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ASYMPTOTICS FOR SHARP SOBOLEV-POINCARÉ INEQUALITIES ON COMPACT RIEMANNIAN MANIFOLDS

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Abstract. Given (M, g) a smooth compact Riemannian manifold of dimension $n \geq 3$, there exist $A, B > 0$ such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq A\|\nabla u\|_2^2 + B\|u\|_1^2,$$

where $H_1^2(M)$ is the standard Sobolev space consisting of functions in $L^2(M)$ whose gradient is also in $L^2(M)$. The best possible A in this inequality is K_n^2, where K_n is the sharp constant K in the Euclidean Sobolev inequality $\|u\|_{2^\star} \leq K\|\nabla u\|_2$. Thanks to previous work by Druet-Hebey-Vaugon and Hebey, it turns out that the above inequality with $A = K_n^2$ is always true when $n = 3$, in other words without any assumption on the manifold, and true when $n = 4$ if the scalar curvature is everywhere negative, or the scalar curvature is nonpositive and the manifold is conformally flat, or the sectional curvature is nonpositive and a local isoperimetric inequality as in the Cartan-Hadamard conjecture holds. On the contrary, thanks to previous works by Druet-Hebey-Vaugon, the inequality with $A = K_n^2$ is false when $n \geq 4$ and the scalar curvature is positive somewhere. Independent considerations give that for any $\varepsilon > 0$ there exists B_ε such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq (K_n^2 + \varepsilon)\|\nabla u\|_2^2 + B_\varepsilon\|u\|_1^2.$$

Defining $B_\varepsilon(g)$ as the smallest B_ε in this inequality, the difficult question we are concerned with in this article is to describe the asymptotic behavior of $B_\varepsilon(g)$ as $\varepsilon \to 0$. A complete answer to this question is given.

Let (M, g) be a smooth compact Riemannian n-manifold, $n \geq 3$, and $H_1^2(M)$ be the standard Sobolev space consisting of functions in $L^2(M)$ whose gradient is also in $L^2(M)$. The Sobolev-Poincaré inequality, actually a currently used form of the Sobolev-Poincaré inequality, asserts that there

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exist positive constants A and B such that for any $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{2^\star} dv_g\Big)^{\frac{2}{2^\star}} \leq A \int_M |\nabla u|^2 dv_g + B\Big(\int_M |u| dv_g\Big)^2, \qquad (0.1)$$

where $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent. The Poincaré version of this inequality appeared first in the Courant-Hilbert monograph [9]. As stated there, the inequality is in Nirenberg [29]. Now we let K_n be the best constant K in the Euclidean Sobolev inequality $\|u\|_{2^\star} \leq K\|\nabla u\|_2$. It is well known (see Aubin [3] and Talenti [36]) that

$$K_n = \sqrt{\frac{4}{n(n-2)\omega_n^{2/n}}}\,,$$

where ω_n is the volume of the unit sphere S^n of $\mathbb{R}^{n+1}$. Rather standard arguments, as developed in Druet-Hebey-Vaugon [16], show that the following holds:

(i) any constant A in (0.1) has to be such that $A \geq K_n^2$, and
(ii) for any $\varepsilon > 0$, there exists $B_\varepsilon > 0$ such that for any $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{2^\star} dv_g\Big)^{\frac{2}{2^\star}} \leq \Big(K_n^2 + \varepsilon\Big) \int_M |\nabla u|^2 dv_g + B_\varepsilon\Big(\int_M |u| dv_g\Big)^2. \qquad (0.2)$$

In particular, the best possible constant A in (0.1) is precisely K_n^2. A natural question is whether or not there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{2^\star} dv_g\Big)^{\frac{2}{2^\star}} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B\Big(\int_M |u| dv_g\Big)^2. \qquad (0.3)$$

Closely related inequalities in the Euclidean context were considered by Brézis-Nirenberg [6] and Brézis-Lieb [5]. It turns out, see Druet-Hebey-Vaugon [16] and Hebey [24], that such a B in (0.3) exists whatever the manifold is when $n = 3$, and in the following cases when $n \geq 4$: when the scalar curvature is everywhere negative, or when the scalar curvature is nonpositive and the manifold is conformally flat, or at last when the sectional curvature is nonpositive and the Cartan-Hadamard conjecture is true. A local version of the Cartan-Hadamard conjecture as in Druet [13] and Johnson and Morgan [26] is actually sufficient. An expository lecture in book form on the Cartan-Hadamard conjecture can be found in Hebey [23]. In particular, when $n = 4$, a dimension where the Cartan-Hadamard conjecture is true thanks to Croke [10], it follows from these results that B exists if either the scalar curvature is everywhere negative, or the scalar curvature is nonpositive and the manifold is conformally flat, or the sectional curvature is nonpositive. Conversely, see once more Druet-Hebey-Vaugon [16] and Hebey [24],

B does not exist, and thus (0.3) is false, either when $n \geq 4$ and the scalar curvature is positive somewhere, or when $n \geq 6$ and the scalar curvature vanishes around one point where the Weyl curvature tensor is not zero.

As an interesting remark, let us consider the standard Sobolev inequality

$$\left(\int_M |u|^{2^\star} dv_g\right)^{\frac{2}{2^\star}} \leq A \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g. \tag{0.4}$$

Here again, easy comparison arguments show that the best A in (0.4) is $A = K_n^2$. On the other hand, contrary to what happens to (0.3), it was shown by Hebey and Vaugon [25] that whatever (M, g) is, $n \geq 3$, there exists B such that for any $u \in H_1^2(M)$,

$$\left(\int_M |u|^{2^\star} dv_g\right)^{\frac{2}{2^\star}} \leq K_n^2 \int_M |\nabla u|^2 dv_g + B \int_M u^2 dv_g. \tag{0.5}$$

In other words, the validity of the sharp Sobolev inequality (0.5) is not affected by the geometry, neither by the dimension. Possible references in book form on sharp Sobolev inequalities are Druet-Hebey [15], and Hebey [23].

Coming back to the Sobolev-Poincaré inequality, let us now define $B_\varepsilon(g)$ as the best B_ε in (0.2). It is clear that (0.2) holds with $B_\varepsilon = B_\varepsilon(g)$. Let us assume that $n \geq 4$. A possible reformulation of the above mentioned results is as follows. On the one hand, $B_\varepsilon(g)$ has a finite limit as $\varepsilon \to 0$ when the scalar curvature is everywhere negative, or when the scalar curvature is nonpositive and the manifold is conformally flat, or at last when the sectional curvature is nonpositive and the Cartan-Hadamard conjecture is true. On the other hand, $B_\varepsilon(g)$ tends to $+\infty$ as $\varepsilon \to 0$ when the scalar curvature is positive somewhere, or when $n \geq 6$ and the scalar curvature vanishes around one point where the Weyl curvature tensor is not zero. Let us assume from now on that the scalar curvature is positive somewhere, and thus that $B_\varepsilon(g)$ tends to $+\infty$ as $\varepsilon \to 0$. The difficult question we are concerned with in this article is to describe the behavior of $B_\varepsilon(g)$ in terms of ε. In particular, to get sharp estimates on $B_\varepsilon(g)$. A somehow similar problem was studied in the very nice Adimurthi-Pacella-Yadava [1]. This paper was concerned with the standard sharp Sobolev inequality with Neumann boundary condition in the Euclidean context. Asymptotic studies were initiated by Atkinson-Peletier [2] and Brézis-Peletier [7]. We refer also to Druet [14], Druet-Robert [17], Han [21], Hebey [22], Hebey-Vaugon [25], Li [27], Li-Zhu [28], Rey [32], Robert [34], and Schoen-Zhang [35].

The answer to our question is given by the following result:

Theorem. *Let (M, g) be a smooth compact Riemannian n-manifold, $n \geq 4$, whose scalar curvature is positive somewhere. Given $\varepsilon > 0$, let $B_\varepsilon(g)$ be the* *smallest* B_ε *such that for any* $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{2^\star} dv_g \Big)^{\frac{2}{2^\star}} \leq \Big(K_n^2 + \varepsilon \Big) \int_M |\nabla u|^2 dv_g + B_\varepsilon \Big(\int_M |u| dv_g \Big)^2 .$$

There exists $C(n) > 0$, *depending only on* n, *such that*

$$B_\varepsilon(g) = C(4) (\max_M S_g)^3 \, |\ln \varepsilon|^3 + o\big(|\ln \varepsilon|^3 \big)$$

if $n = 4$, *and such that*

$$B_\varepsilon(g) = C(n) \big(\max_M S_g \big)^{\frac{n+2}{2}} \varepsilon^{-\frac{(n-4)(n+2)}{2(n-2)}} + o\big(\varepsilon^{-\frac{(n-4)(n+2)}{2(n-2)}} \big)$$

if $n \geq 5$, *where* S_g *is the scalar curvature of* g.

The constant $C(n)$ in the theorem is explicitly known. When $n = 4$ we find that $C(4) = \frac{K_4^2}{2304 \omega_3}$ and when $n \geq 5$, we find that

$$C(n) = \frac{2n(n+2) \omega_n^{2 + \frac{4}{n}} K_n^{\frac{n^2 - 12}{n-2}}}{\Big(4^{n-3} n (n-2)(n-4) \Big)^{\frac{n+2}{n-2}} \omega_{n-1}^{\frac{2n}{n-2}}} .$$

In particular, it follows from the theorem that for any $C > C(n)$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, and any $u \in H_1^2(M)$,

$$\Big(\int_M |u|^{2^\star} dv_g \Big)^{\frac{2}{2^\star}} \leq \big(K_n^2 + \varepsilon \big) \int_M |\nabla u|^2 dv_g + C \big(\max_M S_g \big)^3 |\ln \varepsilon|^3 \Big(\int_M |u| dv_g \Big)^2$$

when $n = 4$, and

$$\Big(\int_M |u|^{2^\star} dv_g \Big)^{\frac{2}{2^\star}} \leq \big(K_n^2 + \varepsilon \big) \int_M |\nabla u|^2 dv_g + \frac{C \big(\max_M S_g \big)^{\frac{n+2}{2}}}{\varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}}} \Big(\int_M |u| dv_g \Big)^2$$

when $n \geq 5$. The rest of the article is devoted to the proof of this theorem. The proof is divided into three main steps, the first of which concerns the study of a closely related problem in the Euclidean context. A test function type argument, based on what is proved in Section 1, is developed in Section 2. The general case of an arbitrary compact Riemannian manifold is treated in Section 3.

1. The Euclidean case

Let $\mathcal{B}$ be the unit ball in $\mathbb{R}^n$, $n \geq 4$, and $\Delta = -\text{div}(\nabla)$ be the Euclidean Laplacian. We let $C_c^\infty(\mathcal{B})$ be the set of smooth functions with compact support in $\mathcal{B}$, and $H_{0,1}^2(\mathcal{B})$ be the standard Sobolev space defined as the completion of $C_c^\infty(\mathcal{B})$ with respect to the norm $\|u\| = \|\nabla u\|_2$. Given $\alpha > 0$ and $B > 0$, we define λ_B by

$$\lambda_B = \inf_{u \in C_c^\infty(\mathcal{B})\backslash\{0\}} \frac{\|\nabla u\|_2^2 - \alpha\|u\|_2^2 + B\|u\|_1^2}{\|u\|_{2^\star}^2}\,. \tag{1.1}$$

For $\delta > 0$ small, let u_δ be the function of $H_{0,1}^2(\mathcal{B})$ defined by

$$u_\delta(x) = \left(\delta + |x|^2\right)^{1-\frac{n}{2}} - \left(\delta + 1\right)^{1-\frac{n}{2}}.$$

Taking the u_δ's as test functions, it is easily seen that for any $B > 0$, $\lambda_B < \frac{1}{K_n^2}$. On such developments, we refer to Druet-Hebey-Vaugon [16] and Hebey [24]. Now we claim that the following holds:

(i) λ_B is continuous in B and increasing in B ,

(ii) $\lambda_B \to \frac{1}{K_n^2}$ as $B \to +\infty$.

Point (i) is easy to get. Just note that if $B_2 = B_1 + \eta$, $\eta \geq 0$, then

$$\lambda_{B_1} \leq \lambda_{B_2} \leq \lambda_{B_1} + \eta V_{\mathcal{B}}^{\frac{2(2^\star-1)}{2^\star}},$$

where $V_{\mathcal{B}}$ is the volume of $\mathcal{B}$. Concerning point (ii), let $B > 0$ be given. Since K_n^{-2} is the minimum energy for blow up, and $\lambda_B < K_n^{-2}$, classical variational methods lead to the existence of a minimizer for λ_B. In particular, see for instance Druet-Hebey-Vaugon [16], there exists $u_B \in C^{1,\delta}(\overline{\mathcal{B}})$, $\delta \in (0,1)$, $u_B \geq 0$ in $\mathcal{B}$ and $u_B = 0$ on $\partial\mathcal{B}$, such that

$$\Delta u_B - \alpha u_B + B\|u_B\|_1 \Sigma_B = \lambda_B u_B^{2^\star-1} \tag{1.2}$$

and $\int_{\mathcal{B}} u_B^{2^\star} dx = 1$, where $\Sigma_B \in L^\infty(\mathcal{B})$ is such that $0 \leq \Sigma_B \leq 1$ and $\Sigma_B u_B = u_B$. Multiplying (1.2) by u_B and integrating over $\mathcal{B}$, we get that $B\|u_B\|_1^2 \leq \lambda_B$. As a consequence, $u_B \to 0$ in $L^1(\mathcal{B})$ as $B \to +\infty$. This implies that blow up occurs as $B \to +\infty$, and thus that $\lambda_B \to K_n^{-2}$ as $B \to +\infty$. Points (i) and (ii) above are proved.

Given $\varepsilon > 0$ small, we let $B_\varepsilon > 0$ be such that

$$\lambda_{B_\varepsilon} = \frac{1-\varepsilon}{K_n^2}\,. \tag{1.3}$$

The goal in this section is to describe the asymptotic behavior of B_ε in terms of ε as $\varepsilon \to 0$. More precisely, we want to prove that

$$\lim_{\varepsilon\to 0} \frac{B_\varepsilon}{|\ln\varepsilon|^3} = \frac{3}{32\omega_3}\alpha^3 \tag{1.4}$$

when $n = 4$, and

$$\lim_{\varepsilon\to 0} B_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = C_n\Big(\frac{4(n-1)}{n-2}\alpha\Big)^{\frac{n+2}{2}} \tag{1.5}$$

when $n \geq 5$, where $C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}}\left(4^{n-3}n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$. By standard symmetrization arguments, based on the co-area formula, functions in (1.1) can be assumed to be radially symmetrical and decreasing. As above, we then get the existence of a decreasing radially symmetrical function $u_\varepsilon \in C^{1,\delta}(\overline{\mathcal{B}})$, $\delta \in (0,1)$, $u_\varepsilon \geq 0$ in $\mathcal{B}$ and $u_\varepsilon = 0$ on $\partial\mathcal{B}$, such that

$$\Delta u_\varepsilon - \alpha u_\varepsilon + B_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star - 1} \tag{1.6}$$

and $\int_{\mathcal{B}} u_\varepsilon^{2^\star} dx = 1$. There, see Druet-Hebey-Vaugon [16], $\Sigma_\varepsilon \in L^\infty(\mathcal{B})$ is such that $\Sigma_\varepsilon = 1$ if $u_\varepsilon > 0$, and $\Sigma_\varepsilon = 0$ if $u_\varepsilon = 0$. In particular, there exists $r_\varepsilon \in (0,1]$ such that $\text{Supp} u_\varepsilon = \mathcal{B}_0(r_\varepsilon)$, where $\mathcal{B}_0(r_\varepsilon)$ is the Euclidean ball of center 0 and radius r_ε. Then,

$$\Sigma_\varepsilon = 1 \text{ in } \mathcal{B}_0(r_\varepsilon) \text{ and } \Sigma_\varepsilon = 0 \text{ in } \overline{\mathcal{B}} \backslash \mathcal{B}_0(r_\varepsilon) \tag{1.7}$$

and, as a consequence, u_ε is smooth around 0. Since for any B, $\lambda_B < K_n^{-2}$, we have that $B_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Independently, multiplying (1.6) by u_ε and integrating over $\mathcal{B}$, we see that $B_\varepsilon \|u_\varepsilon\|_1^2$ is bounded as $\varepsilon \to 0$. By the preceding remark, this implies that $\|u_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$. Thus, blow up must occur, and we are lead to the study of the asymptotic behavior of the u_ε's. As a starting point in the proof of (1.4) and (1.5) we prove weak estimates on the u_ε's.

1.1. **Weak estimates.** We let $\mu_\varepsilon > 0$ be given by

$$u_\varepsilon(0) = \|u_\varepsilon\|_\infty = \mu_\varepsilon^{1-\frac{n}{2}}. \tag{1.8}$$

Then, $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. Since $\Delta u_\varepsilon(0) \geq 0$ and $\Sigma_\varepsilon(0) = 1$, (1.6) gives

$$B_\varepsilon \|u_\varepsilon\|_1 \leq \alpha \mu_\varepsilon^{1-\frac{n}{2}} + \frac{1-\varepsilon}{K_n^2} \mu_\varepsilon^{-1-\frac{n}{2}}.$$

Thus, for $\varepsilon > 0$ sufficiently small,

$$B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{1+\frac{n}{2}} \leq \frac{2}{K_n^2}. \tag{1.9}$$

Now, we let $\tilde{u}_\varepsilon$ be defined by $\tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{n}{2}-1} u_\varepsilon(\mu_\varepsilon x)$. It is easily seen that

$$\Delta \tilde{u}_\varepsilon - \alpha \mu_\varepsilon^2 \tilde{u}_\varepsilon + B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{\frac{n}{2}+1} \tilde{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n^2} \tilde{u}_\varepsilon^{2^\star - 1} \tag{1.10}$$

in $\mathcal{B}_0(\mu_\varepsilon^{-1})$, where $\tilde{\Sigma}_\varepsilon(x) = \Sigma_\varepsilon(\mu_\varepsilon x)$. Noting that $\tilde{u}_\varepsilon \leq 1$, and thanks to (1.9), we get by standard elliptic theory that the $\tilde{u}_\varepsilon$'s are equicontinuous on any compact subset of $\mathbb{R}^n$. By Ascoli's theorem we then get that there exists $u_0 \in C^0(\mathbb{R}^n)$ such that, after passing to a subsequence,

$$\tilde{u}_\varepsilon \to u_0 \text{ in } C^0_{loc}(\mathbb{R}^n). \tag{1.11}$$

Clearly, $u_0(0) = 1$, and we have that $u_0 \in D_1^2(\mathbb{R}^n)$, where $D_1^2(\mathbb{R}^n)$ is the homogeneous Euclidean Sobolev space. Up to a subsequence, we define Σ_0 by

$$\Sigma_0(x) = \lim_{\varepsilon \to 0} B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{1+\frac{n}{2}} \Sigma_\varepsilon(\mu_\varepsilon x).$$

Assuming that $\frac{r_\varepsilon}{\mu_\varepsilon} \to R$ as $\varepsilon \to 0$, and that $B_\varepsilon \|u_\varepsilon\|_1 \mu_\varepsilon^{1+\frac{n}{2}} \to A$ as $\varepsilon \to 0$, we then have that $\Sigma_0 = 0$ if $R = 0$, $\Sigma_0 = A$ if $R = +\infty$, and $\Sigma_0 = A 1\!\!I_{\mathcal{B}_0(R)}$ if $R \in (0, +\infty)$, where $1\!\!I_X$ stands for the characteristic function of a subset X of $\mathbb{R}^n$. It is easily seen that u_0 is a solution in $\mathbb{R}^n$ of the equation

$$\Delta u_0 + \Sigma_0 = \frac{1}{K_n^2} u_0^{2^\star - 1}. \tag{1.12}$$

We claim that this implies $\Sigma_0 = 0$. When $R \in (0, +\infty)$, such a claim easily follows from the Pohozaev identity [31]. Note that in this case, u_0 is compactly supported in $\mathcal{B}_0(R)$. Let us assume now $R = +\infty$ and $A > 0$. Multiplying (1.10) by $\tilde{u}_\varepsilon$ and integrating, we easily get that $u_0 \in L^1(\mathbb{R}^n)$. By standard regularity results, we also have u_0 is $C^{2,k}$, $k \in (0,1)$. We let η be a smooth cut-off function such that $0 \le \eta \le 1$, $\eta = 1$ if $|x| \le 1$, and $\eta = 0$ if $|x| \ge 2$. For $r > 0$, we let also η_r be given by $\eta_r(x) = \eta(\frac{x}{r})$. The Pohozaev identity [31], applied to $\eta_r u_0$, gives

$$2 \int_{\mathbb{R}^n} \Big(\nabla(\eta_r u_0), x\Big) \Delta(\eta_r u_0) dx + (n-2) \int_{\mathbb{R}^n} \eta_r u_0 \Delta(\eta_r u_0) dx \le 0. \tag{1.13}$$

Moreover, $(\nabla \eta_r)(x) = \frac{1}{r} \nabla \eta(\frac{x}{r})$ and $(\Delta \eta_r)(x) = \frac{1}{r^2} \Delta \eta(\frac{x}{r})$. Integrating by parts, using the Lebesgue dominated convergence theorem, and thanks to (1.12),

$$\int_{\mathbb{R}^n} \Big(\nabla(\eta_r u_0), x\Big) \Delta(\eta_r u_0) dx = -\frac{n-2}{2K_n^2} \int_{\mathbb{R}^n} \eta_r^2 u_0^{2^\star} dx + nA \int_{\mathbb{R}^n} \eta_r^2 u_0 dx + o(1)$$
$$\int_{\mathbb{R}^n} \eta_r u_0 \Delta(\eta_r u_0) dx = \frac{1}{K_n^2} \int_{\mathbb{R}^n} \eta_r^2 u_0^{2^\star} dx - A \int_{\mathbb{R}^n} \eta_r^2 u_0 dx + o(1),$$

where $o(1) \to 0$ as $r \to +\infty$. Coming back to (1.13), it follows that

$$A \int_{\mathbb{R}^n} \eta_r^2 u_0 dx \le o(1)$$

and, passing to the limit as $r \to +\infty$, we get a contradiction. Thus, $A = 0$ if $R = +\infty$, and this proves the above claim. In particular, u_0 is a solution of the equation

$$\Delta u_0 = \frac{1}{K_n^2} u_0^{2^\star - 1}.$$

By Caffarelli-Gidas-Spruck [8], and also Obata [30], it follows that

$$u_0(x) = \Big(\frac{1}{1 + \frac{\omega_n^{2/n}}{4} |x|^2} \Big)^{\frac{n-2}{2}}.$$

Noting that $\mathrm{Supp}\tilde{u}_\varepsilon \subset \mathcal{B}_0(\frac{r_\varepsilon}{\mu_\varepsilon})$, we get that $\frac{r_\varepsilon}{\mu_\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Another consequence is that

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \int_{\mathcal{B}_0(R\mu_\varepsilon)} u_\varepsilon^{2^\star} dx = 1. \tag{1.14}$$

Following Druet [12], (1.14) implies in turn that the two following estimates hold. On the one hand, there exists $C > 0$ such that for any $\varepsilon > 0$ and any $x \in \mathcal{B}$,

$$|x|^{\frac{n}{2}-1} u_\varepsilon(x) \le C. \tag{1.15}$$

On the other hand,

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \sup_{\mathcal{B} \backslash \mathcal{B}_0(R\mu_\varepsilon)} |x|^{\frac{n}{2}-1} u_\varepsilon(x) = 0. \tag{1.16}$$

We prove (1.15). Let v_ε be defined by $v_\varepsilon(x) = |x|^{\frac{n}{2}-1} u_\varepsilon(x)$. We assume by contradiction that for some subsequence, $\|v_\varepsilon\|_\infty \to +\infty$ as $\varepsilon \to 0$. Let x_ε be a point in $\mathcal{B}$, where v_ε is maximum. A straightforward consequence of (1.14) is that for $x \neq 0$, and $\delta > 0$ sufficiently small,

$$\int_{\mathcal{B}_x(\delta)} u_\varepsilon^{2^\star} dx \to 0$$

as $\varepsilon \to 0$. Let $x \in \mathcal{B}$, $x \neq 0$, and η be a smooth cut-off function around x. Multiplying (1.6) by $\eta^2 u_\varepsilon^k$, $k \ge 1$, and integrating over $\mathcal{B}$, it is easily seen, see for instance Druet-Hebey-Vaugon [16], that for $\delta > 0$ sufficiently small, the u_ε's are bounded in $L^{(2^\star)^2/2}(\mathcal{B}_x(\delta))$. Since $(2^\star)^2/2 > 2^\star$, it follows from the De Giorgi-Nash-Moser iterative scheme and (1.6) that

$$u_\varepsilon \to 0 \ \text{ in } C^0_{loc}(\mathcal{B} \backslash \{0\}) \tag{1.17}$$

as $\varepsilon \to 0$. In particular, (1.17) implies $x_\varepsilon \to 0$ as $\varepsilon \to 0$. Since $u_\varepsilon(x_\varepsilon) \le u_\varepsilon(0)$ and $\|v_\varepsilon\|_\infty \to +\infty$, we also have

$$\frac{|x_\varepsilon|}{\mu_\varepsilon} \to +\infty \tag{1.18}$$

as $\varepsilon \to 0$, and that $u_\varepsilon(x_\varepsilon) \to +\infty$ as $\varepsilon \to 0$. We set $\Omega_\varepsilon = u_\varepsilon(x_\varepsilon)^{\frac{2}{n-2}}\mathcal{B}_{-x_\varepsilon}(1)$ and for $x \in \Omega_\varepsilon$, we set $\tilde{v}_\varepsilon(x) = u_\varepsilon(x_\varepsilon)^{-1}u_\varepsilon\big(x_\varepsilon + u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}x\big)$. It is easily seen that for $\varepsilon > 0$ small, and all $x \in \mathcal{B}_0(2)$,

$$\left|x_\varepsilon + u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}x\right| \geq \frac{1}{2}\left|x_\varepsilon\right|. \tag{1.19}$$

Then, for all $x \in \mathcal{B}_0(2)$,

$$\begin{aligned}\tilde{v}_\varepsilon(x) &\leq 2^{\frac{n}{2}-1}|x_\varepsilon|^{1-\frac{n}{2}}u_\varepsilon(x_\varepsilon)^{-1}v_\varepsilon\big(x_\varepsilon + u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}x\big)\\ &\leq 2^{\frac{n}{2}-1}|x_\varepsilon|^{1-\frac{n}{2}}u_\varepsilon(x_\varepsilon)^{-1}v_\varepsilon(x_\varepsilon)\end{aligned}$$

so that for $\varepsilon > 0$ small,

$$\sup_{x\in\mathcal{B}_0(2)} \tilde{v}_\varepsilon(x) \leq 2^{\frac{n}{2}-1}. \tag{1.20}$$

Let $R > 0$ be given. By (1.18) and (1.19),

$$\mathcal{B}_{x_\varepsilon}\big(2u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}}\big) \cap \mathcal{B}_0\big(R\mu_\varepsilon\big) = \emptyset \tag{1.21}$$

for $\varepsilon > 0$ small. Noting that

$$\int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star}\,dx = \int_{\mathcal{B}_{x_\varepsilon}(2u_\varepsilon(x_\varepsilon)^{-\frac{2}{n-2}})} u_\varepsilon^{2^\star}\,dx$$

it follows from (1.14) and (1.21) that

$$\int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star}\,dx \to 0 \tag{1.22}$$

as $\varepsilon \to 0$. As is easily checked,

$$\Delta\tilde{v}_\varepsilon - \alpha u_\varepsilon(x_\varepsilon)^{-4/(n-2)}\tilde{v}_\varepsilon \leq \frac{1-\varepsilon}{K_n^2}\tilde{v}_\varepsilon^{2^\star-1}.$$

The De Giorgi-Nash-Moser iterative scheme, (1.20) and (1.22) then give $\sup_{x\in\mathcal{B}_0(1)}\tilde{v}_\varepsilon(x) \to 0$ as $\varepsilon \to 0$. Since $\tilde{v}_\varepsilon(0) = 1$, we get a contradiction. This proves (1.15). The proof of (1.16), that we omit here, goes in the same way. On such a claim, see Druet [12], or Section 3 below. □

Going on with the asymptotic study of the u_ε's, we claim that $r_\varepsilon \to 0$ as $\varepsilon \to 0$. We let $\delta > 0$ and $\eta \in C_c^\infty(\mathcal{B})$ be such that $\eta = 0$ in $\mathcal{B}_0(\frac{\delta}{2})$, $\eta = 1$ in $\mathcal{B}_0(\frac{1}{2})\backslash\mathcal{B}_0(\delta)$. Multiplying (1.6) by η and integrating over $\mathcal{B}$, we get with (1.17)

$$\begin{aligned}B_\varepsilon\|u_\varepsilon\|_1\int_{\mathcal{B}}\eta\Sigma_\varepsilon dx &= \frac{1-\varepsilon}{K_n^2}\int_{\mathcal{B}}\eta u_\varepsilon^{2^\star-1}dx + \alpha\int_{\mathcal{B}}\eta u_\varepsilon dx - \int_{\mathcal{B}}(\Delta\eta)u_\varepsilon dx\\ &= O(\|u_\varepsilon\|_1).\end{aligned}$$

Since $B_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it follows that $\int_{\mathcal{B}} \eta \Sigma_\varepsilon dx \to 0$ as $\varepsilon \to 0$. In particular,

$$\int_{\mathcal{B}_0(\frac{1}{2})\backslash\mathcal{B}_0(\delta)} \Sigma_\varepsilon dx \to 0$$

as $\varepsilon \to 0$, and since this holds for any $\delta > 0$, we get $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Now we prove stronger estimates than (1.15) and (1.16).

