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WEIGHTED NORM ESTIMATES AND MAXIMAL REGULARITY

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Abstract. We give a sufficient condition for maximal regularity of the evolution equation $u'(t) - Au(t) = f(t),\ t \geq 0,\ u(0) = 0$, in L_p-spaces. Our condition is a weighted norm estimate for the semigroup (e^{tA}) and it is strictly weaker than the assumption that the e^{tA} are integral operators whose kernels satisfy Gaussian estimates. As an application we present new results for the maximal regularity of Schrödinger operators with singular potentials, elliptic higher order operators with bounded measurable coefficients, and elliptic second order operators with singular lower order terms. Moreover, we prove a similar result for maximal regularity of the discrete time evolution equation $u_{n+1} - Tu_n = f_n$, $n \in \mathbb{N}_0$, $u_0 = 0$.

1. Introduction and main results

The well-known problem of maximal L_p-regularity for continuous time evolution equations is the following. Let X be a Banach space and A the generator of a bounded analytic semigroup on X. We consider the evolution equation

$$u'(t) - Au(t) = f(t) \quad \text{ for all } t \in \mathbb{R}_+ \quad , \quad u(0) = 0.$$

One says that A *has maximal L_p-regularity* if for every right hand side $f \in L_p(\mathbb{R}_+; X)$ the solution u satisfies $u' \in L_p(\mathbb{R}_+; X)$. This property is equivalent to the existence of a constant $C > 0$ such that

$$\|u'\|_{L_p(\mathbb{R}_+,X)} + \|Au\|_{L_p(\mathbb{R}_+,X)} \leq C\|f\|_{L_p(\mathbb{R}_+,X)}.$$

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An estimate of this form has many applications, in particular in the study of non-autonomous and nonlinear parabolic equations [1], [26]. It is known that the property of maximal L_p-regularity for a given operator does not depend on $p \in (1, \infty)$, and we shall henceforth simply use the term "A has maximal regularity". For a recent survey on maximal regularity we refer to [30, § 1].

Here, we are interested in the case $X = L_q(\Omega)$, $1 < q < \infty$. Since, for $q \neq 2$, not every generator of a bounded analytic semigroup in $L_q(\Omega)$ has maximal regularity (see [18]), it is of great importance to have manageable criteria to check maximal regularity for a given operator A. Based on methods from [15], [14], a very useful one was shown by Hieber/Prüss [17] and then improved by Coulhon/Duong [10]. It is sufficient that the semigroup (e^{tA}) consists of integral operators satisfying Gaussian estimates (see Remark 1.2 below).

Gaussian estimates imply that the semigroup extends consistently to all $L_p(\Omega)$, $1 \leq p \leq \infty$. Hence, it is clear that the result of Hieber/Prüss and Coulhon/Duong cannot be applied in situations where the semigroup does not act on all $L_p(\Omega)$, $1 \leq p \leq \infty$. This happens, e.g., for Schrödinger operators with singular potentials [19], second order elliptic operators with singular lower order terms [24], or higher order elliptic operators with bounded measurable coefficients [13].

Gaussian estimates have been successfully generalized by weighted $L_p(\Omega) \to L_q(\Omega)$-norm estimates in order to treat other problems, e.g. spectral p-independence (see, e.g., [27], [25], [20], [24], [8]), the H^∞ functional calculus [6] and Riesz transforms [7]. The main result of this paper is the following criterion for checking maximal regularity on L_p-spaces in terms of weighted norm estimates.

Theorem 1.1. *Let (Ω_1, d, μ) be a space of dimension $D \geq 1$:*

$$|B_{\Omega_1}(x, \lambda r)| \leq C\lambda^D |B_{\Omega_1}(x,r)| \quad \text{for all } x \in \Omega_1,\ \lambda \geq 1,\ r > 0.$$

Let Ω be a measurable subset of Ω_1, let $1 \leq p < p_o, 2 < q \leq \infty$ and (e^{tA}) be a bounded analytic semigroup on $L_{p_o}(\Omega)$ satisfying the weighted norm estimate

$$||P_{B(x,r_t)}\, e^{tA}\, P_{A(x,r_t,k)}||_{L_p(\Omega)\to L_q(\Omega)} \leq |B_{\Omega_1}(x,r_t)|^{\frac{1}{q}-\frac{1}{p}}\, g(k) \qquad (1.1)$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$, for some $r_t > 0$ and a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\sum_k k^{D-1} g(k) < \infty$. Then, for all $r \in (p, q)$, (e^{tA}) is a bounded analytic semigroup on $L_r(\Omega)$ and A has maximal regularity on $L_r(\Omega)$.

Here, we denote by $B(x, r)$ and $B_{\Omega_1}(x, r)$ closed balls in Ω and Ω_1 as well as by $A(x, r, k)$ the annular regions $A(x, r, k) := B(x,(k+1)r) \setminus B(x, kr)$ in

Ω. Moreover, we write P_E for the projection obtained by multiplying by the characteristic function of a set E.

We give some applications and modifications to indicate the scope of our result.

1.1. **Elliptic operators.** In this subsection we apply Theorem 1.1 to three classes of elliptic operators the first of which is treated in a more general setting. The verification of the hypotheses of Theorem 1.1 is delegated to Section 3.

Remark 1.2. Let Ω and Ω_1 be as in Theorem 1.1 and let (e^{tA}) be a bounded analytic semigroup on $L_2(\Omega)$ which has an integral kernel $P_t(x,y)$ satisfying the following *Gaussian estimates:*

$$|P_t(x,y)| \leq |B_{\Omega_1}(x,t^{1/m})|^{-1}\, g(d(x,y)t^{-1/m}) \quad \text{for all } x,y \in \Omega,\ t > 0,$$

for some $m > 0$ and a decreasing function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\sum_k k^{D-1} g(k) < \infty$. Such a bound is equivalent to the weighted norm estimate (1.1) for $p = 1$, $q = \infty$, $r_t = t^{1/m}$; see Proposition 2.9 below. Therefore, (e^{tA}) satisfies the hypotheses of Theorem 1.1 for $p := 1$, $q := \infty$, $p_o := 2$. Hence, A has maximal regularity on $L_r(\Omega)$ for all $r \in (1,\infty)$.

This result is due Coulhon/Duong [10] who improved a weaker result due to Hieber/Prüss [17]; see [5] for a short proof.

Remark 1.3. Let $H := -\Delta + V$ be a *Schrödinger operator* on $\mathbb{R}^D$, $D \geq 3$, with a real-valued potential V. Then $A := -H$ generates a positive semigroup on a scale of L_p-spaces which depends on the form bound of the potential (cf. [19], [25]). If the semigroup is quasi-contractive in L_p, i.e., if $A - \omega_p$ generates a contraction semigroup in L_p for some $\omega_p \geq 0$, then a result of Lamberton [21] and Weis [30] yields maximal regularity for $A - \omega_p$ on L_p.

It was first observed in [19] that a Schrödinger semigroup with $L_{D/2,weak}$-potential can be defined on L_p for certain p outside the interval of quasi-contractivity. For such p the result of Lamberton/Weis cannot be applied.

Taking a more general point of view, we recall that a method of constructing positive C_0-semigroups corresponding to sesquilinear (not necessarily sectorial) forms in L_2 was developed in [28] and that a precise condition for quasi-contractivity was given there. This can be applied to *elliptic second order operators with singular lower order terms.*

Denoting by I the interval of quasi-contractivity and by $[p_-, p_+]$ its closure, it was shown in [24] that, if the matrix of the second order coefficients is uniformly elliptic, then the semigroup (e^{tA}) extends to an analytic semigroup on L_p for all $p \in (p_{\min}, p_{\max})$, where $p_{\max} := \frac{D}{D-2}$ and $p_{\min} := (\frac{D}{D-2} p'_-)'$. Here, we restrict ourselves to the case $\Omega = \mathbb{R}^D$ for simplicity. The results hold also for arbitrary domains $\Omega \subset \mathbb{R}^D$ under Dirichlet boundary conditions, and they hold for Neumann boundary conditions if $H^1(\Omega) \subset L_{2D/(D-2)}(\Omega)$ (cf. [24, Thm.1.2]).

