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ON AXISYMMETRIC SOLUTIONS OF THE CONFORMAL SCALAR CURVATURE EQUATION ON S^n

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1. Introduction. Consider the sphere $S^n = \{(y_1, \dots, y_{n+1}) : y_1^2 + \dots + y_{n+1}^2 = 1\}$ with the standard metric $ds_0^2 = dy_1^2 + \dots + dy_{n+1}^2$. For $n \geq 3$, we consider the following problem of prescribing scalar curvature on S^n: given a smooth function K on S^n, we want to find a metric ds^2 conformal to ds_0^2 such that the given function K is the scalar curvature of the new metric ds^2. If we write $ds^2 = v^{\frac{4}{n-2}}\, ds_0^2$, then the problem of prescribing scalar curvature on S^n is equivalent to finding a positive smooth solution v of

$$\Delta_{g_0} v - \frac{n(n-2)}{4} v + \frac{n-2}{4(n-1)} K(y) v^{\frac{n+2}{n-2}} = 0 \quad \text{on } S^n , \tag{1.1}$$

where Δ_{g_0} is the Beltrami-Laplace operator associated with ds_0^2. By using $x \in \mathbf{R}^n$ to denote the coordinate of the stereographic projection of a point y on S^n and $u(x) = v(y)(\frac{2}{1+|x|^2})^{\frac{n-2}{2}}$, equation (1.1) reduces to

$$\begin{cases} \Delta u(x) + K(x) u^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbf{R}^n, \\ u(x) = O(|x|^{2-n}) & \text{at } \infty \end{cases} \tag{1.2}$$

after an appropriate scaling. Recently, there have been many works denoted to this problem. We refer interested readers to [1], [2], [3], [9], [10], [11],...,

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and references therein. In particular, we would like to mention that the degree theory, developed by Chang-Gursky-Yang [4] and Li [9], is very useful in finding solutions of (1.1). To apply this degree theory, we need to find a apriori bound for all solutions of (1.1). Assume that u_i is a sequence of solutions on S^n such that $u_i(P_i) \longrightarrow +\infty$ for a sequence $P_i \in S^n$. Let $P_0 = \lim_{i\longrightarrow+\infty} P_i$. Li [9] has shown that P_0 must be a critical point of K. The authors also gave another proof for this fact in [5]. In fact, [5] considers local solutions u_i of (1.1), i.e., u_i are solutions of

$$\Delta u_i + K(x)u_i^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad B_1, \tag{1.3}$$

where B_1 is the unit ball with center 0. Assume $u_i(x_i) \longrightarrow +\infty$ and $|x_i| \le \frac{1}{2}$. Then we proved that the limiting point $x_0 = \lim_{i\longrightarrow+\infty} x_i$ must be a critical point of K. Thus, it is an interesting question how the property of K near its critical points affects the blow-up behavior of a sequence of solutions of (1.3). Assume that 0 is the only critical point of K which satisfies

$$K(x) = K(0) + Q(x) + R(x) \tag{1.4}$$

in a neighborhood of 0, where $Q(x)$ is homogeneous with degree σ and satisfies $c_1|x|^{\sigma-1} \le |\bigtriangledown Q(x)| \le c_2|x|^{\sigma-1}$, and where both $|R(x)||x|^{-\sigma}$ and $|\bigtriangledown R(x)||x|^{-\sigma+1}$ tend to 0 as $|x| \longrightarrow 0$. Let $Q(x)$ satisfy

$$\begin{pmatrix} \int_{\mathbf{R}^n} \bigtriangledown Q(x+\xi)U_0^{\frac{2n}{n-2}}(x)dx \\ \int_{\mathbf{R}^n} Q(x+\xi)U_0^{\frac{2n}{n-2}}(x)dx \end{pmatrix} \ne 0 \quad \text{for all} \quad \xi \in \mathbf{R}^n, \tag{1.5}$$

where U_0 is the positive smooth solution of

$$\begin{cases} \Delta U_0(x) + K(0)U_0^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbf{R}^n, \\ U_0(0) = \max_{\mathbf{R}^n} U_0(x) = 1. \end{cases} \tag{1.6}$$

It was proved in [6] that

Theorem A. *Suppose u_i is a sequence of C^2 solutions of (1.3) and K satisfies the assumptions* (1.4) *and* (1.5) *above with* $1 < \sigma < n$. *Assume that u_i is uniformly bounded in any compact set of $\bar{B}_1\backslash\{0\}$. Then the followings hold.*

(i) If $1 < \sigma < n-2$ and Q satisfies $\int_{\mathbf{R}^n} Q(\xi+x)U_0^{\frac{2n}{n-2}}(x)dx > 0$ whenever $\int_{\mathbf{R}^n} \bigtriangledown Q(\xi+x)U_0^{\frac{2n}{n-2}}(x)dx = 0$ and $\xi \in \mathbf{R}^n$, then u_i is uniformly bounded in $\overline{B_1}$.

(*ii*) *If* $M_i = \max_{\bar{B}_1} u_i \longrightarrow +\infty$ *as* $i \longrightarrow +\infty$ *and* $\frac{n-2}{2} \le \sigma < n$, *then* $m_i = \min_{\bar{B}_{\frac{1}{2}}} u_i \sim M_i^{1-\frac{2\hat{\sigma}}{n-2}}$, *where* $\hat{\sigma} = \min(\sigma, n-2)$ *and* $a_i \sim b_i$ *denotes* a_i/b_i *is bounded between two positive constants independent of* i. *Moreover, if* $u_i(x_i) = M_i$, *then* $\max_{r_i \le |x| \le \frac{1}{2}} u_i(x + x_i) \sim m_i$, *and* $u_i(x + x_i) \sim M_i U_0(M_i^{\frac{2}{n-2}} x)$ *for* $|x| \le r_i$, *where* $r_i \sim M_i^{(\frac{2\hat{\sigma}}{n-2}-1)}$ *and* U_0 *satisfies* (1.6).

(*iii*) *If* $M_i \longrightarrow +\infty$ *as* $i \longrightarrow +\infty$ *and* $\sigma < \frac{n-2}{2}$, *then* $u_i(x)$ *converges to a singular solution of (1.3) in* $B_1 \backslash \{0\}$ *with* $\lim_{i \longrightarrow +\infty} \int_{B_1} u_i^{\frac{2n}{n-2}} dx = +\infty$.

For the case $n-2 \le \sigma < n$, Theorem A was first proved by Li [9]. Based on the method of moving planes, the authors [5] also gave another proof in this situation. For the case $1 < \sigma \le n-2$, Theorem A was proved in [6]. As an application of Theorem A, the following apriori bound holds for all solutions of (1.1).

Theorem 1.1. *Suppose that* K *is a* C^1 *positive function on* S^n *with isolated critical points only. For a critical point* P_0 *of* K, *let* $x = (x_1, \ldots, x_n)$ *denote the stereographic projection of a point on* S^n *with* P_0 *as the South Pole. Assume for any critical point* P_0, $K(x)$ *satisfies the assumption of Theorem A in a neighborhood of* 0 *with* $\frac{n-2}{2} < \sigma = \sigma(P_0) < n-2$. *Then there exists a constant* $C > 0$ *such that for any solution* v *of* (1.1), $\max_{S^n} v \le C$.

At the first sight, we may apply the degree theory, developed by Chang-Gursky-Yang and Li, to obtain solutions of (1.1) if K satisfies the assumption of Theorem 1.1. However, the study for radial solutions suggests that the degree among all solutions might be zero. It is one of the purposes of this article to show that the degree among radial solutions is equal to zero when both of the exponents of the flatness of K at North and South poles is less than $n-2$. For a positive C^1 function K satisfying the assumptions of Theorem 1.1, we suspect the degree of solutions of (1.1) including non-radial ones might be equal to zero. We will continue to study this problem in a forthcoming paper.

Consider the situation when $K(y)$ and solutions v depend on y_{n+1}-axis only. In this case, equation (1.2) becomes

$$\begin{cases} u''(r) + \frac{n-1}{r} u'(r) + K(r) u^{\frac{n+2}{n-2}} = 0 \ \text{ for } r \ge 0, \\ u(r) = O(r^{2-n}) \ \text{ at } \infty. \end{cases} \tag{1.7}$$

Let $K(r)$ satisfy one of the following statements.

$$\begin{cases} \text{(i) } K(r) = K(0) - Ar^{\sigma} + R(r) \text{ in a neighborhood of 0 for} \\ 1 < \sigma \le n \text{ and } A > 0, \text{ where both } R(r)r^{-\sigma} \text{ and } R'(r)r^{-\sigma+1} \\ \text{tend to 0 as } r \longrightarrow 0^+. \\ \text{(ii) } |K(r) - K(0)| \le Ar^{\sigma} \text{ for some } \sigma > n \\ \text{and } \displaystyle\int_0^{\infty} K'(r)r^{-n}dr < 0. \end{cases} \tag{1.8}$$

Let $K^*(r) = K(\frac{1}{r})$. Then we have

Theorem 1.2. *Suppose that both K and K^* satisfy (1.8) with the exponents σ and β respectively. Let $\tilde{\sigma} = \min(\sigma, n)$ and $\tilde{\beta} = \min(\beta, n)$. Assume that $\sigma > \frac{n-2}{2}, \beta > \frac{n-2}{2}$ and $\frac{1}{\tilde{\sigma}} + \frac{1}{\tilde{\beta}} \ne \frac{2}{n-2}$. Then there exists a constant $C > 0$ such that any solution u of* (1.7) *satisfies $u(r) < C(1+r)^{-n+2}$. Moreover, let deg denote the degree among radial solutions of* (1.7), *i.e.,*

$$deg = deg(v + \Delta^{-1}(Kv^{\frac{n+2}{n-2}}), \{v = v(r) \mid \frac{1}{C} < v(r)(1+r)^{n-2} < C\}, 0).$$

