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ON FRONT PROPAGATION PROBLEMS WITH NONLOCAL TERMS

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Abstract. We investigate the evolution of compact hypersurfaces of $\mathbb{R}^N$ depending, not only on terms of curvature of the surface, but also on non local terms such as the measure of the set enclosed by the surface. We present some existence and convexity results for such motions, even for evolutions which do not preserve the inclusion of the initial surfaces.

1. Introduction. In this paper, we study the evolution of compact hypersurfaces Γ_t of $\mathbb{R}^N$ moving according to the general law:

$$V = h(t, x, \Omega_t, \nu_x, A)$$

where V is the normal outward velocity of Γ_t at the point (t, x), h is some evolution law depending on the position (t, x), on the outward normal ν_x to Γ_t at $x \in \Gamma_t$, on the curvature A of Γ_t at x and on the set Ω_t enclosed by Γ_t (we adopt the convention that the curvature is a $(N-1) \times (N-1)$ symmetric non positive matrix, if the set is convex). A typical example is the following evolution law: $h = \text{trace}(A) + \alpha + \beta|\Omega_t|$, studied by Chen, Hilhorst and Logak in [12]:

$$V = \mathbf{K} + \alpha + \beta|\Omega_t|$$

where $\mathbf{K}$ is the mean curvature and $|\Omega_t|$ is the volume enclosed by Γ_t. One of the results of [12] is the existence, on a small interval of time, of an evolution of smooth hypersurfaces evolving according to this law. The main problem

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for defining the front on large intervals of time is that the surface develops singularities in finite time, even in very simple situations. Therefore, there is no smooth solution to this evolution problem. The aim of this paper is to define and to give existence results for motion of nonsmooth hypersurfaces.

The motions of hypersurfaces - or front propagation - studied in this paper are generalizations of the famous motion by mean curvature $V = \mathbf{K}$. (for a general survey of this problem, see Ambrosio's monograph [3]). The motion by mean curvature has been intensively studied in the past years. There are many reasons for studying this problem. The first one is that the evolution of a set by its mean curvature characterizes - in some sense - the asymptotic behaviour of several reaction-diffusion equations (see [1], [9], [18] for the Allen-Cahn equation, but also ([7], [27] and [26]) for other reaction-diffusion equations). Similar motivations do exist for front propagation with nonlocal terms. This is even the main point of [12], where the authors prove that the limiting behaviour of the following reaction-diffusion equation

$$\begin{cases} u_t = \Delta u + \frac{1}{\epsilon^2} f(u, \epsilon \int_\Omega u) & \text{in } \mathbb{R}^+ \times \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial \mathbb{R}^+ \times \Omega \\ u(0,x) = g^\epsilon(x) & \text{for } x \in \Omega \end{cases}$$

is characterized by the motion of hypersurface with a law of the form $V = \mathbf{K} + \alpha + \beta|\Omega_t|$. This convergence is proved on the interval $[0,T]$ on which a smooth solution of the motion exists. It is not known if this convergence result holds true on large intervals, for nonsmooth propagations as we define here.

Another reason for studying motions by mean curvature is that such evolutions are extremely useful for image analysis (see [2]). Here again, motions with nonlocal terms appear naturally: for instance, in [30], the evolution of the front is of non local nature. This motion formalizes a thinning of the initial shape and intends to compute a kind of "skeleton" of the initial shape. Note that such an evolution does not preserve the inclusion of the initial shapes.

One of the most intriguing aspects of the front propagation problems with non local terms concerns the asymptotic behaviour of the hypersurfaces. Indeed, for evolutions such as the mean curvature motion, the front has necessarily a trivial asymptotic behaviour: either it vanishes in finite time, or it explodes as $t \to +\infty$. There is no attractor as in usual dynamical systems. This is due to the inclusion principle, which states that, if Γ_1 and Γ_2 are initial hypersurfaces, boundary of some open sets Ω_1 and Ω_2, and if the following inclusion holds at time $t = 0$: $\overline{\Omega_1} \subset \Omega_2$, then this inclusion is preserved along the time: $\overline{\Omega_{1,t}} \subset \Omega_{2,t}$. In fact, if h does not depend on (t, x), the distance between $\Gamma_{1,t}$ and $\Gamma_{2,t}$ is non decreasing (this is proved for instance in [19] for the mean curvature motion). We extend this result below for motions h which are non decreasing with respect to Ω. Note that this phenomenon prevents non trivial asymptotic behaviours: Every equilibrium is a repulsor. For motions h which are decreasing with respect to Ω, the situation is much more complicated and has not been investigated up to now. In the following plane example, $V = \mathbf{K} + 2 - \beta|\Omega_t|$ where $\beta = 1/\pi$, circles have round evolution, with a radius satisfying the following equation:

$$r'(t) = -\frac{1}{r(t)} - r^2(t) + 2 \; .$$

Therefore, circle with radius 1 are local attractors, for the other circles, of this dynamical system. A natural question is: if Γ_0 is a curve sufficiently close to a circle of radius 1, does the curve Γ_t converge to a circle of radius 1 ? This question, and the related ones, are beyond the scope of this paper, but we intend to investigate it later. Note also that, in this example, the inclusion principle does not hold true (other examples can be found in [12]).

Let us now recall briefly the different approaches for proving the existence of evolution of hypersurfaces. The study of the evolution of a set by its mean curvature starts with the construction, on a small interval of time, of smooth hypersurfaces evolving according to this law ([21], [22], [23],[25]). As in [12], these methods have to face the problem of singularities the front develops in finite time. For overcoming this difficulty, two main approaches have been proposed (other approaches can be found in [9], [14], [15]): Brakke's approach, [8], using the varifold theory, and the level set approach suggested by Osher & Sethian [29] (see also Barles pioneering work [6]). The level set approach has been later justified rigorously by Evans & Spruck [19], [20], by Chen, Giga & Goto [11] and by Giga, Goto, Ishii & Sato [24] who extended it to more general motions of the form $V = h(t, x, \nu, A)$. The level set approach amounts to describe the hypersurface Γ_t as the level set of some function $u(t, x)$: $\Gamma_t = \{x \mid u(t, x) = 0\}$. If the hypersurfaces Γ_t are smooth and evolve according to a mean curvature motion, then the function u satisfies the following degenerate parabolic p.d.e.:

$$u_t = \|\nabla u\| curv(u)$$

where $curv(u) = div(\nabla u/\|\nabla u\|)$. Thanks to the viscosity theory of second order degenerate parabolic equations, it is possible to define a unique generalized solution to this equation. Setting $\Gamma_t = \{x \mid u(t,x) = 0\}$, one can consider Γ_t (which is not smooth any more) as a generalized solution of the motion by mean curvature.

The key assumption for this approach is that the motion h is parabolic, that is, $h(A) \leq h(B)$ if $A \leq B$. This requirement entails the conservation of the inclusion of the sets enclosed by the hypersurfaces through their evolution.

Following the level set approach for motions depending on nonlocal terms yields to p.d.e. with nonlocal terms. For the example of [12], $h = \mathbf{K} + \alpha + \beta|\Omega_t|$, we find

$$u_t = \|\nabla u\| \left[curv(u) - \alpha - \beta |\{x \mid u(t,x) > 0\}| \right] \ .$$

(here the set Ω_t enclosed by Γ_t is equal to $\{x \mid u(t,x) > 0\}$). Up to now, there is no theory for this kind of equation. Actually, if $h(t,x,\Omega_t,\nu,A)$ is non decreasing with respect to the set Ω_t, then the associated p.d.e. satisfies a maximum principle, and therefore, the situation is very close to the usual situation in viscosity theory [13]. However, when h is non increasing with respect to Ω_t, such a maximum principle fails. In this situation, the inclusion principle is not satisfied (see the counter-example above) and the level set approach has to be forsaken.

Another approach to non local evolution problems, inspired by shape derivative (Delfour & Zolesio [16]), is the so-called "mutational equations". It amounts to formalize the evolution of the hypersurface as an o.d.e. in the metric space of compacts of $\mathbb{R}^N$ (see Doyen [17], Aubin [5]). A closely related approach, due to Kurzhanski & Filippova [28], is the idea of "funnel equations". However, both theories require a "smooth" dependence of the velocity with respect to the shape. Such a smoothness is not satisfied by the curvature and, therefore, it is not clear that these approaches can be extended to our problem.

In order to solve our problem, we follow a completely geometric approach. The idea of defining the motion of hypersurfaces in an intrinsic way is already be investigated by Soner in [31] for motions of the form

$$\beta(\nu_x)V + \mathrm{Trace}(G(\nu_x).A) - \alpha \ = \ 0$$

and later by Barles, Soner & Souganidis in [7], for more general evolutions. These works provide an intrinsic definition of the motion: in [31], the author considers the p.d.e. (in the viscosity sense) satisfied by the distance function to the set Γ_t on this set, while in [7], the authors consider the p.d.e. (again in the viscosity sense) satisfied by the characteristic function of Ω_t, where Ω_t is the set enclosed by the hypersurface. In [31] an existence Theorem for the evolution using only geometric arguments is given. However, in [31] as well as in [7], the inclusion principle is obtained by coming back to the level set approach. Our definition follows roughly the ideas of [31] and [7], with some simplifications (see Definition 2.3 below). But, since we cannot come back to the level set approach, we work instead directly on the geometry of the problem for proving existence and, when there is an inclusion principle, to prove this inclusion principle.

This paper is divided in three parts: in the first one, we define the motion of nonsmooth hypersurfaces and justify this definition.

In the second part, we assume that h is non decreasing with respect to the set Ω_t. In this situation, we prove that the fronts satisfy the inclusion principle. Then we extend Soners results stating that there is a maximal and a minimal solution to the motion problem: any solution is contained in the maximal solution and contains the minimal one. In general, the maximal and the minimal solutions are not equal, i.e., there is no uniqueness of the solutions of the motion problem (the level set approach has also to face a similar difficulty, when the set $\{x \mid u(t,x) = 0\}$ has a non empty interior). We establish a relation between uniqueness and the continuity of the largest solution with respect to the initial hypersurface Γ_0. We deduce from this result that there is a "generic" uniqueness of the solutions for a large class of initial data. Note that this result justifies (at least partially) the definition we have proposed. Finally, we prove, under suitable assumptions on h, that the maximal solution remains convex if the initial condition is a convex set.

The third part of the paper deals with the general case, i.e., when h is not necessarily increasing with respect to the set Ω_t. In this situation, we prove the existence of approximate solutions to the problem. Unfortunately, there is little hope that limits of approximate solutions are still solutions to the real problem. We have not enough information on the behaviour of the boundary of the approximate solutions. However, it is possible under suitable assumption on h, to construct approximate solutions which are convex valued. In this situation, we can pass to the limit and obtain existence result of convex solutions. This concerns, in particular, the example of [12]:

$$V = \mathbf{K} + \alpha + \beta|\Omega_t|.$$

We complete this paper by proving that, if there is a smooth solution to the problem on some interval $[0,T]$, then any generalized solution coincides with this solution on $[0,T]$. This again justifies our definition of generalized solution.

2. Definitions. Let us first give some notations needed throughout this paper. For technical reasons, instead of working with hypersurfaces and open sets enclosed by these hypersurfaces, we work with arbitrary subsets of $\mathbb{R}^N$, $\mathcal{K}(t)$, and the desired hypersurfaces are $\Gamma_t = \partial\mathcal{K}(t)$.

To simplify the notations, we consider subsets $\mathcal{K}$ of $\mathbb{R}^+ \times \mathbb{R}^N$. We denote by (t,x) an element of such a set, where $t \in \mathbb{R}^+$ denotes the time and $x \in \mathbb{R}^N$ denotes the space. We set $\mathcal{K}(t) = \mathcal{K} \cap \left(\{t\} \times \mathbb{R}^N\right)$ and we consider $\mathcal{K}(t)$ as a subset of $\mathbb{R}^N$. The closure of the set $\mathcal{K}$ in $\mathbb{R}^{N+1}$ is denoted by $\overline{\mathcal{K}}$. The closure of the complementary of $\mathcal{K}$ is denoted $\widehat{\mathcal{K}}$: $\widehat{\mathcal{K}} = \overline{(\mathbb{R}^+ \times \mathbb{R}^N)\backslash\mathcal{K}}$ and we set $\widehat{\mathcal{K}}(t) = \widehat{\mathcal{K}} \cap \left(\{t\} \times \mathbb{R}^N\right)$ As before, $\widehat{\mathcal{K}}(t)$ is considered as a closed subset of $\mathbb{R}^N$.

We are seeking the solution of our problem in the set of *tubes* of $\mathbb{R}^N$:

Definition 2.1 (Tubes of $\mathbb{R}^N$)**.** A say that a subset $\mathcal{K}$ of $\mathbb{R}^{N+1}$ is a tube if

$$\forall t \geq 0,\ \ \overline{\mathcal{K}}(t) \text{ is a compact subset of } \mathbb{R}^N.$$

The tube is closed if it is closed in $\mathbb{R}^{N+1}$.

We proceed with our list of notations: If $\phi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ is a smooth function, we denote by $\nabla\phi(t,x)$ and $\nabla^2\phi(t,x)$ its gradient and its hessian matrix with respect to (t,x). We also denote by $\phi_x(t,x)$ and $\phi_{xx}(t,x)$ the first and second order partial derivatives of ϕ with respect to x, and by $\phi_t(t,x)$ the first order derivative of ϕ with respect to t. The set of symmetric $N \times N$ matrices is denoted by $\mathcal{S}_N$.

Before giving the exact definition of a solution to the problem, let us explain it heuristically. We are interested in a front $\Gamma(t)$ which is the boundary of some smooth (at this step) closed set $\mathcal{K}(t)$: $\Gamma(t) = \partial\mathcal{K}(t)$. The outward normal velocity V of $\Gamma(t)$ at a point $x \in \Gamma(t)$ is given by

$$V = h(t,x,\mathcal{K}(t),\nu_x,Y), \tag{1}$$

where ν_x is the exterior unit normal vector to $\Gamma(t)$ at x, and Y is the curvature to $\Gamma(t)$ at x. The key assumption of this paper is the map h be elliptic with respect to Y. Namely,

$$\text{if } Y \leq Y',\ \text{ then }\ h(t,x,\mathcal{K}(t),\nu_x,Y) \leq h(t,x,\mathcal{K}(t),\nu_x,Y')\ .$$

To interpret equation (1), we now have to give a definition of normal and of curvature for $\Gamma(t)$. For doing so, let us consider a smooth test function $\phi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}$ and let us assume that ϕ has a local maximum *on the set* $\mathcal{K}$ at some point (t,x) of $\partial\mathcal{K}$, with $t > 0$. Then, there is a neighbourhood $\mathcal{O}$ of (t,x) such that $\mathcal{K} \cap \mathcal{O} \subset \{(s,y) \in \mathcal{O} \mid \phi(s,y) \leq \phi(t,x)\}$. On the one hand, if $\nabla\phi(t,x) \neq 0$, then $\nabla\phi(t,x)$ is a normal to $\mathcal{K}$ at (t,x). Since V is the normal velocity of the front at x, we have

$$< (\phi_t(t,x), \phi_x(t,x)), (1, V\nu_x) > = \ \phi_t(t,x) + \|\phi_x(t,x)\| V \ = \ 0 \, . \qquad (2)$$

Let us denote by E the subset of $\mathbb{R}^N$ defined by $E = \{y \in \mathbb{R}^N \mid \phi(t,y) \leq \phi(t,x)\}$ and let us assume that $\phi_x(t,x) \neq 0$. Then, on another hand, the outward normal to E at x is given by

$$\nu_x = \frac{\phi_x(t,x)}{\|\phi_x(t,x)\|} \, ,$$

while the curvature Y to E at x is the restriction of $-\phi_{xx}(t,x)/\|\phi_x(t,x)\|$ to the subspace $(\phi_x(t,x))^\perp$:

$$Y = -\frac{\phi_{xx}(t,x)}{\|\phi_x(t,x)\|}_{|(\phi_x)^\perp} \, .$$

Since ∂E is tangent to $\Gamma(t) = \partial\mathcal{K}(t)$ at x, it is natural to consider ν as a generalized outward normal to $\mathcal{K}(t)$ at x. Moreover, since $\mathcal{K}(t) \subset E$ (in a neighbourhood of x), Y is not smaller[2] than the "curvature" to $\mathcal{K}(t)$ at x (if it exists). Therefore, we deduce from equation (1) that the normal velocity V of the front at (t,x) satisfies:

$$V \leq h\big(t, x, \mathcal{K}(t), \frac{\phi_x}{\|\phi_x\|}, -\frac{\phi_{xx}}{\|\phi_x\|}_{|(\phi_x)^\perp}\big) \qquad (3)$$

thanks to the ellipticity property of h.

From relations (2) and (3), we finally derive

$$\phi_t(t,x) \ \geq \ -\|\phi_x(t,x)\| h\Big(t, x, \mathcal{K}(t), \frac{\phi_x}{\|\phi_x\|}, -\frac{\phi_{xx}}{\|\phi_x\|}_{|(\phi_x)^\perp}\Big).$$

Let us set, for any $p \neq 0$ and $X \in \mathcal{S}_N$,

$$H(t, x, \mathcal{K}(t), p, X) = -\|p\| h\Big(t, x, \mathcal{K}(t), \frac{p}{\|p\|}, -X_{|p^\perp}\Big),$$

[2] since we adopt the convention that convex sets have negative curvature.

where $X_{|p^\perp}$ is the restriction of X to the subspace $(p)^\perp$. Then the previous inequality becomes

$$H(t,x,\mathcal{K}(t),\phi_x(t,x),\phi_{xx}(t,x)) \leq \phi_t(t,x). \tag{4}$$

A similar formal computation shows that, if a smooth test function $\phi : \mathbb{R}^+\times\mathbb{R}^N \to \mathbb{R}$ has a local maximum on $\widehat{\mathcal{K}}$ at a point $(t,x)\in\partial\widehat{\mathcal{K}}$, then we have, if $\phi_x \neq 0$,

$$H(t,x,\mathcal{K}(t),-\phi_x(t,x),-\phi_{xx}(t,x)) \geq -\phi_t(t,x). \tag{5}$$

Definition 2.2. *Let $\mathcal{C}$ be the set of bounded subsets of $\mathbb{R}^N$ and $\mathcal{S}_N$ be the set of $N\times N$ symmetric matrices. We say that the map $H:\mathbb{R}^+\times\mathbb{R}^N\times\mathcal{C}\times\mathbb{R}^N_*\times\mathcal{S}_N\to\mathbb{R}$ is geometric if*

$$\forall\lambda\geq 0,\ H(t,x,K,\lambda p,\lambda X) \;=\; \lambda H(t,x,K,p,X)$$

and

$$H\big(t,x,K,p,\big(I-\frac{p^tp}{\|p\|^2}\big)X\big(I-\frac{p^tp}{\|p\|^2}\big)\big) = H(t,x,K,p,X)$$

and H is elliptic if

$$\forall(A,B)\in\mathcal{S}_N,\ \ A\leq B \;\Rightarrow\; H(t,x,K,p,A) \;\leq\; H(t,x,K,p,B)\ .$$

Remarks. 1) This is equivalent with saying that there is some map $h:\mathbb{R}^+\times\mathbb{R}^N\times\mathcal{C}\times S\times\mathcal{S}_{N-1}\to\mathbb{R}$ (where S is the $N-1$ dimensional sphere) such that $\forall(t,x,K,p,A)\in\mathbb{R}^+\times\mathbb{R}^N\times\mathcal{K}\times\mathbb{R}^N_*\times\mathcal{S}_N$,

$$H(t,x,K,p,A) = -\|p\|h(t,x,K,\frac{p}{\|p\|},\frac{-A}{\|p\|}_{|_{(p)^\perp}})$$

with h non decreasing with respect to A.

2) For the motion $V=\mathbf{K}+\alpha+\beta|\Omega_t|$, the associated function H is

$$H(K,p,A) = \mathrm{Trace}(A_{|_{p^\perp}}) - (\alpha+\beta|K|)\|p\|\ .$$

If $\mathcal{K}$ is smooth, then $\mathcal{K}$ is a solution to the front propagation problem if and only if $\mathcal{K}$ satisfies conditions (4) and (5). Therefore, it is natural to take these conditions as definition for non smooth solutions. Unfortunately, we cannot do exactly so because, for technical reasons, we need to define H for

$p = 0$ and to regularize the dependence of H with respect to $\mathcal{K}(t)$. For these reasons, we set, for $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^N$, $p \in \mathbb{R}^N$, $X \in \mathcal{S}_N$ and $\mathcal{K}$ a tube of $\mathbb{R}^+ \times \mathbb{R}^N$,

$$H_*(t,x,\mathcal{K}(t),p,X) = \liminf H(t',x',K',p',X'),$$

where this limit is taken over the $(t',t'',x',p',X') \to (t,t,x,p,X)$, $p \neq 0$, $\epsilon \to 0^+$, and K' subset of $\mathbb{R}^N$ with $\mathcal{K}(t'') - \epsilon B \subset K' \subset \mathcal{K}(t) + \epsilon B$, where $K + \epsilon B = \{x \in \mathbb{R}^N \mid d_K(x) \leq \epsilon\}$ and $K - \epsilon B = \{x \in K \mid d_{\partial K}(x) \geq \epsilon\}$ and $d_A(z) = \inf_{a \in A} \|z - a\|$ for any set A.

In the same way, we define

$$H^*(t,x,\mathcal{K}(t),p,X) = \limsup H(t',x',K',p',X'),$$

where this limit is also taken over the $(t',t'',x',p',X') \to (t,t,x,p,X)$, $p \neq 0$, $\epsilon \to 0^+$, and K subset of $\mathbb{R}^N$ with $\mathcal{K}(t'') - \epsilon B \subset K' \subset \mathcal{K}(t) + \epsilon B$. Note that, if $\mathcal{K}$ is a fixed tube, then $(t,x,p,X) \to H_*(t,x,\mathcal{K}(t),p,X)$ is l.s.c. while $(t,x,p,X) \to H^*(t,x,\mathcal{K}(t),p,X)$ is u.s.c.

We are now ready to state the definition of a front:

Definition 2.3. Let K_0 be a bounded subset of $\mathbb{R}^N$ (the initial condition).

i) A tube $\mathcal{K}$ satisfies the external condition if : $\forall (t,x) \in \partial\mathcal{K}$, with $t > 0$, if $\phi \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^N)$ has a local maximum on $\mathcal{K}$ at (t,x), then

$$H_*(t,x,\mathcal{K}(t),\phi_x(t,x),\phi_{xx}(t,x)) \ \leq \ \phi_t(t,x)$$

and it satisfies the external initial condition for K_0 if $\overline{\mathcal{K}}(0) = \overline{K_0}$.

ii) A tube $\mathcal{K}$ satisfies the internal condition if : $\forall (t,x) \in \partial\widehat{\mathcal{K}}$, $t > 0$, if $\phi \in \mathcal{C}^2(\mathbb{R}^+ \times \mathbb{R}^N)$ has a local maximum on $\widehat{\mathcal{K}}$ at (t,x), then

$$H^*(t,x,\mathcal{K}(t'),-\phi_x(t,x),-\phi_{xx}(t,x)) \ \geq \ -\phi_t(t,x)$$

and it satisfies the internal initial condition for K_0 if $\widehat{\mathcal{K}}(0) \subset \overline{\mathbb{R}^N \backslash K_0}$.

iii) A closed tube is a solution to the front propagation problem for K_0 if it satisfies the internal and the external conditions and the internal and the external initial conditions for K_0.