The proof relies heavily on weighted norm estimates (cf. [24, Sect. 2]) and we quote the following: For all $p_{\min} < p < q < p_{\max}$ there are constants $C, \omega > 0$, $\mu \in \mathbb{R}$ such that

$$||e^{-\xi(\cdot)}\, e^{tA}\, e^{\xi(\cdot)}||_{L_p(\mathbb{R}^D) \to L_q(\mathbb{R}^D)} \quad \leq \quad C\, t^{-\frac{D}{2}(\frac{1}{p}-\frac{1}{q})}\, e^{\omega|\xi|^2 t + \mu t}$$

for all $t > 0$, $\xi \in \mathbb{R}^D$. By changing the zero order term we may assume $\mu = 0$. If $2 \in (p_{\min}, p_{\max})$, then the assumptions of Theorem 1.1 are satisfied and A has maximal regularity on L_r for all $r \in (p_{\min}, p_{\max})$. This result is new for $r \in (p_{\min}, p_{\max}) \setminus I$.

If the lower order terms are form-small perturbations of the non-symmetric Dirichlet form which is given by the second order coefficients, then we have $2 \in I$ and bounds of the above form may be found in [25].

Remark 1.4. Let $m \in 2\mathbb{N}$ and A be an *elliptic operator of order m with bounded measurable coefficients* on $\mathbb{R}^D$ as considered, e.g., in [12] or [2]. After a change of the zero order term (e^{tA}) satisfies Gaussian estimates of the type we discussed above, provided that $m \geq D$ [2] [12, § 6].

If $m < D$, then (e^{tA}) satisfies the hypotheses of Theorem 1.1 for $p := \frac{2D}{D+m}$, $q := p'$, $r_t := t^{1/m}$ and all $p_o \in (p, q)$. This will be shown in Section 4 by using results of [12, § 7]. Hence, A has maximal regularity on $L_r(\mathbb{R}^D)$ for all $r \in (p, q)$. Since the operators e^{tA} are neither positive nor quasi-contractive for $r \neq 2$, this result is new.

Recall that for this class of operators the interval $[p, q]$ is optimal for existence of the semigroup [13].

We emphasize that for operators as in Remarks 1.3 and 1.4 more can be said, see Theorem 1.6 below. Remarks 1.3 and 1.4 will be proved by applying the following result involving the so-called Davies-perturbations of the given semigroup (e^{tA}). In contrast to [12] we deal with (in general) unbounded weights $e^{\rho d(x,\cdot)}$.

Proposition 1.5. *Let (Ω_1, d, μ) be a measured metric space and Ω a measurable subset of Ω_1. Let $1 \leq p \leq q \leq \infty$ and (S_t) be a family of operators satisfying*

$$||e^{-\rho d(x,\cdot)}\, S_t\, e^{\rho d(x,\cdot)}||_{L_p(\Omega) \to L_q(\Omega)} \leq C_{t,x} e^{\omega \rho^m t} \tag{1.2}$$

for all $x \in \Omega_1$, $t > 0$, $\rho \geq 0$ *and some* $m > 1$, $\omega > 0$, $C_{t,x} \geq 0$. *Then the following weighted norm estimate holds :*

$$||P_{B(x,t^{1/m})}\, S_t\, P_{A(x,t^{1/m},k)}||_{L_p(\Omega)\to L_q(\Omega)} \;\leq\; 3C_{t,x}\, e^{-b\, k^{m/(m-1)}}$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$ *and some* $b > 0$.

Recently, weighted norm estimates similar to (1.2) were used in [20] (see also [25] and the literature cited in these references) to show p-independence of the spectrum of an operator acting on a scale of L_p-spaces. For the equivalence of (1.2) to other weighted norm estimates we refer to Section 4 and to [8].

1.2. **R-sectoriality.** The ingredients of our proofs are techniques of weighted norm estimates (initiated in [27], [12]) combined with ideas from [15], maximal functions, and Weis' recent characterization of maximal regularity in terms of R-boundedness conditions (see [29] and Theorem A in Section 2).

Weis' result, Theorem A below, shows that if (e^{tA}) is a bounded analytic semigroup on a Banach space X, then A has maximal regularity if and only if the set $\{\, \lambda(\lambda - A)^{-1}\, ;\, |\arg(\lambda)| < \frac{\pi}{2} + \delta \,\}$ is R-bounded for some $\delta > 0$. For some applications (see e.g. [30]) the precise value of the angle δ is important. Under slightly stronger assumptions than in our main result Theorem 1.1, we will show optimality of the angle, i.e., for all $r \in (p,q)$, this angle δ on $X = L_r(\Omega)$ is given by the angle of analyticity of the semigroup (e^{tA}) on $L_{p_o}(\Omega)$.

To this purpose we recall some notation. For any $\psi \in [0,\pi)$, let $\Sigma_\psi := \{\lambda \in \mathbb{C}; |\arg(\lambda)| < \psi\}$. A closed linear and densely defined operator A in a Banach space X is called sectorial if $(-\infty,0) \subset \rho(A)$ and $\{\lambda(\lambda+A)^{-1}; \lambda > 0\}$ is bounded. Then $\{\lambda(\lambda + A)^{-1}; \lambda \in \Sigma_{\pi-\theta}\}$ is bounded for some $\theta \in [0,\pi)$ and the sectoriality angle $\theta_\sigma(A)$ is defined as the inf over all such θ.

If, in these definitions, we replace boundedness by R-boundedness we are led to the concept of R-sectorial operators, i.e., sectorial operators A such that $\{\lambda(\lambda+A)^{-1}; \lambda > 0\}$ is R-bounded. Analogously, the R-sectoriality angle $\theta_R(A)$ is defined as the inf over all $\theta \in [0,\pi)$ such that $\{\lambda(\lambda+A)^{-1}; \lambda \in \Sigma_{\pi-\theta}\}$ is R-bounded.

Note that A is sectorial with $\theta_\sigma(A) < \pi/2$ if and only if $-A$ generates a bounded analytic semigroup of angle $\pi/2 - \theta_\sigma(A)$. Moreover, A is R-sectorial with $\theta_R(A) < \pi/2$ if and only if $-A$ generates an analytic semigroup such that all sets $\{e^{-zA}; |\arg z| = \phi\}$ with $|\phi| < \pi/2 - \theta_R(A)$ are R-bounded [29]. In particular, if X is a UMD space, then A is R-sectorial with $\theta_R(A) < \pi/2$ if and only if $-A$ has maximal regularity (cf. [29] and Theorem A below).

The next theorem states optimality of the R-sectoriality angle under suitable weighted norm estimates.

Theorem 1.6. *Let (Ω_1, d, μ) be a space of dimension $D \geq 1$. Let Ω be a measurable subset of Ω_1, let $1 \leq p < p_o, 2 < q \leq \infty$ and (e^{tA}) be a bounded analytic semigroup of angle $\theta \in (0, \frac{\pi}{2}]$ on $L_{p_o}(\Omega)$ satisfying*

$$||e^{-\rho d(x,\cdot)}\, e^{tA}\, e^{\rho d(x,\cdot)}||_{L_p(\Omega)\to L_q(\Omega)} \;\leq\; C\,|B_{\Omega_1}(x, t^{1/m})|^{\frac{1}{q}-\frac{1}{p}}\, e^{\omega\,\rho^m t} \tag{1.3}$$

for all $x \in \Omega_1$, $t > 0$, $\rho \geq 0$ and some $m > 1$, $\omega, C > 0$. Then, for all $r \in (p, q)$, (e^{tA}) is a bounded analytic semigroup of angle θ on $L_r(\Omega)$, A has maximal regularity on $L_r(\Omega)$ and $-A$ is R-sectorial on $L_r(\Omega)$ with $\theta_R(-A) \leq \pi/2 - \theta$.

Remark 1.7. Theorem 1.6 can be applied to the operators in Remarks 1.3 and 1.4. It may be viewed as an R-boundedness counterpart to [24, Theorem 2.1]. Again we refer to Section 4 for the equivalence of (1.3) to other weighted norm estimates on $\mathbb{R}^D$.

