Advances in Differential Equations Volume 7, Number 12, December 2002, Pages 1479–1512

LIOUVILLE TYPE RESULTS AND EVENTUAL FLATNESS OF POSITIVE SOLUTIONS FOR P-LAPLACIAN EQUATIONS

Y. Du
School of Mathematical and Computer Sciences, University of New England
Armidale, NSW2351, Australia

Z. Guo
Department of Mathematics, Henan Normal University
Xinxiang, 453002, P.R. China

(Submitted by: James Serrin)

Abstract. For a wide class of nonlinearities $f(u)$ satisfying

$$f(0) = f(a) = 0,\ f(u) > 0 \text{ in } (0, a) \text{ and } f(u) < 0 \text{ in } (a, \infty),$$

we study the quasilinear equation $-\Delta_p u = \lambda c(x)f(u)$ over the entire $\mathbb{R}^N$ or over a bounded smooth domain Ω. Such equation covers various models from chemical reaction theory and population biology. We show that any nonnegative solution on the entire $\mathbb{R}^N$ must be a constant, and for large λ, any positive solution on Ω must approach a in compact subsets of Ω, no matter whether or not the solution has a prescribed behavior near the boundary of Ω. We also determine the flat cores of the positive solutions and show that the flat cores enlarge to the whole Ω as λ goes to infinity. Our proof of these results demonstrates the usefulness of boundary blow-up solutions in various classical problems.

1. Introduction

In this paper, we prove some Liouville type results for p-Laplacian equations of the type

$$-\Delta_p u = c(x)f(u) \quad \text{in } \mathbb{R}^N\ (N \geq 2), \tag{1.1}$$

where $\Delta_p u = \operatorname{div}(|Du|^{p-2}Du)$, $p > 1$, $c \in L^\infty(\mathbb{R}^N)$ and $0 < c_1 < c(x) < c_2 < \infty$. More precisely, we show that for a broad class of nonlinearities $f(u)$ satisfying

$$f(0) = f(a) = 0,\ f(u) > 0 \text{ in } (0, a) \text{ and } f(u) < 0 \text{ in } (a, \infty),$$

Accepted for publication: May 2002.
AMS Subject Classifications: 35J15, 35J25, 35J60.

any nonnegative solution of (1.1) must be a constant. By a solution of (1.1) we mean a function $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N} |Du|^{p-2} Du \cdot D\psi dx = \int_{\mathbb{R}^N} c(x) f(u) \psi dx, \ \forall \psi \in C_0^\infty(\mathbb{R}^N).$$

We will also consider the corresponding boundary value problem

$$-\Delta_p u = \lambda c(x) f(u) \quad \text{in } \Omega, \ u|_{\partial\Omega} = \phi \tag{1.2}$$

and the boundary blow-up problem

$$-\Delta_p u = \lambda c(x) f(u) \quad \text{in } \Omega, \ u|_{\partial\Omega} = \infty, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, λ is a positive parameter, and ϕ is a nonnegative function in $C^1(\bar{\Omega})$. By a solution to (1.2), we mean $u \in W^{1,p}(\Omega)$ such that $u - \phi \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega |Du|^{p-2} Du \cdot D\psi dx = \int_\Omega \lambda c(x) f(u) \psi dx, \ \forall \psi \in C_0^\infty(\Omega). \tag{1.4}$$

By a solution to (1.3) we mean $u \in W^{1,p}_{loc}(\Omega) \cap C(\Omega)$ such that (1.4) holds and

$$\lim_{d(x,\partial\Omega) \to 0} u(x) = \infty.$$

We will show that, as λ goes to infinity, any positive solution u of (1.2) or (1.3) must approach, locally uniformly in Ω, the unique positive zero a of $f(s)$; therefore, the solutions eventually become flat (with a boundary layer). Moreover, under suitable further conditions on f near a, for all large λ, any positive solution of (1.2) or (1.3) is shown to have a flat core $\{u = a\}$, and the size of the flat core approaches that of Ω as λ converges to ∞. Thus the solutions become completely flat inside Ω as λ increases to infinity.

Problem (1.2) with zero Dirichlet boundary conditions arises in various models in chemical reaction theory and population biology, and has been extensively studied. We only mention [3, 4, 5, 14, 15, 16, 17, 18, 19, 21, 31] for some relatively recent results. Part of our interest for the study of (1.3) comes from the following fact: Once a property is shown to be shared by both (1.3) and (1.2), then that property is universal in the sense that it is not affected by the boundary condition on $\partial\Omega$. Indeed, it will follow as a simple consequence of our results for (1.2) and (1.3) that any positive solution u_λ of $-\Delta_p u = \lambda c(x) f(u)$ on Ω, without any condition imposed upon u_λ at $\partial\Omega$, possesses the same eventual flatness and flat core properties as that for (1.2) and (1.3). Thus these properties are universal.

To the best of our knowledge, no flat core phenomenon has been observed previously for boundary blow-up solutions; our flat core analysis for (1.3) appears to be the first. Our results on the flat core of (1.2) improve various results in previous studies (see [10, 14, 16, 21]) where more restrictive conditions on $f(u)$ are imposed. Our results on the asymptotic behavior of (1.3) as $\lambda \to \infty$ improve those obtained in [11] and [12]. These previous studies can only deal with a much narrower class of nonlinearities, due to the comparison principle used there. Here we make use of a weak sweeping principle for p-Laplacian equations, which turns out to be more powerful in dealing with this kind of problems. The weak sweeping principle here is a simple variant of the well known sweeping principle introduced by Serrin [28] for Laplacian equations. Problem (1.3) with $p = 2$ has a long history. We refer to [1, 2, 9, 23, 24, 25, 26] for some related recent development on the existence, uniqueness and asymptotic analysis of boundary blow-up solutions.

Special cases of problem (1.1) have been considered in [11] (for $p = 2$) and [12] (for $f(u) = au^{p-1} - bu^q$, $q > p-1$). By making use of boundary blow-up solutions of p-Laplacian equations (see Proposition 3.2 in Section 3) and the weak sweeping principle, we are able to extend these to the more general case here. Our Proposition 3.2 provides more information on boundary blow-up solutions than the classical work of Keller [22] (for the case $p = 2$) and its extension to p-Laplacian equations in [25]; this result is of independent interest. There has been extensive studies on the existence, uniqueness and radial symmetry of ground state solutions of (1.1), for these we refer to [7, 8, 13, 20, 29] and the references therein. For nonlinearities quite different from the ones covered here, various important Liouville theorems have been obtained in the past two decades. We refer to the recent paper [30] and the references therein for more details. Our Liouville type results complement these and the methods are completely different.

We say that $f(s)$ is locally quasi-monotone on $[0, \infty)$ if for any bounded interval $[s_1, s_2] \subset [0, \infty)$, there exists a continuous increasing function $L(s)$ such that $f(s) + L(s)$ is non-decreasing in s for $s \in [s_1, s_2]$. Clearly, this condition is less restrictive than requiring $f(s)$ to be locally Lipschitz continuous on $[0, \infty)$. Our main results are the following.

Theorem 1.1. *Let $f(s)$ be continuous and locally quasi-monotone on $[0, \infty)$ and satisfy the following conditions:*

(F_1) *For some $a > 0$,*

$$f(0) = f(a) = 0,\ f(s) > 0 \ \ \text{in } (0, a),\ f(s) < 0 \ \ \text{in } (0, \infty).$$

(F_2) *For some small* $\delta > 0$*, there exists a constant* $\sigma > 0$ *such that*

$$f(s) \geq \sigma s^{p-1}, \ \forall s \in (0, \delta).$$

(F_3) *For some large* $M > 0$*, there exists a continuous function* $g(s)$ *such that*

$$f(s) \leq g(s) < 0, \ \forall s \in [M, \infty),$$

$$g(s) \text{ is nonincreasing in } [M, \infty) \text{ and } \int_M^\infty \Big[\int_M^u |g(s)| ds \Big]^{-1/p} du < \infty.$$

Then any nonnegative solution of (1.1) *is a constant.*

Remark 1.2. (i) For the special case that $c(x) \equiv 1$ and $f(u) = au^{p-1} - bu^q$, where a and b are positive constants and $q > p-1$, Theorem 1.1 was proved in [12], Theorem 1.1, but the proof there uses in an essential way the special nonlinearity, and is very different from the one in this paper.

(ii) Condition (F_2) is not sharp. For the case $p = 2$ and $c(x) \equiv 1$, [11], Theorem 5.2 shows that the conclusion of Theorem 1.1 holds if (F_2) is replaced by $f(s) \geq \sigma s^{(N+2)/N}$, $\forall s \in (0, \delta)$.

(iii) If $N > p$, then for any $\xi > (p-1)N/(N-p)$,

$$u(x) = (1 + |x|^{p/(p-1)})^{(p-1)/(p-1-\xi)}$$

satisfies

$$-\Delta_p u = c(x) u^\xi,$$

where $c(x) = \big(\frac{p}{\xi-(p-1)}\big)^{p-1}\big[N - (\frac{\xi p}{\xi-(p-1)})\frac{|x|^{p/(p-1)}}{1+|x|^{p/(p-1)}}\big]$. Clearly $c_1 \leq c(x) \leq c_2$, where $c_1 = \big(\frac{p}{\xi-(p-1)}\big)^{p-1}(N - \frac{\xi p}{\xi-(p-1)}) > 0$ and $c_2 = \big(\frac{p}{\xi-(p-1)}\big)^{p-1}N$. Therefore, if we define $f(u) = u^\xi$ for $u \in [0,1]$ and extend $f(u)$ to $u > 1$ so that (F_1) and (F_3) are satisfied for some $a > 1$, we find that Theorem 1.1 does not hold if (F_2) is dropped.

(iv) If $N \leq p$, we will show that condition (F_2) can be dropped from Theorem 1.1. If $N > p$, we conjecture that Theorem 1.1 holds true when (F_2) is replaced by $f(s) \geq \sigma s^{N^*}$, $\forall s \in (0, \delta)$, where $N^* = N(p-1)/(N-p)$.

(v) Condition (F_3) is sharp. We will give an example showing that (1.1) can have an unbounded entire positive solution when (F_3) is violated while (F_1) and (F_2) both hold.

Theorem 1.3. *Suppose that* $f(s)$ *is continuous and locally quasi-monotone on* $[0, \infty)$ *and satisfies* (F_1) *and* (F_2)*. Let* Ω *be a bounded smooth domain in* $\mathbb{R}^N$ *and let* $\phi \in C^1(\bar{\Omega})$ *be nonnegative. Then for all large* $\lambda > 0$*,* (1.2) *has at least one positive solution* $u \in C^1(\bar{\Omega})$. *Moreover, if* u_λ *is an arbitrary positive solution of* (1.2), *then* $u_\lambda \in C^1(\bar{\Omega})$ *and*

$$u_\lambda(x) \to a \quad \text{as } \lambda \to \infty \text{ uniformly on compact subsets of } \Omega.$$

Theorem 1.4. *Suppose that* $f(s)$ *is continuous and locally quasi-monotone on* $[0, \infty)$ *and satisfies* (F_1) *and* (F_3). *Then, for every* $\lambda > 0$, *problem* (1.3) *has a positive solution. Moreover, if* u_λ *denotes an arbitrary nonnegative solution of* (1.3), *then* $u_\lambda(x) \geq a$ *in* Ω *and* $u_\lambda(x) \to a$ *as* $\lambda \to \infty$ *uniformly on compact subsets of* Ω.

To describe our results on flat cores for (1.2) and (1.3), further conditions on f are needed. Suppose that (F_1) is satisfied by f and define $F(s) = \int_0^s f(t)dt$, $\forall s \geq 0$. Then clearly $F(s)$ increases in $[0, a]$ and decreases in $[a, \infty)$ with $F(a) > F(s)$ for $s \neq a$.

Theorem 1.5. *Under the conditions of Theorem* 1.3, *suppose further that*

(F_4) *for some small* $\epsilon > 0$,

$$\int_{a-\epsilon}^{a+\epsilon} [F(a) - F(s)]^{-1/p} ds < \infty.$$

Then given any compact subset K *of* Ω, *one can find a large* $\Lambda > 0$, *such that any positive solution* u_λ *of* (1.2) *satisfies* $u_\lambda \equiv a$ *on* K *provided that* $\lambda \geq \Lambda$.

Theorem 1.6. *Under the conditions of Theorem* 1.4, *suppose further that*

(F_5) *for some small* $\epsilon > 0$,

$$\int_{a}^{a+\epsilon} [F(a) - F(s)]^{-1/p} ds < \infty.$$

Then given any compact subset K *of* Ω, *one can find a large* $\Lambda > 0$, *such that any positive solution* u_λ *of* (1.3) *satisfies* $u_\lambda \equiv a$ *on* K *provided that* $\lambda \geq \Lambda$.

As simple consequences of Theorems 1.3-1.6, we have the following universal results for positive solutions of $-\Delta_p u = \lambda c(x) f(u)$ on an arbitrary bounded domain $\mathcal{D}$, where no condition is imposed upon the solutions at the boundary of $\mathcal{D}$.