1.2. **Strong estimates.** We borrow arguments from Druet and Robert [17], [18]. We define L_ε by

$$L_\varepsilon u = \Delta u - \alpha u - \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star-2} u.$$

Letting $\delta > 0$ sufficiently small so that $\Delta - \alpha$ is coercive on $\mathcal{B}_0(\delta)$, we claim first that L_ε satisfies the maximum principle on $\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)$ for $R > 0$ large and $\varepsilon > 0$ small. Let indeed $z \in C^1\big(\overline{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)}\big)$ be such that $z \geq 0$ on $\partial\big(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)\big)$ and $L_\varepsilon z \geq 0$. Set $z^- = \max(0, -z)$. Then,

$$\begin{aligned} 0 \leq \int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)} z^- L_\varepsilon z dx = &-\|\nabla z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \\ &+ \alpha\|z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} + \frac{1-\varepsilon}{K_n^2}\int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2}(z^-)^2 dx \end{aligned}$$

while, thanks to Hölder's inequality,

$$\int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2}(z^-)^2 dx \leq \|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}\|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}.$$

Thus,

$$\begin{aligned} 0 \leq &-\|\nabla z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} + \alpha\|z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \\ &+ \frac{1-\varepsilon}{K_n^2}\|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}\|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}. \end{aligned} \tag{1.23}$$

By (1.14),

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\|u_\varepsilon\|_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} = 0.$$

It follows that for any $A > 0$, there exist $\varepsilon_A > 0$ and $R_A > 0$ such that for $R \geq R_A$ and $\varepsilon \in (0, \varepsilon_A)$,

$$\|u_\varepsilon\|_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \leq A.$$

Let $B > 0$, given by the coercivity of $L = \Delta - \alpha$ on $\mathcal{B}_0(\delta)$, be such that

$$B\|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} \leq \|\nabla z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))} - \alpha\|z^-\|^2_{L^2(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}.$$

Coming back to (1.23), we have

$$0 \le \|z^-\|^2_{L^{2^\star}(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon))}\Big(\frac{1-\varepsilon}{K_n^2}A^{2^\star-2} - B\Big).$$

Choosing $A > 0$ small, this implies $z^- = 0$. The claim is proved. From now on, we let $c > 0$ be such that $\tilde{L} = \Delta - (\alpha + c)$ is coercive on $\mathcal{B}_0(\delta)$. We let also G be the Green function of $\tilde{L}$ in $\mathcal{B}_0(\delta)$ with zero Dirichlet boundary condition, and set $H(x) = G(0, x)$. We fix $\nu > 0$ small, sufficiently small so that $(1-\nu)c - \nu\alpha > 0$. Then, in $\mathcal{B}_0(\delta)\backslash\{0\}$,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} = \nu(1-\nu)\frac{|\nabla H|^2}{H^2} + \alpha_0 - \frac{1-\varepsilon}{K_n^2}u_\varepsilon^{2^\star-2}, \tag{1.24}$$

where $\alpha_0 > 0$ is given by $\alpha_0 = (1-\nu)c - \nu\alpha$. An easy property of the Green function is that there exists $C_0 > 0$ and $\rho_0 > 0$ such that for $|x| \le \rho_0$,

$$\frac{|\nabla H|^2}{H^2} \ge \frac{C_0}{|x|^2}.$$

Thanks to (1.16), for $R > 0$ large and $\varepsilon > 0$ small,

$$\frac{\nu(1-\nu)C_0}{|x|^2} \ge \frac{1-\varepsilon}{K_n^2}u_\varepsilon^{2^\star-2}$$

in $\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)$. Coming back to (1.24), it follows that $L_\varepsilon H^{1-\nu} \ge 0$ in the annulus $\mathcal{B}_0(\rho_0)\backslash\mathcal{B}_0(R\mu_\varepsilon)$. By (1.17), since $\alpha_0 > 0$, we also have, for $\varepsilon > 0$ small, $L_\varepsilon H^{1-\nu} \ge 0$ outside $\mathcal{B}_0(\rho_0)$. Hence, $L_\varepsilon H^{1-\nu} \ge 0$ in $\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)$ provided that $R > 0$ is large and $\varepsilon > 0$ is small. We fix $R > 0$ large. By (1.15), there exists $C_1 > 0$ such that $u_\varepsilon \le C_1\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)}|x|^{(2-n)(1-\nu)}$ on $\partial\mathcal{B}_0(R\mu_\varepsilon)$. We also have that there exists $C_2 > 0$ such that $H \ge C_2|x|^{2-n}$ around 0, and that there exists $C_3 > 0$ such that $H \le C_3|x|^{2-n}$. Then, since $L_\varepsilon u_\varepsilon = 0$ and $u_\varepsilon = 0$ on $\partial\mathcal{B}_0(\delta)$, we get that there exists $C_4 > 0$ such that

$$L_\varepsilon\big(C_4\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)}H^{1-\nu} - u_\varepsilon\big) \ge 0 \text{ in } \mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)\text{ , and}$$
$$C_4\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)}H^{1-\nu} \ge u_\varepsilon \text{ on } \partial\big(\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)\big).$$

By the maximum principle, $u_\varepsilon \le C_4\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)}H^{1-\nu}$ in $\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)$, and then $u_\varepsilon \le C_5\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)}|x|^{(1-\nu)(2-n)}$ in $\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(R\mu_\varepsilon)$ for some $C_5 > 0$. It is clear that this inequality holds also in $\mathcal{B}_0(R\mu_\varepsilon)$, up to changing C_5. As a consequence, we proved that for $\nu > 0$ small, there exists $C_6 > 0$, such that for $\varepsilon > 0$ small,

$$u_\varepsilon \le C_6\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)}|x|^{(1-\nu)(2-n)} \tag{1.25}$$

in $\mathcal{B}_0(\delta)$, and thus, also in $\mathcal{B}$. Pushing further the analysis, we let now (y_ε) be a sequence of points in $\mathcal{B}_0(\frac{\delta_0}{2})$, and let $\tilde{G}$ be the Green function of $L = \Delta - \alpha$ in $\mathcal{B}_0(\delta_0)$ with zero Dirichlet boundary condition, where $\delta_0 > 0$ is such that $L = \Delta - \alpha$ is coercive on $\mathcal{B}_0(\delta_0)$. Thanks to (1.6), and since $r_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$u_\varepsilon(y_\varepsilon) \leq \frac{1}{K_n^2} \int_{\mathcal{B}_0(\delta_0)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star - 1}(x) dx. \tag{1.26}$$

We set $\Phi_\varepsilon = u_\varepsilon(y_\varepsilon)\mu_\varepsilon^{1-\frac{n}{2}}|y_\varepsilon|^{n-2}$ and distinguish three cases.

Case 1: we assume $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to R$ as $\varepsilon \to 0$, $R \in [0, +\infty)$. Then, thanks to (1.15), (Φ_ε) is bounded.

Case 2: we assume $y_\varepsilon \to y_0$ as $\varepsilon \to 0$, where $y_0 \neq 0$, and let $\delta > 0$ be such that $2\delta \leq |y_0|$. Then,

$$\begin{aligned}
&\int_{\mathcal{B}_0(\delta_0)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star - 1}(x) dx \\
&\leq \int_{\mathcal{B}_0(\delta)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star - 1}(x) dx + \int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_0(\delta)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star - 1}(x) dx \\
&\leq C \int_{\mathcal{B}_0(\delta)} u_\varepsilon^{2^\star - 1} dx + C \int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_0(\delta)} \frac{1}{|y_\varepsilon - x|^{n-2}} u_\varepsilon^{2^\star - 1} dx,
\end{aligned}$$

where $C > 0$ is independent of ε. By (1.25), with $(n+2)\nu < 2$,

$$\int_{\mathcal{B}_0(\delta_0)\backslash\mathcal{B}_0(\delta)} \frac{1}{|y_\varepsilon - x|^{n-2}} u_\varepsilon^{2^\star - 1} dx = o\big(\mu_\varepsilon^{\frac{n}{2}-1}\big).$$

Independently,

$$\int_{\mathcal{B}_0(\delta)} u_\varepsilon^{2^\star - 1} dx = \int_{\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dx + \int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dx.$$

By (1.11),

$$\int_{\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dx = O\big(\mu_\varepsilon^{\frac{n}{2}-1}\big)$$

while by (1.25) where $\nu > 0$ is chosen sufficiently small such that $(n+2)\nu < 2$,

$$\int_{\mathcal{B}_0(\delta)\backslash\mathcal{B}_0(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dx = O\big(\mu_\varepsilon^{\frac{n}{2}-1}\big).$$

By (1.26), this implies (Φ_ε) is bounded.

Case 3: we assume $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to +\infty$ and $|y_\varepsilon| \to 0$ as $\varepsilon \to 0$. Then, by (1.25),

$$\int_{\mathcal{B}_0(\delta_0)} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star - 1}(x) dx$$

$$\leq \int_{\mathcal{B}_{y_\varepsilon}(\frac{|y_\varepsilon|}{2})} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx + \int_{\mathcal{B}_0(\delta_0)\backslash \mathcal{B}_{y_\varepsilon}(\frac{|y_\varepsilon|}{2})} \tilde{G}(y_\varepsilon, x) u_\varepsilon^{2^\star-1}(x)dx$$
$$\leq C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)}|y_\varepsilon|^{(\nu-1)(n+2)}\int_{\mathcal{B}_{y_\varepsilon}(\frac{|y_\varepsilon|}{2})}\frac{1}{|y_\varepsilon - x|^{n-2}}dx + C\frac{1}{|y_\varepsilon|^{n-2}}\int_{\mathcal{B}_0(\delta_0)} u_\varepsilon^{2^\star-1}dx$$
$$\leq C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)}|y_\varepsilon|^{(\nu-1)(n+2)+2} + C\frac{1}{|y_\varepsilon|^{n-2}}\mu_\varepsilon^{\frac{n}{2}-1},$$

where $C > 0$ does not depend on ε. Thanks to (1.26) we then get

$$|y_\varepsilon|^{n-2}\mu_\varepsilon^{1-\frac{n}{2}}u_\varepsilon(y_\varepsilon) \leq C\Big(\frac{\mu_\varepsilon}{|y_\varepsilon|}\Big)^{2-(n+2)\nu} + C$$

and since $\frac{|y_\varepsilon|}{\mu_\varepsilon} \to +\infty$ as $\varepsilon \to 0$, we get (Φ_ε) is bounded.

Summarizing cases 1 to 3, for any sequence (y_ε) in $\mathcal{B}_0(\frac{\delta_0}{2})$, there exists $C > 0$ such that

$$\mu_\varepsilon^{1-\frac{n}{2}}|y_\varepsilon|^{n-2}u_\varepsilon(y_\varepsilon) \leq C.$$

Since the u_ε's are radially decreasing, this implies that there exists $C > 0$ such that for any $x \in \mathcal{B}$ and any $\varepsilon > 0$,

$$\mu_\varepsilon^{1-\frac{n}{2}}|x|^{n-2}u_\varepsilon(x) \leq C. \tag{1.27}$$

An equivalent formulation of (1.27) is that for any $x \in \mathcal{B}$ and any $\varepsilon > 0$,

$$u_\varepsilon(x) \leq C\mu_\varepsilon^{1-\frac{n}{2}}\Big(\frac{1}{1 + \frac{\omega_n^{2/n}}{4\mu_\varepsilon^2}|x|^2}\Big)^{\frac{n-2}{2}}, \tag{1.28}$$

where $C > 0$ is independent of x and ε. □

Going on with the proof of (1.4) and (1.5), the goal of the following subsection is to estimate r_ε in terms of μ_ε. We start with the case $n \geq 5$.

1.3. **Estimating r_ε with respect to μ_ε when $n \geq 5$.** As already mentioned, we want to estimate r_ε in terms of μ_ε. For that purpose, we define the function $\hat{u}_\varepsilon$ by

$$\hat{u}_\varepsilon(x) = r_\varepsilon^{\frac{n}{2}-1}u_\varepsilon(r_\varepsilon x). \tag{1.29}$$

It is easily seen that $\hat{u}_\varepsilon > 0$ in $\mathcal{B}$, $\hat{u}_\varepsilon = 0$ on $\partial\mathcal{B}$,

$$\Delta\hat{u}_\varepsilon - \alpha r_\varepsilon^2\hat{u}_\varepsilon + B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} = \frac{1-\varepsilon}{K_n^2}\hat{u}_\varepsilon^{2^\star-1} \tag{1.30}$$

in $\mathcal{B}$, and

$$\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star}dx = 1. \tag{1.31}$$

Moreover, if we set $\hat{\mu}_\varepsilon = \mu_\varepsilon / r_\varepsilon$, then

$$\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x) \to \Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n-2}{2}} \tag{1.32}$$

in $C^0_{loc}(\mathbb{R}^n)$, and, thanks to (1.27),

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}|x|^{n-2}\hat{u}_\varepsilon(x) \le C \tag{1.33}$$

for any $x \in \mathcal{B}$. By (1.32), $\hat{\mu}_\varepsilon \to 0$ as $\varepsilon \to 0$. As another remark, since u_ε is C^1 in $\mathcal{B}$, we have

$$\partial_\nu \hat{u}_\varepsilon = 0 \ \text{ on } \partial\mathcal{B}. \tag{1.34}$$

Multiplying (1.30) by $\hat{\mu}_\varepsilon^{1-\frac{n}{2}}$ and integrating we get

$$-\alpha r_\varepsilon^2 \hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon dx + B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}|\mathcal{B}| = \frac{1-\varepsilon}{K_n^2}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star-1}dx,$$

where $|\mathcal{B}|$ is the volume of $\mathcal{B}$. Since

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon(x)^{2^\star-1}dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\Big(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\Big)^{2^\star-1}dx$$

we get with (1.32) and (1.33)

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon(x)^{2^\star-1}dx \to \int_{\mathbb{R}^n}\Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n+2}{2}}dx$$

as $\varepsilon \to 0$. Independently, since $r_\varepsilon \to 0$ as $\varepsilon \to 0$, (1.33) gives

$$r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon dx \to 0$$

as $\varepsilon \to 0$. Noting that

$$\frac{1}{K_n^2}\int_{\mathbb{R}^n}\Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n+2}{2}}dx = (n-2)2^{n-2}\omega_n^{\frac{2}{n}-1}\omega_{n-1}$$

it follows that

$$B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \to A_n \tag{1.35}$$

as $\varepsilon \to 0$, where

$$A_n = n(n-2)2^{n-2}\omega_n^{\frac{2}{n}-1}. \tag{1.36}$$

We have

$$\Delta(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon) - \alpha r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon + B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}} = \frac{1-\varepsilon}{K_n^2}\hat{\mu}_\varepsilon^2\Big(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon\Big)^{2^\star-1}$$

and the coefficients in this equation are bounded thanks to (1.35). Since the sequence $(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon)$ is bounded in any compact subset of $\overline{\mathcal{B}}\backslash\{0\}$, we get by standard elliptic theory

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon \to \Phi \text{ in } C^1_{loc}\big(\overline{\mathcal{B}}\backslash\{0\}\big), \tag{1.37}$$

where Φ is a solution of

$$\Delta\Phi + A_n = 0 \tag{1.38}$$

in $\mathcal{B}\backslash\{0\}$. Clearly, Φ is radially symmetrical and decreasing in $\overline{\mathcal{B}}\backslash\{0\}$. Moreover,

$$\Phi = 0 \text{ and } \partial_\nu\Phi = 0 \text{ on } \partial\mathcal{B}.$$

Integrating (1.38) on $\mathcal{B}\backslash\mathcal{B}_0(r)$, we then get

$$\Phi(x) = \frac{A_n}{n(n-2)}\Big(\frac{1}{|x|^{n-2}} - 1\Big) + \frac{A_n}{2n}(|x|^2 - 1). \tag{1.39}$$

Now we apply the Pohozaev identity to $\hat{u}_\varepsilon$ in $\mathcal{B}$. The Pohozaev identity [31] for $\hat{u}_\varepsilon$ in $\mathcal{B}$ states that

$$\begin{aligned}&\int_{\partial\mathcal{B}}(x,\nu)(\partial_\nu\hat{u}_\varepsilon)^2 d\sigma + (n-2)\int_{\partial\mathcal{B}}\hat{u}_\varepsilon(\partial_\nu\hat{u}_\varepsilon)d\sigma\\ &\quad = -2\int_{\mathcal{B}}(\nabla\hat{u}_\varepsilon, x)\Delta\hat{u}_\varepsilon dx - (n-2)\int_{\mathcal{B}}\hat{u}_\varepsilon\Delta\hat{u}_\varepsilon dx,\end{aligned}$$

where ν is the unit outer normal to $\partial\mathcal{B}$. Since $\hat{u}_\varepsilon = 0$ and $\partial_\nu\hat{u}_\varepsilon = 0$ on $\partial\mathcal{B}$, we get with (1.30)

$$\alpha r_\varepsilon^2\int_{\mathcal{B}}\hat{u}_\varepsilon^2 dx = \frac{n+2}{2}B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}}\hat{u}_\varepsilon dx.$$

By (1.33), (1.35), (1.37), and (1.39), this implies

$$\frac{1}{\hat{\mu}_\varepsilon^{n-2}}r_\varepsilon^2\int_{\mathcal{B}}\hat{u}_\varepsilon^2 dx \to \frac{n+2}{2\alpha}A_n\int_{\mathcal{B}}\Phi dx = \frac{A_n^2}{4n\alpha}\omega_{n-1} \tag{1.40}$$

as $\varepsilon \to 0$. Independently,

$$\frac{1}{\hat{\mu}_\varepsilon^2}\int_{\mathcal{B}}\hat{u}_\varepsilon(x)^2 dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\Big(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\Big)^2 dx$$

and by (1.32) and (1.33) we get that when $n \geq 5$,

$$\int_{\mathcal{B}}\hat{u}_\varepsilon(x)^2 dx = \Big(\int_{\mathbb{R}^n}\Big(1 + \frac{\omega_n^{2/n}}{4}|x|^2\Big)^{2-n}dx\Big)\hat{\mu}_\varepsilon^2 + o\Big(\hat{\mu}_\varepsilon^2\Big).$$

It is easily seen, see for instance Demengel and Hebey [11], that

$$\int_{\mathbb{R}^n} \Big(1 + \frac{\omega_n^{2/n}}{4}|x|^2\Big)^{2-n} dx = 2^{n-1}\frac{\omega_{n-1}}{\omega_n}\frac{\Gamma(\frac{n}{2})\Gamma(\frac{n}{2}-2)}{\Gamma(n-2)}\,,$$

where Γ is the Euler function. Since $\Gamma(n) = 2^{n-1}\frac{\omega_{n-1}}{\omega_n}\Gamma(\frac{n}{2})^2$ we get, when $n \geq 5$,

$$\int_{\mathcal{B}} \hat{u}_\varepsilon(x)^2 dx = \frac{4(n-1)}{n-4}\hat{\mu}_\varepsilon^2 + o\Big(\hat{\mu}_\varepsilon^2\Big). \tag{1.41}$$

Combining (1.40) and (1.41), it follows

$$\lim_{\varepsilon\to 0} \frac{r_\varepsilon^2}{\hat{\mu}_\varepsilon^{n-4}} = \frac{(n-4)\omega_{n-1}A_n^2}{16n(n-1)\alpha} \tag{1.42}$$

when $n \geq 5$, where A_n is given by (1.36). □

The goal of the following subsection is to estimate r_ε in terms of μ_ε in the limit case $n = 4$. For that purpose, a stronger estimate than (1.28) is needed.

1.4. **Estimating r_ε with respect to μ_ε when $n = 4$.** We claim, when $n = 4$,

$$\lim_{\varepsilon\to 0} |\ln \hat{\mu}_\varepsilon| r_\varepsilon^2 = \frac{4}{\alpha}. \tag{1.43}$$

In order to prove this claim, we let (y_ε) be a sequence of points in $\mathcal{B}$ such that $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty$ as $\varepsilon \to 0$. We let also $\hat{v}_\varepsilon$ be the function given by $\hat{v}_\varepsilon(x) = |y_\varepsilon|^{\frac{n}{2}-1}\hat{u}_\varepsilon(|y_\varepsilon|x)$. Then,

$$\Delta\hat{v}_\varepsilon - \alpha r_\varepsilon^2|y_\varepsilon|^2\hat{v}_\varepsilon + B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1} = \frac{1-\varepsilon}{K_n^2}\hat{v}_\varepsilon^{2^\star-1}$$

in $\mathcal{B}_0(\frac{1}{|y_\varepsilon|})$, and if $\hat{w}_\varepsilon = \big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\big)^{1-\frac{n}{2}}\hat{v}_\varepsilon$ we get

$$\Delta\hat{w}_\varepsilon - \alpha r_\varepsilon^2|y_\varepsilon|^2\hat{w}_\varepsilon + B_\varepsilon\|u_\varepsilon\|_1\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1} = \frac{1-\varepsilon}{K_n^2}\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^2\hat{w}_\varepsilon^{2^\star-1}. \tag{1.44}$$

By (1.32) and (1.33),

$$\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^{n-2}\hat{w}_\varepsilon\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}x\Big) \to \Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n-2}{2}} \tag{1.45}$$

in $C^0_{loc}(\mathbb{R}^n)$, and

$$|x|^{n-2}\hat{w}_\varepsilon(x) \leq C. \tag{1.46}$$

Integrating (1.44) over $\mathcal{B}_0(\frac{1}{|y_\varepsilon|})$, we get

$$\begin{aligned} &B_\varepsilon\|u_\varepsilon\|_1\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1}\frac{\omega_{n-1}}{n|y_\varepsilon|^n}\\ &\quad= \alpha r_\varepsilon^2|y_\varepsilon|^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)dx + \frac{1-\varepsilon}{K_n^2}\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)^{2^\star-1}dx. \end{aligned} \tag{1.47}$$

By (1.46),

$$|y_\varepsilon|^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)dx = |y_\varepsilon|^{2-n}\int_{\mathcal{B}}\hat w_\varepsilon\Big(\frac{x}{|y_\varepsilon|}\Big)dx \le C\int_{\mathcal{B}}\frac{1}{|x|^{n-2}}dx = C'.$$

Independently,

$$\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)^{2^\star-1}dx = \int_{\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})}\Big(\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^{n-2}\hat w_\varepsilon\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}x\Big)\Big)^{2^\star-1}dx$$

and thanks to (1.45) and (1.46), it follows that

$$\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^2\int_{\mathcal{B}_0(\frac{1}{|y_\varepsilon|})}\hat w_\varepsilon(x)^{2^\star-1}dx \to \int_{\mathbb{R}^n}\Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n+2}{2}}dx$$

as $\varepsilon \to 0$. Coming back to (1.47), and since $|y_\varepsilon| \to 0$ as $\varepsilon \to 0$, we get

$$B_\varepsilon\|u_\varepsilon\|_1\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1} \to 0 \tag{1.48}$$

as $\varepsilon \to 0$. Noting that the sequence $(\hat w_\varepsilon)$ is bounded in any compact subset of $\mathbb{R}^n\backslash\{0\}$, it follows from standard elliptic theory, (1.44), and (1.48), that $\hat w_\varepsilon \to \Psi$ in $C^1_{loc}(\mathbb{R}^n\backslash\{0\})$, where Ψ is a solution of $\Delta\Psi = 0$ in $\mathbb{R}^n\backslash\{0\}$. We let $\delta > 0$ small, and we integrate (1.44) over $\mathcal{B}_0(\delta)$. Then,

$$\begin{aligned} &-\int_{\partial\mathcal{B}_0(\delta)}\partial_\nu\hat w_\varepsilon d\sigma - \alpha r_\varepsilon^2|y_\varepsilon|^2\int_{\mathcal{B}_0(\delta)}\hat w_\varepsilon dx\\ &+ B_\varepsilon\|u_\varepsilon\|_1\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1}|y_\varepsilon|^{\frac{n}{2}+1}|\mathcal{B}_0(\delta)| = \frac{1-\varepsilon}{K_n^2}\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^2\int_{\mathcal{B}_0(\delta)}\hat w_\varepsilon^{2^\star-1}dx. \end{aligned} \tag{1.49}$$

With the same arguments as above, it is easily seen that

$$r_\varepsilon^2|y_\varepsilon|^2\int_{\mathcal{B}_0(\delta)}\hat w_\varepsilon dx \to 0$$

and

$$\Big(\frac{\hat\mu_\varepsilon}{|y_\varepsilon|}\Big)^2\int_{\mathcal{B}_0(\delta)}\hat w_\varepsilon^{2^\star-1}dx \to \int_{\mathbb{R}^n}\Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n+2}{2}}dx$$

as $\varepsilon \to 0$. Since $\hat{w}_\varepsilon \to \Psi$ in $C^1_{loc}(\mathbb{R}^n)$, we also have