1.3. **Discrete maximal regularity.** The concept of weighted norm estimates can also be applied to the maximal regularity problem for the following natural discrete time evolution equation:

$$u_{n+1} - Tu_n = f_n \quad \text{for all } n \in \mathbb{Z}_+\,, \quad u_0 = 0.$$

We say that the powerbounded operator T *has discrete maximal regularity* if for every right hand side $f \in l_p(\mathbb{Z}_+; X)$ the discrete derivative $(u_{n+1} - u_n)$ of the solution u belongs to $l_p(\mathbb{Z}_+; X)$. See [4] and [5] for examples of discrete maximal regularity and connections to the continuous time problem studied above.

It was shown in [4] that if T has discrete maximal regularity, then T is *analytic* in the sense of [11], i.e., $\{\, n(T - I)T^n;\, n \in \mathbb{N}\}$ is bounded. This notion is a discrete analogue of the property "$\{\, tAe^{tA}\,;\, t > 0\,\}$ is bounded" which characterizes the analyticity of a bounded semigroup $(e^{tA})_{t\geq 0}$. The discrete time analogue of Theorem 1.1 is the following criterion for discrete maximal regularity on L_p-spaces in terms of weighted norm estimates.

Theorem 1.8. *Let (Ω_1, d, μ) be a space of dimension $D \geq 1$. Let Ω be a measurable subset of Ω_1, let $1 \leq p < 2 < q \leq \infty$ and the operator T satisfy the weighted norm estimate*

$$||P_{B(x,r_n)} n^j (T - I)^j T^n P_{A(x,r_n,k)}||_{L_p(\Omega)\to L_q(\Omega)} \leq |B_{\Omega_1}(x, r_n)|^{\frac{1}{q}-\frac{1}{p}} g(k) \tag{1.4}$$

for $j = 0, 1$, and for all $x \in \Omega_1$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, for some $r_n > 0$ and a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\sum_k k^{D-1} g(k) < \infty$. Then, for all

$r \in (p,q)$, T is powerbounded and analytic on $L_r(\Omega)$, and T has discrete maximal regularity on $L_r(\Omega)$.

Theorem 1.8 generalizes a result in [5] where it was shown that discrete time Gaussian estimates imply discrete maximal regularity; this is the discrete analogue of the result due to Hieber/Prüss and Coulhon/Duong mentioned in Remark 1.2 above.

The outline of this paper is as follows. In Section 2, we present our main tools involving the R-boundedness and the R_q-boundedness of sets of operators. In Section 3, we give the proofs of the results in Section 1. For convenience of the reader we discuss in Section 4 the equivalence of various weighted norm estimates on $\mathbb{R}^D$ which are studied in the literature. The results are used in the proofs of Remarks 1.2 to 1.4 in Section 3.

2. R_q-BOUNDEDNESS AND WEIGHTED NORM ESTIMATES

The notion of R-boundedness was already implicitly used in [9] and was introduced in [3]. A set τ of bounded linear operators on a Banach space X is called *R-bounded* if there is a constant C such that we have for all $n \in \mathbb{N}$, $T_1, \ldots, T_n \in \tau$ and $x_1, \ldots, x_n \in X$:

$$\int_0^1 || \sum_{j=1}^n r_j(t)\, T_j(x_j)||\, dt \le C \int_0^1 || \sum_{j=1}^n r_j(t)\, x_j||\, dt, \tag{2.1}$$

where (r_j) is a sequence of independent symmetric $\{1,-1\}$-valued random variables on $[0,1]$, e.g. the Rademacher functions. The relevance of R-boundedness is reflected by the following characterization of maximal regularity due to Weis [29].

Theorem A. *Let X be a UMD space and let $(e^{tA})_{t\ge 0}$ be a bounded analytic semigroup on X. Then the following are equivalent:*
(*a*) *A has maximal regularity.*
(*b*) *$\{ e^{zA}\,;\, |arg(z)| = \delta \}$ is R-bounded for some $\delta > 0$.*
(*c*) *$\{ \lambda(\lambda - A)^{-1}\,;\, |\arg(\lambda)| < \frac{\pi}{2} + \delta \}$ is R-bounded for some $\delta > 0$.*

We recall that a Banach space X is a UMD space if and only if the classical Hilbert kernel defines a bounded convolution operator on $L_p(\mathbb{R};X)$ for all $p \in (1,\infty)$.

Note that in a Hilbert space $X = H$ every bounded set $\tau \subset \mathfrak{L}(H)$ is R-bounded, hence Theorem A generalizes the well-known result that, in a Hilbert space, every generator of a bounded holomorphic semigroup has maximal regularity.

By adapting the methods in [29] to the discrete time setting, the following characterization of discrete maximal regularity was proved in [4].

Theorem B. *Let X be a UMD space and let $T \in \mathfrak{L}(X)$ be powerbounded and analytic. Then the following conditions are equivalent:*

(*a*) *T has discrete maximal regularity.*

(*b*) *$\{ T^n, n(T-I)T^n ; n \in \mathbb{N} \}$ is R-bounded.*

In view of (2.1), the following observation using Kahane's inequality [23], Fubini's theorem and Khintchine's inequality [22] is important:

$$\int_0^1 || \sum_j r_j(t) f_j ||_{L_p} \, dt \sim \Big(\int_0^1 || \sum_j r_j(t) f_j ||_{L_p}^p \, dt \Big)^{1/p} \sim || \Big(\sum_j |f_j|^2 \Big)^{1/2} ||_{L_p}.$$

Now (2.1) is recognized as the following square function estimate :

$$|| \Big(\sum_j |T_j f_j|^2 \Big)^{1/2} ||_{L_p} \leq C || \Big(\sum_j |f_j|^2 \Big)^{1/2} ||_{L_p}.$$

This and the purpose to apply interpolation theory to R-boundedness lead Weis to the following definition [30].

Definition 2.1. Let $p, q \in [1, \infty]$. A set τ of sublinear bounded operators on L_p is called R_q-*bounded* if there is a constant C such that for all $N \in \mathbb{N}$, $T_1, \ldots, T_N \in \tau$ and $f_1, \ldots, f_N \in L_p$, we have

$$|| \Big(\sum_j |T_j f_j|^q \Big)^{1/q} ||_{L_p} \leq \; C || \Big(\sum_j |f_j|^q \Big)^{1/q} ||_{L_p} \qquad \text{if } \; 1 \leq q < \infty$$

$$|| \sup_j \, |T_j f_j| \, ||_{L_p} \leq C || \sup_j \, |f_j| \, ||_{L_p} \qquad \text{if } \; q = \infty.$$

The following remarks are taken from [30, 1f) and 4b)].

Remark 2.2. (a) A subset of $\mathfrak{L}(L_p)$ is bounded if and only if it is R_p-bounded.

(b) A subset of $\mathfrak{L}(L_p)$ is R-bounded if and only if it is R_2-bounded.

The following proposition adapts Weis' R_q-boundedness version [30, Proposition 4b)] of the Stein interpolation theorem to our purposes and shall be used as one keystone in the proof of Theorem 1.1.

Proposition 2.3. *Let $p \in [1, \infty]$ and (e^{tA}) a bounded analytic semigroup on L_p. Let $q, q_0 \in [1, \infty]$ be such that $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{p}$ for some $\theta \in (0, 1]$. If $\{ e^{tA} ; t > 0 \}$ is R_{q_0}-bounded, then $\{ e^{zA} ; |arg(z)| = \delta_q \}$ is R_q-bounded for some $\delta_q > 0$.*

Proof. By analyticity and Remark 2.2(a), there exists $\delta_1 > 0$ such that $\{e^{zA}; |\arg(z)| \le \delta_1\}$ is R_p-bounded. Hence we can assume $\theta \in (0,1)$. Then $\{e^{zA}; |\arg(z)| = \theta\delta_1\}$ is R_q-bounded due to [30, Proposition 4b)]. □

The second keystone of the proof of Theorem 1.1 consists in the verification of the hypotheses of the preceding interpolation result. This will be achieved by the aid of the following result of Feffermann and Stein [16] on R_q-boundedness on spaces (Ω, μ, d) of homogenous type. We will use again the standard notation $B(x,r)$ for the ball centered at x with radius r.

