0(x')] + g(0,x') = 0 \\ f, g \in C(Q);\ w_0, \varphi_0 \in C^2(J),\ w_1, \varphi_1 \in C^1(J). \end{cases} \tag{C}$$

Proof. Conditions $(\mathrm{C})_{1,2}$ are deduced by setting $t = 0$ and $x = x'$ in $(\mathrm{P})_{3-6}$. The others are obtained by $(\mathrm{P})_{1,2}$ in a similar way. □

In the next two theorems we will show in which sense problem (P′) can substitute for problem (P). The first one is a direct consequence of the definitions.

Theorem 2.3. *If* (w, φ) *is a solution of* (P), *then* (u_1, u_2, u_3, u_4) *given by* (2.4) *is a solution of* (P′).

Theorem 2.4. *Let problem* (P′) *have a solution* (u_1, u_2, u_3, u_4). *Then conditions* $(\mathrm{C})_{3-5}$ *are satisfied. If in addition* $(\mathrm{C})_{1,2}$ *also hold, then setting*

$$\varphi(t,x) = \int_0^x [u_3(t,y) - u_4(t,y)]dy, \tag{2.5}$$

$$w(t,x) = \int_0^x [u_1(t,y) - u_2(t,y)]dy + \int_0^x \varphi(t,y)dy, \tag{2.6}$$

we obtain a solution of problem (P). *If* (P′) *has a unique solution, then* (P) *has a unique solution.*

Proof. The first part is obtained by setting $t = 0$ and $x = x'$ in the equations of (P′). Suppose now that (C) holds and define φ and w as in (2.5)–(2.6). From $(\mathrm{P}')_{1-4}$, we get

$$(u_3 - u_4)_t = c(u_3 + u_4)_x \tag{2.7}$$

$$(u_1 - u_2)_t = a(u_1 + u_2)_x - c(u_3 + u_4) \tag{2.8}$$

and so by virtue of (2.5), (2.7) and $(\mathrm{P}')_{10}$:

$$\varphi_t(t, x) = \int_0^x (u_3 - u_4)_t(t, y)dy = c\int_0^x (u_3 + u_4)_y(t, y)dy = c(u_3 + u_4)(t, x) \tag{2.9}$$

similarly, (2.6), (2.8), (2.9) and $(\mathrm{P}')_9$ imply

$$\begin{aligned} w_t(t, x) &= \int_0^x (u_1 - u_2)_t(t, y)dy + \int_0^x \varphi_t(t, y)dy \\ &= a\int_0^x (u_1 + u_2)_y(t, y)dy = a(u_1 + u_2)(t, x). \end{aligned} \tag{2.10}$$

Hence, from these, $(\mathrm{P}')_{1-4}$ and (2.5)–(2.6), we get

$$w_{tt} = a(u_1 + u_2)_t = a^2(u_1 - u_2)_x + f = a^2(w_x - \varphi)_x + f$$

$$\varphi_{tt} = c(u_3 + u_4)_t = c^2(u_3 - u_4)_x + e(u_1 - u_2) + g = c^2\varphi_{xx} + e(w_x - \varphi) + g$$

and by virtue of (2.9)–(2.10), $(\mathrm{P}')_{5-8}$

$$w_t(0, x) = a(u_1 + u_2)(0, x) = w_1(x), \quad \varphi_t(0, x) = c(u_3 + u_4)(0, x) = \varphi_1(x).$$

From (2.5), $(\mathrm{P}')_{7-8}$ and $(\mathrm{C})_1$, we get

$$\varphi(0, x) = \int_0^x (u_3 - u_4)(0, y)dy = \int_0^x \varphi_0'(y)dy = \varphi_0(x) \tag{2.11}$$

and from this, (2.6), (2.9)–(2.11), $(\mathrm{P}')_{9-10}$ and $(\mathrm{C})_1$, we obtain

$$\begin{aligned} w(0, x) &= \int_0^x (u_1 - u_2)(0, y)dy + \int_0^x \varphi(0, y)dy \\ &= \int_0^x [w_0'(y) - \varphi_0(y)]dy + \int_0^x \varphi_0(y)dy = w_0(x) \end{aligned}$$

$$\varphi(t, l) = \varphi(0, l) + \int_0^l \varphi_s(s, l)ds = \varphi_0(l) + c\int_0^l (u_3 + u_4)(s, l)ds = 0$$

$$w(t,l) = w(0,l) + \int_0^l w_s(s,l)ds = w_0(l) + a\int_0^l (u_1+u_2)(s,l)ds = 0.$$

As $w(t,0) = \varphi(t,0) = 0$, we have that (w,φ) is a solution of (P). If (w,φ) is a solution of (P) with zero data, then we obtain from the previous theorem a solution (u_1,u_2,u_3,u_4) of (P′) with zero data: hence, if (P′) has a unique solution then (u_1,u_2,u_3,u_4) must be zero: from (2.4) we obtain $w_t = w_x = \varphi_t = \varphi_x = 0$: as $w(t,x)$ and $\varphi(t,x)$ vanish for $x=0$ we conclude that (w,φ) is zero hence the uniqueness for problem (P). □

3. Abstract setting

We will give now an abstract version of problem (P′) by introducing a Banach space of functions defined on $J=[0,l]$. As we look for classical solutions and pointwise estimates we will choose continuous functions with the sup–norm.

Definition 3.1. Let $C(J)$ be the Banach space of continuous functions in $[0,l]\to\mathbb{R}$ with norm $\|u\| = \sup_{x\in J}|u(x)|$ and X the Banach space $X = \{U=(u_1,u_2,u_3,u_4);\ u_i\in C(J);\ i=1,\dots,4\}$, $\|U\| = \max_{1\le i\le 4}\|u_i\|$. Let $A: D(A)\subset X\to X$ be the linear operator with domain

$$\begin{aligned} D(A) = \{(u_1,u_2,u_3,u_4) : u_i\in C^1(J)),\ i=1,\dots,4;\\ u_1(x')+u_2(x') = u_3(x')+u_4(x') = 0,\ x'=0,l\}\end{aligned}$$

and defined as

$$AU = \begin{pmatrix} aD & 0 & 0 & 0\\ 0 & -aD & 0 & 0\\ 0 & 0 & cD & 0\\ 0 & 0 & 0 & -cD\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix} = \begin{pmatrix}au_1'\\-au_2'\\cu_3'\\-cu_4'\end{pmatrix}.$$

Here, $D=\frac{d}{dt}$.

