Advances in Differential Equations Volume 5 (4–6) April–June 2000, pp. 747–772

INSTABILITY OF SPHERICAL INTERFACES IN A NONLINEAR FREE BOUNDARY PROBLEM

X. Chen*
Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260

M. Taniguchi†
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology, Tokyo 152-8552, Japan

(Submitted by: H. Matano)

Abstract. Existence and stability of spherically symmetric stationary interfaces in a two-phase boundary problem are studied in $\mathbf{R}^N$ $(N \geq 2)$. We show that there exist two such solutions: a large ball and a small one. The linearized eigenvalue problem shows that the large ball is unstable with some fastest growing mode. We specify the mode precisely.

1. Introduction. This paper is concerned with a coupled system of a free phase boundary $\Gamma(t)$ and a function $v(x,t)$ in $x \in \mathbf{R}^N$ $(N \geq 2)$, $t > 0$. We consider

$$\tau V_{\Gamma(t)} = c(v) - \varepsilon(N-1)H \quad \text{on } \Gamma(t),\ t>0, \tag{1.1}$$
$$v_t - \Delta v = G^+(v)\chi_{\Omega^+(t)} + G^-(v)\chi_{\Omega^-(t)} \quad \text{in } \mathbf{R}^N,\ t>0, \tag{1.2}$$
$$\Gamma(0) = \Gamma_0,$$
$$v(x,0) = v_0(x) \quad \text{in } \mathbf{R}^N.$$

Here $\Gamma(t)$ is an embedded surface in $\mathbf{R}^N$, and it divides $\mathbf{R}^N$ into two disjoint regions $\Omega_+(t)$ and $\Omega_-(t)$. The free boundary $\Gamma(t)$ is often called an interface. We denote the mean curvature of $\Gamma(t)$ by H, and the normal velocity pointing

*Partially supported by the National Science Foundation Grant DMS–9622872 and the Alfred P. Sloan Research Fellowship.
†Part of this work is completed during the author's visit to the University of Pittsburgh.
Accepted for publication June 1999.
AMS Subject Classifications: 35R35, 35B35.

from $\Omega_+(t)$ into $\Omega_-(t)$ by $V_{\Gamma(t)}$. The letter χ_A stands for the characteristic function of a set $A \subset \mathbf{R}^N$.

This free boundary problem appears in activator-inhibitor models or propagator-controller models in chemical reactions, population biology, morphogenesis and genetics. See, for instance, [3], [8], [7] and the references there. Equation (1.1)–(1.2) can be approximated by a reaction-diffusion equation describing an activator-inhibitor model as was shown by Soravia and Souganidis [11]. See also [1].

This paper studies the existence and stability of spherically symmetric solutions which are stationary to Equation (1.1)–(1.2). Our problem is quite simple, but is not trivial. If $c(v)$, ε and $G^{\pm}(v)$ are approximately fixed, there exist any number of radially symmetric stationary solutions to Equation (1.1)–(1.2). The number of such solution for given $c(v)$, ε and $G^{\pm}(v)$ and their stability should be studied. This paper is devoted to Equation (1.1)–(1.2) under the condition that $\varepsilon > 0$ is small. More general cases without this assumption will be studied in the forthcoming paper [12].

Our standing assumptions are as follows.

(A1) $G^{\pm}(\,\cdot\,) : [0,1] \to \mathbf{R}$ is of class C^1 with $G_v^{\pm}(v) < 0$ for all $v \in [0,1]$, $G^+(1) = 0$, $G^-(0) = 0$.

(A2) $c(\,\cdot\,) : [0,1] \to \mathbf{R}$ is of class C^1 with $c'(v) < 0 \quad$ for all $v \in [0,1]$. There exists $v^* \in (0,1)$ such that $c(v^*) = 0$ holds true.

(A3) $v^* < \widehat{v}$. Here $\widehat{v} \in (0,1)$ is uniquely determined by

$$\int_0^{\widehat{v}} G^-(s)\,ds + \int_{\widehat{v}}^1 G^+(s)\,ds = 0 \tag{1.3}$$

(A4) $\tau > \tau_0$. Here τ_0 is a positive constant given by (4.4) that depends only on $G^{\pm}(\,\cdot\,)$ and $c(\,\cdot\,)$, and is independent of ε.

(A5) $\varepsilon > 0$ is sufficiently small.

Here $G_v^{\pm}(v)$ denote the derivatives of $G^{\pm}(v)$, respectively. It follows from (A1) that $G^+(v) \geq 0$ and $G^-(v) \leq 0$ hold true for all $v \in [0,1]$. The smallness of ε comes from the property: an activator diffuses more slowly and reacts much faster compared with an inhibitor.

The existence of a smooth solution locally in time to (1.1)–(1.2) is studied by X-Y. Chen [2]. The existence of viscosity solutions globally in time is studied by [6].