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma \to \int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma.$$

Passing to the limit as $\varepsilon \to 0$ in (1.49), it follows that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma + \frac{1}{K_n^2}\int_{\mathbb{R}^n}\Big(\frac{1}{1+\frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n+2}{2}} dx = 0$$

and thus

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma + \frac{\omega_{n-1}}{n} A_n = 0, \tag{1.50}$$

where A_n is as in (1.36). In particular, $\Psi \not\equiv 0$. Independently, we have $\Psi \geq 0$, and, thanks to (1.46), there exists $C > 0$ such that

$$|x|^{n-2}\Psi(x) \leq C \tag{1.51}$$

for all $x \in \mathbb{R}^n\backslash\{0\}$. Then the Kelvin transform $\tilde{\Psi}$ of Ψ given by

$$\tilde{\Psi}(x) = \frac{1}{|x|^{n-2}}\Psi\Big(\frac{x}{|x|^2}\Big)$$

is bounded and harmonic in $\mathbb{R}^n\backslash\{0\}$. In particular, $\tilde{\Psi}(x) = o(|x|^{2-n})$ as $|x| \to 0$. Thus, see for instance the excellent paper by Han-Lin [20], 0 is a removable singularity for $\tilde{\Psi}$, and Liouville's theorem implies that $\tilde{\Psi}$ is constant. Hence, there exists $\lambda > 0$ such that $\Psi(x) = \lambda/|x|^{n-2}$ and, thanks to (1.50) we get

$$\Psi(x) = \frac{A_n}{n(n-2)|x|^{n-2}},$$

where A_n is as in (1.36). In particular, taking $x = y_\varepsilon/|y_\varepsilon|$, we get for any sequence (y_ε) in $\mathcal{B}$ such that $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty$ as $\varepsilon \to 0$,

$$|y_\varepsilon|^{n-2}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon(y_\varepsilon) \to \frac{A_n}{n(n-2)} = 2^{n-2}\omega_n^{\frac{2}{n}-1} \tag{1.52}$$

as $\varepsilon \to 0$. Combining (1.32), (1.37), (1.39), and (1.52), it follows that for any $\delta > 0$ and any $x \in \mathcal{B}_0(\delta)$,

$$\frac{1}{C(\delta)}\Big(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n-2}{2}} \leq \hat{u}_\varepsilon(x) \leq C(\delta)\Big(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n-2}{2}} \tag{1.53}$$

for $\varepsilon > 0$ small, where $C(\delta) > 1$ is such that $C(\delta) \to 1$ as $\delta \to 0$. When $n = 4$, and for $\delta > 0$ small, we get with (1.33)

$$\int_{\mathcal{B}\backslash\mathcal{B}_0(\delta)} \hat{u}_\varepsilon^2 dx = O(\hat{\mu}_\varepsilon^2).$$

Thus,

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \int_{\mathcal{B}_0(\delta)} \hat{u}_\varepsilon^2 dx + O(\hat{\mu}_\varepsilon^2). \tag{1.54}$$

Independently,

$$\int_{\mathcal{B}_0(\delta)} \Big(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_4^{1/2}}{4}|x|^2}\Big)^2 dx = \frac{16\omega_3}{\omega_4}\hat{\mu}_\varepsilon^2 \int_0^{\frac{\omega_4^{1/4}\delta}{2\hat{\mu}_\varepsilon}} \Big(1 + r^2\Big)^{-2} r^3 dr$$
$$= \frac{16\omega_3}{\omega_4}\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\Big(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\Big).$$

Then, coming back to (1.54), and thanks to (1.53), we get

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4}\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\Big(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\Big). \tag{1.55}$$

Combining (1.40) and (1.55), this proves (1.43). □

With independent arguments we also have

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}_0(r_\varepsilon)} u_\varepsilon dx = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}} \hat{u}_\varepsilon dx$$

and (1.33), (1.37), and (1.39) imply

$$r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \to \int_{\mathcal{B}} \Phi dx = \frac{A_n \omega_{n-1}}{2n(n+2)}$$

as $\varepsilon \to 0$. By (1.35), we then get

$$B_\varepsilon r_\varepsilon^{n+2} \to \frac{2n(n+2)}{\omega_{n-1}} \tag{1.56}$$

as $\varepsilon \to 0$. As already mentioned, we want to describe the behavior of B_ε in terms of ε as $\varepsilon \to 0$. Thanks to (1.42), (1.43), and (1.56), the question reduces to describing the behavior of $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. This is the subject of the two following subsections.

1.5. **Estimating $\hat{\mu}_\varepsilon$ in terms of ε (Part 1).** We want to describe the behavior of $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. For that purpose, we let

$$\hat{u}_\varepsilon = (1+\theta_\varepsilon)U_\varepsilon + \hat{\mu}_\varepsilon^{\frac{n}{2}-1}(G_\varepsilon + w_\varepsilon), \tag{1.57}$$

where

$$\begin{aligned} U_\varepsilon &= \bar{\mu}_\varepsilon^{\frac{n}{2}-1}\Big(\bar{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}(1-\varepsilon)|x|^2\Big)^{1-\frac{n}{2}}, \\ G_\varepsilon &= \alpha_\varepsilon(|x|^2 - \beta_\varepsilon), \quad \alpha_\varepsilon = \frac{1}{2n} B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \hat{\mu}_\varepsilon^{1-\frac{n}{2}}, \end{aligned} \tag{1.58}$$

θ_ε, β_ε are real numbers, $\bar{\mu}_\varepsilon$ is a positive real number and w_ε is a function. We choose θ_ε and $\bar{\mu}_\varepsilon$ such that

$$\int_{\mathcal{B}} \Big(\nabla U_\varepsilon, \nabla w_\varepsilon\Big) dx = 0, \qquad \int_{\mathcal{B}} \Big(\nabla\Big(x, \nabla U_\varepsilon\Big), \nabla w_\varepsilon\Big) dx = 0. \tag{1.59}$$

Let

$$U_\mu = \mu^{\frac{n}{2}-1}\Big(\mu^2 + \frac{\omega_n^{2/n}}{4}(1-\varepsilon)|x|^2\Big)^{1-\frac{n}{2}}.$$

To get (1.59), it suffices to choose θ_ε and $\bar{\mu}_\varepsilon$ such that they minimize

$$J\Big(\theta,\mu\Big) = \int_{\mathcal{B}} \Big|\nabla\Big(\hat{u}_\varepsilon - (1+\theta)U_\mu\Big) - 2\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\alpha_\varepsilon x\Big|^2 dx$$

among the θ's in $[-\frac{1}{2}, \frac{1}{2}]$ and the μ's in $[\frac{\hat{\mu}_\varepsilon}{2}, 2\hat{\mu}_\varepsilon]$ and to prove that θ_ε and $\bar{\mu}_\varepsilon$ lie in the interior of the interval of constraints for ε small enough. We prove indeed

$$\theta_\varepsilon \to 0 \tag{1.60}$$

as $\varepsilon \to 0$ and

$$\frac{\hat{\mu}_\varepsilon}{\bar{\mu}_\varepsilon} \to 1 \tag{1.61}$$

as $\varepsilon \to 0$. By (1.11), it is clear that $J\big(0,\hat{\mu}_\varepsilon\big) \to 0$ as $\varepsilon \to 0$ so that $J\big(\theta_\varepsilon,\bar{\mu}_\varepsilon\big) \to 0$ as $\varepsilon \to 0$. By (1.11) again, one gets that this enforces the following to happen:

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}} |\nabla\Big((1+\theta_\varepsilon)U_\varepsilon + \hat{\mu}_\varepsilon^{\frac{n}{2}-1}G_\varepsilon\Big)|^2 dx = \frac{1}{K_n^2}$$

and

$$\lim_{\varepsilon\to 0}\int_{\mathcal{B}} \Big(\nabla\Big((1+\theta_\varepsilon)U_\varepsilon + \hat{\mu}_\varepsilon^{\frac{n}{2}-1}G_\varepsilon\Big), \nabla\hat{u}_\varepsilon\Big) dx = \frac{1}{K_n^2},$$

where U_ε and G_ε are as in (1.58). Using (1.11) once again, this is not difficult to check, noting that $\alpha_\varepsilon = O(1)$ thanks to (1.35), these last two relations lead to (1.60) and (1.61). We also choose β_ε such that $w_\varepsilon = 0$ on $\partial\mathcal{B}$. Hence,

$$\alpha_\varepsilon(1-\beta_\varepsilon) + (1+\theta_\varepsilon)\Big(\hat{\mu}_\varepsilon^2 + \frac{(1-\varepsilon)\omega_n^{2/n}}{4}\Big)^{1-\frac{n}{2}} = 0.$$

By (1.35) and (1.61), we have

$$\alpha_\varepsilon \to \frac{A_n}{2n} \text{ and } \beta_\varepsilon \to \frac{n}{n-2} \tag{1.62}$$

as $\varepsilon \to 0$. Thanks to (1.33), (1.37), and (1.39),

$$\int_{\mathcal{B}} w_\varepsilon dx \to 0 \tag{1.63}$$

as $\varepsilon \to 0$. Let W_ε be such that $W_\varepsilon(x) = V_0((1-\varepsilon)^{1/2}x)$, where V_0 is as above. Then,

$$\Delta W_\varepsilon = \frac{1-\varepsilon}{K_n^2} W_\varepsilon^{2^\star-1}$$

and

$$\int_{\mathcal{B}} |\nabla U_\varepsilon|^2 dx = \int_{\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})} |\nabla W_\varepsilon|^2 dx \ , \ \int_{\mathbb{R}^n} |\nabla W_\varepsilon|^2 dx = (1-\varepsilon)^{1-\frac{n}{2}} K_n^{-2}.$$

As an easy consequence, writing

$$\int_{\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})} |\nabla W_\varepsilon|^2 dx = \int_{\mathbb{R}^n} |\nabla W_\varepsilon|^2 dx - \int_{\mathbb{R}^n \setminus \mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})} |\nabla W_\varepsilon|^2 dx$$

we get, thanks to (1.61),

$$\int_{\mathcal{B}} |\nabla U_\varepsilon|^2 dx = \frac{1}{K_n^2} + \frac{n-2}{2K_n^2}\varepsilon - \frac{A_n^2 \omega_{n-1}}{n^2(n-2)} \hat\mu_\varepsilon^{n-2} + o(\hat\mu_\varepsilon^{n-2}) + o(\varepsilon). \quad (1.64)$$

Independent computations give

$$\int_{\mathcal{B}} |\nabla G_\varepsilon|^2 dx = \frac{A_n^2 \omega_{n-1}}{n^2(n+2)} + o(1) \quad (1.65)$$

and

$$\int_{\mathcal{B}} \Big(\nabla G_\varepsilon, \nabla w_\varepsilon\Big) dx = \int_{\mathcal{B}} (\Delta G_\varepsilon) w_\varepsilon dx = o(1) \quad (1.66)$$

thanks to (1.63). Similarly, it is easily seen that

$$\int_{\mathcal{B}} \Big(\nabla G_\varepsilon, \nabla U_\varepsilon\Big) dx = -\frac{A_n^2 \omega_{n-1}}{2n^2} \hat\mu_\varepsilon^{\frac{n}{2}-1} + o(\hat\mu_\varepsilon^{\frac{n}{2}-1}). \quad (1.67)$$

By (1.59), (1.61), and (1.64)-(1.67), we then get

$$\begin{aligned} \int_{\mathcal{B}} |\nabla \hat u_\varepsilon|^2 dx = \frac{1}{K_n^2} + \frac{2\theta_\varepsilon}{K_n^2} + \frac{(n-2)\varepsilon}{2K_n^2} - \frac{A_n^2 \omega_{n-1} \hat\mu_\varepsilon^{n-2}}{n^2-4} \\ + \hat\mu_\varepsilon^{n-2} \int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx + o(\theta_\varepsilon) + o(\varepsilon) + o(\hat\mu_\varepsilon^{n-2}). \end{aligned} \quad (1.68)$$

Now we claim

$$\varepsilon = O(\hat\mu_\varepsilon^{n-2}). \quad (1.69)$$

Applying the sharp Sobolev inequality to $\hat u_\varepsilon$, we get thanks to (1.30) and (1.31)

$$\frac{\varepsilon}{K_n^2} \le \alpha r_\varepsilon^2 \int_{\mathcal{B}} \hat u_\varepsilon^2 dx - B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}} \hat u_\varepsilon dx. \quad (1.70)$$

By (1.40),

$$\alpha r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{\omega_{n-1}}{4n} A_n^2 \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}) \tag{1.71}$$

while (1.33), (1.35), (1.37), and (1.39) imply that

$$B_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}} \hat{u}_\varepsilon dx = \frac{\omega_{n-1}}{2n(n+2)} A_n^2 \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}). \tag{1.72}$$

Combining (1.70)-(1.72), this proves (1.69).

Let us now multiply (1.30) by $\hat{u}_\varepsilon$ and integrate over $\mathcal{B}$. Thanks to (1.68)-(1.72), we get

$$\frac{n\varepsilon}{2K_n^2} + \frac{2\theta_\varepsilon}{K_n^2} + \hat{\mu}_\varepsilon^{n-2} \int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx + o(\theta_\varepsilon) = \frac{\omega_{n-1} A_n^2}{4(n-2)} \hat{\mu}_\varepsilon^{n-2} + o(\hat{\mu}_\varepsilon^{n-2}). \tag{1.73}$$

In particular,

$$\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = o\Big(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}\Big) + O(1)$$

and, thanks to the Sobolev inequality,

$$\Big(\int_{\mathcal{B}} |w_\varepsilon|^{2^\star} dx\Big)^{\frac{2}{2^\star}} = o\Big(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}\Big) + O(1). \tag{1.74}$$

For $1 \le p \le 3$ and X, Y such that $X \ge 0$ and $X + Y \ge 0$,

$$(X+Y)^p = X^p + pX^{p-1}Y + \frac{p(p-1)}{2} X^{p-2} Y^2 + O(|Y|^p)$$

while for $3 \le p \le 4$ and X, Y as above,

$$(X+Y)^p = X^p + pX^{p-1}Y + \frac{p(p-1)}{2} X^{p-2} Y^2 + O(X^{p-3}|Y|^3) + O(|Y|^p).$$

Writing $\int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dx = 1$, we then get

$$\begin{aligned} 1 &= (1+\theta_\varepsilon)^{2^\star} \int_{\mathcal{B}} U_\varepsilon^{2^\star} dx + 2^\star (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx \\ &+ \frac{2^\star(2^\star-1)}{2} \hat{\mu}_\varepsilon^{n-2} (1+\theta_\varepsilon)^{2^\star-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} (G_\varepsilon + w_\varepsilon)^2 dx \\ &+ O\Big(\hat{\mu}_\varepsilon^n \int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\Big) \end{aligned} \tag{1.75}$$

if $n \ge 6$, and

$$\begin{aligned} 1 &= (1+\theta_\varepsilon)^{2^\star} \int_{\mathcal{B}} U_\varepsilon^{2^\star} dx + 2^\star (1+\theta_\varepsilon)^{2^\star-1} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx \\ &+ \frac{2^\star(2^\star-1)}{2} \hat{\mu}_\varepsilon^{n-2} (1+\theta_\varepsilon)^{2^\star-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} (G_\varepsilon + w_\varepsilon)^2 dx \end{aligned} \tag{1.76}$$

$$+ O\Big(\hat{\mu}_\varepsilon^{\frac{3n}{2}-3}\Big(\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\Big)^{3/2^\star}\Big) + O\Big(\hat{\mu}_\varepsilon^n \int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\Big)$$

if $n = 4, 5$. For W_ε as above,

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star} dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon^{2^\star} dx \quad \text{and} \quad \int_{\mathbb{R}^n} W_\varepsilon^{2^\star} dx = \frac{1}{(1-\varepsilon)^{n/2}}.$$

Thanks to (1.61) and (1.69) we then get

$$1 - (1+\theta_\varepsilon)^{2^\star} \int_{\mathcal{B}} U_\varepsilon^{2^\star} dx = -2^\star \theta_\varepsilon - \frac{n}{2}\varepsilon + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}). \tag{1.77}$$

By (1.74) we easily get

$$\hat{\mu}_\varepsilon^n \int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx = o(\hat{\mu}_\varepsilon^{n-2}) + o(\theta_\varepsilon) \tag{1.78}$$

and

$$\hat{\mu}_\varepsilon^{\frac{3n}{2}-3}\Big(\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star} dx\Big)^{3/2^\star} = o(\hat{\mu}_\varepsilon^{n-2}) + o(\theta_\varepsilon). \tag{1.79}$$

Independently, it is easily checked that $\hat{\mu}_\varepsilon^{1-\frac{n}{2}} W_\varepsilon(\frac{1}{\hat{\mu}_\varepsilon}x) = U_\varepsilon(x)$. Hence,

$$\Delta U_\varepsilon = \frac{1-\varepsilon}{K_n^2} U_\varepsilon^{2^\star-1}$$

and, thanks to (1.59), we get

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-1} G_\varepsilon dx.$$

Then,

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx = \hat{\mu}_\varepsilon^{\frac{n}{2}-1} \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon(x)^{2^\star-1} G_\varepsilon(\hat{\mu}_\varepsilon x) dx$$

and we find, thanks to (1.61),

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1}(G_\varepsilon + w_\varepsilon) dx = -\frac{A_n^2 K_n^2 \omega_{n-1}}{2n(n-2)} \hat{\mu}_\varepsilon^{\frac{n}{2}-1} + o(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}). \tag{1.80}$$

Independently, it is easily seen with (1.74)

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon)^2 dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}). \tag{1.81}$$

Coming back to (1.75) and (1.76), we get with (1.77)-(1.81)

$$\hat{\mu}_\varepsilon^{n-2} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx = -\frac{2(n-2)}{n+2}\theta_\varepsilon - \frac{(n-2)^2}{2(n+2)}\varepsilon$$

$$+\frac{A_n^2K_n^2\omega_{n-1}}{n(n+2)}\hat{\mu}_\varepsilon^{n-2}+o(\theta_\varepsilon)+o(\hat{\mu}_\varepsilon^{n-2}). \tag{1.82}$$

On such an assertion, note that

$$\int_{\mathcal{B}}U_\varepsilon^{2^\star-2}w_\varepsilon^2dx=O\Big(\Big(\int_{\mathcal{B}}w_\varepsilon^{2^\star}dx\Big)^{2/2^\star}\Big).$$

Independently, it is easily seen from (1.30)

$$\Delta w_\varepsilon=\frac{1-\varepsilon}{K_n^2}\Big(\hat{u}_\varepsilon^{2^\star-1}-(1+\theta_\varepsilon)U_\varepsilon^{2^\star-1}\Big)\hat{\mu}_\varepsilon^{1-\frac{n}{2}}+\alpha r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon$$

in $\mathcal{B}$. By (1.59) we then get that

$$\int_{\mathcal{B}}|\nabla w_\varepsilon|^2dx=\frac{1-\varepsilon}{K_n^2}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star-1}w_\varepsilon dx+\alpha r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon w_\varepsilon dx. \tag{1.83}$$

Now we want to estimate the terms in the right hand side of (1.83). By (1.37) and (1.39), $|x|^{n-2}w_\varepsilon(x)\to 0$ in $C^0_{loc}(\overline{\mathcal{B}}\backslash\{0\})$ as $\varepsilon\to 0$. Independently, it follows from (1.53) that for $|x|\le\delta$ and ε small, $|x|^{n-2}w_\varepsilon(x)\le\varepsilon(\delta)$, where $\varepsilon(\delta)\to 0$ as $\delta\to 0$. Hence,

$$|x|^{n-2}w_\varepsilon(x)\to 0 \text{ in } C^0(\overline{\mathcal{B}}) \tag{1.84}$$

as $\varepsilon\to 0$. We now write

$$\int_{\mathcal{B}}\frac{\hat{u}_\varepsilon}{|x|^{n-2}}dx=\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\frac{\hat{u}_\varepsilon}{|x|^{n-2}}dx+\int_{\mathcal{B}\backslash\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\frac{\hat{u}_\varepsilon}{|x|^{n-2}}dx.$$

Thanks to (1.42) and (1.43), it follows from (1.32)

$$r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\frac{\hat{u}_\varepsilon}{|x|^{n-2}}dx=O(1)$$

and from (1.33)

$$r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}\backslash\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\frac{\hat{u}_\varepsilon}{|x|^{n-2}}dx=O(1).$$

Then, (1.84) implies

$$r_\varepsilon^2\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon w_\varepsilon dx=o(1). \tag{1.85}$$

For X,Y such that $X\ge 0$ and $X+Y\ge 0$ we write now

$$(X+Y)^{2^\star-1}=X^{2^\star-1}+(2^\star-1)X^{2^\star-2}Y+f(n)O(X^{2^\star-3}Y^2)+O(|Y|^{2^\star-1}),$$

where $f(n)=1$ if $n=4,5$, and $f(n)=0$ if $n\ge 6$. Then

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}\hat{u}_\varepsilon^{2^\star-1}w_\varepsilon dx=(1+\theta_\varepsilon)^{2^\star-1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}}U_\varepsilon^{2^\star-1}w_\varepsilon dx$$

$$+ (2^\star - 1)(1+\theta_\varepsilon)^{2^\star-2}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon)w_\varepsilon dx \tag{1.86}$$
$$+ \hat{\mu}_\varepsilon^{\frac{n}{2}-1} O\Big(\int_{\mathcal{B}} U_\varepsilon^{2^\star-3}(G_\varepsilon + w_\varepsilon)^2|w_\varepsilon|dx\Big)f(n)$$
$$+ \hat{\mu}_\varepsilon^2 O\Big(\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star-1}|w_\varepsilon|dx\Big).$$

As when proving (1.81), it follows from (1.74)

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}(G_\varepsilon + w_\varepsilon)w_\varepsilon dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}). \tag{1.87}$$

Still thanks to (1.74), we easily get

$$\int_{\mathcal{B}} |G_\varepsilon + w_\varepsilon|^{2^\star-1}|w_\varepsilon|dx = O(1) + O(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{1.88}$$

and

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-3}(G_\varepsilon + w_\varepsilon)^2|w_\varepsilon|dx = O(1) + O(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}) \tag{1.89}$$

when $n = 4, 5$. We have already seen that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-1} w_\varepsilon dx = 0. \tag{1.90}$$

Combining (1.86)-(1.90), it follows that

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star-1}w_\varepsilon dx = \frac{n+2}{n-2}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}). \tag{1.91}$$

Coming back to (1.83), we get with (1.85) and (1.91)

$$\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = \frac{n+2}{(n-2)K_n^2}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx + o(1) + o(\theta_\varepsilon \hat{\mu}_\varepsilon^{2-n}). \tag{1.92}$$

Then, combining (1.82) with (1.92),

$$\hat{\mu}_\varepsilon^{n-2}\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx = -\frac{2}{K_n^2}\theta_\varepsilon - \frac{n-2}{2K_n^2}\varepsilon + \frac{A_n^2\omega_{n-1}}{n(n-2)}\hat{\mu}_\varepsilon^{n-2} + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}) \tag{1.93}$$

and coming back to (1.73), we get

$$\varepsilon = \frac{(n-4)\omega_{n-1}}{4n(n-2)}A_n^2K_n^2\hat{\mu}_\varepsilon^{n-2} + o(\theta_\varepsilon) + o(\hat{\mu}_\varepsilon^{n-2}), \tag{1.94}$$

where A_n is given by (1.36). □

As already mentioned, we want to express $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. Thanks to (1.94), if we prove that $\theta_\varepsilon = O(\hat{\mu}_\varepsilon^{n-2})$, then we get a description of $\hat{\mu}_\varepsilon$ in terms of ε as $\varepsilon \to 0$. The following section is devoted to this estimation of θ_ε in terms of $\hat{\mu}_\varepsilon$.