Remarks. 1) We could have defined the solution of the front propagation problem in many other ways: via the distance function, as in [31], via a Hamilton-Jacobi equation satisfied by the characteristic function of the front as in [7]. All these definitions are more or less equivalent, although we think that our definition is simpler for practical computations. Let us also point out that we often use the properties of the distance function throughout the proofs.

2) It is customary (see [31]) to define the extinction time of a front, i.e.,

$$T^* = \inf\{t \geq 0 \mid \mathcal{K}(t) = \emptyset\}.$$

With our definition, there is no need to introduce this notation. Let us point out however that, if $\mathcal{K}$ is a solution to the front propagation problem, it is possible that $\mathcal{K}(t) = \emptyset$ for $t \geq T$ for some $T > 0$. We still say that $\mathcal{K}$ is a solution on $[0, +\infty)$.

3) For justifying rigorously such a definition, we have to prove existence and uniqueness of solutions. The problem of existence is considered in Theorem 3.5 and 3.9 for front satisfying the inclusion principle, and in Theorem 4.7 for the existence of convex solutions without inclusion principle. It is well-known that uniqueness cannot be expected in general. However, as we prove in Corollary 3.12 in the case of fronts satisfying the inclusion principle, there is a generic uniqueness for compact sets with Lipschitz boundary. Although we are not able to prove a similar result in the general case, we prove in Theorem 4.8 that, if a solution is smooth on some interval $[0, T]$, then any other solution, with the same initial data, coincides with the smooth solution on $[0, T]$.

3. Fronts satisfying an inclusion principle. In this section, we consider fronts for motions H which are non increasing with respect to K. In this case, the fronts satisfy an inclusion principle (Theorem 3.1). Moreover, there is a largest and a smallest solution to the front propagation problem (Theorems 3.5 and 3.9), the maximal and the minimal solution, denoted respectively by $S(K_0)$ and $s(K_0)$. Then we study the map which associates to any initial set K_0 the solution $S(K_0)$, and prove that this map is upper semi-continuous in the Kuratowski sense. We also prove that continuity of the map $K_0 \to S(K_0)$ at some K_0 implies the uniqueness of the solution of the problem with initial condition K_0. These results entail the generic uniqueness of the solution for a large class of initial conditions. We finally prove that, under suitable assumptions on H, the maximal solution is convex valued provided that the initial condition is a convex set.

3.1. The inclusion principle and its applications. We start by proving the inclusion principle, from which we derive a priori estimates on the tubes satisfying the external condition.

3.1.1. The inclusion principle. Before stating the result, we introduce some assumptions on H:

$$\begin{cases} \text{i)} & H \text{ is geometric and elliptic} \\ \text{ii)} & H(\cdot,\cdot,K,\cdot,\cdot) \text{ is continuous on } \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N_* \times \mathcal{S}_N \\ & \text{uniformly with respect to } K \text{ for } K \subset B(0,R), \text{ for any } R > 0 \\ \text{iii)} & \text{If } 2\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq 6\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \text{then} \\ & H(t,x,K,-\frac{2}{3}(x-y),X) - H(t,y,K,-\frac{2}{3}(x-y),-Y) \geq -\ell\|y-x\|^2 \\ \text{iv)} & \text{for any compact set } K,\ H_*(t,x,K,0,0) = H^*(t,x,K,0,0) = 0 \\ \text{v)} & \text{If } K' \subset K, \text{ then } H(t,x,K',p,X) \geq H(t,x,K,p,X) \end{cases} \tag{6}$$

The key assumption is assumption (v), which means that H is non increasing with respect to K. This assumption entails the inclusion principle.

Examples. We give two examples of maps H satisfying (6): 1)

$$H(t,x,K,p,A) = \text{Trace} A_{|_{p^\perp}} + (\alpha - \beta|K|)\|p\|,$$

where $|K|$ is the Lebesgue measure of K, $\alpha \in \mathbb{R}$ and $\beta \geq 0$ (see the example of [12]). 2) More generally,

$$H(t,x,K,p,A) \ = \ \text{Trace}\left(\sigma^T A_{|_{p^\perp}} \sigma\right) - \|p\|\beta\Big[\int_K \phi(t,x,y,\frac{p}{\|p\|})dy\Big],$$

where $\sigma = \sigma(t,x) \in \mathcal{C}^2(\mathbb{R}^{N+1}, \mathbb{R}^{N-1} \times M)$, $\phi \in \mathcal{C}^1(\mathbb{R}^{3N+1}, \mathbb{R}^+)$ and $\beta \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ is non decreasing.

Theorem 3.1 (Inclusion principle)**.** *Assume that H satisfies* (6)*. Let K_0^1 and K_0^2 be bounded subsets of $\mathbb{R}^N$ such that $\overline{K_0^1} \subset Int(K_0^2)$. Let $\mathcal{K}_1$ be a subset of $\mathbb{R}^+ \times \mathbb{R}^N$ satisfying the external condition on the interval $[0,T)$ for some $T > 0$, and the external initial condition K_0^1, and let $\mathcal{K}_2$ be a subset of $\mathbb{R}^+ \times \mathbb{R}^N$ satisfying the internal condition on $[0,T)$, with internal initial condition K_0^2. If $\mathcal{K}_1(t)$ and $\mathcal{K}_2(t)$ are non empty and bounded for $t \in [0,T)$, then $\forall t \in [0,T)$, $\overline{\mathcal{K}_1}(t) \subset Int(\mathcal{K}_2(t))$. More precisely, if we set*

$$d(t) = \inf_{y_1 \in \overline{\mathcal{K}_1}(t),\ y_2 \in \widehat{\mathcal{K}_2}(t)} \|y_1 - y_2\|,$$

then $\forall t \in [0,T)$, $d(t) \geq d(0)e^{-(3\ell t)/2}$, where ℓ is the constant given by (6-iii)*.*

Remark. If H is does not depend on x, then we can choose $\ell = 0$. In this case, d is non decreasing. This result is proved for the motion by mean curvature in [19].

Proof. 1) Note that $\overline{\mathcal{K}_1}$ satisfies the external condition because so does $\mathcal{K}_1$ and because H is non increasing with respect to K. So, working with $\overline{\mathcal{K}_1}$ instead of $\mathcal{K}_1$, we can assume that $\mathcal{K}_1$ is a closed subset of $\mathbb{R}^{N+1}$, as well as K_0^1 is compact.

2) Let us define

$$\rho(t) := \inf_{x_1,x_2\in\mathbb{R}^N}\left[d^2_{\mathcal{K}_1(t)}(x_1) + d^2_{\widehat{\mathcal{K}_2}(t)}(x_2) + \|x_1 - x_2\|^2\right].$$

Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are bounded on $[0,T]$, $\rho(t)$ is positive if and only if $d(t)$ is positive. Moreover, for any $t \geq 0$ such that $d(t) > 0$, there are x_1, x_2, $y_1 \in \mathcal{K}_1(t)$ and $y_2 \in \widehat{\mathcal{K}_2}(t)$ such that

$$\rho(t) = \|x_1 - y_1\|^2 + \|x_2 - y_2\|^2 + \|x_1 - x_2\|^2.$$

Since $d(t) > 0$, we have $y_1 \neq y_2$ and therefore, a simple computation shows that

$$x_1 = \frac{2}{3}y_1 + \frac{1}{3}y_2 \text{ and } x_2 = \frac{1}{3}y_1 + \frac{2}{3}y_2\,.$$

Thus,

$$\rho(t) \;=\; \frac{1}{3}\|y_1 - y_2\|^2 \;\geq\; \frac{1}{3}d^2(t).$$

We can prove in the same way the opposite inequality. So finally, $\rho(t) = \frac{1}{3}d^2(t)$ for any $t \geq 0$.

From now on, we work with ρ instead of d. Note that we have to prove that $\forall t \in [0,T)$, $\rho(t) \geq \rho(0)e^{-3\ell t}$.

3) Let us set

$$T^* := \sup\{t \in [0,T] : \forall s \in [0,t),\ \rho(s) > 0\}.$$

We have to prove that $T^* = T$. It is easy to check that $\rho(\cdot)$ is lower semi-continuous. Moreover, since $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy respectively the external and the internal initial condition, and since $K_0^1 \subset Int(K_0^2)$, we have $\rho(0) > 0$ and so $T^* > 0$.

We now establish that ρ is continuous on the left:

$$\forall t \in (0,T),\quad \lim_{t'\to t^-}\rho(t') = \rho(t)\ .$$

Indeed, assume on the contrary that there is some $t \in (0,T]$ such that $\lim_{t' \to t^-} \rho(t') > \rho(t)$. This means that either $d_{\mathcal{K}_1(\cdot)}(x)$ or $d_{\widehat{\mathcal{K}_2}(\cdot)}(x)$ is not continuous on the left at t.

If $d_{\mathcal{K}_1(\cdot)}(x)$ is not continuous on the left at t, there is some $y \in \mathcal{K}_1(t)$ and some $\epsilon > 0$ with $\forall s \in (t-\epsilon, t)$, $B(y,\epsilon) \cap \mathcal{K}_1(s) = \emptyset$. Therefore, the map $\phi(s,z) = -s$ has a local maximum on $\mathcal{K}_1$ at (t,y). Since $\mathcal{K}_1$ satisfies the external condition, we have

$$H_*(t,y,\mathcal{K}_1(t),\phi_x(t,y),\phi_{xx}(t,y)) \le \phi_t(t,y).$$

This yields to a contradiction since $H_*(t,y,\mathcal{K}_1(t),0,0) = 0$ from assumption (6-iv) and $\phi_t(t,y) = -1$.

If $d_{\widehat{\mathcal{K}_2}(\cdot)}(x)$ is not continuous on the left at t, then there is some $y \in \widehat{\mathcal{K}_2}(t)$ such that

$$\forall t - \epsilon < s < t, \;\; B(y,\epsilon) \cap \widehat{\mathcal{K}_2}(s) = \emptyset\,.$$

Therefore, the map $\phi(s,z) = -s$ has a local maximum on $\widehat{\mathcal{K}_2}$ at (t,y). Since $\mathcal{K}_2$ satisfies the internal condition, we have

$$H^*(t,y,\mathcal{K}_2(t),-\phi_x(t,y),-\phi_{xx}(t,y)) \ge -\phi_t(t,y).$$

This yields to a contradiction since $H^*(t,y,\widehat{\mathcal{K}_2}(t),0,0) = 0$ from assumption (6-iv) and $\phi_t(t,y) = -1$.

4) Estimates of the variations of $\rho(\cdot)$ are quite difficult to obtain. Therefore, we define, for any $\epsilon > 0$, the map

$$\rho^\epsilon(t) = \inf_{x_1,x_2 \in \mathbb{R}^N} \big(d_1^\epsilon(t,x_1) + d_2^\epsilon(t,x_2) + \|x_1 - x_2\|^2\big),$$

where we have set

$$d_1^\epsilon(t,x) = \min_{(s,y)\in\mathcal{K}_1} \big[\frac{1}{\epsilon^2}(s-t)^2 + \|y-x\|^2\big]$$

$$d_2^\epsilon(t,x) = \min_{(s,y)\in\widehat{\mathcal{K}_2}} \big[\frac{1}{\epsilon^2}(s-t)^2 + \|y-x\|^2\big].$$

We also define

$$\tau^\epsilon = \sup\big\{t \in [0,T^*] \mid \forall s \in [0,t],\; \rho^\epsilon(s) > 0 \;\text{ and }\; \rho^\epsilon(t) \ge \frac{1}{\epsilon^2}t^2\big\},$$

$$T^\epsilon = \inf\{t \in [0,T^*]|\, \forall s \in [0,t],\; \rho^\epsilon(s) > 0 \text{ and } \rho^\epsilon(t) \ge \frac{1}{\epsilon^2}(T-t)^2\}.$$

(We set $T^\epsilon = T^*$ if the set in the right-hand side is empty).

5) Since $d_1^\epsilon(t,x) \leq d_{\mathcal{K}_1(t)}(x)$ and $d_2^\epsilon(t,x) \leq d_{\widehat{\mathcal{K}_2(t)}}(x)$, we clearly have that

$$\forall t \in [0,T], \quad \rho^\epsilon(t) \leq \rho(t) \ \text{ and } \lim_{\epsilon \to 0^+} \rho^\epsilon(t) = \rho(t) \ .$$

From now on, we fix $\delta > 0$. We shall choose δ in step (12). Using standard arguments and the definition of τ^ϵ and T^ϵ, we can prove that there is some $\epsilon_0 = \epsilon_0(\delta) > 0$ such that, for any $\epsilon \in (0, \epsilon_0)$,

i) $T^\epsilon > T^* - \delta$,

ii) for any fixed $t \in [\delta, T^* - \delta]$, there are some $x_1^\epsilon \in \mathbb{R}^N$ and $x_2^\epsilon \in \mathbb{R}^N$ with

$$\rho^\epsilon(t) = d_1^\epsilon(t, x_1^\epsilon) + d_2^\epsilon(t, x_2^\epsilon) + \|x_1^\epsilon - x_2^\epsilon\|^2 \,,$$

iii) If t, x_1^ϵ and x_2^ϵ are as above and if $(s_1^\epsilon, y_1^\epsilon) \in \mathcal{K}_1$ and $(s_2^\epsilon, y_2^\epsilon) \in \widehat{\mathcal{K}_2}$ are such that

$$\forall i \in \{1,2\}, \ \ d_i^\epsilon(t, x_i^\epsilon) = \frac{1}{\epsilon^2}(s_i^\epsilon - t)^2 + \|y_i^\epsilon - x_i^\epsilon\|^2 \,,$$

then $x_1^\epsilon \neq y_1^\epsilon$, $x_2^\epsilon \neq y_2^\epsilon$ and $s_i^\epsilon \in (0,T)$ for $i = 1,2$ and $|s_1^\epsilon - s_2^\epsilon| \leq \delta$. As an application, it is easy checked that

$$\liminf_{\epsilon \to 0^+,\, t' \to t} \rho^\epsilon(t') \ \geq \ \rho(t).$$

6) We keep the notations of the previous step. We fix $t \in (0, T^* - \delta)$, $\epsilon \in (0, \epsilon_0)$, x_1^ϵ and x_2^ϵ as before.

We now prove that there are points (t, x_1^n) and (t, x_2^n) converging to (t, x_1^ϵ) and (t, x_2^ϵ), at which respectively $d_1^\epsilon(t,\cdot)$ and $d_2^\epsilon(t,\cdot)$ are twice differentiable, and where their first derivatives p_1^n and p_2^n and their second derivatives A_1^n and A_2^n converge to some limits p_1^ϵ, p_2^ϵ, A_1^ϵ and A_2^ϵ satisfying respectively $p_1^\epsilon = -p_2^\epsilon = -2(x_1^\epsilon - x_2^\epsilon)$ and

$$2\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \leq \begin{pmatrix} A_1^\epsilon & 0 \\ 0 & A_2^\epsilon \end{pmatrix} \leq 6 \begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix}.$$

Proof. Since ϵ is fixed, we set $x_1 = x_1^\epsilon$ and $x_2 = x_2^\epsilon$. Let us define

$$r^\epsilon(y_1, y_2) = d_1^\epsilon(t, y_1) + d_2^\epsilon(t, y_2) + \|y_1 - y_2\|^2 \,.$$

Since the maps $d_1^\epsilon(t,\cdot)$ and $d_2^\epsilon(t,\cdot)$ are semi-concave, r^ϵ is also semi-concave. More precisely, r^ϵ is the sum of a concave function and of the map $(y_1, y_2) \to$

$\|y_1\|^2 + \|y_2\|^2 + \|y_1 - y_2\|^2$. For later use, let us point out that the second order derivative of this smooth map is not larger than $6I_{2N}$.

We now apply the minimum principle for semi-concave functions (see Theorem 9.12 in [3]) at the minimum (x_1, x_2). There are (x_1^n, x_2^n) points of twice differentiability of r^ϵ converging to (x_1, x_2) and such that

$$\lim_n \nabla r^\epsilon(x_1^n, x_2^n) = 0 \text{ and } \nabla^2 r^\epsilon(x_1^n, x_2^n) \geq -\theta_n I_{2N},$$

where $\theta_n \to 0^+$. Since r^ϵ is twice differentiable at (x_1^n, x_2^n), $d_1^\epsilon(t,\cdot)$ and $d_2^\epsilon(t,\cdot)$ are twice differentiable at x_1^n and x_2^n. We denote by p_1^n and A_1^n, and by p_2^n and A_2^n their first and second derivatives at these points. Then $\nabla r^\epsilon(x_1^n, x_2^n) = (p_1^n + 2(x_1^n - x_2^n), p_2^n - 2(x_1^n - x_2^n))$ converges to 0. Therefore, $p_1^\epsilon = \lim_n p_1^n \ = \ -2(x_1 - x_2) = -\lim_n p_2^n = -p_2^\epsilon$. Moreover,

$$\nabla^2 r^\epsilon(x_1^n, x_2^n) = \begin{pmatrix} A_1^n + 2I & -2I \\ -2I & A_2^n + 2I \end{pmatrix} \ \geq \ -\theta_n I_{2N}.$$

From the very definition of d_1^ϵ and d_2^ϵ which are semi-concave, A_1^n and A_2^n are bounded from above. Moreover, an easy computation shows that A_1^n and A_2^n are bounded by $2I$. The previous inequality states that A_1^n and A_2^n are bounded from below. Thus A_1^n and A_2^n converges (up to a subsequence) to some symmetric matrices A_1 and A_2 such that

$$2\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \ \leq \ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \ \leq \ 6\begin{pmatrix} I_N & 0 \\ 0 & I_N \end{pmatrix}.$$

This completes our statement.

7) We keep the notations of step (6). Let $(s_1^n, y_1^n) \in \mathcal{K}_1$ be such that $d_1^\epsilon(t, x_1^n) = \frac{1}{\epsilon^2}(s_1^n - t)^2 + \|y_1^n - x_1^n\|^2$. Then we have $p_1^n = 2(x_1^n - y_1^n)$ and

$$H_*(s_1^n, y_1^n, \mathcal{K}_1(s_1^n), p_1^n, A_1^n) \leq \frac{2}{\epsilon^2}(t - s_1^n)\ .$$

Proof. Since we do the proof for fixed n and for the index 1, we omit the indices 1, ϵ and n. The claim $p_1^n = 2(x_1^n - y_1^n)$ is a well-known property of derivatives of functions defined by minimization at their points of differentiability. Let us now establish that, for any fixed $\gamma > 0$, the map

$$\begin{aligned}\psi(\tau, z) = &-\frac{1}{\epsilon^2}(t-\tau)^2 + \frac{1}{\epsilon^2}(t-s)^2 + < p, z - y > \\ &+ \frac{1}{2} < A(z-y), (z-y) > - \frac{\gamma}{2}\|(\tau - s, z - y)\|^2\end{aligned}$$

has a local maximum at $(s,y)\in\partial\mathcal{K}_1$ on $\mathcal{K}_1$. Indeed, otherwise, there are $(s_k,y_k)\in\mathcal{K}_1$ converging to (s,y) such that $\psi(s_k,y_k)>\psi(s,y)$.

From the twice differentiability of $d_1(t,\cdot)$ at x, we have

$$\begin{array}{rcl} d_1(t,x+y_k-y) &=& d_1(t,x)+<p,y_k-y> \\ &+& \frac{1}{2}<A(y_k-y),(y_k-y)>+\|y_k-y\|^2\epsilon(y_k-y). \end{array}$$

Since $(s_k,y_k)\in\mathcal{K}_1$,

$$\begin{array}{rcl} d_1(t,x+y_k-y) &\leq& \frac{1}{\epsilon^2}(s_k-t)^2+\|y_k-(x+y_k-y)\|^2 \\ &\leq& d_1(t,x)+\frac{1}{\epsilon^2}(t-s_k)^2-\frac{1}{\epsilon^2}(t-s)^2. \end{array}$$

Putting the previous inequalities together yields on the one hand

$$\begin{array}{rcl} 0 &\geq& -\frac{1}{\epsilon^2}(t-s_k)^2+\frac{1}{\epsilon^2}(t-s)^2+<p,y_k-y>+ \\ && \frac{1}{2}<A(y_k-y),(y_k-y)>+\|y_k-y\|^2\epsilon(y_k-y). \end{array}$$

On another hand, since $\psi(s_k,y_k)>\psi(s,y)$, one has,

$$\begin{array}{l} -\frac{1}{\epsilon^2}(t-s_k)^2+\frac{1}{\epsilon^2}(t-s)^2+<p,y_k-y> \\ \quad +\frac{1}{2}<A(y_k-y),(y_k-y)>-\frac{\gamma}{2}\|(s_k-s,y_k-y)\|^2>0. \end{array}$$

Thus,

$$0>\frac{\gamma}{2}\|(s_k-s,y_k-y)\|^2+\|y_k-y\|^2\epsilon(y_k-y)$$

which is clearly impossible. So ψ has a local maximum at (s,y).

From the first step, $s_1\in(0,T)$, so that, for n large enough, $s_1^n\in(0,T)$. Since $\mathcal{K}_1$ satisfies the external condition, and since $\psi_x(s,y)=p$, $\psi_t(s,y)=\frac{2}{\epsilon^2}(t-s)$, $\psi_{xx}(s,y)=A-\gamma I$, one has

$$H_*(s,y,\mathcal{K}(s),p,A-\gamma I)\leq\frac{2}{\epsilon^2}(t-s).$$

Letting $\gamma\to 0^+$ gives the desired result.