Theorem 1.1 ([2]). *Let Γ_0 be a compact surface in $\mathbf{R}^N$ of class $C^{2+\theta_0}$ with θ_0 with $0 < \theta_0 < 1$, and let $v_0(x)$ belong to $C^{1+\theta_0}(\mathbf{R}^N)$ with $\|v_0\|_{C^{1+\theta_0}(\mathbf{R}^N)} < +\infty$. Then there exists a solution (v, Γ) in $[0, T]$ with $T = T(\Gamma, v_0) > 0$. The solution $\Gamma(t)$ is a C^2-surface and of class C^1 in t. On the other hand v is of class C^1 in t, and it is of class C^2 in x in the set $\bigcup_{0\le t\le T}[(\Omega_+(t) \cup \Omega_-(t)) \cup \{t\}]$.*

Remark 1.1. The solution is unique. See [1] for the proof.

In this paper, we show that there exist exactly two spherically (or radially) symmetric solutions that are stationary to (1.1)–(1.2). One is a small ball of radius R^{S} with $R^{\mathrm{S}}(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. The radius of the other ball, say, a large ball has a positive limit value $\lim_{\varepsilon\to 0} R^{\mathrm{L}}(\varepsilon) > 0$. We show that this large ball is unstable by studying the linearized eigenvalue problem. This instability was pointed out by [8] for some special case. Here we will study instability in more general cases, and study the distribution of eigenvalues more precisely. The eigenvalues in the right-hand side in the complex plane turn out to be all real. If we parameterize them by some parameter, they lie on a graph of a strictly concave non-negative function. The parameter corresponds to the order of spherical harmonics of the associated eigenfunction. We study the maximum point, that is, the largest eigenvalue, and study the maximizer precisely. The result is that the largest eigenvalue is positive and is equal to $\sigma/\tau + O(\varepsilon^{\frac{1}{3}})$, and that the order of the spherically harmonic is $O(\varepsilon^{-\frac{1}{3}})$. This result is similar to the instability of planar interfaces as in [13].

Interfaces may form a various kind of shapes. Among them, this paper studies only spherical interfaces which are stationary. However to study these interfaces is important in the following senses. Using the stability analysis in this paper, one can classify all the spherical interfaces of equilibrium state into the following four types in the forthcoming paper [12]. The spherical interfaces of the first type are stable for all perturbations; those of the second type are stable for all radially symmetric perturbations but unstable for some non-radial ones; those of the third type are unstable for some radial perturbations, and in this case there exists only one positive eigenvalue in the right half-plane $\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \ge 0\}$ and the associated eigenfunction is radially symmetric except the zero eigenvalue corresponding the phase shift; those of the fourth type is an exceptional case. The large equilibrium ball in the present paper is a typical example of the second type. Suppose that a stationary ball of the second type are randomly perturbed in the initial time and that it does not remain the same, then they do not spherically shrink nor expand, but lose the symmetry as it evolves and changes its shape with some characteristic pattern. We will study this characteristic pattern for the large stationary ball in Theorem 2.1 in this paper. In [12], the small equilibrium ball of Theorem 2.1 turns out to belong to the third type. If it is randomly perturbed and if it does not remain the same, then it shrinks or expands spherically with spherical symmetry. This small ball seems to be a threshold for nucleation and growth of an activated small region.

This paper is organized as follows. Section 2 is devoted to the construction of the radially symmetric stationary solutions. In Section 3, we study the linearized eigenvalue problem for the large equilibrium ball and derives ordinary differential equation for the eigenvalues. Section 4 is devoted to the analysis of this equation, which gives the distribution of the eigenvalues in the complex plane.

2. The stationary radially symmetric solutions. In this section, we study the stationary solutions of (1.1)–(1.2) with spherical interfaces; namely, we are looking for (v, R) with $R \in (0,\infty), v \in L^\infty(\mathbf{R}^N)$ such that

$$\begin{cases} -\Delta v = G^R(v,x) \stackrel{\text{def}}{=} G^+(v)\chi_{B(R)} + G^-(v)\chi_{B(R)^{\mathrm{c}}} \quad \text{in } \mathbf{R}^N, \\ c(v(x))\Big|_{|x|=R} = (N-1)\varepsilon/R. \end{cases} \tag{2.1}$$

holds true. We shall prove the following theorem.

Theorem 2.1. *Assume that* $G^\pm(\,\cdot\,) \in C^1(\mathbf{R})$ *satisfies* $G^\pm_v(v) < 0$ *in* $\mathbf{R}$ *and* $G^-(0) = 0 = G^+(1)$. *Also assume that* $c(\,\cdot\,) \in C^1(\mathbf{R})$ *satisfies* $c'(v) < 0$ *in* $\mathbf{R}$ *and* $c(v^*) = 0$ *for some* $v^* \in (0, \widehat{v})$ *where* $\widehat{v}$ *is the unique constant determined by* (1.3). *Then for every small positive* ε , *Problem* (2.1) *has exactly two solutions:* $(v^S(\,\cdot\,,\varepsilon), R^S(\,\cdot\,,\varepsilon))$ *and* $(v^L(\,\cdot\,,\varepsilon), R^L(\varepsilon))$. *These two solutions have the following properties:*