1.6. **Estimating $\hat{\mu}_\varepsilon$ in terms of ε (Part 2).** After (1.94), we claim

$$\theta_\varepsilon = O(\hat{\mu}_\varepsilon^{n-2}). \tag{1.95}$$

We prove (1.95) by contradiction. We assume

$$|\theta_\varepsilon|\hat{\mu}_\varepsilon^{2-n} \to +\infty \tag{1.96}$$

as $\varepsilon \to 0$. Then, by (1.69), (1.73) and (1.82),

$$\lim_{\varepsilon\to 0} \frac{\int_{\mathcal{B}} |\nabla w_\varepsilon|^2 dx}{\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} w_\varepsilon^2 dx} = \frac{2^\star - 1}{K_n^2}\,. \tag{1.97}$$

We contradict (1.97). For that purpose we consider the eigenvalue problem

$$\begin{aligned} &\Delta\varphi_{i,\varepsilon} = \mu_{i,\varepsilon} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon} \text{ in } \mathcal{B} \\ &\varphi_{i,\varepsilon} = 0 \text{ on } \partial\mathcal{B}, \end{aligned} \tag{1.98}$$

where $\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon}\varphi_{j,\varepsilon}dx = \delta_{ij}$ and $\mu_{1,\varepsilon} \le \dots \le \mu_{i,\varepsilon} \le \dots$. Let V_0 be as above, given by

$$V_0(x) = \left(1 + \frac{\omega_n^{2/n}}{4}|x|^2\right)^{1-\frac{n}{2}}.$$

We claim that for any $i \ge 1$,

$$\mu_{i,\varepsilon} \to \mu_i \tag{1.99}$$

as $\varepsilon \to 0$, $\mu_1 \le \dots \le \mu_i \le \dots$, and that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\left(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon}\right)^2 dx \to 0 \tag{1.100}$$

as $\varepsilon \to 0$ for functions $\psi_{i,\varepsilon}$ satisfying

$$\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\psi_{i,\varepsilon}(\hat{\mu}_\varepsilon x) \to \psi_i(x)$$

in $C^0_{loc}(\mathbb{R}^n) \cap L^{2^\star}(\mathbb{R}^n)$ as $\varepsilon \to 0$, where the ψ_i's are such that

$$\Delta\psi_i = \mu_i V_0^{2^\star-2}\psi_i \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} V_0^{2^\star-2}\psi_i^2 dx < +\infty. \tag{1.101}$$

We prove (1.99) and (1.100) by induction. When $i = 1$,

$$\mu_{1,\varepsilon} = \inf_{\{\varphi\in C_c^\infty(\mathcal{B}),\varphi\not\equiv 0\}} \frac{\int_{\mathcal{B}} |\nabla\varphi|^2 dx}{\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi^2 dx}\,.$$

On the one hand, taking $\varphi = U_\varepsilon - U_\varepsilon(1)$, we get $\limsup_{\varepsilon\to 0} \mu_{1,\varepsilon} \le \frac{1}{K_n^2}$. On the other hand, thanks to the sharp Sobolev inequality,

$$\Big(\int_{\mathcal{B}} |\varphi_{1,\varepsilon}|^{2^\star} dx\Big)^{\frac{2}{2^\star}} \le K_n^2 \int_{\mathcal{B}} |\nabla\varphi_{1,\varepsilon}|^2 dx = K_n^2 \mu_{1,\varepsilon} \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{1,\varepsilon}^2 dx$$
$$\le K_n^2 \mu_{1,\varepsilon}\Big(\int_{\mathcal{B}} U_\varepsilon^{2^\star} dx\Big)^{1-\frac{2}{2^\star}}\Big(\int_{\mathcal{B}} |\varphi_{1,\varepsilon}|^{2^\star} dx\Big)^{\frac{2}{2^\star}}$$

and we get that

$$\liminf_{\varepsilon\to 0} \mu_{1,\varepsilon} \ge \frac{1}{K_n^2}.$$

Hence,

$$\mu_{1,\varepsilon} \to \mu_1 = \frac{1}{K_n^2} \tag{1.102}$$

as $\varepsilon \to 0$, and we also have

$$\int_{\mathcal{B}} |\varphi_{1,\varepsilon}|^{2^\star} dx \to 1 \ \text{ and } \ \int_{\mathcal{B}} |\nabla\varphi_{1,\varepsilon}|^2 dx \to \frac{1}{K_n^2}$$

as $\varepsilon \to 0$, since $\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{1,\varepsilon}^2 dx = 1$. We let W_ε be as above, given by

$$W_\varepsilon(x) = \Big(1 + \frac{\omega_n^{2/n}}{4}(1-\varepsilon)|x|^2\Big)^{1-\frac{n}{2}}$$

and let $\hat\varphi_{1,\varepsilon}$ be given by

$$\hat\varphi_{1,\varepsilon}(x) = \hat\mu_\varepsilon^{\frac{n}{2}-1}\varphi_{1,\varepsilon}(\hat\mu_\varepsilon x) \quad \text{in } \mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon})$$
$$\hat\varphi_{1,\varepsilon}(x) = 0 \quad \text{in } \mathbb{R}^n\backslash\mathcal{B}_0(\frac{1}{\hat\mu_\varepsilon}).$$

It is easily seen that

$$\int_{\mathbb{R}^n} W_\varepsilon^{2^\star-2}\hat\varphi_{1,\varepsilon}^2 dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{1,\varepsilon}^2 dx = 1$$

and that the $\hat\varphi_{1,\varepsilon}$'s are bounded in $D_1^2(\mathbb{R}^n)$. We may therefore assume that the $\hat\varphi_{1,\varepsilon}$'s converge weakly to some ψ_1 in $D_1^2(\mathbb{R}^n)$ as $\varepsilon \to 0$. In particular, it is easily seen that

$$\int_{\mathbb{R}^n} \psi_1^{2^\star} dx \le 1. \tag{1.103}$$

For $R > 0$, we write

$$1 = \int_{\mathbb{R}^n} W_\varepsilon^{2^\star-2}\hat\varphi_{1,\varepsilon}^2 dx = \int_{\mathbb{R}^n} (W_\varepsilon^{2^\star-2} - V_0^{2^\star-2})\hat\varphi_{1,\varepsilon}^2 dx$$

$$+\int_{\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx + \int_{\mathbb{R}^n\backslash\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx.$$

By Hölder's inequality,

$$\int_{\mathbb{R}^n}(W_\varepsilon^{2^\star-2}-V_0^{2^\star-2})\hat{\varphi}_{1,\varepsilon}^2 dx \le C\Big(\int_{\mathbb{R}^n}\Big|W_\varepsilon^{2^\star-2}-V_0^{2^\star-2}\Big|^{2^\star/(2^\star-2)}dx\Big)^{1-\frac{2}{2^\star}}$$

and

$$\int_{\mathbb{R}^n\backslash\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx \le C\Big(\int_{\mathbb{R}^n\backslash\mathcal{B}_0(R)} V_0^{2^\star}dx\Big)^{(2^\star-2)/2^\star}.$$

Hence,

$$\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^n}(W_\varepsilon^{2^\star-2}-V_0^{2^\star-2})\hat{\varphi}_{1,\varepsilon}^2 dx = 0$$

$$\lim_{R\to+\infty}\limsup_{\varepsilon\to 0}\int_{\mathbb{R}^n\backslash\mathcal{B}_0(R)} V_0^{2^\star-2}\hat{\varphi}_{1,\varepsilon}^2 dx = 0$$

and we get that

$$\int_{\mathbb{R}^n} V_0^{2^\star-2}\psi_1^2 dx = \lim_{R\to+\infty}\int_{\mathcal{B}_0(R)} V_0^{2^\star-2}\psi_1^2 dx = 1.$$

By Hölder's inequality,

$$\int_{\mathbb{R}^n} V_0^{2^\star-2}\psi_1^2 dx \le \Big(\int_{\mathbb{R}^n} V_0^{2^\star}dx\Big)^{(2^\star-2)/2^\star}\Big(\int_{\mathbb{R}^n}\psi_1^{2^\star}dx\Big)^{2/2^\star} \tag{1.104}$$

and since V_0 is of norm 1 in $L^{2^\star}(\mathbb{R}^n)$, we get from (1.103) and (1.104) that ψ_1 is of norm 1 in $L^{2^\star}(\mathbb{R}^n)$ and that $\psi_1 = V_0$. Then ψ_1 is a solution of (1.101). Writing that

$$\int_{\mathbb{R}^n}|\nabla(\hat{\varphi}_{1,\varepsilon}-\psi_1)|^2 dx = \int_{\mathbb{R}^n}|\nabla\hat{\varphi}_{1,\varepsilon}|^2 dx + \int_{\mathbb{R}^n}|\nabla\psi_1|^2 dx - \frac{2}{K_n^2}\int_{\mathbb{R}^n}\hat{\varphi}_{1,\varepsilon}\psi_1^{2^\star-1}dx$$

we also get that the $\hat{\varphi}_{1,\varepsilon}$'s converge strongly to ψ_1 in $D_1^2(\mathbb{R}^n)$, and in particular that the $\hat{\varphi}_{1,\varepsilon}$'s converge strongly to ψ_1 in $L^{2^\star}(\mathbb{R}^n)$. Then, (1.100) is proved and we get the result for $i=1$. Let us now assume that (1.99)-(1.101) hold for $i=1,\ldots,p$. We have

$$\mu_{p+1,\varepsilon} = \inf_{\varphi\in\mathcal{H}}\int_{\mathcal{B}}|\nabla\varphi|^2 dx,$$

where $\mathcal{H}$ is the set of the functions $\varphi\in C_c^\infty(\mathcal{B})$ which are such that

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi^2 dx = 1 \ \text{ and } \ \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}\varphi_{i,\varepsilon}\varphi dx = 0$$

for all $i=1,\ldots,p$. We claim first that the $\mu_{p+1,\varepsilon}$'s are bounded. It is easily seen that the $\hat{\varphi}_{i,\varepsilon}$'s, $i=1,\ldots,p$, are bounded in $D_1^2(\mathbb{R}^n)$. Then, it follows from (1.100) that the $\hat{\varphi}_{i,\varepsilon}$'s, $i = 1, \dots, p$, converge to ψ_i weakly in $D_1^2(\mathbb{R}^n)$. We let $f \in C_c^\infty(\mathbb{R}^n)$ be such that

$$\int_{\mathbb{R}^n} V_0^{2^\star-2} f \psi_i dx = 0$$

for all $i = 1, \dots, p$. We set $f_\varepsilon(x) = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} f(\frac{1}{\hat{\mu}_\varepsilon} x)$ and

$$\tilde{f}_\varepsilon = f_\varepsilon - \sum_{i=1}^{p} \Big(\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon \varphi_{i,\varepsilon} dx \Big) \varphi_{i,\varepsilon}.$$

For $\varepsilon > 0$ sufficiently small, $\tilde{f}_\varepsilon \in C_c^\infty(\mathcal{B})$, and since $\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} \varphi_{i,\varepsilon} \varphi_{j,\varepsilon} dx = \delta_{ij}$, we have

$$\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} \tilde{f}_\varepsilon \varphi_{i,\varepsilon} dx = 0 \tag{1.105}$$

for all $i = 1, \dots, p$. It is easily checked that

$$\begin{aligned} &\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} \tilde{f}_\varepsilon^2 dx = \int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon^2 dx - \sum_{i=1}^{p} \Big(\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon \varphi_{i,\varepsilon} dx \Big)^2 \\ &\int_{\mathcal{B}} |\nabla \tilde{f}_\varepsilon|^2 dx = \int_{\mathcal{B}} |\nabla f_\varepsilon|^2 dx - \sum_{i=1}^{p} \Big(\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon \varphi_{i,\varepsilon} dx \Big)^2 \mu_{i,\varepsilon} \end{aligned} \tag{1.106}$$

for all $\varepsilon > 0$, and that

$$\begin{aligned} &\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon^2 dx = \int_{\mathbb{R}^n} W_\varepsilon^{2^\star-2} f^2 dx \to \int_{\mathbb{R}^n} V_0^{2^\star-2} f^2 dx \\ &\int_{\mathcal{B}} U_\varepsilon^{2^\star-2} f_\varepsilon \varphi_{i,\varepsilon} dx = \int_{\mathbb{R}^n} W_\varepsilon^{2^\star-2} f \hat{\varphi}_{i,\varepsilon} dx \to \int_{\mathbb{R}^n} V_0^{2^\star-2} f \psi_i dx = 0 \end{aligned} \tag{1.107}$$

as $\varepsilon \to 0$. Since,

$$\int_{\mathcal{B}} |\nabla f_\varepsilon|^2 dx = \int_{\mathbb{R}^n} |\nabla f|^2 dx \tag{1.108}$$

for all $\varepsilon > 0$, we get by combining (1.105)-(1.108) that for $C > 1$ and $\varepsilon > 0$ small,

$$\mu_{p+1,\varepsilon} \le C \frac{\int_{\mathbb{R}^n} |\nabla f|^2 dx}{\int_{\mathbb{R}^n} V_0^{2^\star-2} f^2 dx}.$$

In particular, the $\mu_{p+1,\varepsilon}$'s are bounded, and this proves the above claim. We may then assume that $\mu_{p+1,\varepsilon} \to \mu_{p+1}$ as $\varepsilon \to 0$, where $\mu_{p+1} \ge \mu_p$. As above, the $\hat{\varphi}_{p+1,\varepsilon}$'s are bounded in $D_1^2(\mathbb{R}^n)$. We may therefore assume that the $\hat{\varphi}_{p+1,\varepsilon}$'s converge weakly to some ψ_{p+1} in $D_1^2(\mathbb{R}^n)$. The $\hat{\varphi}_{p+1,\varepsilon}$'s are solutions of

$$\Delta \hat{\varphi}_{p+1,\varepsilon} = \mu_{p+1,\varepsilon} W_\varepsilon^{2^\star-2} \hat{\varphi}_{p+1,\varepsilon}$$

in $\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})$. It is then clear that ψ_{p+1} is a solution of (1.101). Now we write

$$\int_{\mathbb{R}^n} W_\varepsilon^{2^\star-2}(\hat{\varphi}_{p+1,\varepsilon}-\psi_{p+1})^2dx = \int_{\mathcal{B}_0(R)} W_\varepsilon^{2^\star-2}(\hat{\varphi}_{p+1,\varepsilon}-\psi_{p+1})^2dx + o_{R,\varepsilon}(1),$$

where $\lim_{R\to+\infty}\lim_{\varepsilon\to 0} o_{R,\varepsilon}(1)=0$. We may assume that the $\hat{\varphi}_{p+1,\varepsilon}$'s converge to ψ_{p+1} in $L^2_{loc}(\mathbb{R}^n)$. Hence,

$$\int_{\mathbb{R}^n} W_\varepsilon^{2^\star-2}(\hat{\varphi}_{p+1,\varepsilon}-\psi_{p+1})^2dx \to 0$$

as $\varepsilon\to 0$, and this clearly proves that (1.100) holds. By induction, it follows that (1.99)-(1.101) hold for all i. Now, as shown by Bianchi-Egnell [4] and Rey [33], the eigenvalue problem

$$\Delta\psi = \nu V_0^{2^\star-2}\psi \text{ in } \mathbb{R}^n, \quad \int_{\mathbb{R}^n} V_0^{2^\star-2}\psi^2dx < +\infty \tag{1.109}$$

has a discrete spectrum $\nu_1\le\dots\le\nu_i\le\dots$ such that $\nu_1=\frac{1}{K_n^2}$, $\nu_2=\dots=\nu_{n+2}=\frac{2^\star-1}{K_n^2}$, $\nu_{n+3}>\frac{2^\star-1}{K_n^2}$ and the eigenspaces corresponding to the eigenvalues $\frac{1}{K_n^2}$ and $\frac{2^\star-1}{K_n^2}$ are $\mathcal{E}_1=\text{Span}\{V_0\}$ and $\mathcal{E}_2=\text{Span}\{\Phi_j, j=0,\dots,n\}$, where

$$\Phi_0=\Big(1+\frac{\omega_n^{2/n}}{4}|x|^2\Big)^{-\frac{n}{2}}\Big(1-\frac{\omega_n^{2/n}}{4}|x|^2\Big) \text{ and } \Phi_j=\Big(1+\frac{\omega_n^{2/n}}{4}|x|^2\Big)^{-\frac{n}{2}}x_j$$

for $j=1,\dots,n$. Coming back to our problem, we let k_0 be such that $\mu_{k_0+1}>\frac{2^\star-1}{K_n^2}$, and write

$$w_\varepsilon=\sum_{i=1}^{k_0}\alpha_{i,\varepsilon}\varphi_{i,\varepsilon}+R_\varepsilon, \tag{1.110}$$

where w_ε is given by (1.58), and

$$\alpha_{i,\varepsilon}=\frac{\int_{\mathcal{B}}\Big(\nabla w_\varepsilon,\nabla\varphi_{i,\varepsilon}\Big)dx}{\int_{\mathcal{B}}|\nabla\varphi_{i,\varepsilon}|^2dx}=\frac{1}{\mu_{i,\varepsilon}}\int_{\mathcal{B}}\Big(\nabla w_\varepsilon,\nabla\varphi_{i,\varepsilon}\Big)dx.$$

We write

$$\begin{aligned}\Big(\int_{\mathcal{B}}\Big(\nabla w_\varepsilon,\nabla\varphi_{i,\varepsilon}\Big)dx\Big)^2 &\le 2\Big(\int_{\mathcal{B}}\Big(\nabla w_\varepsilon,\nabla\psi_{i,\varepsilon}\Big)dx\Big)^2\\ &+2\Big(\int_{\mathcal{B}}\Big(\nabla w_\varepsilon,\nabla(\varphi_{i,\varepsilon}-\psi_{i,\varepsilon})\Big)dx\Big)^2,\end{aligned}$$

where $\psi_{1,\varepsilon} = U_\varepsilon$ and $\psi_{i,\varepsilon}(x) = \bar{\mu}_\varepsilon^{1-\frac{n}{2}}\psi_i(\frac{\sqrt{1-\varepsilon}}{\bar{\mu}_\varepsilon}x)$ for $2 \le i \le k_0$, so that the $\psi_{i,\varepsilon}$'s when $2 \le i \le k_0$ are linear combinations of the functions $\hat{\Phi}_j$ given by

$$\hat{\Phi}_j(x) = \bar{\mu}_\varepsilon^{1-\frac{n}{2}}\Phi_j(\frac{\sqrt{1-\varepsilon}}{\bar{\mu}_\varepsilon}x).$$

By (1.59), and since the w_ε's are radially symmetrical,

$$\int_{\mathcal{B}}\Big(\nabla w_\varepsilon, \nabla\psi_{i,\varepsilon}\Big)dx = 0$$

for $1 \le i \le k_0$. Hence,

$$\alpha_{i,\varepsilon}^2 \le \frac{2}{\mu_{i,\varepsilon}^2}\int_{\mathcal{B}}|\nabla w_\varepsilon|^2dx\int_{\mathcal{B}}|\nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon})|^2dx. \tag{1.111}$$

When $i = 1$, we have seen that the $\hat{\varphi}_{1,\varepsilon}$'s converge strongly to V_0 in $D_1^2(\mathbb{R}^n)$ as $\varepsilon \to 0$. Hence,

$$\int_{\mathcal{B}}|\nabla(\varphi_{1,\varepsilon} - \psi_{1,\varepsilon})|^2dx \to 0 \tag{1.112}$$

as $\varepsilon \to 0$. We claim now for $2 \le i \le k_0$,

$$\int_{\mathcal{B}}|\nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon})|^2dx \to 0. \tag{1.113}$$

It is easily seen that

$$\begin{aligned}\int_{\mathcal{B}}|\nabla(\varphi_{i,\varepsilon} - \psi_{i,\varepsilon})|^2dx &= \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}|\nabla(\hat{\varphi}_{i,\varepsilon} - \psi_i)|^2dx\\ &= \mu_{i,\varepsilon} + \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}|\nabla\psi_i|^2dx - 2\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}\Big(\nabla\hat{\varphi}_{i,\varepsilon}, \nabla\psi_i\Big)dx.\end{aligned} \tag{1.114}$$

Similarly,

$$\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}W_\varepsilon^{2^\star-2}\psi_i^2dx = \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}V_0^{2^\star-2}\psi_i^2dx + o(1) \tag{1.115}$$

and

$$\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}W_\varepsilon^{2^\star-2}\hat{\varphi}_{i,\varepsilon}\psi_idx = \int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}V_0^{2^\star-2}\hat{\varphi}_{i,\varepsilon}\psi_idx + o(1). \tag{1.116}$$

Independently, since the ψ_i's are linear combinations of the Φ_j's, we get

$$\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}|\nabla\psi_i|^2dx = \mu_i\int_{\mathcal{B}_0(\frac{1}{\bar{\mu}_\varepsilon})}V_0^{2^\star-2}\psi_i^2dx + o(1). \tag{1.117}$$

At last, it follows from (1.100)

$$1+\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon^{2^\star-2}\psi_i^2 dx = 2\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} W_\varepsilon^{2^\star-2}\hat{\varphi}_{i,\varepsilon}\psi_i dx + o(1). \tag{1.118}$$

Noting that

$$\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\Big(\nabla\hat{\varphi}_{i,\varepsilon},\nabla\psi_i\Big)dx = \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\hat{\varphi}_{i,\varepsilon}\Delta\psi_i dx \tag{1.119}$$

we get by combining (1.114)-(1.119) that (1.113) holds. Coming back to (1.111), it follows from (1.112) and (1.113) that

$$\alpha_{i,\varepsilon} = o\Big(\int_{\mathcal{B}}|\nabla w_\varepsilon|^2 dx\Big). \tag{1.120}$$

Then, by (1.110) and (1.120),

$$\int_{\mathcal{B}}|\nabla w_\varepsilon|^2 dx \geq o\Big(\int_{\mathcal{B}}|\nabla w_\varepsilon|^2 dx\Big) + \mu_{k_0+1,\varepsilon}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dx. \tag{1.121}$$

Independently, it is easily seen that

$$\begin{aligned}\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx &= \sum_{i=1}^k \alpha_{i,\varepsilon}^2 + \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dx\\ &= o\Big(\int_{\mathcal{B}}|\nabla w_\varepsilon|^2 dx\Big) + \int_{\mathcal{B}} U_\varepsilon^{2^\star-2}R_\varepsilon^2 dx.\end{aligned} \tag{1.122}$$

Then, it follows from (1.121) and (1.122) that

$$\liminf_{\varepsilon\to 0}\frac{\int_{\mathcal{B}}|\nabla w_\varepsilon|^2 dx}{\int_{\mathcal{B}} U_\varepsilon^{2^\star-2}w_\varepsilon^2 dx} \geq \mu_{k_0+1} \tag{1.123}$$

and since $\mu_{k_0+1} > \frac{2^\star-1}{K_n^2}$, (1.123) is in contradiction with (1.97). In particular, (1.95) is proved. □

The final argument in the proof of (1.4) and (1.5) goes as follows. Combining (1.94) and (1.95) we get

$$\lim_{\varepsilon\to 0}\varepsilon\hat{\mu}_\varepsilon^{2-n} = \frac{(n-4)\omega_{n-1}}{4n(n-2)}A_n^2. \tag{1.124}$$

By (1.42),

$$\alpha\lim_{\varepsilon\to 0}\hat{\mu}_\varepsilon^{4-n}r_\varepsilon^2 = \frac{(n-4)\omega_{n-1}A_n^2}{16n(n-1)} \tag{1.125}$$

when $n\geq 5$, and by (1.56),

$$\lim_{\varepsilon\to 0}B_\varepsilon r_\varepsilon^{n+2} = \frac{2n(n+2)}{\omega_{n-1}}. \tag{1.126}$$

It follows from (1.124)-(1.126) that

$$\lim_{\varepsilon\to 0} B_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = \frac{2n(n+2)\omega_n^{2(n+2)/n}}{\omega_{n-1}^{2n/(n-2)}}\Big(4^{n-3}n(n-2)(n-4)\Big)^{-\frac{n+2}{n-2}}\Big(\frac{4(n-1)}{n-2}\alpha\Big)^{\frac{n+2}{2}}$$

when $n \geq 5$. This proves (1.5). We need some more work to get (1.4). Assuming $n = 4$, it follows from (1.69) that $\varepsilon = O(\hat{\mu}_\varepsilon^2)$. Hence,

$$\limsup_{\varepsilon\to 0} \frac{|\ln \hat{\mu}_\varepsilon|}{|\ln \varepsilon|} \leq \frac{1}{2}\,. \tag{1.127}$$

By (1.43) and (1.56),

$$\lim_{\varepsilon\to 0} |\ln \hat{\mu}_\varepsilon| r_\varepsilon^2 = \frac{4}{\alpha} \quad\text{and}\quad \lim_{\varepsilon\to 0} B_\varepsilon r_\varepsilon^6 = \frac{48}{\omega_3}\,. \tag{1.128}$$

Writing that

$$\frac{B_\varepsilon}{|\ln \varepsilon|^3} = \frac{B_\varepsilon r_\varepsilon^6}{\Big(r_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\Big)^3}\Big(\frac{|\ln \hat{\mu}_\varepsilon|}{|\ln \varepsilon|}\Big)^3$$

it follows from (1.127) and (1.128) that

$$\limsup_{\varepsilon\to 0} \frac{B_\varepsilon}{|\ln \varepsilon|^3} \leq \frac{3\alpha^3}{32\omega_3}\,. \tag{1.129}$$

Conversely, we claim that

$$\liminf_{\varepsilon\to 0} \frac{B_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{3\alpha^3}{32\omega_3}\,. \tag{1.130}$$

For that purpose, we let f_ε be the function given by

$$f_\varepsilon(x) = \frac{\lambda_\varepsilon}{\lambda_\varepsilon^2 + \frac{\sqrt{\omega_4}}{4}|x|^2} + a_\varepsilon \lambda_\varepsilon\Big(|x|^2 - b_\varepsilon\Big), \tag{1.131}$$

where λ_ε, a_ε, and b_ε are real numbers. Given $k_\varepsilon > 0$, we let also $\tilde{f}_\varepsilon$ be the function given by

$$\begin{aligned} \tilde{f}_\varepsilon(x) &= \frac{1}{k_\varepsilon} f_\varepsilon\Big(\frac{1}{k_\varepsilon}x\Big) \quad \text{in } \mathcal{B}_0(k_\varepsilon) \\ \tilde{f}_\varepsilon(x) &= 0 \quad \text{in } \mathcal{B}\backslash\mathcal{B}_0(k_\varepsilon). \end{aligned} \tag{1.132}$$

We choose k_ε such that

$$k_\varepsilon^2 = \frac{8}{\alpha|\ln \varepsilon|} \tag{1.133}$$

and $\lambda_\varepsilon > 0$ small such that

$$\varepsilon = \frac{\lambda_\varepsilon^2}{|\ln \lambda_\varepsilon|}\,. \tag{1.134}$$

Moreover, we choose a_ε and b_ε such that $\tilde f_\varepsilon$ is C^1 in $\mathcal{B}$, and hence such that

$$a_\varepsilon = \frac{\sqrt{\omega_4}}{4\Big(\lambda_\varepsilon^2 + \frac{\sqrt{\omega_4}}{4}\Big)^2} \quad \text{and} \quad b_\varepsilon = \frac{1}{a_\varepsilon\Big(\lambda_\varepsilon^2 + \frac{\sqrt{\omega_4}}{4}\Big)} + 1. \tag{1.135}$$

In particular,

$$a_\varepsilon \to \frac{4}{\sqrt{\omega_4}} \quad \text{and} \quad b_\varepsilon \to 2 \tag{1.136}$$

as $\varepsilon \to 0$. Noting that $f_\varepsilon \geq 0$ in $\mathcal{B}$, we write now that for any $\varepsilon > 0$,

$$\int_{\mathcal{B}} |\nabla \tilde f_\varepsilon|^2 dx - \alpha \int_{\mathcal{B}} \tilde f_\varepsilon^2 dx + B_\varepsilon \Big(\int_{\mathcal{B}} \tilde f_\varepsilon dx\Big)^2 \geq \frac{1-\varepsilon}{K_4^2}\Big(\int_{\mathcal{B}} \tilde f_\varepsilon^4 dx\Big)^{1/2}. \tag{1.137}$$

Easy computations give

$$\int_{\mathcal{B}} \tilde f_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon| + o(k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon|) \tag{1.138}$$

and, thanks to (1.136), that

$$\int_{\mathcal{B}} \tilde f_\varepsilon dx = \frac{2\omega_3 k_\varepsilon^3 \lambda_\varepsilon}{3\sqrt{\omega_4}}\Big(1 + o(1)\Big). \tag{1.139}$$

Similarly, we find with (1.136) that

$$\int_{\mathcal{B}} |\nabla \tilde f_\varepsilon|^2 dx = \frac{1}{K_4^2} - \frac{256\omega_3}{3\omega_4}\lambda_\varepsilon^2 + o(\lambda_\varepsilon^2) \tag{1.140}$$

and that

$$\Big(\int_{\mathcal{B}} \tilde f_\varepsilon^4 dx\Big)^{1/2} = 1 - \frac{128\omega_3 K_4^2}{\omega_4}\lambda_\varepsilon^2 + o(\lambda_\varepsilon^2). \tag{1.141}$$

Plugging (1.138)-(1.141) into (1.137), it follows that

$$\begin{aligned} &\frac{128\omega_3}{3\omega_4}\lambda_\varepsilon^2 + \frac{4\omega_3^2}{9\omega_4} B_\varepsilon \Big(1 + o(1)\Big) k_\varepsilon^6 \lambda_\varepsilon^2 + \frac{\varepsilon}{K_4^2} \\ &\geq \frac{16\alpha\omega_3}{\omega_4} k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon| + o(k_\varepsilon^2 \lambda_\varepsilon^2 |\ln \lambda_\varepsilon|) + o(\lambda_\varepsilon^2). \end{aligned} \tag{1.142}$$

By (1.133) and (1.134), $\varepsilon = o(\lambda_\varepsilon^2)$ and $k_\varepsilon^2 |\ln \lambda_\varepsilon| = \frac{4}{\alpha} + o(1)$. We then get with (1.142), $\frac{B_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{3\alpha^3}{32\omega_3} + o(1)$ and (1.130) is proved. Then, thanks to (1.129) and (1.130), $\lim_{\varepsilon \to 0} \frac{B_\varepsilon}{|\ln \varepsilon|^3} = \frac{3\alpha^3}{32\omega_3}$ and (1.4) is also proved.