8. We still assume that ϵ is fixed, and we keep the notations of step (6). Let $(s_2^n,y_2^n)\in\widehat{\mathcal{K}_2}$ be such that

$$d_2^\epsilon(t,x_2^n)=\frac{1}{\epsilon^2}(s_2^n-t)^2+\|y_2^n-x_2^n\|^2.$$

Since $\mathcal{K}_2$ satisfies the internal condition, we can prove as in the previous step that $p_2^n=2(x_2^n-y_2^n)$ and

$$H^*(s_2^n,y_2^n,\mathcal{K}_2(s_2^n),-p_2^n,-A_2^n)\geq-\frac{2}{\epsilon^2}(t-s_2^n).$$

9. Let $(s_1^\epsilon, y_1^\epsilon)$, $(s_2^\epsilon, y_2^\epsilon)$ be the limits (up to a subsequence) of (s_1^n, y_1^n) and (s_2^n, y_2^n) defined in steps (7) and (8). Then it is easy to check that $(s_1^\epsilon, y_1^\epsilon) \in \partial\mathcal{K}_1$, $(s_2^\epsilon, y_2^\epsilon) \in \partial\widehat{\mathcal{K}_2}$ and, for $i \in \{1,2\}$,

$$d_i^\epsilon(t, x_i^\epsilon) = \frac{1}{\epsilon^2}(s_i^\epsilon - t)^2 + \|y_i^\epsilon - x_i^\epsilon\|^2.$$

Moreover, from steps (6), (7) and (8), we have the equalities

$$p_1^\epsilon = 2(x_1^\epsilon - y_1^\epsilon) = -p_2^\epsilon = -2(x_2^\epsilon - y_2^\epsilon) = -2(x_1^\epsilon - x_2^\epsilon)$$

and thus

$$p_1^\epsilon = -p_2^\epsilon = -\frac{2}{3}(y_1^\epsilon - y_2^\epsilon)\ . \tag{7}$$

Since $\rho^\epsilon(t) = \frac{1}{\epsilon^2}[(s_1^\epsilon - t)^2 + (s_2^\epsilon - t)^2] + \|y_1^\epsilon - x_1^\epsilon\|^2 + \|y_2^\epsilon - x_2^\epsilon\|^2 + \|x_1^\epsilon - x_2^\epsilon\|^2$, we have,

$$\rho^\epsilon(t) \ \geq\ \frac{1}{3}\|y_1^\epsilon - y_2^\epsilon\|^2. \tag{8}$$

From step (7) again, we know that

$$H_*(s_1^n, y_1^n, \mathcal{K}_1(s_1^n), p_1^n, A_1^n) \leq \frac{2}{\epsilon^2}(t - s_1^n).$$

Letting $n \to +\infty$ gives

$$H_*(s_1^\epsilon, y_1^\epsilon, \mathcal{K}_1(s_1^\epsilon), p_1^\epsilon, A_1^\epsilon) \ \leq\ \frac{2}{\epsilon^2}(t - s_1^\epsilon). \tag{9}$$

In the same way, from step (8), we have

$$H^*(s_2^\epsilon, y_2^\epsilon, \mathcal{K}_2(s_2^\epsilon), -p_2^\epsilon, -A_2^\epsilon) \geq -\frac{2}{\epsilon^2}(t - s_2^\epsilon). \tag{10}$$

10. It is easy checked that ρ^ϵ is Lipschitz continuous, and so absolutely continuous. Let t be a point at which ρ^ϵ is derivable. We have

$$\begin{aligned}
&\rho^\epsilon(t-h) - \rho^\epsilon(t)\\
&\quad \leq\ \tfrac{1}{\epsilon^2}\left[(t-h-s_1^\epsilon)^2 + (t-h-s_2^\epsilon)^2\right] + \|y_1^\epsilon - x_1^\epsilon\|^2 + \|y_2^\epsilon - x_2^\epsilon\|^2\\
&\quad\quad -\tfrac{1}{\epsilon^2}\left[(t-s_1^\epsilon)^2 + (t-s_2^\epsilon)^2\right] - \|y_1^\epsilon - x_1^\epsilon\|^2 - \|y_2^\epsilon - x_2^\epsilon\|^2\\
&\quad \leq\ -\tfrac{2h}{\epsilon^2}\left[(t-s_1^\epsilon) + (t-s_2^\epsilon) - 2h\right].
\end{aligned}$$

Dividing these inequalities by $h > 0$ and letting $h \to 0^+$, we obtain

$$(\rho^\epsilon)'(t) \geq \frac{2}{\epsilon^2}\left[(s_1^\epsilon - t) + (s_2^\epsilon - t)\right] \ .$$

From inequalities (9) and (10) , we have

$$(\rho^\epsilon)'(t) \geq H_*(s_1^\epsilon, y_1^\epsilon, \mathcal{K}_1(s_1^\epsilon), p_1^\epsilon, A_1^\epsilon) - H^*(s_2^\epsilon, y_2^\epsilon, \mathcal{K}_2(s_2^\epsilon), -p_2^\epsilon, -A_2^\epsilon). \quad (11)$$

11. Before going further, we need to compare the sets $\mathcal{K}_1(s_1')$ and $\mathcal{K}_2(s_2')$ for s_1' and s_2' sufficiently close to s_1^ϵ and s_2^ϵ. We claim that there are $\bar{\sigma} > 0$ and $\theta > 0$ such that

$$\forall s_1' \in (s_1^\epsilon - \theta, s_1^\epsilon + \theta),\ \forall s_2' \in (s_2^\epsilon - \theta, s_2^\epsilon + \theta),\ \ \mathcal{K}_1(s_1') + \bar{\sigma}B\ \subset\ \mathcal{K}_2(s_2') - \bar{\sigma}B\ .$$

Proof. We argue by contradiction. Assume that there are τ_1^n and τ_2^n converging respectively to s_1^ϵ and s_2^ϵ and some z^n with

$$z^n \in [\mathcal{K}_1(s_1^n) + \frac{1}{n}B] \backslash [\mathcal{K}_2(s_2^n) - \frac{1}{n}B]\ .$$

Then, there are $z_1^n \in \mathcal{K}_1(\tau_1^n)$ and $z_2^n \in \widehat{\mathcal{K}_2}(\tau_2^n)$ such that

$$\|z^n - z_1^n\| \leq \frac{1}{n} \text{ and } \|z^n - z_2^n\| \leq \frac{1}{n}\ .$$

Let z be the limit, up to a subsequence, of (z^n), (z_1^n) and (z_2^n). Clearly, z belongs to $\mathcal{K}_1(s_1^\epsilon)$ and to $\widehat{\mathcal{K}_2}(s_2^\epsilon)$. Therefore,

$$\begin{array}{rl} \rho^\epsilon(t) = & \frac{1}{\epsilon^2}(s_1^\epsilon - t)^2 + \|y_1^\epsilon - x^\epsilon\|^2 + \frac{1}{\epsilon^2}(s_2^\epsilon - t)^2 + \|y_2^\epsilon - x^\epsilon\|^2 \\ \leq & d_1^\epsilon(s_1^\epsilon, z) + d_2^\epsilon(s_2^\epsilon, z) \leq \frac{1}{\epsilon^2}\left[(s_1^\epsilon - t)^2 + (s_2^\epsilon - t)^2\right]. \end{array}$$

Thus, one has necessarily $x^\epsilon = y_1^\epsilon = y_2^\epsilon$ which is impossible from step (5). So there is a contradiction and our claim is proved.

12. Since H is continuous, for any fixed $\gamma > 0$, there is some $\delta > 0$ such that $|H(s_1, x, K, p, A) - H(s_2, x, K, p, A)| \leq \gamma$ for any bounded x, K, p, A and any bounded s_1, s_2 with $|s_1 - s_2| \leq 2\delta$. From now on, we fix γ and we choose δ as above. From step (5), we can choose $\epsilon > 0$ arbitrary small such that $|s_1^\epsilon - t| \leq \delta$ and $|s_2^\epsilon - t| \leq \delta$, i.e., $|s_1^\epsilon - s_2^\epsilon| \leq 2\delta$.

Now recall that H^* is defined in the following way:

$$H^*(s_2^\epsilon, y_2^\epsilon, \mathcal{K}_2(s_2^\epsilon), -p_2^\epsilon, -A_2^\epsilon) = \limsup_{\substack{(t', t'', y', p', X') \\ \sigma > 0,\ K'}} H(t', x', K', -p', -X')$$

where $(t', t'', y', p', X') \to (s_2^\epsilon, s_2^\epsilon, y_2^\epsilon, p_2^\epsilon, A_2^\epsilon)$, $p' \neq 0$, $\sigma \to 0^+$ and K is a compact subset of $\mathbb{R}^N$ with $\mathcal{K}_2(t'') - \sigma B \subset K' \subset \mathcal{K}_2(s_2^\epsilon) + \sigma B$. Thanks to assumption (6-ii) and since $p_2^\epsilon \neq 0$, we have

$$H^*(s_2^\epsilon, y_2^\epsilon, \mathcal{K}_2(s_2^\epsilon), -p_2^\epsilon, -A_2^\epsilon) = \limsup_{\substack{t'' \to s_2^\epsilon,\ \sigma \to 0^+ \\ K' \text{ compact}}} H(s_2^\epsilon, y_2^\epsilon, K', -p_2^\epsilon, -A_2^\epsilon),$$

where the upper limit is taken over all K' such that $\mathcal{K}_2(t'') - \sigma B \subset K' \subset \mathcal{K}_2(s_2^\epsilon) + \sigma B$. In the same way,

$$H_*(s_1^\epsilon, y_1^\epsilon, \mathcal{K}_1(s_1^\epsilon), p_1^\epsilon, A_1^\epsilon) = \liminf_{\substack{t'' \to s_1^\epsilon,\ \sigma \to 0^+ \\ K' \text{ compact}}} H(s_1^\epsilon, y_1^\epsilon, K', p_1^\epsilon, A_1^\epsilon),$$

where the lower limit is taken over all K' such that $\mathcal{K}_1(t'') - \sigma B \ \subset K' \subset \mathcal{K}_1(s_1^\epsilon) + \sigma B$. Therefore, there are some $t_2'' \in (s_2^\epsilon - \theta, s_2^\epsilon + \theta)$ and some compact K_2' such that $\mathcal{K}_2(t_2'') - \bar{\sigma} B \subset K_2' \subset \mathcal{K}_2(s_2^\epsilon) + \bar{\sigma} B$ with

$$H^*(s_2^\epsilon, y_2^\epsilon, \mathcal{K}_2(s_2^\epsilon), -p_2^\epsilon, -A_2^\epsilon) \le H(s_2^\epsilon, y_2^\epsilon, K_2', -p_2^\epsilon, -A_2^\epsilon) + \gamma\ .$$

In the same way, there are some $t_1'' \in (s_1^\epsilon - \theta, s_1^\epsilon + \theta)$ and some compact K_1' such that $\mathcal{K}_1(t_1'') - \bar{\sigma} B \subset K_1' \subset \mathcal{K}_1(s_1^\epsilon) + \bar{\sigma} B$ with

$$H_*(s_1^\epsilon, y_1^\epsilon, \mathcal{K}_1(s_1^\epsilon), p_1^\epsilon, A_1^\epsilon) \ge H(s_1^\epsilon, y_1^\epsilon, K_1', p_1^\epsilon, A_1^\epsilon) - \gamma\ .$$

Thanks to the previous step, we have $\mathcal{K}_1(t_1'') + \bar{\sigma} B \subset \mathcal{K}_2(t_2'') - \bar{\sigma} B$ so that we have the inclusion: $K_1' \subset K_2'$. Therefore,

$$\begin{aligned} &H_*(s_1^\epsilon, y_1^\epsilon, \mathcal{K}_1(s_1^\epsilon), p_1^\epsilon, A_1^\epsilon) \ge H(s_1^\epsilon, y_1^\epsilon, K_1', p_1^\epsilon, A_1^\epsilon) - \gamma \\ &\ge H(s_1^\epsilon, y_1^\epsilon, K_2', p_1^\epsilon, A_1^\epsilon) - \gamma \ \text{ thanks to (6-v)} \\ &\ge H(s_2^\epsilon, y_1^\epsilon, K_2', p_1^\epsilon, A_1^\epsilon) - 2\gamma, \end{aligned}$$

because $|s_1^\epsilon - s_2^\epsilon| \le 2\delta$. Then, from inequality (11), we have

$$(\rho^\epsilon)'(t) \ge H(s_2^\epsilon, y_1^\epsilon, K_2', p_1^\epsilon, A_1^\epsilon) - H(s_2^\epsilon, y_2^\epsilon, K_2', -p_2^\epsilon, -A_2^\epsilon) - 3\gamma$$

with (thanks to equality (7)) $p_1^\epsilon = -p_2^\epsilon = -\frac{2}{3}(y_1^\epsilon - y_2^\epsilon)$ and, from step 6,

$$2\begin{pmatrix} -I & I \\ I & -I \end{pmatrix} \le \begin{pmatrix} A_1^\epsilon & 0 \\ 0 & A_2^\epsilon \end{pmatrix} \le 6\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

From assumption (6-iii), we have $(\rho^\epsilon)'(t) \geq -\ell\|y_1^\epsilon - y_2^\epsilon\|^2 - 3\gamma$. So we have, for almost every $t \in [\tau^\epsilon, T^* - \delta]$, $(\rho^\epsilon)'(t) \geq -3\ell\rho^\epsilon(t) - 3\gamma$ thanks to inequality (8).

13. We deduce from the previous step that

$$\forall t \in [\tau^\epsilon, T^* - \delta], \quad \rho^\epsilon(t) \geq \rho^\epsilon(\tau^\epsilon) e^{-3\ell(t-\tau^\epsilon)} - (\gamma/\ell)(1 - e^{-3\ell(t-\tau^\epsilon)}).$$

Since $\tau^\epsilon \to 0$ and $\liminf \rho^\epsilon(\tau^\epsilon) \geq \rho(0)$ (see step (5)), letting $\epsilon \to 0^+$, we obtain

$$\forall t \in [0, T^* - \delta], \quad \rho(t) \geq \rho(0) e^{-3\ell t} - (\gamma/\ell)(1 - e^{-3\ell t}).$$

Finally, $\gamma > 0$ being arbitrary and $\delta = \delta(\gamma)$ converging to 0 when $\gamma \to 0^+$, we have $\forall t \in [0, T^*)$, $\rho(t) \geq \rho(0) e^{-3\ell t}$. Recall now that ρ is left continuous on $(0, T)$ (from step 3). So, if $T^* < T$, we obtain $\rho(T^*) \geq \rho(0) e^{-3\ell T^*} > 0$. Therefore, $\rho(t) > 0$ on some interval $[0, T^* + \alpha]$, with $\alpha > 0$, which is impossible from the very definition of T^*. So, we necessarily have $T^* = T$ and we have proved the desired result $\forall t \in [0, T)$, $\rho(t) \geq \rho(0) e^{-3\ell t}$ $\square$

3.1.2. Applications. We deduce from Theorem 3.1 Lemma 3.2 below. We need this Lemma for proving existence of maximal solutions and for proving the existence of convex solutions for problems without inclusion principle:

Lemma 3.2. *Assume that H satisfies the following conditions*

$$\begin{cases} i) & H \text{ is geometric and elliptic} \\ ii) & H(\cdot,\cdot,K,\cdot,\cdot) \text{ is continuous on } \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N_* \times \mathcal{S}_N \\ & \text{uniformly with respect to } K \text{ for } K \subset B(0,R), \text{ for any } R > 0 \\ iii) & \forall R > 0,\ \lim_{(p,A)\to 0} \sup_{t,x,K} |H(t,x,K,p,A)| = 0 \\ & \text{where the supremum is taken over all } \|(t,x)\| \leq R \\ & \text{and } K \text{ with } K \subset B(0,R). \end{cases} \tag{12}$$

Let K_0 be a bounded subset of $\mathbb{R}^N$ and let $x_0 \notin \overline{K_0}$. Then there is some $\epsilon > 0$ such that, for any tube $\mathcal{K} \subset \mathbb{R}^+ \times \mathbb{R}^N$ satisfying the external condition and the external initial condition for some set $\overline{\mathcal{K}}(0) \subset \overline{K_0}$, we have $d_{\mathcal{K}}((0, x_0)) \geq \epsilon$, where $d_{\mathcal{K}}(\xi)$ denotes the distance of the point ξ to the set $\mathcal{K}$.

Proof. Let us denote by $\tilde{H}$ the following function:

$$\tilde{H}(t, K, p, A) = \inf_{x \in K,\ K' \subset K} H(t, x, K', p, A) \ .$$

From the assumptions on H, $\tilde{H}$ satisfies assumptions (6).

The basic idea of the proof amounts to construct a tube $\mathcal{K}_2$ and an initial condition K_0^2 such that $\mathcal{K}_2$ satisfies the internal condition for $\tilde{H}$ and the internal initial condition for K_0^2 and such that, on the one hand, $\overline{K_0} \subset Int(K_0^2)$ and on another hand, $d_{\mathcal{K}_2}((0,x_0)) = \epsilon > 0$. Let $\mathcal{K}$ be a tube satisfying the external condition for H and such that $\overline{\mathcal{K}}(0) \subset \overline{K_0}$. Then $\mathcal{K}$ also satisfies the external condition for $\tilde{H}$. From the inclusion principle (Theorem 3.1), we have therefore, $\overline{\mathcal{K}}(t) \subset Int(\mathcal{K}_2(t))$, so that $d_{\mathcal{K}}((0,x_0)) \geq d_{\mathcal{K}_2}((0,x_0)) \geq \epsilon$. So if we construct a tube satisfying the required properties, then the Lemma is proved.

Let R and $\bar{r}$ be such that $d_{K_0}(x_0) > \bar{r}$ and $K_0 \subset \overset{o}{B}(0,R)$, where $\overset{o}{B}(x,r)$ denotes the open ball centered at x and of radius r. We denote by $B(x,r)$ the closed ball centered at x and of radius r. Let us also set

$$C(r) = B(0, R+r) \setminus \overset{o}{B}(x_0, \bar{r} - r)$$

and let us define

$$\theta = \sup_{0 \leq t \leq 1,\ \|p\|=1,\ \|A\| \leq 2/\bar{r}} |\tilde{H}(t, B(0, R + \bar{r}/2), p, A)|.$$

Note that θ is finite. We now consider the tube $\mathcal{K}_2$ defined by $\mathcal{K}_2(t) = C(\theta t)$. Let us prove that $\mathcal{K}_2$ satisfies the required properties for $t \in [0, \frac{\bar{r}}{2\theta}]$.

For proving that $\mathcal{K}_2$ enjoys the internal condition, let us consider a test function ϕ and $(t,x) \in \partial\widehat{\mathcal{K}_2}$ such that ϕ has a local maximum at (t,x) on $\widehat{\mathcal{K}_2}$. There are two cases.

Case 1: $\|x\| = R + \theta t$. Then from Lemma 5.1 given in appendix, there is some $\lambda \geq 0$ such that $\phi_x(t,x) = -2\lambda x$ and $\phi_t(t,x) = 2\lambda(R+\theta t)\theta$ while $\phi_{xx}(t,x) \leq 2\lambda I$ on $(x)^\perp$. Thus,

$$\begin{aligned} &\tilde{H}^*(t, \mathcal{K}_2(t), -\phi_x(t,x), -\phi_{xx}(t,x)) \geq \tilde{H}^*(t, B(0, R+\bar{r}/2), 2\lambda x, -2\lambda I) \\ &\geq 2\lambda(R+\theta t)\tilde{H}^*(t, x, B(0, R+\bar{r}/2), \tfrac{x}{R+\theta t}, -\tfrac{I}{R+\theta t}) \geq -2\lambda(R+\theta t)\theta, \end{aligned}$$

because $\|\frac{x}{R+\theta t}\| = 1$ and $\|-\frac{I}{R+\theta t}\| \leq 2/\bar{r}$. So we have

$$\tilde{H}^*(t, \mathcal{K}_2(t), -\phi_x(t,x), -\phi_{xx}(t,x)) \geq -\phi_t(t,x)\,.$$

Case 2: $\|x - x_0\| = \bar{r} - \theta t$. Then from Lemma 5.1 again, there is some $\lambda \geq 0$ such that $\phi_x(t,x) = 2\lambda(x - x_0)$ and $\phi_t(t,x) = -2\lambda(\bar{r} - \theta t)\theta$, while

$$\phi_{xx}(t,x) \leq -2\lambda I \ \text{ on } \ (x - x_0)^\perp.$$

Thus, as in the first step,

$$\tilde{H}^*(t, \mathcal{K}_2(t), -\phi_x(t,x), -\phi_{xx}(t,x)) \geq \tilde{H}^*(t, B(0, R+\bar{r}/2), -2\lambda x, 2\lambda I)$$
$$\geq 2\lambda(\bar{r}-\theta t)\tilde{H}^*(t, B(0,R+\bar{r}/2), \tfrac{x}{\bar{r}-\theta t}, -\tfrac{I}{\bar{r}-\theta t}) \geq -2\lambda(\bar{r}-\theta t)\theta \geq -\phi_t(t,x)\,.$$

In both cases, we have proved that $\mathcal{K}_2$ satisfies the internal condition for $\tilde{H}$. Moreover, if we set $K_0^2 = C(0)$, then $\widehat{\mathcal{K}_2}(0) = \widehat{K_0^2}$ and $K_0 \subset Int(K_0^2)$ So $\mathcal{K}_2$ satisfies the internal initial condition for K_0^2. □

We also have the following growth estimate on the solutions:

Proposition 3.3. *Assume that H satisfies assumptions* (12) *and the following growth condition:*

$$\begin{array}{c}\exists c \in \mathcal{C}^0(\mathbb{R}^+),\ \forall R \geq 1,\ \forall x \in \mathbb{R}^N,\ \text{with } \|x\| \leq R,\ \forall K \subset B(0,R),\\ H(t,x,K,x,I_N) \geq -c(t)R.\end{array} \tag{13}$$

For any $R > 0$, for any tube $\mathcal{K}$ satisfying the external condition and such that $\mathcal{K}(0) \subset B(0,R)$ (where $B(0,R)$ denotes the ball centered at 0 *and of radius R), we have*

$$\forall t \geq 0,\ \ \mathcal{K}(t) \subset B(0, Re^{\int_0^t c(s)ds})\ .$$

Proof of Proposition 3.3. Let us denote by $\tilde{H}$ the following function:

$$\tilde{H}(t,K,p,A) = \inf_{x\in K,\ K'\subset K} H(t,x,K',p,A)\ .$$

From the assumptions on H, $\tilde{H}$ satisfies assumptions (6). Let $r_\epsilon(t)$ be defined by $r_\epsilon(t) = (R+\epsilon)e^{\int_0^t c(s)ds}$. We are going to prove that $\mathcal{K}_2 = \{(t,x) \mid \|x\| \leq r_\epsilon(t)\}$ satisfies the internal condition for $\tilde{H}$. If this is the case, we claim that Proposition 3.3 is proved. Indeed, let $\mathcal{K}$ be a tube satisfying the external condition for H and such that $\mathcal{K}(0) \subset B(0,R) \subset Int(\mathcal{K}_2(0))$. Then $\mathcal{K}$ also satisfies the external condition for $\tilde{H}$. So we can apply the inclusion principle (Theorem 3.1) which states that $\forall t \geq 0$, $\mathcal{K}(t) \subset Int(\mathcal{K}_2(t)) = \overset{o}{B}(0, r_\epsilon(t))$. The proposition is proved by letting $\epsilon \to 0^+$.

Let us now prove that $\mathcal{K}_2$ satisfies the internal condition for $\tilde{H}$. Let us set $\phi(t,x) = \|x\|^2 - (r_\epsilon(t))^2$. Then $\mathcal{K}_2 = \{\phi(t,x) \leq 0\}$. Since $\mathcal{K}_2$ is smooth, $\mathcal{K}_2$ satisfies the internal condition if and only if

$$\forall (t,x) \text{ with } \phi(t,x) = 0,\ \ \tilde{H}^*(t, \mathcal{K}_2(t), \phi_x(t,x), \phi_{xx}(t,x)) \geq \phi_t(t,x).$$

Here we have: $\phi_t(t,x) = -2r_\epsilon(t)r'_\epsilon(t) = -2c(t)(r_\epsilon(t))^2$, $\phi_x(t,x) = -2x = -2r_\epsilon(t)\frac{x}{\|x\|}$ and $\phi_{xx}(t,x) = -2I_N$. Thus

$$\begin{aligned}\tilde{H}^*(t,\mathcal{K}_2(t),\phi_x(t,x),\phi_{xx}(t,x)) = &\quad 2r_\epsilon(t)\tilde{H}^*(t,B(0,r(t)),\tfrac{x}{\|x\|},I_N)\\ \geq &\quad -2c(t)(r(t))^2 \geq \phi_t(t,x).\end{aligned}$$

Therefore, $\mathcal{K}_2$ satisfies the internal condition for $\tilde{H}$. $\square$

Lemma 3.4 below will be used for the proof of Theorem 3.9 and in the proof of the existence of convex solution without inclusion principle:

Lemma 3.4. *Let H satisfy assumption* (12). *Let $x_0 \in \mathbb{R}^N$ and $R > 0$ and $r > 0$ be fixed. Then there is some positive ϵ such that, for any bounded set K_0 with $B(x_0,r) \subset K_0$, for any tube $\mathcal{K}$ satisfying the internal condition and the internal initial condition for K_0, and such that $\forall t \in [0,r]$, $\mathcal{K}(t) \subset B(0,R)$, we have $d_{\widehat{\mathcal{K}}}(0,x_0) \geq \epsilon$.*

Proof. Let us denote by $\tilde{H}$ the following function:

$$\tilde{H}(t,p,A) = \sup_{x\in B(0,R),\ K'\subset B(0,R)} H(t,x,K',p,A)\ .$$

From the assumptions on H, $\tilde{H}$ satisfies assumptions (6). Let us define the constant M by setting $M = \sup_{|s|\leq r, \|p\|\leq r,\ p\neq 0} \tilde{H}(s,p,I)$. Note that $M < +\infty$ thanks to assumption (12-i and iii).