Corollary 1.7. *Suppose that* $f(s)$ *is continuous and locally quasi-monotone on* $[0, \infty)$ *and satisfies* $(F_1) - (F_3)$. *Let* $\mathcal{D}$ *be an arbitrary bounded domain in* R^N. *Then for any given* $\epsilon > 0$ *and compact subset* K *of* $\mathcal{D}$, *there exists* $\Lambda > 0$ *such that when* $\lambda > \Lambda$, *any positive solution of* $-\Delta_p u = \lambda c(x) f(u)$ *in*

$\mathcal{D}$ (i.e., $u \in C(\mathcal{D}) \cap W^{1,p}_{loc}(\mathcal{D})$ satisfying (1.4) with Ω replaced by $\mathcal{D}$) satisfies $|u(x) - a| < \epsilon$ for all $x \in K$.

Corollary 1.8. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$ and satisfies $(F_1) - (F_4)$. Let $\mathcal{D}$ be an arbitrary bounded domain in R^N. Then for any give compact subset K of $\mathcal{D}$, there exists $\Lambda > 0$ such that when $\lambda > \Lambda$, any positive solution of $-\Delta_p u = \lambda c(x) f(u)$ in $\mathcal{D}$ satisfies $u \equiv a$ on K.*

Remark 1.9. By the strong maximum principle in [27], condition (F_4) is also necessary for the existence of a flat core in Theorem 1.5. When $p \le 2$, condition (F_4) can never be satisfied if $f(u)$ is locally Lipschitz continuous near $u = a$. For example, if $f(u) = u|a-u|^{q-1}(a-u)$, then (F_4) is satisfied if and only if $q < p-1$. Similar comments also apply to (F_5).

Remark 1.10. There is a great difference between the case where $f(u)$ is first positive and then negative, as discussed in this paper, and the reverse case where $f(u)$ is first negative and then positive. For example, the problem

$$-\Delta u = u - u^q, \quad x \in R^N$$

with $q > 1$ has only constant nonnegative solutions by applying Theorem 1.1, but in contrast, it is well known that the problem

$$-\Delta u = u^q - u, \quad x \in R^N \tag{1.5}$$

with $1 < q < (N+2)/(N-2)$ has a ground state solution (i.e., a positive solution which decays to zero at infinity). Generally speaking, problems of the type (1.5) are more difficult to handle. A remarkable result of Serrin and Zou (see Theorem IV(a) in [30]) shows that for a class of more general equations including (1.5), the nonnegative solutions possess a universal bound.

Remark 1.11. The conjecture in Remark 1.2 (iv) has been confirmed recently by Dancer and Du [6].

The rest of the paper is organized as follows. In Section 2, we prove the weak sweeping principle and make use of it to obtain various estimates for positive solutions of (1.1) and (1.2); Theorem 1.3 will be proved here. Section 3, discusses boundary blow-up solutions and their applications to (1.1), where we generalize the classical result of Keller [22] and prove Theorems 1.1 and 1.4. Section 4, is devoted to the proof of Theorems 1.5 and 1.6; indeed, a better estimate on the location of the flat cores will be provided. The simple proof for Corollaries 1.7 and 1.8 is given at the end of Section 4.

2. Weak sweeping principle and estimate from below

In this section, we first prove a simple yet very useful weak sweeping principle and then use it to obtain a lower bound for solutions to (1.1). We also prove Theorem 1.3.

Lemma 2.1. (Weak sweeping principle) *Suppose that $\mathcal{D}$ is a bounded smooth domain in $\mathbb{R}^N$, $h(x,s)$ is measurable in $x \in \mathcal{D}$, continuous and locally quasi-monotone (uniformly in x) with respect to $s \in (-\infty, \infty)$. Let u and v_t, $t \in [t_1, t_2]$, be functions in $W^{1,p}(\mathcal{D}) \cap C(\overline{\mathcal{D}})$ and satisfy in the weak sense, for some $\epsilon > 0$,*

$$-\Delta_p u \geq h(x,u), \ -\Delta_p v_t \leq h(x, v_t) - \epsilon \ \text{in } \mathcal{D}, \ \forall t \in [t_1, t_2],$$

$$u \geq v_t + \epsilon \ \ \text{on } \partial\mathcal{D}, \ \forall t \in [t_1, t_2].$$

Moreover, suppose that $u \geq v_{t_0}$ in $\mathcal{D}$ for some $t_0 \in [t_1, t_2]$ and $t \to v_t$ is continuous from the finite closed interval $[t_1, t_2]$ to $C(\overline{\mathcal{D}})$. Then

$$u \geq v_t \ \ \text{on } \mathcal{D}, \ \forall t \in [t_1, t_2].$$

Proof. Denote $T = \{t \in [t_1, t_2] : u \geq v_t \text{ on } \mathcal{D}\}$. Clearly $t_0 \in T$ and T is a closed set. We show that T is relatively open in $[t_1, t_2]$, which implies $T = [t_1, t_2]$, as required.

Since v_t varies continuously with t, it is easily seen that there exist finite numbers $s_1 < s_2$ such that $u(x), v_t(x) \in [s_1, s_2]$ for all $x \in \mathcal{D}$ and all $t \in [t_1, t_2]$. Since $h(x,s)$ is locally quasi-monotone in s, we can find a continuous increasing function $L(s)$ such that $\tilde{h}(x,s) := h(x,s) + L(s)$ is nondecreasing in s for all $x \in \mathcal{D}$ and $s \in [s_1, s_2]$.

Let $\delta > 0$ be sufficiently small. Then, for any $t \in T$,

$$-\Delta_p u + L(u) \geq \tilde{h}(x,u) \geq \tilde{h}(x, v_t) \geq -\Delta_p v_t + L(v_t) + \epsilon$$

$$\geq -\Delta_p(v_t + \delta) + L(v_t + \delta) \ \text{ in } \mathcal{D},$$

and

$$u \geq v_t + \delta \ \text{ on } \partial\mathcal{D}.$$

By the weak maximum principle (see, e.g., [10], Theorem 4.9) we obtain $u \geq v_t + \delta$ in $\mathcal{D}$. Thus, for all $s \in [t_1, t_2]$ with $|s - t|$ small, $u \geq v_s$. This shows that T is relatively open in $[t_1, t_2]$. The proof is complete. □

Lemma 2.2. *Suppose f satisfies (F_1) and u is a locally bounded nonnegative solution of (1.1) or (1.2). Then either $u \equiv 0$ or $u > 0$ in the interior of the underlying domain.*

Proof. Define $\tilde{f}$ on $[0,\infty)$ by

$$\tilde{f}(u) = 0 \text{ on } [0,a],\ \tilde{f}(u) = c_2 f(u) \text{ on } [a,\infty).$$

Then any nonnegative solution u of (1.1) or (1.2) satisfies

$$-\Delta_p u \geq \tilde{f}(u).$$

Since u is locally bounded, by standard regularity theory (see [33]) u is C^1 in the interior of the underlying domain. Thus, by the strong maximum principle (see [27] Theorem 1), either $u \equiv 0$ or $u > 0$ in the interior of the underlying domain. □

Lemma 2.2 shows that under condition (F_1) a nonnegative solution of (1.1) or (1.2) is a positive solution unless it is identically zero. Therefore, without loss of generality, we will from now on be only concerned with positive solutions of (1.1) and (1.2). We will always assume that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$ so that $h(x,s) := c(x)f(u)$ meets the basic requirement in Lemma 2.1.

Lemma 2.3. *Let (F_1) be satisfied and let $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be a solution of* (1.1) *satisfying $u(x) \geq \delta$ for all $x \in \mathbb{R}^N$ and some $\delta > 0$. Then $u(x) \geq a$ on $\mathbb{R}^N$.*

Proof. We may assume that $\delta < a$ for otherwise there is nothing to prove.

Let $B_r(x_0)$ denote the open ball in $\mathbb{R}^N$ with center x_0 and radius r. We denote by ϕ the unique solution to

$$-\Delta_p \phi = 1 \text{ in } B_1(0),\ \phi|_{\partial B_1(0)} = 0.$$

It is easily seen that ϕ is radially symmetric, is positive in $B_1(0)$ and $\phi(0) = \|\phi\|_\infty$. For any given $\eta > 0$ such that $2\eta < \min\{a-\delta, \delta\}$, any $x_0 \in \mathbb{R}^N$, any $\lambda > 0$ and any $t \in [t_1, t_2] := [0, (a-\delta)/\phi(0)]$, we define

$$\phi_{\lambda,t}(x) = (\delta - \eta) + t\phi(\lambda^{-1}(x - x_0)),\ x \in B_\lambda(x_0).$$

A simple calculation gives

$$-\Delta_p \phi_{\lambda,t} = t^{p-1}/\lambda^p \leq t_2^{p-1}/\lambda^p \text{ in } B_\lambda(x_0),\ \phi_{\lambda,t}|_{\partial B_\lambda(x_0)} = \delta - \eta,\ \forall t \in [t_1, t_2]. \tag{2.1}$$

By (F_1), $\sigma := c_1 \min_{s\in[\delta-\eta, a-\eta]} f(s) > 0$. Since

$$\delta - \eta \leq \phi_{\lambda,t}(x) \leq a - \eta,\ \forall x \in B_\lambda(x_0),\ \forall t \in [t_1,t_2],\ \forall \lambda > 0,$$

we obtain

$$c(x) f(\phi_{\lambda,t}(x)) \geq \sigma,\ \forall x \in B_\lambda(x_0),\ \forall t \in [t_1,t_2],\ \forall \lambda > 0.$$

Thus, by (2.1), for all $\lambda \geq (2/\sigma)^{1/p} t_2^{1-1/p}$ and $t \in [t_1, t_2]$,

$$-\Delta \phi_{\lambda,t} \leq \sigma/2 \leq c(x) f(\phi_{\lambda,t}) - \sigma/2, \ \forall x \in B_\lambda(x_0).$$

We can now apply Lemma 2.1 with $\mathcal{D} = B_\lambda(x_0)$, $v_t = \phi_{\lambda,t}$ and $\epsilon = \min\{\sigma/2, \eta\}$ to conclude that $u \geq \phi_{\lambda,t}$ in $B_\lambda(x_0)$ for all $t \in [t_1, t_2]$ and all large λ. In particular, $u(x_0) \geq \phi_{\lambda,t_2}(x_0) = a - \eta$. Since $x_0 \in \mathbb{R}^N$ and $\eta \in (0, \min\{a - \delta, \delta\}/2)$ are arbitrary, this implies that $u \geq a$ on $\mathbb{R}^N$. □

Our next result reveals the role that condition (F_2) is played in our analysis.

Proposition 2.4. *Let (F_1) and (F_2) be satisfied and let $u \in W^{1,p}_{loc}(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ be an arbitrary positive solution of* (1.1). *Then $u \geq a$ in $\mathbb{R}^N$.*

Proof. It is well known that for any bounded domain $\mathcal{D}$ of $\mathbb{R}^N$,

$$\lambda_1(\mathcal{D}) := \inf \Big\{ \int_{\mathcal{D}} |Du|^p dx : u \in W_0^{1,p}(\mathcal{D}), \ \int_{\mathcal{D}} |u|^p dx > 0 \Big\}$$

is achieved by some positive function ϕ which satisfies

$$-\Delta_p \phi = \lambda_1(\mathcal{D}) \phi^{p-1} \text{ in } \mathcal{D}, \ \phi|_{\partial \mathcal{D}} = 0.$$

Moreover, such ϕ is unique if we require $\phi(0) = 1$.

Taking $\mathcal{D} = B_1(0)$, then ϕ is radially symmetric and $\phi(0) = \|\phi\|_\infty$. We assume also $\phi(0) = 1$.

For any given $\eta \in (0, a)$, by (F_1) and (F_2), we can find $\sigma_\eta > 0$ such that

$$c_1 f(s) \geq \sigma_\eta s^{p-1}, \ \forall s \in [0, a - \eta]. \tag{2.2}$$

We now fix $\lambda > 0$ large enough so that $\lambda^{-p} \lambda_1(B_1(0)) < \sigma_\eta/2$. For arbitrarily given $x_0 \in \mathbb{R}^N$, since $u > 0$ on $B_\lambda(x_0)$, there exists $\delta \in (0, a - \eta)$ such that $u(x) \geq \delta$, $\forall x \in B_\lambda(x_0)$. We now let $t_1 = \delta$, $t_2 = a - \eta$, and for $t \in [t_1, t_2]$, we define $v_t(x) = t\phi(\lambda^{-1}(x - x_0))$, $x \in B_\lambda(x_0)$. Clearly,

$$0 \leq v_t(x) \leq a - \eta, \ \forall x \in B_\lambda(x_0), \forall t \in [t_1, t_2].$$

Let $\delta_1 \in (0, 1)$ be so small that $t_2 \phi(x) < \delta/2$ whenever $1 \geq |x| \geq 1 - \delta_1$. Then

$$v_t(x) \leq \delta/2, \ \forall x \in \partial B_{\lambda(1-\delta_1)}(x_0), \forall t \in [t_1, t_2].$$

Moreover, by the definition of v_t and (2.2), we obtain, for $t \in [t_1, t_2]$,

$$-\Delta_p v_t = \lambda^{-p} \lambda_1(B_1(0)) v_t^{p-1} \leq (1/2) \sigma_\eta v_t^{p-1}$$

$$\leq \sigma_\eta v_t^{p-1} - \zeta \leq c(x) f(v_t) - \zeta, \ \forall x \in B_{\lambda(1-\delta_1)}(x_0),$$

where $\zeta = \min_{x\in\overline{B_{\lambda(1-\delta_1)}(x_0)}}(1/2)\sigma_\eta v_{t_1}^{p-1} > 0$. Therefore, we can apply Lemma 2.1 with $\mathcal{D} = B_{\lambda(1-\delta_1)}(x_0)$ and $\epsilon = \min\{\delta/2, \zeta\}$ to conclude that

$$u \geq v_t,\ \forall x \in B_{\lambda(1-\delta_1)}(x_0),\ \forall t \in [t_1, t_2].$$

In particular, $u(x_0) \geq v_{t_2}(x_0) = a - \eta$. Since x_0 and η are arbitrary, this implies that $u \geq a$ in $\mathbb{R}^N$. □

We now prove Theorem 1.3 whose statement we repeat below.