Then, (*i*) *when $\frac{1}{\tilde{\sigma}} + \frac{1}{\tilde{\beta}} < \frac{2}{n-2}$, $deg = -1$;* (*ii*) *when $\dfrac{1}{\tilde{\sigma}} + \dfrac{1}{\tilde{\beta}} > \dfrac{2}{n-2}$, $deg = 0$.*

We are going to see that Theorem A is the key point to derive the apriori bound. In general situation, the proof of Theorem A is very complicated. However, for the radial case, we will present a much simple proof of Theorem A, based on an estimate by Cheng-Chern (Theorem 3.3 in [3].) The details of the arguments will be given in Section 3. For the proof of last part of Theorem 1.2, we need to consider a particular K. Define

$$K(r) = \begin{cases} 1 - (Ar)^{\sigma} & \text{for } 0 \le r \le A^{-1}, \\ 0 & \text{for } A^{-1} \le r \le B, \\ 1 - (Br^{-1})^{\beta} & \text{for } r \ge B. \end{cases} \tag{1.9}$$

Theorem 1.3. *Let K satisfy* (1.9) *with $\sigma > 1, \beta > 1$ and $\frac{1}{\widetilde{\sigma}} + \frac{1}{\widetilde{\beta}} > \frac{2}{n-2}$, where $\widetilde{\sigma} = \min(\sigma, n)$ and $\widetilde{\beta} = \min(\beta, n)$. There exists a constant $C > 0$ such that if $AB \ge C$, then* (1.7) *has no solution.*

Under the assumption $\frac{n(n-2)}{n+2} < \sigma, \beta \le n$ and $\frac{1}{\widetilde{\sigma}} + \frac{1}{\widetilde{\beta}} \ge \frac{2}{n-2}$, Theorem 1.3 was proved by Bianchi-Egnell [2]. We extend their result to allow the exponents less than $\frac{n(n-2)}{n+2}$. Our approach is based on the apriori estimates mentioned earlier, and is different from the method of phase planes by Bianchi-Engnell.

The paper is organized as follows. In Section 2, Theorem 1.1 is proved as an application of Theorem A. In Section 3, we are going to derive Theorem A for the radial solutions, based on a previous result by Cheng and Chern [CC]. Finally, both Theorem 1.2 and Theorem 1.3 are proved in Section 4.

2. An apriori bound.

Proof of Theorem 1.1. Suppose v_i is a sequence of solutions of (1.1) on S^n and $\max_{S^n} v_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$. Let $m_i = \min_{S^n} v_i$ and $w_i(y) = m_i^{-1} v_i(y)$. Let $\Gamma = \{P \in S^n \mid P \text{ is a critical point of } K\}$. By the result mentioned in Introduction, $v_i(y)$ is uniformly bounded in any compact set of $S^n \backslash \Gamma$. By the Harnack inequality for linear elliptic equations, $w_i(y)$ is uniformly bounded in any compact set of $S^n \backslash \Gamma$. By Theorem A, we have $v_i(y) \longrightarrow 0$ as $i \longrightarrow +\infty$ in any compact set of $S^n \backslash \Gamma$. Therefore after passing to a subsequence, $w_i(y)$ converges to a function $h(y)$ in $C^2_{loc}(S^n \backslash \Gamma)$, where h satisfies

$$\Delta_{g_0} h - \frac{1}{4} n(n-2) h = 0 \quad \text{in} \quad S^n \backslash \Gamma. \tag{2.1}$$

We claim that, in fact, h is a smooth solution of (2.1) on S^n.

To see the claim, let $P_0 \in \Gamma$ and $x = (x_1, \ldots, x_n)$ be the stereographic projection of S^n with P_0 as the South Pole. Let $u_i(x) = v_i(y) (\frac{2}{1+|x|^2})^{\frac{n-2}{2}}$. Then u_i satisfies

$$\Delta u_i + K(x) u_i^{\frac{n+2}{n-2}} = 0 \quad \text{in} \quad \mathbf{R}^n.$$

By the assumptions, we may assume that $K(x)$ has no other critical points in $|x| < 2\delta_0$ except the origin, and u_i is uniformly bounded in any compact subset of $\overline{B}_{2\delta_0} \backslash \{0\}$ with $M_i = \max_{\bar{B}_{2\delta_0}} u_i \longrightarrow +\infty$. Let $\tilde{u}_i(x) = u_i(x + x_i)$, where $u_i(x_i) = M_i$. By Theorem A, we have

$$\begin{cases} \tilde{u}_i(x) \simeq m_i \simeq M_i^{1-\frac{2\sigma}{n-2}} & \text{for } r_i \le |x| \le \frac{\delta_0}{2} \\ \tilde{u}_i(x) \simeq M_i U_0(M_i^{\frac{2}{n-2}} x) & \text{for } |x| \le r_i, \end{cases} \tag{2.2}$$

where $r_i = c M_i^{-\frac{2}{n-2} + \frac{2\sigma}{(n-2)^2}}$ and U_0 is the solution of (1.6). Therefore, we can write

$$
\begin{aligned}
& m_i^{-1}\int_{|x|\le\delta_0} K(x)u_i^{\frac{n+2}{n-2}}(x)dx \\
\le\ & m_i^{-1}\int_{|x|\le r_i} K(x)\widetilde{u}_i^{\frac{n+2}{n-2}}(x)dx + m_i^{-1}\int_{r_i\le|x|\le 2\delta_0} K(x)\widetilde{u}_i^{\frac{n+2}{n-2}}(x)dx.
\end{aligned}
$$

By the Harnack inequality and (2.2), the two integrals can be estimated by

$$
m_i^{-1}\int_{|x|\le r_i} K(x)\widetilde{u}_i^{\frac{n+2}{n-2}}(x)dx \sim M_i^{\frac{2\sigma}{n-2}-2} \quad \text{and} \tag{2.3}
$$

$$
m_i^{-1}\int_{r_i\le|x|\le 2\delta_0} \widetilde{u}_i^{\frac{n+2}{n-2}}(x)dx \sim m_i^{\frac{4}{n-2}}. \tag{2.4}
$$

Suppose $h_0(x) = h(y)(\frac{2}{1+|x|^2})^{\frac{n-2}{2}} = a|x|^{2-n} + b(x)$ in B_{δ_0} for some $a \ge 0$ and for a smooth harmonic function $b(x)$. Then

$$
\begin{aligned}
-a|S^{n-1}|(n-2) &= \int_{|x|=\delta_0} \frac{\partial h_0}{\partial r}(x)d\sigma_x \\
&= -\lim_{i\longrightarrow+\infty} m_i^{-1}\int_{|x|\le\delta_0} K(x)u_i^{\frac{n+2}{n-1}}(x)dx \quad = \quad 0
\end{aligned}
$$

by (2.3) and (2.4). Thus, h is smooth at P_0. The claim is proved. Since the conformal Laplacian $\Delta_{g_0} - \frac{1}{4}n(n-2)$ has a positive first eigenvalue, we conclude that $h(y) \equiv 0$ on S^n. But, by our construction, $\min_{S^n} h(y) \ge 1$, which yields a contradiction.

3. Estimates of lower bounds. Following the conventional notations, we let $v(r,\alpha)$ denote the solution of

$$
v''(r) + \frac{n-1}{r}v' + K(r)v^{\frac{n+2}{n-2}} = 0, \quad v(0,\alpha) = \alpha,\ v'(0,\alpha) = 0. \tag{3.1}
$$

Lemma 3.1. *Suppose that K is bounded in $[0,1]$ and satisfies $|K(r) - K(0)| \le Cr^\sigma$ for $0 \le r \le \delta_0$ with some constants $K(0) > 0$ and $\sigma > 0$. Then there is $\alpha_0 > 0$ such that for each $\alpha \ge \alpha_0$, we have $v(r;\alpha) > 0$ for $0 \le r \le \delta(\alpha,\sigma)$ and*

$$
|v(r,\alpha) - v_0(r,\alpha)| \le C_1\alpha G(\alpha,\sigma) \tag{3.2}
$$

for $0 \le r \le \delta(\alpha, \sigma)$*, where*

$$v_0(r, \alpha) = \alpha(1 + B^2 r^2)^{-\frac{n-2}{2}}, \tag{3.3}$$

$$B^2 = \frac{K(0)\alpha^{\frac{4}{n-2}}}{n(n-2)}, \tag{3.4}$$

$$G(\alpha, \sigma) = \begin{cases} B^{-\sigma} & \textit{if } 0 < \sigma < n, \\ B^{-\sigma}\log(B) & \textit{if } \sigma = n, \\ B^{-n} & \textit{if } \sigma > n, \end{cases} \tag{3.5}$$

$$\delta(\alpha, \sigma) = \begin{cases} \delta_1 B^{-1+\frac{\sigma}{n-2}} & \textit{if } 0 < \sigma \le n-2, \\ 1 & \textit{if } \sigma > n-2, \end{cases} \tag{3.6}$$

and constants C_1 *and* δ_1 *depend on* $\frac{C}{K(0)}$, δ_0 *and* n *only.*

Proof. (3.2) was due to Cheng-Chern (see Theorem 3.3 in [3]). However, $\delta(\alpha, \sigma)$ is not stated sharply as (3.6). For the reader's convenience, we present a brief sketch of the proof here. We refer [3] for more detailed arguments.