Let us define $\mathcal{K}_1 = \{(s,y) \in \mathbb{R}^+ \times \mathbb{R}^N : (s+M)^2 + \|y-x_0\|^2 \leq M^2 + \frac{r^2}{4}\}$. We claim that $\mathcal{K}_1$ satisfies the external condition for $\tilde{H}$, with external initial condition $B(x_0,r)$. Let us set $\phi(s,y) = (s+M)^2 + \|y-x_0\|^2$. Since $\mathcal{K}_1$ is smooth, it is enough to check that $\tilde{H}_*(s,\phi_x(s,y),\phi_{xx}(s,y)) \leq \phi_t(s,y)$ for any $(s,y) \in \partial\mathcal{K}_1$. Since $\phi_x(s,y) = 2(y-x_0)$, $\phi_{xx}(s,y) = 2I$ and $\phi_t(s,y) = 2(s+M)$, we have $\tilde{H}_*(s,\phi_x(s,y),\phi_{xx}(s,y)) \leq 2\tilde{H}_*(s,(y-x_0),I)$, where $\|y-x_0\| \leq r/2$ and $|s| \leq r/2$. Therefore, $\tilde{H}_*(s,\phi_x(s,y),\phi_{xx}(s,y)) \leq 2M \leq \phi_t(s,y)$, because $s \geq 0$. So we have proved that $\mathcal{K}_1$ satisfies the external condition for $\tilde{H}$.

Let $\mathcal{K}$ satisfy the internal condition and the internal initial condition for some set K_0 with $B(x_0,r) \subset K_0$, and be such that $\forall t \in [0,r]$, $\mathcal{K}(t) \subset B(0,R)$. Then, from the very definition of $\tilde{H}$, $\mathcal{K}$ also satisfies the internal condition for $\tilde{H}$ on $[0,r]$. Thanks to the inclusion principle, we have $\forall t \geq 0$, $\mathcal{K}_1(t) \subset Int(\mathcal{K}(t))$, because $\mathcal{K}_1(0) = B(x_0,r/2) \subset Int(B(x_0,r)) \subset Int(\mathcal{K}(0))$. Therefore, if we set $\epsilon = d_{\widehat{\mathcal{K}_1}}(0,x_0) > 0$, we have $d_{\widehat{\mathcal{K}}}(0,x_0) \geq d_{\widehat{\mathcal{K}_1}}(0,x_0) = \epsilon$. $\square$

3.2. The maximal and the minimal solutions.

3.2.1. The maximal solution.

Theorem 3.5. *Assume that H satisfies* (12), (13) *and that H is non increasing with respect to K. For any initial position K_0, there is a maximal solution to the front propagation problem, i.e., a solution which contains any other solution to the problem. This solution is denoted by $S(K_0)$.*

Remarks. 1) When H is independent of K, it can be proved that this solution is actually the solution given by the level set approach. 2) The proof follows Perron's method already used in [31].

Proof. 1. Let $\mathcal{E}$ be the set of tubes $\mathcal{A}$ satisfying the external condition and the external initial condition for some initial position $\mathcal{A}(0)$ with $\overline{\mathcal{A}}(0) \subset \overline{K_0}$. Let us set $\mathcal{K} = \overline{\bigcup_{\mathcal{A}\in\mathcal{E}} \mathcal{A}}$. If we prove that $\mathcal{K}$ is a solution, then $\mathcal{K}$ is clearly maximal. Note that $\mathcal{K}$ is a tube, thanks to Proposition 3.3. Let us now prove that $\mathcal{K}$ is a solution.

2. Note first that $\mathcal{E}$ is not empty because $\mathcal{A} = \{0\} \times K_0$ belongs to $\mathcal{E}$.

3. We now prove that $\mathcal{K}$ satisfies the external condition. Let $(t, x) \in \partial\mathcal{K}$, with $t > 0$, and ϕ be such that (t, x) is a local maximum of ϕ on $\mathcal{K}$. Without loss of generality, we can assume that (t, x) is a strict local maximum (otherwise, we prove the result for $\phi(s, y) + \epsilon\|(s, y) - (t, x)\|^2$, which has a strict local maximum on $\mathcal{K}$ at (t, x), and we let $\epsilon \to 0^+$). From the very definition of $\mathcal{K}$, there are $\mathcal{A}_n \in \mathcal{E}$ such that

$$\lim_{n\to+\infty} d_{\mathcal{A}_n}(t, x) = 0.$$

Then, since (t, x) is a strict local maximum, there are (t_n, x_n) local maxima of ϕ on $\overline{\mathcal{A}}_n$ which converge to (t, x), and we have

$$H_*(t_n, x_n, \mathcal{A}_n(t_n), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \leq \phi_t(t_n, x_n),$$

because the $\mathcal{A}_n$ satisfy the external condition. Since $\mathcal{A}_n \subset \mathcal{K}$, we have

$$\begin{aligned} &H_*(t_n, x_n, \mathcal{K}(t_n), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \\ &\qquad \leq H_*(t_n, x_n, \mathcal{A}_n(t_n), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \end{aligned}$$

because H is non increasing with respect to K. From the lower semi continuity of the map $(t, x, p, X) \to H_*(t, x, \mathcal{K}(t), p, X)$, we get:

$$H_*(t, x, \mathcal{K}(t), \phi_x(t, x), \phi_{xx}(t, x)) \leq \phi_t(t, x).$$

So $\mathcal{K}$ satisfies the external condition.

4. We now prove that $\mathcal{K}$ satisfies the external initial condition, i.e., $\mathcal{K}(0) = \overline{K_0}$ (recall that $\mathcal{K}$ is closed). Since $\mathcal{A} = \{0\} \times K_0$ belongs to $\mathcal{E}$, we have $\overline{K_0} \subset \mathcal{K}(0)$. Let us prove the converse inclusion. Let $x_0 \notin \overline{K_0}$ and let us prove that $x_0 \notin \mathcal{K}(0)$. From Lemma 3.2, there is some positive ϵ such that, for any $\mathcal{A} \in \mathcal{E}$, $d_{\mathcal{A}}((0, x_0)) \geq \epsilon$. Thus, we have necessarily that $d_{\mathcal{K}}((0, x_0)) \geq \epsilon$, so that $x_0 \notin \mathcal{K}(0)$. Therefore, $\mathcal{K}$ satisfies the external initial condition for K_0.

5. Next we prove that $\mathcal{K}$ satisfies the internal condition. Let (t, x), with $t > 0$, belong to $\partial\widehat{\mathcal{K}}$ and assume that a map ϕ has a local maximum at (t, x) on $\widehat{\mathcal{K}}$. Without loss of generality, we assume that (t, x) is the unique maximum of ϕ on $\widehat{\mathcal{K}}$. Assume that, contrary to our claim,

$$H^*(t, x, \mathcal{K}(t), -\phi_x(t, x), -\phi_{xx}(t, x)) \; < \; -\phi_t(t, x)\,.$$

Since $(s, y, p, X) \to H^*(s, y, \mathcal{K}(s), p, X)$ is u.s.c., there is some $r > 0$ such that

$$\begin{array}{c} \text{if } \|(s, y) - (t, x)\| \leq r, \quad \text{then} \\ H^*(s, y, \mathcal{K}(s), -\phi_x(s, y), -\phi_{xx}(s, y)) \; < \; -\phi_t(s, y). \end{array} \tag{14}$$

We also choose r sufficiently small so that if $(s, y) \in B((t, x), r)$, one has $s > 0$. Let us now point out that (t, x) is not a local minimum of ϕ. Indeed, otherwise, there would be some open neighbourhood W of (t, x) such that $W \cap \widehat{\mathcal{K}} = \{(t, x)\}$, because (t, x) is a strict maximum of ϕ on $\widehat{\mathcal{K}}$. This is in contradiction with the fact that $\widehat{\mathcal{K}}$, from its construction, is equal to the closure of its interior. Set $D_\epsilon = \{(s, y) \in B((t, x), r) : \phi(s, y) \geq \phi(t, x) - \epsilon\}$. Then we have

$$\bigcap_{\epsilon > 0} D_\epsilon \cap \widehat{\mathcal{K}} = \{(t, x)\},$$

because (t, x) is the unique maximum of ϕ on $\widehat{\mathcal{K}} \cap B((t, x), r)$. Thus, for ϵ sufficiently small (say $\epsilon \in (0, \epsilon_0)$), $D_\epsilon \backslash \widehat{\mathcal{K}}$ is contained in the interior of $B((t, x), r)$. If $\phi_x(t, x) \neq 0$, we also choose $\epsilon_0 > 0$ sufficiently small so that ϕ_x is non zero in $B((t, x), r)$. Note that (t, x) belongs to the interior of D_ϵ since (t, x) is not a local minimum of ϕ. From Sard Theorem, we can choose $\epsilon \in (0, \epsilon_0)$ such that the level $\phi(t, x) - \epsilon$ is not critical for ϕ.

Let us now consider $\mathcal{K}_\epsilon = \mathcal{K} \cup D_\epsilon$. Note that $\mathcal{K}_\epsilon$ strictly contains $\mathcal{K}$ because $\mathcal{K}_\epsilon$ contains an open neighbourhood of (t, x) and that $(t, x) \in \partial\widehat{\mathcal{K}}$. We now prove that $\mathcal{K}_\epsilon$ satisfies the external condition. Let ψ be a test function such that ψ has a local maximum at some point $(t',x') \in \partial\mathcal{K}_\epsilon$ on $\mathcal{K}_\epsilon$. We have to prove that $\psi_t(t',x') \geq H_*(t',x',\mathcal{K}_\epsilon(t'),\psi_x(t',x'),\psi_{xx}(t',x'))$.

If (t',x') belongs to $\mathcal{K}$, then the result holds true because ψ has also a local maximum at (t',x') on $\mathcal{K}$ ($\mathcal{K} \subset \mathcal{K}_\epsilon$) and so

$$\begin{array}{rcl}\psi_t(t',x') & \geq & H_*(t',x',\mathcal{K}(t'),\psi_x(t',x'),\psi_{xx}(t',x')) \\ & \geq & H_*(t',x',\mathcal{K}_\epsilon(t'),\psi_x(t',x'),\psi_{xx}(t',x')),\end{array}$$

because H is non increasing with respect to K.

Let us now assume that (t',x') belongs to $\mathcal{K}_\epsilon \backslash \mathcal{K}$. Then (t',x') belongs to the interior of the ball $B((t,x),r)$ and ψ has a local maximum on the set D_ϵ. Moreover, since $(t',x') \in \partial\mathcal{K}_\epsilon$, (t',x') belongs to the boundary of D_ϵ and so $\phi(t',x') = \phi(t,x) - \epsilon$.

Note that $\nabla\phi(t',x') \neq 0$ since (t',x') belongs to the level set $\phi(t,x) - \epsilon$ for ϕ, which is not critical. Therefore, Lemma 5.1 given in appendix states that there is some $\lambda \geq 0$ such that

$$\nabla\psi(t',x') + \lambda\nabla\phi(t',x') = 0$$

and

$$\nabla^2\psi(t',x')_{|_{(\nabla\phi(t',x'))^\perp}} \leq -\lambda\nabla^2\phi(t',x')_{|_{(\nabla\phi(t',x'))^\perp}}.$$

If $\phi_x(t',x') \neq 0$, then the previous inequality implies that

$$\psi_{xx}(t',x')_{|_{(\phi_x(t',x'))^\perp}} \leq -\lambda\phi_{xx}(t',x')_{|_{(\phi_x(t',x'))^\perp}},$$

while if $\phi_x(t',x') = 0$, then

$$\psi_{xx}(t',x') \leq -\lambda\phi_{xx}(t',x').$$

Let us now distinguish two cases. If $\lambda = 0$, then there is nothing to prove since

$$\psi_t(t',x') = 0 = H_*(t',x',\mathcal{K}_\epsilon(t'),0,0) \geq H_*(t',x',\mathcal{K}_\epsilon(t'),\psi_x(t',x'),\psi_{xx}(t',x'))$$

thanks to assumption (12-iii) and the ellipticity of H.

Secondly, if $\lambda > 0$, then we have,

$$\begin{array}{ll}\psi_t(t',x') = -\lambda\phi_t(t',x') & \\ \geq \lambda H^*(t',x',\mathcal{K}(t'),-\phi_x(t',x'),-\phi_{xx}(t',x')) & \text{thanks to inequality (14)} \\ \geq H^*(t',x',\mathcal{K}(t'),-\lambda\phi_x(t',x'),-\lambda\phi_{xx}(t',x')) & \text{because } H \text{ is geometric} \\ \geq H^*(t',x',\mathcal{K}_\epsilon(t'),\psi_x(t',x'),\psi_{xx}(t',x')) & \\ \geq H_*(t',x',\mathcal{K}_\epsilon(t'),\psi_x(t',x'),\psi_{xx}(t',x')), & \end{array}$$

because H is non increasing with respect to K and from the fact that

$$\psi_{xx}(t',x')_{|_{(\phi_x(t',x'))^\perp}} \le -\lambda\phi_{xx}(t',x')_{|_{(\phi_x(t',x'))^\perp}} .$$

In particular, this proves that the external condition is fulfilled for $\mathcal{K}_\epsilon$. The external initial condition is also fulfilled because $\mathcal{K}_\epsilon(0) = \mathcal{K}(0) = \overline{K_0}$. So $\mathcal{K}_\epsilon$ satisfies the external condition and contains strictly $\mathcal{K}$. This is impossible from the very definition of $\mathcal{K}$. Therefore, we have a contradiction with the assumption that $H^*(t,x,\mathcal{K}(t),-\phi_x(t,x),-\phi_{xx}(t,x)) < -\phi_t(t,x)$. So we have proved the desired inequality:

$$H^*(t,x,\mathcal{K}(t),-\phi_x(t,x),-\phi_{xx}(t,x)) \;\ge\; -\phi_t(t,x)\,.$$

6. We finally prove that the internal initial condition $\widehat{\mathcal{K}}(0) \subset \overline{\mathbb{R}^N\backslash K_0}$ is satisfied. Let $x_0 \notin \widehat{K_0}$. We claim that, for $\epsilon > 0$ sufficiently small and $\rho > 0$ sufficiently large, the set

$$\mathcal{A}(\epsilon,\rho) = \left\{(s,y) \in \mathbb{R}^+ \times \mathbb{R}^N \mid (s+\rho)^2 + \|y-x_0\|^2 \le (\rho+\epsilon)^2\right\}$$

belongs to $\mathcal{E}$. If our claim is proved, then we have $\mathcal{A}(\epsilon,\rho) \subset \mathcal{K}$, so that $\widehat{\mathcal{K}} \subset \widehat{\mathcal{A}(\epsilon,\rho)}$. Then, since $x_0 \notin \widehat{\mathcal{A}(\epsilon,\rho)}(0)$, the point x_0 does not belong to $\widehat{\mathcal{K}}(0)$ either. So, if we prove that $\mathcal{A}(\epsilon,\rho)$ belongs to $\mathcal{E}$, then we have proved that $\mathcal{K}$ satisfies the internal initial condition for K_0. Let us set

$$\rho = \sup_{\|(s,y)-(0,x_0)\|\le 1,\ s>0,\ y\neq x_0} H(s,y,\emptyset,y-x_0,I)$$

which is bounded from assumptions (12-*ii* and *iii*). We choose $\epsilon > 0$ sufficiently small in such a way that $\mathcal{A}(\epsilon,\rho) \subset (K_0 \times \mathbb{R}^+) \cap B((0,x_0),\frac{1}{2})$. Let us now prove that $\mathcal{A}(\epsilon,\rho)$ satisfies the external condition. Let ϕ be a test function which has a local maximum at some point $(t',x') \in \partial\mathcal{A}(\epsilon,\rho)$ on $\mathcal{A}(\epsilon,\rho)$ with $t' > 0$. Then from Lemma 5.1, there is $\lambda \ge 0$ with $\phi_x(t',x') = \lambda(x'-x_0)$, $\phi_t(t',x') = \lambda(t'+\rho)$ and

$$\phi_{xx}(t',x') \le \lambda I \text{ on } (x'-x_0)^\perp.$$

As usual, if $\lambda = 0$, there is nothing to prove. If $\lambda > 0$, then

$$\begin{aligned}&H_*(t',x',\mathcal{A}(\epsilon,\rho)(t'),\phi_x(t',x'),\phi_{xx}(t',x'))\\ &\le \lambda H_*(t',x',\mathcal{A}(\epsilon,\rho)(t'),(x'-x_0),I)\\ &\le \lambda H_*(t',x',\emptyset,(x'-x_0),I) \le \lambda\rho \le \lambda(\rho+t')\end{aligned}$$

from the very definition of ρ. Therefore, we finally have

$$H_*(t', x', \mathcal{A}(\epsilon,\rho)(t'), \phi_x(t', x'), \phi_{xx}(t', x')) \leq \phi_t(t', x').$$

So $\mathcal{A}(\epsilon,\rho)$ satisfies the external condition. The external initial condition $\mathcal{A}(\epsilon,\rho)(0) \subset K_0$ is fulfilled by construction. So $\mathcal{A}(\epsilon,\rho)$ belongs to $\mathcal{E}$ and $(0, x_0) \in Int(\mathcal{A}(\epsilon,\rho)) \subset Int(\mathcal{K})$, which proves that $x_0 \notin \widehat{\mathcal{K}}(0)$. So $\mathcal{K}$ satisfies the external initial condition for K_0. In conclusion, $\mathcal{K}$ satisfies the external and internal conditions and external and internal initial conditions. So $\mathcal{K}$ is a solution of the front propagation problem which, by construction, contains any other solution of the problem. □

From the very construction of $S(K_0)$, we have:

Corollary 3.6. *Let K_0 be a closed subset of $\mathbb{R}^N$ and H satisfy condition* (12), (13) *and be non increasing with respect to K. Then $S(K_0)$ is a closed subset of $\mathbb{R}^{N+1}$ and contains any tube satisfying the external condition and the external initial condition for K_0^1 with $K_0^1 \subset K_0$.*

3.2.2. Stability properties of the solutions. The solutions to the front propagation problem are "stable" in the following sense:

Proposition 3.7. (Stability Property) *Assume that H satisfies* (12), (13) *and that H is non increasing with respect to K. Let $\mathcal{K}^n$ be a sequence of tubes of $\mathbb{R}^+ \times \mathbb{R}^N$ satisfying the external condition and such that the initial conditions $\mathcal{K}^n(0)$ are uniformly bounded*[3]*. Set $\mathcal{K} = \operatorname{Limsup} \mathcal{K}^n (\subset \mathbb{R}^{N+1})$ and $K_0 = \operatorname{Limsup} \mathcal{K}^n(0) (\subset \mathbb{R}^N)$. Then $\mathcal{K}$ satisfies the external condition with external initial condition K_0.*

Proof. Thanks to Proposition 3.3, $\mathcal{K}$ is a tube. Let (t, x) belong to the upper limit $\mathcal{K}$, with $t > 0$, and ϕ reach a local maximum at (t, x) on $\mathcal{K}$. We have to prove that $H_*(t, x, \mathcal{K}(t), \phi_x(t, x), \phi_{xx}(t, x)) \leq \phi_t(t, x)$.

We can assume as usually that (t, x) is a strict local maximum. Then there exist a subsequence $\mathcal{K}_{n'}$ and points $(t_{n'}, x_{n'}) \in \partial\mathcal{K}_{n'}$ local maxima of ϕ on $\overline{\mathcal{K}_{n'}}$ with $(t_{n'}, x_{n'})$ converging to (t, x).

Since $\mathcal{K}_n$ satisfies the external condition,

$$H_*(t_n, x_n, \mathcal{K}_n(t), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \leq \phi_t(t_n, x_n)\ .$$

[3] The Limsup denotes the Kuratowski upper-limit of sets, i.e., the set of cluster points of sequences in $\mathcal{K}^n$. Recall that this set is closed.

Moreover, since $\mathcal{K}_n$ converges to $\mathcal{K}$, for any $\epsilon > 0$, there is some n_0 such that if $n \geq n_0$, then $\forall t' \in [0, t+1]$, $\mathcal{K}_n(t') \subset \mathcal{K}(t') + \epsilon B$. Thus, since H is non decreasing with respect to K, we have

$$H_*(t_n, x_n, \mathcal{K}_n(t_n), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \geq \\ H_*(t_n, x_n, \mathcal{K}(t_n) + \epsilon B, \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \; .$$

From Lemma 5.2 given in appendix, we have

$$\liminf_{n \to +\infty, \epsilon \to 0^+} H_*(t_n, x_n, \mathcal{K}(t_n) + \epsilon B, \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \\ \geq \; H_*(t, x, \mathcal{K}(t), \phi_x(t, x), \phi_{xx}(t, x))$$

so that $H_*(t, x, \mathcal{K}(t), \phi_x(t, x), \phi_{xx}(t, x)) \leq \phi_t(t, x)$. Thus, $\mathcal{K}$ satisfies the external condition.

We can prove that $\mathcal{K}$ satisfies the external initial condition for K_0 by following step (4) of the proof of Theorem 3.5. $\square$

Corollary 3.8. *The maps $K_0 \to S(K_0)$ is upper semi-continuous, i.e., if* $\text{Limsup}_n K_0^n = K_0$, *then* $\text{Limsup}_n S(K_0^n) \subset S(K_0)$.

Proof. Since the upper-limit $\mathcal{K}^\infty$ of solutions $S(K_0^n)$ satisfies the external condition and the external initial condition for $K_0 = \text{Limsup}_n K_0^n$, we deduce from Corollary 3.6 that $\mathcal{K}^\infty$ is contained in $S(K_0)$. $\square$

3.2.3. The minimal solution.

Theorem 3.9. *Assume that H satisfies* (6), (12), (13) *and that H is non increasing with respect to K. For any initial position K_0, there is a minimal solution to the front propagation problem, i.e., a solution which is contained in any other solution to the problem. This solution is denoted by $s(K_0)$.*

Remark. Following the proof of Theorem 3.5, it is actually possible to establish the existence of a smallest closed solution of the front propagation problem, under the assumptions (12), (13) and that H is non increasing with respect to K.

Proof. Let $D_n = K_0 - \frac{1}{n}B$. Then $D_n \subset Int(K_0)$. Let us set

$$s(K_0) = [\{0\} \times K_0] \cup \Big[\bigcup_n S(D_n)\Big] \; .$$

From the inclusion principle, $s(K_0)$ is contained in any other solution of the front propagation problem with initial condition K_0. Let us now prove that $s(K_0)$ is a solution.

First, $s(K_0)$ satisfies the external initial condition for K_0, thanks to Lemma 3.2. Let us now point out that $s(K_0)$ is a tube. Indeed, from the inclusion principle, we have $\forall n \geq 1$, $S(D_n) \subset S(K_0)$, so that $s(K_0) \subset S(K_0)$. Since $S(K_0)$ is a tube, so is $s(K_0)$.