We consider the Hardy-Littlewood p-maximal operator M_p defined by

$$M_p f(x) := \sup_{r>0} \Big(|B(x,r)|^{-1} \int_{B(x,r)} |f(y)|^p \, d\mu(y) \Big)^{1/p}.$$

Proposition 2.4. *Let $p \in [1,\infty)$ and Ω be a space of homogenous type:*

$$|B(x,2s)| \ \le \ C_d \, |B(x,s)| \quad \textit{for all } x \in \Omega_1, \ s > 0.$$

Then the Hardy-Littlewood p-maximal operator M_p on Ω satisfies

$$\{ M_p \} \ \textit{ is } R_s\textit{-bounded on } L_{p_o}(\Omega) \textit{ for all } p_o, s \in (p,\infty).$$

Proof. This is shown in [16] for the case $\Omega = \mathbb{R}^n$, $p = 1$, and the proof given there extends easily to the general case. □

As we cannot deal with pointwise estimates in the proof of Theorem 1.1 (since we have no kernels), we shall need a somehow dual property of the following family of operators $N_{q,r}$, $r > 0$, over which one takes the sup in the definition of the q-maximal operator M_q:

$$N_{q,r} f(x) := \Big(|B(x,r)|^{-1} \int_{B(x,r)} |f(y)|^q \, d\mu(y) \Big)^{1/q}$$

Lemma 2.5. *Let Ω be a space of homogenous type and $q \in (1,\infty]$. Then one has for all $1 < p_o, s < q$ and all sequences (r_j) in $\mathbb{R}_+$:*

$$|| \Big(\sum_j |f_j|^s \Big)^{1/s} ||_{L_{p_o}} \le C_{p_o,s} || \Big(\sum_j |N_{q,r_j} f_j|^s \Big)^{1/s} ||_{L_{p_o}}$$

Proof. We fix a sequence (r_j). Denoting by $\| \cdot \|_{t,x,j}$ the L_t-norm in the (probability) space $(B(x,r_j), |B(x,r_j)|^{-1} dy)$ and using Proposition 2.4 we obtain, for all $1 \le t < u, v < \infty$, the estimate

$$\| x \mapsto \Big(\sum_j \|f_j\|_{t,x,j}^v \Big)^{1/v} \|_u = || \Big(\sum_j |N_{t,r_j} f_j|^v \Big)^{1/v} ||_u$$

$$\leq ||\Big(\sum_j |M_t f_j|^v\Big)^{1/v}||_u \leq C_{u,v,t}||\Big(\sum_j |f_j|^v\Big)^{1/v}||_u$$

for the linear map $(f_j) \mapsto (f_j)$ from $L_u(\Omega; l_v)$ into the space of all families of measurable functions on Ω for which the left hand side is finite. By dualization this bound yields the claim for $p_o = u'$, $s = v'$ and $q = t'$, i.e., for all $q \in (1, \infty]$ and $1 < p_o, s < q$. □

The preceding lemma generalizes the evident, but important observation that R_s-boundedness is preserved by pointwise domination (the case $q = \infty$).

The following lemma can be seen as a generalization of the result in [15] that a semigroup of integral operators satisfying Gaussian estimates [see Remark 1.2] is pointwisely bounded by the maximal operator M_1.

Lemma 2.6. *Let $p \in [1, \infty)$, $q \in [1, \infty]$ and Ω be a space of dimension $D \geq 1$:*

$$|B(x, \lambda s)| \leq C_0\, \lambda^D\, |B(x, s)| \quad \text{for all } x \in \Omega,\ \lambda \geq 1,\ s > 0.$$

Let S be a linear operator such that for some $r > 0$, we have

$$||P_{B(x,r)}\, S\, P_{A(x,r,k)}||_{L_p(\Omega)\to L_q(\Omega)} \leq |B(x, r)|^{\frac{1}{q}-\frac{1}{p}}\, g(k) \tag{2.2}$$

for all $x \in \Omega$, $k \in \mathbb{N}_0$ and a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying

$$K := \sum_{k=0}^{\infty} (k + 1)^{D-1}\, g(k) < \infty.$$

Then we have for all $x \in \Omega$: $N_{q,r} Sf(x) \leq C_1\, K M_p f(x)$. Here the constant C_1 depends on p, q, C_0, D and on nothing else.

Proof. Observe that

$$\begin{aligned} N_{q,r} Sf(x) &= |B(x, r)|^{-1/q} ||P_{B(x,r)} Sf||_q \\ &\leq |B(x, r)|^{-1/q} \sum_{k=0}^{\infty} ||P_{B(x,r)} S\, P_{A(x,r,k)} f||_q \\ &\leq |B(x, r)|^{-1/p} \sum_{k=0}^{\infty} g(k) ||f||_{L_p(A(x,r,k))}. \end{aligned}$$

Hence, by using Hölder's inequality, it suffices to show:

$$|B(x, r)|^{-1} \sum_{k=0}^{\infty} g(k)\, ||f||^p_{L_p(A(x,r,k))} \leq C_1\, K\, M_p f(x)^p\ .$$

The latter is seen as follows:

$$
\begin{aligned}
&|B(x,r)|^{-1} \sum_{k=0}^{\infty} g(k)\, ||f||^p_{L_p(A(x,r,k))} \\
&= |B(x,r)|^{-1} \sum_{k=1}^{\infty} ||f||^p_{L_p(B(x,kr))} \Big(g(k-1) - g(k) \Big) \\
&\le |B(x,r)|^{-1}\, M_p f(x)^p \sum_{k=1}^{\infty} |B(x,kr)| \Big(g(k-1) - g(k) \Big) \\
&\le C_0\, M_p f(x)^p \sum_{k=1}^{\infty} k^D \Big(g(k-1) - g(k) \Big) \le C_1\, K\, M_p f(x)^p.
\end{aligned}
$$

A combination of the three preceding results yields a powerful criterion for checking R_q-boundedness. The result shall be used as the other keystone of the proof of Theorem 1.1.

Corollary 2.7. *Let $1 \le p < q \le \infty$ and Ω be a space of dimension $D \ge 1$. Let $(S_\lambda)_{\lambda\in\Lambda}$ be a family of linear operators such that for some $r_\lambda > 0$ we have*

$$||P_{B(x,r_\lambda)}\, S_\lambda\, P_{A(x,r_\lambda,k)}||_{L_p(\Omega)\to L_q(\Omega)} \le |B(x,r_\lambda)|^{\frac{1}{q}-\frac{1}{p}}\, g(k)$$

for all $\lambda \in \Lambda$, $x \in \Omega$, $k \in \mathbb{N}_0$ and a function $g : \mathbb{N}_0 \to \mathbb{R}_+$ satisfying $\sum_{k=0}^{\infty} (k+1)^{D-1} g(k) < \infty$. Then we have for all $p_o, s \in (p,q)$:

$$\{\, S_\lambda\, ;\, \lambda \in \Lambda \,\} \quad \text{is } R_s\text{-bounded on } L_{p_o}(\Omega).$$

Proof. For all sequences (λ_j) in Λ and (f_j) in $L_{p_o}(\Omega)$, we have due to Lemma 2.5, Lemma 2.6 and Proposition 2.4 :

$$
\begin{aligned}
&||\Big(\sum_j |S_{\lambda_j} f_j|^s\Big)^{1/s}||_{L_{p_o}} \le C||\Big(\sum_j |N_{q,r_{\lambda_j}} S_{\lambda_j} f_j|^s\Big)^{1/s}||_{L_{p_o}} \\
&\le C'||\Big(\sum_j |M_p f_j|^s\Big)^{1/s}||_{L_{p_o}} \le C''||\Big(\sum_j |f_j|^s\Big)^{1/s}||_{L_{p_o}}.
\end{aligned}
$$

Corollary 2.8. *Let $1 \le p < q \le \infty$ and Ω be a space of dimension $D \ge 1$. Let $(S_t)_{t>0}$ be a family of linear operators such that*

$$||P_{B(x,t^{1/m})} S_t P_{A(x,t^{1/m},k)}||_{L_p(\Omega)\to L_q(\Omega)} \le C|B(x,t^{1/m})|^{\frac{1}{q}-\frac{1}{p}} e^{-bk^{m/(m-1)}}$$

for all $t > 0$, $x \in \Omega$, $k \in \mathbb{N}_0$ and some $m > 1$, $b > 0$. For each $\rho \in \mathbb{R}$ let $w_\rho : \Omega^2 \to \mathbb{C}$ be a measurable function satisfying $|w_\rho(x,y)| \le e^{|\rho| d(x,y)}$ for all $x, y \in \Omega$.