Following the notations in [3], we let $\eta = Br$ and $w(\eta) = \alpha^{-1}v(r, \alpha)$. Set $z(\eta)$ by $w(\eta) = w_0(\eta)z(\eta)$, where $w_0(\eta) = (1+|\eta|^2)^{-\frac{n-2}{2}}$. Then $z(\eta)$ satisfies

$$\begin{cases} z'' + (\frac{n-1}{\eta} + \frac{2w_0'}{w_0})z' + n(n-2)w_0^{\frac{4}{n-2}}\tilde{K}(\eta)z^{\frac{n+2}{n-2}} - n(n-2)w_0^{\frac{4}{n-2}}z = 0, \\ z(0) = 1 \quad \text{and} \quad z'(0) = 0, \end{cases}$$

where $\tilde{K}(\eta) = K(0)^{-1}K(B^{-1}\eta)$. By the assumptions,

$$|\tilde{K}(\eta) - 1| \le \frac{C}{K(0)} B^{-\sigma}|\eta|^{\sigma}$$

for $0 \le \eta \le B\delta_0$. Assume

$$|z(\eta) - 1| \le \varepsilon_1 \tag{3.7}$$

for $0 \le \eta \le \eta_0$, where ε_1 is a small positive number and $\eta_0 \le B\delta_0$. Cheng-Chern proved that if ε_1 is sufficiently small (depending on n only) and (3.7) holds for $0 \le \eta \le \eta_0$, then

$$|z(\eta) - 1| \le C_1 G(\alpha, \sigma) w_0^{-1}(\eta)$$

for $0 \le \eta \le \eta_0$, where C_1 is a constant depending on $\frac{C}{K(0)}$ and n only (Please see the argument in [3] from (3.21) up to (3.31)). Let $\tilde{\eta_0} = \sup\{\eta_0 :$ (3.7) holds for $0 \le \eta \le \eta_0\}$. Choose $\delta(\alpha, \sigma)$ as in (3.6). If α is large, we have $B\delta(\alpha, \sigma) \ge 1$. It is obvious to see that

$$C_1 G(\alpha, \sigma) w_0^{-1}(B\delta(\alpha, \sigma)) \le C_1 G(\alpha, \alpha)(B\delta)^{n-2} \le \varepsilon_1$$

provided that δ_1 is small when $0 < \sigma \le n-2$ and α is large when $\sigma > n-2$. Hence $B\delta(\alpha, \sigma) \le \tilde{\eta_0}$. The proof of Lemma 3.1 is complete. □

Suppose $v(r)$ is a solution of (3.1). Set $w(t) = v(r) r^{\frac{n-2}{2}}$ and $t = \log r$. Then $w(t)$ satisfies

$$w'' - (\frac{n-2}{2})^2 w + \tilde{K}(t) w^{\frac{n+2}{n-2}}(t) = 0 \tag{3.8}$$

for $t \le 0$, where $\tilde{K} = K(e^t)$.

Lemma 3.2. *Assume $|K'(r)| \le C r^{\sigma-1}$ for some $\sigma > 1$ and $K(r)$ is nonincreasing in $[0, 1]$. Then there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that the following holds.*

(i) Assume $w(t)$ is nonincreasing in $[t_0, t_1]$ and $t_1 \le \log \delta_0$, then

$$t_1 - t_0 \le \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)} + c_1 \tag{3.9}$$

for some constant c_1 depending on n only. Furthermore, if t_1 is a local minimum point, then

$$t_1 - t_0 \ge \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)}. \tag{3.10}$$

(ii) Assume that $w(t)$ is nondecreasing in $[t_1, t_2]$ and $w(t_2) \le \varepsilon_0$, then

$$t_2 - t_1 \le \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)} + c_1 \tag{3.11}$$

for some constant c_1. Furthermore, if t_1 is a local minimum point, then

$$t_2 - t_1 \ge \frac{2}{n-2} \log \frac{w(t_2)}{w(t_1)}. \tag{3.12}$$

Proof. Let

$$E(t) = {w'}^2 - (\frac{n-2}{2})^2 w^2 + \frac{n-2}{n}\tilde{K}(t)w^{\frac{2n}{n-2}}. \tag{3.13}$$

By (3.8), we have $E'(t) = \frac{n-2}{n}\tilde{K}'(t)w^{\frac{2n}{n-2}} \leq 0$. Since $E(-\infty) = 0$, $E(t) < 0$ for all $t \leq 0$. Hence $w(t) \leq C_n$ for some constant $C_n > 0$.

Let ε_0 be small and will be chosen later. Assume $w(t_0) \leq \varepsilon_0$. By (3.8), we have

$$(\frac{n-2}{2})^2 w - c_1 w^{\frac{n+2}{n-2}} \leq w_{tt} \leq (\frac{n-2}{2})^2 w^2 \tag{3.14}$$

By the first inequality above, we have that ${w'}^2 - (\frac{n-2}{2})^2 w^2 + \frac{c_1(n-2)}{n} w^{\frac{2n}{n-2}}$ is nonincreasing for $t_0 \leq t \leq t_1$. Thus for $t_0 \leq t \leq t_1$,

$${w'}^2 - g(w) \geq -g(w(t_1)), \tag{3.15}$$

where $g(w) = (\frac{n-2}{2})^2 w^2 - \frac{(n-2)c_1}{n} w^{\frac{2n}{n-2}}$. Integrating (3.15), we have

$$t_1 - t_0 \leq \int_{w(t_1)}^{w(t_0)} \frac{dw}{\sqrt{g(w) - g(w(t_1))}} = \int_1^{\frac{w(t_0)}{w(t_1)}} \frac{d\eta}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}} \tag{3.16}$$

where $\bar{g}(\eta) = (\frac{n-2}{2})^2 \eta^2 - \frac{(n-2)c_1}{n} w^{\frac{4}{n-2}}(t_1)\eta^{\frac{2n}{n-2}}$. Choose $\varepsilon_0 > 0$ small such that

$$\frac{1}{\sqrt{\bar{g}(\eta) - \bar{g}(1)}} \leq \frac{2}{n-2}\frac{1}{\sqrt{\eta^2 - 1}} + c_1 \frac{w^{\frac{4}{n-2}}(t_1)\eta^{\frac{4}{n-2}}}{\sqrt{\eta^2 - 1}} \tag{3.17}$$

for $0 \leq \eta \leq \frac{\varepsilon_0}{w(t_1)}$. Then, by (3.16)

$$\begin{aligned} t_1 - t_0 &\leq \frac{2}{n-2}\int_1^{\frac{w(t_0)}{w(t_1)}} \frac{d\eta}{\sqrt{\eta^2 - 1}} + c_1 w^{\frac{4}{n-2}}(t_1) \int_1^{\frac{w(t_0)}{w(t_1)}} \frac{\eta^{\frac{4}{n-2}}}{\sqrt{\eta^2 - 1}} d\eta \\ &\leq \frac{2}{n-2} \log \frac{w(t_0)}{w(t_1)} + c_2, \end{aligned} \tag{3.18}$$

where c_2 is an universal constant. It is easy to see that (3.9) holds true if $w(t_1) \geq \varepsilon_0$. Without loss of generality, we may assume $w(t_1) \leq \varepsilon_0$ and $w(t_0) \geq \varepsilon_0$. Since $w(-\infty) = 0$, let $T_0 < t_1$ denote the maximum point of

w which is closest to t_1. To prove (3.9), it suffices to show that $t_0 - T_0$ is bounded by a constant depending on n only if $t_1 \le \log \delta_0$ for some small $\delta_0 > 0$. By (3.13),

$$-E(T_0) = \frac{n-2}{n} \int_{-\infty}^{T_0} |\tilde{K}'(t)| w^{\frac{2n}{n-2}}(t) dt \le c_1 \int_{-\infty}^{T_0} \exp(\sigma t) dt \le c_1 e^{\sigma T_0}.$$

Choose δ_0 so small such that

$$f(w(T_0)) = -E(T_0) + \frac{n-2}{n}(\tilde{K}(T_0) - \tilde{K}(t_1)) w^{\frac{2n}{n-2}}(T_0) \le c_1 e^{\sigma t_1} \le f(\varepsilon_0), \tag{3.19}$$

where $f(w) = (\frac{n-2}{2})^2 w^2 - \frac{n-2}{n} \tilde{K}(t_1) w^{\frac{2n}{n-2}}$. If ε_0 is small enough, then $w(T_0) > w_1$, where w_1 is the only local maximum point of f. Thus, we have $f(w) \ge f(w(T_0))$ for all $\varepsilon_0 \le w \le w(T_0)$. In particular, we have $f(w(t)) \ge f(w(T_0))$ for $T_0 \le t \le t_0$. Since

$$\frac{d}{dt}(w'^2 - (\frac{n-2}{2})^2 w^2 + \frac{n-2}{n} \tilde{K}(t_1) w^{\frac{2n}{n-2}}) = 2w'(\tilde{K}(t_1) - \tilde{K}(t)) w^{\frac{n+2}{2}} \ge 0$$

for $T_0 \le t \le t_1$, we have $w'^2(t) \ge f(w(t)) - f(w(T_0))$. Therefore,

$$t_0 - T_0 \le \int_{\varepsilon_0}^{w(T_0)} \frac{dw}{\sqrt{f(w) - f(w(T_0))}}. \tag{3.20}$$

It is not difficult to show that the right hand side is bounded by a constant depending on ε_0 and n only. Hence (3.9) is proved.

Suppose t_1 is a local minimum. By the second inequality of (3.14), $w'^2 - (\frac{n-2}{2})^2 w^2$ is nondecreasing in (t_0, t_1). Hence

$$w'^2(t) - (\frac{n-2}{2})^2 w^2(t) \le -(\frac{n-2}{2})^2 w^2(t_1). \tag{3.21}$$

Integrating (3.21), we obtain (3.10).