1. *Both* v^S *and* v^L *are radially symmetric, monotone decreasing in* $|x|$, *and take values in* $(0,1)$.
2. *The solution* (v^S, R^S) *is a small solution in the sense that as* $\varepsilon \searrow 0$, $R^S(\varepsilon) = (N-1)\varepsilon/c(0) + O(\varepsilon^3|\ln\varepsilon|)$, $\|v^S(\,\cdot\,,\varepsilon)\|_{L^\infty(\mathbf{R}^N)} = O(\varepsilon^2|\ln\varepsilon|)$ *hold true.*
3. *The solution* (v^L, R^L) *satisfies, as* $\varepsilon \searrow 0$, $R^L(\varepsilon) = R^* + O(\varepsilon)$, $v^L(\,\cdot\,,\varepsilon) = V(\,\cdot\,) + O(\varepsilon)$ *in* $L^\infty(\mathbf{R}^N)$, *where* (V, R^*) *is the unique solution to* (2.1)

with $\varepsilon = 0$*:*

$$\begin{cases} -\Delta V = G^+(V)\chi_{B(R^*)} + G^-(V)\chi_{B(R^*)^c} & \textit{in } \mathbf{R}^N, \\ c(V(R^*)) = 0 \qquad (\textit{equivalent to } V(R^*) = v^*). \end{cases} \tag{2.2}$$

Remark 2.1. (1) One of the authors was informed that the existence of $(v^{\mathrm{L}}, R^{\mathrm{L}})$ in Theorem 2.1 is obtained in [14]. The method in this paper seems to be simpler and more elementary than that of [14].

(2) Since a bounded solution v to the first equation in (2.1) for any given $R \in [0,\infty)$ satisfies $0 \le v \le 1$, the values of $G^{\pm}(\,\cdot\,)$ and $c(\,\cdot\,)$ outside of the interval $[0,1]$ is irrelevant to the solution. Here extending them to functions of $\mathbf{R}$ is only for convenience.

The idea of the proof of Theorem 2.1 is as follows. For each fixed $R > 0$, consider

$$-\Delta w = G^R(w,x) \overset{\text{def}}{=} G^+(w)\chi_{B(R)} + G^-(w)\chi_{B(R)^c} \qquad \text{in } \ \mathbf{R}^N. \tag{2.3}$$

We shall show that there exists a unique bounded solution $w = w(\,\cdot\,,R)$, and the solution is radially symmetric. Hence, if we define $h(R) = w(x,R)|_{|x|=R}$, then $(R, v \overset{\text{def}}{=} w(\,\cdot\,,R))$ is a solution to (2.1) if and only if R solves

$$H(R) \overset{\text{def}}{=} c(h(R))R = (N-1)\varepsilon. \tag{2.4}$$

We show that for every small positive ε, this equation has exactly two solutions, by proving the following

1) $h'(R) > 0$ for all $R \in (0,\infty)$;
2) $h(R) \to \widehat{v}$ as $R \to \infty$;
3) $h(R) = O(R^2|\ln R|)$ and $h'(R) = O(R|\ln R|)$ as $R \searrow 0$.

Now we begin to prove Theorem 2.1. First, we establish the existence of a bounded solution to (2.3).

Lemma 2.2. *For every* $R > 0$, (2.3) *has a unique bounded solution* $w = w(x,R)$*. In addition, the solution is radially symmetric, and takes values only in* $(0,1)$*.*

Proof. Define $W^R(w,x) = \int_w^1 G^+(s)ds$ for $|x| \le R$ and $W^R(w,x) = -\int_0^w G^-(s)ds$ for $|x| > R$. Then $W^R(\,\cdot\,,x)$ is non-negative and strictly convex (i.e., $W^R_{ww} > 0$). Consider the functional $\int_{\mathbf{R}^N}[\frac{1}{2}|\nabla w|^2 + W^R(w,|x|)]dx$ in

$H^1(\mathbf{R}^N)$. This functional is non-negative and convex. Also it is bounded for the function $w \equiv 0$. Hence, by a classical various technique, this functional has a unique minimizer in $H^1(\mathbf{R}^N)$. Denote this minimizer by $w(x,R)$. Then it solves (2.3). In addition, w is confined in $[0,1]$. Indeed, otherwise a function $\max\{\min\{w(x),1\},0\}$ would have strictly smaller energy than that of w. Since $G_v^R(v,x) < 0$ in $\mathbf{R}\times\mathbf{R}^N$, a strong maximum principle (cf. [10]) is applicable to (2.3). It says that $w \in (0,1)$ and is the only bounded solution to (2.3). Consequently, w is radially symmetric, because otherwise a rotation of the coordinates for w would yield another solution. □

Next, we study certain properties of the solution w. Since w is radially symmetric, we shall often write $w(x,R)$ as $w(r,R)$ with $r = |x|$. Since we are interested in $h(R) \overset{\text{def}}{=} w(R,R)$, it is convenient to introduce $u(s,R)$ defined by $u(s,R) \overset{\text{def}}{=} w(Rs,R)$, $s \in [0,\infty)$. Then $h(R) = u(1,R)$, and u satisfies, for $s \in (0,\infty)$,

$$\begin{cases} -\Delta_s u = R^2 G(u,s) \overset{\text{def}}{=} R^2\big[G^+(u)\chi_{\{(0,1)\}} + G^-(u)\chi_{\{(1,\infty)\}}\big], \\ 0 < u(s) < 1, \quad s \in (0,\infty), \\ u_s(0,R) = 0, \end{cases} \tag{2.5}$$

where $\Delta_s = \partial_{ss} + \frac{N-1}{s}\partial_s$. Here $\partial_{ss} = (\partial_s)^2$.