(a) For all $\rho \in \mathbb{R}$, $t > 0$, $x \in \Omega$ *and some* $\omega > 0$, *we have*

$$N_{q,t^{1/m}} S_t(w_\rho(x,\cdot)f)(x) \le C\, e^{\omega|\rho|^m t}\, M_p f(x)\ .$$

(b) For all $p_o, s \in (p,q)$ *and* $\omega' := \omega + 1$, *we have*

$\{e^{-\omega'|\rho|^m t} e^{-\rho d(x,\cdot)} S_t e^{\rho d(x,\cdot)}; t > 0, \rho \in \mathbb{R}, x \in \Omega\}$ *is* R_s*-bounded on* $L_{p_o}(\Omega)$.

Proof. Let $g(k) := Ce^{-bk^{m/(m-1)}}$, $C_1 := \sum_k g(k)$. The proof of Lemma 2.6 shows

$$N_{q,t^{1/m}} S_t f(x)^p \le C_1^{p/p'} |B(x,t^{1/m})|^{-1} \sum_{k=1}^{\infty} ||f||^p_{L_p(B(x,kt^{1/m}))}(g(k-1)-g(k)).$$

Apply this to $w_\rho(x,\cdot)f$ instead of f and observe that by assumption on w_ρ

$$||w_\rho(x,\cdot)f||_{L_p(B(x,r))} \;\le\; e^{|\rho|r}\, ||f||_{L_p(B(x,r))} \quad \text{for all } \rho \in \mathbb{R},\ r > 0.$$

We then obtain (a) by using the fact that Ω is of dimension D :

$$\begin{aligned} &N_{q,t^{1/m}} S_t(w_\rho(x,\cdot)f)(x)^p \\ &\le C_1^{p/p'} C_0 M_p f(x)^p \sum_{k=1}^{\infty} e^{p|\rho|kt^{1/m}} k^D \big(g(k-1)-g(k)\big) \\ &\le C_2 M_p f(x)^p e^{p\omega|\rho|^m t} \sum_{k=1}^{\infty} e^{-(k-1)} = C_3 M_p f(x)^p e^{p\omega|\rho|^m t}. \end{aligned}$$

For the proof of (b) we let $\omega' := \omega + 1$ and $p_o, s \in (p,q)$. We take sequences (f_j), (ρ_j), (t_j), (z_j) in $L_{p_o}(\Omega)$, $\mathbb{R}$, $\mathbb{R}_+$, and Ω, respectively, and let $w_{j,\rho}(x,\cdot) := e^{-\rho d(z_j,\cdot)}\, e^{\rho d(z_j,x)} \le e^{|\rho| d(x,\cdot)}$ for all $\rho \in \mathbb{R}$ and $x \in \Omega$. We will apply part (a) to those functions but we shall write w_j instead of w_{j,ρ_j}. Now we obtain part (b) as an application of Lemma 2.5, part (a) and Proposition 2.4:

$$\begin{aligned} &||\Big(\sum_j |e^{-\omega'|\rho_j|^m t_j} e^{-\rho_j d(z_j,\cdot)} S_{t_j} e^{\rho_j d(z_j,\cdot)} f_j|^s\Big)^{1/s}||_{L_{p_o}(\Omega)} \\ &\le C_{p_o,s,q} ||\Big(\sum_j |N_{q,t_j^{1/m}} e^{-\omega'|\rho_j|^m t_j} e^{-\rho_j d(z_j,\cdot)} S_{t_j} e^{\rho_j d(z_j,\cdot)} f_j|^s\Big)^{1/s}||_{L_{p_o}(\Omega)} \\ &= C_{p_o,s,q} ||x \mapsto \Big(\sum_j |e^{-\omega'|\rho_j|^m t_j} N_{q,t_j^{1/m}} \big(w_j(x,\cdot) S_{t_j} w_j(x,\cdot)^{-1} f_j\big)(x)|^s\Big)^{1/s}||_{L_{p_o}(\Omega)} \\ &\le C_{p_o,s,q} ||x \mapsto \Big(\sum_j |e^{-\omega'|\rho_j|^m t_j} N_{q,t_j^{1/m}} \big(e^{|\rho_j| d(x,\cdot)} S_{t_j} w_j(x,\cdot)^{-1} f_j\big)(x)|^s\Big)^{1/s}||_{L_{p_o}(\Omega)} \\ &\le C_{p_o,s,q} ||x \mapsto \Big(\sum_j |e^{-\omega'|\rho_j|^m t_j + |\rho_j| t_j^{1/m}} N_{q,t_j^{1/m}} \big(S_{t_j} w_j(x,\cdot)^{-1} f_j\big)(x)|^s\Big)^{1/s}||_{L_{p_o}(\Omega)} \end{aligned}$$

$$\leq eC_{p_o,s,q}||x \mapsto \Big(\sum_j |e^{-\omega|\rho_j|^m t_j} N_{q,t_j^{1/m}}\big(S_{t_j} w_j(x,\cdot)^{-1} f_j\big)(x)|^s\Big)^{1/s}||_{L_{p_o}(\Omega)}$$

$$\leq eC_{p_o,s,q}||x \mapsto \Big(\sum_j |e^{-\omega|\rho_j|^m t_j} N_{q,t_j^{1/m}}\big(S_{t_j} w_{j,-\rho_j}(x,\cdot) f_j\big)(x)|^s\Big)^{1/s}||_{L_{p_o}(\Omega)}$$

$$\leq CeC_{p_o,s,q}||x \mapsto \Big(\sum_j |M_p f_j(x)|^s\Big)^{1/s}||_{L_{p_o}(\Omega)} \leq C_{p,p_o,s,q}||\Big(\sum_j |f_j|^s\Big)^{1/s}||_{L_{p_o}(\Omega)}.$$

In the special case $p = 1, q = \infty$, the weighted norm estimates appearing in our main results characterize the fact that the corresponding operators have integral kernels satisfying Gaussian estimates of the type mentioned in Section 1. This can be seen from the following observation.

Proposition 2.9. *Let (Ω_1, d, μ) be a space of homogenous type :*

$$|B_{\Omega_1}(x,2s)| \;\leq\; C_d\,|B_{\Omega_1}(x,s)| \quad \text{for all } x \in \Omega_1,\ s > 0.$$

Let Ω be a measurable subset of Ω_1 and let $S \in \mathfrak{L}(L_1(\Omega), L_\infty(\Omega))$ have the integral kernel $K \in L_\infty(\Omega^2)$. Furthermore, let $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a decreasing function and $r > 0$. Then the following are equivalent:
(a) For all $x \in \Omega_1$, $k \in \mathbb{R}_{\geq 0}$ we have

$$||P_{B(x,r)}\, S\, P_{A(x,r,k)}||_{L_1(\Omega)\to L_\infty(\Omega)} \leq |B_{\Omega_1}(x,r)|^{-1}\, g(k)\,.$$

(b) For all $x, y \in \Omega$ we have

$$|K(x,y)| \leq |B_{\Omega_1}(x,r)|^{-1}\, g(d(x,y)r^{-1}).$$

Here the statement is written modulo identification of g and $\widetilde{g}$, where $\widetilde{g}(s) := C_d\, g((s-1)_+)$ and C_d is the doubling constant from above.