Theorem 2.5. *Let (F_1) and (F_2) be satisfied, let Ω be a bounded smooth domain in $\mathbb{R}^N$ and let $\phi \in C^1(\bar{\Omega})$ be nonnegative. Then for all large $\lambda > 0$,* (1.2) *has at least one positive solution $u \in C^1(\bar{\Omega})$. Moreover, if u_λ is an arbitrary positive solution of* (1.2), *then $u_\lambda \in C^1(\bar{\Omega})$ and*

$$u_\lambda(x) \to a \quad \text{as } \lambda \to \infty \text{ uniformly on compact subsets of } \Omega. \tag{2.3}$$

Proof. Let ψ be given by

$$-\Delta_p \psi = \lambda_1(\Omega)\psi^{p-1} \text{ in } \Omega;\ \psi|_{\partial\Omega} = 0,\ \max_{\bar{\Omega}} \psi = 1.$$

For $\xi \in (0, \delta)$, we have $0 \leq \xi\psi(x) < \delta$ in Ω and thus by (F_2),

$$-\Delta_p(\xi\psi) = \lambda_1(\Omega)(\xi\psi)^{p-1} \leq \lambda_1(\Omega)(\sigma c_1)^{-1}c(x)f(\xi\psi),\ \forall x \in \Omega;\ \xi\psi|_{\partial\Omega} = 0.$$

It follows that $\psi_1 := \xi\psi$ is a sub-solution to (1.2) when $\lambda \geq \lambda_1(\Omega)(\sigma c_1)^{-1}$. Clearly, $\psi_2 := \max\{a, \delta, \|\phi\|_\infty\}$ is a super-solution to (1.2). Hence, by the well-known sub- and super-solution method (see [10]), (1.2) has at least one solution satisfying $\psi_1 \leq u \leq \psi_2$ when $\lambda \geq \lambda_1(\Omega)(\sigma c_1)^{-1}$. By standard regularity results ([32, 33]), $u \in C^1(\bar{\Omega})$.

We now set to prove (2.3). We first claim that any positive solution u_λ of (1.2) satisfies

$$u_\lambda \leq M := \max\{a, \|\phi\|_{L^\infty(\partial\Omega)}\}. \tag{2.4}$$

Indeed, by (F_1), we see that $\mu := \max_{s\geq 0} f(s)$ is achieved at some $s_0 \in (0, a)$. Let ϕ_λ denote the unique solution to

$$-\Delta_p \phi_\lambda = \lambda c_2 \mu \ \text{ in } \Omega,\ \phi_\lambda|_{\partial\Omega} = \|\phi\|_{L^\infty(\partial\Omega)}.$$

Then,

$$-\Delta_p \phi_\lambda = \lambda c_2 \mu \geq \lambda c(x) f(u_\lambda) = -\Delta_p u_\lambda,\ \forall x \in \Omega,$$

and $\phi_\lambda \geq \phi$ on $\partial\Omega$. Thus, by the weak maximum principle for the p-Laplacian operator, we have $u_\lambda \leq \phi_\lambda$ in Ω. By standard regularity results, we have $\phi_\lambda \in C^1(\bar{\Omega})$. Thus, u_λ belongs to $L^\infty(\Omega)$ and hence, by standard regularity theory again, $u_\lambda \in C^1(\bar{\Omega})$.

We now use the weak sweeping principle to prove (2.4). Let ψ^* be the unique positive solution to

$$-\Delta_p \psi^* = 1 \text{ in } \Omega,\ \psi^*|_{\partial\Omega} = 0.$$

For any given $\eta > 0$ and $t \geq 0$, define $\psi_t := M + \eta + t\psi^*$, where M is given in (2.4). It is easily seen that there exists $t_2 > 0$ such that $\psi_{t_2} \geq u_\lambda$ on Ω. By (F_1),

$$\delta_\eta := - \max_{s\in[a+\eta,\|\psi_{t_2}\|_\infty]} \lambda c_1 f(s) > 0.$$

Thus, if we let $t_1 = 0$, then for $t \in [t_1, t_2]$,

$$a + \eta \leq \psi_t(x) \leq \|\psi_{t_2}\|_\infty,\ \forall x \in \Omega,$$

$$-\Delta_p \psi_t = t^{p-1} \geq 0 \geq \lambda c(x) f(\psi_t) + \delta_\eta,\ \forall x \in \Omega,$$

$$\psi_t \geq u_\lambda + \eta,\ \forall x \in \partial\Omega.$$

Therefore, we can apply Lemma 2.1 with $h(x, s) = -\lambda c(x) f(-s)$ and $\epsilon = \min\{\delta_\eta, \eta\}$ to $u = -u_\lambda$ and $v_t = -\psi_t$ to conclude that $\psi_t \geq u_\lambda$ on Ω for all $t \in [t_1, t_2]$. In particular, $u_\lambda \leq \psi_{t_1} = M + \eta$ on Ω. Since $\eta > 0$ is arbitrary, this proves (2.4).

Let K be an arbitrary compact subset of Ω and u_λ be an arbitrary positive solution of (1.2). We show next that

$$\overline{\lim}_{\lambda\to\infty} u_\lambda(x) \leq a \text{ uniformly for } x \in K. \tag{2.5}$$

By (2.4), this is not trivial only if $\|\phi\|_{L^\infty(\partial\Omega)} > a$. Hence, we assume this holds from now on, and thus $M = \|\phi\|_{L^\infty(\partial\Omega)} > a$. Choose $r \in (0, d(K, \partial\Omega))$ and let ϕ^0 denote the unique positive solution of

$$-\Delta_p \phi^0 = 1 \text{ in } B_r(0),\ \phi^0|_{\partial B_r(0)} = 0.$$

Clearly ϕ^0 is radially symmetric and $\phi^0(0) = \|\phi^0\|_\infty$.

Now for arbitrary $x_0 \in K$, $\eta \in (0, M - a)$ and $t \geq 0$, we define

$$\phi_t(x) = M + \eta - t\phi^0(x - x_0),\ x \in B_r(x_0).$$

Set $t_1 = 0$ and $t_2 = (M - a)/\phi^0(0)$. Clearly

$$a + \eta \leq \phi_t(x) \leq M + \eta,\ \forall x \in B_r(x_0),\ \forall t \in [t_1, t_2],$$

$$-\Delta_p \phi_t = -t^{p-1} \geq -t_2^{p-1},\ \forall x \in B_r(x_0), \forall t \in [t_1, t_2].$$

As $\zeta_\eta := -\max_{s\in[a+\eta,M+\eta]} c_1 f(s) > 0$, when $\lambda \geq 2t_2^{p-1}/\zeta_\eta$, we have

$$\lambda c(x) f(\phi_t) \leq \lambda c_1 f(\phi_t) \leq -\lambda\zeta_\eta \leq -2t_2^{p-1}, \forall x \in B_r(x_0), \forall t \in [t_1, t_2].$$

Thus, whenever $\lambda \geq 2t_2^{p-1}/\zeta_\eta$,

$$-\Delta_p \phi_t \geq \lambda c(x) f(\phi_t) + t_2^{p-1},\ \forall x \in B_r(x_0),\ \forall t \in [t_1, t_2].$$

By (2.4), we have

$$u_\lambda \leq M \leq \phi_{t_1} \text{ on } \Omega,\ u_\lambda \leq M = \phi_t - \eta \text{ on } \partial B_r(x_0),\ \forall t \in [t_1, t_2].$$

Thus, we can use Lemma 2.1 to conclude that $u_\lambda \leq \phi_t$ on $B_r(x_0)$ for all $t \in [t_1, t_2]$. In particular, $u_\lambda(x_0) \leq \phi_{t_2}(x_0) = a + \eta$ provided that $\lambda \geq 2t_2^{p-1}/\zeta_\eta$. Since $x_0 \in K$ and $\eta \in (0, M-a)$ are arbitrary, this implies (2.5).

It is now clear that to prove (2.3), it suffices to show

$$\underline{\lim}_{\lambda\to\infty} u_\lambda(x) \geq a \text{ uniformly for } x \in K. \tag{2.6}$$

For arbitrarily given $\eta \in (0, a)$, by (F_1) and (F_2), there exists $\sigma_\eta > 0$ such that

$$c_1 f(s) \geq \sigma_\eta s^{p-1},\ \forall s \in [0, a-\eta). \tag{2.7}$$

Fix $\lambda \geq \lambda_\eta := 2\lambda_1(B_r(0))\sigma_\eta^{-1}$ and let x_0 be an arbitrary point in K. Since $u_\lambda(x) > 0$ in the closure of $B_r(x_0)$ and by regularity it is C^1 on $\bar{\Omega}$, there exists $\delta^* \in (0, a-\eta)$ such that $u_\lambda(x) \geq \delta^*$ on $B_r(x_0)$.

Let $v(x)$ be the positive eigenfunction corresponding to $\lambda_1(B_r(0))$ with $v(0) = 1$. It is well-known that v is radially symmetric and $v(0) = \|v\|_\infty$. We now set $t_1 = \delta^*$, $t_2 = a - \eta$ and $v_t(x) = tv(x - x_0)$, $\forall x \in B_r(x_0)$. Clearly

$$0 \leq v_t \leq a - \eta,\ \forall x \in B_r(x_0), \forall t \in [t_1, t_2].$$

Let $\delta_1 \in (0,1)$ be small enough such that $t_2 v(x) \leq \delta^*/2$ when $r \geq |x| \geq r(1-\delta_1)$. Then

$$v_t \leq \delta^*/2,\ \forall x \in \partial B_{r(1-\delta_1)}(x_0), \forall t \in [t_1, t_2].$$

Moreover, by (2.7) and the fact that $\lambda \geq \lambda_\eta$, we deduce

$$-\Delta_p v_t = \lambda_1(B_r(0)) v_t^{p-1} \leq \lambda c(x) f(v_t) - \zeta,\ \forall x \in B_{r(1-\delta_1)}(x_0), \forall t \in [t_1, t_2],$$

where $\zeta := \lambda_1(B_r(0)) \min_{x \in \overline{B_{r(1-\delta_1)}(x_0)}} v_{t_1}^{p-1} > 0$. Thus, we can apply Lemma 2.1 to conclude that

$$u_\lambda \geq v_t,\ \forall x \in B_{r(1-\delta_1)}(x_0), \forall t \in [t_1, t_2].$$

In particular, $u_\lambda(x_0) \geq v_{t_2}(x_0) = a - \eta$ provided that $\lambda \geq \lambda_\eta$. This clearly implies (2.6). The proof of Theorem 2.5 is now complete. □

Remark 2.6. (i) In Theorem 2.5, condition (F_2) cannot be dropped. If

$$\lim_{s\to 0^+} f(s)/s^{p-1} = 0,$$

then (1.2) with $\phi \equiv 0$ may have a positive solution u_λ satisfying $\lim_{\lambda\to\infty} u_\lambda < a$. See [15] for more details.

(ii) If $\phi > 0$ on $\partial\Omega$, then it is easy to see from the maximum principle that $u_\lambda \geq \min\{\min_{\partial\Omega}\phi, a\}$ and hence Theorem 2.5 remains true when condition (F_2) is dropped.

3. Boundary blow-up solutions and estimate from above

In this section, we show how boundary blow-up solutions can be used to guarantee global boundedness of positive solutions of (1.1). Once we know a positive solution of (1.1) is globally bounded, then we can apply the weak sweeping principle to obtain a better bound from above, as the following result shows.

As in the last section, we always assume that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$.

Proposition 3.1. *Let (F_1) be satisfied and u be a globally bounded positive solution of* (1.1). *Then $u \leq a$ in $\mathbb{R}^N$.*

Proof. Let $M > 0$ be such that $u(x) \leq M$ on $\mathbb{R}^N$. We may assume that $M > a$ for otherwise there is nothing to prove.