Lemma 2.3. *The solution $u(s,R)$ to (2.5) has the following properties:*

1. *For every $R > 0$, $u_s(\,\cdot\,,R) < 0$ in $(0,\infty)$ and $\lim_{s\to\infty} u(s,R) = 0$ hold true.*
2. *For all $s > 0$ and $R > 0$, $u_R(s,R) < 2R^{-1}u(s,R)$ holds true.*
3. *For all $s \geq 0$ and $R > 0$, $(s-1)u_s < R\, u_R$ holds true, and consequently, $h(R) = u(1,R)$ satisfies $h'(R) > 0$ for all $R > 0$.*

Proof. 1. First we show that $u_s < 0$. When $s \in (0,1)$, $(s^{N-1}u_s)_s = -s^{N-1}R^2G^+(u) < 0$ so that $u_s < 0$ in $(0,1]$. When $s \in (1,\infty)$, $(s^{N-1}u_s)_s = -s^{N-1}R^2G^-(u) > 0$. This implies $u_s < 0$ in $(1,\infty)$, because u is bounded. In conclusion, $u_s < 0$ in $(0,\infty)$, and $\max_{s>0}(s^{N-1}|u_s|) = |u_s(1,R)|$. Since u monotonically decreases in s, $\lim_{s\to\infty} u(s,R)$ exists. As $u \in H^1(\mathbf{R}^N)$, the limit must be zero.

2. We show that u_R exists and belongs to $H^1(\mathbf{R}^N)$. For any given $R > 0$ and $R' > 0$, we have

$$\begin{aligned} &-\Delta_s(u(s,R') - u(s,R)) - R^2[G(u(s,R'),s) - G(u(s,R),s)] \\ &= [(R')^2 - R^2]G(u(s,R),s). \end{aligned}$$

Thus for some $\theta_0 \in (0,1)$, we have

$$\begin{aligned}&\left(-\Delta_s - R^2 G_v(u(s,R)+\theta_0(u(s,R')-u(s,R)),s)\right)(u(s,R')-u(s,R))\\&= [(R')^2 - R^2]G(u(s,R),s). \qquad (2.6)\end{aligned}$$

Note that $0 < a_0 < -G_v(u(s,R)+\theta_0(u(s,R')-u(s,R)))$ for all $s>0$ and $R>0$, where $a_0 \overset{\text{def}}{=} \min\{\min_{0\le y\le 1}|G_v^+(y)|, \min_{0\le y\le 1}|G_v^-(y)|\}$. We multiply (2.6) by $s^{N-1}(u(s,R')-u(s,R))$ and integrate it over $(0,\infty)$ in s. Then using

$$\begin{aligned}&\int_0^\infty s^{N-1}G(u(s,R),s)(u(s,R')-u(s,R))\,ds\\&\le \left(\int_0^\infty s^{N-1}G(u(s,R),s)^2\,ds\right)^{\frac12}\left(\int_0^\infty s^{N-1}(u(s,R')-u(s,R))^2\,ds\right)^{\frac12},\end{aligned}$$

we see that $u(s,R')$ converges to $u(s,R)$ in $H^1(\mathbf{R}^N)$, that is,

$$\begin{aligned}&\lim_{R'\to R}\int_0^\infty s^{N-1}\left(u(s,R')-u(s,R)\right)^2\,ds\\&=\lim_{R'\to R}\int_0^\infty s^{N-1}\left(u_s(s,R')-u_s(s,R)\right)^2\,ds = 0. \qquad (2.7)\end{aligned}$$

From (2.5) we have $|u(s,R)-u(0,R)| \le \frac{R^2G^+(0)}{2N}s^2$ for all $s>0$ and $R>0$. From this inequality and (2.7), $\lim_{R'\to R}\sup_{0<s<\infty}|u(s,R')-u(s,R)| = 0$ holds true. Define

$$f(s,R',R) = -R^2G_v(u(s,R)+\theta_0(u(s,R')-u(s,R)),s) + R^2G_v(u(s,R),s).$$

Then we have

$$\lim_{R'\to R}\sup_{0<s<\infty}|f(s,R',R)| = 0. \qquad (2.8)$$

Define $U(s,R',R) = (u(s,R')-u(s,R))/(R'-R)$. From (2.6), we have

$$\begin{aligned}&\left(-\Delta_s - R^2G_v(u(s,R),s)\right)U(s,R',R)\\&= (R'+R)G(u(s,R),s) - f(s,R',R)U(s,R',R). \qquad (2.9)\end{aligned}$$