Secondly, $s(K_0)$ satisfies the external condition. For doing so, let us first point out that the non decreasing sequence of sets $\overline{S(D_n)}$ converges to $\overline{s(K_0)}$. Then, if a test function ϕ has a strict maximum at some point (t,x) of $\overline{s(K_0)}$, with $t > 0$, there are (t_n, x_n) local maxima of ϕ on $\overline{S(D_n)}$ which converge to (t,x). Since $S(D_n)$ are solutions, we have

$$H_*(t_n, x_n, S(D_n)(t_n), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \leq \phi_t(t_n, x_n)$$

and thus,

$$H_*(t_n, x_n, s(K_0)(t_n), \phi_x(t_n, x_n), \phi_{xx}(t_n, x_n)) \leq \phi_t(t_n, x_n)$$

because H is non increasing with respect to K. Letting $n \to +\infty$ gives the desired result.

For proving that $s(K_0)$ satisfies the internal condition, let us first point out that the limit of the $\widehat{S(D_n)}$ is equal to $\widehat{s(K_0)}$. Moreover, the $S(D_n)$ satisfy the internal condition. Thus, arguing as the proof of Proposition 3.7 and using Lemma 5.3 instead of Lemma 5.2, we can check that $s(K_0)$ satisfies the internal condition.

Finally, let us prove that $s(K_0)$ satisfies the internal initial condition for K_0. Let $x_0 \in Int(K_0)$. We have to prove that $x_0 \notin \widehat{s(K_0)}(0)$. There is some n sufficiently large such that $x_0 \in Int(D_n)$. Thus $x_0 \notin \widehat{S(D_n)}(0)$ because $S(D_n)$ satisfies the internal initial condition. Since $\widehat{s(K_0)}(0) \subset \widehat{S(D_n)}(0)$, the desired result is proved. □

From the very construction of $s(K_0)$, we have:

Corollary 3.10. *Let K_0 be a closed subset of $\mathbb{R}^N$ and H satisfy condition* (12), (13) *and be non increasing with respect to K. Then $s(K_0)$ is contained into any tube satisfying the internal condition and the internal initial condition for K_0.*

3.2.4. Remarks on the uniqueness problem. Usually, one cannot expect that the maximal and the minimal solutions coincide, even for the motion by mean curvature. This is one of the most intriguing question on the subject. This question is known as the "fattening problem" in the literature of the level-set approach. Several conditions guaranteeing uniqueness can be found in [31] and in [7]. However, the situation is far from being understood. We prove here that uniqueness of front propagation problem is related with some continuity property of the maximal solution $S(\cdot)$.

Proposition 3.11. *Assume that H satisfies* (6), (12) *and* (13) *and that H is non increasing with respect to K. Let K_0 be a compact subset of $\mathbb{R}^N$ which is equal to the closure of its interior. Then there is a unique solution to the front propagation problem, in the sense that any solution $\mathcal{K}$, satisfies $\overline{\mathcal{K}} = S(K_0)$, if and only if, the maximal solution $\Delta \to S(\Delta)$ is continuous at $\Delta = K_0$ in the following sense:*

$$\text{If } \operatorname*{Lim}_n \Delta_n = K_0 \quad \textit{and} \quad \operatorname*{Lim}_n \partial\Delta_n = \partial K_0, \text{ then } \operatorname*{Lim}_n S(\Delta_n) = S(K_0)\ .$$

(*where* Lim *denotes the Kuratowski limit*[4]*.*)

Proof : Assume that the set-valued map $\Delta \to S(\Delta)$ is continuous at K_0. Let $D_n = K_0 - \frac{1}{n}B$. Then $D_n \subset Int(K_0)$. Let $\mathcal{K}$ be any solution to the front propagation problem starting from K_0. Thanks to the inclusion principle, we have $\forall n \geq 0, \forall t \geq 0$, $S(D_n)(t) \subset Int(\mathcal{K}(t))$. Since K_0 is equal to the closure of its interior, D_n converges to K_0 in the sense of the Proposition. Thus, $S(D_n)$ converges to $S(K_0)$ from the continuity assumption of $\Delta \to S(\Delta)$. Therefore, $S(K_0) = \mathrm{Lim}_n S(D_n) \subset \overline{\mathcal{K}} \subset S(K_0)$, which means that $\mathcal{K} = S(K_0)$. In conclusion, there is a unique solution starting from K_0.

Conversely, let us assume that there is a unique solution to the front propagation problem. Let us set as before $D_n = K_0 - \frac{1}{n}B$. We already know that $S(K_0) = \overline{s(K_0)} = \overline{\bigcup_n S(D_n)}$. Assume that Δ_n converge to K_0 in the sense of the Proposition. From Corollary 3.8, we know that

$$\mathrm{Limsup}_n S(\Delta_n) \subset S(K_0)\ .$$

From the definition of the convergence of the Δ_n, for n sufficiently large, there is some k_n such that $D_{k_n} \subset Int(\Delta_n)$. Note that $k_n \to +\infty$. From the inclusion principle, we obtain that $S(D_{k_n}) \subset S(\Delta_n)$. Therefore

$$S(K_0) = \mathrm{Liminf}_n S(D_{k_n}) \subset \mathrm{Liminf}_n S(\Delta_n)$$

[4] Recall that the Kuratowski upper limit of sets A_n is the set of cluster points of sequences of A_n. The Kuratowski lower limit of the sets A_n is the set of limits of sequences $a_n \in A_n$. We say that the sequence of sets A_n has a Kuratowski limit if the upper limit of the A_n is equal to the lower limit of the A_n.

In conclusion, $S(\Delta_n)$ converges to $S(K_0)$. □

Let us denote by $\mathcal{C}_r$, for any $r>0$, the set of the compact subsets of $\mathbb{R}^N$ with the following cone property: K belongs to $\mathcal{C}_r$ if

$$\forall x \in \partial K,\ \exists v \in \mathbb{R}^N,\ \|v\| = 1,\ \text{with } \forall s \in (0,r)\ ,\ \forall \|u\| \le 1\ ,\\ x + s(v+ru) \in K \text{ and } x + s(-v+ru) \notin K.$$

If $\mathcal{C}_r$ is endowed with the Hausdorf topology, then $\mathcal{C}_r$ is a complete separable metric space. Moreover, it is equivalent for a sequence $K_n \in \mathcal{C}_r$ to converge for the Hausdorf topology and to converge in the sense of Proposition 3.11.

Corollary 3.12. *For any positive r, there is a residual E of $\mathcal{C}_r$ such that, for any $K_0 \in E$, there is uniqueness of the front starting from K_0.*

Proof. The map $S : \mathcal{C}_r \to \mathbb{R}^+ \times \mathbb{R}^N$ which associates to any initial condition K_0 the closed set $S(K_0)$ is upper semi-continuous (Proposition 3.8). Thus, from Theorem 1.4.13 of [4], there is a residual E on which S is continuous. Proposition 3.11 gives the conclusion. □

3.3. Convexity of the solutions. We now investigate fronts with convex initial data. We prove that, if H satisfies some convexity properties, then the maximal solution remains convex. Our result is an extension of results given in [24] for motion of the form $H(p, A)$. Let us point out that the proof is completely different.

For any compact set A, we denote by $\mathrm{Co}(A)$ the convex hull of the set A. Let us start with a Lemma, which is an improvement of Lemma 3.3. This Lemma states that, if $\mathcal{K}$ satisfies the external condition, then $\mathcal{K}$ has a finite velocity growth at the extremal points of $\mathrm{Co}[\mathcal{K}(t)]$.

Lemma 3.13. *Let H satisfy condition* (12) *and the following growth condition: There is some $c : \mathbb{R}^+ \to \mathbb{R}^+$ continuous such that*

$$\forall R > 0,\ \forall (x,p) \in \mathbb{R}^{2N},\ \text{with } \|x\| \le R,\ \text{and } \|p\| = 1,\ \forall K \subset B(0,R),\\ H(t,x,K,p,0) \ge -c(t)R. \quad (15)$$

Let $\mathcal{K}$ be a tube satisfying the external condition. Then we have,

$$\forall 0 \le s \le t,\ \ \mathrm{Co}(\mathcal{K}(t)) \ \subset\ \mathrm{Co}(\mathcal{K}(s)) + \big[(\sup_{y\in\mathcal{K}(s)} \|y\|)(e^{\int_s^t c(\tau)d\tau} - 1)\big]B\ .$$

Proof. Let us denote by $\tilde{H}$ the following function:

$$\tilde{H}(t,K,p,A) = \inf_{x\in K,\, K'\subset K} H(t,x,K',p,A)\ .$$

From the assumptions on H, $\tilde{H}$ satisfies assumptions (6).

We fix $s \geq 0$ and $K_0 = \mathrm{Co}[\mathcal{K}(s)]$. We also set $R = \sup_{y\in\mathcal{K}(s)} \|y\|$. Let ϵ be positive and $r(t)$ be defined by $r_\epsilon(t) = (R+\epsilon)e^{\int_s^t c(\tau)d\tau} - R$. Then satisfies $r'_\epsilon(t) = c(t)(R + r_\epsilon(t))$. We are going to prove that $\mathcal{K}_2 = \{(t,x) : d_{K_0}(t) \leq r_\epsilon(t)\}$ satisfies the internal condition for $\tilde{H}$.

We claim that, if this is the case, then the Proposition is proved. Indeed, let $\mathcal{K}$ be a tube satisfying the external condition for H and the external initial condition for K_0. Then $\mathcal{K}$ also satisfies the external condition for $\tilde{H}$. So we can apply the inclusion principle (Theorem 3.1) on the interval $[s,+\infty)$ to $\mathcal{K}$ and $\mathcal{K}_2$ because $\mathrm{Co}[\mathcal{K}(s)] \subset Int(K_0 + \epsilon B)$ and $\mathcal{K}_2$ satisfies the internal condition for $K_0 + \epsilon B$. Then we obtain $\forall t \geq s$, $\mathcal{K}(t) \subset \mathrm{Co}[\mathcal{K}(s)] + r_\epsilon(t)B$ so that $\forall t \geq s$, $\mathrm{Co}[\mathcal{K}(t)] \subset \mathrm{Co}[\mathcal{K}(s)] + r_\epsilon(t)B$. The result is proved by letting $\epsilon \to 0^+$.

Let us now prove that $\mathcal{K}_2$ satisfies the internal condition for $\tilde{H}$ on $(s,+\infty)$. Let ϕ have a maximum at some point $(t,x) \in \partial\widehat{\mathcal{K}_2}$ (with $t > s$) on $\widehat{\mathcal{K}_2}$. Let y be a projection of x onto K_0 and let us set

$$\pi = \frac{x-y}{\|x-y\|}.$$

Let us point out that $\|x-y\| = r_\epsilon(t) > \epsilon$ and that π belongs to the normal cone to K_0 at y. So, for any τ, the point $(\tau, y + r_\epsilon(\tau)\pi)$ belongs to $\widehat{\mathcal{K}_2}(\tau)$, and we have $\phi(\tau, y + r_\epsilon(\tau)\pi) \leq \phi(t,x)$. Therefore, we have

$$\phi_t(t,x) + r'_\epsilon(t) < \phi_x(t,x), \pi >= 0\ .$$

On another hand, for any $v \in \mathbb{R}^N$ such that $< v, \pi >\geq 0$, the point $(t, x+v)$ belongs to $\widehat{\mathcal{K}_2}(t)$ by convexity, and therefore, $\phi_x(t,x) = -\|\phi_x(t,x)\|\pi$ and $\phi_{xx}(t,x) \leq 0$ on $\pi^\perp$. Thus,

$$\begin{aligned}\tilde{H}^*(t,\mathcal{K}_2(t),-\phi_x(t,x),-\phi_{xx}(t,x)) &\geq \|\phi_x(t,x)\|\tilde{H}^*(t,B(0,R+r_\epsilon(t)),\pi,0)\\ &\geq -\|\phi_x(t,x)\|c(t)(R+r_\epsilon(t))\end{aligned}$$

from assumptions (12) and (15). Since $\phi_t(t,x) = r'_\epsilon(t)\|\phi_x(t,x)\|$, and since $r'_\epsilon(t) = c(t)(R + r_\epsilon(t))$, we have the desired result

$$\tilde{H}^*(t,\mathcal{K}_2(t),-\phi_x(t,x),-\phi_{xx}(t,x)) \geq -\phi_t(t,x)\ .$$

Theorem 3.14. *Assume that H depends only on (K,p,A), that H satisfies assumptions* (12), *growth assumption* (15) *and the following convexity condition:*

$$A \to H(K,p,A) \text{ is convex} \tag{16}$$

for any fixed $p \in \mathbb{R}^N$ and any convex compact set K. Then, for any convex compact initial condition K_0, the maximal solution $S(K_0)$ has convex values:

$$\forall t \geq 0, \quad S(K_0)(t) \text{ is convex .}$$

Remarks. 1) A slight modification of the proof shows that the result still holds true for motions $H = H(t,x,K,p,A)$ such that $(x,A) \to H(t,x,K,p,A)$ is convex for any fixed $(t,p) \in \mathbb{R}^{1+N}$ and any convex compact set K. We shall not prove it for simplicity.

2) From the construction of the minimal solution $s(K_0)$, this solution is also convex valued.

Proof of Theorem 3.14. The key point of the proof is Lemma 3.15:

Lemma 3.15. *Under the assumptions of Theorem* 3.14, *if $\mathcal{K}$ is a closed tube satisfying the external condition, then the tube $\mathcal{K}_1$ defined by*

$$\forall t \geq 0, \quad \mathcal{K}_1(t) = \mathrm{Co}\,[\mathcal{K}(t)]$$

also satisfies the external condition.

Before proving Lemma 3.15, we complete the proof of Theorem 3.14. Let $\mathcal{K} = S(K_0)$ and let us define, as in the Lemma $\forall t \geq 0$, $\mathcal{K}_1(t) = \mathrm{Co}\,[\mathcal{K}(t)]$. Thanks to the Lemma, $\mathcal{K}_1$ satisfies the external condition. Moreover, $\mathcal{K}_1$ also satisfies the external initial condition for K_0 because K_0 is convex. Then Corollary 3.6 states that $\mathcal{K}_1$ is contained in $S(K_0)$. From the very definition of $\mathcal{K}_1$, the converse inclusion also holds true. So $\mathcal{K}_1 = S(K_0)$, and $S(K_0)$ is convex valued. □

Proof of Lemma 3.15. 1. We are going to prove that the tube $\mathcal{K}_{1,\epsilon}$ defined by $\forall t \geq 0$, $\mathcal{K}_{1,\epsilon}(t) = \mathrm{Co}[\mathcal{K}(t) + \epsilon B]$ satisfies the external condition. Then we let $\epsilon \to 0^+$ and obtain the desired result.

2. Let us first prove that the tube $\mathcal{K}_\epsilon$ defined by $\forall t \geq 0$, $\mathcal{K}_\epsilon(t) = \mathcal{K}(t) + \epsilon B$ satisfies the external condition. For doing so, let ϕ be a test function with a strict local maximum on $\mathcal{K}_\epsilon$ at some point $(t,x) \in \partial\mathcal{K}_\epsilon$ with $t > 0$. Let y belong to the projection of x onto $\mathcal{K}(t)$. We claim that the test function

$\forall (s,z) \in \mathbb{R}^{N+1}$, $\psi(s,z) = \phi(s, z+x-y)$ has a local maximum on $\mathcal{K}$ at (t,y). Indeed, for any (s,z) belonging to $\mathcal{K}$, the point $(s, z+x-y)$ belongs to $\mathcal{K}_\epsilon$ because $\|x-y\| \le \epsilon$. Therefore,

$$\psi(s,z) = \phi(s, z+x-y) \le \phi(t,x) = \psi(t,y) \ .$$

Thus, ψ has a local maximum at (t,y), so that

$$H_*(\mathcal{K}(t), \psi_x(t,y), \psi_{xx}(t,y)) \le \psi_t(t,y) \ .$$

Since $\psi_x(t,y) = \phi_x(t,x)$, $\psi_{xx}(t,y) = \phi_{xx}(t,x)$ and $\psi_t(t,x) = \phi_t(t,x)$, we obtain the desired result

$$H_*(\mathcal{K}_\epsilon(t), \phi(t,x), \phi_{xx}(t,x)) \le \phi_t(t,x)$$

because H is non increasing with respect to K.

3. We now prove that $\mathcal{K}_{1,\epsilon}(t) = \mathrm{Co}[\mathcal{K}_\epsilon(t)]$ satisfies the external condition. Let ϕ be a test function with a strict local maximum on $\mathcal{K}_{1,\epsilon}$ at some point $(t,x) \in \partial\mathcal{K}_{1,\epsilon}$, with $t > 0$. We have to prove that

$$H_*(\mathcal{K}_{1,\epsilon}(t), \phi_x(t,x), \phi_{xx}(t,x)) \le \phi_t(t,x) \ .$$

For doing so, we introduce the following perturbed function:

$$\psi^\alpha(\tilde{s}, \tilde{\lambda}, \tilde{y}) = \phi\Big(\sum_i \lambda_i s_i, \sum_i \lambda_i y_i\Big) - \frac{1}{\alpha}\sum_i \lambda_i d^2_{\mathcal{K}_\epsilon}(s_i, y_i) - \frac{1}{\alpha}\sum_{i,j} \lambda_i\lambda_j (s_i - s_j)^2$$

where $\tilde{s} = (s_0, \ldots, s_N)$ belongs to $(\mathbb{R}^+)^{N+1}$, $\tilde{\lambda} = (\lambda_0, \ldots, \lambda_N)$ belongs to $[0,1]^{N+1}$ with $\sum_i \lambda_i = 1$, and $\tilde{y} = (y_0, \ldots, y_N)$ belongs to $(\mathbb{R}^N)^{N+1}$.

4. Following standard arguments in perturbation methods, there is some $(\tilde{s}^\alpha, \tilde{\lambda}^\alpha, \tilde{y}^\alpha)$ local maximum of ψ^α such that, up to a subsequence,

$$\forall i \in \{0, \ldots, N\},\ (s_i^\alpha, y_i^\alpha) \to (t, y_i) \in \mathcal{K}_\epsilon,\ \lambda_i^\alpha \to \lambda_i \in [0,1]$$

with $\sum_i \lambda_i y_i = x$ and $\sum_i \lambda_i = 1$.

5. Since $d^2_{\mathcal{K}_\epsilon}$ is semi-concave, the function ψ^α is semi-convex. From the maximum principle for semi-convex functions (see Theorem 9.12 in [3]), there are $(s_i^{n,\alpha}, y_i^{n,\alpha})$ points of twice differentiability of $d^2_{\mathcal{K}_\epsilon}$, converging to (s_i^α, y_i^α) such that, for any $i = 0, \ldots, N$, we have

$$\lambda_i^\alpha \phi_x(\sum_i \lambda_i^\alpha s_i^{n,\alpha}, \sum_i \lambda_i^\alpha y_i^{n,\alpha}) - \frac{1}{\alpha}\lambda_i^\alpha p_i^{n,\alpha} \ \ \to 0,$$

where $p_i^{n,\alpha} = \frac{\partial}{\partial x} d^2_{\mathcal{K}_\epsilon}(s_i^{n,\alpha}, y_i^{n,\alpha})$, and

$$\lambda_i^\alpha \phi_t(\sum_i \lambda_i^\alpha s_i^{n,\alpha}, \sum_i \lambda_i^\alpha y_i^{n,\alpha}) - \frac{1}{\alpha}\lambda_i^\alpha q_i^{n,\alpha} - \frac{2}{\alpha}\lambda_i \sum_j \lambda_j (s_i^{n,\alpha} - s_j^{n,\alpha}) \;\; \to \;\; 0,$$

where $q_i^{n,\alpha} = \frac{\partial}{\partial t} d^2_{\mathcal{K}_\epsilon}(s_i^{n,\alpha}, y_i^{n,\alpha})$, and

$$\begin{pmatrix} (\lambda_0^\alpha)^2\phi_{xx} & \dots & \lambda_1^\alpha\lambda_N^\alpha\phi_{xx} \\ & \dots & \\ \dots & \lambda_i^\alpha\lambda_j^\alpha\phi_{xx} & \dots \\ & \dots & \\ \lambda_0^\alpha\lambda_N^\alpha\phi_{xx} & \dots & (\lambda_N^\alpha)^2\phi_{xx} \end{pmatrix} - \frac{1}{\alpha}\begin{pmatrix} \lambda_0 A_0^{n,\alpha} & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \lambda_i A_i^{n,\alpha} & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & \lambda_N A_N^{n,\alpha} \end{pmatrix} \leq \alpha_n I$$

where $\alpha_n \to 0$, $A_i^{n,\alpha} = \frac{\partial^2}{\partial x^2} d^2_{\mathcal{K}_\epsilon}(s_i^{n,\alpha}, y_i^{n,\alpha})$ and

$$\phi_{xx} = \phi_{xx}(\sum \lambda_i^\alpha s_i^{n,\alpha}, \sum \lambda_i^\alpha y_i^{n,\alpha}) \; .$$

6. Up to a subsequence, $(p_i^{n,\alpha}, q_i^{n,\alpha}, A_i^{n,\alpha})$ converge to some $(p_i^\alpha, q_i^\alpha, A_i^\alpha)$ with, for any i such that $\lambda_i^\alpha \neq 0$,

$$\phi_x(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha) - \frac{1}{\alpha} p_i^\alpha \;\; = \;\; 0$$

and

$$\phi_t(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha) - \frac{1}{\alpha} q_i^\alpha - \frac{2}{\alpha}\sum_j \lambda_j^\alpha (s_i^\alpha - s_j^\alpha) \;\; = \;\; 0$$

and

$$\begin{pmatrix} (\lambda_0^\alpha)^2\phi_{xx} & \dots & \lambda_0^\alpha\lambda_N^\alpha\phi_{xx} \\ & \dots & \\ \dots & \lambda_i^\alpha\lambda_j^\alpha\phi_{xx} & \dots \\ & \dots & \\ \lambda_0^\alpha\lambda_N^\alpha\phi_{xx} & \dots & (\lambda_N^\alpha)^2\phi_{xx} \end{pmatrix} \;\leq\; \frac{1}{\alpha}\begin{pmatrix} \lambda_0 A_0^\alpha & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & \lambda_i A_i^\alpha & \vdots \\ 0 & \dots & 0 \\ 0 & \dots & \lambda_N A_N^\alpha \end{pmatrix}$$

where $\phi_{xx} = \phi_{xx}(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha)$. Let us point out for later use that this matrix inequality implies that

$$\phi_{xx}(\sum \lambda_i^\alpha s_i^\alpha, \sum \lambda_i^\alpha y_i^\alpha) \;\leq \frac{1}{\alpha}\sum_i \lambda_i^\alpha A_i^\alpha \; .$$

7. Arguing as in the proof of Theorem 3.1, steps 7 and 8, there are, for any i, some $(\tau_i^\alpha, z_i^\alpha) \in \mathcal{K}_\epsilon$ belonging to the projection of (s_i^α, y_i^α) onto $\mathcal{K}_\epsilon$, such that $H_*(\mathcal{K}_\epsilon(\tau_i^\alpha), p_i^\alpha, A_i^\alpha) \le q_i^\alpha$. Note that $(\tau_i^\alpha, z_i^\alpha)$ converges to (t, y_i) for any i because (s_i^α, y_i^α) converges to $(t, y_i) \in \mathcal{K}_\epsilon$.

8. Let us fix $\gamma > 0$. For $\alpha > 0$ sufficiently small, we have $\forall i \in \{0, \ldots, N\}$, $\mathcal{K}_\epsilon(\tau_i^\alpha) \subset \mathcal{K}_{1,\epsilon}(t) + \gamma B$, because $\mathcal{K}_{1,\epsilon}$ has a closed graph and that $\forall t \ge 0$, $\mathcal{K}_\epsilon(t) \subset \mathcal{K}_{1,\epsilon}(t)$.