Proof. (b)⇒(a) We fix $z \in \Omega_1$, $k \in \mathbb{R}_{\geq 0}$ and observe that the operator $P_{B(z,r)} S P_{A(z,r,k)}$ has the integral kernel $(x,y) \mapsto \chi_{B(z,r)}(x) K(x,y) \chi_{A(z,r,k)}(y)$. Hence, we can estimate as follows:

$$||P_{B(z,r)} S P_{A(z,r,k)}||_{L_1(\Omega)\to L_\infty(\Omega)} = \sup_{x,y\in\Omega} \chi_{B(z,r)}(x) |K(x,y)| \chi_{A(z,r,k)}(y)$$

$$\leq \sup_{x\in B(z,r)} \sup_{y\in B(x,(k-1)r)^c} |K(x,y)|$$

$$\leq \sup_{x\in B(z,r)} |B_{\Omega_1}(x,r)|^{-1} g((k-1)_+) \qquad [\text{ by (b) }]$$

$$\leq |B_{\Omega_1}(z,r)|^{-1} \widetilde{g}(k).$$

(a)⇒(b) A reformulation of (a) is the following :

$$\sup_{z,y\in\Omega}\ \chi_{B(x,r)}(z)\, |K(z,y)|\, \chi_{A(x,r,k)}(y) \leq |B_{\Omega_1}(x,r)|^{-1}\, g(k)$$

for all $x \in \Omega_1$, $k \in \mathbb{R}_{\geq 0}$. Applying this for $z = x$ and $k = d(x,y)\,r^{-1}$ yields

$$|K(x,y)| \leq |B_{\Omega_1}(x,r)|^{-1}\, g(d(x,y)\,r^{-1}).$$

3. Proofs of the main results

Proof of Theorem 1.1. Recall that (e^{tA}) is a bounded analytic semigroup on $L_{p_o}(\Omega)$ satisfying the weighted norm estimate

$$||P_{B(x,r_t)}\, e^{tA}\, P_{A(x,r_t,k)}||_{L_p(\Omega)\to L_q(\Omega)} \leq |B_{\Omega_1}(x,r_t)|^{\frac{1}{q}-\frac{1}{p}}\; g(k^m) \tag{3.1}$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$, for some $r_t > 0$ and a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying $\sum_k k^{D-1} g(k^m) < \infty$. We consider the linear operators $S_t, t > 0$, on Ω_1 defined by

$$S_t f(x) \quad := \quad \begin{cases} e^{tA}(f|_\Omega)(x) & x \in \Omega \\ 0 & x \in \Omega_1 \setminus \Omega \end{cases} .$$

The weighted norm estimate (3.1) for the e^{tA} on Ω implies easily the corresponding weighted norm estimate for the S_t on Ω_1 :

$$||P_{B_{\Omega_1}(x,r_t)}\, S_t\, P_{A_{\Omega_1}(x,r_t,k)}||_{L_p(\Omega_1)\to L_q(\Omega_1)} \;\leq\; |B_{\Omega_1}(x,r_t)|^{\frac{1}{q}-\frac{1}{p}}\; g(k^m)$$

for all $x \in \Omega_1$, $t > 0$, $k \in \mathbb{N}_0$. Since Ω_1 is a space of homogenous type and dimension D, we can apply Corollary 2.7 and obtain that the set $\{S_t\,;\, t > 0\}$ is R_s-bounded on $L_r(\Omega_1)$ for all $r, s \in (p,q)$. But this implies that

$$\{\, e^{tA}\,;\, t > 0 \,\} \text{ is } R_s\text{-bounded on } L_r(\Omega) \text{ for all } r, s \in (p,q)\,. \tag{3.2}$$

Indeed, denoting for a function f on Ω by $\widetilde{f}$ its 0-extension to Ω_1, we see

$$\begin{aligned} ||\Big(\sum_j |e^{t_j A} f_j|^s\Big)^{1/s}||_{L_r(\Omega)} &= ||\Big(\sum_j |S_{t_j} \widetilde{f}_j|^s\Big)^{1/s}||_{L_r(\Omega_1)} \\ \leq C||\Big(\sum_j |\widetilde{f}_j|^s\Big)^{1/s}||_{L_r(\Omega_1)} &= C||\Big(\sum_j |f_j|^s\Big)^{1/s}||_{L_r(\Omega)}. \end{aligned}$$

Applying (3.2) for $r = s$ and Remark 2.2(a) shows that (e^{tA}) is bounded on $L_r(\Omega)$ for all $r \in (p,q)$. Hence, since $L_r(\Omega)$ is reflexive, strong continuity on $L_r(\Omega)$ follows from the strong continuity on $L_{p_o}(\Omega)$. Thus, (e^{tA}) is a bounded C_0-semigroup on $L_r(\Omega)$ for all $r \in (p,q)$. Since (e^{tA}) is bounded analytic on $L_{p_o}(\Omega)$, we obtain by Stein interpolation that (e^{tA}) is bounded analytic on $L_r(\Omega)$ for all $r \in (p,q)$.

Now we fix $r \in (p,q)$. Due to the hypothesis $2 \in (p,q)$ we can apply (3.2) for some $s \neq 2$ such that 2 is between r and s. Hence, since (e^{tA}) is a bounded analytic semigroup on $L_r(\Omega)$, we can apply interpolation in form of Proposition 2.3 and obtain

$$\{ e^{zA} ; |arg(z)| = \delta \} \text{ is } R_2\text{-bounded on } L_r(\Omega) \text{ for some } \delta > 0.$$

Hence, A has maximal regularity on $L_r(\Omega)$ by Remark 2.2(b) and Weis' characterization [29, Thm. 4.2] cited (partly) as Theorem A in Section 2 of this paper. □

For the rest of this section we denote by $\|T\|_{p\to q}$ the norm of $T : L_p \to L_q$.

Proof of Proposition 1.5. Recall that the operators S_t satisfy

$$||e^{-\rho d(x,\cdot)}\, S_t\, e^{\,\rho d(x,\cdot)}||_{p\to q} \le C_{t,x}\; e^{\omega\rho^m t}$$

for all $x \in \Omega_1$, $t > 0$, $\rho \ge 0$ and some $m > 1$, $\omega > 0$, $C_{t,x} \ge 0$. Hence, we have

$$\begin{aligned}
&||P_{B(x,t^{1/m})}\, S_t\, P_{A(x,t^{1/m},k)}||_{p\to q} \\
&= ||e^{\rho d(x,\cdot)} P_{B(x,t^{1/m})}\;\; e^{-\rho d(x,\cdot)} S_t\, e^{\rho d(x,\cdot)}\;\; e^{-\rho d(x,\cdot)} P_{A(x,t^{1/m},k)}||_{p\to q} \\
&\le ||e^{\rho d(x,\cdot)}||_{L_\infty(B(x,t^{1/m}))}\;\; ||e^{-\rho d(x,\cdot)}\, S_t\, e^{\rho d(x,\cdot)}||_{p\to q}\;\; ||e^{-\rho d(x,\cdot)}||_{L_\infty(\Omega\setminus B(x,kt^{1/m}))} \\
&\le \exp(\rho t^{1/m} - \rho k t^{1/m})\; C_{t,x}\; e^{\omega\rho^m t} \\
&\le e\, C_{t,x}\; e^{(\omega+1)\rho^m t - \rho k t^{1/m}}\; .
\end{aligned}$$

Now optimizing with respect to $\rho \ge 0$ yields

$$||P_{B(x,t^{1/m})}\, S_t\, P_{A(x,t^{1/m},k)}||_{p\to q} \le e\, C_{t,x} e^{-bk^{m/(m-1)}}$$

for all $k \in \mathbb{N}_0$, where $b := (\omega + 1)^{-1/(m-1)}(m - 1)m^{-m/(m-1)}$. □

The proof of Remark 1.2 is a combination of Proposition 2.9 and Theorem 1.1.