Let ψ^0 be the unique positive solution of

$$-\Delta_p \psi^0 = 1 \;\text{ in } B_1(0),\; \psi^0|_{\partial B_1(0)} = 0.$$

Then ψ^0 is radially symmetric and $\psi^0(0) = \|\psi^0\|_\infty$. For any given $\eta > 0$, by (F_1),

$$\delta_\eta := -\max_{s\in[a+\eta,M+\eta]} c_1 f(s) > 0. \tag{3.1}$$

For arbitrary $x_0 \in \mathbb{R}^N$, $t \geq 0$ and $\lambda > 0$, we define

$$\psi_{t,\lambda}(x) = M + \eta - t\psi^0(\lambda^{-1}(x - x_0)),\; x \in B_\lambda(x_0).$$

Let $t_1 = 0$ and $t_2 = (M - a)/\psi^0(0)$. Clearly, for $t \in [t_1, t_2]$,

$$a + \eta \leq \psi_{t,\lambda}(x) \leq M + \eta,\; \forall x \in B_\lambda(x_0),$$

and

$$-\Delta_p\psi_{t,\lambda} = -t^{p-1}/\lambda^p \geq -t_2^{p-1}/\lambda^p,\; \forall x \in B_\lambda(x_0).$$

In view of (3.1), it follows that

$$-\Delta_p\psi_{t,\lambda} \geq c(x)f(\psi_{t,\lambda}) + \delta_\eta/2,\; \forall x \in B_\lambda(x_0), \forall t \in [t_1, t_2],$$

provided that $\lambda \geq (2/\delta_\eta)^{1/p} t_2^{1-1/p}$. Clearly,

$$\psi_{t_1,\lambda}(x) = M + \eta \geq u(x),\ \forall x \in B_\lambda(x_0),$$
$$\psi_{t,\lambda}(x) = M + \eta \geq u + \eta,\ \forall x \in \partial B_\lambda(x_0), \forall t \in [t_1, t_2].$$

Thus, we can apply Lemma 2.1 to conclude that $u \leq \psi_{t,\lambda}$ on $B_\lambda(x_0)$ for all $t \in [t_1, t_2]$ provided that $\lambda \geq (2/\delta_\eta)^{1/p} t_2^{1-1/p}$. In particular, $u(x_0) \leq \psi_{t_2,\lambda}(x_0) = a + \eta$. Since $x_0 \in \mathbb{R}^N$ and $\eta > 0$ are arbitrary, this implies that $u(x) \leq a$ on $\mathbb{R}^N$. □

We now consider boundary blow-up solutions. First, we extend the classical results of Keller [22] to p-Laplacian equations. This has essentially been done by Matero [25], however, we will obtain more information than [22] and [25]. We will also tide up some of the results in [25] (see Proposition 3.3 and Remark 3.4 below).

Proposition 3.2. *Suppose that $h(s)$ is a continuous, positive and non-decreasing function for $s \in (-\infty, \infty)$. Moreover,*

$$\int_0^\infty H(t)^{-1/p} dt < \infty,\ \text{where } H(t) = \int_0^t h(s)ds. \tag{3.2}$$

Then the following conclusions hold.

(i) For any $R > 0$, the problem

$$\Delta_p u = h(u) \text{ in } B_R(0),\ u|_{\partial B_R(0)} = \infty \tag{3.3}$$

has a minimal solution $u_R(x)$ and maximal solution $U_R(x)$; moreover, $u_R(x)$ and $U_R(x)$ have the following properties:

(a) *$u_R(x)$ and $U_R(x)$ are radially symmetric:*

$$u_R(x) = u_R(r), U_R(x) = U_R(r),\ r = |x|,$$

(b) *$u_R'(r) > 0$, $U_R'(r) > 0$ when $r \in (0, R)$,*

(c) *$u_R(0)$ and $U_R(0)$ converge to ∞ as $R \to 0$, $u_R(0)$ and $U_R(0)$ converge to $-\infty$ as $R \to \infty$.*

(ii) Let $\mathcal{D}$ be any bounded domain in $\mathbb{R}^N$ and $u \in W^{1,p}(\mathcal{D}) \cap C(\bar{\mathcal{D}})$ satisfy in the weak sense

$$\Delta_p u \geq h(u) \text{ in } \mathcal{D}.$$

Then

$$u(x) \leq u_{d(x,\partial\mathcal{D})}(0),\ \forall x \in \mathcal{D},$$

where $u_R(r)$ is given in (i) above.

(iii) If we further assume that $h(s)$ is locally Lipschitz continuous, then for almost every $R > 0$, (3.3) has a unique solution, and if (3.3) has more

than one solution at some $R_0 > 0$*, then* (3.3) *has a continuum of radially symmetric solutions on* $B_{R_0}(0)$*.*

Proof. Part of our argument below is rather standard, but as it is needed in our maximal and minimal solution analysis, the details are included.

Let α be an arbitrary positive number and denote $B = B_R(0)$ for simplicity. Then due to our assumptions on h, a simple sub- and super solution argument shows that the boundary value problem

$$\Delta_p v = h(v) \text{ in } B,\ v|_{\partial B} = \alpha \tag{3.4}$$

has at least one solution. For example, large positive constants can be used as super-solutions, while for $\zeta > [h(0)]^{1/(p-1)}$ and ψ given by

$$\Delta_p \psi = 1 \text{ in } B,\ \psi|_{\partial B} = 0,$$

$\zeta\psi$ is a sub-solution, due to $\zeta^{p-1} > h(0) \geq h(\zeta\psi(x))$. The solution is unique since $h(s)$ is non-decreasing in s. This implies that the solution must be radially symmetric. We denote this solution by $v_{R,\alpha}$.

If $0 < \alpha' < \alpha$, then $v_{R,\alpha'}$ serves as a sub-solution to equation (3.4) and any large positive constant can be used as a super-solution. It follows that the unique solution $v_{R,\alpha}$ must satisfy $v_{R,\alpha} \geq v_{R,\alpha'}$. That is to say that $v_{R,\alpha}(x)$ is non-decreasing with α.

Write $v_{R,\alpha}(x) = v_{R,\alpha}(r)$ with $r = |x|$, and write this by $v(r)$ when confusion is unlikely caused. By standard regularity theory we know $v'(r)$ exists everywhere and

$$|v'(r)|^{p-2}v'(r) = r^{1-N}\int_0^r s^{N-1}h(v(s))ds, \forall r \in [0, R].$$

It follows immediately that $v'(r) > 0$ for $r \in (0, R]$. Thus, we have

$$\begin{aligned} v'(r)^{p-1} &= r^{1-N}\int_0^r s^{N-1}h(v(s))ds \\ &\leq r^{1-N}h(v(r))\int_0^r s^{N-1}ds = (r/N)h(v(r)),\ \forall r \in [0, R]. \end{aligned}$$

From this we deduce

$$[(v')^{p-1}]' + \frac{N-1}{r}(v')^{p-1} = h(v), \forall r \in (0, R],$$

and using $0 \leq (v')^{p-1} \leq (r/N)h(v)$, we finally obtain

$$(1/N)h(v)v' \leq \frac{p-1}{p}[(v')^p]' \leq h(v)v'.$$

Denoting $\xi = \xi_\alpha = \xi_{R,\alpha} = v_{R,\alpha}(0)$, we obtain

$$(1/N)\int_\xi^{v(r)} h(s)ds \leq \frac{p-1}{p}[v'(r)]^p \leq \int_\xi^{v(r)} h(s)ds.$$

Writing $H(z,\xi) = \frac{p}{p-1}\int_\xi^z h(s)ds$, we deduce

$$\int_\xi^{v(r)} H^{-1/p}(s,\xi)ds \leq r \leq N^{1/p}\int_\xi^{v(r)} H^{-1/p}(s,\xi)ds, \ \forall r \in (0,R].$$

In particular,

$$\int_{\xi_\alpha}^{\alpha} H^{-1/p}(s,\xi_\alpha)ds \leq R \leq N^{1/p}\int_{\xi_\alpha}^{\alpha} H^{-1/p}(s,\xi_\alpha)ds. \tag{3.5}$$

Since $v_{R,\alpha}$ is non-decreasing with α, so is ξ_α, and hence $\xi_\infty := \lim_{\alpha\to\infty}\xi_\alpha$ exists, with the possibility that $\xi_\infty = \infty$. We show next that actually $\xi_\infty < \infty$.

Since $h(s)$ is non-decreasing in s, for any $\eta > 0$,

$$H(\eta+t,\eta) = \frac{p}{p-1}\int_\eta^{\eta+t} h(s)ds \geq H(t,0), \ \forall t \geq 0,$$

$$H(\eta+t,\eta) = \frac{p}{p-1}\int_\eta^{\eta+t} h(s)ds \geq \frac{p}{p-1}h(\eta)t, \ \forall t \geq 0.$$

Therefore, for any $M > 0$,

$$\begin{aligned}
&\int_\eta^\infty H^{-1/p}(s,\eta)ds = \int_0^\infty H^{-1/p}(\eta+s,\eta)ds\\
&= \int_0^M H^{-1/p}(\eta+s,\eta)ds + \int_M^\infty H^{-1/p}(\eta+s,\eta)ds\\
&\leq \int_0^M \big[\frac{p}{p-1}h(\eta)\big]^{-1/p} s^{-1/p}ds + \int_M^\infty H^{-1/p}(s,0)ds\\
&= \big[\frac{p}{p-1}h(\eta)\big]^{-1/p}\frac{M^{1-1/p}}{1-1/p} + \int_M^\infty H^{-1/p}(s,0)ds.
\end{aligned}$$

Our assumption (3.2) and the monotonicity of $h(s)$ imply that $h(s) \to \infty$ as $s \to \infty$. Therefore,

$$\overline{\lim}_{\eta\to\infty}\int_\eta^\infty H^{-1/p}(s,\eta)ds \leq \int_M^\infty H^{-1/p}(s,0)ds.$$

Due to (3.2), sending $M \to \infty$ we obtain

$$\lim_{\eta\to\infty}\int_\eta^\infty H^{-1/p}(s,\eta)ds = 0. \tag{3.6}$$

By (3.5),

$$\int_{\xi_\alpha}^{\infty} H^{-1/p}(s,\xi_\alpha)ds \geq RN^{-1/p} > 0,\ \forall \alpha > 0.$$

Hence, we must have $\xi_\infty < \infty$. Moreover, letting $\alpha \to \infty$ in (3.5), we deduce

$$\int_{\xi_\infty}^{\infty} H^{-1/p}(s,\xi_\infty)ds \leq R \leq N^{1/p}\int_{\xi_\infty}^{\infty} H^{-1/p}(s,\xi_\infty)ds. \tag{3.7}$$

To stress the dependence of ξ_∞ on R, we denote $\xi_\infty = \xi(R)$. From (3.6) and (3.7), we find that $\xi(R) \to \infty$ as $R \to 0$. We show next that $\xi(R) \to -\infty$ as $R \to \infty$. Otherwise, we can find $R_n \to \infty$ such that $\xi(R_n) \geq M > -\infty$ for some constant M and all $n \geq 1$. From (3.6) and (3.7) we know that $\{\xi(R_n)\}$ must be bounded from above. Thus, by passing to a subsequence we may assume that $\xi(R_n) \to \xi^*$ as $n \to \infty$. But then, by (3.7) and (3.2),

$$\infty = \lim_{n\to\infty} R_n \leq N^{1/p}\int_{\xi^*}^{\infty} H^{-1/p}(s,\xi^*)ds < \infty.$$

This contradiction proves that $\xi(R) \to -\infty$ as $R \to \infty$.

Consider now u satisfying $\Delta_p u \geq h(u)$ in $\mathcal{D}$. For any given $x_0 \in \mathcal{D}$, denote $R = d(x_0, \partial\mathcal{D})$ and consider $v_{R,\alpha}(x - x_0)$, which satisfies

$$\Delta_p v = h(v) \text{ in } B_R(x_0),\ v|_{\partial B_R(x_0)} = \alpha.$$

If $\alpha \geq \|u\|_{L^\infty(\mathcal{D})}$, then clearly u is a sub-solution of the above problem and any large constant is a super-solution. Hence, $u(x) \leq v_{R,\alpha}(x-x_0)$ on $B_R(x_0)$. In particular, $u(x_0) \leq v_{R,\alpha}(0) = \xi_{R,\alpha}$ for all large α. Letting $\alpha \to \infty$ we obtain $u(x_0) \leq \xi(R) = \xi(d(x_0, \partial\mathcal{D}))$.

We now come back to analyze the unique solution $v_\alpha = v_{R,\alpha}$ of (3.4). We already know that v_α is non-decreasing in α. By what have been proved above, for any $R_0 \in (0, R)$, $v_\alpha(x) \leq \xi(R - R_0) < \infty$ for all $\alpha > 0$ and $x \in B_{R_0}(0)$. It follows that $u_R(x) := \lim_{\alpha\to\infty} v_{R,\alpha}(x)$ is well-defined. Moreover, by standard regularity theory, u_R is a solution to (3.3).

Since each v_α is radially symmetric, so is u_R. Using (3.3) and $h(s) > 0$ we also deduce $u_R'(r) > 0$ for $r \in (0, R)$. Clearly $\xi(R) = u_R(0)$.