We have

$$\|G(u(s,R),s)\|\,\|U(s,R',R)\| \le k_0\|U(s,R',R)\|^2 + \frac{1}{4k_0}\|G(u(s,R),s)\|^2$$

for any $k_0 > 0$, where $\|\cdot\|$ is the $L^2(\mathbf{R}^N)$ norm:

$$\|u(s,R)\|^2 = \int_0^\infty |u(s,R)|^2 s^{N-1}\,ds.$$

We multiply (2.9) by $s^{N-1}U(s,R',R)$ and integrate it over $(0,\infty)$. Then using the above inequality and (2.8), we see that $U(s,R',R)$ is bounded in $H^1(\mathbf{R}^N)$ as $R' \to R$. Thus the right-hand side of (2.9) converges to $2RG(u(s,R),s)$ in $L^2(\mathbf{R}^N)$ as $R' \to R$. From the Lax-Milgram theorem, $\left(-\Delta_s - R^2G_v(u(s,R),s)\right)^{-1}$ is a bounded linear operator from $L^2(\mathbf{R}^N)$ to $H^1(\mathbf{R}^N)$. Hence when $R' \to R$, $U(s,R',R)$ converges to

$$2R\left(-\Delta_s - R^2G_v(u(s,R),s)\right)^{-1}G(u(s,R),s)$$

in $H^1(\mathbf{R}^N)$. Thus we obtain

$$[-\Delta_s - R^2G_v(u,s)]u_R = 2\,R\,G(u,s) \quad \text{ in } \mathbf{R}^N.$$

Since u_R belongs to $H^1(\mathbf{R}^N)$, $\lim_{s\to\infty} u_R(s,R) = 0$ holds true. It follows that

$$[-\Delta_s - R^2G_v(u,s)](2R^{-1}u - u_R) = -2uRG_v(u,s) > 0$$

in $\mathbf{R}^N$. Since $G_v(u)$ is negative and uniformly apart from zero, by a maximum principle for elliptic equations [10], we obtain $2R^{-1}u - u_R > 0$ in $\mathbf{R}^N$.

3. Consider the function $R\,u_R(s,R) + (1-s)u_s(s,R)$. It is of class C^1 in the variable s in $(0,\infty)$. Also, direct calculation shows that

$$\left[-\Delta_s - R^2G_v(u,s)\right]\left(R\,u_R - su_s + u_s\right) = -(N-1)s^{-2}u_s > 0 \text{ for all } s \in (0,\infty).$$

Therefore, by a maximum principle, the function $R\,u_R(s,R) + (1-s)u_s(s,R)$ cannot obtain a negative minimum in $(0,\infty)$. Also, since

$$[R\,u_R + (1-s)u_s]_s|_{s=0} = u_{ss}(0,R) = -R^2G^+(u(0,R))/N < 0,$$

the function $R\,u_R + (1-s)u_s$ cannot obtain a negative minimum at $s = 0$. Because the limit, as $s \to \infty$, of $R\,u_R + (1-s)u_s$ is zero, we then conclude that $R\,u_R + (1-s)u_s > 0$ for all $s \geq 0$. Consequently, $h'(R) = u_R(1,R) > 0$. This completes the proof of the lemma. □

Next we study the behavior of $h(R)$ as $R \to \infty$. To do this, consider the ordinary differential equation

$$-Q''(z) = G^+(Q)\chi_{\{z<0\}} + G^-(Q)\chi_{\{z>0\}}, \qquad z \in \mathbf{R}. \tag{2.10}$$

If $Q(z)$ is a bounded and non-increasing solution, then $Q(\pm\infty) \stackrel{\text{def}}{=} \lim_{z\to\pm\infty} Q(z)$ exists, and by the differential equation, $Q(\infty) = 0$ and $Q(-\infty) = 1$ hold true. In addition, multiplying (2.10) by $2Q'$ and integrating over $(-\infty, z]$ for $z \le 0$ and over $[z, \infty)$ for $z \ge 0$, we obtain

$$\Big(Q'(z)\Big)^2 = 2\int_{Q(z)}^{1} G^+(v)dv \quad \text{if} \quad z \le 0, \tag{2.11}$$

$$\Big(Q'(z)\Big)^2 = 2\int_{0}^{Q(z)} |G^-(v)|dv \quad \text{if} \quad z \ge 0. \tag{2.12}$$

Taking $z = 0$ we see that $Q(0) = \widehat{v}$ where $\widehat{v}$ is defined in (1.3). On the other hand, if we set $Q(0) = \widehat{v}$ and integrate (2.11) or (2.12) for $z < 0$ and $z > 0$ respectively, we obtain a bounded and non-increasing solution of (2.10).

In conclusion, Equation (2.10) has a unique bounded and non-increasing solution, and the solution satisfies $Q(0) = \widehat{v}$.

Now, we study $h(R)$ for large R.

Lemma 2.4. *Let $w(r, R)$ be the solution of* (2.3) *and $h(R) = w(R, R)$. Then*

$$\lim_{R\to\infty} h(R) = \widehat{v}, \qquad \lim_{R\to\infty} w(R + z, R) = Q(z) \quad \textit{for all } z \in \mathbf{R},$$

where $\widehat{v}$ is as in (1.3) *and $Q(\,\cdot\,)$ is the unique bounded and non-increasing solution to* (2.10).