2. A test function type argument

We let (M, g) be a smooth compact Riemannian manifold, $n \geq 4$, whose scalar curvature is positive somewhere. We let also $x_0 \in M$ be such that $S_g(x_0) = \max_{x \in M} S_g(x)$, where S_g is the scalar curvature of g. For $\delta > 0$ small, we consider $\mathcal{B}_0(\delta)$ the Euclidean ball of center 0 and radius δ, and we still denote by g the metric $\exp_{x_0}^\star g$. Let us assume that for any $u \in C_c^\infty(\mathcal{B}_0(\delta))$,

$$\int_{\mathcal{B}_0(\delta)} |\nabla u|^2 dv_g + \hat{B}_\varepsilon \Big(\int_{\mathcal{B}_0(\delta)} |u| dv_g \Big)^2 \geq \frac{1-\varepsilon}{K_n^2} \Big(\int_{\mathcal{B}_0(\delta)} |u|^{2^\star} dv_g \Big)^{2/2^\star}. \quad (2.1)$$

The goal in this section is to prove

$$\liminf_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{1}{2304 \omega_3} \Big(\max_{x \in M} S_g \Big)^3 \quad (2.2)$$

when $n = 4$, and

$$\liminf_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \geq C_n \Big(\max_{x \in M} S_g \Big)^{\frac{n+2}{2}} \quad (2.3)$$

when $n \geq 5$, where $C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3} n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$ is as in Section 1. For that purpose, we let B_ε and the u_ε's be as in Section 1, where α is given by $\alpha = \frac{n-2}{4(n-1)} S_g(0) \delta^2$. Then,

$$\Delta u_\varepsilon - \alpha u_\varepsilon + B_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star - 1} \text{ in } \mathcal{B} \quad (2.4)$$

$$u_\varepsilon = 0 \text{ on } \partial\mathcal{B}, \; \int_{\mathcal{B}} u_\varepsilon^{2^\star} dx = 1.$$

We let z_ε be the function in $\mathcal{B}_0(\delta)$ given by

$$z_\varepsilon(x) = \delta^{1 - \frac{n}{2}} u_\varepsilon\Big(\frac{1}{\delta} x\Big). \quad (2.5)$$

Then,

$$\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dv_g + \hat{B}_\varepsilon \Big(\int_{\mathcal{B}_0(\delta)} z_\varepsilon dv_g \Big)^2 \geq \frac{1-\varepsilon}{K_n^2} \Big(\int_{\mathcal{B}_0(\delta)} z_\varepsilon^{2^\star} dv_g \Big)^{2/2^\star}. \quad (2.6)$$

Thanks to the Cartan expansion of a metric in geodesic normal coordinates,

$$\begin{aligned} \int_{\mathcal{B}_0(\delta)} z_\varepsilon dv_g &= \int_{\mathcal{B}_0(\delta)} z_\varepsilon dx + O\Big(\int_{\mathcal{B}_0(\delta)} |x|^2 z_\varepsilon dx \Big) \\ &= \delta^{\frac{n}{2}+1} \int_{\mathcal{B}} u_\varepsilon dx + \delta^{\frac{n}{2}+3} O\Big(\int_{\mathcal{B}} |x|^2 u_\varepsilon dx \Big). \end{aligned}$$

We have seen in Section 1 that u_ε has its support in $\mathcal{B}_0(r_\varepsilon)$ where $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Hence, we can write that

$$\Big(\int_{\mathcal{B}_0(\delta)} z_\varepsilon dv_g\Big)^2 = \delta^{n+2}\Big(1+o(1)\Big)\Big(\int_{\mathcal{B}} u_\varepsilon dx\Big)^2. \tag{2.7}$$

Still thanks to the Cartan expansion of a metric in geodesic normal coordinates, and since z_ε is radially symmetrical,

$$\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dv_g = \int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dx - \frac{1}{6}R_{ij}(0)\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 x^i x^j dx$$
$$+O\Big(\int_{\mathcal{B}_0(\delta)} |x|^4 |\nabla z_\varepsilon|^2 dx\Big),$$

where R stands for the Ricci curvature of g. By (2.4), we get

$$\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dx = \int_{\mathcal{B}} |\nabla u_\varepsilon|^2 dx = \frac{1-\varepsilon}{K_n^2} - B_\varepsilon\Big(\int_{\mathcal{B}} u_\varepsilon dx\Big)^2 + \alpha\int_{\mathcal{B}} u_\varepsilon^2 dx.$$

Independently, since u_ε is radially symmetrical,

$$\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 x^i x^j dx = \delta^2\int_{\mathcal{B}} |\nabla u_\varepsilon|^2 x^i x^j dx = \delta^2\frac{1}{n}\delta^{ij}\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx.$$

Noting that u_ε has its support in $\mathcal{B}_0(r_\varepsilon)$, where $r_\varepsilon \to 0$ as $\varepsilon \to 0$, we also have

$$\int_{\mathcal{B}} |x|^4|\nabla z_\varepsilon|^2 dx = \delta^4\int_{\mathcal{B}} |x|^4|\nabla u_\varepsilon|^2 dx = o\Big(\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx\Big).$$

Hence,

$$\begin{aligned}\int_{\mathcal{B}_0(\delta)} |\nabla z_\varepsilon|^2 dv_g &= \frac{1-\varepsilon}{K_n^2} - B_\varepsilon\Big(\int_{\mathcal{B}} u_\varepsilon dx\Big)^2 + \alpha\int_{\mathcal{B}} u_\varepsilon^2 dx \\ &- \frac{\delta^2}{6n}S_g(0)\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx + o\Big(\int_{\mathcal{B}} |x|^2|\nabla u_\varepsilon|^2 dx\Big).\end{aligned} \tag{2.8}$$

Similar arguments give

$$\int_{\mathcal{B}_0(\delta)} z_\varepsilon^{2^\star} dv_g = 1 - \frac{\delta^2}{6n}S_g(0)\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + o\Big(\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx\Big)$$

and hence that

$$\Big(\int_{\mathcal{B}_0(\delta)} z_\varepsilon^{2^\star} dv_g\Big)^{2/2^\star} = 1 - \frac{(n-2)\delta^2}{6n^2}S_g(0)\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + o\Big(\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx\Big). \tag{2.9}$$

Coming back to (2.6), we get with (2.7)-(2.9)

$$\alpha\int_{\mathcal{B}} u_\varepsilon^2 dx + \Big(\hat{B}_\varepsilon\Big(1+o(1)\Big)\delta^{n+2} - B_\varepsilon\Big)\Big(\int_{\mathcal{B}} u_\varepsilon dx\Big)^2$$

$$
\begin{aligned}
&- \frac{\delta^2}{6n} S_g(0) \int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx + o\Big(\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx \Big) \\
&\geq - \frac{(n-2)\delta^2}{6n^2 K_n^2} S_g(0) \int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + o\Big(\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx \Big).
\end{aligned} \tag{2.10}
$$

With the notations of Section 1,

$$
\Big(\int_{\mathcal{B}} u_\varepsilon dx \Big)^2 = \frac{A_n^2 \omega_{n-1}^2}{4n^2(n+2)^2} \hat{\mu}_\varepsilon^{n-2} r_\varepsilon^{n+2} + o\Big(\hat{\mu}_\varepsilon^{n-2} r_\varepsilon^{n+2} \Big). \tag{2.11}
$$

We also have

$$
\int_{\mathcal{B}} u_\varepsilon^2 dx = r_\varepsilon^2 \int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx.
$$

Hence, by (1.41) and (1.55),

$$
\int_{\mathcal{B}} u_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\Big(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| \Big) \tag{2.12}
$$

if $n = 4$, and

$$
\int_{\mathcal{B}} u_\varepsilon^2 dx = \frac{4(n-1)}{n-4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\Big(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 \Big) \tag{2.13}
$$

if $n \geq 5$. Similarly,

$$
\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx = r_\varepsilon^2 \int_{\mathcal{B}} |x|^2 \hat{u}_\varepsilon^{2^\star} dx = r_\varepsilon^2 \hat{\mu}_\varepsilon^2 \int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |x|^2 \Big(\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \hat{u}_\varepsilon(\hat{\mu}_\varepsilon x) \Big)^{2^\star} dx
$$

and thanks to (1.32) and (1.33), we have

$$
\int_{\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} |x|^2 \Big(\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \hat{u}_\varepsilon(\hat{\mu}_\varepsilon x) \Big)^{2^\star} dx \to \int_{\mathbb{R}^n} |x|^2 V_0^{2^\star} dx
$$

as $\varepsilon \to 0$. It is easily seen, see for instance Demengel and Hebey [11],

$$
\int_{\mathbb{R}^n} |x|^2 V_0^{2^\star} dx = 2^{n+1} \omega_n^{-\frac{n+2}{n}} \omega_{n-1} \frac{\Gamma(\frac{n+2}{2}) \Gamma(\frac{n-2}{2})}{\Gamma(n)}
$$

and since $\Gamma(n) = \frac{2^{n-1}\omega_{n-1}}{\omega_n} \Gamma(\frac{n}{2})^2$, we have

$$
\int_{\mathbb{R}^n} |x|^2 V_0^{2^\star} dx = \frac{4n}{(n-2)\omega_n^{2/n}} .
$$

Hence,

$$
\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx = \frac{4n}{(n-2)\omega_n^{2/n}} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\Big(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 \Big). \tag{2.14}
$$

Integrating by parts, and thanks to (2.4), we also have

$$
\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx = n \int_{\mathcal{B}} u_\varepsilon^2 dx + \int_{\mathcal{B}} |x|^2 u_\varepsilon \Delta u_\varepsilon dx
$$

$$= n\int_{\mathcal{B}} u_\varepsilon^2 dx + \frac{1-\varepsilon}{K_n^2}\int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx - B_\varepsilon \|u_\varepsilon\|_1 \int_{\mathcal{B}} |x|^2 u_\varepsilon^{2^\star} dx + \alpha \int_{\mathcal{B}} |x|^2 u_\varepsilon^2 dx.$$

Noting that

$$\int_{\mathcal{B}} |x|^2 u_\varepsilon dx = r_\varepsilon^2 O\Big(\int_{\mathcal{B}} u_\varepsilon dx\Big) \quad \text{and} \quad \int_{\mathcal{B}} |x|^2 u_\varepsilon^2 dx = o\Big(\int_{\mathcal{B}} u_\varepsilon^2 dx\Big)$$

it follows that

$$\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx = \frac{64\omega_3}{\omega_4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \mu_\varepsilon| + o\Big(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \mu_\varepsilon|\Big) + O\Big(B_\varepsilon r_\varepsilon^8 \hat{\mu}_\varepsilon^2\Big) \tag{2.15}$$

when $n = 4$, and

$$\int_{\mathcal{B}} |x|^2 |\nabla u_\varepsilon|^2 dx = \frac{n(n^2-4)}{n-4} r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o\Big(r_\varepsilon^2 \hat{\mu}_\varepsilon^2\Big) + O\Big(B_\varepsilon r_\varepsilon^{n+4} \hat{\mu}_\varepsilon^{n-2}\Big) \tag{2.16}$$

when $n \geq 5$. Let us assume first $n \geq 5$. Plugging (2.11)-(2.16) into (2.10), and thanks to the choice we made for α, we get

$$\begin{aligned}&\big(\hat{B}_\varepsilon \delta^{n+2} - B_\varepsilon\big) \frac{A_n^2 \omega_{n-1}^2}{4n^2(n+2)^2} r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2} \\ &\quad \geq o\big(r_\varepsilon^2 \hat{\mu}_\varepsilon^2\big) + o\big(\hat{B}_\varepsilon r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2}\big) + O\big(B_\varepsilon r_\varepsilon^{n+4} \hat{\mu}_\varepsilon^{n-2}\big)\end{aligned}$$

so that

$$\big(\frac{\hat{B}_\varepsilon}{B_\varepsilon}\delta^{n+2} - 1\big) \frac{A_n^2 \omega_{n-1}^2}{4n^2(n+2)^2} \geq o\big(r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n} B_\varepsilon^{-1}\big) + o\big(\frac{\hat{B}_\varepsilon}{B_\varepsilon}\big) + O\big(r_\varepsilon^2\big). \tag{2.17}$$

By (1.42) and (1.56), $r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n} B_\varepsilon^{-1} = O(1)$. Hence, (2.17) gives

$$\liminf_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{B_\varepsilon} \delta^{n+2} \geq 1$$

and thanks to (1.5) we get

$$\liminf_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \geq C_n S_g(0)^{\frac{n+2}{2}},$$

where $C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3} n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$. This proves (2.3). Let us assume $n = 4$. Plugging (2.11)-(2.16) into (2.10), and thanks to the choice we made for α, we get

$$\frac{\hat{B}_\varepsilon}{B_\varepsilon}\delta^6 - 1 \geq o\Big(\frac{\hat{B}_\varepsilon}{B_\varepsilon}\Big) + o\Big(B_\varepsilon^{-1} r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon|\Big) + O\Big(r_\varepsilon^2\Big). \tag{2.18}$$

By (1.43) and (1.56), $B_\varepsilon^{-1} r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon| = O(1)$. Hence, (2.18) gives

$$\liminf_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{B_\varepsilon} \delta^6 \geq 1$$

and thanks to (1.4) we get $\liminf_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} \geq \frac{1}{2304\omega_3} S_g(0)^3$. This proves (2.4).

3. The Riemannian case

As in Section 2, we let (M, g) be a smooth compact Riemannian manifold of dimension $n \geq 4$. We assume that the scalar curvature S_g of g is such that $\max_{x \in M} S_g > 0$. For $\varepsilon > 0$ small, we let $\hat{B}_\varepsilon$ be the smallest B such that for all $u \in C^\infty(M)$,

$$\frac{1-\varepsilon}{K_n^2} \|u\|_{2^\star}^2 \leq \|\nabla u\|_2^2 + B\|u\|_1^2.$$

As in Section 1, it can be proved that

$$\inf_{u \in C^\infty(M) \setminus \{0\}} \frac{\|\nabla u\|_2^2 + \hat{B}_\varepsilon \|u\|_1^2}{\|u\|_{2^\star}^2} = \frac{1-\varepsilon}{K_n^2}\,. \tag{3.1}$$

With respect to the notations of the introduction, we have that

$$\hat{B}_\varepsilon = \frac{1-\varepsilon}{K_n^2} B_{\frac{K_n^2 \varepsilon}{1-\varepsilon}}(g) \quad \text{and} \quad B_\varepsilon(g) = \left(K_n^2 + \varepsilon\right) \hat{B}_{\frac{\varepsilon}{K_n^2+\varepsilon}}\,. \tag{3.2}$$

The goal in this section is to prove that

$$\limsup_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} \leq \frac{1}{2304\omega_3} \left(\max_{x \in M} S_g\right)^3 \tag{3.3}$$

when $n = 4$, and that

$$\limsup_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \leq C_n \left(\max_{x \in M} S_g\right)^{\frac{n+2}{2}} \tag{3.4}$$

when $n \geq 5$, where $C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3} n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}$ is as in Sections 1 and 2. As indicated at the end of this section, the theorem follows from (3.3), (3.4), and what is proved in Section 2.

Thanks to Druet, Hebey, and Vaugon [16], $\hat{B}_\varepsilon \to +\infty$ as $\varepsilon \to 0$. Independently, the LHS in (3.1) being less than $1/K_n^2$, it easily follows from standard variational arguments, as in Section 1, that there exists a minimizer for the infimum in (3.1). With no risk of confusion with the notations of Section 1, we denote by u_ε this minimizer. We then get that for any $\varepsilon > 0$, there exists $u_\varepsilon \in C^{1,\beta}(M)$, $0 < \beta < 1$, such that

$$\Delta_g u_\varepsilon + \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon = \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star - 1} \tag{3.5}$$

and

$$\int_M u_\varepsilon^{2^\star} dv_g = 1 \ , \ \ u_\varepsilon \geq 0 \text{ in } M, \tag{3.6}$$

where $\Delta_g = -div_g(\nabla)$ is the Riemannian Laplacian, and $\Sigma_\varepsilon \in L^\infty(M)$, $0 \leq \Sigma_\varepsilon \leq 1$, is such that $\Sigma_\varepsilon u_\varepsilon = u_\varepsilon$. We let x_ε be a point where u_ε is maximum, and set

$$\mu_\varepsilon^{1-\frac{n}{2}} = \|u_\varepsilon\|_\infty = u_\varepsilon(x_\varepsilon). \tag{3.7}$$

Multiplying (3.5) by u_ε and integrating over M, we get with (3.6) that

$$\hat{B}_\varepsilon \|u_\varepsilon\|_1^2 \leq \frac{1}{K_n^2}.$$

Since $\hat{B}_\varepsilon \to +\infty$ as $\varepsilon \to 0$, it follows that $\|u_\varepsilon\|_1 \to 0$ as $\varepsilon \to 0$. In particular, by Hölder's inequality and (3.6), $\|u_\varepsilon\|_2 \to 0$ as $\varepsilon \to 0$. Noting that

$$1 = \int_M u_\varepsilon^{2^\star} dv_g \leq \mu_\varepsilon^{-\frac{n+2}{2}} \int_M u_\varepsilon dv_g,$$

we also have that

$$\lim_{\varepsilon \to 0} \mu_\varepsilon = 0. \tag{3.8}$$

Independently, by Hebey and Vaugon [25], there exists $B > 0$ such that for any $u \in H_1^2(M)$,

$$\|u\|_{2^\star}^2 \leq K_n^2 \|\nabla u\|_2^2 + B\|u\|_2^2.$$

Taking $u = u_\varepsilon$ in this inequality,

$$1 - B\|u_\varepsilon\|_2^2 \leq K_n^2 \|\nabla u_\varepsilon\|_2^2 = 1 - \varepsilon - K_n^2 \hat{B}_\varepsilon \|u_\varepsilon\|_1^2$$

and it follows that

$$\lim_{\varepsilon \to 0} \hat{B}_\varepsilon \|u_\varepsilon\|_1^2 = 0. \tag{3.9}$$

As in Druet, Hebey and Vaugon [16], there exists $x_0 \in M$ such that for any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \int_{B_{x_0}(\delta)} u_\varepsilon^{2^\star} dv_g = 1$$

and

$$u_\varepsilon \to 0 \ \text{ in } \ C^0_{loc}(M \backslash \{x_0\}) \tag{3.10}$$

as ε goes to 0. According to what we just said, $x_\varepsilon \to x_0$ and $\mu_\varepsilon \to 0$ as $\varepsilon \to 0$. By (3.9), noting that $1 = \|u_\varepsilon\|_{2^\star}^{2^\star} \le \|u_\varepsilon\|_\infty^{2^\star-1}\|u_\varepsilon\|_1$, we get

$$\lim_{\varepsilon\to 0} \hat{B}_\varepsilon \mu_\varepsilon^{\frac{n+2}{2}} \|u_\varepsilon\|_1 = 0. \tag{3.11}$$

In what follows we let $\exp_{x_\varepsilon}$ be the exponential map at x_ε. There clearly exists $\delta > 0$, independent of ε, such that for any ε, $\exp_{x_\varepsilon}$ is a diffeomorphism from $\mathcal{B}_0(\delta) \subset \mathbb{R}^n$ onto $B_{x_\varepsilon}(\delta)$. As a starting point in the proof of (3.3) and (3.4), we prove weak estimates on the u_ε's.

3.1. **Weak Estimates.** For $x \in \mathcal{B}_0(\mu_\varepsilon^{-1}\delta)$, we set

$$\tilde{g}_\varepsilon(x) = \big(\exp_{x_\varepsilon}^\star g\big)(\mu_\varepsilon x) \quad , \quad \tilde{u}_\varepsilon(x) = \mu_\varepsilon^{\frac{n}{2}-1} u_\varepsilon\big(\exp_{x_\varepsilon}(\mu_\varepsilon x)\big)$$

and $\tilde{\Sigma}_\varepsilon(x) = \Sigma_\varepsilon\big(\exp_{x_\varepsilon}(\mu_\varepsilon x)\big)$. It is easily seen that

$$\Delta_{\tilde{g}_\varepsilon} \tilde{u}_\varepsilon + \hat{B}_\varepsilon \mu_\varepsilon^{\frac{n+2}{2}} \|u_\varepsilon\|_1 \tilde{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n^2} \tilde{u}_\varepsilon^{2^\star-1}. \tag{3.12}$$

Moreover,

$$\tilde{u}_\varepsilon(0) = \|\tilde{u}_\varepsilon\|_\infty = 1 \tag{3.13}$$

and if ξ stands for the Euclidean metric of $\mathbb{R}^n$,

$$\lim_{\varepsilon\to 0} \tilde{g}_\varepsilon = \xi \quad \text{in } C^2(K) \tag{3.14}$$

for any compact subset K of $\mathbb{R}^n$. Thanks to (3.11)-(3.14), we get by standard elliptic theory, as developed in Gilbarg-Trudinger [19], that there exists some $\tilde{u} \in C^1(\mathbb{R}^n)$ such that for any compact subset K of $\mathbb{R}^n$,

$$\lim_{\varepsilon\to 0} \tilde{u}_\varepsilon = \tilde{u} \quad \text{in } C^1(K). \tag{3.15}$$

Clearly, $\tilde{u}(0) = 1$ and $\tilde{u} \not\equiv 0$. Moreover, it is easily seen that $\tilde{u} \in D_1^2(\mathbb{R}^n)$, where $D_1^2(\mathbb{R}^n)$ is the homogeneous Euclidean Sobolev space. By passing to the limit as ε goes to 0 in (3.12), according to (3.11), (3.14), and (3.15), we get $\tilde{u}$ is a solution of

$$\Delta \tilde{u} = \frac{1}{K_n^2} \tilde{u}^{2^\star-1}.$$

By Caffarelli-Gidas-Spruck [8], and also Obata [30],

$$\tilde{u}(x) = \Big(1 + \frac{\omega_n^{2/n}}{4}|x|^2\Big)^{1-\frac{n}{2}}. \tag{3.16}$$

Noting that $\tilde{u}$ is of norm 1 in $L^{2^\star}(\mathbb{R}^n)$, and that for any $R > 0$,

$$\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star} dv_g = \int_{\mathcal{B}_0(R)} \tilde{u}_\varepsilon^{2^\star} dv_{\tilde{g}_\varepsilon}$$

we get

$$\lim_{\varepsilon\to 0}\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star}\,dv_g = 1-\int_{\mathbb{R}^n\setminus\mathcal{B}_0(R)}\tilde{u}^{2^\star}\,dx.$$

Hence,

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\int_{B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star}\,dv_g = 1. \tag{3.17}$$

We claim now that the two following estimates hold. On the one hand there exists $C>0$, such that for any ε, and any x,

$$d_g(x_\varepsilon,x)^{\frac{n}{2}-1}u_\varepsilon(x)\le C \tag{3.18}$$

where d_g is the distance with respect to g. On the other hand

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{x\in M\setminus B_{x_\varepsilon}(R\mu_\varepsilon)} d_g(x_\varepsilon,x)^{\frac{n}{2}-1}u_\varepsilon(x)=0. \tag{3.19}$$

In order to prove (3.18), we set $v_\varepsilon(x)=d_g(x_\varepsilon,x)^{\frac{n}{2}-1}u_\varepsilon(x)$ and assume by contradiction that for some subsequence,

$$\lim_{\varepsilon\to 0}\|v_\varepsilon\|_\infty=+\infty. \tag{3.20}$$

Let y_ε be some point in M where v_ε is maximum. By (3.10), $y_\varepsilon\to x_0$ as $\varepsilon\to 0$, while by (3.20),

$$\lim_{\varepsilon\to 0}\frac{d_g(x_\varepsilon,y_\varepsilon)}{\mu_\varepsilon}=+\infty. \tag{3.21}$$

Fix now $\delta>0$ small, and set $\Omega_\varepsilon=u_\varepsilon(y_\varepsilon)^{\frac{2}{n-2}}\exp_{y_\varepsilon}^{-1}\big(B_{x_\varepsilon}(\delta)\big)$. For $x\in\Omega_\varepsilon$, define

$$\tilde{v}_\varepsilon(x)=u_\varepsilon(y_\varepsilon)^{-1}u_\varepsilon\big(\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\big),\quad h_\varepsilon(x)=\big(\exp_{y_\varepsilon}^\star g\big)\big(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x\big).$$

It easily follows from (3.20), since M is compact, that $u_\varepsilon(y_\varepsilon)\to+\infty$ as $\varepsilon\to 0$. Hence,

$$\lim_{\varepsilon\to 0}h_\varepsilon=\xi\quad\text{in } C^2\big(\mathcal{B}_0(2)\big), \tag{3.22}$$

where ξ is the Euclidean metric. Independently, we have

$$\Delta_{h_\varepsilon}\tilde{v}_\varepsilon\le\frac{1-\varepsilon}{K_n^2}\tilde{v}_\varepsilon^{2^\star-1}. \tag{3.23}$$

Since $v_\varepsilon(y_\varepsilon)$ goes to $+\infty$, for ε small, and all $x\in\mathcal{B}_0(2)$,

$$d_g\big(x_\varepsilon,\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\big)\ge\frac{1}{2}d_g(x_\varepsilon,y_\varepsilon). \tag{3.24}$$

This implies

$$\tilde{v}_\varepsilon(x)\le 2^{\frac{n}{2}-1}d_g(x_\varepsilon,y_\varepsilon)^{1-\frac{n}{2}}u_\varepsilon(y_\varepsilon)^{-1}v_\varepsilon\big(\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\big)$$

$$\leq 2^{\frac{n}{2}-1} d_g(x_\varepsilon, y_\varepsilon)^{1-\frac{n}{2}} u_\varepsilon(y_\varepsilon)^{-1} v_\varepsilon(y_\varepsilon)$$

so that for ε small,

$$\sup_{x\in\mathcal{B}_0(2)} \tilde{v}_\varepsilon(x) \leq 2^{\frac{n}{2}-1}. \tag{3.25}$$

By (3.21) and (3.24), given $R > 0$, and for ε small,

$$B_{y_\varepsilon}(2u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}) \bigcap B_{x_\varepsilon}(R\mu_\varepsilon) = \emptyset. \tag{3.26}$$

Noting that

$$\int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star} dv_{h_\varepsilon} = \int_{B_{y_\varepsilon}(2u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}})} u_\varepsilon^{2^\star} dv_g$$

it follows from (3.17) and (3.26) that

$$\lim_{\varepsilon\to 0} \int_{\mathcal{B}_0(2)} \tilde{v}_\varepsilon^{2^\star} dv_{h_\varepsilon} = 0. \tag{3.27}$$

By (3.22), (3.23), (3.25), (3.27), and the De Giorgi-Nash-Moser iterative scheme we get $\lim_{\varepsilon\to 0}\sup_{x\in\mathcal{B}_0(1)} \tilde{v}_\varepsilon(x) = 0$. But $\tilde{v}_\varepsilon(0) = 1$, so (3.20) must be false. This proves (3.18). In order to prove (3.19), we let v_ε be as above, and proceed once more by contradiction. Then there exists $y_\varepsilon \in M$ and $k_0 > 0$ such that

$$\lim_{\varepsilon\to 0} \frac{d_g(x_\varepsilon, y_\varepsilon)}{\mu_\varepsilon} = +\infty \text{ and } v_\varepsilon(y_\varepsilon) \geq k_0.$$

As above, we fix $\delta > 0$ small, and set $\Omega_\varepsilon = u_\varepsilon(y_\varepsilon)^{\frac{2}{n-2}} \exp_{y_\varepsilon}^{-1}\big(B_{x_\varepsilon}(\delta)\big)$. For $x \in \Omega_\varepsilon$, we define

$$\tilde{v}_\varepsilon(x) = u_\varepsilon(y_\varepsilon)^{-1} u_\varepsilon\big(\exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)\big)$$

and

$$h_\varepsilon(x) = \big(\exp_{y_\varepsilon}^\star g\big)\big(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x\big).$$

Once again

$$\Delta_{h_\varepsilon} \tilde{v}_\varepsilon \leq \frac{1-\varepsilon}{K_n^2} \tilde{v}_\varepsilon^{2^\star-1}.$$

As when proving (3.18), for any $x \in \mathcal{B}_0(\frac{1}{2}k_0^{\frac{2}{n-2}})$, $d_g(x_\varepsilon, z_\varepsilon) \geq \frac{1}{2}d_g(x_\varepsilon, y_\varepsilon)$ and $\tilde{v}_\varepsilon(x) = u_\varepsilon(y_\varepsilon)^{-1} v_\varepsilon(z_\varepsilon) d_g(x_\varepsilon, z_\varepsilon)^{1-\frac{n}{2}}$, where $z_\varepsilon = \exp_{y_\varepsilon}(u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}x)$. It follows from (3.18) that $\tilde{v}_\varepsilon(x) \leq C2^{\frac{n}{2}-1}k_0^{-1}$. Noting that for $R > 0$, and for ε small,

$$B_{y_\varepsilon}\big(\frac{1}{2}k_0^{\frac{2}{n-2}} u_\varepsilon(y_\varepsilon)^{-\frac{2}{n-2}}\big) \bigcap B_{x_\varepsilon}(R\mu_\varepsilon) = \emptyset$$

we conclude as when proving (3.18) that (3.19) holds. □

Now we need stronger estimates than (3.18) and (3.19). This is the subject of the following subsection.