(ii) follows immediately from similar arguments as in the proof of (i) if t is replaced by $2t_1 - t$. □

As an application of Lemma 3.1, we consider a sequence of solutions $v_i(x) = v_i(|x|)$ of

$$\Delta v_i + K_i(|x|) v_i^{\frac{n+2}{n-2}} = 0$$

for $0 \le |x| \le 1$, where K_i satisfies (1.8) uniformly, i.e.,

$$(1.8)' \begin{cases} \text{There are positive constants } A,\ B \text{ independent of } i \text{ and } K_i(0) \\ \text{with } \lim_{i\longrightarrow\infty} K_i(0) > 0 \text{ such that one of the following holds:} \\ \text{(i)} \ \ K_i(r) = K_i(0) - A_i r^{\sigma_i} + R_i(r) \ \text{ for } 0 \le r \le 1 \text{ with} \\ A \le A_i \le A^{-1}, 1 < \sigma_i \le n, \ \text{ where } R_i(r) r^{-\sigma_i} \text{ and } R_i'(r) r^{-\sigma_i+1} \\ \text{converge to 0 as } r \longrightarrow 0 \text{ uniformly in } i. \\ \text{(ii)} \ \ |K_i(r) - K_i(0)| \le A r^{\sigma_i} \ \text{ for } 0 \le r \le 1 \text{ with } \sigma_i > n, \\ A > 0 \text{ and } \displaystyle\int_0^\infty K_i'(r) r^{-n} dr \le -B < 0. \end{cases}$$

Let $M_i = v_i(0)$ and $\sigma = \lim_{i\longrightarrow+\infty} \sigma_i > 1$. Assume $M_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$. By Lemma 3.1, we have

$$M_i G(M_i, \sigma_i) \sim \begin{cases} o(1) M_i^{-1} & \text{if } \sigma > n-2, \\ M_i^{-1} & \text{if } \sigma_i \ge n-2 \text{ and } \sigma = n-2, \\ M_i^{1-\frac{2\sigma_i}{n-2}} & \text{if } \sigma_i < n-2. \end{cases} \tag{3.22}$$

Let

$$r_i = \begin{cases} \delta_2 M_i^{(-1+\frac{\sigma_i}{n-2})\frac{2}{n-2}} & \text{if } 0 < \sigma \le n-2, \\ 1 & \text{if } \sigma > n-2, \end{cases} \tag{3.23}$$

then

$$v_0(r_i, M_i) \sim \begin{cases} \delta_2^{-(n-2)} M_i^{(1-\frac{2\sigma_i}{n-2})} & \text{if } 0 < \sigma \le n-2, \\ M_i^{-1} & \text{if } \sigma > n-2. \end{cases} \tag{3.24}$$

Therefore, if $\delta_2 < \delta_1$ is sufficient small (recall that δ_1 is the number in (3.6)), we have

$$v_i(r_i) \sim \begin{cases} M_i^{(1-\frac{2\sigma_i}{n-2})} & \text{if } 0 < \sigma \le n-2, \\ M_i^{-1} & \text{if } \sigma > n-2. \end{cases} \tag{3.25}$$

From now on, δ_2 is chosen small so that (3.25) holds, and then δ_2 is fixed throughout this section.

Lemma 3.3. *Suppose K_i and v_i are stated as above. Then the following statements hold*:

$$\begin{cases} \text{(i)} \quad \text{If } \sigma_i \ge n-2, \text{ then } v_i(1) \sim M_i^{-1}. \\ \text{(ii)} \quad \text{If } \frac{n-2}{2} \le \sigma_i < n-2, \text{ then } v_i(1) \sim M_i^{1-\frac{2\sigma_i}{n-2}}. \\ \text{(iii)} \quad \text{If } \sigma_i < \frac{n-2}{2}, \text{ then } v_i(1) \sim 1. \end{cases} \tag{3.26}$$

Proof. If $\sigma = \lim_{i\longrightarrow\infty}\sigma_i > n-2$, then by (3.25), $v_i(1) \sim M_i^{-1}$. If $\sigma_i \geq n-2$ and $\sigma = n-2$, then by (3.3) and (3.22), $v_i(\delta_2) \sim M_i^{-1}$. By the Harnack inequality, $v_i(1) \sim v_i(\delta_2) \sim M_i^{-1}$.

For the case $\sigma_i < \frac{n-2}{2}$, (3.26) was proved in [7] (See Theorem 1.5 in [7]). Hence we consider the case $\frac{n-2}{2} \leq \sigma_i < n-2$. By (3.25), we have

$$v_i(r_i) \sim M_i^{1-\frac{2\sigma_i}{n-2}} \tag{3.27}$$

Let $w_i(t) = v_i(r)r^{\frac{n-2}{2}}$ and $t = \log r$. It is easy to see that w_i has its first local maximum at T_i, where

$$T_i = \frac{-2}{n-2}\log M_i + C + o(1) \tag{3.28}$$

for some constant $C \in \mathbf{R}$ as $i \longrightarrow +\infty$. Let t_i be the first local minimum of w_i after T_i. Two cases happen.

Case 1. Assume $t_i \geq T_0$ for some T_0. Suppose K_i is decreasing in $r \leq r_0$. Without loss of generality, we may assume $T_0 \leq \log r_0$ and $r_0 = 1$. By (3.9), we have $w_i(T_0)e^{\frac{n-2}{2}T_0} \leq c_1 w_i(T_i)e^{\frac{n-2}{2}T_i} \leq c_2 M_i^{-1}$. Applying the Harnack inequality and the gradient estimates, we have $v_i(r_0) \leq c_3 v_i(e^{T_0}) \leq c_4 M_i^{-1}$. On the other hand, (3.13) implies

$$(\frac{n-2}{n})\int_{-\infty}^{T_0} |\tilde{K}_i'(t)| w_i^{\frac{2n}{n-2}}(t)dt \leq (\frac{n-2}{2})^2 w_i^2(T_0).$$

By Lemma 3.1 and the estimates above,

$$c_5 M_i^{-\frac{2\sigma_i}{n-2}} \leq \int_{-\infty}^{T_i} |\tilde{K}_i'(t)| w_i^{\frac{2n}{n-2}}(t)dt \leq c_6 M_i^{-2}$$

where $\tilde{K}_i(t) = K_i(e^t)$. Hence we conclude $M_i^{2-\frac{2\sigma_i}{n-2}}$ is bounded. By (3.27) and the Harnack inequality, $v_i(1) \sim M_i^{-1} \sim M_i^{1-\frac{2\sigma_i}{n-2}}$.

Case 2. Assume $\lim_{i\longrightarrow\infty} t_i = -\infty$. By (3.23) and (3.25), we have $w_i(\log r_i) \sim M_i^{-\frac{\sigma_i}{n-2}}$. Hence

$$w_i(t_i) \leq c_1 M_i^{-\frac{\sigma_i}{n-2}}. \tag{3.29}$$

On the other hand, by Lemma 3.1 again, we have

$$c_2 M_i^{-\frac{2\sigma_i}{n-2}} \le \frac{n-2}{n}\int_{-\infty}^{t_i} |\tilde{K}_i'(t)| w_i^{\frac{2n}{n-2}} dt \tag{3.30}$$
$$= w_i^2(t_i) - \frac{n-2}{n}\tilde{K}_i(t_i) w_i^{\frac{2n}{n-2}} \le \frac{1}{4} w_i^2(t_i).$$

Together with (3.29), (3.10) implies $w_i(t_i) \sim M_i^{-\frac{\sigma_i}{n-2}}$. By (3.9) of Lemma 3.2, $w_i(t_i)e^{\frac{n-2}{2}t_i} \sim w(T_i)e^{\frac{n-2}{2}T_i} \sim M_i^{-1}$. Therefore,

$$e^{t_i} \sim M_i^{\frac{2}{n-2}(-1+\frac{\sigma_i}{n-2})}, \tag{3.31}$$

which in turn implies

$$v_i(e^{t_i}) \sim M_i^{1-\frac{2\sigma_i}{n-2}}. \tag{3.32}$$

Let ε_0 be the positive constant in (ii) of Lemma 3.2. Let $\bar{t}_i > t_i$ satisfy $w_i(\bar{t}_i) = \varepsilon_0 \le 0$ and $w_i'(t) \ge 0$ for $t \in (t_i, \bar{t}_i)$. Assume $w_i(\bar{t}_i) = \varepsilon_0$. By (3.12), $\bar{t}_i - t_i \ge \frac{2}{n-2}\log\frac{\varepsilon_0}{w_i(t_i)}$. Let $t_i^* < t_i$ satisfy $w_i(t_i^*) = \varepsilon_0$ and $w_i'(t) \le 0$ for $t_i^* \le t \le t_i$. By (3.10), $t_i - t_i^* \ge \frac{2}{n-2}\log\frac{\varepsilon_0}{w(t_i)}$. Putting these two inequalities together, we have

$$\bar{t}_i - t_i^* \ge \frac{4}{n-2}\log\frac{\varepsilon_0}{w_i(t_i)}. \tag{3.33}$$

On the other hand, we have for i large

$$\frac{1}{2}w_i^2(t_i) \le \frac{4}{n(n-2)}\Big(\int_{-\infty}^{t_i^*} |\tilde{K}_i'(t)| w_i^{\frac{2n}{n-2}} dt + \int_{t_i^*}^{t_i} |\tilde{K}_i'(t)| w_i^{\frac{2n}{n-2}}(t) dt\Big) \tag{3.34}$$

For the first term of the right hand side, we have

$$\int_{-\infty}^{t_i^*} |\tilde{K}_i^1(t)| w_i^{\frac{2n}{n-2}} dt \le c_1 A \int_{-\infty}^{t_i^*} e^{\sigma_i t} dt = \frac{c_1 A}{\sigma_i} e^{\sigma_i t_i^*}.$$

For the estimate of the second term, we note that $w_i(t) \sim w_i(t_i^*)e^{\frac{n-2}{2}t_i^*}e^{-\frac{n-2}{2}t}$ for $t \in (t_i^*, t_i)$. Hence,

$$\int_{t_i^*}^{t_i} |\tilde{K}_i'(t)| w_i^{\frac{2n}{n-2}}(t) dt \le c_2 (w_i(t_i^*)e^{\frac{n-2}{2}t_i^*})^{\frac{2n}{n-2}} \int_{t_i^*}^{t_i} e^{(\sigma_i - n)t} dt$$
$$\le c_3 (w_i(t_i^*)e^{\frac{n-2}{2}t_i^*})^{\frac{2n}{n-2}} e^{(\sigma_i - n)t_i^*} = c_3 \varepsilon_0^{\frac{2n}{n-2}} e^{\sigma_i t_i^*}.$$

By (3.34), we have $w(t_i) \le c_4 e^{\frac{\sigma_i}{2} t_i^*}$. Therefore, we have from (3.33), $\bar{t}_i - t_i^* \ge \frac{4}{n-2}(-\frac{\sigma_i}{2} t_i^*) - c_5$, i.e.,

$$\bar{t}_i + (\frac{2\sigma_i}{n-2} - 1)t_i^* \ge -c_5. \tag{3.35}$$

If $\sigma = \lim_{i \longrightarrow \infty} \sigma_i > \frac{n-2}{2}$, then $t_i^* \ge -c_6$, which yields a contradiction to $\lim_{i \longrightarrow +\infty} t_i^* \le \lim_{i \longrightarrow +\infty} t_i = -\infty$.