Proof. Define $Q^R(z) = w(R + z, R)$. Then

$$-Q^R_{zz} - \frac{N-1}{R+z} Q^R_z = G^+(Q^R)\chi_{\{z<0\}} + G^-(Q^R)\chi_{\{z>0\}} \tag{2.13}$$

for all $z \in (-R, \infty)$. As Q^R is non-increasing and takes values in $[0, 1]$, there exists a subsequence $R_j \to \infty$ such that $Q^{R_j}(z) \to \tilde{Q}(z)$ is valid for some bounded and non-increasing function $\tilde{Q}$.

Since both $\Delta w(\cdot, , R)$ and $w(\cdot, R)$ are bounded uniformly in R, local elliptic estimates imply that $\{w(\cdot, R)\}_{R>0}$ is uniformly bounded in $C^{1+\frac{1}{2}}_{\text{loc}}[0, \infty)$. Hence, the convergence $Q^{R_j}(z) \to \tilde{Q}(z)$ is in $C^1[-M, M]$ for every $M > 0$. It then follows from (2.13) that $\tilde{Q}$ is a bounded and non-increasing solution to (2.10). Since solutions to (2.10) are unique, $\tilde{Q}$ is independent of the sequence $\{Q^{R_j}\}$. Hence, as $R \to \infty$, $Q^R(z) \to Q(z)$ in $C^1[-M, M]$ for every $M > 0$. This completes the proof of the lemma. □

Finally we study the behavior of $h(R)$ as $R \searrow 0$. To get an accurate estimate, consider the following linear ordinary differential equation:

$$B_{zz}(z) + \frac{N-1}{z} B_z(z) = B(z) \qquad \text{for all } z \in (0,\infty). \tag{2.14}$$

This equation has an analytic solution $B(z) = K(z)$ defined by $K(z) = \sum_{n=0}^{\infty} a_n z^{2n}$ for all $z \geq 0$, where $a_0 = 1$, $a_n = \frac{a_{n-1}}{2n(2n+N-2)}$. This solution has the property that all derivatives are positive. By variation of constants, there is another solution $B(z) = J(z)$ given by

$$J(z) = K(z) \int_z^{\infty} \frac{1}{s^{N-1} K^2(s)} ds. \tag{2.15}$$

Clearly $J(\cdot)$ is positive and $J(z) \to 0$ as $z \to \infty$. Since $J(z)$ cannot obtain a positive local maximum, $J_z(z) < 0$ for all $z > 0$. In addition, using the expression of K, we can calculate, as $z \searrow 0$,

$$J(z) = \begin{cases} \frac{1}{N-2} z^{-(N-2)} (1 + O(z^2 |\ln z|)) & \text{if } N > 3, \\ \frac{1}{N-2} z^{-(N-2)} (1 + O(z)) & \text{if } N = 3, \\ |\ln z| + O(1) & \text{if } N = 2, \end{cases} \tag{2.16}$$

and

$$\frac{J(z)}{z J_z(z)} = \begin{cases} -\frac{1}{N-2} + O(z^2 |\ln z|) & \text{if } N > 3, \\ -\frac{1}{N-2} + O(z) & \text{if } N = 3, \\ -|\ln z| + O(1) & \text{if } N = 2. \end{cases} \tag{2.17}$$

Now we set

$$I(R) \stackrel{\text{def}}{=} \left[0, (2N)^{-1} G^+(0)\right] \times \left[R \min_{0 \leq y \leq 1} |G_v^-(y)|^{\frac{1}{2}}, R \max_{0 \leq y \leq 1} |G_v^-(y)|^{\frac{1}{2}}\right].$$

For $(a,b) \in I(R)$, consider a function $v(a,b,s)$ defined by

$$v(a,b,s) = \begin{cases} R^2 a \left\{(1-s^2) + 2J(b)/\left[b|J_z(b)|\right]\right\} & \text{if } \ s \in [0,1], \\ 2R^2 a J(bs)/\left[b|J_z(b)|\right] & \text{if } \ s \in (1,\infty). \end{cases} \tag{2.18}$$

Clearly $v \in C^1[0,\infty)$ and $v_s|_{s=0} = 0$. Also, v is positive-valued, and decreases monotonically in s. Define $\widetilde{v}(R)$ and $v_0(R)$ as

$$\widetilde{v}(R) \stackrel{\text{def}}{=} \sup_{(a,b) \in I(R)} v(a,b,1), \quad v_0(R) \stackrel{\text{def}}{=} \sup_{(a,b) \in I(R)} v(a,b,0).$$

Then from (2.17) and (2.18), the following estimates

$$\begin{aligned} &\widetilde{v}(R) = O(R^2),\ v_0(R) = O(R^2) && \text{if } N \geq 3, \\ &\widetilde{v}(R) = O(R^2|\ln R|),\ v_0(R) = O(R^2|\ln R|) && \text{if } N = 2 \end{aligned} \tag{2.19}$$

hold true as $R \to 0$. Using the relation $J_{zz}(z) + \frac{N-1}{z} J_z(z) = J(z)$, we obtain

$$-\Delta_s v - R^2 G(v) = \begin{cases} R^2[2Na - G^+(v)] & \text{if } \ s \in (0,1), \\ -b^2 v - R^2 G^-(v) & \text{if } \ s \in (1,\infty). \end{cases} \tag{2.20}$$