Proof of Remark 1.3. We base the proof on the estimate

$$||e^{-\xi(\cdot)}\, e^{tA}\, e^{\xi(\cdot)}||_{p\to q} \le C\, t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\; e^{\omega|\xi|_2^m t}\,, \tag{3.3}$$

where $m > 1$ and $|\cdot|_2$ denotes the Euclidean norm. By Section 4, this type of estimate is equivalent to a similar estimate for the so-called Davies-perturbations of our given semigroup (e^{tA}). Indeed, Proposition 4.1 yields

$$\begin{aligned}
||e^{-\rho|x-\cdot|_2}\, e^{tA}\, e^{\rho|x-\cdot|_2}||_{p\to q} &\le 3^{D/q} \sup_{y\in\mathbb{R}^D} ||e^{-c\rho|y-\cdot|_\infty}\, e^{tA}\, e^{c\rho|y-\cdot|_\infty}||_{p\to q} \\
&\le 3^{D/q} 2D \sup_{|\xi|_2 = c\rho} ||e^{-\xi(\cdot)}\, e^{tA}\, e^{\xi(\cdot)}||_{p\to q} \le 3^{D/q} 2DC t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})} e^{\omega c^m \rho^m t}.
\end{aligned}$$

Now we can apply Proposition 1.5 and obtain that the weighted norm estimate (1.1) in the hypothesis of Theorem 1.1 is satisfied:

$$||P_{B(x,t^{1/m})}\, e^{tA}\, P_{A(x,t^{1/m},k)}||_{p\to q} \le C' t^{-\frac{D}{m}(\frac{1}{p}-\frac{1}{q})}\, e^{-bk^{m/(m-1)}}.$$

Hence, A has maximal regularity on $L_r(\mathbb{R}^D)$ for all $r \in (p_{\min}, p_{\max})$. □

Proof of Remark 1.4. Let $p := 2D/(D+m)$ and $q := 2D/(D-m)$ ($p = q_c$ and $q = p_c$ in the notation of [12]). By applying [12, Lem. 24] once directly and once in the dual situation, we obtain (after a suitable change of the zero order term) the bound (note that $\frac{D}{m}(\frac{1}{p}-\frac{1}{q}) = 1$)

$$||e^{-\lambda\phi}\, e^{tA}\, e^{\lambda\phi}||_{p\to q} \le C\, t^{-1}\, e^{\omega\lambda^m t}$$

for some $C, \omega > 0$ and all $\lambda \in \mathbb{R}$, $\phi \in \mathcal{E}_m$, where $\mathcal{E}_m$ denotes the set of all bounded real-valued C^∞-functions on $\mathbb{R}^D$ such that $\|D^\alpha\phi\|_\infty \le 1$ for all α satisfying $1 \le |\alpha| \le \frac{m}{2}$. These arguments carry over to the non-symmetric situation (cf. [24, Sect. 6]). An approximation argument as in [12, p. 148] proves the bound (3.3), and we verify the hypotheses of Theorem 1.1 as in the previous proof. □

Proof of Theorem 1.6. We first show

$$||e^{-\rho d(x,\cdot)}\, e^{tA}\, e^{\rho d(x,\cdot)}||_{u\to v} \;\le\; C_{u,v}\, |B_{\Omega_1}(x, t^{1/m})|^{\frac{1}{u}-\frac{1}{v}}\, e^{\omega_{u,v}\rho^m t} \tag{3.4}$$

for all $p < u \le v < q$, $x \in \Omega_1$, $t > 0$, $\rho \ge 0$ and some $m > 1$, $\omega_{u,v} > 0$, $C_{u,v} \ge 0$. Note that $u = p$, $v = q$ this is our hypothesis (1.3). Proposition 1.5 and Corollary 2.8 give (3.4) for all $p < u = v < q$. Now interpolation gives (3.4) for all $p < u \le v < q$.

Fix $r \in (p, q)$. For the proof of the theorem it suffices to show

$$\{\, e^{zA}\, ;\, z \in \Sigma_\alpha \,\} \quad \text{is } R\text{-bounded on } L_r(\Omega) \text{ for all } \alpha \in (0, \theta). \tag{3.5}$$

We fix $\tilde p$ and $\tilde q$ satisfying $p < \tilde p < r < \tilde q < q$. Then (3.5) follows from Proposition 1.5 and Corollary 2.7 once we prove for all $\alpha \in (0, \theta)$:

$$||e^{-\rho d(x,\cdot)}\, e^{zA}\, e^{\rho d(x,\cdot)}||_{\tilde p\to\tilde q} \;\le\; C_\alpha\, |B_{\Omega_1}(x, Re(z)^{1/m})|^{\frac{1}{\tilde q}-\frac{1}{\tilde p}}\, e^{\omega_\alpha \rho^m Re z} \tag{3.6}$$

for all $z \in \Sigma_\alpha$, $\rho \ge 0$. To this purpose, we fix $0 < \alpha < \alpha_1 < \theta$. By Stein interpolation as in [24, Prop. 2.5] between (3.4) for $u = v = p_o$ and the analyticity assumption on $L_{p_o}(\Omega)$, one obtains

$$||e^{-\rho d(x,\cdot)}\, e^{zA}\, e^{\rho d(x,\cdot)}||_{p_o\to p_o} \;\le\; C\, e^{\omega\rho^m Re\, z} \quad \text{for all } z \in \Sigma_{\alpha_1},\ \rho \ge 0\,.$$

Taking now $\delta \in (0,1)$ such that $z - 2\delta Re\, z \in \Sigma_{\alpha_1}$ for all $z \in \Sigma_\alpha$, we see (3.6) as follows:

$$\|e^{-\rho d(x,\cdot)}\, e^{zA}\, e^{\rho d(x,\cdot)}\|_{\tilde p\to\tilde q}$$

$$\leq \|e^{-\rho d(x,\cdot)} e^{(\delta Re\, z)A} e^{\rho d(x,\cdot)}\|_{p_o \to \tilde{q}} \; \|e^{-\rho d(x,\cdot)} e^{(z-2\delta Re\, z)A} e^{\rho d(x,\cdot)}\|_{p_o \to p_o} \times$$
$$\times \; \|e^{-\rho d(x,\cdot)} e^{(\delta Re\, z)A} e^{\delta(x,\cdot)}\|_{\tilde{p} \to p_o}$$
$$\leq C_{p_o,\tilde{q}} C\, C_{\tilde{p},p_o} |B_{\Omega_1}(x, Re(\delta z)^{1/m})|^{\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}}} e^{\omega_{p_o,\tilde{q}} \rho^m \delta Rez} e^{\omega \rho^m (1-2\delta) Rez} e^{\omega_{\tilde{p},p_o} \rho^m \delta Rez}$$
$$\leq C' \, |B_{\Omega_1}(x, Re(z)^{1/m})|^{(\frac{1}{\tilde{q}} - \frac{1}{\tilde{p}})} e^{\omega' \rho^m Re\, z}.$$

The **Proof of Theorem 1.8** is very similar to the proof of Theorem 1.1, so that we only give the main steps.

We define the following linear operators $S_n^{(j)}$, $n \in \mathbb{N}$, $j = 0, 1$, on $L_{p_o}(\Omega_1)$:

$$S_n^{(j)} f(x) \quad := \quad \begin{cases} n^j (T-I)^j T^n (f|_\Omega)(x) & x \in \Omega \\ 0 & x \in \Omega_1 \setminus \Omega \end{cases} .$$

The weighted norm estimate (1.4) for the $n^j (T-I)^j T^n$ on Ω implies easily the corresponding weighted norm estimate for the $S_n^{(j)}$ on Ω_1 :

$$||P_{B(x,r_n)} \, S_n^{(j)} \, P_{A(x,r_n,k)}||_{L_p(\Omega_1) \to L_q(\Omega_1)} \; \leq \; |B_{\Omega_1}(x, r_n)|^{\frac{1}{q} - \frac{1}{p}} \; g(k^m)$$

for $j = 0, 1$, and for all $x \in \Omega_1$, $n \in \mathbb{N}$, $k \in \mathbb{N}_0$. Now we can apply Corollary 2.7 and obtain that the set $\{ S_n^{(j)} \, ; \, n \in \mathbb{N}, \, j = 0, 1 \}$ is R_s-bounded on $L_{p_o}(\Omega_1)$ for all $s \in (p, q)$, hence $\{ n^j (T-I)^j T^n \, ; \, n \in \mathbb{N}, \, j = 0, 1 \}$ is R_s-bounded on $L_{p_o}(\Omega)$ for all $s \in (p, q)$. Therefore, T is powerbounded and analytic on $L_{p_o}(\Omega)$ and has discrete maximal regularity on $L_{p_o}(\Omega)$ by Remark 2.2 and [4, Thm. 1.1], the latter being cited (partly) as Theorem B at the beginning of Section 2 of this paper. □