Hence, when $\sigma > \frac{n-2}{2}, w_i(t) \le \varepsilon_0$ for all $t_i \le t \le 0$. (Note that we assume $K_i' \le 0$ for $r \le 1$.) By (3.11) and (3.12), we have $v_i(1) \sim v_i(e^{t_i}) \sim M_i^{1-\frac{2\sigma_i}{n-2}}$.

If $\sigma = \frac{n-2}{2}$, then we may assume $w_i(\bar{t}_i) = \varepsilon_0$, otherwise we have $w_i(t) \le \varepsilon_0$ for all $t_i \le t \le 0$ and again $v_i(1) \sim v_i(e^{t_i}) \sim M_i^{1-\frac{2\sigma_i}{n-2}}$ by (3.11) and (3.12). By (3.35), we have $\bar{t}_i \ge -c_5$ and, by (3.32) and (3.12), $(\frac{n-2}{2} - \sigma_i)\log M_i$ is bounded between two positive constants. It follows from (3.11) and (3.12), $v_i(e^{\tilde{t}_i}) \sim v_i(e^{t_i}) \sim M_i^{1-\frac{2\sigma_i}{n-2}} \sim 1$. Therefore, $v_i(1) \sim 1$. Lemma 3.3 is completely proved.

4. Proofs of Main Theorems. In this section, we first prove Theorem 1.3. Let K be defined as in (1.9) and u be a solution. Set $v(y) = A^{-\frac{n-2}{2}} u(A^{-1}y)$. Then v satisfies

$$\Delta v + K_1(|y|)v^{\frac{n+2}{n-2}} = 0 \quad \text{for} \quad |y| \le 1, \tag{4.1}$$

where $K_1(y) = 1 - |y|^\sigma$ for $|y| \le 1$. Let $u^*(r) = r^{2-n}u(\frac{1}{r})$ be the Kelvin transformation and $v^*(y) = B^{-\frac{n-2}{2}} u^*(B^{-1}y)$. Then v^* satisfies

$$\Delta u^* + K_1^*(|y|){v^*}^{\frac{n+2}{n-2}}(y) = 0 \quad \text{for} \quad |y| \le 1, \tag{4.2}$$

where $K_1^*(y) = 1 - |y|^\beta$.

Proof of Theorem 1.3. Suppose u is a solution of (1.7) with this specified K in (1.9). Then $u(r) = cr^{2-n} + d$ for $A^{-1} \le r \le B$. Since $u(A^{-1}) = A^{\frac{n-2}{2}} v(1)$ and $u'(A^{-1}) = A^{\frac{n}{2}} v'(1)$, c, d can be solved in terms of $v(1)$ and $v'(1)$ by the following:

$$c = \frac{A^{-\frac{n-2}{2}} v'(1)}{2-n}, \quad d = A^{\frac{n-2}{2}}(v(1) + \frac{1}{n-2} v'(1)). \tag{4.3}$$

Since c and d can be solved in terms of $v^*(1)$ and $v^{*'}(1)$ also, we have

$$\begin{cases} \frac{A^{-\frac{n-2}{2}}v'(1)}{2-n} = B^{\frac{n-2}{2}}(v^*(1) + \frac{1}{n-2}v^{*'}(1)), \\ A^{\frac{n-2}{2}}(v(1) + \frac{1}{n-2}v'(1)) = \frac{B^{-\frac{n-2}{2}}v^{*'}(1)}{2-n}. \end{cases} \tag{4.4}$$

Now suppose there exists a sequence of solutions u_i of (1.7) with $A_iB_i \longrightarrow +\infty$. Let v_i and v_i^* be the corresponding solutions of (4.1) and (4.2). Since it is well known that $v_i(r) \le cr^{\frac{2-n}{2}}$ and $|v_i'(r)| \le cr^{-\frac{n}{2}}$ for some constant $c > 0$, we have

$$v_i^*(1) + \frac{1}{n-2}v_i^{*'}(1) = \frac{(A_iB_i)^{-\frac{n-2}{2}}v_i'(1)}{n-2} \longrightarrow 0$$

as $i \longrightarrow +\infty$. The same conclusion for $v_i(1) + \frac{1}{n-2}v_i'(1)$ holds also.

Applying the Pohozaev identity, we have

$$\frac{n-2}{2n}\int_0^1 K_1'(r)v_i^{\frac{2n}{n-2}}r^n dr = \frac{v_i'(1)}{2}(v_i'(1) + (n-2)v_i(1)) \longrightarrow 0$$

as $i \longrightarrow +\infty$. Hence, we have for any $1 \ge r_0 > 0$,

$$cv_i^{\frac{2n}{n-2}}(r_0) \le \int_0^{r_0} r^{\sigma+n-1}v_i^{\frac{2n}{n-2}}(r)dr \le \int_0^1 |K_1'(r)|r^n v_i^{\frac{2n}{n-2}}dr.$$

Therefore,

$$\lim_{i\longrightarrow\infty} v_i(r_0) = 0 \quad \text{and} \quad \lim_{i\longrightarrow+\infty} v_i^*(r_0) = 0. \tag{4.5}$$

Suppose $\sigma \le \beta$. By the assumptions, $\sigma < n-2$. We claim

$$|v_i'(1)| = o(1)v_i(1) \tag{4.6}$$

for $i \longrightarrow +\infty$.

We consider two cases separately.

Case 1. Assume that $v_i(0)$ is bounded. Since $v_i(1) \longrightarrow 0$, we have also $v_i(0) \longrightarrow 0$ as $i \longrightarrow 0$ by the Harnack inequality. Integrating (4.1) gives us

$$|v_i'(1)| = \int_0^1 K_1(r)v_i^{\frac{n+2}{n-2}}(r)r^{n-1}dr = o(1)v_i(0) = o(1)v_i(1).$$

Case 2. Assume that $v_i(0) \longrightarrow +\infty$ as $i \longrightarrow +\infty$. Then, by Lemma 3.3 and (4.5), we have $\frac{n-2}{2} < \sigma < n-2$ and

$$|v_i'(1)|v_i(1)^{-1} \sim v_i(0)^{(\frac{2\sigma}{n-2}-1)} \int_0^1 K_1(r) v_i^{\frac{n+2}{n-2}}(r)dr \tag{4.7}$$
$$= v_i(0)^{\frac{2\sigma}{n-2}-1}(\int_0^{r_i} K_1(s) v_i^{\frac{n+2}{n-2}}(s)ds + \int_{r_i}^1 K_1(s) v_i^{\frac{n+2}{n-2}}(s)ds)$$

where $r_i \sim v_i(0)^{(\frac{\sigma}{n-2}-1)\frac{2}{n-2}}$ by (3.23). By Lemma 3.1, we have

$$v_i(r) \sim v_i(0)(1+(B_i r)^2)^{-\frac{n-2}{2}}, \tag{4.8}$$

where $B_i \sim v_i(0)^{\frac{2}{n-2}}$ for $r \le r_i$, and

$$v_i(r) \sim v_i(r_i) \sim v_i(0)^{1-\frac{2\sigma}{n-2}} \tag{4.9}$$

for $r_i \le r \le 1$. Using (4.8) and (4.9), we have

$$|v_i'(1)|v_i(1)^{-1} \sim v_i(0)^{(\frac{2\sigma}{n-2}-2)} + v_i(0)^{(1-\frac{2\sigma}{n-2})\frac{4}{n-2}}$$

which tends to 0 as $u \longrightarrow +\infty$. Hence (4.6) is proved.

Suppose both σ and $\beta < n-2$, then, by (4.4) and (4.6) (note that in this case, (4.6) also holds for v_i^*),

$$\begin{aligned} v_i(1) &= O(1)(A_iB_i)^{-\frac{n-2}{2}}|v_i^{*\prime}(1)| = o(1)(A_iB_i)^{-\frac{n-2}{2}} v_i^*(1) \\ &= o(1)(A_iB_i)^{-(n-2)} v_i'(1) \end{aligned}$$

which yields a contradiction.