Theorem 3.2. *Let $\lambda\in\mathbb{R}$, $\lambda\neq 0$, then there exists $(\lambda-A)^{-1}\in\mathcal{L}(X)$ and*

$$\|\lambda(\lambda-A)^{-1}\|_{\mathcal{L}(X)} \le 1.$$

Proof. Let $F=(f_1,f_2,f_3,f_4)\in X$, then problem

$$\lambda U - AU = F,\quad U\in D(A)$$

is equivalent to

$$\begin{cases} \lambda u_1(x) - a u_1'(x) = f_1(x) \\ \lambda u_2(x) + a u_2'(x) = f_2(x) \\ u_1(x') + u_2'(x') = 0 \\ \lambda u_3(x) - c u_3'(x) = f_3(x) \\ \lambda u_4(x) + c u_4'(x) = f_4(x) \\ u_3(x') + u_4(x') = 0 \end{cases} \tag{R}$$

for $x \in J$ and $x' = 0, l$. It can be checked that setting $\mu = \frac{\lambda}{a}$ the problem $(\mathrm{R})_{1-3}$ has a unique solution given by

$$u_1(x) = e^{\mu x} u_0 - a^{-1} \int_0^x e^{\mu(x-y)} f_1(y) dy$$
$$u_2(x) = -e^{-\mu x} u_0 + a^{-1} \int_0^x e^{-\mu(x-y)} f_2(y) dy$$

with

$$u_0 = (e^{\mu l} - e^{-\mu l})^{-1} a^{-1} \int_0^l [e^{\mu(l-y)} f_1(y) - e^{-\mu(l-y)} f_2(y)] dy.$$

By substituting a, f_1, f_2 with c, f_3, f_4, we obtain the solution of $(\mathrm{R})_{4-6}$. From these formulas we can get as in the proof of Theorem 2.1 of [4] the estimates

$$\|u_1\|, \|u_2\| \leq \frac{1}{|\lambda|} \max(\|f_1\|, \|f_2\|), \quad \|u_3\|, \|u_4\| \leq \frac{1}{|\lambda|} \max(\|f_3\|, \|f_4\|)$$

and the conclusion follows. □

Definition 3.3. Let $B : X \to X$ be defined as

$$BU = \begin{pmatrix} 0 & 0 & -\frac{c}{2} & -\frac{c}{2} \\ 0 & 0 & \frac{c}{2} & \frac{c}{2} \\ \frac{e}{2c} & -\frac{e}{2c} & 0 & 0 \\ \frac{e}{2c} & -\frac{e}{2c} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -c(u_3 + u_4) \\ c(u_3 + u_4) \\ \frac{e}{c}(u_1 - u_2) \\ \frac{e}{c}(u_1 - u_2) \end{pmatrix}$$

and $\Lambda : D(\Lambda) \subset X \to X$ be given by

$$D(\Lambda) = D(A), \qquad \Lambda U = AU + BU.$$

Theorem 3.4. *Let*

$$\omega = \|B\| = \sup(c, ce^{-1}). \tag{3.2}$$

If $\lambda > \omega$, then $(\lambda - \Lambda)^{-1} \in \mathcal{L}(x)$ and

$$\|(\lambda - \Lambda)^{-1}\| \leq \frac{1}{\lambda - \omega}. \tag{3.3}$$

Proof. Let $\lambda > \|B\|$. As $\|B(\lambda - A)^{-1}\| \leq \lambda^{-1}\|B\| < 1$ there exists $[I - B(\lambda - A)^{-1}]^{-1}$ and

$$\|(I - B(\lambda - A)^{-1})^{-1}\| \leq \frac{1}{1 - \lambda^{-1}\|B\|}$$

hence, $(\lambda - A)^{-1}[I - B(\lambda - A)^{-1}]^{-1} \in \mathcal{L}(X)$. This implies that $\lambda - \Lambda = [I - B(\lambda - A)^{-1}](\lambda - A)$ has a bounded inverse in X and

$$\|(\lambda-\Lambda)^{-1}\| = \|(\lambda-A)^{-1}[I-B(\lambda-A)^{-1}]^{-1}\| \leq \frac{1}{\lambda(1 - \lambda^{-1}\|B\|)} = \frac{1}{\lambda - \|B\|}.$$

In this way we have proved that Λ is a Hille–Yosida operator and so we are in position to give conditions for the well–posedness of the abstract initial value problem for the unknown $U : \mathbb{R}_+ \to X$

$$U'(t) = \Lambda U(t) + F(t),\ t \geq 0, \qquad U(0) = U_0. \qquad (P')_X$$

Theorem 3.5. *Given $U_0 \in D(\Lambda)$ and $F \in C^1(\mathbb{R}_+; X)$ such that*

$$U_1 = \Lambda U_0 + F(0) \in \overline{D(A)} \qquad (3.4)$$

problem $(P')_X$ has a unique solution $U \in C^1(\mathbb{R}_+; X)$. In addition the following estimates hold for each $t \geq 0$

$$\|U(t)\| \leq e^{\omega t}(\|U_0\| + \int_0^t \|e^{-\omega s}F(s)\|ds), \qquad (3.5)$$

$$\|U'(t)\| \leq e^{\omega t}(\|U_1\| + \int_0^t \|e^{-\omega s}F'(s)\|ds). \qquad (3.6)$$

Proof. Theorem 3.4 shows that Λ is a Hille–Yosida operator (with non dense domain $D(\Lambda) = D(A)$). Hence, we can apply the temporal regularity results for problem $(\mathrm{P}')_X$ by using Theorems A1–A3 of [1] with $M = 1$ and ω given by (3.2).

4. Classical solutions of the Timoshenko system

We are able now to prove the existence and uniqueness of a C^2–solution of problem (P).

Theorem 4.1. *Let $w_0, w_1, \varphi_0, \varphi_1 : J \to \mathbb{R}$ and $f, g : Q \to \mathbb{R}$ satisfy conditions* (C) *and also*

$$f_t, g_t \in C(Q). \qquad (4.1)$$

Then problem (P) *has a unique solution $(w, \varphi) \in [C^2(Q)]^2$.*

Proof. By virtue of Theorem 2.4 if we can prove that problem (P′) has a unique solution $(u_1, u_2, u_3, u_4) \in [C^1(Q)]^4$, then (w, φ) defined by (2.5)–(2.6) is a solution of (P).