We show next that u_R is the minimal solution to (3.3). Let u be an arbitrary solution to (3.3). Then for each $\alpha > 0$ we can find $R_\alpha \in (0, R)$ such that $v_{R,\alpha}(x) < u(x)$ when $|x| \in (R_\alpha, R)$ due to the behavior of u near $\partial B_R(0)$. Since $h(s)$ is non-decreasing with s, this implies, by the weak maximum principle, $v_{R,\alpha}(x) \leq u(x)$ in $B_R(0)$. It follows that $u_R(x) = \lim_{\alpha\to\infty} v_{R,\alpha}(x) \leq u(x)$ in $B_R(0)$. Therefore, u_R is the minimal solution to (3.3).

To prove that (3.3) has a maximal solution, we first observe that $u_R(x)$ is non-increasing with R, that is, $R_1 < R_2$ implies $u_{R_1}(x) \geq u_{R_2}(x)$ for all $x \in B_{R_1}(0)$. Indeed, let u be any solution of (3.3) with $R = R_2$, then we can find $R_0 \in (0, R_1)$ such that $u(x) \leq u_{R_1}(x)$ when $R_0 < |x| < R_1$. Thus, as before the weak maximum principle implies $u(x) \leq u_{R_1}(x)$ on $B_{R_1}(0)$. In particular $u_{R_2}(x) \leq u_{R_1}(x)$ in $B_{R_1}(0)$, as claimed. Thus,

$$U_R(x) = \lim_{R' \to R-0} u_{R'}(x), \; x \in B_R(0) \tag{3.8}$$

is well-defined and $U_R(x) \geq u(x)$ in $B_R(0)$ for any solution u of (3.3). Moreover, a standard regularity consideration shows that U_R is a solution to (3.3), hence the maximal solution. $U_R(x)$ is radially symmetric because it is the limit of such functions. $U_R(x)$ is non-increasing with R, as $R_1 < R_2$ implies

$$U_{R_1}(x) = \lim_{R' \to R_1 - 0} u_{R'}(x) \geq \lim_{R' \to R_1 - 0} u_{R' + (R_2 - R_1)}(x) = U_{R_2}(x), \; \forall x \in B_{R_1}(0).$$

A similar consideration shows that $\lim_{R' \to R+0} U_{R'}(x)$ is a minimal solution to (3.3) and thus necessarily $u_R(x) = \lim_{R' \to R+0} U_{R'}(x)$. Analogously, $\lim_{R' \to R+0} u_{R'}(x)$ is a minimal solution of (3.3) and $\lim_{R' \to R-0} U_{R'}(x)$ is a maximal solution of (3.3). Hence,

$$\lim_{R' \to R+0} u_{R'}(x) = u_R(x), \; \lim_{R' \to R-0} U_{R'}(x) = U_R(x). \tag{3.9}$$

As $U_R(x)$ is radially symmetric, from the ordinary differential equation we find that $U_R'(r) > 0$ when $r \in (0, R)$. Moreover, it is easily seen that (3.7) holds when we replace ξ_∞ by $U_R(0)$. Therefore, the conclusion that $U_R(0) \to \infty$ as $R \to 0$ and $U_R(0) \to -\infty$ as $R \to \infty$ follows from the same argument as that for $u_R(0)$. This completes our proof for conclusions (i) and (ii) in the proposition.

To prove (iii), we assume further that $h(s)$ is locally Lipschitz continuous. By Proposition A4 in [13], for any $\xi \in (-\infty, \infty)$, the initial value problem

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' - h(u) = 0, \; u(0) = \xi, u'(0) = 0 \tag{3.10}$$

has a unique solution as long as the solution exists. Thus, whenever $u_R(0) = U_R(0)$, we must have $u_R(r) \equiv U_R(r)$ which implies that (3.3) has a unique solution. By (3.8) and (3.9), we find that $u_{R_0}(0) = U_{R_0}(0)$ whenever $u_R(0)$ is continuous at R_0. As $R \to u_R(0)$ is a non-increasing function (actually it is strictly decreasing by the uniqueness result of [13]), it is continuous almost everywhere. Hence, (3.3) has a unique solution for almost every $R > 0$.

If (3.3) has more than one solution for some $R = R_0$, then by what has just been proved, we must have $U_{R_0}(0) > u_{R_0}(0)$. Consider now the initial value problem (3.10) with $\xi \in (u_{R_0}(0), U_{R_0}(0))$. It has a unique solution $v_\xi(r)$ defined in some small interval $[0, r_0)$. Since $h(s)$ is non-decreasing, $v_\xi(r)$ must stay between $u_{R_0}(r)$ and $U_{R_0}(r)$ and therefore it is defined in $[0, R_0)$ and thus solves (3.3). Hence, (3.3) has a continuum of radially symmetric solutions. The proof is now complete. □

Proposition 3.2 can be used to deduce various existence results for boundary blow-up problems on arbitrary bounded domains. When the bounded domain has smooth boundary, the blow-up rate of the boundary blow-up solutions can sometimes be obtained. The following result is due to Matero [25], Theorems 3.3 and 4.4.

Proposition 3.3. *Suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary, and $h(s)$ is as in Proposition* 3.2. *Then the boundary blow-up problem*

$$\Delta_p u = h(u) \text{ in } \Omega,\ u|_{\partial\Omega} = \infty$$

has at least one solution. Moreover, any solution u of this problem satisfies

$$\lim_{d(x,\partial\Omega)\to 0} \frac{\Psi(u(x))}{d(x,\partial\Omega)} = 1, \tag{3.11}$$

where $\Psi(t)$ is given by

$$\Psi(t) = \Big(\frac{p}{p-1}\Big)^{-1/p} \int_t^\infty H(s)^{-1/p} ds,\ \forall t > 0.$$

Remark 3.4. (i) Theorem 3.3 in [25] as stated there is not quite correct. In [25], the function $h(s)$ is only defined on $R^+ = [0, \infty)$ and condition (A_1) there assumes that $h(s)$ is continuous, positive and non-decreasing on R^+. We now show that, contrary to what is implied by Theorem 3.3 of [25], this condition and (3.2), which is exactly condition (A_2) in [25], do not guarantee the existence of a nonnegative boundary blow-up solution. Indeed, by our Proposition 3.2 above, if R is large enough, then the maximal solution U_R of (3.3) satisfies $U_R(0) < 0$, and hence (3.3) cannot have a solution which is nonnegative.

(ii) This oversight in [25] cannot be fixed by simply assuming that $h(s)$ is also defined for $s < 0$. Theorem 4 in [1] shows that for any bounded C^2 convex domain Ω, one can find $h(s)$ such that h is continuous and positive on $(-\infty, \infty)$ and satisfies conditions (A_1) and (A_2) of [25] in $[0, \infty)$ but $\Delta u = h(u)$ has no boundary blow-up solution on Ω.

(iii) If (A_1) in [25] is strengthened to: $h(s)$ is continuous, positive and non-decreasing in $(-\infty, \infty)$ (as in Proposition 3.3 above), or it is replaced by: $h(s_0) = 0$ for some $s_0 > -\infty$ and $h(s)$ is continuous, positive and non-decreasing in (s_0, ∞), then Theorem 3.3 in [25] (and hence Proposition 3.3 above) holds.

We now come back to problems (1.1) and (1.3).

Proposition 3.5. *Suppose that $f(s)$ satisfies (F_1) and (F_3). Then any positive solution of* (1.1) *is globally bounded.*

Proof. Define $h(s) = -c_1 g(s)$ for $s \geq M$ and $h(s) = -c_1 g(M)$ for $s < M$. Then $h(s)$ satisfies all the conditions of Proposition 3.2. Therefore, we can find $R > 0$ such that problem (3.3) has a minimal solution v satisfying $v(x) \geq v(0) > M$ in $B_R(0)$. Note that $v \in C^1(B_R(0))$ by standard regularity theory. Let u be a positive solution of (1.1). We show that $u(x) \leq v(0)$ in $\mathbb{R}^N$. Otherwise, we can find $x_0 \in \mathbb{R}^N$ such that $u(x_0) > v(0)$. Then letting $v_0(x) = v(x - x_0)$, we find $v_0 \in C^1(B_R(x_0))$,

$$-\Delta_p v_0 = -h(v_0) = c_1 g(v_0) \text{ in } B_R(x_0),\ v_0|_{\partial B_R(x_0)} = \infty,$$

and $u(x_0) > v_0(x_0)$. Let $\mathcal{D}$ be a component of the set $\{x \in B_R(x_0) : u(x) > v_0(x)\}$. We find $\bar{\mathcal{D}} \in B_R(x_0)$, $u(x) > v_0(x)$ in $\mathcal{D}$ and $u = v_0$ on $\partial\mathcal{D}$. Thus,

$$-\Delta_p u = c(x) f(u) \leq c_1 f(u) \leq c_1 g(u),\ -\Delta_p v_0 = c_1 g(v_0) \text{ in } \mathcal{D},$$

$$u = v_0 \text{ on } \partial\mathcal{D}.$$

Since $g(s)$ is non-increasing in s, by the weak maximum principle, we deduce $u \leq v_0$ in $\mathcal{D}$. This contradiction completes our proof. □

Clearly, Theorem 1.1 is a direct consequence of Lemma 2.2 and Propositions 2.4, 3.1 and 3.5. To see that condition (F_2) can be dropped from Theorem 1.1 in the case $N \leq p$, as indicated in Remark 1.2 (iv), we first observe that by Propositions 3.1 and 3.5, under the conditions (F_1) and (F_3), any nonnegative solution u of (1.1) satisfies $0 \leq u \leq a$. Therefore, $-\Delta_p u \geq 0$ in $\mathbb{R}^N$. Since $N \leq p$, by Theorem II (a) of [30], u must be a constant.

Next we construct the counter-example mentioned in Remark 1.2(v). We first prove the following result.

Proposition 3.6. *Suppose that $g(s)$ is positive and continuous on $[s_0, \infty)$. If*

$$\int_{s_0}^{\infty} G(t)^{-1/p} dt = \infty,\ \text{where } G(t) = \int_{s_0}^{t} g(s) ds, \tag{3.12}$$

then for any $\xi > s_0$, the problem

$$\Delta_p u = g(u),\ x \in \mathbb{R}^N$$

has a radially symmetric solution $u(|x|)$ satisfying $u(0) = \xi$, $u'(r) > 0$ for $r > 0$ and $u(r) \to \infty$ as $r \to \infty$.

Proof. By Proposition A1 in [13], for any $\xi > s_0$, the initial value problem (3.10) with $h(u)$ replaced by $g(u)$ has a solution $u(r)$ which exists as long as the solution remains bounded.

The analysis in the proof of Proposition 3.2 which leads to (3.5) now gives

$$\int_{\xi}^{u(r)} G^{-1/p}(s,\xi)ds \le r \le N^{1/p} \int_{\xi}^{u(r)} G^{-1/p}(s,\xi)ds, \ G(s,\xi) = G(s) - G(\xi),$$

for all those $r > 0$ such that $u(r)$ is defined. By (3.12), this inequality indicates that $u(r)$ is finite if and only if r is finite. Therefore, $u(r)$ is defined for all $r > 0$ and $u(r) \to \infty$ as $r \to \infty$. Since $g(s) > 0$ for $s \ge s_0$, we find $u'(r) > 0$ by using

$$|u'(r)|^{p-2}u'(r) = r^{1-N} \int_0^r s^{N-1} g(u(s))ds.$$

The proof is complete. □

Now, if we take $s_0 > 0$ in Proposition 3.6, let $f(s) = -g(s)$ for $s \ge s_0$ and extend $f(s)$ to $s < s_0$ such that (F_1) and (F_2) are satisfied, then only (F_3) is violated by f, yet the problem

$$-\Delta_p u = f(u)$$

has infinitely many radially symmetric solutions given by Proposition 3.6. This gives the counter-example.

Let us now prove Theorem 1.4 which we restate below.

Theorem 3.7. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$ and satisfies (F_1) and (F_3). Then, for every $\lambda > 0$, problem (1.3) has a positive solution. Moreover, if u_λ denotes an arbitrary nonnegative solution of (1.3), then $u_\lambda(x) \ge a$ in Ω and*

$$u_\lambda(x) \to a \text{ as } \lambda \to \infty \text{ uniformly on compact subsets of } \Omega. \tag{3.13}$$

Proof. Let $\lambda > 0$. For any positive integer $k > a$, the problem

$$-\Delta_p u = \lambda c(x) f(u) \text{ in } \Omega, u|_{\partial\Omega} = k \tag{3.14}$$

has a solution $u_k \ge a$. This follows from a simple sub- and super-solution argument by using a and k, respectively, as sub- and super-solutions. By standard regularity theory, $u_k \in C^1(\bar{\Omega})$.