9. Let us now distinguish two cases: We first assume that $\phi_x(t,x) \neq 0$. For any $\alpha > 0$ sufficiently small, we have $H_*(\mathcal{K}_{1,\epsilon}(t) + \gamma B, p_i^\alpha, A_i^\alpha) \le q_i^\alpha$. Using the convexity assumption and the fact that

$$p_i^\alpha = \alpha \phi_x(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha) \ \neq \ 0 \ ,$$

we obtain

$$H_*(\mathcal{K}_{1,\epsilon}(t) + \gamma B, \alpha\phi_x, \sum_i \lambda_i^\alpha A_i^\alpha) \le \sum_i \lambda_i^\alpha q_i^\alpha$$

where $\phi_x = \phi_x(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha)$. Now recall that

$$\phi_{xx}(\sum \lambda_i^\alpha s_i^\alpha, \sum \lambda_i^\alpha y_i^\alpha) \ \le \ \frac{1}{\alpha} \sum_i \lambda_i^\alpha A_i^\alpha$$

and that

$$\phi_t(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha) = \frac{1}{\alpha} q_i^\alpha + \frac{2}{\alpha} \sum_j \lambda_j^\alpha (s_i^\alpha - s_j^\alpha).$$

Then we obtain, since H is geometric, $H_*(\mathcal{K}_{1,\epsilon}(t) + \gamma B, \phi_x, \phi_{xx}) \le \phi_t$, where $\phi_x = \phi_x(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha)$, $\phi_{xx} = \phi_{xx}(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha)$ and $\phi_t = \phi_t(\sum_i \lambda_i^\alpha s_i^\alpha, \sum_i \lambda_i^\alpha y_i^\alpha)$. Letting $\alpha \to 0^+$ gives

$$H_*(\mathcal{K}_{1,\epsilon}(t) + \gamma B, \phi_x(t,x), \phi_{xx}(t,x)) \le \phi_t(t,x),$$

because $x = \sum_i \lambda_i y_i$. Letting $\gamma \to 0^+$, we finally obtain the desired result

$$H_*(\mathcal{K}_{1,\epsilon}(t), \phi_x(t,x), \phi_{xx}(t,x)) \le \phi_t(t,x)$$

thanks to Lemma 5.2 in appendix.

10. We now assume that $\phi_x(t,x) = 0$. A simple calculus shows that $\forall t \ge 0$, $\mathcal{K}_{1,\epsilon}(t) = \mathrm{Co}[\mathcal{K}(t)] + \epsilon B$. Let y belong to the projection of x onto

$\mathrm{Co}[\mathcal{K}(t)]$. Then, we have, $\|y-x\| = \epsilon$ and $B(y,\epsilon) \subset \mathcal{K}_{1,\epsilon}(t)$. In particular, for any $v \in \mathbb{R}^N$ such that $< v, x-y > < 0$, the point $x + \sigma v$ belongs to $\mathcal{K}_{1,\epsilon}(t)$ for $\sigma > 0$ sufficiently small. Since ϕ has a local maximum on $\mathcal{K}_{1,\epsilon}$ at (t,x) and since $\phi_x(t,x) = 0$, we have

$$\phi(t, x+\sigma v) = \phi(t,x) + \frac{\sigma^2}{2} < \phi_{xx}(t,x)v, v > + \sigma^2 \epsilon(\sigma) \leq \phi(t,x)$$

and thus $\phi_{xx}(t,x) \leq 0$. From Lemma 3.13,

$$\forall 0 \leq s \leq t, \;\; \mathcal{K}_{1,\epsilon}(t) \; \subset \; \mathcal{K}_{1,\epsilon}(s) + M(e^{\int_s^t c(\tau)d\tau} - 1)B,$$

where $M = \sup_{s\leq t} \sup_{y \in \mathcal{K}_\epsilon(s)} \|y\|$. So, for any $s < t$, there is some $y_s \in \mathcal{K}_{1,\epsilon}(s)$ with $\|y_s - x\| \leq M(e^{\int_s^t c(\tau)d\tau} - 1)$. Then

$$\phi(s, y_s) = \phi(t,x) + \phi_t(t,x)(s-t) + \phi_x(t,x)(y_s - x) + (t-s)\epsilon(t-s)$$

and, since $\phi(s, y_s) \leq \phi(t,x)$ and $\phi_x(t,x) = 0$, we have $\phi_t(t,x) \geq 0$. In particular,

$$H_*(\mathcal{K}_{1,\epsilon}(t), \phi_x, \phi_{xx})) \; \leq H_*(\mathcal{K}_{1,\epsilon}(t), 0, 0) = 0 \leq \phi_t(t,x) \; .$$

11. So we have proved that $\mathcal{K}_{1,\epsilon}$ satisfies the external condition. Using Lemma 3.7, we can let $\epsilon \to 0^+$ and obtain that $\mathcal{K}_1$ also satisfies the external condition, which is the desired result. □

4. Fronts without inclusion principle. We now start the study of fronts for which the inclusion principle fails (i.e., H is not necessarily decreasing with respect to K). We first give an existence result of approximate solutions, called $\epsilon-$solutions (Proposition 4.6). Unfortunately, it is not clear that limits, when $\epsilon \to 0^+$, of $\epsilon-$solutions are solutions. For getting such a result, we would need estimates on the behaviour of the boundary of the $\epsilon-$solutions when $\epsilon \to 0^+$. This behaviour is not yet understood in the general case.

However, under suitable assumptions on H, we can construct convex valued $\epsilon-$solutions. For these $\epsilon-$solutions, we have enough estimates to let $\epsilon \to 0^+$ and get solutions at the limit (Theorem 4.7).

We complete the paper by proving that, if there is some solution to the front which is smooth on some interval $[0,T]$, then any other solution coincides with this solution on $[0,T]$ (Theorem 4.8).

4.1. Semi-group property. We first give two results on the concatenation of external (or internal) solutions.

Lemma 4.1. *Let $T>0$ and $\mathcal{K}$ be a closed tube. We assume that $\mathcal{K}$ satisfies the external condition on $(0,T)$ and that $\mathcal{K}(T) = \mathrm{Limsup}_{t\to T^-}\ (\mathcal{K}(t)) \neq \emptyset$. Then the tube $\mathcal{K}$ satisfies the external condition up to time T.*

Proof. Let ϕ be a test function with a strict maximum on $\mathcal{K}$ at some point $(T,x)\in\partial\mathcal{K}$. Then there is some $r\in(0,1)$ such that

$$\forall (s,y)\in B((T,x),r)\cap\mathcal{K},\ \text{with } (s,y)\neq(T,x),\ \ \phi(s,y)\ <\ \phi(T,x)\ .$$

Let us consider the following perturbation of ϕ:

$$\phi^\epsilon(s,y) = \phi(s,y) + \epsilon\log(T-s).$$

We claim that, for any $\epsilon>0$ sufficiently small, ϕ^ϵ has a local maximum on $\mathcal{K}$ at some point $(s^\epsilon,y^\epsilon)\in B((T,x),r)\cap\mathcal{K}$. Indeed, let us set

$$\theta = \inf_{(s,y)\in\partial B((T,x),r)\cap\mathcal{K}} [\phi(T,x)-\phi(s,y)]\ >\ 0.$$

Since

$$\mathcal{K}(T)\ =\ \mathrm{Limsup}_{t\to T^-}\ (\mathcal{K}(t))\,,$$

there is some $(\bar{s},\bar{y})\in\mathcal{K}$, with $\bar{s}<T$, such that

$$\|(\bar{s},\bar{y})-(T,x)\|\le r/2 \text{ and } |\phi(\bar{s},\bar{y})-\phi(T,x)|\le\theta/3\ .$$

Let $\epsilon_0>0$ be defined by $\epsilon_0 = \frac{-\theta}{3\log(T-\bar{s})}$. Then, for any $\epsilon\in(0,\epsilon_0)$, for any $(s,y)\in\partial B((T,x),r)\cap\mathcal{K}$, with $s<T$, we have

$$\begin{aligned}\phi^\epsilon(s,y) &= \phi(s,y)+\epsilon\log(T-s) \le \phi(T,x)-\theta+\epsilon\log(T-s)\\ &\le \phi(\bar{s},\bar{y})-2\theta/3+\epsilon\log(T-s)\le\phi^\epsilon(\bar{s},\bar{y})-\theta/3+\epsilon\log(T-s)\\ &< \phi^\epsilon(\bar{s},\bar{y})-\theta/3.\end{aligned}$$

Therefore, there is some point $(\bar{s},\bar{y})\in B((T,x),r/2)\cap\mathcal{K}$ such that

$$\sup_{(s,y)\in\partial B((T,x),r)\cap\mathcal{K}}\phi^\epsilon(s,y)\ \le\ \phi^\epsilon(\bar{s},\bar{y})-\theta/3$$

and so ϕ^ϵ has a local maximum on $\mathcal{K}$ at some point $(s^\epsilon,y^\epsilon)\in\mathcal{K}\cap B((T,x),r)$.

Let us now point out that (s^ϵ, y^ϵ) converges to (T,x) when $\epsilon \to 0^+$. Indeed, let (s,y) be a cluster point of (s^ϵ, y^ϵ) when $\epsilon \to 0^+$. For any $\gamma > 0$, there is some $(\bar{s},\bar{y}) \in \mathcal{K}$, with $\bar{s} < T$, such that $\|(\bar{s},\bar{y}) - (T,x)\| \leq r/2$ and $|\phi(\bar{s},\bar{y}) - \phi(T,x)| \leq \gamma$. Then

$$\begin{aligned}\phi(s^\epsilon, y^\epsilon) &\geq \phi^\epsilon(s^\epsilon, y^\epsilon) \geq \phi^\epsilon(\bar{s},\bar{y}) \geq \phi(\bar{s},\bar{y}) + \epsilon \log(T-\bar{s}) \\ &\geq \phi(T,x) - \gamma + \epsilon \log(T-\bar{s}).\end{aligned}$$

Letting $\epsilon \to 0^+$, we obtain $\phi(s,y) \geq \phi(T,x) - \gamma$. Since $\gamma > 0$ is arbitrary, we have, on the one hand, $\phi(s,y) \geq \phi(T,x)$ and, on the other hand, that $(s,y) \in B((t,x),r) \cap \mathcal{K}$. Thus, $(s,y) = (T,x)$ and our claim is proved.

Since $\mathcal{K}$ satisfies the external condition on $(0,T)$ and since (s^ϵ, y^ϵ) is a local maximum of ϕ^ϵ on $\mathcal{K}$, we have

$$H_*(s^\epsilon, y^\epsilon, \mathcal{K}(s^\epsilon), \phi_x(s^\epsilon, y^\epsilon), \phi_{xx}(s^\epsilon, y^\epsilon)) \leq \phi_t(s^\epsilon, y^\epsilon) - \frac{\epsilon}{T - s^\epsilon} \leq \phi_t(s^\epsilon, y^\epsilon).$$

Letting $\epsilon \to 0^+$, we obtain the desired result:

$$H_*(T,x,\mathcal{K}(T), \phi_x(T,x), \phi_{xx}(T,x)) \leq \phi_t(T,x),$$

because $(s,y,p,A) \to H_*(s,y,\mathcal{K}(s),p,A)$ is l.s.c. $\square$

Lemma 4.2. *Let $T > 0$ and $\mathcal{K}$ be a closed tube. We assume that $\mathcal{K}$ satisfies the internal condition on $(0,T)$ and that*

$$\left[\overline{([T,+\infty) \times \mathbb{R}^N) \backslash \mathcal{K}}\right](T) \ \subset \ \overline{\mathbb{R}^N \backslash \mathcal{K}(T)}\ .$$

Then the tube $\mathcal{K}$ satisfies the internal condition up to time T.

Proof. Let ϕ be a test function with a strict maximum on $\widehat{\mathcal{K}}$ at some point $(T,x) \in \partial\widehat{\mathcal{K}}$. Then there is some $r \in (0,1)$ such that

$$\forall (s,y) \in B((T,x),r) \cap \widehat{\mathcal{K}},\ \text{with } (s,y) \neq (T,x), \ \ \phi(s,y) \ < \ \phi(T,x)\ .$$

We first prove that, for any $\gamma > 0$, there is some $(\bar{s},\bar{y}) \in \widehat{\mathcal{K}}$, with $\bar{s} < T$, such that $\|(\bar{s},\bar{y}) - (T,x)\| \leq r/2$ and $|\phi(\bar{s},\bar{y}) - \phi(T,x)| \leq \gamma$. Indeed, assume that, contrary to our claim, there is some $\eta > 0$ such that, for any $(s,y) \in B((T,x),\eta)$, with $s < T$, (s,y) does not belong to $\widehat{\mathcal{K}}$, i.e.,

$$\{(s,y) \in B((T,x),\eta) : s < T\} \ \subset \ \mathcal{K}\ .$$

Then $B(x,\eta) \subset \mathcal{K}(T)$, because $\mathcal{K}$ has a closed graph. In particular,

$$B(x,\eta/2) \cap \overline{\mathbb{R}^N \backslash \mathcal{K}(T)} = \emptyset \ .$$

Since

$$\left[\overline{([T,+\infty) \times \mathbb{R}^N) \backslash \mathcal{K}}\right](T) \subset \overline{\mathbb{R}^N \backslash \mathcal{K}(T)} \ ,$$

we have

$$B(x,\eta/3) \cap \left[\overline{([T,+\infty) \times \mathbb{R}^N) \backslash \mathcal{K}}\right](T) = \emptyset \ .$$

So, there is some $\eta_1 \in (0,\eta)$ such that $\{(s,y) \in B((T,x),\eta_1) : s > T\} \subset \mathcal{K}$. In conclusion, we have $B((T,x),\eta_1) \subset \mathcal{K}$. In particular, $(T,x) \notin \widehat{\mathcal{K}}$, which in contradiction with the assumption. Thus we have proved that, for any $\gamma > 0$, there is some $(\bar{s},\bar{y}) \in \widehat{\mathcal{K}}$, with $\bar{s} < T$, such that $\|(\bar{s},\bar{y}) - (T,x)\| \leq r/2$ and $|\phi(\bar{s},\bar{y}) - \phi(T,x)| \leq \gamma$.

Let us now consider the following perturbation of ϕ:

$$\phi^\epsilon(s,y) = \phi(s,y) + \epsilon \log(T-s).$$

Following the proof of Lemma 4.1, we can check that, for any $\epsilon > 0$ sufficiently small, ϕ^ϵ has a local maximum on $\widehat{\mathcal{K}}$ at some point $(s^\epsilon,y^\epsilon) \in B((T,x),r) \cap \widehat{\mathcal{K}}$. Moreover, we prove as before that (s^ϵ,y^ϵ) converge to (T,x) when $\epsilon \to 0^+$.

Since $\widehat{\mathcal{K}}$ satisfies the internal condition on $(0,T)$ and since (s^ϵ,y^ϵ) is a local maximum of ϕ^ϵ on $\widehat{\mathcal{K}}$, we have

$$H^*(s^\epsilon,y^\epsilon,\mathcal{K}(s^\epsilon),-\phi_x(s^\epsilon,y^\epsilon),-\phi_{xx}(s^\epsilon,y^\epsilon)) \geq -\phi_t(s^\epsilon,y^\epsilon) + \frac{\epsilon}{T-s^\epsilon} \geq -\phi_t(s^\epsilon,y^\epsilon).$$

Letting $\epsilon \to 0^+$, we obtain the desired result:

$$H^*(T,x,\mathcal{K}(T),-\phi_x(T,x),-\phi_{xx}(T,x)) \geq -\phi_t(T,x),$$

because $(s,y,p,A) \to H^*(s,y,\mathcal{K}(s),p,A)$ is u.s.c. $\square$

As a straightforward application of the Lemmas, we have

Corollary 4.3. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed tubes satisfying the external condition respectively on the interval $[0,T)$ and $(T,+\infty)$. Assume that*

$$\text{Limsup}_{t \to T^-} \mathcal{K}_1(t) = \mathcal{K}_2(T) \ .$$

Then the tube $\mathcal{K}_3$ defined by

$$\mathcal{K}_3(t) = \begin{cases} \mathcal{K}_1(t) & \text{if } t \in [0,T] \\ \mathcal{K}_2(t) & \text{if } t \in [T,+\infty) \end{cases}$$

is satisfies the external condition.

Corollary 4.4. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be closed tubes satisfying the internal condition respectively on the interval $[0,T)$ and $(T,+\infty)$. Assume that*

$$\left[\overline{([T+\infty)\times\mathbb{R}^N)\backslash\mathcal{K}_2}\right](T)\subset\overline{\mathbb{R}^N\backslash\mathcal{K}_1(T)}\ .$$

Then the tube $\mathcal{K}_3$ defined by

$$\mathcal{K}_3(t)=\begin{cases}\mathcal{K}_1(t) & \text{if } t\in[0,T)\\ \mathcal{K}_2(t) & \text{if } t\in(T,+\infty)\\ \mathrm{Limsup}_{t\to T, t\neq T}\mathcal{K}_3(t) & \text{if } t{=}T\end{cases}$$

is satisfies the external condition.

4.2. Existence of $\epsilon-$solutions. We now construct approximate solutions, called $\epsilon-$solutions.

Definition 4.5. Let $\epsilon>0$ be fixed. We say that a tube $\mathcal{K}$ satisfies the $\epsilon-$external condition if $\mathcal{K}$ satisfies the external condition for $H^{\flat,\epsilon}$ defined by

$$H^{\flat,\epsilon}(t,x,K,p,A)=\inf_{K-\epsilon B\subset K'\subset K+\epsilon B}H(t,x,K',p,A)\ .$$

We say that a tube $\mathcal{K}$ satisfies the $\epsilon-$internal condition if $\mathcal{K}$ satisfies the internal condition for $H^{\sharp,\epsilon}$ defined by

$$H^{\sharp,\epsilon}(t,x,K,p,A)=\sup_{K-\epsilon B\subset K'\subset K+\epsilon B}H(t,x,K',p,A)\ .$$

We say that a tube $\mathcal{K}$ is an $\epsilon-$solution of the front propagation problem with initial condition K_0 if $\mathcal{K}$ satisfies the $\epsilon-$external and the $\epsilon-$internal condition, as well as the internal and the external initial condition for K_0.

Let us introduce some assumptions on H.

$$\left\{\begin{array}{rl} i) & H \text{ is geometric and elliptic}\\ ii) & H(\cdot,\cdot,K,\cdot,\cdot) \text{ is continuous on } \mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N_*\times\mathcal{S}_N\\ & \text{uniformly with respect to } K \text{ for } K\subset B(0,R)\text{, for any } R>0\\ iii) & \forall R>0,\ \lim_{(p,A)\to 0}\sup_{t,x,K}|H(t,x,K,p,A)|\ =\ 0\\ & \text{where the supremum is taken over all } \|(t,x)\|\leq R\\ & \text{and } K \text{ with } K\subset B(0,R)\\ iv) & \exists c(t)\in\mathcal{C}^0(\mathbb{R}^+),\ \ \forall(x,p)\in\mathbb{R}^{2N} \text{ with } \|x\|\leq R \text{ and } \|p\|=1,\\ & \forall K\subset B(0,R),\ \ H(t,x,K,p,I_N)\geq -c(t)R.\end{array}\right. \tag{17}$$

Examples. If H is of the form

$$H(t,x,K,p,A) = \mathrm{Tr}\left(\sigma^T\left(A_{|_{p^\perp}}\right)\sigma\right) + \|p\|\int_K \phi(t,x,y,\frac{p}{\|p\|})dy \qquad (18)$$

where $\sigma = \sigma(t,x) \in \mathcal{C}^2(\mathbb{R}^{N+1},\mathbb{R}^N)$, $\phi \in \mathcal{C}^1(\mathbb{R}^{3N+1},\mathbb{R})$ (not necessarily positive), then H satisfies (17). This contains the case $H = \mathrm{Tr}A_{|_{p^\perp}} + (\alpha + \beta|K|)\|p\|$ (for all $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^-$) of [12].

Proposition 4.6. *Let H satisfy assumption* (17) *and K_0 be a compact subset of $\mathbb{R}^N$. Then, for any $\epsilon > 0$, there is an $\epsilon-$solution to the front propagation problem with initial condition K_0.*

Proof of Proposition 4.6. 1. From now on $\epsilon > 0$ is fixed. We first construct an $\epsilon-$solution defined on some small interval $[0,T)$, with $T > 0$, with initial condition K_0. For doing so, let us set $K_\epsilon = (K_0 - \epsilon B) + \epsilon B$. Let $\mathcal{K}$ be the maximal solution of the front propagation problem with initial condition K_0 for H_ϵ defined by $H_\epsilon(t,x,p,A) = H(t,x,K_\epsilon,p,A)$. Recall that $\mathcal{K}$ is closed. We are going to prove that there is some $T > 0$ such that $\mathcal{K}$ is an $\epsilon-$solution for H on $[0,T)$.

2. For doing so, we first claim that there is some $T > 0$ such that

$$\forall t \in [0,T),\ \ \mathcal{K}(t) - \epsilon B\ \subset\ K_\epsilon\ \subset\ \mathcal{K}(t) + \epsilon B.$$

Proof. Assume first that, contrary to our claim, there is some $t_n \to 0^+$ and some y_n with $y_n \in [\mathcal{K}(t_n) - \epsilon B]\backslash K_\epsilon$. In particular, $y_n + \epsilon B \subset \mathcal{K}(t_n)$ and $y_n \notin K_\epsilon$. Since $\mathcal{K}$ is a closed tube, the sequence y_n is bounded, and thus converges, up to a subsequence, to some y. Note that $y \in \overline{\mathbb{R}^N\backslash K_\epsilon}$ and $y + \epsilon B \subset K_0$, because $\mathcal{K}(0) = K_0$. Therefore, y belongs to $K_0 - \epsilon B$, so that $y + \epsilon B \subset K_\epsilon$. This is in contradiction with $y \in \overline{\mathbb{R}^N\backslash K_\epsilon}$. Thus, there is some $T_1 > 0$ such that $\forall t \in [0,T_1]$, $\mathcal{K}(t) - \epsilon B \subset K_\epsilon$. In the same way, assume that there is some $t_n \to 0^+$ and some y_n such that $y_n \in K_\epsilon\backslash[\mathcal{K}(t_n) + \epsilon B]$. Then $y_n + \epsilon B \subset \widehat{\mathcal{K}}(t_n)$.

Since K_ϵ is compact, y_n converges, up to a subsequence, to some $y \in K_\epsilon$. Since $\widehat{\mathcal{K}}$ has a closed graph, we have $y + \epsilon B \subset \widehat{\mathcal{K}}(0) \subset \widehat{K_0}$, because $\mathcal{K}$ satisfies the internal initial condition for K_0.

Now recall that y belongs to K_ϵ. From the very definition of K_ϵ, there is some $z \in K_0 - \epsilon B$ such that $\|y - z\| \le \epsilon$. Then z belongs to $\widehat{K_0}$, because $y + \epsilon B \subset \widehat{K_0}$, and z belongs to $K_0 - \epsilon B$. So there is a contradiction.

Therefore, there is some $T_2 > 0$ such that $\forall t \in [0, T_2]$, $K_\epsilon \subset \mathcal{K}(t) + \epsilon B$. Setting $T = \inf\{T_1, T_2\}$, we obtain the desired result.