3.2. **Strong estimates 1.** In order to get stronger estimates than (3.18) and (3.19), we let $h_0 \in C^\infty(M)$ be such that $h_0 \geq 0$, $h_0 \not\equiv 0$, $h_0 \equiv 0$ in $B_{x_0}(\delta_0)$. As a remark, it is easy to check such a choice of h_0 implies $\Delta_g + h_0$ is coercive. We define L_ε by

$$L_\varepsilon u = \Delta_g u + h_0 u - \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star - 2} u$$

and claim that L_ε satisfies the maximum principle in $M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)$ for $R > 0$ large and $\varepsilon > 0$ small. Let indeed $z \in C^1(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))$ be such that $z \geq 0$ on $\partial B_{x_\varepsilon}(R\mu_\varepsilon)$ and $L_\varepsilon z \geq 0$. Set $z^- = \max(0, -z)$. Then,

$$\begin{aligned} 0 \leq \int_{M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)} z^- L_\varepsilon z dv_g &= - \int_{M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)} |\nabla z^-|^2 dv_g \\ - \int_{M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)} h_0 (z^-)^2 dv_g &+ \frac{1-\varepsilon}{K_n^2} \int_{M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star-2} (z^-)^2 dv_g \end{aligned}$$

while, thanks to Hölder's inequality,

$$\int_{M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)} u_\varepsilon^{2^\star - 2} (z^-)^2 dv_g \leq \|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \|z^-\|^2_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))}.$$

Thus,

$$\begin{aligned} 0 \leq &- \|\nabla z^-\|^2_{L^2(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} - \|\sqrt{h_0} z^-\|^2_{L^2(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \\ &+ \frac{1-\varepsilon}{K_n^2} \|u_\varepsilon\|^{2^\star-2}_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \|z^-\|^2_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))}. \end{aligned} \tag{3.28}$$

By (3.17),

$$\lim_{R \to +\infty} \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} = 0.$$

It follows that for any $A > 0$, there exists $\varepsilon_A > 0$ and $R_A > 0$ such that for $R \geq R_A$ and $\varepsilon \in (0, \varepsilon_A)$, $\|u_\varepsilon\|_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \leq A$. Let $\lambda > 0$, given by the coercivity of $\Delta_g + h_0$, be such that

$$\lambda \|z^-\|^2_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \leq \|\nabla z^-\|^2_{L^2(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} + \|\sqrt{h_0} z^-\|^2_{L^2(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))}.$$

Coming back to (3.28), we get that

$$0 \leq \|z^-\|^2_{L^{2^\star}(M \backslash B_{x_\varepsilon}(R\mu_\varepsilon))} \big(\frac{1-\varepsilon}{K_n^2} A^{2^\star - 2} - \lambda \big).$$

Choosing $A > 0$ small, this implies that $z^- \equiv 0$. The claim is proved. Now, thanks to the De Giorgi-Nash-Moser iterative scheme applied to $\Delta_g u_\varepsilon \leq K_n^{-2} u_\varepsilon^{2^\star-1}$, we have that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$\sup_{M\backslash B_{x_0}(\delta)} u_\varepsilon \leq C_\delta \|u_\varepsilon\|_1. \tag{3.29}$$

Taking $\delta = \delta_0$, this implies that

$$L_\varepsilon u_\varepsilon \leq 0 \quad \text{in } M. \tag{3.30}$$

We let $\varepsilon_0 > 0$ be such that $\Delta_g + h_0 - \varepsilon_0$ is still coercive in M, and let $G(x,y)$ be the Green function of this operator. We set $H(x) = G(x_\varepsilon, x)$. Given $\nu \in (0,1)$, we have that

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} = \nu(1-\nu)\frac{|\nabla H|^2}{H^2} + \hat{h}_0 - \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star-2}, \tag{3.31}$$

where $\hat{h}_0 = (1-\nu)\varepsilon_0 + \nu h_0$. A standard property of the Green function (F. Robert, private communication) is that there exists $\rho > 0$ and $C > 0$ such that for any $x \in B_{x_\varepsilon}(\rho)\backslash\{x_\varepsilon\}$,

$$\frac{|\nabla G(x_\varepsilon, x)|}{G(x_\varepsilon, x)} \geq C \frac{1}{d_g(x_\varepsilon, x)}\,,$$

where d_g is the distance with respect to g. We also have that

$$\left| d_g(x_\varepsilon, x)^{n-2} G(x_\varepsilon, x) \right| \leq C$$

for any $x \neq x_\varepsilon$, where $C > 0$ does not depend on ε, and that

$$d_g(x_\varepsilon, x)^{n-2} G(x_\varepsilon, x) \geq C$$

as soon as $d_g(x_\varepsilon, x) \leq r_0$, where $r_0 > 0$ and $C > 0$ do not depend on ε. Then, for $x \in B_{x_\varepsilon}(\rho)\backslash\{x_\varepsilon\}$,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}}(x) \geq d_g(x_\varepsilon, x)^{-2}\left(C\nu(1-\nu) - \frac{1-\varepsilon}{K_n^2} d_g(x_\varepsilon, x)^2 u_\varepsilon^{2^\star-2} \right)$$

and thanks to (3.19) we get that for $R > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} \geq 0 \quad \text{in } B_{x_\varepsilon}(\rho)\backslash B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3.32}$$

In $M\backslash B_{x_\varepsilon}(\rho)$, we have (3.10). Thus, by (3.31), and for $\varepsilon > 0$ small,

$$\frac{L_\varepsilon H^{1-\nu}}{H^{1-\nu}} \geq \hat{h}_0 - \frac{1-\varepsilon}{K_n^2} u_\varepsilon^{2^\star-2} \geq 0 \quad \text{in } M\backslash B_{x_\varepsilon}(\rho). \tag{3.33}$$

Summarizing, it follows from (3.30), (3.32), and (3.33), that there exists $R > 0$, depending on ν, such that

$$L_\varepsilon u_\varepsilon \le 0 \le L_\varepsilon H^{1-\nu} \quad \text{in } M \backslash B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3.34}$$

By (3.15) and (3.16), there exists $C > 0$ such that

$$u_\varepsilon \le C\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} H^{1-\nu} \quad \text{on } \partial B_{x_\varepsilon}(R\mu_\varepsilon). \tag{3.35}$$

The maximum principle, (3.34), and (3.35), then give

$$u_\varepsilon(x) \le C\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} d_g(x_\varepsilon, x)^{(2-n)(1-\nu)}$$

in $M \backslash B_{x_\varepsilon}(R\mu_\varepsilon)$. Noting that this inequality is satisfied in $B_{x_\varepsilon}(R\mu_\varepsilon)$ thanks to (3.15) and (3.16), we have proved that for any $\nu \in (0, 1)$, there exists $C(\nu) > 0$ such that

$$u_\varepsilon(x) \le C\mu_\varepsilon^{(\frac{n}{2}-1)(1-2\nu)} d_g(x_\varepsilon, x)^{(2-n)(1-\nu)} \tag{3.36}$$

for any $\varepsilon > 0$ and any $x \in M\backslash\{x_\varepsilon\}$. Now we claim there actually exists $C > 0$ such that for any $\varepsilon > 0$ and any $x \in M$,

$$\mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, x)^{n-2} u_\varepsilon(x) \le C. \tag{3.37}$$

In other words, we claim (3.36) holds with $\nu = 0$. We let $G_0(x, y)$ be the Green function of $\Delta_g + h_0$. Thanks to (3.29), noting $\Sigma_\varepsilon(x) = 1$ if $u_\varepsilon(x) \ne 0$,

$$h_0 u_\varepsilon - \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon \le 0$$

in M when $\varepsilon > 0$ is small. For (y_ε) a sequence in M, we can then write that for $\varepsilon > 0$ small,

$$\begin{aligned}
u_\varepsilon(y_\varepsilon) &= \int_M G_0(x, y_\varepsilon)\Big(\Delta_g u_\varepsilon + h_0 u_\varepsilon\Big)(x) dv_g(x) \\
&= \frac{1-\varepsilon}{K_n^2} \int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star-1} dv_g \\
&\quad + \int_M G_0(x, y_\varepsilon)\Big(h_0 u_\varepsilon - \hat{B}_\varepsilon \|u_\varepsilon\|_1 \Sigma_\varepsilon\Big)(x) dv_g(x) \\
&\le \frac{1}{K_n^2} \int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star-1} dv_g.
\end{aligned} \tag{3.38}$$

We set $\Phi_\varepsilon = u_\varepsilon(y_\varepsilon)\mu_\varepsilon^{1-\frac{n}{2}} d_g(x_\varepsilon, y_\varepsilon)^{n-2}$ and let H_ε be such that $H_\varepsilon(x) = G_0(x, y_\varepsilon)$. We distinguish three cases.

Case 1: We assume $\mu_\varepsilon^{-1} d_g(x_\varepsilon, y_\varepsilon) \to R$ as $\varepsilon \to 0$, $R \in [0, +\infty)$. Then thanks to (3.18), (Φ_ε) is bounded.

Case 2: We assume $y_\varepsilon \to y_0$ as $\varepsilon \to 0$, where $y_0 \neq x_0$. We let $\delta > 0$ be such that $2\delta \leq d_g(x_0, y_0)$, and write

$$\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star - 1} dv_g \leq \int_{B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g + \int_{M \setminus B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g.$$

As above, standard properties of the Green function give

$$\int_{B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g \leq C \int_{B_{x_\varepsilon}(\delta)} u_\varepsilon^{2^\star - 1} dv_g$$

and

$$\int_{M \setminus B_{x_\varepsilon}(\delta)} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g \leq C \int_{M \setminus B_{x_\varepsilon}(\delta)} d_g(x, y_\varepsilon)^{2-n} u_\varepsilon^{2^\star - 1} dv_g,$$

where $C > 0$ is independent of ε. Thanks to (3.36),

$$\int_{M \setminus B_{x_\varepsilon}(\delta)} d_g(x, y_\varepsilon)^{2-n} u_\varepsilon^{2^\star - 1} dv_g = o\Big(\mu_\varepsilon^{\frac{n}{2} - 1}\Big).$$

Independently, we can write

$$\int_{B_{x_\varepsilon}(\delta)} u_\varepsilon^{2^\star - 1} dv_g = \int_{B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dv_g + \int_{B_{x_\varepsilon}(\delta) \setminus B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dv_g.$$

By (3.15),

$$\int_{B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dv_g = O\Big(\mu_\varepsilon^{\frac{n}{2} - 1}\Big)$$

while by (3.36), taking $\nu > 0$ sufficiently small,

$$\int_{B_{x_\varepsilon}(\delta) \setminus B_{x_\varepsilon}(\mu_\varepsilon)} u_\varepsilon^{2^\star - 1} dv_g = O\Big(\mu_\varepsilon^{\frac{n}{2} - 1}\Big).$$

Coming back to (3.38), we get (Φ_ε) is bounded.

Case 3: We assume that $\mu_\varepsilon^{-1} d_g(x_\varepsilon, y_\varepsilon) \to +\infty$ and that $d_g(x_\varepsilon, y_\varepsilon) \to 0$ as $\varepsilon \to 0$. We write that

$$\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star - 1} dv_g \leq \int_{\Omega_\varepsilon} H_\varepsilon u_\varepsilon^{2^\star - 1} + \int_{M \setminus \Omega_\varepsilon} H_\varepsilon u_\varepsilon^{2^\star - 1} dv_g,$$

where $\Omega_\varepsilon = B_{y_\varepsilon}(\frac{d_g(x_\varepsilon, y_\varepsilon)}{2})$. As above, standard properties of the Green function and (3.36) give that

$$\int_M G_0(x, y_\varepsilon) u_\varepsilon^{2^\star - 1} dv_g \leq C \mu_\varepsilon^{\frac{n+2}{2}(1 - 2\nu)} d_g(x_\varepsilon, y_\varepsilon)^{(n+2)(\nu - 1)} \int_{\Omega_\varepsilon} d_g(x, y_\varepsilon)^{2-n} dv_g$$
$$+ C d_g(x_\varepsilon, y_\varepsilon)^{2-n} \int_M u_\varepsilon^{2^\star - 1} dv_g.$$

Then,

$$\int_M G_0(x,y_\varepsilon)u_\varepsilon^{2^\star-1}dv_g \le C\mu_\varepsilon^{\frac{n+2}{2}(1-2\nu)}d_g(x_\varepsilon,y_\varepsilon)^{(n+2)(\nu-1)+2} + Cd_g(x_\varepsilon,y_\varepsilon)^{2-n}\mu_\varepsilon^{\frac{n}{2}-1},$$

where $C>0$ does not depend on ε. Coming back to (3.38), we get that

$$u_\varepsilon(y_\varepsilon)\mu_\varepsilon^{1-\frac{n}{2}}d_g(x_\varepsilon,y_\varepsilon)^{n-2} \le C\Big(\frac{\mu_\varepsilon}{d_g(x_\varepsilon,y_\varepsilon)}\Big)^{2-(n+2)\nu} + C$$

and since $\mu_\varepsilon^{-1}d_g(x_\varepsilon,y_\varepsilon)\to+\infty$ as $\varepsilon\to 0$, taking $\nu<\frac{2}{n+2}$, we get once again (Φ_ε) is bounded.

Summarizing cases 1 to 3, we have proved that for any sequence (y_ε) in M, there exists $C>0$, independent of ε, such that

$$u_\varepsilon(y_\varepsilon)\mu_\varepsilon^{1-\frac{n}{2}}d_g(x_\varepsilon,y_\varepsilon)^{n-2}\le C.$$

This proves (3.37). □

Thanks to (3.15), (3.16), and (3.37), integrating (3.5) over M and letting $\varepsilon\to 0$ give

$$\lim_{\varepsilon\to 0}\hat B_\varepsilon\|u_\varepsilon\|_1\|\Sigma_\varepsilon\|_1\mu_\varepsilon^{1-\frac{n}{2}} = \frac{\omega_{n-1}}{n}A_n, \tag{3.39}$$

where A_n is given by (1.36). Independently, let $\delta>0$ small, and η be a smooth function such that $\eta=0$ in $B_{x_0}(\frac{\delta}{2})$ and $\eta=1$ in $M\backslash B_{x_0}(\delta)$. Multiplying (3.5) by η, and integrating over M, we get with (3.10)

$$\hat B_\varepsilon\|u_\varepsilon\|_1\int_M \eta\Sigma_\varepsilon dv_g = O\Big(\|u_\varepsilon\|_1\Big).$$

Since $\hat B_\varepsilon\to+\infty$ as $\varepsilon\to 0$, it follows that $\int_M\eta\Sigma_\varepsilon dv_g\to 0$ as $\varepsilon\to 0$. In particular,

$$\int_{M\backslash B_{x_0}(\delta)}\Sigma_\varepsilon dv_g\to 0$$

as $\varepsilon\to 0$. Since this holds for any $\delta>0$ small, and since $0\le\Sigma_\varepsilon\le 1$, we have proved

$$\int_M\Sigma_\varepsilon dv_g\to 0 \tag{3.40}$$

as $\varepsilon\to 0$. We let $r_\varepsilon>0$ be such that

$$\int_M\Sigma_\varepsilon dv_g = \frac{\omega_{n-1}}{n}r_\varepsilon^n. \tag{3.41}$$

Then, $r_\varepsilon\to 0$ as $\varepsilon\to 0$, and thanks to (3.39),

$$\lim_{\varepsilon\to 0}\hat B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^n\mu_\varepsilon^{1-\frac{n}{2}} = A_n. \tag{3.42}$$

Now, for $x \in \mathcal{B}_0(\delta r_\varepsilon^{-1})$, $\delta > 0$ small, we define $\hat{g}_\varepsilon(x) = \big(\exp^\star_{x_\varepsilon} g\big)(r_\varepsilon x)$, $\hat{u}_\varepsilon(x) = r_\varepsilon^{\frac{n}{2}-1} u_\varepsilon\big(\exp_{x_\varepsilon}(r_\varepsilon x)\big)$ and $\hat{\Sigma}_\varepsilon(x) = \Sigma_\varepsilon\big(\exp_{x_\varepsilon}(r_\varepsilon x)\big)$. Then

$$\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon + \hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n^2}\hat{u}_\varepsilon^{2^\star-1}. \tag{3.43}$$

Thanks to (3.41),

$$\limsup_{\varepsilon\to 0}\int_{\mathcal{B}_0(\delta r_\varepsilon^{-1})}\hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon} \le \frac{\omega_{n-1}}{n}\,. \tag{3.44}$$

We set

$$\hat{\mu}_\varepsilon = \frac{\mu_\varepsilon}{r_\varepsilon}\,. \tag{3.45}$$

It follows from (3.11) and (3.42) that $\hat{\mu}_\varepsilon \to 0$ as $\varepsilon \to 0$. Independently, (3.37) gives

$$|x|^{n-2}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon(x) \le C, \tag{3.46}$$

where $C > 0$ is independent of ε. By (3.43),

$$\Delta_{\hat{g}_\varepsilon}(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon) + \hat{B}_\varepsilon\|u_\varepsilon\|_1\hat{\mu}_\varepsilon^{1-\frac{n}{2}} r_\varepsilon^{\frac{n}{2}+1}\hat{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n^2}\hat{\mu}_\varepsilon^2(\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon)^{2^\star-1} \tag{3.47}$$

and (3.42) gives

$$\hat{B}_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \to A_n \tag{3.48}$$

as $\varepsilon \to 0$. Since $r_\varepsilon \to 0$ we also have that $\hat{g}_\varepsilon$ converges C^2 to the Euclidean metric in any compact subset of the Euclidean space. By (3.46), the $\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon$'s are bounded in any compact subset of $\mathbb{R}^n\backslash\{0\}$. By (3.47) and (3.48) they satisfy an equation with bounded coefficients. We can therefore assume that, up to a subsequence,

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\hat{u}_\varepsilon \to H \quad \text{in } C^1_{loc}(\mathbb{R}^n\backslash\{0\}), \tag{3.49}$$

where H is a solution of

$$\Delta H + A_n\hat{\Sigma} = 0 \quad \text{in } \mathbb{R}^n\backslash\{0\} \tag{3.50}$$

and $\hat{\Sigma}$ is such that $\hat{\Sigma}_\varepsilon \rightharpoonup \hat{\Sigma}$ in $L^p(\mathbb{R}^n)$ for any $p \ge 1$. Thanks to (3.44), and since $\hat{\Sigma}_\varepsilon \le 1$, $\|\hat{\Sigma}\|_\infty \le 1$. By (3.46) we clearly have

$$|x|^{n-2}H(x) \le C \tag{3.51}$$

for any $x \in \mathbb{R}^n\backslash\{0\}$, where $C > 0$ is independent of x.

We claim now that H can be computed explicitly. This is the subject of the following subsection.

3.3. **An explicit expression for H.** We claim first that H can be expressed as

$$H(x) = \frac{\lambda}{|x|^{n-2}} + \overline{H}, \tag{3.52}$$

where λ is real and $\overline{H}$ is smooth. For the sake of completeness, we prove this elementary claim by using basic notions from the theory of harmonic functions. A possible reference for such notions is the excellent paper by Han and Lin [20]. As a preliminary remark, we claim that a bounded harmonic function in $\mathbb{R}^n\backslash\mathcal{B}$, $n \geq 3$, has a limit at infinity. In order to prove this preliminary claim, one may proceed as follows. Let u be harmonic and bounded in $\mathbb{R}^n\backslash\mathcal{B}$. Up to replacing u by $u + A$, $A > 0$ a suitable constant, we can assume that u is nonnegative. Given $R > 1$, we let v_R be the smooth function in $\mathcal{B}_0(R)$ such that $\Delta v_R = 0$ in $\mathcal{B}_0(R)$ and $v_R = u$ on $\partial\mathcal{B}_0(R)$. When $|x| < R$, $v_R(x)$ is given by the Poisson integral formula

$$v_R(x) = \int_{\partial\mathcal{B}_0(R)} K(x,y)u(y)d\sigma(y).$$

The Poisson kernel K is such that $K \geq 0$ and

$$\int_{\partial\mathcal{B}_0(R)} K(x,y)d\sigma(y) = 1$$

for all $|x| < R$. In particular, we get v_R is nonnegative and such that for any $x \in \mathcal{B}_0(R)$, $|v_R| \leq K$, where K is a bound for $|u|$ in $\mathbb{R}^n\backslash\mathcal{B}$. Given x and y two points in $\mathbb{R}^n$, and R large, the Harnack inequality for harmonic functions gives that

$$\lim_{R\to+\infty} \frac{v_R(y)}{v_R(x)} = 1. \tag{3.53}$$

We set now $w = u - v_R$, and let $r > 1$. Clearly, $|w(x)| \leq K_r r^{n-2}|x|^{2-n}$ on $\partial\mathcal{B}_0(r)$, where K_r is the maximum of $|w|$ over $\partial\mathcal{B}_0(r)$. Since w and $\frac{1}{|x|^{n-2}}$ are harmonic in $\mathcal{B}_0(R)\backslash\mathcal{B}_0(r)$, the maximum principle gives $|w(x)| \leq K_r r^{n-2}|x|^{2-n}$ in $\mathcal{B}_0(R)\backslash\mathcal{B}_0(r)$. According to what we said above, $K_r \leq 2K$. Hence,

$$|u(x) - v_R(x)| \leq \frac{2Kr^{n-2}}{|x|^{n-2}} \tag{3.54}$$

in $\mathcal{B}_0(R)\backslash\mathcal{B}_0(r)$. We fix x in $\mathbb{R}^n$. Since the $v_R(x)$'s are bounded, there exists a sequence (R_k), with the property that $R_k \to +\infty$ as $k \to +\infty$, and there exists $\lambda \in \mathbb{R}$, such that $v_{R_k}(x) \to \lambda$ as $k \to +\infty$. Thanks to (3.53), we get for any $x \in \mathbb{R}^n$, $v_{R_k}(x) \to \lambda$ as $k \to +\infty$. Coming back to (3.54), taking

$R = R_k$, and passing to the limit $k \to +\infty$, we get for any $x \in \mathbb{R}^n \backslash \mathcal{B}_0(r)$,

$$|u(x) - \lambda| \le \frac{2Kr^{n-2}}{|x|^{n-2}}\,.$$

Then, $u(x) \to \lambda$ as $|x| \to +\infty$, and this proves our preliminary claim. We let now $u \in C^1(\mathcal{B})$ be such that $\Delta u = -A_n \hat{\Sigma}$ in $\mathcal{B}$, and set $\tilde{H} = H - u$. Then $\Delta \tilde{H} = 0$ in $\mathcal{B}\backslash\{0\}$. We let $\hat{H}$ be the Kelvin transform of $\tilde{H}$ given by

$$\hat{H}(x) = \frac{1}{|x|^{n-2}} \tilde{H}\Big(\frac{x}{|x|^2}\Big).$$

It is easily seen that $\Delta \hat{H} = 0$ in $\mathbb{R}^n \backslash \mathcal{B}$. Moreover, thanks to (3.51), $\hat{H}$ is bounded. The preliminary claim we just proved then gives that there exists λ real such that $\lim_{x\to 0} |x|^{n-2}\tilde{H}(x) = \lambda$. Let Φ be given by $\Phi(x) = \tilde{H}(x) - \frac{\lambda}{|x|^{n-2}}$. It is easily seen that Φ is harmonic in $\mathcal{B}\backslash\{0\}$, and, thanks to what we just proved, $\Phi(x) = o(|x|^{2-n})$. Standard arguments, see Han and Lin [20], then give 0 is a removable singularity for Φ. This proves H can be expressed as in (3.52), and thus our claim.