We show next that (3.14) has a maximal solution. To this end, we use the assumption that $f(s)$ is locally quasi-monotone and consider the following iteration process:

$$-\Delta_p w_n + \lambda c(x) L(w_n) = \lambda c(x)[f(w_{n-1}) + L(w_{n-1})] \text{ in } \Omega,\ w_n|_{\partial\Omega} = k,$$

$n = 1, 2, ...,$ where $w_0 = k$ and $f(s) + L(s)$ is non-decreasing in $[a, k]$. Given w_{n-1} satisfying $k \geq w_{n-1} \geq a$, the existence of w_n follows from a sub- and super-solution argument, with a and k, respectively, as the sub- and super-solutions. Since $L(s)$ is increasing in s, such w_n is unique. By the weak maximum principle and an induction argument, it is easily seen that

$$k = w_0 \geq w_1 \geq \cdots \geq w_{n-1} \geq w_n \geq \cdots \geq a \text{ in } \Omega.$$

Hence, $w := \lim_{k\to\infty} w_k$ exists. Moreover, a standard regularity consideration shows that w is a solution of (3.14).

We show that w is the maximal solution to (3.14). Let u be an arbitrary solution of (3.14). We first observe that $u \geq a$ in Ω. Otherwise, $\alpha := \min_{x\in\Omega} u(x) < a$ and we can find a ball $B_r(x_0)$ lying in Ω such that $u < a$ in $B_r(x_0)$, $u(x_0) = \alpha$ and $u(x_1) = a$ for some $x_1 \in \partial B_r(x_0)$. Then $v := u - \alpha$ satisfies

$$-\Delta_p v = \lambda c(x) f(u) \geq 0 \text{ in } B_r(x_0).$$

By the strong maximum principle ([27]), it follows from $v(x_0) = 0$ that $v \equiv 0$ in $B_r(x_0)$, that is, $u \equiv \alpha$ in $B_r(x_0)$. This is in contradiction to $u(x_1) = a$. Thus, we have proved that $u \geq a$ in Ω.

We claim next that $u \leq k$ in Ω. Given $\eta > 0$, we define $v_t = t$ for $t \in [t_1, t_2] := [k + \eta, k + \eta + \|u\|_{L^\infty(\Omega)}]$. Clearly, $v_t \geq u + \eta$ on $\partial\Omega$ for all $t \in [t_1, t_2]$. Moreover,

$$-\Delta_p v_t = 0 \geq \lambda c(x) f(v_t) + \epsilon \text{ in } \Omega,\ \forall t \in [t_1, t_2],$$

where $\epsilon := \lambda c_1 \min_{s\in[t_1,t_2]}[-f(s)] > 0$. Hence, we can apply Lemma 2.1 to conclude that $u \leq v_{t_1} = k + \eta$ in Ω. As $\eta > 0$ is arbitrary, it follows that $u \leq k$ in Ω. As $a \leq u \leq k$, and

$$-\Delta_p u + \lambda c(x) L(u) = \lambda c(x)[f(u) + L(u)] \text{ in } \Omega,\ u|_{\partial\Omega} = k,$$

from the weak maximum principle we deduce, by induction, $w_n \geq u$ for all $n \geq 1$. Hence, $w \geq u$. This shows that w is the maximal solution of (3.14).

We assume from now on that u_k is the maximal solution of (3.14). Since u_{k-1} is a sub-solution to (3.14) and any large constant M is a super-solution, (3.14) has a solution satisfying $u_{k-1} \leq u \leq M$. Therefore, its maximal solution u_k satisfies $u_k \geq u \geq u_{k-1}$. An application of Proposition 3.2 in the spirit of the proof of Proposition 3.5 shows that $\{u_k(x)\}$ is bounded from above for x in compact subsets of Ω. Hence, $u(x) := \lim_{k\to\infty} u_k(x)$ exists. A standard regularity consideration shows that u is a solution of (1.3).

We observe that any nonnegative solution u_λ of (1.3) satisfies $u_\lambda \geq a$ in Ω, as the argument which proves this property for solutions of (3.14) above works as well for u_λ.

It remains to prove (3.13). Let K be an arbitrary compact subset of Ω, and u_λ an arbitrary nonnegative solution of (1.3). We already have $u_\lambda \geq a$ in Ω.

Let $\delta > 0$ be small enough so that $K \subset \Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) > \delta\}$. Replacing Ω by Ω_δ in (3.14) we find, by the same argument, that for any $\lambda > 0$ and $k > a$, there is a maximal solution $w_{k,\lambda}$ of

$$-\Delta_p w = \lambda c(x) f(w) \text{ in } \Omega_\delta,\ w|_{\partial\Omega_\delta} = k. \tag{3.15}$$

Moreover, $w_{k,\lambda}$ is non-decreasing with k, $a \leq w_{k,\lambda} \leq k$ in Ω_δ, $w_\lambda := \lim_{k\to\infty} w_{k,\lambda}$ exists and solves

$$-\Delta_p w = \lambda c(x) f(w) \text{ in } \Omega_\delta,\ w|_{\partial\Omega_\delta} = \infty.$$

Clearly, $w_\lambda \geq w_{k,\lambda}$ in Ω_δ for all $k > a$.

For every large enough k such that $u_\lambda < k$ on $\partial\Omega_\delta$, u_λ is a sub-solution to (3.15). As any large constant is a super-solution, we conclude that the maximal solution $w_{k,\lambda}$ of (3.15) satisfies $w_{k,\lambda} \geq u_\lambda$ in Ω_δ. It follows that $w_\lambda \geq u_\lambda$ in Ω_δ, $\forall \lambda > 0$.

We claim that w_λ is non-increasing with λ. By the definition of w_λ, it suffices to prove this property for every $w_{k,\lambda}$. Since $w_{k,\lambda} \geq a$ in Ω_δ, we find that $\lambda c(x) f(w_{k,\lambda}) \leq \lambda' c(x) f(w_{k,\lambda})$ in Ω_δ whenever $0 < \lambda' < \lambda$. This implies that $w_{k,\lambda}$ is a sub-solution for the equation satisfied by $w_{k,\lambda'}$. From this, it is easily seen that $w_{k,\lambda'} \geq w_{k,\lambda}$ in Ω_δ. Thus, every $w_{k,\lambda}$ is non-increasing with λ, as we wanted. For fixed $\lambda_0 > 0$, we now have

$$u_\lambda \leq w_\lambda \leq w_{\lambda_0},\ \forall x \in \Omega_\delta, \forall \lambda > \lambda_0.$$

We choose $\delta_1 > \delta$ but very close to δ so that $K \subset \Omega_{\delta_1}$. Then we can find $k > a$ large enough such that $w_{\lambda_0} < k$ on $\partial\Omega_{\delta_1}$. For $\lambda > \lambda_0$, replacing Ω by Ω_{δ_1} in (3.14), we know that there is a maximal solution v_λ to

$$-\Delta_p v = \lambda c(x) f(v) \text{ in } \Omega_{\delta_1},\ v|_{\partial\Omega_{\delta_1}} = k.$$

Since $u_\lambda \leq w_\lambda \leq w_{\lambda_0} < k$ on $\partial\Omega_{\delta_1}$, u_λ is a sub-solution to the above equation for v_λ. From this one deduces $u_\lambda \leq v_\lambda$ in Ω_{δ_1}, $\forall \lambda > \lambda_0$.

Since $v_\lambda \geq u_\lambda \geq a$ on Ω_{δ_1}, by (2.5) in the proof of Theorem 2.5 (with Ω replaced by Ω_{δ_1}), where condition (F_2) is not needed, we find that $v_\lambda \to a$ uniformly in K as $\lambda \to \infty$. In view of $a \leq u_\lambda \leq v_\lambda$ in $\Omega_{\delta_1} \supset K$, (3.13) now follows immediately. □

4. Flat core analysis

The main purpose of this section is to prove Theorems 1.5 and 1.6. The proof of Theorem 1.6 will largely be reduced to an application of Theorem 1.5. The strategy for the proof of Theorem 1.5 is the following.

Corresponding to (1.2) we consider two auxiliary problems:

$$-\Delta_p u = \lambda c(x) f(u) \text{ in } \Omega,\ u|_{\partial\Omega} = 0, \tag{4.1}$$

and

$$-\Delta_p u = \lambda c(x) f(u) \text{ in } \Omega,\ u|_{\partial\Omega} = \alpha, \tag{4.2}$$

where $\alpha > \max\{a, \|\phi\|_{L^\infty(\Omega)}\}$.

For large λ, problem (4.1) has a minimal positive solution, say v_λ, and (4.2) has a maximal solution, say w_λ. We will show that v_λ and w_λ both have flat cores for large λ, and the flat cores $\{v_\lambda = a\}$ and $\{w_\lambda = a\}$ enlarge to Ω as $\lambda \to \infty$. As any positive solution u_λ of (1.2) must satisfy $v_\lambda \le u_\lambda \le w_\lambda$, the conclusion in Theorem 1.5 then follows.

Lemma 4.1. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$ and satisfies (F_1). Then*

(i) *for any $\alpha > a$ and any $\lambda > 0$, problem (4.2) has a maximal positive solution w_λ, and it satisfies $a \le w_\lambda \le \alpha$ in Ω;*
(ii) *under the further assumption that f satisfies (F_2), for all large $\lambda > 0$, problem (4.1) has a minimal positive solution v_λ, and it satisfies $0 \le v_\lambda \le a$ in Ω.*

Proof. Part (i) follows from the same argument involving the iteration process used in the proof of Theorem 3.7. We now prove part (ii).

As in the proof of Theorem 2.5, we let ψ denote the normalized (in the L^∞-norm) positive eigenfunction corresponding to $\lambda_1(\Omega)$. We fix $\xi \in (0,\delta)$ and show that $u \ge \xi\psi$ in Ω for any positive solution u of (4.1), provided that $\lambda > \lambda_1(\Omega)(\sigma c_1)^{-1}$, where δ and σ are given in (F_2).

Otherwise, for some $\lambda > \lambda_1(\Omega)(\sigma c_1)^{-1}$, (4.1) has a solution u satisfying $u(x_0) < \xi\psi(x_0)$ at some $x_0 \in \Omega$. Denote by $\mathcal{D}$ the component of the set $\{x \in \Omega : u(x) < \xi\psi(x)\}$ that contains x_0. We find that $u(x) < \xi\psi(x)$ in $\mathcal{D}$ and $u(x) = \xi\psi(x)$ on $\partial\mathcal{D}$. Thus, by (F_2),

$$-\Delta_p u = \lambda c(x) f(u) \ge \lambda c_1 \sigma u^{p-1} > \lambda_1(\Omega) u^{p-1} \text{ in } \mathcal{D}.$$

Since $-\Delta_p(\xi\psi) = \lambda_1(\Omega)(\xi\psi)^{p-1}$ in $\mathcal{D}$ and $\lambda]1(\Omega) \le \lambda_1(\mathcal{D})$, we can now apply Corollary 2.4 of [12] to conclude that either $u \ge \xi\psi$ in $\mathcal{D}$ or $\lambda_1(\Omega) = \lambda_1(\mathcal{D})$ and both $\xi\psi$ and u are eigenfunctions corresponding to $\lambda_1(\mathcal{D})$. The first alternative is not possible due to $u < \xi\psi$ in $\mathcal{D}$. Since $-\Delta_p u > \lambda_1(\Omega)u^{p-1}$ in $\mathcal{D}$, the second alternative is also impossible. Therefore, we must have $u \geq \xi\psi$ in Ω, as we wanted.

For fixed $\lambda > \lambda_1(\Omega)(\sigma c_1)^{-1}$, it is easily checked that $\xi\psi$ is a sub-solution of (4.1). Let $L(s)$ be an increasing function such that $f(s) + L(s)$ is non-decreasing over $[0, a]$, and then consider the iteration process:

$$-\Delta_p u_n + \lambda c(x)L(u_n) = \lambda c(x)[f(u_{n-1}) + L(u_{n-1})] \text{ in } \Omega,\ u_n|_{\partial\Omega} = 0,$$

$n = 1, 2, ...,$ where $u_0 = \xi\psi$. Given u_{n-1} satisfying $\xi\psi \leq u_{n-1} \leq a$, the existence of u_n follows from a sub- and super-solution argument, with $\xi\psi$ and a, respectively, as the sub- and super-solutions. Since $L(s)$ is increasing in s, such u_n is unique. By the weak maximum principle and an induction argument, it is easily seen that

$$\xi\psi = u_0 \leq u_1 \leq ... \leq u_{n-1} \leq u_n \leq ... \leq a \text{ in } \Omega.$$

Hence, $u^* := \lim_{n\to\infty} u_n$ exists. Moreover, a standard regularity consideration shows that u^* is a solution of (4.1). Clearly, $0 \leq u^* \leq a$ in Ω.

We claim that u^* is the minimal positive solution of (4.1). Indeed, let u be an arbitrary positive solution of (4.1). By what was already proved, we know $u \geq \xi\psi = u_0$ in Ω. By a simple induction argument we deduce from $u_0 \leq u$ that $u_n \leq u$ for all n. Hence, $u^* \leq u$ in Ω, that is, u^* is the minimal solution of (4.1). □

Remark 4.2. (i) It is easily seen from the proof of Lemma 4.1 that its conclusions remain valid in dimension 1, that is, when Ω is an interval.