3. Let us now prove that $\mathcal{K}$ satisfies the $\epsilon-$external condition on $[0, T)$.

Proof. Let ϕ be a test function with a local maximum on $\mathcal{K}$ at some point (t, x) with $t \in (0, T)$. Since $\mathcal{K}$ is a solution of the front propagation problem for H_ϵ, we have

$$\liminf_{\substack{(t',x',p',A') \to (t,x,\phi_x,\phi_{xx}) \\ p \neq 0}} H(t', x', K_\epsilon, p', A') \leq \phi_t,$$

where $\phi_x = \phi_x(t, x)$, $\phi_t = \phi_t(t, x)$ and $\phi_{xx} = \phi_{xx}(t, x)$. Since, from the previous step, $\forall t \in [0, T]$, $\mathcal{K}(t) - \epsilon B \subset K_\epsilon \subset \mathcal{K}(t) + \epsilon B$, we obtain

$$H_*^{\flat,\epsilon}(t, x, \mathcal{K}(t), \phi_x, \phi_{xx}) \leq \liminf_{\substack{(t',x',p',A') \to (t,x,\phi_x,\phi_{xx}) \\ p \neq 0}} H(t', x', K_\epsilon, p', A') \leq \phi_t.$$

Therefore, $\mathcal{K}$ satisfies the $\epsilon-$external condition.

4. In the same way, we can prove that $\mathcal{K}$ satisfies the $\epsilon-$internal condition. Since $\mathcal{K}$ satisfies the initial external and internal condition for K_0 by construction, we have proved that $\mathcal{K}$ is an $\epsilon-$solution to the front propagation problem on $[0, T)$ for some $T > 0$.

5. Now using Zorn Lemma, we can prove, as in the theory of o.d.e., that there is some maximal solution to the problem, i.e., a solution defined on some interval $[0, T^*)$, which cannot be extended on a larger interval. We claim that, actually, $T^* = +\infty$, so that $\mathcal{K}$ is an $\epsilon-$solution on $[0, +\infty)$.

Proof. Let $\mathcal{K}$ be a maximal solution, defined on $[0, T^*)$. Assume that, contrary to our claim, $T^* < +\infty$. We are going to extend $\mathcal{K}$ on a larger interval $[0, T^* + T)$ for some $T > 0$.

We first remark that $\mathcal{K}$ is uniformly bounded on this interval: this comes from the the growth assumption (17-iv) and Proposition 3.3.

Let us define $\mathcal{K}(T^*) = \text{Limsup}_{t \to T^*,\, t < T^*} \mathcal{K}(t)$. As in the first step, we can construct an $\epsilon-$solution $\mathcal{K}_1$, defined on $[T^*, T^* + T)$ with initial condition $\mathcal{K}(T^*)$. Let set

$$\mathcal{K}_2(t) \;=\; \begin{cases} \mathcal{K}(t) & \text{if } \; t \leq T^* \\ \mathcal{K}_1(t) & \text{if } \; t \in [T^*, T^* + T). \end{cases}$$

Note that $\mathcal{K}_2$ is well defined because $\mathcal{K}_1(T^*) = \mathcal{K}(T^*)$. Thanks to Corollaries 4.3 and 4.4, $\mathcal{K}_2$ satisfies the $\epsilon-$external and the $\epsilon-$internal conditions on

$[0, T^* + T)$ because so does $\mathcal{K}$ on $[0, T^*)$ and so does $\mathcal{K}_1$ on $(T^*, T^* + T)$. Moreover, $\mathcal{K}_2$ satisfies the initial condition for K_0 because so does $\mathcal{K}$. Therefore, we have constructed a strict extension of the maximal solution, which is impossible by definition. So $T^* = +\infty$. □

4.3. Existence of convex solutions. We now give an existence result for a front propagation problem without inclusion principle. As already mentioned, we have to restrict our study to convex evolutions. For simplicity, we assume that H does not depend on (t, x).

Theorem 4.7. *Assume that H does not depend on (t, x), that H satisfies assumptions* (17), (15) *and the convexity condition* (16). *Then, for any convex compact initial condition K_0, there is a convex solution to the front propagation problem.*

Remark. For instance $H(t, x, K, p, A) = Trace(A_{|_{p^\perp}}) + (\alpha + \beta|K|)\|p\|$ satisfies the previous assumptions.

Proof of the Theorem. 1. We claim that, from any convex compact initial condition K_0, for any $\epsilon > 0$, starts an $\epsilon-$solution with convex values.

Proof. The proof is exactly the same than that of Proposition 4.6. Note that we can choose these solutions with convex values thanks to Theorem 3.14 which states that under the assumptions on $(t, x, p, A) \to H(t, x, K, p, A)$, (for some fixed convex compact set K) the maximal solution is convex.

2. Let now, for any $\epsilon > 0$, $\mathcal{K}_\epsilon$ be such an $\epsilon-$solution with convex values starting from a convex compact set K_0. Since any $\mathcal{K}_\epsilon$ is in particular a $1-$solution, Lemma 3.13 states that $\mathcal{K}_\epsilon(t) \subset B(0, Re^{\int_0^t c(s)ds})$, where $R = \sup_{y \in K_0} \|y\|$. Therefore, the sets $\mathcal{K}_\epsilon^T = \mathcal{K}_\epsilon \cap ([0, T] \times \mathbb{R}^N)$ are uniformly bounded. Then, from using Ascoli Theorem on the distance functions, one can see easily that the sets $\mathcal{K}_\epsilon^T$ converge, for the Hausdorff topology, and up to a subsequence, to some compact set $\mathcal{K}$. Let us denote by $\mathcal{K}_n = \mathcal{K}_{\epsilon_n}^T$ some converging subsequence, associated with some ϵ_n. We are going to prove that $\mathcal{K}$ is convex valued and is a solution to the front propagation problem starting from K_0 on $[0, T)$.

3. We first prove that $\mathcal{K}$ is convex valued. Let $t \in [0, T]$ and $x_1, x_2 \in \mathcal{K}(t)$, $\lambda \in [0, 1]$. There are $(t_{1,n}, x_{1,n}) \in \mathcal{K}_n$, $(t_{2,n}, x_{2,n}) \in \mathcal{K}_n$ which converge respectively to (t, x_1) and (t, x_2). We assume, for instance, that $t_{1,n} \leq t_{2,n}$. Let us set $M = (\sup_{y \in K_0} \|y\|) e^{\int_0^t} c(s)ds$. From Lemma 3.13, we have, for any n, for any $0 \leq s \leq s'$, $\mathcal{K}_n(s') \subset \mathcal{K}_n(s) + M(e^{\int_s^{s'} c(u)du} - 1)B$. Therefore, there

are $y_{2,n} \in \mathcal{K}_n(t_{1,n})$ such that

$$\|y_{2,n} - x_{2,n}\| \leq M(e^{\int_{t_{1,n}}^{t_{2,n}} c(u)du} - 1) \ .$$

In particular, $y_{2,n}$ converges to x_2. Since $\mathcal{K}_n$ is convex valued, we have $\lambda x_{1,n} + (1-\lambda) y_{2,n} \in \mathcal{K}_n(t_{1,n})$. Letting $n \to +\infty$, we finally obtain $\lambda x_1 + (1-\lambda) x_2 \in \mathcal{K}(t)$. So $\mathcal{K}$ is convex valued.

4. Let us prove that $\mathcal{K}$ satisfies the external initial condition for K_0. Since $\mathcal{K}_\epsilon(0) = K_0$, we have $K_0 \subset \mathcal{K}(0)$. Let us prove the converse inclusion. From Lemma 3.2, for any $x_0 \notin K_0$, there is some $\gamma > 0$ such that, for any n, $d_{\mathcal{K}_n}(0, x_0) \geq \gamma$, because all the $\mathcal{K}_n$ are $1-$external solutions. Note that the constant $\gamma > 0$ can be chosen independent of n. Thus, letting $n \to +\infty$, we obtain $d_{\mathcal{K}}(0, x_0) \geq \gamma$, so that $x_0 \notin \mathcal{K}(0)$. Therefore, $\mathcal{K}(0) = K_0$.

5. We prove in the same way, using Lemma 3.4 instead of Lemma 3.2, that $\mathcal{K}$ satisfies the internal initial condition.

6. It remains to prove that $\mathcal{K}$ satisfies the external and internal conditions. For doing so, we need the following key result: For any $\epsilon > 0$, there is some $\gamma = \gamma(\epsilon) > 0$, such that, for any $t \in (0, T)$, for any $t_n \to t$, for any $s \in (t, t+\gamma)$, for any n sufficiently large, we have $\mathcal{K}_n(t) \subset \mathcal{K}(t) + \epsilon B$ and $\mathcal{K}(s) - 3\epsilon B \subset \mathcal{K}_n(t_n) - \epsilon B$.

Proof. The first inclusion is an obvious consequence of the convergence of $\mathcal{K}_n$ to $\mathcal{K}$. Let us prove the second inclusion. Let x which does not belong to $\mathcal{K}_{n_k}(t_{n_k}) - \epsilon B$ for some subsequence n_k with $n_k \to +\infty$. We are going to prove that x does not belong to $\mathcal{K}(s) - 3\epsilon B$ for any $s \in (t, t+\gamma)$ for some $\gamma > 0$ defined later. Since the $\mathcal{K}_{n_k}(t_{n_k})$ are convex sets, there are some y_k with $d_{\mathcal{K}_{n_k}(t_{n_k})}(y_k) \geq \epsilon$ and $\|y_k - x\| \leq 2\epsilon$. From Lemma 3.13, we have $\forall s \geq t_{n_k}$, $\mathcal{K}_n(s) \subset \mathcal{K}_n(t_n) + M(e^{\int_{t_n}^{s} c(u)du} - 1)B$. Choosing $\gamma > 0$ sufficiently small so that $M(e^{\int_\tau^s c(u)du} - 1)$ is smaller than $\epsilon/2$ for any $\tau \in [0, T]$ and any $s \in [\tau, \tau + 2\gamma]$, we obtain $\forall s \in [t_n, t_n + 2\gamma]$, $d_{\mathcal{K}_{n_k}(s)}(y_k) \geq \epsilon/2$, i.e.,

$$([t_n, t_n + 2\gamma] \times B(y_k, \epsilon/3)) \cap \mathcal{K}_{n_k} = \emptyset \ .$$

Since the sequence $\mathcal{K}_n$ converges to $\mathcal{K}$ for the Hausdorff topology, we have, letting $n \to +\infty$,

$$[(t, t+2\gamma) \times B(y, \epsilon/4)] \cap \mathcal{K} = \emptyset,$$

where y is the limit, up to a subsequence, of y_k. Since $\|y - x\| \leq 2\epsilon$, the point x does not belong to $\mathcal{K}(s+\gamma) - 3\epsilon B$ for $s \in (t, t+\gamma)$. So we have proved the desired result.

7. Let us now prove that the external condition is satisfied. Let ϕ have a strict maximum on $\mathcal{K}$ at some point $(t,x)\in\partial\mathcal{K}$ with $t\in(0,T)$.

Since the $\mathcal{K}_n$ converge to $\mathcal{K}$, there are (t_n,x_n) points of local maximum of ϕ on $\mathcal{K}_n$. Since the $\mathcal{K}_n$ are ϵ_n-solutions, we have

$$H_*^{\flat,\epsilon_n}(t_n,x_n,\mathcal{K}_n(t_n),\phi_x(t_n,x_n),\phi_{xx}(t_n,x_n))\leq\phi_t(t_n,x_n)\ .$$

Therefore, for any $n>0$, there are (s_n,s_n',p_n,A_n) with $p_n\neq 0$, such that

$$\|(s_n,s_n',y_n,p_n,A_n)-(t_n,t_n,x_n,\phi_x(t_n,x_n),\phi_{xx}(t_n,x_n))\|\leq\frac{1}{n}$$

and there are K_n' such that $\mathcal{K}_n(s_n')-\epsilon_nB\subset K_n'\subset\mathcal{K}_n(s_n)+\epsilon_nB$ and

$$H_*^{\flat,\epsilon_n}(t_n,x_n,\mathcal{K}_n(t_n),\phi_x(t_n,x_n),\phi_{xx}(t_n,x_n))\geq H(s_n,y_n,K_n',p_n,A_n)-\frac{1}{n}.$$

From step 6, there is some $\gamma>0$ such that, for n large enough, for any $s\in(t,t+\gamma)$, $\mathcal{K}(s)-3\epsilon B\subset K_n'\subset\mathcal{K}(t)+2\epsilon B$. So

$$\begin{aligned}&H_*(t,x,\mathcal{K}(t),\phi_x(t,x),\phi_{xx}(t,x))\leq\liminf_n H(s_n,y_n,K_n',p_n,A_n)\\&\leq\liminf_n H_*^{\flat,\epsilon_n}(t_n,x_n,\mathcal{K}_n(t_n),\phi_x(t_n,x_n),\phi_{xx}(t_n,x_n))\leq\phi_t(t,x).\end{aligned}$$

Therefore, $\mathcal{K}$ satisfies the external condition.

8. We now prove that $\mathcal{K}$ satisfies the internal condition. For doing so, we first prove that the limit of $\widehat{\mathcal{K}_n}$ is equal to $\widehat{\mathcal{K}}$. Let (t_n,x_n) belong to $\widehat{\mathcal{K}_n}$ and converge to some point (t,x). We have to prove that (t,x) belongs to $\widehat{K}$. Since $(t_n,x_n)\in\widehat{\mathcal{K}_n}$, for any $\epsilon>0$, we have $(t-\frac{1}{n},t+\frac{1}{n})\times B(x_n,\epsilon)\not\subset\mathcal{K}_n$. So there is some $s_n\in(t-\frac{1}{n},t+\frac{1}{n})$ such that $B(x_n,\epsilon)\not\subset\mathcal{K}_n(s_n)$. Note that $s_n\to t$. So, from step 6, there is some $\gamma>0$ such that, for n sufficiently large, for any $s\in(t,t+\gamma)$, we have $\mathcal{K}(s)-3\epsilon B\subset\mathcal{K}_n(s_n)-\epsilon B$. In particular, for n sufficiently large, $B(x_n,3\epsilon)\not\subset\mathcal{K}(s)$ for any $s\in(t,t+\gamma)$. In particular, for any $s\in(t,t+\gamma)$, $d_{\widehat{\mathcal{K}}}(s,x_n)\leq 3\epsilon$. Letting $n\to+\infty$, and $s\to t$ gives $d_{\widehat{\mathcal{K}}}(t,x)\leq 3\epsilon$. Finally, letting $\epsilon\to 0^+$, we finally obtain that (t,x) belongs to $\widehat{\mathcal{K}}$. So we have proved that the upper limit of the $\widehat{\mathcal{K}_n}$ is contained in $\widehat{\mathcal{K}}$.

We now prove that the lower limit of the $\widehat{\mathcal{K}_n}$ contains $\widehat{\mathcal{K}}$. Let (t,x) belong to $\widehat{\mathcal{K}}$. There is, for any $\epsilon>0$, some (s,y) with $(s,y)\notin\mathcal{K}$ and $\|(s,y)-(t,x)\|\leq\epsilon$. For n sufficiently large, (s,y) does not belong to $\mathcal{K}_n$ because $\mathcal{K}_n$ converges to $\mathcal{K}$. Therefore, (s,y) belongs to $\widehat{\mathcal{K}_n}$ for any n sufficiently large and, thus, (t,x) belongs to the lower limit of the $\widehat{\mathcal{K}_n}$.

In conclusion, the limit of $\widehat{\mathcal{K}_n}$ is equal to $\widehat{\mathcal{K}}$. Following the previous step, we prove exactly in the same way that $\mathcal{K}$ satisfies the internal condition on $(0,T)$.

9. Therefore, we have constructed a solution on the interval $[0,T)$. Using Corollaries 4.3 and 4.4, we can extend the solution on $[0,+\infty)$. $\Box$

4.4. Agreement with a smooth flow. We complete this paper by proving that if there is a solution to the front propagation problem which is smooth on some interval $[0,T]$, then any solution to the front propagation problem coincides with this solution on $[0,T]$.

For doing so, let us introduce some new assumptions on H.

(**H1**) We first assume that the non local term in H is Lipschitz continuous in a neighbourhood of smooth compact sets:

If K is a compact set with $\mathcal{C}^2$ boundary and if $M>0$, there are $k(K,M)$ and $\alpha(K,M)>0$ such that for any set K' with

$$\mathcal{H}(K,K')+\mathcal{H}(\widehat{K},\widehat{K'})\leq\alpha(K,M)\ ,$$

(where $\mathcal{H}$ denotes the Hausdorf distance) we have:

$$|H(t,x,K,p,A)-H(t,x,K',p,A)|\leq k(K,M)[\mathcal{H}(K,K')+\mathcal{H}(\widehat{K},\widehat{K'})] \quad (19)$$

for any (t,x,p,A) such that $\|(t,x)\|\leq M$, $\|p\|=1$, $\|A\|\leq M$. Moreover, the constants $k(K,M)$ and $\alpha(K,M)$ depend continuously on K for the $\mathcal{C}^2$ norm.

(**H2**) We also assume a Lipschitz dependence of H with respect to x and A: For any $M>0$, there some Lipschitz constant $\ell(M)$ such that

$$|H(t,x,K,p,A)-H(t,x',K,p,A')|\leq\ell(M)\left[\|x-x'\|+\|A-A'\|\right] \quad (20)$$

for any (t,x,x',K,p,A,A') such that $\|(t,x,x')\|\leq M$, $\|p\|=1$, $\|A,A'\|\leq M$, $\sup_{y\in K}\|y\|\leq M$.

Remark. If H is of the form (18), then H satisfies (**H1**) and (**H2**).

Theorem 4.8. *Assume that H satisfies assumptions* (17), (**H1**) *and* (**H2**). *Let K_0 be a compact subset of $\mathbb{R}^N$ with $\mathcal{C}^3$ boundary and $\mathcal{K}_r$ be a solution to the front propagation starting from K_0. If $\mathcal{K}_r$ is a classical solution* (*with a $\mathcal{C}^3$ regularity*) *to the front propagation problem on $[0,T]$ for some $T>0$, then any other solution $\mathcal{K}$ to the front propagation problem starting from K_0 coincides with $\mathcal{K}_r$ on $[0,T]$ in the following sense*: $\forall t\geq 0$, $\partial\mathcal{K}(t)=\partial\mathcal{K}_r(t)$.

Remark. We could also prove, in the same way, that the $\epsilon-$solutions converge to the regular solution as $\epsilon \to 0^+$.

Proof. 1. Let $\mathcal{K}$ be another solution to the front propagation problem and $\phi(t,x)$ be the signed distance to the regular front:

$$\phi(t,x) = \begin{cases} d_{\mathcal{K}_r(t)}(x) & \text{if } x \notin \mathcal{K}_r(t) \\ -d_{\partial\mathcal{K}_r(t)}(x) & \text{otherwise.} \end{cases}$$

Let us consider $d_1(t) = \sup_{x\in\mathcal{K}(t)} \phi(t,x)$ and $d_2(t) = \sup_{x\in\widehat{\mathcal{K}}(t)} -\phi(t,x)$. For proving that $\partial\mathcal{K}(t)$ coincides with $\partial\mathcal{K}_r(t)$ on $[0,T]$, it is enough to verify that d_1 and d_2 are non positive on $[0,T]$.

2. As in the proof of Theorem 3.1, estimates of the variations of d_1 and d_2 are difficult to get. Therefore, we introduce, for $\epsilon > 0$,

$$d_1^\epsilon(t) = \sup_{(s,y)\in\overline{\mathcal{K}}} \big(\phi(s,y) - \frac{1}{\epsilon}(s-t)^2\big)$$

$$d_2^\epsilon(t) = \sup_{(s,y)\in\widehat{\mathcal{K}}} \big(-\phi(s,y) - \frac{1}{\epsilon}(s-t)^2\big).$$

We also set $d^\epsilon(t) = \sup\{0, d_1^\epsilon(t), d_2^\epsilon(t)\}$. Using standard arguments in perturbation methods, we can easily prove that d_1^ϵ and d_2^ϵ converge to d_1 and d_2 as ϵ tends to 0. Moreover, since $\overline{\mathcal{K}}(0) = \mathcal{K}_r(0)$, for any fixed $\gamma > 0$, there is some $\bar{\epsilon} = \bar{\epsilon}(\gamma) > 0$ such that, for any $\epsilon \in (0,\bar{\epsilon})$, for any $t \in (0, T-\gamma)$, we have: If (s,y) is a maximum for $d_1^\epsilon(t)$ with $d_1^\epsilon(t) > 0$ (resp. d_2^ϵ with $d_2^\epsilon(t) > 0$), then $s \in (0,T)$ and $\frac{1}{\epsilon}(s-t) \le \gamma/2$.

3. Since $\mathcal{K}_r$ is a smooth solution, the map ϕ is $\mathcal{C}^3$ in a neighbourhood of the boundary of $\mathcal{K}_r$: there is some $\alpha_1 > 0$ such that ϕ is $\mathcal{C}^3$ on

$$\mathcal{O} = \{(t,x) \in (0,T)\times\mathbb{R}^N \mid |\phi(t,x)| < \alpha_1\} .$$

Let us also set

$$M = \sup\{\|(t,x)\|,\ |\phi_t(t,x)|\ ,\ \|\phi_{xx}(t,x)\|\ , \|\phi_{xxx}(t,x)\|\ ,\ \text{for } (t,x)\in\mathcal{O}\} .$$

Since $\mathcal{K}_r(t)$ depends continuously on t for the $\mathcal{C}^2$ norm, we can define

$$k = \sup_{t\in[0,T]} k(\mathcal{K}_r(t), M) < +\infty \text{ and } \alpha_2 = \inf_{t\in[0,T]} \alpha(\mathcal{K}_r(t), M) > 0.$$

We finally set $\alpha = \inf\{\alpha_1, \alpha_2\}$. For later use, we fix $\gamma \in (0, \inf\{\frac{\alpha}{8}, \frac{\alpha}{8(e^{k'T}-1)}\})$ and $\epsilon_0 \in (0, \bar{\epsilon}(\gamma))$ in such a way that, for any $\epsilon \in (0,\epsilon_0)$, we have $d^\epsilon(0) \le$

$\frac{\alpha}{8}e^{-k'T}$, where $k' = \ell M + \ell + k$. This choice is possible because $d_1^\epsilon(0) \to d_1(0) = 0$ and $d_2^\epsilon(0) \to d_2(0) = 0$ since $\overline{\mathcal{K}}(0) = \mathcal{K}_r(0)$.