Hence, if we define

$$\overline{a} \stackrel{\text{def}}{=} (2N)^{-1} G^+(0), \qquad \overline{b} \stackrel{\text{def}}{=} R \min_{0 \leq y \leq \widetilde{v}(R)} |G_v^-(y)|^{\frac{1}{2}},$$

$$\underline{a} \stackrel{\text{def}}{=} (2N)^{-1} \min_{0 \leq y \leq v_0(R)} G^+(y), \qquad \underline{b} \stackrel{\text{def}}{=} R \max_{0 \leq y \leq \widetilde{v}(R)} |G_v^-(y)|^{\frac{1}{2}},$$

then $-\Delta v(\overline{a}, \overline{b}, \cdot\,) - R^2 G(v(\overline{a}, \overline{b}, \cdot\,)) \geq 0$ and $-\Delta v(\underline{a}, \underline{b}, \cdot\,) - R^2 G(v(\underline{a}, \underline{b}, \cdot\,)) \leq 0$ in $[0,\infty)$. Hence, by comparison principle,

$$v(\underline{a}, \underline{b}, s) \leq u(s, R) \leq v(\overline{a}, \overline{b}, s) \qquad \text{for all } s \geq 0. \tag{2.21}$$

Using the asymptotic behavior of $J(z)$ for small z in (2.17), we have the following:

1) when $N \geq 3$, $\overline{a}, \underline{a} = (2N)^{-1} G^+(0) + O(R^2)$, and $\overline{b}, \underline{b} = R|G_v^-(0)|^{\frac{1}{2}} + o(R)$ as $R \to 0$;
2) when $N = 2$, $\overline{a}, \underline{a} = (2N)^{-1} G^+(0) + O(R^2|\ln R|)$, and $\overline{b}, \underline{b} = R|G_v^-(0)|^{\frac{1}{2}}$ $+ o(R)$ as $R \to 0$.

Thus, we have the following lemma.

Lemma 2.5. *As* $R \searrow 0$, $h(R) \stackrel{\text{def}}{=} w(R,R) = u(1,R)$ *satisfies*

$$h(R) = \frac{G^+(0)}{N(N-2)} R^2 + O(R^3) \quad \textit{if } N \geq 3, \tag{2.22}$$

$$h(R) = \frac{G^+(0)}{N} R^2 |\ln R| + O(R^2) \quad \textit{if } N = 2, \tag{2.23}$$

$$0 < h'(R) < 2h(R)/R = O(R|\ln R|). \tag{2.24}$$

Observe that the estimate (2.24) follows from Lemma 2.3. Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1. Since $h'(\,\cdot\,) > 0$, $h(+0) = 0$, and $h(\infty) = \widehat{v} > v^*$ where $c(v^*) = 0$, there exists a unique $R^* > 0$ such that $h(R^*) = v^*$ holds true. That is, (2.2) has a unique solution, and it is given by $R = R^*, V = w(\,\cdot\,, R^*)$.

Now we solve (2.1). We know that $(R, v \overset{\text{def}}{=} w(\,\cdot\,, R))$ is a solution to (2.1) if and only if R satisfies (2.4).

When $R \in [0, \sqrt[3]{\varepsilon}]$, we have $H'(R) = c(h(R)) + R\,c'(h(R))\,h'(R) = c(0) + O(R^2|\ln R|) > c(0)/2$. Therefore, as $H(0) = 0$, there exists a unique $R^{\mathrm{S}} \in (0, \sqrt[3]{\varepsilon})$ with $H(R^{\mathrm{S}}) = (N-1)\varepsilon$. In addition,

$$R^{\mathrm{S}} = \frac{(N-1)\varepsilon}{c(0) + O((R^{\mathrm{S}})^2|\ln R^{\mathrm{S}}|)} = \frac{(N-1)\varepsilon}{c(0)} + O(\varepsilon^3|\ln \varepsilon|). \tag{2.25}$$

When $R \in [\sqrt[3]{\varepsilon}, R^* - \sqrt{\varepsilon}]$, we have $H(R) > (N-1)\varepsilon$ since $h'(\,\cdot\,) > 0$ and $c'(\,\cdot\,) < 0$. Hence, there is no solution to $H(R) = (N-1)\varepsilon$ when $R \in [\sqrt[3]{\varepsilon}, R^* - \sqrt{\varepsilon}]$. When $R \in [R^* - \sqrt{\varepsilon}, R^*]$, we have

$$H'(R) = c(h(R)) + R\,c'(h(R))\,h'(R) = R^* c'(v^*) h'(R^*) + o(1) < 0,$$

so that there exists a unique $R^{\mathrm{L}} \in (R^* - \sqrt{\varepsilon}, R^*)$ such that $H(R^{\mathrm{L}}) = \varepsilon(N-1)$ holds true. In addition,

$$R^{\mathrm{L}} = R^* + \frac{(N-1)\varepsilon}{c'(v^*)\,h'(R^*)} + o(\varepsilon). \tag{2.26}$$

In the interval (R^*, ∞), we have $c(h(R)) < 0$ so that there is no solution to $H(R) = \varepsilon(N-1)$. In conclusion, (2.1) has exactly two solutions.