Suppose $\beta \ge n-2$. Then, by (4.4) and (4.6), we still have

$$v_i(1) = O(1)(A_iB_i)^{-\frac{n-2}{2}}|v_i^{*\prime}(1)|. \tag{4.10}$$

If $v_i^*(0) \longrightarrow 0$ as $i \longrightarrow +\infty$, then the same arguments as in Case 1 above yield a contradiction again. Hence, we may suppose $v_i^*(0) \longrightarrow \infty$ as $i \longrightarrow +\infty$. By Pohozaev's identity, we have

$$\int_o^{A^{-1}} K'(r) r^n u_i^{\frac{2n}{n-2}}(r)dr = -\int_B^\infty K'(r) u_i^{\frac{2n}{n-2}} r^n dr.$$

By scaling, the identity becomes

$$\int_0^1 K_1'(s) v_i^{\frac{2n}{n-2}}(s) s^n ds = \int_0^1 K_1^{*\prime}(s) v_i^{*\frac{2n}{n-2}} s^n ds. \tag{4.11}$$

Suppose $v_i(0) \longrightarrow 0$. Then, obviously

$$\int_0^1 s^n |K_1'(s)| v_i^{\frac{2n}{n-2}}(s) ds \sim v_i^{\frac{2n}{n-2}}(1). \tag{4.12}$$

By Lemma 3.1 and similar arguments as in (4.8) and (4.9), we have

$$\int_0^1 |K_1^{*\prime}(s)| v_i^{*\frac{2n}{n-2}} s^n ds \sim \begin{cases} v_i^*(0)^{-\frac{2\beta}{n-2}} & \text{if } n-2 \le \beta < n, \\ v_i^*(0)^{-\frac{2n}{n-2}} \log v_i^*(0) & \text{if } \beta = n, \\ (v_i^*(0))^{-\frac{2n}{n-2}} & \text{if } \beta > n. \end{cases} \tag{4.13}$$

Since $v_i^*(1) \sim v_i^*(0)^{-1}$, we have by (4.12) and (4.13),

$$v_i(1)^{\frac{2n}{n-2}} \sim \begin{cases} v_i^*(1)^{\frac{2\beta}{n-2}} & \text{if } n-2 \le \beta < n, \\ v_i^*(1)^{\frac{2n}{n-2}} \log \frac{1}{v_i^*(1)} & \text{if } \beta = n, \\ (v_i^*(1))^{\frac{2n}{n-2}} & \text{if } \beta > n. \end{cases} \tag{4.14}$$

By (4.10), (4.14) implies

$$c_1 v_i^*(1)^{\frac{2\tilde{\beta}}{n-2}} \le v_i(1)^{\frac{2n}{n-2}} \sim (A_i B_i)^{-n} |v_i^{*\prime}(1)|^{\frac{2n}{n-2}} \sim (A_i B_i)^{-n} |v_i^*(1)|^{\frac{2n}{n-2}}$$

for some constant $c_1 > 0$. Hence $A_i B_i$ is bounded, which yields a contradiction.

Suppose $v_i(0) \longrightarrow +\infty$. By (4.5) and Lemma 3.3, we have $\frac{n-2}{2} < \sigma < n-2$. By similar considerations as in (4.8) and (4.9), we can prove

$$\int_0^1 s^n |K_1'(s)| v_i^{\frac{2n}{n-2}}(s) ds = \int_0^{r_i} s^n |K_1'(s)| v_i^{\frac{2n}{n-2}}(s) ds + \int_{r_i}^1 s^n |K_1'(s)| v_i^{\frac{2n}{n-2}}(s) ds,$$

$$\int_0^{r_i} s^n |K_1'(s)| v_i^{\frac{2n}{n-2}}(s) ds \sim v_i(0)^{-\frac{2\sigma}{n-2}}, \tag{4.15}$$

$$\int_{r_i}^1 s^n |K_1'(s)| v_i^{\frac{2n}{n-2}}(s) ds \sim v_i(0)^{(1-\frac{2\sigma}{n-2})\frac{2n}{n-2}}. \tag{4.16}$$

Hence, we have

$$\int_0^1 s^n|K_1'(s)|v_i^{\frac{2n}{n-2}}(s)ds \sim \begin{cases} v_i(0)^{-\frac{2\sigma}{n-2}} & \text{if } \frac{n(n-2)}{n+2} \le \sigma < n-2, \\ (v_i(0))^{(1-\frac{2\sigma}{n-2})(\frac{2n}{n-2})} & \text{if } \frac{n-2}{2} < \sigma \le \frac{n(n-2)}{n+2}. \end{cases} \tag{4.17}$$

If $\sigma \ge \frac{n(n-2)}{n+2}$, then by (4.10), (4.11), (4.13) and (4.17)

$$\begin{aligned} c_1 v_i^*(0)^{-\frac{2\tilde{\beta}}{n-2}} &\le v_i(0)^{-\frac{2\sigma}{n-2}} \sim v_i(1)^{(\frac{2\sigma}{n-2}-1)^{-1}\frac{2\sigma}{n-2}} \\ &\sim (A_iB_i)^{-\tau} v_i^*(1)^{(\frac{2\sigma}{n-2}-1)^{-1}\frac{2\sigma}{n-2}} \end{aligned} \tag{4.18}$$

where $\tau = \frac{n-2}{2}\frac{2\sigma}{n-2}(\frac{2\sigma}{n-2}-1) = \sigma(\frac{2\sigma}{n-2}-1) > 0$. Since $\frac{1}{\sigma} + \frac{1}{\tilde{\beta}} \ge \frac{2}{n-2}$, we have $(\frac{2\sigma}{n-2}-1)^{-1}\frac{2\sigma}{n-2} \ge \frac{2\tilde{\beta}}{n-2}$. Then (4.18) implies A_iB_i is bounded, a contradiction.

If $\frac{n-2}{2} < \sigma \le \frac{n(n-2)}{n+2}$, then

$$c_1 v_i^*(0)^{-\frac{2\tilde{\beta}}{n-2}} \le v_i(0)^{(1-\frac{2\sigma}{n-2})(\frac{2n}{n-2})} \sim v_i(1)^{\frac{2n}{n-2}} \sim (A_iB_i)^{-n} v_i^*(1)^{\frac{2n}{n-2}}.$$

Since $\tilde{\beta} \le n$, we have A_iB_i is bounded again, which yields a contradiction. Therefore, the proof of Theorem 1.3 is completely finished. □

Proof of Theorem 1.2 Without loss of generality, we may assume that both K and K^* satisfy

$$K(r) = K(0) - Ar^\sigma + R(r) \tag{4.19}$$

for some $A > 0$ and $\sigma > 1$. When $\sigma > n$, we further assume

$$\int_0^\infty K'(r)r^{-n}dr < 0. \tag{4.20}$$

We consider a family of functions $K_t(r)$, continuous in both t and r for $0 \le t \le 1$ and $r \ge 0$, with $K_1(r) = K(r)$. If $\frac{1}{\tilde{\sigma}} + \frac{1}{\tilde{\beta}} < \frac{2}{n-2}$, we set $K_0(r)$ as

$$\begin{cases} K_0(r) = K(0) - \varepsilon_0 r^{n-2} + R_0(r)\,, \\ K_0^*(r) = K(\infty) - \varepsilon_0 r^{n-2} + R_1(r) \end{cases} \tag{4.21}$$

in a neighborhood of 0, where $\varepsilon_0 > 0$ is small but a fixed positive number. If $\dfrac{1}{\tilde{\sigma}} + \dfrac{1}{\tilde{\beta}} > \dfrac{2}{n-2}$, we let $K_0(r)$ satisfy the form in (1.9). Furthermore, both

$K_t(r)$ and $K_t^*(r)$ satisfy (4.19) and (4.20) (when $\sigma > n$) uniformly in t, i.e.,

$$\begin{cases} K_t(r) = K(0) - A(t)r^{\sigma(t)} + R_t(r) \text{ for small } r > 0, \\ K_t^*(r) = K(\infty) - B(t)r^{\beta(t)} + R_t^*(r) \text{ for small } r > 0, \\ \displaystyle\int_0^\infty K_t'(r)r^{-n}dr < -C < 0, \quad \text{and} \\ \displaystyle\int_0^\infty K_t^{*\prime}(r)\, r^{-n}dr < -C < 0. \end{cases} \tag{4.22}$$

where $A(t)$, $B(t)$ and C are bounded between two positive constants, and where $|R_t(r)|r^{-\sigma(t)} + |R_t^*(r)|r^{-\beta(t)}$ and $|R_t'(r)|r^{-\sigma(t)+1} + |R_t^{*\prime}(r)|r^{-\beta(t)+1}$ converge to 0 as $r \longrightarrow 0$ uniformly in t. We also assume

$$\frac{1}{\tilde{\sigma}(t)} + \frac{1}{\tilde{\beta}(t)} \neq \frac{2}{n-2} \tag{4.23}$$

for $t > 0$, and assume $\tilde{\sigma}(t) > n-2$ and $\tilde{\beta}(t) > n-2$ for small $t > 0$ in case $\frac{1}{\tilde{\sigma}} + \frac{1}{\tilde{\beta}} < \frac{2}{n-2}$.

We want to prove there exists a constant $C > 0$, independent of t, such that any solution $u(r)$ of

$$\begin{cases} \Delta u(|x|) + K_t(|x|)u^{\frac{n+2}{n-2}} = 0 & \text{in } \mathbf{R}^n, \\ u(x) = O(|x|^{2-n}) & \text{at } \infty \end{cases} \tag{4.24}$$

satisfies

$$u(|x|) \leq C(1 + |x|)^{-n+2}. \tag{4.25}$$

Then we can employ the method of continuity to find the degree of radial solutions. Suppose the conclusion is not true. Then there exists a sequence of solutions u_i of (4.24) with $t_i \longrightarrow t_0$ as $i \longrightarrow +\infty$ (hence, $K_i = K_{t_i} \longrightarrow K_{t_0}$, σ_i and $\beta_i \longrightarrow \sigma_0$ and β_0 respectively), and with either $u_i(0) \longrightarrow +\infty$ or $u_i^*(0) \longrightarrow +\infty$.

Suppose $u_i(0) = M_i \longrightarrow +\infty$. We discuss two cases separately.

(I) Either $\sigma_0 > n-2$, or, $\sigma_0 = n-2$ and $M_i^{1-\frac{\sigma_i}{n-2}}$ is bounded.