For convenience, we write

$$H(x) = \frac{\lambda}{|x|^{n-2}} + \frac{A_n}{2n}|x|^2 + H_0(x), \tag{3.55}$$

where $H_0 \in C^1(\mathbb{R}^n)$ is such that

$$\Delta H_0 = A_n(1 - \hat{\Sigma}). \tag{3.56}$$

Let $r > 0$. By (3.15), (3.16) and (3.37),

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}_0(r)} \hat{u}_\varepsilon^{2^\star-1} dv_{\hat{g}_\varepsilon} \to \int_{\mathbb{R}^n} \tilde{u}^{2^\star-1} dx = \frac{\omega_{n-1}}{n} K_n^2 A_n$$

as $\varepsilon \to 0$. Integrating (3.47) over $\mathcal{B}_0(r)$ and passing to the limit as $\varepsilon \to 0$ we then get

$$-\int_{\partial\mathcal{B}_0(r)} \partial_\nu H d\sigma + A_n \int_{\mathcal{B}_0(r)} \hat{\Sigma} dx = \frac{\omega_{n-1}}{n} A_n. \tag{3.57}$$

Since $\int_{\mathcal{B}_0(r)} \hat{\Sigma} dx \to 0$ as $r \to 0$, and

$$-\int_{\partial\mathcal{B}_0(r)} \partial_\nu H d\sigma \to (n-2)\lambda\omega_{n-1}$$

as $r \to 0$, it follows from (3.57)

$$\lambda = \frac{A_n}{n(n-2)}\,. \tag{3.58}$$

Noting that $H \geq 0$, we get with (3.58), $H_0 \geq -\frac{A_n}{2(n-2)}$ on $\partial\mathcal{B}$. By (3.56) and the maximum principle, since $\|\hat{\Sigma}\|_\infty \leq 1$, we have

$$H_0 \geq -\frac{A_n}{2(n-2)} \quad \text{in } \mathcal{B}.$$

In particular, $H(x) > 0$ for any $x \in \mathcal{B}\backslash\{0\}$, and it follows from (3.49) that for any r_1 and r_2 such that $0 < r_1 < r_2 < 1$, $\hat{u}_\varepsilon > 0$ in $\mathcal{B}_0(r_2)\backslash\mathcal{B}_0(r_1)$ for $\varepsilon > 0$ small. Then, $\hat{\Sigma}_\varepsilon = 1$ in $\mathcal{B}_0(r_2)\backslash\mathcal{B}_0(r_1)$ for $\varepsilon > 0$ small, and we get

$$\int_{\mathcal{B}_0(r_2)\backslash\mathcal{B}_0(r_1)} \hat{\Sigma} dx = \left|\mathcal{B}_0(r_2)\backslash\mathcal{B}_0(r_1)\right|,$$

where $|\mathcal{B}_0(r_2)\backslash\mathcal{B}_0(r_1)|$ is the Euclidean volume of $\mathcal{B}_0(r_2)\backslash\mathcal{B}_0(r_1)$. Letting $r_1 \to 0$ and $r_2 \to 1$, we then get

$$\int_{\mathcal{B}} \hat{\Sigma} dx = \frac{\omega_{n-1}}{n}.$$

We have $\|\hat{\Sigma}\|_\infty \leq 1$, and $\int_{\mathbb{R}^n} |\hat{\Sigma}| dx \leq n^{-1}\omega_{n-1}$. Thus,

$$\hat{\Sigma} = 1 \ \text{ in } \mathcal{B}, \qquad \hat{\Sigma} = 0 \ \text{ in } \mathbb{R}^n\backslash\mathcal{B}. \tag{3.59}$$

In particular, for any annulus $A \subset \mathbb{R}^n\backslash\mathcal{B}$, we get with (3.46)

$$\hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_A \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_A \hat{\Sigma}_\varepsilon \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} \leq C \int_A \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon}$$

and since $\int_A \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon} \to \int_A \hat{\Sigma} dx$ as $\varepsilon \to 0$, we get with (3.59) that $H = 0$ in $\mathbb{R}^n\backslash\mathcal{B}$. Since H is C^1 in $\mathbb{R}^n\backslash\{0\}$, this implies

$$\begin{aligned} &H(x) = \frac{A_n}{n(n-2)}\left(|x|^{2-n} - 1\right) + \frac{A_n}{2n}\left(|x|^2 - 1\right) \ \text{ in } \mathcal{B} \\ &H(x) = 0 \ \text{ in } \mathbb{R}^n\backslash\mathcal{B} \end{aligned} \tag{3.60}$$

and we have an explicit expression for H. □

Thanks to (3.37),

$$\begin{aligned} r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} u_\varepsilon dv_g &= r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} \Sigma_\varepsilon u_\varepsilon dv_g \\ &\leq C r_\varepsilon^{-2} \int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} \frac{\Sigma_\varepsilon}{d_g(x_\varepsilon, x)^{n-2}} dv_g \leq C r_\varepsilon^{-n} \int_{M\backslash B_{x_\varepsilon}(r_\varepsilon)} \Sigma_\varepsilon dv_g \end{aligned}$$

and since

$$r_\varepsilon^{-n} \int_{B_{x_\varepsilon}(r_\varepsilon)} \Sigma_\varepsilon dv_g = \int_{\mathcal{B}} \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon} \to \int_{\mathcal{B}} \hat{\Sigma} dx = \frac{\omega_{n-1}}{n}$$

as $\varepsilon \to 0$, we get with (3.41)

$$r_\varepsilon^{-n} \int_{M \backslash B_{x_\varepsilon}(r_\varepsilon)} \Sigma_\varepsilon dv_g \to 0$$

as $\varepsilon \to 0$. Hence,

$$r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{M \backslash B_{x_\varepsilon}(r_\varepsilon)} u_\varepsilon dv_g \to 0 \tag{3.61}$$

as $\varepsilon \to 0$. Independently, given $\delta > 0$,

$$r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B}_0(\delta)} \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}$$

and it follows from (3.46)

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g = 0. \tag{3.62}$$

At last,

$$r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{B_{x_\varepsilon}(r_\varepsilon) \backslash B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g = \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{\mathcal{B} \backslash \mathcal{B}_0(\delta)} \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}$$

and we get with (3.49) that for any $\delta \in (0, 1)$,

$$r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \int_{B_{x_\varepsilon}(r_\varepsilon) \backslash B_{x_\varepsilon}(\delta r_\varepsilon)} u_\varepsilon dv_g \to \int_{\mathcal{B} \backslash \mathcal{B}_0(\delta)} H dx \tag{3.63}$$

as $\varepsilon \to 0$. Combining (3.61)-(3.63), letting $\delta \to 0$, we get with (3.60)

$$r_\varepsilon^{-1-\frac{n}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \to \int_{\mathcal{B}} H dx = \frac{\omega_{n-1}}{2n(n+2)} A_n \tag{3.64}$$

as $\varepsilon \to 0$. Then, combining (3.48) and (3.64),

$$\hat{B}_\varepsilon r_\varepsilon^{n+2} \to \frac{2n(n+2)}{\omega_{n-1}} \tag{3.65}$$

as $\varepsilon \to 0$. Going on with the asymptotic study of $\hat{u}_\varepsilon$, we prove sharp asymptotic estimates in the following subsection.

3.4. **Strong estimates 2.** We claim that for any $\delta > 0$ there exists $C(\delta) > 1$ such that for $\varepsilon > 0$ small and any $x \in \mathcal{B}_0(\delta)$,

$$\frac{1}{C(\delta)} \Big(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4} |x|^2} \Big)^{\frac{n-2}{2}} \le \hat{u}_\varepsilon(x) \le C(\delta) \Big(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4} |x|^2} \Big)^{\frac{n-2}{2}} \tag{3.66}$$

with the property that $C(\delta) \to 1$ as $\delta \to 0$. Let us define U_ε by

$$U_\varepsilon(x) = \Big(\frac{\hat{\mu}_\varepsilon}{\hat{\mu}_\varepsilon^2 + \frac{\omega_n^{2/n}}{4}|x|^2}\Big)^{\frac{n-2}{2}}$$

and let (y_ε) be a sequence in $\mathcal{B}$. Suppose that $y_\varepsilon \to y_0$ as $\varepsilon \to 0$, $y_0 \neq 0$. Then, thanks to (3.49) and (3.60),

$$\frac{\hat{u}_\varepsilon(y_\varepsilon)}{U_\varepsilon(y_\varepsilon)} = 1 + 0(|y_\varepsilon|^{n-2}).$$

In order to prove (3.66) it thus suffices to proves that if $y_\varepsilon \to 0$ as $\varepsilon \to 0$, then

$$\lim_{\varepsilon\to 0} \frac{\hat{u}_\varepsilon(y_\varepsilon)}{U_\varepsilon(y_\varepsilon)} = 1. \tag{3.67}$$

If $|y_\varepsilon| \leq C\hat{\mu}_\varepsilon$, (3.67) follows from (3.15). In order to prove (3.67), and so (3.66), we are therefore left with the case where $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty$ as $\varepsilon \to 0$. Let $\hat{v}_\varepsilon$ be given by $\hat{v}_\varepsilon(x) = |y_\varepsilon|^{\frac{n}{2}-1}\hat{u}_\varepsilon(|y_\varepsilon|x)$ and let $\hat{h}_\varepsilon(x) = \hat{g}_\varepsilon(|y_\varepsilon|x)$, $\sigma_\varepsilon(x) = \hat{\Sigma}_\varepsilon(|y_\varepsilon|x)$. It is easily seen that

$$\Delta_{\hat{h}_\varepsilon}\hat{v}_\varepsilon + \hat{B}_\varepsilon(r_\varepsilon|y_\varepsilon|)^{\frac{n+2}{2}}\|u_\varepsilon\|_1\sigma_\varepsilon = \frac{1-\varepsilon}{K_n^2}\hat{v}_\varepsilon^{2^\star-1}$$

and that $\hat{h}_\varepsilon \to \xi$ in $C^1_{loc}(\mathbb{R}^n)$ as $\varepsilon \to 0$. We set $\hat{w}_\varepsilon(x) = (\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|})^{1-\frac{n}{2}}\hat{v}_\varepsilon$. Then,

$$\Delta_{\hat{h}_\varepsilon}\hat{w}_\varepsilon + |y_\varepsilon|^n\hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1\sigma_\varepsilon = \frac{1-\varepsilon}{K_n^2}\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^2\hat{w}_\varepsilon^{2^\star-1}. \tag{3.68}$$

Thanks to (3.15) and (3.37),

$$\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^{n-2}\hat{w}_\varepsilon\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}x\Big) \to \tilde{u}(x) \tag{3.69}$$

in $C^0_{loc}(\mathbb{R}^n)$ as $\varepsilon \to 0$, and

$$|x|^{n-2}\hat{w}_\varepsilon(x) \leq C. \tag{3.70}$$

Thanks to (3.64) and (3.65), we also have

$$|y_\varepsilon|^n\hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}}\hat{\mu}_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1 \to 0 \tag{3.71}$$

as $\varepsilon \to 0$. Noting that by (3.70), $\hat{w}_\varepsilon$ is bounded in any compact subset of $\mathbb{R}^n\backslash\{0\}$, it follows from standard elliptic theory, from (3.68), and (3.71), that

$\hat{w}_\varepsilon \to \Psi$ in $C^1_{loc}(\mathbb{R}^n\backslash\{0\})$, where Ψ is a solution of $\Delta\Psi = 0$ in $\mathbb{R}^n\backslash\{0\}$. We let $\delta > 0$ small, and we integrate (3.68) over $\mathcal{B}_0(\delta)$. Then

$$\begin{aligned} &-\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma_{\hat{h}_\varepsilon} + |y_\varepsilon|^n \hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \int_{\mathcal{B}_0(\delta)} \sigma_\varepsilon dv_{\hat{h}_\varepsilon} \\ &\quad = \frac{1-\varepsilon}{K_n^2}\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1} dv_{\hat{h}_\varepsilon}. \end{aligned} \tag{3.72}$$

It is easily seen that

$$\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1} dv_{\hat{h}_\varepsilon} = \int_{\mathcal{B}_0(\delta\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon})} \Big(\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^{n-2} \hat{w}_\varepsilon\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}x\Big)\Big)^{2^\star-1} dv_{\tilde{g}_\varepsilon}.$$

Thanks to (3.69) and (3.70) we then get that

$$\Big(\frac{\hat{\mu}_\varepsilon}{|y_\varepsilon|}\Big)^2 \int_{\mathcal{B}_0(\delta)} \hat{w}_\varepsilon^{2^\star-1} dv_{\hat{h}_\varepsilon} \to \int_{\mathbb{R}^n} \tilde{u}^{2^\star-1} dx \tag{3.73}$$

as $\varepsilon \to 0$. Thanks to (3.71) we have that

$$|y_\varepsilon|^n \hat{B}_\varepsilon r_\varepsilon^{\frac{n+2}{2}} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \|u_\varepsilon\|_1 \int_{\mathcal{B}_0(\delta)} \sigma_\varepsilon dv_{\hat{h}_\varepsilon} \to 0 \tag{3.74}$$

as $\varepsilon \to 0$, and since $\hat{h}_\varepsilon \to \xi$ as $\varepsilon \to 0$, we also have that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \hat{w}_\varepsilon d\sigma_{\hat{h}_\varepsilon} \to \int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma_\xi \tag{3.75}$$

as $\varepsilon \to 0$. Combining (3.72)-(3.75), it follows that

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma_\xi + \frac{1}{K_n^2}\int_{\mathbb{R}^n} \tilde{u}^{2^\star-1} dx = 0$$

and thus

$$\int_{\partial\mathcal{B}_0(\delta)} \partial_\nu \Psi d\sigma_\xi + \frac{\omega_{n-1}}{n} A_n = 0. \tag{3.76}$$

As in Section 1, (3.70) and (3.76) imply that

$$\Psi(x) = \frac{A_n}{n(n-2)|x|^{n-2}}\,.$$

In particular, taking $x = y_\varepsilon/|y_\varepsilon|$, we get that for any sequence (y_ε) such that $y_\varepsilon \to 0$ and $\frac{|y_\varepsilon|}{\hat{\mu}_\varepsilon} \to +\infty$ as $\varepsilon \to 0$,

$$|y_\varepsilon|^{n-2} \hat{\mu}_\varepsilon^{1-\frac{n}{2}} \hat{u}_\varepsilon(y_\varepsilon) \to \frac{A_n}{n(n-2)} = 2^{n-2}\omega_n^{\frac{2}{n}-1} \tag{3.77}$$

as $\varepsilon \to 0$. This proves (3.67), and thus also (3.66). □

From now on, we let $\mathcal{B}_2$ be the Euclidean ball $\mathcal{B}_0(2)$, and let $\eta \in C_c^\infty(\mathcal{B}_2)$ be a radially symmetrical function such that $\eta = 1$ in $\mathcal{B}$. We want to estimate

$$I = \int_{\mathcal{B}_2} (\eta\hat{u}_\varepsilon)^{2^\star} dx \quad \text{and} \quad J = \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2 dx. \tag{3.78}$$

We start with I.

3.5. **An expansion for** $\int_{\mathcal{B}_2}(\eta\hat{u}_\varepsilon)^{2^\star}dx$ **as** $\varepsilon \to 0$**.** We write

$$\begin{aligned}\int_{\mathcal{B}_2} (\eta\hat{u}_\varepsilon)^{2^\star} dx &= \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dx + \int_{\mathcal{B}_2\setminus\mathcal{B}} (\eta\hat{u}_\varepsilon)^{2^\star} dx \\ &= \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star}(1 - \sqrt{|\hat{g}_\varepsilon|})dx + \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2\setminus\mathcal{B}} (\eta\hat{u}_\varepsilon)^{2^\star} dx,\end{aligned}$$

where $|\hat{g}_\varepsilon|$ is the determinant of the components of $\hat{g}_\varepsilon$ in Euclidean coordinates. Thanks to the Cartan expansion of a metric in geodesic normal coordinates, we can write $\sqrt{|\hat{g}_\varepsilon|} = 1 - \frac{r_\varepsilon^2}{6} R_{ij}(x_\varepsilon)x^i x^j + r_\varepsilon^3 O(|x|^3)$, where the R_{ij}'s are the components of the Ricci curvature of g in the exponential chart at x_ε. Then,

$$\begin{aligned}\int_{\mathcal{B}_2} (\eta\hat{u}_\varepsilon)^{2^\star} dx = &\int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} + \frac{r_\varepsilon^2}{6} R_{ij}(x_\varepsilon) \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} x^i x^j dx \\ &+ r_\varepsilon^3 O\Big(\int_{\mathcal{B}} |x|^3 \hat{u}_\varepsilon^{2^\star} dx\Big) + \int_{\mathcal{B}_2\setminus\mathcal{B}} (\eta\hat{u}_\varepsilon)^{2^\star} dx.\end{aligned} \tag{3.79}$$

Thanks to (3.46),

$$\int_{\mathcal{B}_2\setminus\mathcal{B}} (\eta\hat{u}_\varepsilon)^{2^\star} dx = O(\hat{\mu}_\varepsilon^n) = o(\hat{\mu}_\varepsilon^{n-2}). \tag{3.80}$$

Similarly, it is easily seen that

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} = 1 + o(\hat{\mu}_\varepsilon^{n-2}). \tag{3.81}$$

By (3.15),

$$\hat{\mu}_\varepsilon^{\frac{n}{2}-1} \hat{u}_\varepsilon(\hat{\mu}_\varepsilon x) \to \tilde{u}(x) \tag{3.82}$$

in $C^1_{loc}(\mathbb{R}^n)$, where $\tilde{u}$ is the fundamental solution given by (3.16). Combining this estimate with (3.46), it follows that

$$r_\varepsilon^3 \int_{\mathcal{B}} |x|^3 \hat{u}_\varepsilon^{2^\star} dx = o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) \tag{3.83}$$

and that

$$\frac{r_\varepsilon^2}{6} R_{ij}(x_\varepsilon) \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} x^i x^j dx = \frac{S_g(x_0)}{6n} \Big(\int_{\mathbb{R}^n} |x|^2 \tilde{u}^{2^\star} dx\Big) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2).$$

Noting that

$$\int_{\mathbb{R}^n} |x|^2 \tilde{u}^{2^\star} dx = \frac{4n}{(n-2)\omega_n^{2/n}} = n^2 K_n^2$$

we get

$$\frac{r_\varepsilon^2}{6} R_{ij}(x_\varepsilon) \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} x^i x^j dx = \frac{nK_n^2}{6} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2). \tag{3.84}$$

Combining (3.79)-(3.81), (3.83), and (3.84), it follows that

$$I = 1 + \frac{nK_n^2}{6} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}). \tag{3.85}$$

This is the expansion we were looking for. □

We now compute an expansion for J. This is the subject of the following subsection.

3.6. **An expansion for** $\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2 dx$ **as** $\varepsilon \to 0$**.** We write

$$\begin{aligned} \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_\xi^2 dx &= \int_{\mathcal{B}_2} (\delta^{ij} - \hat{g}_\varepsilon^{ij}) \partial_i(\eta\hat{u}_\varepsilon) \partial_j(\eta\hat{u}_\varepsilon) dx \\ &+ \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 (1 - \sqrt{|\hat{g}_\varepsilon|}) dx + \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon}, \end{aligned} \tag{3.86}$$

where the subscripts ξ and $\hat{g}_\varepsilon$ refer to the metric with respect to which the expression has to be understood. Thanks to (3.43), namely the equation satisfied by the $\hat{u}_\varepsilon$'s, we can write

$$\begin{aligned} \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2 dv_{\hat{g}_\varepsilon} &= \int_{\mathcal{B}_2} |\nabla\eta|^2 \hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon \Delta_{\hat{g}_\varepsilon} \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} \\ &= \frac{1-\varepsilon}{K_n^2} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} - \hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2} |\nabla\eta|^2 \hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon}. \end{aligned} \tag{3.87}$$

Thanks to (3.46) and (3.59),

$$\int_{\mathcal{B}_2} |\nabla\eta|^2 \hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon} = O\Big(\int_{\mathcal{B}_2\backslash\mathcal{B}} \hat{\Sigma}_\varepsilon \hat{u}_\varepsilon^2 dv_{\hat{g}_\varepsilon}\Big) = \hat{\mu}_\varepsilon^{n-2} O\Big(\int_{\mathcal{B}_2\backslash\mathcal{B}} \hat{\Sigma}_\varepsilon dv_{\hat{g}_\varepsilon}\Big) = o\Big(\hat{\mu}_\varepsilon^{n-2}\Big). \tag{3.88}$$

Independently, we can write

$$\int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} = \int_{\mathcal{B}} \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} + \int_{\mathcal{B}_2\backslash\mathcal{B}} \eta^2 \hat{u}_\varepsilon^{2^\star} dv_{\hat{g}_\varepsilon} = 1 + o(\hat{\mu}_\varepsilon^{n-2}). \tag{3.89}$$

At last, we write that

$$\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} \tag{3.90}$$

$$= \hat\mu_\varepsilon^{n-2}\left[r_\varepsilon^{-1-\frac{n}{2}}\hat\mu_\varepsilon^{1-\frac{n}{2}}\|u_\varepsilon\|_1\right]\left(\hat B_\varepsilon r_\varepsilon^{n+2}\right)\hat\mu_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_2}\eta^2\hat u_\varepsilon dv_{\hat g_\varepsilon}.$$

By (3.46),

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\hat\mu_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_0(\delta)}\hat u_\varepsilon dv_{\hat g_\varepsilon} = 0.$$

With such a relation, it easily follows from (3.49) and (3.60) that

$$\lim_{\varepsilon\to 0}\hat\mu_\varepsilon^{1-\frac{n}{2}}\int_{\mathcal{B}_2}\eta^2\hat u_\varepsilon dv_{\hat g_\varepsilon} = \int_{\mathcal{B}} H dx = \frac{\omega_{n-1}}{2n(n+2)}A_n. \tag{3.91}$$

By (3.64), (3.65), and (3.91), coming back to (3.90), we then get

$$\hat B_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}\int_{\mathcal{B}_2}\eta^2\hat u_\varepsilon dv_{\hat g_\varepsilon} = \frac{\omega_{n-1}}{2n(n+2)}A_n^2\hat\mu_\varepsilon^{n-2} + o(\hat\mu_\varepsilon^{n-2}). \tag{3.92}$$

Finally, thanks to (3.88), (3.89), and (3.92), we get with (3.87) that

$$\int_{\mathcal{B}_2}|\nabla(\eta\hat u_\varepsilon)|^2 dv_{\hat g_\varepsilon} = \frac{1-\varepsilon}{K_n^2} - \frac{\omega_{n-1}}{2n(n+2)}A_n^2\hat\mu_\varepsilon^{n-2} + o(\hat\mu_\varepsilon^{n-2}). \tag{3.93}$$

Concerning the first term in the RHS of (3.86), we claim that

$$\int_{\mathcal{B}_2}(\delta^{ij} - \hat g_\varepsilon^{ij})\partial_i(\eta\hat u_\varepsilon)\partial_j(\eta\hat u_\varepsilon)dx = o(r_\varepsilon^2\hat\mu_\varepsilon^2) \tag{3.94}$$

if $n \ge 5$, and

$$\int_{\mathcal{B}_2}(\delta^{ij} - \hat g_\varepsilon^{ij})\partial_i(\eta\hat u_\varepsilon)\partial_j(\eta\hat u_\varepsilon)dx = o(r_\varepsilon^2\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|) \tag{3.95}$$

if $n = 4$. We assume first $n \ge 5$. Thanks to the Cartan expansion of a metric in geodesic normal coordinates, we can write

$$\hat g_\varepsilon^{ij} = \delta^{ij} - \frac{r_\varepsilon^2}{3}R^i{}_{\alpha\beta}{}^j(x_\varepsilon)x^\alpha x^\beta + r_\varepsilon^3 O(|x|^3),$$

where the R_{ijkl}'s are the components of the Riemann curvature tensor of g in the exponential chart at x_ε, and an index is raised with the metric. Let $\tilde u$ be given by (3.16). Since η and $\tilde u$ are radially symmetrical,

$$R^i{}_{\alpha\beta}{}^j(x_\varepsilon)\partial_i\big(\eta(\hat\mu_\varepsilon x)\tilde u(x)\big)\partial_j\big(\eta(\hat\mu_\varepsilon x)\tilde u(x)\big)x^\alpha x^\beta = 0. \tag{3.96}$$

Let $R > 0$. Thanks to (3.82), writing that $\int_{\mathcal{B}_2} = \int_{\mathcal{B}_0(R\hat\mu_\varepsilon)} + \int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat\mu_\varepsilon)}$, it is easily seen with (3.96) that that for any $R > 0$,

$$\int_{\mathcal{B}_2}(\delta^{ij} - \hat g_\varepsilon^{ij})\partial_i(\eta\hat u_\varepsilon)\partial_j(\eta\hat u_\varepsilon)dx \le Cr_\varepsilon^2\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat\mu_\varepsilon)}|x|^2|\nabla(\eta\hat u_\varepsilon)|^2_{\hat g_\varepsilon}dv_{\hat g_\varepsilon} + o(r_\varepsilon^2\hat\mu_\varepsilon^2), \tag{3.97}$$

where $C > 0$ does not depend on ε and R. We write

$$\begin{aligned}&\frac{1}{2}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}\\&\le\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla\eta|^2_{\hat{g}_\varepsilon}\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}+\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2|\nabla\hat{u}_\varepsilon|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}.\end{aligned}\tag{3.98}$$

As in (3.88),

$$\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2|\nabla\eta|^2_{\hat{g}_\varepsilon}\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}=o(\hat{\mu}_\varepsilon^{n-2}).\tag{3.99}$$

Independently, thanks to (3.82),

$$\int_{\partial\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon|\partial_\nu\hat{u}_\varepsilon|d\sigma_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2\ \text{ and }\ \int_{\partial\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|\hat{u}_\varepsilon^2d\sigma_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2,$$

where ε_R is such that $\lim_{R\to+\infty}\lim_{\varepsilon\to0}\varepsilon_R=0$. Hence,

$$\begin{aligned}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2|\nabla\hat{u}_\varepsilon|^2_{\hat{g}_\varepsilon}dv_{\hat{g}_\varepsilon}&=\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}\\&-\frac{1}{2}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Delta_{\hat{g}_\varepsilon}(|x|^2\eta^2)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}+\varepsilon_R\hat{\mu}_\varepsilon^2.\end{aligned}\tag{3.100}$$

Thanks to (3.43), namely the equation satisfied by the $\hat{u}_\varepsilon$'s,

$$\begin{aligned}\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}&\le C\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\hat{u}_\varepsilon^{2^\star}dx\\&=C\hat{\mu}_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)}|x|^2\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^{2^\star}dx\end{aligned}$$

so that, by (3.46),

$$\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}|x|^2\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon dv_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2,\tag{3.101}$$

where ε_R is as above. Similarly,

$$\begin{aligned}\left|\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Delta_{\hat{g}_\varepsilon}(|x|^2\eta^2)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}\right|&\le C\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\hat{u}_\varepsilon^2dx\\&=C\hat{\mu}_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)}\left(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)\right)^2dx\end{aligned}$$

and still thanks to (3.46), since $n\ge5$, we get

$$\int_{\mathcal{B}_2\backslash\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Delta_{\hat{g}_\varepsilon}(|x|^2\eta^2)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}=\varepsilon_R\hat{\mu}_\varepsilon^2,\tag{3.102}$$

where ε_R is as above. Combining (3.97)-(3.102) we get that

$$\int_{\mathcal{B}_2} (\delta^{ij} - \hat{g}_\varepsilon^{ij}) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) dx = \varepsilon_R r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2)$$

and since R is arbitrary, this proves (3.94). In order to prove (3.95), there we have $n = 4$, we need to be more subtle. Still thanks to the Cartan expansion of a metric in geodesic normal coordinates, we can write

$$\begin{aligned} &\int_{\mathcal{B}_2} (\delta^{ij} - \hat{g}_\varepsilon^{ij}) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) dx \\ &\le C r_\varepsilon^2 \int_{\mathcal{B}_2} R^i{}_{\alpha\beta}{}^j(x_\varepsilon) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) x^\alpha x^\beta dv_{\hat{g}_\varepsilon} + r_\varepsilon^2 o\Big(\int_{\mathcal{B}_2} |x|^2 |\nabla(\eta \hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon} \Big), \end{aligned} \tag{3.103}$$

where $C > 0$ does not depend on ε. Similar developments to the ones we made when $n \ge 5$ give

$$\int_{\mathcal{B}_2} |x|^2 |\nabla(\eta \hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon} = O\Big(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|\Big) \tag{3.104}$$

when $n = 4$. Independently, thanks to (3.96),

$$\begin{aligned} &\int_{\mathcal{B}_2} R^i{}_{\alpha\beta}{}^j(x_\varepsilon) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) x^\alpha x^\beta dv_{\hat{g}_\varepsilon} \\ &\quad \le \int_{\mathcal{B}_2 \backslash \mathcal{B}_0(R\hat{\mu}_\varepsilon)} R^i{}_{\alpha\beta}{}^j(x_\varepsilon) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) x^\alpha x^\beta dv_{\hat{g}_\varepsilon} + o(\hat{\mu}_\varepsilon^2), \end{aligned} \tag{3.105}$$

where $R > 0$ is fixed. Combining (3.103)-(3.105), we then get, for $R > 0$,

$$\begin{aligned} &\int_{\mathcal{B}_2} (\delta^{ij} - \hat{g}_\varepsilon^{ij}) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) dx \\ &\le C r_\varepsilon^2 \int_{\mathcal{B}_2 \backslash \mathcal{B}_0(R\hat{\mu}_\varepsilon)} R^i{}_{\alpha\beta}{}^j(x_\varepsilon) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) x^\alpha x^\beta dv_{\hat{g}_\varepsilon} + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|). \end{aligned} \tag{3.106}$$