To this end let us check that the assumptions of Theorem 3.5 are satisfied if we define for $t \geq 0$:

$$U_0 = \frac{1}{2}\begin{pmatrix} a^{-1}w_1 + w_0' - \varphi_0 \\ a^{-1}w_1 - w_0' + \varphi_0 \\ c^{-1}\varphi_1 + \varphi_0' \\ c^{-1}\varphi_1 - \varphi_0' \end{pmatrix}, \quad F(t) = \frac{1}{2}\begin{pmatrix} a^{-1}f(t,\cdot) \\ a^{-1}f(t,\cdot) \\ c^{-1}g(t,\cdot) \\ c^{-1}g(t,\cdot). \end{pmatrix} \tag{4.2}$$

From (4.1) we deduce that $F \in C^1(\mathbb{R}_+; X)$ and (C) implies that $U_0 \in D(\Lambda)$. As we have $\overline{D(A)} = \{(u_1, u_2, u_3, u_4) : u_i \in C(J),\ i = 1, \ldots, 4,\ u_1(x') + u_2(x') = u_3(x') + u_4(x') = 0,\ x' = 0, l\}$ and

$$U_1 = \Lambda U_0 + F(0) = \frac{1}{2}\begin{pmatrix} w_1' + aw_0'' - a\varphi_0' - \varphi_1 + a^{-1}f(0,\cdot) \\ -w_1' + aw_0'' - a\varphi_0' + \varphi_1 + a^{-1}f(0,\cdot) \\ \varphi_1' + c\varphi_0'' + ec^{-1}(w_0' - \varphi_0) + c^{-1}g(0,\cdot) \\ -\varphi_1' + c\varphi_0'' + ec^{-1}(w_0' - \varphi_0) + c^{-1}g(0,\cdot) \end{pmatrix} \tag{4.3}$$

we see that $\Lambda U_0 + F(0) \in \overline{D(A)}$. Therefore, we can apply Theorem 3.5 and deduce the existence and uniqueness of a solution $U \in C^1(\mathbb{R}_+, X)$ of problem $(\mathrm{P}')_X$. Let $u_i : Q \to \mathbb{R}$ $(i = 1, \ldots, 4)$ be such that

$$U(t) =: (u_1(t,\cdot)\ ,\ u_2(t,\cdot)\ ,\ u_3(t,\cdot)\ ,\ u_4(t,\cdot)). \tag{4.4}$$

By using the definition of $\Lambda = A + B$, we deduce from $(\mathrm{P}')_X$ that (u_1, u_2, u_3, u_4) is a solution of (P′). Conversely, if (u_1, u_2, u_3, u_4) is a solution of (P′), then U defined as in (4.4) is a solution of $(\mathrm{P}')_X$ (with U_0 and F given by (4.2), in conclusion (P′) and so (P) has a unique solution. □

Remark 4.2. From the results of [1] we see that condition (4.1) can be weakened by supposing

$$f, g \in W^{1,1}(0, T; C(J)) \quad \text{for each } T > 0. \tag{4.5}$$

The $W^{1,1}$–dependence on t could be substituted by a C^1–regularity in x (with suitable modifications of the statement of Theorem 4.1) but one cannot expect to get C^2–solutions by assuming only the continuity of f and g (see e.g. [7] p. 106).

5. Pointwise estimates

In this section we derive pointwise estimates for the deflection w, the rotation φ and the shear angle s of the beam. They are a consequence of the abstract estimates of Theorem 3.5 and the sup–norm given to the Banach space introduced in Definition 3.1.

It must be observed that in these estimates all the constants can be exactly calculated from the data.

Theorem 5.1. *Let* (w, φ) *be the solution of problem* (P) *given by Theorem* 4.1. *The following estimates hold for* $t \geq 0$ *and* $x \in [0, l]$*:*

$$|\varphi(t,x)| \leq x e^{\omega t}(c_0 + \int_0^t e^{-\omega s} h(s) ds), \tag{5.1}$$

$$\|\varphi(t,\cdot)\| \leq \|\varphi_0\| + c(c_0 \tilde{\omega}(t) + \int_0^t \tilde{\omega}(t-s) h(s) ds), \tag{5.2}$$

$$|w(t,x)| \leq x\big(1 + \frac{x}{2}\big) e^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s} h(s) ds\Big), \tag{5.3}$$

$$\|w(t,\cdot)\| \leq \|w_0\| + a\Big(c_0 \tilde{\omega}(t) + \int_0^t \tilde{\omega}(t-s) h(s) ds\Big), \tag{5.4}$$

$$\|s(t,\cdot)\| \leq e^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s} h(s) ds\Big), \tag{5.5}$$

where s *is defined in* (2.2) *and we have set*

$$\omega = \sup(c, ec^{-1}) \quad , \quad \tilde{\omega}(t) = \int_0^t e^{\omega s} ds \tag{5.6}$$

$$c_0 = \max(\|a^{-1}w_1 + w_0' - \varphi_0\|, \ \|a^{-1}w_1 - w_0' + \varphi_0\|, \ \|c^{-1}\varphi_1 + \varphi_0'\|, \ \|c^{-1}\varphi_1 - \varphi_0'\|) \tag{5.7}$$

$$h(t) = \max(\|a^{-1} f(t,\cdot)\| \ , \ \|c^{-1} g(t,\cdot)\|). \tag{5.8}$$

Proof. By using the data we define U_0 and F as in (4.2) hence $c_0 = 2\|U_0\|$ and $h(t) = 2\|F(t)\|$, $t \geq 0$. From the proof of Theorem 4.1 we deduce that if (w, φ) is a solution of (P) then the function $U : \mathbb{R}_+ \to X$ given by

$$U(t) = \frac{1}{2}\begin{pmatrix} a^{-1} w_t(t,\cdot) + s(t,\cdot) \\ a^{-1} w_t(t,\cdot) - s(t,\cdot) \\ c^{-1}\varphi_t(t,\cdot) + \varphi_x(t,\cdot) \\ c^{-1}\varphi_t(t,\cdot) - \varphi_x(t,\cdot) \end{pmatrix} \tag{5.9}$$

is a solution of the abstract Cauchy problem $(\mathrm{P}')_X$ for which Theorem 3.5 yields the estimate

$$2\|U(t)\| \leq 2e^{\omega t}\Big(\|U_0\| + \int_0^t \|e^{-\omega s} F(s)\| ds\Big) = e^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s} h(s) ds\Big). \tag{5.10}$$