(II) Either $\frac{n-2}{2} < \sigma_0 < n-2$, or, $\sigma_0 = n-2$ and $M_i^{1-\frac{\sigma_i}{n-2}} \longrightarrow +\infty$ as $i \longrightarrow +\infty$.

Suppose u_i satisfies (I). Then by Lemma 3.1 and (3.24), we have

$$u_i(1) \sim M_i^{-1} \quad \text{and} \tag{4.26}$$

$$u_i(r) = M_i(1 + B_i^2 r^2)^{-\frac{n-2}{2}}(1 + o(1)) \tag{4.27}$$

for $0 \le r \le r_0$, where $r_0 > 0$ is independent of i and

$$B_i^2 = \frac{K_i(0)M_i^{\frac{4}{n-2}}}{n(n-2)}. \tag{4.28}$$

Using (4.26) $\sim$ (4.28), simple calculations can show that

$$\text{if } \sigma_0 < n, \quad -\int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr \sim M_i^{-\frac{2\sigma_i}{n-2}} \tag{4.29}$$

$$\text{if } \sigma_0 = n,\ M_i^{\frac{2\sigma_i}{n-2}} \int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr \longrightarrow -\infty \tag{4.30}$$

as $i \longrightarrow +\infty$. If $\sigma_0 \ge n$, then for any $0 < \tau < n$, we have

$$\int_0^1 |K_i'(r)|u_i^{\frac{2n}{n-2}}(r)r^n dr = o(1)M_i^{-\frac{-2\tau}{n-2}}. \tag{4.31}$$

If $\sigma_0 > n$ and $M_i u_i(r) \longrightarrow ar^{2-n}$ for some $a > 0$ as $i \longrightarrow +\infty$, then

$$M_i^{\frac{2n}{n-2}} \int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr \longrightarrow a^{\frac{2n}{n-2}} \int_0^1 K_{t_0}'(r)r^{-n}dr, \tag{4.32}$$

as $i \longrightarrow +\infty$.

Suppose u_i satisfies (II). By Lemma 3.1 and (3.24), we have

$$u_i(r) \sim M_i^{1-\frac{2\sigma_i}{n-2}} \tag{4.33}$$

for $r_i \le r \le 1$ where r_i is given in (3.23), and

$$u_i(r) = M_i(1 + B_i^2|x|^2)^{-\frac{n-2}{2}}(1 + o(1)) \tag{4.34}$$

for $0 \le r \le r_i$. Using (4.33) and (4.34), we have if $\sigma_0 > \frac{n(n-2)}{n+2}$, then

$$-\int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr \sim M_i^{-\frac{2\sigma_i}{n-2}}. \tag{4.35}$$

If $\frac{n-2}{2} < \sigma_0 < \frac{n(n-2)}{n+2}$, then we can show

$$\int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr = M_i^{-\tau_i} b^{\frac{2n}{n-2}} \int_0^1 K_i'(r)r^n dr(1+o(1)), \tag{4.36}$$

where $\tau_i = (\frac{2\sigma_i}{n-2} - 1)\frac{2n}{n-2}$ and $b = \lim_{i \longrightarrow +\infty} M_i^{(\frac{2\sigma_i}{n-2}-1)} u_i(1)$.

To prove (4.36), we let $w_i(r) = M_i^{(\frac{2\sigma_i}{n-2}-1)} u_i(r)$. The function $w_i(r)$ is uniformly bounded in any compact set of $(0, \infty)$. Therefore, a subsequence of w_i (still denoted by w_i) converges to $h(r)$. Since w_i satisfies

$$\Delta w_i + K_i(r) u_i^{\frac{4}{n-2}} w_i = 0$$

with $u_i(r) \longrightarrow 0$ in any compact set of $(0, \infty)$, we have $h(r) = \frac{a}{r^{n-2}} + b$ for some $a, b \geq 0$ and $a + b > 0$. The number a can be determined by

$$(2-n)a = \lim_{i \longrightarrow +\infty} w_i'(1) = -M_i^{\frac{2\sigma_i}{n-2}-1} \int_0^1 K_i(r) u_i^{\frac{n+2}{n-2}}(r) r^{n-1} dr.$$

Applying (4.34), we have

$$M_i \int_0^{r_i} K_i(r) u_i^{\frac{n+2}{n-2}} r^{n-1} dr \sim 1.$$

Hence, together with (4.33), it implies

$$\lim_{i \longrightarrow +\infty} M_i^{\frac{2\sigma_i}{n-2}-1} \int_0^1 K_i r^{n-1} u_i^{\frac{n+2}{n-2}}(r) dr = 0$$

since (II) is satisfied. Thus, $a = 0$ (here we only assume (II) is satisfied). By (4.34), it is elementary to compute

$$\int_0^{r_i} |K'(r)| u_i^{\frac{2n}{n-2}}(r) r^n dr \sim M_i^{-\frac{2\sigma_i}{n-2}}. \tag{4.37}$$

Since $\sigma < \frac{n(n-2)}{n+2}$, we have

$$M_i^{(\frac{2\sigma_i}{n-2}-1)\frac{2n}{n-2}} \int_0^{r_i} |K_i'(r)| u_i^{\frac{2n}{n-2}}(r) r^n dr \longrightarrow 0 \tag{4.38}$$

as $i \longrightarrow +\infty$. Thus,

$$\lim_{i\longrightarrow+\infty} M_i^{(\frac{2\sigma_i}{n-2}-1)\frac{2n}{n-2}} \int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr = b^{\frac{2n}{n-2}} \int_0^1 K_t'(r)r^n dr,$$

from which (4.36) follows.

If $\sigma = \frac{n(n-2)}{n+2}$, we have

$$\int_0^1 |K_i'(r)|u_i^{\frac{2n}{n-2}}(r)r^n dr = O(M_i^{-\frac{2\sigma_i}{n-2}} + M_i^{(1-\frac{2\sigma_i}{n-2})\frac{2n}{n-2}}). \tag{4.39}$$

Now suppose both $u_i(0) = M_i$ and $u_i^*(0) = N_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$.

Case 1. Both u_i and u_i^* satisfy (I), where $u_i^*(r) = r^{2-n}u(\frac{1}{r})$ is the Kelvin transformation of u_i.

In this case, we have $M_i^{-1} \sim u_i(1) = u_i^*(1) \sim N_i^{-1}$. Hence $M_i u_i(x)$ converges to $a|x|^{2-n} + b$ for some $a, b > 0$ as $i \longrightarrow +\infty$. Without loss of generality, we may assume $M_i \geq N_i$. We can compute a and b by

$$(2-n)a = \lim_{i\longrightarrow+\infty} M_i u_i'(1) = \lim_{i\longrightarrow+\infty} M_i \int_0^1 K_i(x)u_i^{\frac{n+2}{n-2}}(r)r^{n-1}dr \tag{4.40}$$

$$= K(0)^{1-\frac{n}{2}}(n(n-2))^{\frac{n}{2}} \int_0^\infty \frac{r^{n-1}}{(1+r^2)^{\frac{n+2}{2}}}dr,$$

$$(n-2)b = \lim_{i\longrightarrow+\infty}(-M_i u_i^{*\prime}(1)) = \lim_{i\longrightarrow+\infty}(\frac{M_i}{N_i})N_i \int_0^1 K_i^*(x)u_i^{*\frac{n+2}{n-2}}(r)r^{n-1}dr$$

$$\geq (K(\infty))^{1-\frac{n}{2}}(n(n-2))^{\frac{n}{2}} \int_0^\infty \frac{s^{n-1}}{(1+s^2)^{\frac{n+2}{2}}}ds. \tag{4.41}$$

If one of σ_0 and β_0 is bigger than $n-2$, say, $\sigma_0 > n-2$, then

$$\frac{n-2}{2n}\int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr = P(1,u_i) = -\frac{(n-2)^2}{2}abM_i^{-2}(1+o(1)), \tag{4.42}$$

where $P(1,u_i) = \frac{n-2}{2}u_i\ u_i'(1) + \frac{1}{2}u_i'^2(1) + \frac{2n}{n-2}K_i(1)u_i^{\frac{2n}{n-2}}(1)$. However, by (4.29) and (4.31), we have the left-hand side $= o(1)M_i^{-2}$, which yields a contradiction. If $\sigma_0 = n-2$ and $\beta_0 = n-2$, the absolute value of the left hand side $= (\frac{K(0)}{n(n-2)})^{-(n-1)}\ \frac{(n-2)^2\varepsilon_0}{2n}\ M_i^{-\frac{2\sigma_i}{n-2}}(1+o(1)) < \frac{(n-2)^2}{2}abM_i^{-2}$ by

(4.40) and (4.41), provided that ε_0 is small, (we note that by our assumption on K_t, we must have $t_0 = 0, \sigma_i \geq n-2$ and $\beta_i \geq n-2$), which again leads to a contradiction.

Case 2. Assume u_i satisfies (I) and u_i^* satisfies (II). In this situation, we have $\frac{1}{\tilde{\sigma}_0} + \frac{1}{\tilde{\beta}_0} \neq \frac{2}{n-2}$.