4. Note that d^ϵ as well as d_1^ϵ and d_2^ϵ are absolutely continuous. We are going to estimate the derivative of d^ϵ for $t \in (\gamma, T^\epsilon)$ where $T^\epsilon = T^\epsilon(\gamma)$ is defined by $T^\epsilon = \sup\{t \in [0, T-\gamma] : \forall s \in [0,t],\ d^\epsilon(s) < \alpha/4\}$. We have, for almost every t,

$$\frac{d}{dt}d^\epsilon(t) \leq \sup\{0,\ \frac{d}{dt}d_1^\epsilon(t),\ \frac{d}{dt}d_2^\epsilon(t)\}.$$

5. Let us start with an estimation of $\frac{d}{dt}d_1^\epsilon(t)$. Let $t \in (0, T^\epsilon)$ be a point where d_1^ϵ is derivable, and $(s,y) \in \overline{\mathcal{K}}$ be a point where the maximum is reached for $d_1^\epsilon(t)$. Then

$$\frac{d}{dt}d_1^\epsilon(t) = \frac{2}{\epsilon}(s-t)\ .$$

Since the smooth test function $\psi(\sigma, z) = \phi(\sigma, z) - \frac{1}{\epsilon}(\sigma - t)^2$ has a maximum on $\overline{\mathcal{K}}$ at (s,y), we have

$$H_*(s, y, \mathcal{K}(s), \phi_x(s,y), \phi_{xx}(s,y)) \leq \phi_t(s,y) - \frac{2}{\epsilon}(s-t).$$

Therefore,

$$\frac{d}{dt}d_1^\epsilon(t) \leq \phi_t(s,y) - H_*(s, y, \mathcal{K}(s), \phi_x(s,y), \phi_{xx}(s,y))\ .$$

6. Before going further, let us recall some known facts about the distance function. Let $(s,y) \notin \mathcal{K}_r$ and x be the projection of y on $\mathcal{K}_r(s)$. Then, we have

$$\phi_t(s,y) = \phi_t(s,x),\ \phi_x(s,y) = \phi_x(s,x) = \frac{y-x}{\|y-x\|}$$

and, since $\|\phi_{xxx}\| \leq M$,

$$\|\phi_{xx}(s,y) - \phi_{xx}(s,x)\| \leq M\|y-x\|\ .$$

So we have, thanks to assumption (**H2**),

$$\frac{d}{dt}d_1^\epsilon(t) \leq \phi_t(s,x) - H_*(s, x, \mathcal{K}(s), \phi_x(s,x), \phi_{xx}(s,x)) + (M+1)\ell\|y-x\| \tag{21}$$

where x is a projection of y onto $\mathcal{K}_r(s)$.

7. Our objective is now to compare $H_*(s,x,\mathcal{K}(s),\phi_x(s,x),\phi_{xx}(s,x))$ and $H_*(s,x,\mathcal{K}_r(s),\phi_x(s,x),\phi_{xx}(s,x))$ and to estimate $\|y-x\|$. For doing so, we wish to apply Lemma 5.4 in appendix.

We have to estimate the distance between $\mathcal{K}(s')$ and $\mathcal{K}_r(s')$ for s' close to s. From the very definition of $d_1^\epsilon(t)$, we have $\forall\sigma>0$, $\forall z\in\overline{\mathcal{K}}(\sigma)$, $d_{\mathcal{K}_r(\sigma)}(z)-\frac{1}{\epsilon}(\sigma-t)^2\le d_1^\epsilon(t)$, i.e., $\forall\sigma>0$, $\overline{\mathcal{K}}(\sigma)\subset\mathcal{K}_r(\sigma)+(d_1^\epsilon(t)+\frac{1}{\epsilon}(\sigma-t)^2)B$.

Let us now recall that $\frac{1}{\epsilon}(s-t)^2\le\gamma/2$. Thus we can find $\eta\in(0,1)$ such that $\forall\sigma\in(s-\eta,s+\eta)$, $\frac{1}{\epsilon}(\sigma-t)^2+M\eta\le\gamma$. Note that $\forall\sigma\in(s-\eta,s+\eta)$, $\mathcal{K}_r(\sigma)\subset\mathcal{K}_r(s)+M|\sigma-s|B\subset\mathcal{K}_r(s)+M\eta B$, because $|\phi_t|\le M$. Thus, for any $\sigma\in(s-\eta,s+\eta)$, we obtain $\overline{\mathcal{K}}(\sigma)\subset\mathcal{K}_r(s)+(d^\epsilon(t)+\gamma)B$, because $d_1^\epsilon\le d^\epsilon$. In particular, with $\sigma=s$, we have

$$\|y-x\|\le d^\epsilon(t)+\gamma, \tag{22}$$

because $y\in\overline{\mathcal{K}}(s)$ and x is the projection of x on $\mathcal{K}_r(s)$. Using the same argument for $d_2^\epsilon(t)$, we obtain: for any $\sigma\in(s-\eta,s+\eta)$ $\widehat{\mathcal{K}}(\sigma)\subset\widehat{\mathcal{K}_r}(s)+(d^\epsilon(t)+\gamma)B$. Note also that $\widehat{\mathcal{K}}(s)=\widehat{\mathcal{K}(s)}$ because $\mathcal{K}_r$ is smooth.

8. From the definition of H_*, assumption (17-ii) and the fact that $\phi_x(t,x)\neq 0$, there are $s_n\to s$, $\epsilon_n\to 0^+$ and compact subsets K_n of $\mathbb{R}^N$ with $\mathcal{K}(s_n)-\epsilon_nB\subset K_n\subset\mathcal{K}(s)+\epsilon_nB$ and

$$H_*(s,x,\mathcal{K}(s),\phi_x(s,x),\phi_{xx}(s,x))\ge H(s,x,K_n,\phi_x(s,x),\phi_{xx}(s,x))-\frac{1}{n}\,.$$

We have proved, on the one hand, that, for any $\sigma\in(s-\eta,s+\eta)$, $\overline{\mathcal{K}}(\sigma)\subset\mathcal{K}_r(s)+(d^\epsilon(t)+\gamma)B$ and $\widehat{\mathcal{K}}(\sigma)\subset\widehat{\mathcal{K}_r(s)}+(d^\epsilon(t)+\gamma)B$. On another hand, we have $d^\epsilon(t)\le\alpha/4$ because $t\in(0,T^\epsilon)$ and $\gamma\le\alpha/8$. So, for ϵ_n sufficiently small, $d^\epsilon(t)+\gamma+\epsilon_n<\alpha/2$. We have chosen α in such a way that $\mathcal{K}_r(s)$ is smooth on $|d_{\mathcal{K}_r(s)}|\le\alpha$. So we are now ready to use Lemma 5.4 in appendix: since $\mathcal{K}(s_n)-\epsilon_nB\subset K_n\subset\mathcal{K}(s)+\epsilon_nB$ we have $\mathcal{H}(K_n,\mathcal{K}_r(s))\le d^\epsilon(t)+\gamma+\epsilon_n$ and $\mathcal{H}(\widehat{K_n},\widehat{\mathcal{K}_r}(s))\le d^\epsilon(t)+\gamma+\epsilon_n$. Thus,

$$\mathcal{H}(K_n,\mathcal{K}_r(s))+\mathcal{H}(\widehat{K_n},\widehat{\mathcal{K}_r}(s))\le\alpha.$$

So we are in position to apply assumption **H1**:

$$\begin{aligned}&H(s,x,K_n,\phi_x(s,x),\phi_{xx}(s,x))\\&\quad\ge H(s,x,\mathcal{K}_r(t),\phi_x(s,x),\phi_{xx}(s,x))-k(d^\epsilon(t)+\gamma+\epsilon_n)\end{aligned}$$

Letting $n \to +\infty$, we have

$$\begin{aligned} & H_*(s,x,\mathcal{K}(s),\phi_x(s,x),\phi_{xx}(s,x)) \\ & \quad \geq H(s,x,\mathcal{K}_r(t),\phi_x(s,x),\phi_{xx}(s,x)) - k(d^\epsilon(t)+\gamma). \end{aligned}$$

Using (21) and (22), we get

$$\frac{d}{dt}d_1^\epsilon(t) \leq \phi_t(s,x) - H(s,x,\mathcal{K}_r(s),\phi_x(s,x),\phi_{xx}(s,x)) + k'(d^\epsilon(t)+\gamma),$$

where $k' = k+\ell+M\ell$. Since $\mathcal{K}_r$ is a regular solution to the front propagation problem, we have $\phi_t(s,x) = H(s,x,\mathcal{K}_r(s),\phi_x(s,x),\phi_{xx}(s,x))$. So we have proved that

$$\frac{d}{dt}d_1^\epsilon(t) \leq k'(d^\epsilon(t)+\gamma)\ .$$

9. Using the same arguments for $d_2^\epsilon(t)$ gives:

$$\frac{d}{dt}d_2^\epsilon(t) \leq k'(d^\epsilon(t)+\gamma)\ .$$

Therefore,

$$\frac{d}{dt}d^\epsilon(t) \leq k'(d^\epsilon(t)+\gamma)$$

for almost every $t \in (0,T^\epsilon)$. Thus $d^\epsilon(t) \leq d^\epsilon(0)e^{k't} + \gamma(e^{k't}-1)$. From the very choice of γ and of $\epsilon \in (0,\epsilon_0)$ (see step (3)), we have for any $t \in [0,T^*]$,

$$d^\epsilon(t) \leq d^\epsilon(0)e^{k'T} + \gamma(e^{k'T}-1) \ \leq \ \frac{\alpha}{4}\ .$$

Therefore, from the definition of T^*, we have $T^* = T-\gamma$. Letting $\epsilon \to 0^+$ gives $\forall t \in [0,T-\gamma]$, $d(t) \leq \gamma(e^{k't}-1)$ where $d(t) = \sup\{0, d_1(t), d_2(t)\}$. Now, since γ is arbitrary, we can let $\gamma \to 0^+$, and thus $\forall t \in [0,T]$, $d(t) = 0$ which completes the proof. □

5. Appendix.

Lemma 5.1. *Let $f : \mathbb{R}^N \to \mathbb{R}$ and $g : \mathbb{R}^N \to \mathbb{R}$ be two $\mathcal{C}^2$ functions. Assume that x is a solution of the problem*

$$(\mathcal{P}) \qquad \max_{\{y \mid g(y)\geq 0\}} f(y)$$

such that $g(x) = 0$ and that $\nabla g(x) \neq 0$. Then there is some $\lambda \geq 0$ such that

$$\nabla f(x) + \lambda \nabla g(x) = 0$$

and

$$(I-\frac{vv^t}{\|v\|^2})\nabla^2 f(x)(I-\frac{vv^t}{\|v\|^2}) \le -\lambda(I-\frac{vv^t}{\|v\|^2})\nabla^2 g(x)(I-\frac{vv^t}{\|v\|^2}),$$

where $v=\nabla g(x)$ *and where* v^t *denotes the transpose of* v.

Proof. A necessary condition for x to be a solution of the problem $(\mathcal{P})$ is the existence of $\lambda \ge 0$ such that $\nabla f(x)+\lambda g'(x)=0$. Since $f'(x)\neq 0$ from assumption, $\lambda>0$. Let now $a\in\mathbb{R}^N$ be such that $<a,\nabla g(x)>=0$. Let us also consider

$$b_\epsilon = -(\frac{1}{2}\nabla^2 g(x)(a,a)-\epsilon)\frac{\nabla g(x)}{\|\nabla g(x)\|^2}.$$

Then it is easy to check that $g(x+ha+h^2b_\epsilon)>0$ for $h>0$ sufficiently small, so that $f(x+ha+h^2b_\epsilon)\le f(x)$ for $h>0$ sufficiently small. Thus,

$$<\nabla f(x), b_\epsilon> + \frac{1}{2}\nabla^2 f(x)(a,a)\le 0$$

because $\nabla f(x)(a)=0$. From the very definition of b_ϵ, we conclude that

$$a\perp \nabla g(x) \quad \Rightarrow \quad \lambda(\nabla^2 g(x)(a,a)-\epsilon)+\nabla^2 f(x)(a,a)\le 0.$$

Letting $\epsilon\to 0^+$ yields Lemma 5.1. □

Lemma 5.2. *Let* $\mathcal{K}_n$ *be a non increasing sequence of closed tubes and let us set* $\mathcal{K}=\bigcap_n \mathcal{K}_n$. *Let* (t_n,x_n,p_n,A_n) *converge to some* (t,x,p,A). *Then*

$$\liminf_n H_*(t_n,x_n,\mathcal{K}_n(t_n),p_n,A_n)\ge H_*(t,x,\mathcal{K}(t),p,A),$$

$$\limsup_n H^*(t_n,x_n,\mathcal{K}_n(t_n),p_n,A_n)\le H^*(t,x,\mathcal{K}(t),p,A).$$

Proof of Lemma 5.2. 1. We do the proof for H_*, the case of H^* being similar. Without loss of generality, we assume that the lower limit is actually a limit.

2. Let us fix $\epsilon>0$. From the very definition of H_*, there are sequences $(t_n',t_n'',x_n',p_n',A_n')$ such that

$$\|(t_n',t_n'',x_n',p_n',A_n')-(t_n,t_n,x_n,p_n,A_n)\|\le\frac{1}{n}$$

and $p_n'\neq 0$ and there is a sequence K_n of compact sets such that $\mathcal{K}_n(t_n'')-(\epsilon/2)B\subset K_n\subset\mathcal{K}_n(t_n)+(\epsilon/2)B$ and

$$H_*(t_n,x_n,\mathcal{K}_n(t_n),p_n,A_n)\ge H(t_n',x_n',K_n,p_n',A_n')-\frac{1}{n}.$$

3. Let us now prove that, if n is sufficiently large, then $\mathcal{K}(t_n'')-\epsilon B\subset K_n\subset\mathcal{K}(t)+\epsilon B$. Indeed, the upper-limit of $\mathcal{K}_n(t_n)$ is a subset of $\mathcal{K}(t)$ because t_n tends to

t and $\mathcal{K}_n$ to $\mathcal{K}$. Therefore, for n sufficiently large, we have $\mathcal{K}_n(t_n) \subset \mathcal{K}(t) + \frac{\epsilon}{2}B$. On another hand, we have $\mathcal{K}(t''_n) \subset \mathcal{K}_n(t''_n)$ because the sequence $\mathcal{K}_n$ is non increasing. Thus, for n sufficiently large, we have $\mathcal{K}(t''_n) - \epsilon B \subset K_n \subset \mathcal{K}(t) + \epsilon B$.

4. We have constructed a sequence $(t'_n, t''_n, x'_n, p'_n, A'_n)$ which converges to (t, t, x, p, A) and a sequence K_n such that $\mathcal{K}(t''_n) - \epsilon B \subset K_n \subset \mathcal{K}(t) + \epsilon B$ with the following property:

$$\liminf_n H(t'_n, x'_n, K_n, p'_n, A'_n) \ \leq\ \lim_n H_*(t_n, x_n, \mathcal{K}(t_n), p_n, A_n)\ .$$

If we let $\epsilon \to 0^+$, we obtain

$$\begin{array}{rcl} H_*(t, x, \mathcal{K}(t), p, A) & \leq & \liminf_{\epsilon\to 0^+} \liminf_n H(t'_n, x'_n, K_n, p'_n, A'_n) \\ & \leq & \lim_n H_*(t_n, x_n, \mathcal{K}(t_n), p_n, A_n) \end{array}$$

which is the desired result. □

We can prove in the same way:

Lemma 5.3. *Let $\mathcal{K}_n$ be a non decreasing sequence of tubes and let us set $\mathcal{K} = \bigcup_n \mathcal{K}_n$. Let us assume that $\mathcal{K}$ is also a tube. Let (t_n, x_n, p_n, A_n) converge to some (t, x, p, A). Then*

$$\liminf_n H_*(t_n, x_n, \mathcal{K}_n(t_n), p_n, A_n) \geq H_*(t, x, \mathcal{K}(t), p, A),$$

$$\limsup_n H^*(t_n, x_n, \mathcal{K}_n(t_n), p_n, A_n) \leq H^*(t, x, \mathcal{K}(t), p, A).$$

Lemma 5.4. *Let C be a compact subset with $\mathcal{C}^2$ boundary and $\sigma > 0$ be such that the signed distance function d_C is smooth on $\{x \in \mathbb{R}^N : |d_C(x)| < \sigma\}$. Let $\epsilon \in (0, \sigma)$ and $\mathcal{K}$ be a tube defined on some open interval I and such that, $\forall t \in I$, $\mathcal{K}(t) \subset C + \epsilon B$ and $\widehat{\mathcal{K}}(t) \subset \widehat{C} + \epsilon B$. Then, for any $(s, t) \in I$, for any $\theta > 0$ with $\epsilon + \theta < \sigma$, for any compact set K such that $\mathcal{K}(s) - \theta B \subset K \subset \mathcal{K}(t) + \theta B$, we have $\mathcal{H}(K, C) \leq \epsilon + \theta$ and $\mathcal{H}(\widehat{K}, \widehat{C}) \leq \epsilon + \theta$.*

Proof of the Lemma. 1. We claim that $K \subset C + (\epsilon + \theta)B$. Indeed, $K \subset \mathcal{K}(t) + \theta B$ and $\mathcal{K}(t) \subset C + \epsilon B$. Therefore, $K \subset C + (\epsilon + \theta)B$.

2. We claim that $C \subset K + (\epsilon + \theta)B$.

Proof. Let $x \in C$. Recall that d_C is smooth on $\{y \mid |d_C(y)| \leq \sigma\}$ and that $\epsilon + \theta < \sigma$. Thus, for any $\theta' > \theta$, we have $B(x, \epsilon + \theta') \not\subset \widehat{C} + (\epsilon + \theta)B$, because $x \in C$. Since $\widehat{\mathcal{K}}(s) \subset \widehat{C} + \epsilon B$, we have $B(x, \epsilon + \theta') \not\subset \widehat{\mathcal{K}}(s) + \theta B$. Thus, $B(x, \epsilon + \theta') \cap [\mathcal{K}(s) - \theta B] \neq \emptyset$, which implies that $B(x, \epsilon + \theta') \cap K \neq \emptyset$. Therefore, we have $x \in K + (\epsilon + \theta)B$.

3. We claim that $\widehat{K} \subset \widehat{C} + (\epsilon + \theta)B$.

Proof. Let $x \in \widehat{K}$. Since $x \in \widehat{K}$, x does not belong to the interior of K. Thus, for any $\gamma > 0$, $B(x, \gamma) \not\subset K$. Since $\mathcal{K}(s) - \theta B \subset K$, we have $B(x, \gamma) \not\subset \mathcal{K}(s) - \theta B$ which means that $B(x, \gamma + \theta) \not\subset \mathcal{K}(s)$. Thus, since $\mathbb{R}^N \backslash \mathcal{K}(s) \subset \widehat{\mathcal{K}}(s)$, we have

$B(x,\gamma+\theta)\cap\widehat{\mathcal{K}}(s)\neq\emptyset$. But $\widehat{\mathcal{K}}(s)\subset\widehat{C}+\epsilon B$, and so $B(x,\gamma+\theta)\cap[\widehat{C}+\epsilon B]\neq\emptyset$. Therefore, $d_{\widehat{C}}(x)\leq\gamma+\theta+\epsilon$ because C is smooth, and, since γ is arbitrary, we have the desired result: $x\in\widehat{C}+(\epsilon+\theta)B$.

4. Let us finally prove that $\widehat{C}\subset\widehat{K}+(\epsilon+\theta)B$.

Proof. Let $x\in\widehat{C}$. Since x does not belong to the interior of C and since d_C is smooth on $\{|d_C(y)|<\sigma\}$, where $\theta+\epsilon<\sigma$, for any $\theta'>\theta$, the ball $B(x,\theta'+\epsilon)$ is not contained in $C+(\theta+\epsilon)B$. Thus, $B(x,\theta'+\epsilon)\not\subset\mathcal{K}(t)+\theta B$, because $\mathcal{K}(t)\subset K+\epsilon B$. Therefore, $B(x,\theta'+\epsilon)\not\subset K$, because $K\subset\mathcal{K}(t)+\theta B$. Thus, $B(x,\theta'+\epsilon)\cap\widehat{K}\neq\emptyset$, i.e., $x\in\widehat{K}+(\epsilon+\theta')B$. Since $\theta'>\theta$ is arbitrary, we have the desired result $x\in\widehat{K}+(\epsilon+\theta)B$.

REFERENCES

[1] S.M. Allen and J.W Cahn, *A macroscopic theory for antiphase boundary and its application to antiphase domain coarsening*, Acta Metal., 27 (1979), 1085-1095.

[2] L. Alvarez, E. Guichard, P.-L. Lions, and J.-M. Morel, *Axioms and fundamental equations of image processing*, Arch. Rat. Mec., 123(3) (1993), 199-257.

[3] L. Ambrosio, "Lecture Notes on Geometric Evolution Problems, Distance Function and Viscosity Solutions", preprint, 1997.

[4] J.-P. Aubin and H. Frankowska, "Set-Valued Analysis", Birkäuser, Boston, 1990.

[5] J.-P. Aubin, "Mutational and Morphological Analysis. Tools for Shape Evolution and Morphogenesis", Birkäuser, 1998.

[6] G. Barles, *Remarks on a flame propagation model*, I.N.R.I.A., Rapport 464 (1985).

[7] G. Barles, H.M. Soner, and P.E. Souganidis, *Front propagation and phase field theory*, SIAM J. Control and Opti., 31(2) (1993), 439-469.

[8] K. Brakke, "The Motion of a Surface by its Mean Curvature", Princeton University Press, Princeton, N.J, 1978.

[9] L. Bronsard and R. Kohn, *Motion by mean curvature as the singular limit of Ginzburg-Landau model*, J. Differential Equations, 90 (1991), 211-237.

[10] P. Cardaliaguet and D. Pasquignon, (in preparation).

[11] Y.-G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom., 33 (1991), 749-786.

[12] X. Chen., D. Hilhorst, and E. Logak, *Asymptotic behavior of solutions of an Allen-Cahn equation with a nonlocal term*, Nonlinear Anal., T.M.A., 28 (1997), 1283-1298.

[13] M. Crandall, H. Ishii and, P.-L. Lions, *User's guide to viscosity solution of second order partial differential equations*, Bulletin of the American Mathematical Society, 27 (1992), 1-67.

[14] E. De Giorgi, *Some conjectures on flows by mean curvature*, Proc. Capri Workshop, (1990), Benevento, Bruno, and Sbardone eds.

[15] E. De Giorgi, *Barrier, boundaries, motion of manifolds*, Preliminary draft, (1994), Pavia.

[16] M. Delfour and J.-P. Zolesio, *Shape analysis via oriented distance function*, J. Funct. Anal., 123 (1994), 129-201.

[17] L. Doyen, *Filippov and Invariance Theorems for mutational inclusion for tubes*, Set-Valued Anal., 1 (1993), 289-303.

[18] L.C. Evans, H.M. Soner and P.E. Souganidis, *Phase transition and motion by mean curvature*, Comm. Pure Appl. Math., 45 (1992), 1097-1123.

[19] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature I*, J. Differential Geom., 33 (1991), 635-681.

[20] L.C. Evans and J. Spruck, *Motion of level sets by mean curvature II*, Trans. AMS, 330 (1992), 321-332.

[21] M. Gage, *An isoperimetric inequality with application to curve shortening*, Duke Math. J., 50 (1983), 1225-1229.

[22] M. Gage, *Curve shortening makes convex curves circular*, Inventiones Math., 76 (1984), 357-364.

[23] M. Gage and R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom., 23 (1986), 69-96.

[24] Y. Giga, S. Goto, H. Ishii and M.-H. Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J., 40 (1990), 443-470.

[25] M. Grayson, *The heat equation shricks embeded plane curves to round points*, J. Differential Geom., 26 (1987), 285-314.

[26] R.L. Jerrard, *Fully nonlinear phase field equations and generalized mean curvature motion*, Comm. Partial Diff. Eq., 20 (1995), 233-265.

[27] M.A. Katsoulakis and P. E. Souganidis, *Generalized motion by mean curvature as a macroscopic limit of stochastic Ising models with long range interactions and Glauber dynamics*, Comm. Mth. Phys., 169 (1995), 61-97.

[28] A.B. Kurzhanski and T.F. Fillipova, *On the description of the set of viable trajectories of a differential inclusion*, Soviet. Dokl., 34 (1987), 30-33.

[29] S. Osher and J.A. Sethian, *Front propagation with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comp.Phys., 79 (1988), 12-49.

[30] D. Pasquignon, *Computation of skeleton by p.d.e.*, IEEE-ICIP (1995), Washington.

[31] H.M. Soner, *Motion of a set by the mean curvature of its boundary*, J. Diff. Equations, 101 (1993), 313-372.