Now, (5.9) implies for $(t,x) \in Q$

$$2|s(t,x)| \leq \|s(t,\cdot) + a^{-1}w_t(t,\cdot)\| + \|s(t,\cdot) - a^{-1}w_t(t,\cdot)\| \leq 4\|U(t)\|. \quad (5.11)$$

Similarly, we obtain

$$|\varphi_x(t,x)| \leq 2\|U(t)\|, \quad (5.12)$$

$$|\varphi_t(t,x) \leq 2c\|U(t)\|, \quad (5.13)$$

$$|w_t(t,x)| \leq 2a\|U(t)\|. \quad (5.14)$$

Estimate (5.5) is a consequence of (5.10) and (5.11). To prove the others let us first observe that for each $t \geq 0$ we get from (5.10)

$$\begin{aligned} 2\int_0^t \|U(s)\|ds &\leq 2\|U_0\|\int_0^t e^{\omega s}ds + 2\int_0^t e^{-\omega\sigma}\|F(\sigma)\|d\sigma \int_\sigma^t e^{\omega s}ds \\ &= c_0\tilde{\omega}(t) + \int_0^t \tilde{\omega}(t-s)h(s)ds \end{aligned} \quad (5.15)$$

and from (5.12)

$$|\varphi(t,x)| = \Big|\int_0^x \varphi_y(t,y)dy\Big| \leq x\|\varphi_y(t,\cdot)\| \leq 2x\|U(t)\| \quad (5.16)$$

and so by virtue of (5.10) we obtain estimate (5.1). We also have from (5.13)

$$|\varphi(t,x)| = \Big|\varphi(0,x) + \int_0^t \varphi_s(s,x)ds\Big| \leq \|\varphi_0\| + 2c\int_0^t \|U(s)\|ds. \quad (5.17)$$

By using (5.15), we deduce (5.2).

From (5.11) and (5.16), we get

$$\begin{aligned} |w(t,x)| &= \Big|\int_0^x w_y(t,y)dy\Big| = \Big|\int_0^x [s(t,y) + \varphi(t,y)]dy\Big| \\ &\leq 2\int_0^x (\|U(t)\| + y\|U(t)\|)dy = x(x+2)\|U(t)\| \end{aligned}$$

and so from (5.10) we obtain (5.3).

Finally, by virtue of (5.13), we have

$$|w(t,x)| = \Big|w(0,x) + \int_0^t w_s(s,x)ds\Big| \leq \|w_0\| + 2a\int_0^t \|U(s)\|ds$$

hence, (5.15) implies (5.4). □

In the next theorem we will use the same methods to obtain pointwise estimates of all the derivatives of φ and w.

Theorem 5.3. *Under the assumptions of Theorem* 5.2 *the following estimates hold*

$$\|\varphi_t(t,\cdot)\| \le ce^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big), \tag{5.18}$$

$$\|w_t(t,\cdot)\| \le ae^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big), \tag{5.19}$$

$$\|\varphi_x(t,\cdot)\| \le e^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big), \tag{5.20}$$

$$|w_x(t,x)| \le (1+x)\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big), \tag{5.21}$$

$$\begin{aligned}\|w_x(t,\cdot)\| \le \|\varphi_0\| &+ e^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big)\\ &+ c\Big(c_0\tilde{\omega}(t) + \int_0^t \tilde{\omega}(t-s)h(s)ds\Big),\end{aligned} \tag{5.22}$$

$$\|\varphi_{tt}(t,\cdot)\| \le ce^{\omega t}\Big(c_1 + \int_0^t e^{-\omega s}k(s)ds\Big), \tag{5.23}$$

$$\|w_{tt}(t,\cdot)\| \le ae^{\omega t}\Big(c_1 + \int_0^t e^{-\omega s}k(s)ds\Big), \tag{5.24}$$

$$\begin{aligned}\|\varphi_{xx}(t,\cdot)\| \le ec^{-2}e^{\omega t}&\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big)\\ &+ c^{-1}e^{\omega t}\Big(c_1 + \int_0^t e^{-\omega s}k(s)ds\Big) + c^{-2}\|g(t,\cdot)\|,\end{aligned} \tag{5.25}$$

$$\begin{aligned}\|w_{xx}(t,\cdot)\| \le e^{\omega t}&\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big)\\ &+ a^{-1}e^{\omega t}\Big(c_1 + \int_0^t e^{-\omega s}k(s)ds\Big) + a^{-2}\|f(t,\cdot)\|,\end{aligned} \tag{5.26}$$

$$\|\varphi_{tx}(t,\cdot)\| \le e^{\omega t}\Big(c_1 + \int_0^t e^{-\omega s}k(s)ds\Big), \tag{5.27}$$

$$\|w_{tx}(t,\cdot)\| \le ce^{\omega t}\Big(c_0 + \int_0^t e^{-\omega s}h(s)ds\Big) + e^{\omega t}\Big(c_1 + \int_0^t e^{-\omega s}k(s)ds\Big), \tag{5.28}$$

where ω, $\tilde{\omega}(\cdot)$, c_0, *and* h *are given by* (5.6)–(5.8) *and we have set*

$$c_1 = \max(\|w_1' + aw_0'' - a\varphi_0' - \varphi_1 + a^{-1}f(0,\cdot)\|,$$

$$\begin{aligned}&\|-w_1'+aw_0''-a\varphi_0'+\varphi_1++a^{-1}f(0,\cdot)\|,\\&\|\varphi_1'+c\varphi_0''+ec^{-1}(w_0'-\varphi_0)+c^{-1}g(0,\cdot)\|,\\&\|-\varphi_1'+c\varphi_0''+ec^{-1}(w_0'-\varphi_0)+c^{-1}g(0,\cdot)\|), \qquad (5.29)\end{aligned}$$

$$k(t)=\max(\|a^{-1}f_t(t,\cdot)\|,\ \|c^{-1}g_t(t,\cdot)\|). \qquad (5.30)$$

Proof. (5.18) and (5.19) follow from (5.13), (5.14) and (5.10). From (5.9) we deduce, in the same way we obtained (5.11), that

$$\|\varphi_x(t,\cdot)\|\le 2\|U(t)\| \qquad (5.31)$$

hence, (5.10) implies (5.20). By virtue of (5.11), (5.16) or (5.17), we have

$$|w_x(t,x)|\le|s(t,x)|+|\varphi(t,x)|\le 2(1+x)\|U(t)\|,$$

$$|w_x(t,x)|\le|s(t,x)|+|\varphi(t,x)|\le 2\|U(t)\|+\|\varphi_0\|+2c\int_0^t\|U(s)\|ds,$$

hence, (5.10) and (5.15) yield (5.21) and (5.22).