(ii) A simple variant of the proof of Lemma 4.1 shows that (4.2) has a minimal solution among those satisfying $u \geq a$ in Ω, and for all large λ, (4.1) has a maximal solution among those satisfying $u \leq a$ in Ω.

Suppose that $f(s)$ satisfies (F_1) and (F_5), and $\alpha > a$. We now consider the function

$$l(\tau) := \int_\tau^\alpha [F(\tau) - F(s)]^{-1/p} ds,\ \tau \in (a, \alpha).$$

Clearly, $l(\tau)$ is a continuous function on (a, α). By (F_5),

$$l(a) := \lim_{\tau\to a} l(\tau) = \int_a^\alpha [F(a) - F(s)]^{-1/p} ds < \infty.$$

For $s \in (\tau, \alpha)$, as $\tau \to \alpha$, $F(\tau) - F(s) = [|f(\alpha)| + o(1)](s - \tau)$. Thus, as $\tau \to \alpha$,

$$l(\tau) = \int_\tau^\alpha [F(\tau) - F(s)]^{-1/p} ds = [|f(\alpha)| + o(1)]^{-1/p} \int_\tau^\alpha (s-\tau)^{-1/p} ds \to 0.$$

It follows that $\gamma^* := \sup_{\tau \in (a,\alpha)} l(\tau) < \infty$.

To analyze the flat core for w_λ, we consider several special cases of (4.2). We start with the following one dimensional problem

$$-(|u'|^{p-2}u')' = \lambda f(u), \quad 0 < x < \ell, \quad u(0) = u(\ell) = \alpha, \tag{4.3}$$

where $\ell > 0$ is independent of λ.

Lemma 4.3. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0, \infty)$, and satisfies (F_1) and (F_5). Then for $\lambda > \frac{p-1}{p}(2\gamma^*/\ell)^p$, (4.3) has a unique positive solution u_λ, and $\mathcal{E}_\lambda = \{x \in (0, \ell) : v_\lambda(x) = a\} = [d(\lambda), \ell - d(\lambda)]$, where $d(\lambda) = \lambda^{-1/p}(1 - 1/p)^{1/p} l(a)$.*

Proof. By Lemma 4.1, we know that (4.3) always has a positive solution. Moreover, the argument in the proof of Theorem 3.7 to show that any solution of (3.14) is bounded from below by a works as well for (4.3) and hence any positive solution of (4.3) is bounded from below by a. We claim that if $u_\lambda \in C^1[0, \ell]$ is a positive solution of (4.3), then u_λ is symmetric about $x = \ell/2$. In fact, the first integral of (4.3) gives

$$|u_\lambda'(x)|^p + p'\lambda F(u_\lambda(x)) = C, \quad \forall x \in [0, \ell], \tag{4.4}$$

where p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. Let $\tau_\lambda = \inf_{0<x<\ell} u_\lambda(x)$. Then $\tau_\lambda \geq a$ and it follows from (4.4) that

$$|u_\lambda'|^p = p'\lambda(F(\tau_\lambda) - F(u_\lambda)). \tag{4.5}$$

On the other hand, we easily see from (4.4) that τ_λ is the only critical value of $u_\lambda(x)$. Therefore, if $x_1^\lambda = \min\{x : u_\lambda(x) = \tau_\lambda\}$, $x_2^\lambda = \max\{x : u_\lambda(x) = \tau_\lambda\}$, then u_λ decreases before x_1^λ, and increases after x_2^λ, while $u_\lambda \equiv \tau_\lambda$ when $x_1^\lambda \leq x \leq x_2^\lambda$. Thus, it follows from (4.5) that

$$\int_{u_\lambda(x)}^\alpha \frac{ds}{(F(\tau_\lambda) - F(s))^{1/p}} = (p'\lambda)^{1/p} x, \quad 0 < x < x_1^\lambda$$

and

$$\int_{u_\lambda(x)}^\alpha \frac{ds}{(F(\tau_\lambda) - F(s))^{1/p}} = (p'\lambda)^{1/p}(\ell - x), \quad x_2^\lambda < x < \ell.$$

Therefore, u_λ is symmetric with respect to $\ell/2$ and

$$\int_{\tau_\lambda}^{\alpha} \frac{ds}{(F(\tau_\lambda)-F(s))^{1/p}} = (p'\lambda)^{1/p} x_1^\lambda. \tag{4.6}$$

If $\tau_\lambda > a$, then we necessarily have $x_1^\lambda = x_2^\lambda = \ell/2$, for otherwise we deduce from (4.3) that $0 = \lambda f(\tau_\lambda) < 0$. Then (4.6) and the definition of γ^* yield $\gamma^* \geq (p'\lambda)^{1/p}\ell/2$. Therefore, we must have $\tau_\lambda = a$ and $x_1^\lambda < \ell/2$ when $\lambda > (2\gamma^*/\ell)^p/p'$. Moreover, by (4.6), we obtain $x_1^\lambda = d(\lambda)$. The uniqueness of the solution for such λ now follows from the identity

$$\int_{u(x)}^{\alpha} (F(a)-F(s))^{-1/p} ds = (p'\lambda)^{1/p} x,\ 0 < x < d(\lambda),$$

whenever u is a positive solution of (4.3). □

We next consider the problem

$$-\Delta_p u = \lambda f(u) \ \text{ in } A,\ u|_{\partial A} = \alpha, \tag{4.7}$$

where $A := \{x \in \mathbb{R}^N :\ 0 < R_1 < |x| < R_2\}$ is an annulus.

Lemma 4.4. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$, and satisfies (F_1) and (F_5). Then for any $\lambda > 0$, (4.7) has a maximal positive solution u_λ. Moreover, u_λ is radially symmetric, and if $G_\lambda := \{r \in (R_1,R_2) : u_\lambda(r) = a\}$, then for all large λ, there exists $\delta_1^\lambda > 0$ and $\delta_2^\lambda > 0$ such that $G_\lambda = [R_1+\delta_1^\lambda, R_2-\delta_2^\lambda]$, and*

$$\lim_{\lambda\to\infty} \delta_1^\lambda/d(\lambda) = 1,\ \lim_{\lambda\to\infty} \delta_2^\lambda/d(\lambda) = 1,$$

where $d(\lambda) = \lambda^{-1/p}(1-1/p)^{1/p} l(a)$ is given in Lemma 4.3.

Proof. The existence of a maximal positive solution u_λ to (4.7) follows from an application of Lemma 4.1. The maximality of u_λ forces it to be radially symmetric: $u_\lambda(x) = u_\lambda(r)$, $R_1 < r < R_2$, $r = |x|$. Thus, u_λ solves the problem

$$-(r^{N-1}|u'|^{p-2}u')' = \lambda r^{N-1} f(u),\ \ R_1 < r < R_2,\ \ u(R_1) = u(R_2) = \alpha.$$

Since $u_\lambda \geq a$, we have $f(u_\lambda) \leq 0$ and hence $(r^{N-1}|u_\lambda'|^{p-2}u_\lambda')' \geq 0$. It follows that if $u_\lambda'(r_0) = 0$ for some $r_0 \in (R_1,R_2)$, then $u_\lambda'(r) \leq 0$ on $[R_1,r_0]$ and $u_\lambda'(r) \geq 0$ on $[r_0,R_2]$. Thus, G_λ is an interval whenever it is non-empty. Hence, we can write $G_\lambda = [R_1+\delta_1^\lambda, R_2-\delta_2^\lambda]$ when non-empty. Setting

$$\rho = g(r) = \begin{cases} \frac{1}{1-\theta}[r^{1-\theta} - R_1^{1-\theta}], & p \neq N, \\ \log(\frac{r}{R_1}), & p = N, \end{cases}$$

where $\theta = \frac{N-1}{p-1}$, and $z_\lambda(\rho) = u_\lambda(g^{-1}(\rho))$, we find z_λ satisfies

$$-(|z'|^{p-2}z')' = \lambda(g^{-1}(\rho))^{p\theta} f(z), \quad 0 < \rho < T, \quad z(0) = z(T) = \alpha,$$

where $' = \frac{d}{d\rho}$, $T = g(R_2)$.

Now we consider the problem

$$-(|v'|^{p-2}v')' = \lambda R_1^{p\theta} f(v), \quad 0 < \rho < T, \quad v(0) = v(T) = \alpha. \tag{4.8}$$

Lemma 4.3 implies that for all large $\lambda > 0$, (4.8) has a unique positive solution $Z_\lambda(\rho)$ and

$$Z_\lambda(\rho) = a \text{ if and only if } \rho \in [d(\lambda R_1^{p\theta}), T - d(\lambda R_1^{p\theta})]. \tag{4.9}$$

It is clear that z_λ is a sub-solution to (4.8) and thus,

$$a \le z_\lambda(\rho) \le Z_\lambda(\rho), \quad 0 < \rho < T. \tag{4.10}$$

This implies that, for all large λ, z_λ has a flat core and thus, u_λ has a flat core. Moreover, from (4.9), (4.10) and $u_\lambda(r) = z_\lambda(\rho)$, we find $u_\lambda(r) = a$ when $r \in (r_1, r_2)$, where r_1 and r_2 are determined by

$$d(\lambda R_1^{p\theta}) = g(r_1), \ g(R_2) - d(\lambda R_1^{p\theta}) = g(r_2).$$

From this and a direct calculation, we obtain that $r_1 = R_1 + \bar{\delta}_1^\lambda$, $r_2 = R_2 - \bar{\delta}_2^\lambda$, with $\bar{\delta}_1^\lambda$ and $\bar{\delta}_2^\lambda$ satisfying

$$\lim_{\lambda\to\infty} \bar{\delta}_1^\lambda / d(\lambda) = 1, \ \lim_{\lambda\to\infty} \bar{\delta}_2^\lambda / d(\lambda) = R_2/R_1.$$

Clearly we have $\delta_1^\lambda \le \bar{\delta}_1^\lambda$ and $\delta_2^\lambda \le \bar{\delta}_2^\lambda$.

Similarly, the problem

$$-(|v'|^{p-2}v')' = \lambda R_2^{p\theta} f(v), \quad 0 < \rho < T, \quad v(0) = v(T) = \alpha. \tag{4.11}$$

has a unique positive solution $V_\lambda(\rho)$ for all large λ and

$$V_\lambda(\rho) = a \text{ if and only if } \rho \in [d(\lambda R_2^{p\theta}), T - d(\lambda R_2^{p\theta})]. \tag{4.12}$$

Now z_λ is a super-solution to (4.11) while the constant a is a sub-solution. It follows that $V_\lambda \le z_\lambda$. From this and (4.12), we deduce $G_\lambda \subset [\tilde{r}_1, \tilde{r}_2]$ with $\tilde{r}_1$ and $\tilde{r}_2$ determined by

$$d(\lambda R_2^{p\theta}) = g(\tilde{r}_1), \ g(R_2) - d(\lambda R_2^{p\theta}) = g(\tilde{r}_2).$$

It follows that $\tilde{r}_1 = R_1 + \underline{\delta}_1^\lambda$, $\tilde{r}_2 = R_2 - \underline{\delta}_2^\lambda$ with

$$\lim_{\lambda\to\infty} \underline{\delta}_1^\lambda / d(\lambda) = R_1/R_2, \ \lim_{\lambda\to\infty} \underline{\delta}_2^\lambda / d(\lambda) = 1.$$

We must have $\delta_i^\lambda \geq \underline{\delta}_i^\lambda$, $i = 1, 2$. Summarizing, we obtain

$$\overline{\lim}_{\lambda\to\infty}\frac{\delta_1^\lambda}{d(\lambda)} \leq 1, \quad \underline{\lim}_{\lambda\to\infty}\frac{\delta_1^\lambda}{d(\lambda)} \geq R_1/R_2,$$

$$\overline{\lim}_{\lambda\to\infty}\frac{\delta_2^\lambda}{d(\lambda)} \leq R_2/R_1, \quad \underline{\lim}_{\lambda\to\infty}\frac{\delta_2^\lambda}{d(\lambda)} \geq 1.$$

To obtain a better estimate for δ_1^λ, we observe that for $0 < \rho < \rho_\lambda := g(R_1 + \delta_1^\lambda)$, $z_\lambda'(\rho) < 0$ and

$$-(|z_\lambda'|^{p-2}z_\lambda')' = \lambda(g^{-1}(\rho))^{p\theta}f(z_\lambda) \geq \lambda(R_1 + \delta_1^\lambda)^{p\theta}f(z_\lambda).$$

Using this and $z_\lambda(0) = \alpha,\ z_\lambda(\rho_\lambda) = a$, we deduce

$$\int_a^\alpha [F(a) - F(s)]^{-1/p}ds \leq [p'\lambda(R_1 + \delta_1^\lambda)^{p\theta}]^{1/p}\rho_\lambda.$$

That is, $l(a) \leq (p'\lambda)^{1/p}(R_1+\delta_1^\lambda)^\theta g(R_1+\delta_1^\lambda)$, or, $(R_1+\delta_1^\lambda)^\theta g(R_1+\delta_1^\lambda) \geq d(\lambda)$. From our earlier estimate for δ_1^λ and a simple calculation, we easily obtain

$$\lim_{\lambda\to\infty}(R_1 + \delta_1^\lambda)^\theta g(R_1 + \delta_1^\lambda)/\delta_1^\lambda = 1.$$

It follows that

$$\underline{\lim}_{\lambda\to\infty}\delta_1^\lambda/d(\lambda) \geq 1.$$

Thus, we must have

$$\lim_{\lambda\to\infty}\delta_1^\lambda/d(\lambda) = 1.$$

A similar consideration of z_λ over $\rho \in (g(R_2 - \delta_2^\lambda), g(R_2))$ gives

$$\overline{\lim}_{\lambda\to\infty}\delta_2^\lambda/d(\lambda) \leq 1,$$

which combined with our earlier estimate gives

$$\lim_{\lambda\to\infty}\delta_2^\lambda/d(\lambda) = 1. \qquad \square$$

We are now ready to estimate the flat core of the maximal solution of (4.2). For $\xi > 0$, we will use the notation $\Omega_\xi := \{x \in \Omega : d(x, \partial\Omega) > \xi\}$.