4. Weighted norm estimates on $\mathbb{R}^D$

For the verification of the hypothesis (1.1) in our main result Theorem 1.1, i.e., of the weighted norm estimate

$$||P_{B(x,t^{1/m})} \, e^{tA} \, P_{A(x,t^{1/m},k)}||_{L_p(\Omega) \to L_q(\Omega)} \quad \leq \quad |B_{\Omega_1}(x, t^{1/m})|^{\frac{1}{q} - \frac{1}{p}} \; g(k^m) \; ,$$

our Proposition 1.5 shows that it suffices to check the following estimate for the so-called Davies-perturbations of the given semigroup (e^{tA}) :

$$||e^{-\rho d(x,\cdot)} \, e^{tA} \, e^{\rho d(x,\cdot)}||_{L_p(\Omega) \to L_q(\Omega)} \quad \leq \quad C \, |B_{\Omega_1}(x, t^{1/m})|^{\frac{1}{q} - \frac{1}{p}} \; e^{\omega \rho^m t} \; . \qquad (4.1)$$

On the other hand, for the case $\Omega_1 = \mathbb{R}^D = \Omega$ also estimates of the following type are intensively studied in the literature :

$$||e^{-\xi(\cdot)} \, e^{tA} \, e^{\xi(\cdot)}||_{L_p(\mathbb{R}^D) \to L_q(\mathbb{R}^D)} \quad \leq \quad C \, t^{-\frac{D}{m}(\frac{1}{p} - \frac{1}{q})} \; e^{\omega |\xi|^m t} \; . \qquad (4.2)$$

The purpose of this section is to show that the estimates (4.1) on $\mathbb{R}^D$ and (4.2) are (more or less) equivalent. This fact is not new but we shall prove it here for the convenience of the reader.

For the rest of this section, we denote by $|\cdot|_p$ for $p \in [1,\infty]$ the usual l_p-norm on $\mathbb{R}^D$. Moreover, we use the symbol $\|T\|_{p\to q}$ for the norm of $T : L_p(\mathbb{R}^D) \to L_q(\mathbb{R}^D)$. We will show the following result.

Proposition 4.1. *Let $\|\cdot\|$ and $|\cdot|$ be two norms on $\mathbb{R}^D$ and $1 \le p \le q \le \infty$. Then there exists $c > 0$ such that we have for all operators T and for all $x, \xi \in \mathbb{R}^D$, $\rho \ge 0$:*

$$||e^{-\rho||x-\cdot||}\, T\, e^{\rho||x-\cdot||}||_{p\to q} \le 3^{D/q} \sup_{y\in\mathbb{R}^D} ||e^{-c\rho|y-\cdot|}\, T\, e^{c\rho|y-\cdot|}||_{p\to q}$$

$$||e^{-\xi(\cdot)}\, T\, e^{\xi(\cdot)}||_{p\to q} \le 3^{D/q} \sup_{y\in\mathbb{R}^D} ||e^{-c|\xi|_2\,|y-\cdot|}\, T\, e^{c|\xi|_2\,|y-\cdot|}||_{p\to q}$$

$$||e^{-\rho|x-\cdot|_\infty}\, T\, e^{\rho|x-\cdot|_\infty}||_{p\to q} \le 2D \sup_{|\zeta|_2=\rho} ||e^{-\zeta(\cdot)}\, T\, e^{\zeta(\cdot)}||_{p\to q}$$

Our main tool for the proof will be the following lemma (cf. [20, Prop. 13(i)]).

Lemma 4.2. *Let $|\cdot|$ be a norm on $\mathbb{R}^D$ and let $w : \mathbb{R}^D \to (0,\infty)$ be a weight function satisfying $w(x)/w(y) \le Me^{\mu|x-y|}$ for all $x, y \in \mathbb{R}^D$ and some $M, \mu > 0$. If $T : L_{\infty,c} \to L_{1,loc}$ is a linear operator, then*

$$||w^{-1}Tw||_{p\to q} \le M^2 3^{D/q} \sup_{z\in\mathbb{R}^D} ||e^{-2\mu|z-(\cdot)|}\, T\, e^{2\mu|z-(\cdot)|}||_{p\to q}\,.$$

Proof. For any function $f \in L_q(\mathbb{R}^D)$ and any $\varepsilon > 0$, Fubini's theorem yields the following equality (cf. [20, Lem. 12]) :

$$\|x \mapsto \|e^{-\varepsilon|x-(\cdot)|} f\|_q\|_q = \|e^{-\varepsilon|\cdot|}\|_q \|f\|_q.$$

Denoting $C := \sup_z \|e^{-2\mu|z-(\cdot)|} T e^{2\mu|z-(\cdot)|}\|_{p\to q}$ we deduce for $f \in L_{\infty,c}(\mathbb{R}^D)$, $\varepsilon := 3\mu$:

$$\begin{aligned}
&\|w^{-1}Twf\|_q = \|e^{-\varepsilon|\cdot|}\|_q^{-1} \|x \mapsto \|e^{-3\mu|x-(\cdot)|} w^{-1} w(x) T w(x)^{-1} wf\|_q\|_q \\
&\le M\|e^{-\varepsilon|\cdot|}\|_q^{-1} \|x \mapsto \|e^{-2\mu|x-(\cdot)|} T e^{2\mu|x-(\cdot)|}(e^{-2\mu|x-(\cdot)|} w(x)^{-1} wf)\|_q\|_q \\
&\le MC\|e^{-\varepsilon|\cdot|}\|_q^{-1} \|x \mapsto \|e^{-2\mu|x-(\cdot)|} w(x)^{-1} wf\|_p\|_q \\
&\le M^2 C\|e^{-\varepsilon|\cdot|}\|_q^{-1} \|x \mapsto \|e^{-\mu|x-(\cdot)|} f\|_p\|_q \\
&\le M^2 C\|e^{-3\mu|\cdot|}\|_q^{-1} \|e^{-\mu|x-(\cdot)|}\|_q \|f\|_p.
\end{aligned}$$

We used a Young-type inequality for integral operators in the last step. Recalling $\|e^{-\mu|\cdot|}\|_q = \mu^{-D/q}\|e^{-|\cdot|}\|_q$ proves the claim. □

Proof of Proposition 4.1. Choose $c > 0$ such that $|y|_2 \leq \frac{c}{2}|y|$ and $||y|| \leq \frac{c}{2}|y|$ for all $y \in \mathbb{R}^D$. Then the first estimate follows from Lemma 4.2, applied to $w = e^{\rho||x-\cdot||}$ and $\mu := \frac{c}{2}\rho$. The second estimate follows from Lemma 4.2, applied to $w := e^{\xi(\cdot)}$ and $\mu := \frac{c}{2}|\xi|_2$. The third estimate is seen as follows (cf. also [20, Sect. 2.2]). We let, for $k \in \{1, \ldots, D\}$, $\zeta_{2k-1} := e_k$ and $\zeta_{2k} := -e_k$, where e_k denotes the k-th unit vector. Then, for any $z, x \in \mathbb{R}^D$ and $\rho > 0$, we have

$$e^{-\rho|x-z|_\infty} = \inf_j e^{-\rho\zeta_j(x-z)} \quad \text{and} \quad e^{\rho|x-z|_\infty} = \sup_j e^{\rho\zeta_j(x-z)} \leq \sum_j e^{\rho\zeta_j(x-z)}.$$

Thus, we can estimate as follows:

$$\begin{aligned}
&\|e^{-\rho|x-\cdot|_\infty} T e^{\rho|x-\cdot|_\infty}\|_{p\to q} \\
&\leq \|e^{-\rho|x-\cdot|_\infty}\, T \sum_j e^{\rho\zeta_j(x-\cdot)}\|_{p\to q} \cdot \|e^{\rho|x-\cdot|_\infty}(\sum_j e^{\rho\zeta_j(x-\cdot)})^{-1}\|_\infty \\
&\leq \sum_j \|e^{-\rho|x-\cdot|_\infty}\, T\, e^{\rho\zeta_j(x-\cdot)}\|_{p\to q} \leq \sum_j \|e^{-\rho\zeta_j(x-\cdot)}\, T\, e^{\rho\zeta_j(x-\cdot)}\|_{p\to q} \\
&\leq 2D \sup_{|\zeta|_2=\rho} ||e^{-\zeta(\cdot)}\, T\, e^{\zeta(\cdot)}||_{p\to q}.
\end{aligned}$$

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