From the proof of Theorem 5.1, we deduce that

$$U'(t)=\frac{1}{2}\begin{pmatrix}a^{-1}w_{tt}(t,\cdot)+s_t(t,\cdot)\\a^{-1}w_{tt}(t,\cdot)-s_t(t,\cdot)\\c^{-1}\varphi_{tt}(t,\cdot)+\varphi_{xt}(t,\cdot)\\c^{-1}\varphi_{tt}(t,\cdot)-\varphi_{xt}(t,\cdot)\end{pmatrix} \qquad (5.32)$$

$c_1=2\|U_1\|$, $K(t)=2\|F'(t)\|$ and so by proceeding as in the proof of (5.11), we get

$$\|\varphi_{tt}(t,\cdot)\|\le 2c\|U'(t)\|, \qquad (5.33)$$

$$\|w_{tt}(t,\cdot)\|\le 2a\|U'(t)\|, \qquad (5.34)$$

$$\|\varphi_{xt}(t,\cdot)\|\le 2\|U'(t)\|, \qquad (5.35)$$

$$\|s_t(t,\cdot)\|\le 2\|U'(t)\|. \qquad (5.36)$$

Now, from (3.6) and (4.2), we have

$$2\|U'(t)\|\le 2e^{\omega t}\Big(\|U_1\|+\int_0^t\|e^{-\omega s}F'(s)\|ds\Big)=e^{\omega t}\Big(c_1+\int_0^t e^{-\omega s}k(s)ds\Big) \qquad (5.37)$$

hence, (5.33), (5.34) and (5.35) imply (5.23), (5.24) and (5.27).

From the equations of problem (P), we obtain by using (5.33) and (5.11)

$$\begin{aligned}\|\varphi_{xx}(t,\cdot)\|&=\|c^{-2}\varphi_{tt}(t,\cdot)-ec^{-2}s(t,\cdot)-c^{-2}g(t,\cdot)\|\\&\le 2c^{-1}\|U'(t)\|+2ec^{-2}\|U(t)\|+c^{-2}\|g(t,\cdot)\|\end{aligned}$$

and by virtue of (5.34) and (5.12)

$$\begin{aligned}\|w_{xx}(t,\cdot)\| &= \|a^{-2}w_{tt}(t,\cdot)+\varphi_x(t,\cdot)-a^{-2}f(t,\cdot)\| \\ &\le 2a^{-1}\|U'(t)\|+2\|U(t)\|+a^{-2}\|f(t,\cdot)\|.\end{aligned}$$

These estimates (by virtue of (5.10) and (5.37)) imply (5.25) and (5.26).

Finally, from (5.36) and (5.13), we have

$$\|w_{tx}(t,\cdot)\| = \|s_t(t,\cdot)+\varphi_t(t,\cdot)\| \le 2\|U'(t)\|+2c\|U(t)\|$$

hence, (5.37) and (5.10) yield (5.28). □

6. L^2–ESTIMATES

For the purposes illustrated in the introduction, we will substitute the Banach space X of continuous functions (of the space variable x) with a Hilbert space H of square integrable functions. In this space the operator formally identical to Λ is the generator of a unitary group so that the solution of the abstract equation corresponding to $(\mathrm{P}')_X$ satisfies estimates (3.5)–(3.6) (in the H–norm) with $\omega = 0$.

When the data are regular (as in our assumptions) the solution of the abstract equation in H coincides with the solution found in Section 3. So the L^2–estimates (with $\omega = 0$) can be used in particular to prove that the classical solution (w,φ) of problem (P) is bounded in Q when $f = g = 0$.

Definition 6.1. Let $(\cdot,\cdot)_2$ and $\|\cdot\|_2$ be the scalar product and the norm in the Hilbert space $L^2(J)$.

Let us define the Hilbert space $H = [L^2(J)]^4$ with scalar product

$$(U,V) = \frac{e}{2c}[(u_1,v_1)_2+(u_2,v_2)_2]+\frac{c}{2}[(u_3,v_3)_2+(u_4,v_4)_2],$$

where $U = (u_1,u_2,u_3,u_4)$, $V = (v_1,v_2,v_3,v_4) \in H$, $\|U\|_2 = (U,U)^{1/2}$ and the linear operators $L : D(L) \subset H \to H$, $M : H \to H$

$$D(L) = \{U \in [W^{1,2}(J)]^4 : u_1(x')+u_2(x') = u_3(x')+u_4(x') = 0;\ x' = 0,l\}$$

$$LU = \begin{pmatrix} aD & 0 & 0 & 0 \\ 0 & -aD & 0 & 0 \\ 0 & 0 & cD & 0 \\ 0 & 0 & 0 & -cD \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} au_1' \\ -au_2' \\ cu_3' \\ -cu_4' \end{pmatrix} \tag{6.1}$$

and D is the derivative a.e. in J

$$MU = \begin{pmatrix} 0 & 0 & -\frac{c}{2} & -\frac{c}{2} \\ 0 & 0 & \frac{c}{2} & \frac{c}{2} \\ \frac{e}{2c} & -\frac{e}{2c} & 0 & 0 \\ \frac{e}{2c} & -\frac{e}{2c} & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -c(u_3+u_4) \\ c(u_3+u_4) \\ \frac{e}{c}(u_1-u_2) \\ \frac{e}{c}(u_1-u_2) \end{pmatrix}. \tag{6.2}$$

Theorem 6.1. *L is the generator of a unitary group in H.*

Proof. Let us first note that for each $\lambda > 0$, $\lambda - L$ is surjective, i.e., given $F = (f_1, f_2, f_3, f_4) \in H$ there exists $U = (u_1, u_2, u_3, u_4)$ with $u_i \in W^{1,2}(J)$ $(i = 1, \ldots, 4)$ verifying for $x \in J$ a.e. the system (R) of Theorem 3.2. This can be proved by using the same definitions of u_i given in the proof of that theorem.