Suppose $\sigma_0 < n$ and $\beta_0 > \frac{n(n-2)}{n+2}$. Then the Pohozaev identity implies

$$\int_0^1 K_i'(r) u_i^{\frac{2n}{n-2}}(r) r^n dr = -\int_1^\infty K_i'(r) u_i^{\frac{2n}{n-2}} r^n dr = \int_0^1 K_i^{*\prime} u_i^{*\frac{2n}{n-2}}(r) r^n dr. \tag{4.43}$$

Since

$$M_i^{-1} \sim u_i(1) = u_i^*(1) \sim N_i^{1-\frac{2\beta_i}{n-2}}, \tag{4.44}$$

we have from (4.29), (4.35) and (4.43),

$$N_i^{(1-\frac{2\beta_i}{n-2})\frac{2\sigma_i}{n-2}} \sim M_i^{-\frac{2\sigma_i}{n-2}} \sim N_i^{-\frac{2\beta_i}{n-2}} \tag{4.45}$$

which implies

$$0 = \lim_{i \longrightarrow +\infty} [\frac{2\beta_i}{n-2} + (1 - \frac{2\beta_i}{n-2})\frac{2\sigma_i}{n-2}] = \frac{2\beta_0}{n-2} + (1 - \frac{2\beta_0}{n-2})\frac{2\sigma_0}{n-2},$$

i.e., $\frac{1}{\alpha_0} + \frac{1}{\beta_0} = \frac{2}{n-2}$, which yields a contradiction.

Suppose $\sigma_0 < n$ and $\beta_0 < \frac{n(n-2)}{n+2}$. Then by (4.29), (4.36) and (4.43), we have $N_i^{(1-\frac{2\beta_i}{n-2})\frac{2\sigma_i}{n-2}} \sim M_i^{-\frac{2\sigma_i}{n-2}} \sim N_i^{(1-\frac{2\beta_i}{n-2})\frac{2n}{n-2}}$ which implies $\sigma_0 = n$, a contradiction again.

Suppose $\beta_0 = \frac{n(n-2)}{n+2}$. Since $\frac{1}{\tilde{\alpha}_0} + \frac{1}{\tilde{\beta}_0} \neq \frac{2}{n-2}$, we must have $\sigma_0 < n$, in this case, we have $N_i^{(1-\frac{2\beta_i}{n-2})\frac{2\sigma_i}{n-2}} \sim M_i^{-\frac{2\sigma_i}{n-2}} \leq c(N_i^{(1-\frac{2\beta_i}{n-2})\frac{2n}{n-2}} + N_i^{\frac{-2\beta_i}{n-2}})$, which implies

$$\begin{aligned} 0 \geq \frac{2\beta_0}{n-2} + (1 - \frac{2\beta_0}{n-2})\frac{2\sigma_0}{n-2} &= (1 - \frac{2\beta_0}{n-2})(\frac{2(\sigma_0 - n)}{n-2}) \\ &= \frac{2(n-\sigma_0)}{n-2}(\frac{2\beta_0}{n-2} - 1) > 0, \end{aligned}$$

a contradiction.

Suppose $\sigma_0 \geq n$. Then β_0 satisfies either $\beta_0 > \frac{n(n-2)}{n+2}$ or $\beta_0 < \frac{n(n-2)}{n+2}$. If $\beta_0 > \frac{n(n-2)}{n+2}$, by (4.31) and (4.43), we have

$$N_i^{-\frac{2\beta_i}{n-2}} \leq o(1) M_i^{-\frac{2\tau}{n-2}} = o(1) N_i^{(1-\frac{2\beta_i}{n-2})\frac{2\tau}{n-2}}$$

for any $\tau < n$. Hence, it leads to

$$0 \geq \lim_{i \longrightarrow +\infty} [(\frac{2\beta_i}{n-2} - 1)\frac{2n}{n-2} - \frac{2\beta_i}{n-2}] = \frac{2(n+2)}{(n-2)^2}(\beta_0 - \frac{n(n-2)}{n+2}) > 0,$$

a contradiction.

If $\beta_0 < \frac{n(n-2)}{n+2}$, then, by (4.36) and (4.43), the identity

$$\begin{aligned} & b^{\frac{2n}{n-2}} N_i^{(1-\frac{2\beta_i}{n-2})\frac{2n}{n-2}} \int_0^1 K_i^{*\prime}(r) r^n dr (1 + o(1)) \\ & = \int_0^1 K_i'(r) u_i^{\frac{2n}{n-2}}(r) r^n dr = N_i^{(1-\frac{2\beta_i}{n-2})\frac{2n}{n-2}} b^{\frac{2n}{n-2}} \int_0^1 K_i'(r) r^{-n} dr (1 + o(1)) \end{aligned} \tag{4.46}$$

holds, where $b = \lim_{i \longrightarrow +\infty} N_i^{(\frac{2\beta_i}{n-2}-1)} u_i^*(1)$. To see this, from the proof of (4.36), $N_i^{(\frac{2\beta_i}{n-2}-1)} u_i^*(r) \longrightarrow b$ in any compact set of $(0, \infty)$. By Kelvin transformations, it implies $N_i^{(\frac{2\beta_i}{n-2}-1)} u_i(r) \longrightarrow b r^{2-n}$ in any compact set of $(0, \infty)$, and then (4.46) follows immediately (we note that $\int_0^1 K_i'(r) r^{-n} dr$ might be $-\infty$ if $\sigma = n$). Therefore,

$$\int_0^1 K_i'(r) r^{-n} dr = \int_0^1 K_i^{*\prime}(r) r^n dr = -\int_1^\infty K_i'(r) r^{-n} dr,$$

i.e.,

$$\int_0^\infty K_i'(r) r^{-n} dr = 0,$$

which yields a contradiction to our assumption. Hence, the proof for Case 2 is complete.

Case 3. Both u_i and u_i^* satisfies (II). In this case, we have

$$M_i^{1-\frac{2\sigma_i}{n-2}} \sim u_i(1) = u_i^*(1) \sim N_i^{1-\frac{2\beta_i}{n-2}}.$$

From the proof of (4.36), we see that both $M_i^{\frac{2\sigma_i}{n-2}-1}u_i(r)$ and $M_i^{\frac{2\sigma_i}{n-2}-1}u_i^*(r)$ converges to positive constants in any compact set. On the the other hand, $M_i^{\frac{2\sigma_i}{n-2}-1}u_i^*(r) = M_i^{\frac{2\sigma_i}{n-2}-1}u_i(\frac{1}{r})r^{2-n}$ converges to ar^{2-n} for some $a > 0$. Hence we obtain a contradiction.

Next, we assume $M_i \longrightarrow +\infty$ as $i \longrightarrow +\infty$ and $N_i = u_i^*(0)$ is bounded. Since $u_i(1) \sim M_i^{1-\frac{2\hat{\sigma}_i}{n-2}}$ where $\hat{\sigma}_i = \sigma_i$ if (II) is satisfied, and $\hat{\sigma}_i = n-2$ if (I) is satisfied, we have $u_i^*(0) \sim u_i(1) \sim M_i^{1-\frac{2\hat{\sigma}_i}{n-2}}$, i.e., there is some constants $c > 0$ such that $M_i^{\frac{2\hat{\sigma}_i}{n-2}-1}u_i(r) \leq cr^{2-n}$ for all $r > 0$ if (I) is satisfied, for $r \geq r_i$ with r_i given in (3.23) if (II) is satisfied.

Suppose (II) is satisfied for u_i. Since $M_i^{\frac{2\sigma_i}{n-2}-1}u_i(r)$ converges to a positive constant as $i \longrightarrow +\infty$ from the proof of (4.36), we obtain a contradiction. Suppose (I) is satisfied for u_i and $\sigma < n$,

$$M_i^{-\frac{2\sigma_i}{n-2}} \sim -\int_0^1 K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr = \int_1^\infty K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr \leq cM_i^{-\frac{2n}{n-2}},$$

which yields a contradiction. If $\sigma \geq n$, then we obtain

$$0 = \lim_{i\longrightarrow+\infty} M_i^{\frac{2n}{n-2}} \int_0^\infty K_i'(r)u_i^{\frac{2n}{n-2}}(r)r^n dr = a^{\frac{2n}{n-2}} \int_0^\infty K_t'(r)r^{-n}dr < 0,$$

which yields a contradiction, where $M_i u_i(r) \longrightarrow ar^{2-n}$ as $i \longrightarrow +\infty$ is used. Therefore, the proof of (4.25) is completely finished. And then, the conclusion of Theorem 1.2 follows immediately from Theorem 0.7 of [10] for part(i) and from Theorem 1.3 above for part (ii).

REFERENCES

[1] A. Bahri and J. M. Coron, *The scalar-curvature problem on standard three-dimensional sphere,* J. of Func. Anal., 95 (1991), 106-172.

[2] G. Bianchi and H. Egnell, *An ODE approach to the equation* $\Delta u + Ku^{\frac{n+2}{n-2}} = 0$ *in* $\mathbf{R}^n$, Math. Z., 21 (1992), 137-166.

[3] C. S. Cheng and J. L. Chern, *Conformal Metrics with Prescribed Curvature on* S^n, J. of Diff. Eqs., 106 (1993), 155-185.

[4] S. Y. Chang, M. J. Gursky, and P. Yang, *The scalar curvature equation on 2 and 3-sphere,* Calculus of Variations and PDE, 1 (1993), 205-229.

[5] C. C. Chen and C. S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes,* Comm. Pure Appl. Math., (1997), 971–1017.

[6] C. C. Chen and C. S. Lin, *Estimates of the conformal scalar curvature equation via the method of moving planes,* II, J. Diff. Geom., 49 (1998), 115–178.

[7] C. C. Chen and C. S. Lin, *On the asymptotic symmetry of singular solutions of the conformal scalar curvature equation.*

[8] C. C. Chen and C. S. Lin, *Local behavior of singular positive solutions of semilinear elliptic equations with Sobolev exponent,* Duke Math. J., 78 (1995), 315-334.

[9] Y. Y. Li, *Prescribing scalar curvature on S^n and related problems, part I,* J. Diff. Equations, 120 (1995), 319-410.

[10] Y. T. Li, *Prescribing scalar curvature on S^n and related problems, part II: Existence and Compactness,* Comm. Pure Appl. Math., 49 (1996), 541-597.

[11] R. Schoen and D. Zhang, *Prescribing scalar curvature on S^n,* Calculus of Variation and PDEs, 4 (1996), 1-25.