Proposition 4.5. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0, \infty)$ and satisfies (F_1) and (F_5). Let w_λ be the maximal solution of (4.2) and $O_\lambda := \{x \in \Omega : w_\lambda(x) = a\}$. Then for any c_1^* and c_2^* satisfying $0 < c_1^* < c_1 \leq c(x) \leq c_2 < c_2^* < \infty$, and all large λ, we have $\Omega_{d(c_1^*\lambda)} \subset O_\lambda \subset \Omega_{d(c_2^*\lambda)}$, where*

$$d(t) = t^{-1/p}(1 - 1/p)^{1/p}\int_a^\alpha [F(a) - F(s)]^{-1/p}ds.$$

Proof. Since $\partial\Omega$ is smooth, it satisfies a uniform interior and exterior sphere condition, that is, there exist R and R_1 such that any $x_0 \in \partial\Omega$ can be touched by a ball $B_R(y_0)$ of radius R lying inside Ω and by a ball $B_{R_1}(z_0)$ of radius R_1 lying outside Ω. Clearly, the centers y_0 and z_0 of the balls are determined uniquely by x_0 and the three points x_0, y_0 and z_0 lie on a straight line.

We now choose R_2 large enough such that the annulus $A_1 := B_{R_2}(z_0) \setminus \overline{B}_{R_1}(z_0)$ always contains Ω. Clearly, the annulus $A_2 := B_R(y_0) \setminus \overline{B}_{R/2}(y_0)$ is always contained in Ω.

Let u^1_λ and u^2_λ be the maximal solutions of (4.7) with $A = A_1$ and $A = A_2$, respectively. Since $w_\lambda \le \alpha$ on $\partial A_2 \subset \bar{\Omega}$ and $\lambda c(x) f(w_\lambda) \le \lambda c_1 f(w_\lambda)$, w_λ is a sub-solution to (4.7) with $A = A_2$ and with λ replaced by λc_1. It follows that $u^2_{\lambda c_1} \ge w_\lambda$ in A_2. On the other hand, $u^1_{\lambda c_2}$ is a sub-solution to (4.2) due to $u^1_{\lambda c_2} \le \alpha$ on $\partial\Omega \subset \bar{A}_1$ and $\lambda c_2 f(u^1_{\lambda c_2}) \le \lambda c(x) f(u^1_{\lambda c_2})$. Thus, we have $u^1_{\lambda c_2} \le w_\lambda$ in Ω. In particular, we have

$$u^1_{\lambda c_2} \le w_\lambda \le u^2_{\lambda c_1} \ \text{ in } A_2. \tag{4.13}$$

Consider now the line segment

$$L_{x_0 y_0} := \{y_0 + \frac{t}{R}(x_0 - y_0) : t \in [\frac{2}{3}R, R]\} \subset A_2 \subset A_1.$$

By Lemma 4.4 applied to A_2, for all large λ, $u^2_{\lambda c_1}(x) = a$ if $x = y_0 + \frac{t}{R}(x_0 - y_0)$ with $t \in [\frac{2}{3}R, R - \delta_2^{\lambda c_1}]$, where $\delta_2^{\lambda c_1}/d(\lambda c_1) \to 1$ as $\lambda \to \infty$.

Applied to A_1, Lemma 4.4 implies that for all large λ, $u^1_{\lambda c_2}(x) > a$ if $x = y_0 + \frac{t}{R}(x_0 - y_0)$ with $t \in [R - \delta_1^{\lambda c_2}, R]$, where $\delta_1^{\lambda c_2}/d(\lambda c_2) \to 1$ as $\lambda \to \infty$.

By (4.13) we find that $w_\lambda(x) = a$ when $t \in [\frac{2}{3}R, R - \delta_2^{\lambda c_1}]$, and $w_\lambda(x) > a$ when $t \in [R - \delta_1^{\lambda c_2}, R]$. As

$$d(\lambda c_1^*) = (c_1^*/c_1)^{-1/p} d(\lambda c_1),\ d(\lambda c_2^*) = (c_2^*/c_2)^{-1/p} d(\lambda c_2),$$

we find that for all large λ, $\delta_2^{\lambda c_1} < d(\lambda c_1^*)$, $\delta_1^{\lambda c_2} > d(\lambda c_2^*)$. Therefore, for all large λ,

$$w_\lambda(x) = a,\ \forall t \in [\tfrac{2}{3}R, R - d(\lambda c_1^*)];\ w_\lambda(x) > a,\ \forall t \in [R - d(\lambda c_2^*), R].$$

Letting x_0 run through $\partial\Omega$, our above analysis implies that $w_\lambda(x) = a$ when $\frac{1}{3}R \ge d(x, \partial\Omega) \ge d(\lambda c_1^*)$, $w_\lambda(x) > a$ when $d(x, \partial\Omega) \le d(\lambda c_2^*)$. Now by a simple application of the maximum principle, or by comparing w_λ with maximal solutions of (4.7) with suitable A, it is easily seen that $w_\lambda(x) = a$ when $x \in \Omega_{R/3}$ and λ is large. Thus, for all large λ, $\Omega_{d(c_1^*\lambda)} \subset O_\lambda \subset \Omega_{d(c_2^*\lambda)}$. The proof is complete. $\square$

To estimate the flat core of the minimal solution of (4.1), we can carry out an analogous analysis to the above, with α replaced by 0 everywhere. Since the modifications needed are minor, we omit the details and only state the final result.

Proposition 4.6. *Suppose that $f(s)$ is continuous and locally quasi-monotone on $[0,\infty)$ and satisfies (F_1) and*

(F_6) *for some small $\epsilon > 0$,*

$$\int_{a-\epsilon}^{a} [F(a) - F(s)]^{-1/p} ds < \infty.$$

Let v_λ be the minimal positive solution of (4.1) and $\tilde{O}_\lambda := \{x \in \Omega : v_\lambda(x) = a\}$. Then for any c_1^* and c_2^* satisfying $0 < c_1^* < c_1 \le c(x) \le c_2 < c_2^* < \infty$, and all large λ, we have $\Omega_{\tilde{d}(c_1^*\lambda)} \subset \tilde{O}_\lambda \subset \Omega_{\tilde{d}(c_2^*\lambda)}$, where

$$\tilde{d}(t) = t^{-1/p}(1-1/p)^{1/p} \int_0^a [F(a) - F(s)]^{-1/p} ds.$$

As mentioned at the beginning of this section, Theorem 1.5 follows directly from Propositions 4.5 and 4.6.

We now consider Theorem 1.6 and show that it follows from an application of Theorem 1.5. Let K be an arbitrary compact subset of Ω and u_λ an arbitrary positive solution of (1.3). By Theorem 1.4, we know $u_\lambda \ge a$ in Ω. By the proof of Theorem 3.7, we know that for any small $\delta_1 > 0$ such that $\Omega_{\delta_1} \supset K$, one can find α large enough so that the maximal positive solution z_λ of (4.2) with Ω replaced by Ω_{δ_1} satisfies $z_\lambda \ge u_\lambda \ge a$ on Ω_{δ_1}, for all large λ. By Theorem 1.5, for all large λ, $z_\lambda = a$ on K. Thus, so is u_λ. This proves Theorem 1.6.

A better estimate for the flat core of an arbitrary positive solution u_λ of (1.3) can be obtained. We sketch a proof below. For any small $\delta > 0$, let Z_λ^α be the maximal solution of (4.2) with Ω replaced by Ω_δ and with large enough α so that $u_\lambda \le Z_\lambda^\alpha$ on Ω_δ, for all large λ. We also consider the minimal positive solution z_λ^α of (4.2) that satisfies $z_\lambda^\alpha \ge a$ in Ω (cf. Remark 4.2 (ii)). It is easy to see that $z_\lambda^\alpha \le u_\lambda$ on Ω for any $\alpha > a$. By Proposition 4.5, $Z_\lambda^\alpha(x) = a$ when

$$d(x, \partial\Omega_\delta) \ge (\lambda c_1^*)^{-1/p}(1-1/p)^{1/p} \int_a^\alpha [F(a) - F(s)]^{-1/p} ds.$$

By (F_3),

$$\int_a^\infty [F(a) - F(s)]^{-1/p} ds < \infty.$$

Thus, $Z_\lambda^\alpha(x) = a$ for all large α when

$$d(x, \partial\Omega_\delta) \geq (\lambda c_1^*)^{-1/p}(1-1/p)^{1/p}\int_a^\infty [F(a)-F(s)]^{-1/p}ds.$$

It follows that

$$u_\lambda(x) = a \ \text{ if } d(x, \partial\Omega_\delta) \geq (\lambda c_1^*)^{-1/p}(1-1/p)^{1/p}\int_a^\infty [F(a)-F(s)]^{-1/p}ds.$$

Letting $\delta \to 0$, we obtain

$$u_\lambda(x) = a \ \text{ if } d(x, \partial\Omega) \geq (\lambda c_1^*)^{-1/p}(1-1/p)^{1/p}\int_a^\infty [F(a)-F(s)]^{-1/p}ds.$$

On the other hand, with minor modifications of the proof of Proposition 4.5 one finds that its conclusion also holds for the minimal positive solution z_λ^α. Thus, $z_\lambda^\alpha(x) > a$ if

$$d(x, \partial\Omega) \leq (\lambda c_2^*)^{-1/p}(1-1/p)^{1/p}\int_a^\alpha [F(a)-F(s)]^{-1/p}ds.$$

It follows that

$$u_\lambda(x) > a \ \text{ if } d(x, \partial\Omega) \leq (\lambda c_2^*)^{-1/p}(1-1/p)^{1/p}\int_a^\alpha [F(a)-F(s)]^{-1/p}ds.$$

Letting $\alpha \to \infty$ we obtain

$$u_\lambda(x) > a \ \text{ if } d(x, \partial\Omega) \leq (\lambda c_2^*)^{-1/p}(1-1/p)^{1/p}\int_a^\infty [F(a)-F(s)]^{-1/p}ds.$$

Therefore, if we denote $H_\lambda := \{x \in \Omega : u_\lambda(x) = a\}$, then, for all large λ, $\Omega_{\xi_1\lambda^{-1/p}} \subset H_\lambda \subset \Omega_{\xi_2\lambda^{-1/p}}$, where

$$\xi_i = (c_i^*)^{-1/p}(1-1/p)^{1/p}\int_a^\infty [F(a)-F(s)]^{-1/p}ds, \ i = 1, 2.$$

We would like to remark that, in contrast, we only have a lower estimate for the flat core of the positive solutions of (1.2) by using Propositions 4.5 and 4.6. The boundary function ϕ has some important influence on the flat core near the boundary. For example, if $\phi \equiv a$, then the flat function $u = a$ is a solution to (1.2) for all λ.

Finally, we show how Corollaries 1.7 and 1.8 follow easily from Theorems 1.3-1.6. Let K be a compact subset of $\mathcal{D}$. Then we can find a smooth domain Ω such that $K \subset \Omega$ and $\bar{\Omega} \subset \mathcal{D}$. By Lemma 4.1, (4.1) has a minimal positive solution v_λ for all large $\lambda > 0$. It follows easily that any positive solution u_λ of $-\Delta_p u = \lambda c(x) f(u)$ in $\mathcal{D}$ satisfies $u_\lambda \geq v_\lambda$ in Ω.

Let z_λ denote the boundary blow-up solution of (1.3) obtained as the limit of the maximal solution u_k of (3.14) in the proof of Theorem 3.7. We claim that $u_\lambda \le z_\lambda$ in Ω. Indeed, let $M_\lambda := \max_{x\in\partial\Omega} u_\lambda(x)$. We have $M_\lambda < \infty$ and hence $u_\lambda \le u_k$ in Ω whenever $k \ge M_\lambda$. As a consequence, $u_\lambda \le z_\lambda$ in Ω. Now from $v_\lambda \le u_\lambda \le z_\lambda$ in $\Omega \supset K$, the conclusions of Corollaries 1.7 and 1.8 follow immediately from Theorems 1.3-1.6.

Acknowledgment: We thank Professors James Serrin and Honghui Zou for sending us [30]. We are grateful for several useful suggestions from Professor Serrin on improving the presentation of the paper.

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