By virtue of the boundary conditions we have $(LU, U) = 0$ for $U \in D(L)$ and so for each $\lambda > 0$:

$$\lambda \|U\|_2^2 = (\lambda U \pm LU, U) \leq \|\lambda U \pm LU\|_2 \|U\|_2$$

hence,

$$\|U\|_2 \leq \frac{1}{\lambda} \|(\lambda \pm L)U\|_2.$$

This proves that L and $-L$ are generators of contraction semigroups and so L generates a unitary group (see [2] Section 3.11 of Chapter 2). □

Definition 6.2. Let $\Gamma : D(\Gamma) \subset H \to H$ be the linear operator defined by $D(\Gamma) = D(L)$, $\Gamma U = LU + MU$, $U \in D(L)$.

Theorem 6.3. *Γ is the generator of a unitary group in H.*

Proof. By virtue of Stone's theorem (see [3] Theorem 4.7 p. 32) we deduce from Theorem 6.1 that $L^* = -L$ (where L^* denotes the adjoint operator of L); now it can be checked that for each $U, V \in H$ we have $(MU, V) = -(U, MV)$ hence, $M^* = -M$. From these results one can deduce that $\Gamma^* = -\Gamma$ and so by using the Stone's theorem we get the conclusion. □

Theorem 6.4. *Let Γ be defined as above. If $U_0 \in D(L)$ and $F \in C^1(\mathbb{R}_+; H)$, then there exists a unique solution $U \in C^1(\mathbb{R}_+; H)$ of problem*

$$U'(t) = \Gamma U(t) + F(t), \ t \geq 0, \quad U(0) = U_0 \tag{P''_H}$$

and in addition

$$\|U(t)\|_2 \leq \|U_0\|_2 + \int_0^t \|F(s)\|_2 ds, \quad t \geq 0 \tag{6.3}$$

$$\|U'(t)\|_2 \leq \|U_1\|_2 + \int_0^t \|F'(s)\|_2 ds, \quad t \geq 0. \tag{6.4}$$

When $F = 0$, we have

$$\|U(t)\|_2 = \|U_0\|_2 \quad , \quad t \geq 0. \tag{6.5}$$

Proof. From the preceding theorem we know that Γ generates a semigroup $T(t)$ such that for each $U_0 \in H$ we have $\|T(t)U_0\|_2 = \|U_0\|_2$, $t \geq 0$. Hence, the result follows from the classical theory of semigroups (see e.g. [6] Proposition 4.2, p. 298). □

We could give applications of this abstract result to problem (P) when f and g are discontinuous in x but we prefer to obtain more information about the classical solution found in Section 4 by assuming the regularity of the data as in Theorem 4.1.

Theorem 6.5. *Let $U_0 \in D(\Lambda)$ and $F \in C^1(\mathbb{R}_+; X)$ be such that $\Lambda U_0 + F(0) \in \overline{D(\Lambda)}$. Then the solutions of problem $(P')_X$ and $(P'')_H$ coincide.*

Proof. By virtue of Theorem 3.5, problem $(\mathrm{P}')_X$ has a solution $U \in C^1(\mathbb{R}_+, X)$. As $X \hookrightarrow H$, $D(A) \subset D(L)$ and $\Gamma|_{D(\Lambda)} = \Lambda$, we have $U \in C^1(\mathbb{R}_+, H)$ and U verifies $(\mathrm{P}'')_H$. By the uniqueness of the solution of $(\mathrm{P}'')_H$ the conclusion follows. □

Remark 6.6. Given $w_0, w_1, \varphi_0, \varphi_1, f, g$ as in Theorem 4.1 and defined U_0, U_1 and F as in (4.2)–(4.3) we have for $t \geq 0$

$$\|U_0\|_2 = \Big(\frac{e}{4a^2c}\|w_1\|_2^2 + \frac{e}{4c}\|w_0' - \varphi_0\|_2^2 + \frac{1}{4c}\|\varphi_1\|_2^2 + \frac{c}{4}\|\varphi_0'\|_2^2\Big)^{1/2}, \tag{6.6}$$

$$\begin{aligned}\|U_1\|_2 = \Big(&\frac{e}{4c}\|w_1' - \varphi_1\|_2^2 + \frac{e}{4c}\|aw_0'' - a\varphi_0' + a^{-1}f(0,\cdot)\|_2^2 \\ &+ \frac{c}{4}\|\varphi_1'\|_2^2 + \frac{c}{4}\|c\varphi_0'' + \frac{e}{c}(w_0' - \varphi_0) + c^{-1}g(0,\cdot)\|_2^2\Big)^{1/2},\end{aligned} \tag{6.7}$$

$$\|F(t)\|_2 = \Big(\frac{e}{4a^2c}\|f(t,\cdot)\|_2^2 + \frac{1}{4c}\|g(t,\cdot)\|_2^2\Big)^{1/2}, \tag{6.8}$$

$$\|F'(t)\|_2 = \Big(\frac{e}{4a^2c}\|f_t(t,\cdot)\|_2^2 + \frac{1}{4c}\|g_t(t,\cdot)\|_2^2\Big)^{1/2}. \tag{6.9}$$

If (w, φ) is the solution of (P) and U the solution of $(\mathrm{P}')_X$ or $(\mathrm{P}'')_H$, we know from the proof of Theorem 4.1 that (5.9) and (5.32) hold. So, we have

$$\|U(t)\|_2 = \Big(\frac{e}{4a^2c}\|w_t(t,\cdot)\|_2^2 + \frac{e}{4c}\|s(t,\cdot)\|_2^2 + \frac{1}{4c}\|\varphi_t(t,\cdot)\|_2^2 + \frac{c}{4}\|\varphi_x(t,\cdot)\|_2^2\Big)^{\frac{1}{2}}, \tag{6.10}$$

$$\|U'(t)\|_2 = \Big(\frac{e}{4a^2c}\|w_{tt}(t,\cdot)\|_2^2 + \frac{e}{4c}\|s_t(t,\cdot)\|_2^2 + \frac{1}{4c}\|\varphi_{tt}(t,\cdot)\|_2^2 + \frac{c}{4}\|\varphi_{xt}(t,\cdot)\|_2^2\Big)^{\frac{1}{2}}. \tag{6.11}$$

We are in position now to prove pointwise estimates for w, φ and their first derivatives with constants independent of t, when $f = g = 0$.