Let $K > 0$ be an upper bound for the sectional curvature of g. Then

$$\begin{aligned} &\int_{\mathcal{B}_2 \backslash \mathcal{B}_0(R\hat{\mu}_\varepsilon)} R^i{}_{\alpha\beta}{}^j(x_\varepsilon) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) x^\alpha x^\beta dv_{\hat{g}_\varepsilon} \\ &\le K \int_{\mathcal{B}_2 \backslash \mathcal{B}_0(R\hat{\mu}_\varepsilon)} \big(|\nabla(|x| \eta \hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 - (\nabla(|x| \eta \hat{u}_\varepsilon), \nu)_{\hat{g}_\varepsilon}^2 \big) dv_{\hat{g}_\varepsilon} + o\big(\int_{\mathcal{B}_2} |x|^2 |\nabla(\eta \hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon} \big), \end{aligned}$$

where $\nu = \frac{x}{|x|}$, and thanks to (3.104), we get

$$\begin{aligned} &\int_{\mathcal{B}_2 \backslash \mathcal{B}_0(R\hat{\mu}_\varepsilon)} R^i{}_{\alpha\beta}{}^j(x_\varepsilon) \partial_i(\eta \hat{u}_\varepsilon) \partial_j(\eta \hat{u}_\varepsilon) x^\alpha x^\beta dv_{\hat{g}_\varepsilon} \\ &\le K \int_{\mathcal{B}_2 \backslash \mathcal{B}_0(R\hat{\mu}_\varepsilon)} \Big(|\nabla(|x| \eta \hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 - (\nabla(|x| \eta \hat{u}_\varepsilon), \nu)_{\hat{g}_\varepsilon}^2 \Big) dv_{\hat{g}_\varepsilon} + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|). \end{aligned} \tag{3.107}$$

It is easily seen that

$$\begin{aligned}&\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\Big(|\nabla(|x|\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\eta\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\Big)dv_{\hat{g}_\varepsilon}\\&=\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\eta^2\Big(|\nabla(|x|\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\Big)dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2).\end{aligned}\tag{3.108}$$

Combining (3.106)-(3.108), it follows that

$$\begin{aligned}&\int_{\mathcal{B}_2}(\delta^{ij}-\hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx\\&\le Cr_\varepsilon^2\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\eta^2\Big(|\nabla(|x|\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\Big)dv_{\hat{g}_\varepsilon}+o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|).\end{aligned}\tag{3.109}$$

Letting $\tilde{u}_\varepsilon$ be as in (3.12), $\tilde{u}_\varepsilon$ is given by $\tilde{u}_\varepsilon(x)=\hat{\mu}_\varepsilon^{\frac{n}{2}-1}\hat{u}_\varepsilon(\hat{\mu}_\varepsilon x)$ we have

$$\begin{aligned}&\int_{\mathcal{B}_2\setminus\mathcal{B}_0(R\hat{\mu}_\varepsilon)}\eta^2\Big(|\nabla(|x|\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}-(\nabla(|x|\hat{u}_\varepsilon),\nu)^2_{\hat{g}_\varepsilon}\Big)dv_{\hat{g}_\varepsilon}\\&=\hat{\mu}_\varepsilon^2\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\Big(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon}-(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}\Big)dv_{\tilde{g}_\varepsilon}.\end{aligned}\tag{3.110}$$

We write now

$$\begin{aligned}&\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\Big(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon}-(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}\Big)dv_{\tilde{g}_\varepsilon}\\&\le C\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi+\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2|x|\tilde{u}_\varepsilon\Delta_{\tilde{g}_\varepsilon}(|x|\tilde{u}_\varepsilon)dv_{\tilde{g}_\varepsilon}\\&\quad-\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}dv_{\tilde{g}_\varepsilon}+C\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\tilde{u}_\varepsilon^2dv_{\tilde{g}_\varepsilon}\end{aligned}$$

and since

$$\Delta_{\tilde{g}_\varepsilon}(|x|\tilde{u}_\varepsilon)=|x|\Delta_{\tilde{g}_\varepsilon}\tilde{u}_\varepsilon+\tilde{u}_\varepsilon\Delta_{\tilde{g}_\varepsilon}|x|-\frac{2}{|x|}(\nabla\tilde{u}_\varepsilon,x)_{\tilde{g}_\varepsilon}$$

we get

$$\begin{aligned}&\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2\Big(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon}-(\nabla(|x|\tilde{u}_\varepsilon),\nu)^2_{\tilde{g}_\varepsilon}\Big)dv_{\tilde{g}_\varepsilon}\\&\le C\int_{\partial\mathcal{B}_0(R)}|\nabla(|x|^2\tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi+\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2|x|^2\tilde{u}_\varepsilon\Delta_{\tilde{g}_\varepsilon}\tilde{u}_\varepsilon dv_{\tilde{g}_\varepsilon}\\&\quad+\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)}\eta(\hat{\mu}_\varepsilon x)^2|x|\tilde{u}_\varepsilon^2\Delta_{\tilde{g}_\varepsilon}(|x|)dv_{\tilde{g}_\varepsilon}+C\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})}\tilde{u}_\varepsilon^2dv_{\tilde{g}_\varepsilon}\end{aligned}$$

$$- 2 \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \eta(\hat{\mu}_\varepsilon x)^2 \tilde{u}_\varepsilon (\nabla \tilde{u}_\varepsilon, x)_{\tilde{g}_\varepsilon} dv_{\tilde{g}_\varepsilon}$$
$$- \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \eta(\hat{\mu}_\varepsilon x)^2 \Big((\nabla \tilde{u}_\varepsilon, x)_{\tilde{g}_\varepsilon} + \tilde{u}_\varepsilon \Big)^2 dv_{\tilde{g}_\varepsilon}.$$

Noting that $|x|\Delta_{\tilde{g}_\varepsilon}(|x|) \le -(n-1) + C\mu_\varepsilon^2 |x|^2$ and since

$$\Delta_{\tilde{g}_\varepsilon} \tilde{u}_\varepsilon + \hat{B}_\varepsilon \mu_\varepsilon^{\frac{n+2}{2}} \|u_\varepsilon\|_1 \tilde{\Sigma}_\varepsilon = \frac{1-\varepsilon}{K_n^2} \tilde{u}_\varepsilon^{2^\star - 1}$$

it follows from the above computations that

$$\begin{aligned}
&\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \eta(\hat{\mu}_\varepsilon x)^2 \Big(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon} - (\nabla(|x|\tilde{u}_\varepsilon), \nu)^2_{\tilde{g}_\varepsilon} \Big) dv_{\tilde{g}_\varepsilon} \\
&\le C \int_{\partial\mathcal{B}_0(R)} |\nabla(|x|^2 \tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi + C \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} |x|^2 \tilde{u}_\varepsilon^{2^\star} dv_{\tilde{g}_\varepsilon} \\
&\quad + C r_\varepsilon^2 \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon} + C \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon} \\
&\quad -(n-4) \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \eta(\hat{\mu}_\varepsilon x)^2 \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon} \\
&\quad - \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \eta(\hat{\mu}_\varepsilon x)^2 \Big((\nabla \tilde{u}_\varepsilon, x)_{\tilde{g}_\varepsilon} + 2\tilde{u}_\varepsilon \Big)^2 dv_{\tilde{g}_\varepsilon}
\end{aligned}$$

and hence that

$$\begin{aligned}
&\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \eta(\hat{\mu}_\varepsilon x)^2 \Big(|\nabla(|x|\tilde{u}_\varepsilon)|^2_{\tilde{g}_\varepsilon} - (\nabla(|x|\tilde{u}_\varepsilon), \nu)^2_{\tilde{g}_\varepsilon} \Big) dv_{\tilde{g}_\varepsilon} \\
&\le C \int_{\partial\mathcal{B}_0(R)} |\nabla(|x|^2 \tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi + C \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} |x|^2 \tilde{u}_\varepsilon^{2^\star} dv_{\tilde{g}_\varepsilon} \\
&\quad + C r_\varepsilon^2 \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon} + C \int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon}.
\end{aligned} \tag{3.111}$$

By (3.82),

$$\int_{\partial\mathcal{B}_0(R)} |\nabla(|x|^2 \tilde{u}_\varepsilon^2)|_\xi d\sigma_\xi \le C \tag{3.112}$$

while by (3.43),

$$\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\backslash\mathcal{B}_0(R)} |x|^2 \tilde{u}_\varepsilon^{2^\star} dv_{\tilde{g}_\varepsilon} \le C \tag{3.113}$$

$$\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(R)} \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon} = O(|\ln\hat{\mu}_\varepsilon|)$$

$$\int_{\mathcal{B}_0(\frac{2}{\hat{\mu}_\varepsilon})\setminus\mathcal{B}_0(\frac{1}{\hat{\mu}_\varepsilon})} \tilde{u}_\varepsilon^2 dv_{\tilde{g}_\varepsilon} \le C$$

when $n = 4$. Combining (3.109)-(3.113), it follows that when $n = 4$,

$$\int_{\mathcal{B}_2} (\delta^{ij} - \hat{g}_\varepsilon^{ij})\partial_i(\eta\hat{u}_\varepsilon)\partial_j(\eta\hat{u}_\varepsilon)dx = o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|)$$

and this proves (3.95). Still when estimating J, we now have to deal with the second term in the RHS of (3.86). We claim here that

$$\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2(1-\sqrt{|\hat{g}_\varepsilon|})dx = \frac{(n-2)(n+2)}{6(n-4)}S_g(x_0)r_\varepsilon^2\hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) \quad (3.114)$$

when $n \ge 5$, and that

$$\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2(1-\sqrt{|\hat{g}_\varepsilon|})dx = \frac{8\omega_3}{3\omega_4}S_g(x_0)r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon| + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|) \quad (3.115)$$

when $n = 4$. In order to prove this claim, we write

$$\sqrt{|\hat{g}_\varepsilon|} = 1 - \frac{r_\varepsilon^2}{6}R_{ij}(x_\varepsilon)x^i x^j + r_\varepsilon^3 O(|x|^3),$$

where the R_{ij}'s are the components of the Ricci curvature of g in the exponential chart at x_ε. Then

$$\begin{aligned}&\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2(1-\sqrt{|\hat{g}_\varepsilon|})dx \qquad (3.116)\\ &= \frac{r_\varepsilon^2}{6}R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 x^i x^j dv_{\hat{g}_\varepsilon} + r_\varepsilon^3 O\Big(\int_{\mathcal{B}_2}|x|^2|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon}\Big).\end{aligned}$$

As above,

$$\int_{\mathcal{B}_2} |x|^2|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon} = O(\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|) \quad (3.117)$$

when $n = 4$, and

$$\int_{\mathcal{B}_2} |x|^2|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 dv_{\hat{g}_\varepsilon} = O(\hat{\mu}_\varepsilon^2) \quad (3.118)$$

when $n \ge 5$. Similarly, it is easily seen that

$$R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}|\nabla(\eta\hat{u}_\varepsilon)|_{\hat{g}_\varepsilon}^2 x^i x^j dv_{\hat{g}_\varepsilon} = R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2|\nabla\hat{u}_\varepsilon|_{\hat{g}_\varepsilon}^2 x^i x^j dv_{\hat{g}_\varepsilon} + o(\hat{\mu}_\varepsilon^2). \quad (3.119)$$

Then

$$R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2|\nabla\hat{u}_\varepsilon|^2_{\hat{g}_\varepsilon}x^ix^jdv_{\hat{g}_\varepsilon} \tag{3.120}$$
$$= R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon x^ix^jdv_{\hat{g}_\varepsilon} - \frac{1}{2}\int_{\mathcal{B}_2}\Delta_{\hat{g}_\varepsilon}(\eta^2R_{ij}(x_\varepsilon)x^ix^j)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon}.$$

By (3.43),

$$R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon x^ix^jdv_{\hat{g}_\varepsilon}$$
$$=\frac{1-\varepsilon}{K_n^2}R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon^{2^\star}x^ix^jdv_{\hat{g}_\varepsilon} - \hat{B}_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{\Sigma}_\varepsilon\hat{u}_\varepsilon x^ix^jdv_{\hat{g}_\varepsilon}.$$

As when getting (3.92),

$$\hat{B}_\varepsilon\|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1}R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{\Sigma}_\varepsilon\hat{u}_\varepsilon x^ix^jdv_{\hat{g}_\varepsilon} = o(\hat{\mu}_\varepsilon^2)$$

while, thanks to (3.46) and (3.82),

$$R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon^{2^\star}x^ix^jdv_{\hat{g}_\varepsilon} = \frac{S_g(x_0)}{n}\int_{\mathbb{R}^n}|x|^2\tilde{u}^{2^\star}dx + o(\hat{\mu}_\varepsilon^2).$$

Noting that $\int_{\mathbb{R}^n}|x|^2\tilde{u}^{2^\star}dx = n^2K_n^2$ it follows that

$$R_{ij}(x_\varepsilon)\int_{\mathcal{B}_2}\eta^2\hat{u}_\varepsilon\Delta_{\hat{g}_\varepsilon}\hat{u}_\varepsilon x^ix^jdv_{\hat{g}_\varepsilon} = nK_n^2S_g(x_0)\hat{\mu}_\varepsilon^2 + o(\hat{\mu}_\varepsilon^2). \tag{3.121}$$

Independently,

$$\int_{\mathcal{B}_2}\Delta_{\hat{g}_\varepsilon}(\eta^2R_{ij}(x_\varepsilon)x^ix^j)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon} = -2S_g(x_0)\Big(1+o(1)\Big)\int_{\mathcal{B}}\hat{u}_\varepsilon^2dx + o(\hat{\mu}_\varepsilon^2).$$

When $n \geq 5$, we get with (3.46) and (3.82) that

$$\int_{\mathcal{B}}\hat{u}_\varepsilon^2dx = \hat{\mu}_\varepsilon^2\int_{\mathbb{R}^n}\tilde{u}^2dx + o(\hat{\mu}_\varepsilon^2).$$

Since, see (1.41),

$$\int_{\mathbb{R}^n}\tilde{u}^2dx = \frac{4(n-1)}{n-4}$$

it follows that when $n \geq 5$,

$$\int_{\mathcal{B}_2}\Delta_{\hat{g}_\varepsilon}(\eta^2R_{ij}(x_\varepsilon)x^ix^j)\hat{u}_\varepsilon^2dv_{\hat{g}_\varepsilon} = -\frac{8(n-1)}{n-4}S_g(x_0)\hat{\mu}_\varepsilon^2 + o(\hat{\mu}_\varepsilon^2). \tag{3.122}$$

Combining (3.116) and (3.118)-(3.122), we get, when $n \geq 5$,

$$\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}(1-\sqrt{|\hat{g}_\varepsilon|})dx = \frac{n^2-4}{6(n-4)} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2)$$

and this proves (3.114). When $n = 4$, we use (3.66). As in (1.55), it follows from (3.66) that

$$\int_{\mathcal{B}} \hat{u}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4}\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o\Big(\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|\Big). \tag{3.123}$$

Combining (3.116), (3.117), (3.119)-(3.121), and (3.123), we get, when $n = 4$,

$$\int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2_{\hat{g}_\varepsilon}(1-\sqrt{|\hat{g}_\varepsilon|})dx = \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon| + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|)$$

and this proves (3.115). Summarizing, it follows from (3.86), (3.93)-(3.95), (3.114), and (3.115) that

$$J = \frac{1-\varepsilon}{K_n^2} - \frac{\omega_{n-1}}{2n(n+2)} A_n^2 \hat{\mu}_\varepsilon^{n-2} + \frac{n^2-4}{6(n-4)} S_g(x_0) r_\varepsilon^2\hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \tag{3.124}$$

when $n \geq 5$, and that

$$\begin{aligned} J = {} & \frac{1-\varepsilon}{K_n^2} - \frac{\omega_{n-1}}{2n(n+2)} A_n^2 \hat{\mu}_\varepsilon^{n-2} \\ & + \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln\hat{\mu}_\varepsilon| + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{3.125}$$

when $n = 4$. □

Now the proof of (3.4) and (3.5) proceeds as follows. We write

$$\Big(\int_{\mathcal{B}_2} (\eta\hat{u}_\varepsilon)^{2^\star} dx\Big)^{\frac{2}{2^\star}} \leq K_n^2 \int_{\mathcal{B}_2} |\nabla(\eta\hat{u}_\varepsilon)|^2 dx$$

namely, $I^{2/2^\star} \leq K_n^2 J$. Thanks to (3.85), (3.124), and (3.125), we then get

$$\varepsilon + \frac{\omega_{n-1}}{2n(n+2)} A_n^2 K_n^2 \hat{\mu}_\varepsilon^{n-2} \leq \frac{n-2}{n-4} K_n^2 S_g(x_0) r_\varepsilon^2\hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \tag{3.126}$$

when $n \geq 5$, and

$$\varepsilon + \frac{\omega_3}{48} K_4^2 A_4^2 \hat{\mu}_\varepsilon^2 \leq \frac{8\omega_3}{3\omega_4} K_4^2 S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon| + o(r_\varepsilon^2\hat{\mu}_\varepsilon^2|\ln\hat{\mu}_\varepsilon|) + o(\hat{\mu}_\varepsilon^2) \tag{3.127}$$

when $n = 4$. A direct consequence of (3.126) and (3.127) is $S_g(x_0) \geq 0$. We claim $S_g(x_0) > 0$. Let us assume first that $n \geq 5$. Writing

$$\big(\hat{B}_\varepsilon r_\varepsilon^{n+2}\big)^{\frac{2}{n+2}} = \big(\varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}}\hat{B}_\varepsilon\big)^{\frac{2}{n+2}} \varepsilon^{\frac{4-n}{n-2}} r_\varepsilon^2$$

it follows from (3.65) and (2.3)

$$\limsup_{\varepsilon\to 0} \varepsilon^{\frac{4-n}{n-2}} r_\varepsilon^2 < +\infty. \tag{3.128}$$

Thanks to (3.126), assuming $S_g(x_0) = 0$, we then get with (3.128)

$$\varepsilon\hat\mu_\varepsilon^{2-n} = o\big(r_\varepsilon^2 \hat\mu_\varepsilon^{4-n}\big) = o\big((\varepsilon\hat\mu_\varepsilon^{2-n})^{\frac{n-4}{n-2}}\big).$$

Hence, $\varepsilon\hat\mu_\varepsilon^{2-n} \to 0$ as $\varepsilon \to 0$, so that $r_\varepsilon^2\hat\mu_\varepsilon^{4-n} = o\big(r_\varepsilon^2\varepsilon^{\frac{4-n}{n-2}}\big)$ and $r_\varepsilon^2\hat\mu_\varepsilon^{4-n} \to 0$ as $\varepsilon \to 0$ thanks to (3.128). Coming back to (3.126), we get a contradiction. This proves the claim that $S_g(x_0) > 0$ in the case $n \geq 5$. When $n = 4$, (3.65) and (2.2) give

$$\limsup_{\varepsilon\to 0} r_\varepsilon^2 |\ln \varepsilon| < +\infty. \tag{3.129}$$

Combining (3.127) and (3.129), we can write $\varepsilon|\ln\varepsilon| \leq C\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|$ and this implies that

$$\frac{1}{|\ln\varepsilon|} = O\Big(\frac{1}{|\ln\hat\mu_\varepsilon|}\Big). \tag{3.130}$$

Coming back to (3.127), assuming $S_g(x_0) = 0$, we get

$$\frac{\omega_3}{48} K_4^2 A_4^2 + o(1) \leq o\Big(\frac{|\ln\hat\mu_\varepsilon|}{|\ln\varepsilon|}\Big)$$

a contradiction thanks to (3.130). This proves the claim that $S_g(x_0) > 0$ in the case $n = 4$. Then it follows from (3.126) and (3.127) that

$$\liminf_{\varepsilon\to 0} r_\varepsilon^2 \hat\mu_\varepsilon^{4-n} > 0 \tag{3.131}$$

when $n \geq 5$, and

$$\liminf_{\varepsilon\to 0} r_\varepsilon^2 |\ln\hat\mu_\varepsilon| > 0 \tag{3.132}$$

when $n = 4$. In particular, $\hat\mu_\varepsilon^{n-2} = O(r_\varepsilon^2\hat\mu_\varepsilon^2)$ when $n \geq 5$, and $\hat\mu_\varepsilon^2 = O(r_\varepsilon^2\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|)$ when $n = 4$. Coming back to (3.126) and (3.127) we then get

$$\varepsilon = O(r_\varepsilon^2\hat\mu_\varepsilon^2) \text{ when } n \geq 5 \text{ and } \varepsilon = O(r_\varepsilon^2\hat\mu_\varepsilon^2|\ln\hat\mu_\varepsilon|) \text{ when } n = 4. \tag{3.133}$$

We now consider the sharp inequality of Section 1. We choose α to be given by $\alpha = \frac{n-2}{4(n-1)}S_g(x_0)$, and apply this inequality to the function

$$\varphi_\varepsilon(x) = \eta(\frac{x}{r_\varepsilon})u_\varepsilon\big(\exp_{x_\varepsilon}(x)\big)$$

where η is as above. The change of variable $x = r_\varepsilon y$ then gives

$$\frac{1-\varepsilon}{K_n^2}\Big(\int_{\mathcal{B}_2}(\eta\hat u_\varepsilon)^{2^\star}dx\Big)^{\frac{2}{2^\star}} \tag{3.134}$$

$$\leq \int_{\mathcal{B}_2} |\nabla(\eta \hat{u}_\varepsilon)|^2 dx - \frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx + B_\varepsilon r_\varepsilon^{n+2} \Big(\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx \Big)^2.$$

Thanks to (3.85)-(3.89), (3.94), (3.95), (3.114), and (3.115), it follows from (3.134) that

$$\begin{aligned} &\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} - B_\varepsilon \Big(\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx \Big)^2 r_\varepsilon^{n+2} \\ &\leq \frac{n-2}{n-4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 - \frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{3.135}$$

when $n \geq 5$, and

$$\begin{aligned} &\hat{B}_\varepsilon \|u_\varepsilon\|_1 r_\varepsilon^{\frac{n}{2}+1} \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} - B_\varepsilon \Big(\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx \Big)^2 r_\varepsilon^{n+2} \leq \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| \\ &- \frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) + o(\hat{\mu}_\varepsilon^{n-2}) \end{aligned} \tag{3.136}$$

when $n = 4$. We have already seen, see (3.122) and (3.123), that

$$\int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{4(n-1)}{n-4} \hat{\mu}_\varepsilon^2 + o(\hat{\mu}_\varepsilon^2)$$

when $n \geq 5$, and

$$\int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{16\omega_3}{\omega_4} \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(\hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|)$$

when $n = 4$. Hence,

$$\frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{n-2}{n-4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2) \tag{3.137}$$

when $n \geq 5$, and

$$\frac{n-2}{4(n-1)} S_g(x_0) r_\varepsilon^2 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon^2 dx = \frac{8\omega_3}{3\omega_4} S_g(x_0) r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon| + o(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|) \tag{3.138}$$

when $n = 4$. Independently, similar computations to the ones we made to get (3.64) give

$$r_\varepsilon^{\frac{n}{2}+1} \|u_\varepsilon\|_1 \int_{\mathcal{B}_2} \eta^2 \hat{u}_\varepsilon dv_{\hat{g}_\varepsilon} = \Big(\int_{\mathcal{B}} H dx \Big)^2 r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2} + o(r_\varepsilon^{n+2} \hat{\mu}_\varepsilon^{n-2}) \tag{3.139}$$

$$\int_{\mathcal{B}_2} \eta \hat{u}_\varepsilon dx = \Big(\int_{\mathcal{B}} H dx \Big) \hat{\mu}_\varepsilon^{\frac{n}{2}-1} + o(\hat{\mu}_\varepsilon^{\frac{n}{2}-1}). \tag{3.140}$$

We have already seen that $\int_{\mathcal{B}} H dx > 0$. We also have $\hat{\mu}_\varepsilon^{n-2} = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2)$, when $n \geq 5$, and $\hat{\mu}_\varepsilon^2 = O(r_\varepsilon^2 \hat{\mu}_\varepsilon^2 |\ln \hat{\mu}_\varepsilon|)$ when $n = 4$. Combining (3.135)-(3.140) we then get

$$\hat{B}_\varepsilon - B_\varepsilon + o(B_\varepsilon) \leq o(r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n}) \tag{3.141}$$

when $n \geq 5$, and

$$\hat{B}_\varepsilon - B_\varepsilon + o(B_\varepsilon) \leq o(r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon|) \tag{3.142}$$

when $n = 4$. It easily follows from (3.131) and (3.133) that

$$r_\varepsilon^{-n} \hat{\mu}_\varepsilon^{4-n} \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = O(1) \tag{3.143}$$

when $n \geq 5$, and it easily follows from (3.132) and (3.133) that

$$r_\varepsilon^{-4} |\ln \hat{\mu}_\varepsilon| |\ln \varepsilon|^{-3} = O(1) \tag{3.144}$$

when $n = 4$. Combining (3.140)-(3.144), we then get with (1.4) and (1.5)

$$\limsup_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} \leq C_n S_g(x_0)^{\frac{n+2}{2}} \tag{3.145}$$

when $n \geq 5$, and

$$\limsup_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} \leq \frac{1}{2304 \omega_3} S_g(x_0)^3 \tag{3.146}$$

when $n = 4$, where

$$C_n = \frac{2n(n+2)\omega_n^{2+\frac{4}{n}}}{\omega_{n-1}^{\frac{2n}{n-2}} \left(4^{n-3} n(n-2)(n-4)\right)^{\frac{n+2}{n-2}}}.$$

Thanks to the results of Section 2, namely (2.2) and (2.3), it follows from (3.145) and (3.146) that

$$S_g(x_0) = \max_{x \in M} S_g(x). \tag{3.147}$$

Combining (3.145)-(3.147) we then have (3.3) and (3.4) are proved.

It is easily seen that the theorem follows from the results of Sections 2 and 3. Combining (2.2)-(2.3) and (3.3)-(3.4), we indeed do get

$$\lim_{\varepsilon \to 0} \frac{\hat{B}_\varepsilon}{|\ln \varepsilon|^3} = \frac{1}{2304 \omega_3} \Big(\max_{x \in M} S_g \Big)^3$$

when $n = 4$, and

$$\lim_{\varepsilon \to 0} \hat{B}_\varepsilon \varepsilon^{\frac{(n-4)(n+2)}{2(n-2)}} = C_n \Big(\max_{x \in M} S_g \Big)^{\frac{n+2}{2}}$$

when $n \geq 5$. Thanks to (3.2), this ends the proof of the theorem.

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