Theorem 6.7. *Under the assumptions of Theorem* 4.1*, the following estimates hold for the solution* (w, φ) *of problem* (P) *for* $t \geq 0$ *and* $x \in J$*:*

$$|\varphi(t,x)| \leq c_1\Big(\|U_0\|_2 + \int_0^t \|F(s)\|_2 ds\Big), \tag{6.12}$$

$$|w(t,x)| \leq c_2\Big(\|U_0\|_2 + \int_0^t \|F(s)\|_2 ds\Big), \tag{6.13}$$

$$|\varphi_t(t,x)| \leq c_3\Big(\|U_1\|_2 + \int_0^t \|F'(s)\|_2 ds\Big), \tag{6.14}$$

$$|w_t(t,x)| \leq c_4'\Big(\|U_0\|_2 + \int_0^t \|F(s)\|_2 ds\Big) + c_4''\Big(\|U_1\|_2 + \int_0^t \|F'(s)\|_2 ds\Big), \tag{6.15}$$

$$\begin{aligned}|\varphi_x(t,x)| \leq{}& c_5'\Big(\|U_0\|_2 + \int_0^t \|F(s)\|_2 ds\Big) \\ &+ c_5''\Big(\|U_1\|_2 + \int_0^t \|F'(s)\|_2 ds\Big) + c_5'''\|g(t,\cdot)\|_2,\end{aligned} \tag{6.16}$$

$$\begin{aligned}|w_x(t,x)| \leq{}& c_6'\Big(\|U_0\|_2 + \int_0^t \|F(s)\|_2 ds\Big) \\ &+ c_6''\Big(\|U_1\|_2 + \int_0^t \|F'(s)\|_2 ds\Big) + c_6'''\|f(t,\cdot)\|_2,\end{aligned} \tag{6.17}$$

$$\begin{aligned}|s(t,x)| \leq{}& c_7'\Big(\|U_0\|_2 + \int_0^t \|F(s)\|_2 ds\Big) \\ &+ c_7''\Big(\|U_1\|_2 + \int_0^t \|F'(s)\|_2 ds\Big) + c_7'''\|f(t,\cdot)\|_2,\end{aligned} \tag{6.18}$$

where

$$\begin{aligned}
&c_1 = 2\sqrt{c^{-1}l}, \quad c_2 = 2\sqrt{ce^{-1}l} + 2l\sqrt{c^{-1}l} \quad , \quad c_3 = 2\sqrt{c^{-1}l} \\
&c_4' = 2\sqrt{cl}, \quad c_4'' = 2\sqrt{ce^{-1}l} \\
&c_5' = 2\sqrt{c^{-1}l^{-1}} + 2ec^{-2}\sqrt{ce^{-1}l}, \quad c_5'' = 2c^{-2}\sqrt{cl}, \quad c_5''' = c^{-2}\sqrt{l} \\
&c_6' = 2\sqrt{ce^{-1}l^{-1}} + 4\sqrt{c^{-1}l}, \quad c_6'' = 2a^{-1}\sqrt{ce^{-1}l}, \quad c_6''' = a^{-2}\sqrt{l} \\
&c_7' = 2\sqrt{ce^{-1}l} + 6\sqrt{c^{-1}l}, \quad c_7'' = 2a^{-1}\sqrt{ce^{-1}l}, \quad c_7''' = a^{-2}\sqrt{l}
\end{aligned} \tag{6.19}$$

and $\|U_0\|_2$, $\|U_1\|_2$, $\|F(t)\|_2$ *and* $\|F'(t)\|_2$ *are given by* (6.6)–(6.9).

Proof. From (6.10) and (6.11), we have for $t \geq 0$

$$\|w_t(t,\cdot)\|_2 \leq 2a\sqrt{ce^{-1}}\ \|U(t)\|_2 \tag{6.20}$$

$$\|s(t,\cdot)\|_2 \leq 2\sqrt{ce^{-1}}\ \|U(t)\|_2 \tag{6.21}$$

$$\|\varphi_t(t,\cdot)\|_2 \leq 2\sqrt{c}\ \|U(t)\|_2 \tag{6.22}$$

$$\|\varphi_x(t,\cdot)\|_2 \leq 2\sqrt{c^{-1}}\ \|U(t)\|_2 \tag{6.23}$$

$$\|w_{tt}(t,\cdot)\|_2 \leq 2a\sqrt{ce^{-1}}\ \|U'(t)\|_2 \tag{6.24}$$

$$\|s_t(t,\cdot)\|_2 \leq 2\sqrt{ce^{-1}}\ \|U'(t)\|_2 \tag{6.25}$$

$$\|\varphi_{tt}(t,\cdot)\|_2 \leq 2\sqrt{c}\ \|U'(t)\|_2 \tag{6.26}$$

$$\|\varphi_{xt}(t,\cdot)\|_2 \leq 2\sqrt{c^{-1}}\ \|U'(t)\|_2 \tag{6.27}$$

and from equations in (P) and (6.21), (6.24), (6.26), we get

$$\|s_x(t,\cdot)\|_2 \leq 2a^{-1}\sqrt{ce^{-1}}\ \|U'(t)\|_2 + a^{-2}\|f(t,\cdot)\|_2 \tag{6.28}$$

$$\|\varphi_{xx}(t,\cdot)\|_2 \leq 2ec^{-2}\sqrt{ce^{-1}}\ \|U(t)\|_2 + 2c^{-2}\sqrt{c}\ \|U'(t)\|_2 + c^{-2}\|g(t,\cdot)\|_2. \tag{6.29}$$

From (6.23), we have

$$|\varphi(t,x)| = \Big|\int_0^x \varphi_y(t,y)dy\Big| \leq \sqrt{l}\ \|\varphi_x(t,\cdot)\|_2 \leq 2\sqrt{c^{-1}l}\ \|U(t)\|_2 \tag{6.30}$$

hence, by using also (6.21), we get

$$\begin{aligned}|w(t,x)| &= \Big|\int_0^x w_y(t,y)dy\Big| = \Big|\int_0^x s(t,y)dy + \int_0^x \varphi(t,y)dy\Big|\\ &\leq 2(\sqrt{ce^{-1}l} + l\sqrt{c^{-1}l})\|U(t)\|_2.\end{aligned}$$

By virtue of (6.27), (6.25), and (6.22), we deduce

$$\begin{aligned}|\varphi_t(t,x)| &= \Big|\int_0^x \varphi_{ty}(t,y)dy\Big| \leq 2\sqrt{c^{-1}l}\ \|U'(t)\|_2\\ |w_t(t,x)| &= \Big|\int_0^x w_{ty}(t,y)dy\Big| = \Big|\int_0^x s_t(t,y)dy + \int_0^x \varphi_t(t,y)dy\Big|\\ &\leq 2\sqrt{ce^{-1}l}\ \|U'(t)\|_2 + 2\sqrt{cl}\ \|U(t)\|_2.\end{aligned}$$

Let us observe now that for each $v \in C^2(J)$ and $x, y, \in J$, we have

$$|v(x)| \leq |v(y)| + \int_J |v'(t)|dt.$$

By integrating with respect to y, we get

$$l|v(x)| \le \int_J |v(y)|dy + l\int_J |v'(t)|dt \le \sqrt{l}\ \|v\|_2 + l\sqrt{l}\ \|v'\|_2$$

and so

$$|v(x)| \le \sqrt{l^{-1}}\ \|v\|_2 + \sqrt{l}\ \|v'\|_2 \quad , \quad x \in J. \tag{6.31}$$

By choosing $v = \varphi_x(t,\cdot)$ and using (6.23) and (6.29), we get

$$\begin{aligned}&|\varphi_x(t,x)| \le \sqrt{l^{-1}}\|\varphi_x(t,\cdot)\|_2 + \sqrt{l}\|\varphi_{xx}(t,\cdot)\|_2\\ &\le 2\left(\sqrt{c^{-1}l^{-1}} + ec^{-2}\sqrt{ce^{-1}l}\right)\|U(t)\|_2 + 2c^{-2}\sqrt{cl}\|U'(t)\|_2 + c^{-2}\sqrt{l}\|g(t,\cdot)\|_2.\end{aligned}$$

Setting $v = w_x(t,\cdot)$ in (6.31) and using (6.21), (6.30), (6.28), and (6.23), we obtain

$$\begin{aligned}&|w_2(t,x)| \le \sqrt{l^{-1}}\|w_t(t,\cdot)\|_2 + \sqrt{l}\|w_{xx}(t,\cdot)\|_2\\ &\le \sqrt{l^{-1}}(\|s(t,\cdot)\|_2 + \|\varphi(t,\cdot)\|_2) + \sqrt{l}(\|s_x(t,\cdot)\|_2 + \|\varphi_x(t,\cdot)\|_2\\ &\le (2\sqrt{ce^{-1}l^{-1}} + 4\sqrt{c^{-1}l})\|U(t)\|_2 + 2a^{-1}\sqrt{ce^{-1}l}\|U'(t)\|_2 + \sqrt{l}\ a^{-2}\|f(t,\cdot)\|_2.\end{aligned}$$

From this and (6.30), we deduce

$$\begin{aligned}|s(t,x)| &\le |w_x(t,x)| + |\varphi(t,x)|\\ &\le (2\sqrt{ce^{-1}l^{-1}} + 4\sqrt{c^{-1}l})\|U(t)\|_2 + 2a^{-1}\sqrt{ce^{-1}l}\ \|U'(t)\|_2\\ &+ a^{-2}\sqrt{l}\ \|f(t,\cdot)\|_2 + 2\sqrt{c^{-1}l}\ \|U(t)\|_2.\end{aligned}$$

From the above estimates, by virtue of (6.3) and (6.4) we can deduce (6.12)–(6.18). □

We do not write explicitly the L^2–estimates for (w,φ) and its derivatives which can be obtained by (6.20)–(6.29) and (6.3)–(6.4). We want to just examine the situation in which no exterior force is exerted.

Theorem 6.8. *Let $f = g = 0$ and $w_0, w_1, \varphi_0, \varphi_1$ verify the assumptions of Theorem* 4.1. *Then the solution (w,φ) of problem* (P) *and its first derivatives are bounded in Q. In addition for each $t \ge 0$, we have*

$$\begin{aligned}e\|w_t(t,\cdot)\|_2^2 + ea^2\|s(t,\cdot)\|_2^2 + a^2\|\varphi_t(t,\cdot)\|_2^2 + a^2c^2\|\varphi_x(t,\cdot)\|_2^2\\ = e\|w_1\|_2^2 + ea^2\|w_0' - \varphi_0\|_2^2 + a^2\|\varphi_1\|_2^2 + a^2c^2\|\varphi_0'\|_2^2.\end{aligned}$$

Proof. The first statement is a consequence of (6.12)–(6.18) as $F \equiv 0$. The identity (6.32) is equivalent to (6.5) by virtue of (6.6) and (6.10).

References

[1] G. Da Prato and E. Sinestrari, *Differential operators with nondense domain*, Annali Scuola Norm. Sup. Pisa, 14 (1987), 285–344.

[2] K.-J. Engel and R. Nagel, "One-Parameter Semigroups for Linear Evolution Equations," Springer, 2000.

[3] J.A. Goldstein, "Semigroups of Operators and Applications," Oxford University Press, 1985.

[4] R. Grimmer and E. Sinestrari, *Maximum norm in one-dimensional hyperbolic problems*, Diff. Int. Equ., 5 (1992), 421–432.

[5] J.U. Kim and Y. Renardy, *Boundary control of the Timoshenko beam*, SIAM J. Control and Optimization, 25 (1987) 1417–1429.

[6] R.H. Martin, "Nonlinear Operators and Differential Equations in Banach Spaces," Wiley, 1976.

[7] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer, 1993.

[8] S.P. Timoshenko, *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*, Phil. Mag., 41 (1921), 744–746.