Using (2.25) and (2.21), we obtain $\|v^{\mathrm{S}}(\,\cdot\,, \varepsilon)\|_{L^\infty(\mathbf{R}^N)} = O(\varepsilon^2|\ln \varepsilon|)$. Finally we estimate $V - v^{\mathrm{L}}$. Recalling that $(s-1)u_s < Ru_R < 2u$ and that $\max_s |s^{N-1}u_s| \le |u_s(1, s)| = R^2 \int_0^1 s^{N-1} G^+(u)\,ds$ as in Lemma 2.3 and its proof, we can derive that $|u_R|$ is uniformly bounded. It then follows that

$$\|u(\,\cdot\,, R^*) - u(\,\cdot\,, R^{\mathrm{L}})\|_{C^0[0,\infty)} \le \|u_R\|_{L^\infty([0,\infty)\times[r_0, R^*])} |R^* - R^{\mathrm{L}}| = O(\varepsilon).$$

Here $r_0 > 0$ is any small number. This completes the proof of Theorem 2.1.

3. Linearization. In the sequel, we are only interested in the radially symmetric stationary solution $(v^{\mathrm{L}}, R^{\mathrm{L}})$ given in Theorem 2.1. For the stability of $(v^{\mathrm{S}}(\,\cdot\,,\varepsilon), R^{\mathrm{S}}(\,\cdot\,,\varepsilon))$, one can refer to [12]. Although $(v^{\mathrm{L}}, R^{\mathrm{L}})$ depends on ε, we shall not write this ε–dependency explicitly. Hence, we rewrite it as $(v_0(x), R_0)$. Often we write v_0 as $v_0(r)$ where $r = |x|$. We have

$$\left\{\begin{array}{rcl} -\Delta v_0 &=& G^+(v_0)\chi_{B(R_0)} + G^-(v_0)\chi_{B(R_0)^{\mathrm{c}}} \quad \text{in } \mathbf{R}^N, \\ c(v_0)|_{|x|=R_0} &=& (N-1)\varepsilon/R_0. \end{array}\right. \tag{3.1}$$

We seek solutions of (1.1), (1.2) of the form

$$\left\{\begin{array}{l} \Gamma(t) = \big\{[R_0 + \eta\rho(\xi)e^{\mu t} + O(\eta^2)]\xi : \xi \in S^{N-1} \subset \mathbf{R}^N\big\}, \\ v(x,t) = v_0(x) + \eta w(x)e^{\mu t} + O(\eta^2). \end{array}\right. \tag{3.2}$$

We are interested in non-trivial solutions (μ, ρ, w) such that $\mathrm{Re}(\mu) \geq 0$ is valid since they correspond to perturbations that grow exponentially fast.

We now derive equations for (μ, ρ, w). Denote by $\delta(|x| - R_0)$ the following distribution concentrated at the sphere $r = R_0$; namely,

$$\iint_{\mathbf{R}^N} \delta(|x| - R_0)\zeta(x)dx = R_0^{N-1}\int_{S^{N-1}} \zeta(R_0\xi)d\xi, \qquad \text{for all } \zeta \in C_0^1(\mathbf{R}^N).$$

Then, substituting the expression of (ρ, v) in (3.2) into (1.2), dividing both sides by $\eta e^{\mu t}$, and sending η to 0, we obtain

$$\begin{aligned} \mu w - \Delta w = &\big[G_v^+(v_0)\chi_{B(R_0)} + G_v^-(v_0)\chi_{B(R_0)^c}\big]w \\ &+ [G^+(v_0(R_0)) - G^-(v_0(R_0))]\rho(x/|x|)\delta(|x| - R_0) \end{aligned} \tag{3.3}$$

in $\mathbf{R}^N$. To simplify Equation (1.1), we first note that, for every $x = [R_0 + \eta e^{\mu t}\rho(\xi) + O(\eta^2)]\xi \in \Gamma(t)$,

$$c(v(x)) = c(v_0(R_0)) + \eta e^{\mu t}\, c'(v_0(R_0))[v_{0,r}(R_0)\,\rho(\xi) + w(R_0\xi)] + o(\eta).$$

To find the normal velocity V and the mean curvature H of $\Gamma(t)$, it is convenience to express $\Gamma(t)$ as the zero level set of

$$\Phi(x,t) \stackrel{\text{def}}{=} |x| - R_0 - \eta e^{\mu t}\widetilde{\rho}\big(\tfrac{x}{|x|}\big) + O(\eta^2)$$

where $\widetilde{\rho}(y) : \mathbf{R}^N \to \mathbf{R}$ is an arbitrarily fixed smooth extension of $\rho(\xi) : S^{N-1} \to \mathbf{R}$. Under this setting,

$$H = \frac{1}{N-1}\mathrm{div}\Big(\frac{\nabla\Phi}{|\nabla\Phi|}\Big), \quad V = -\frac{\Phi_t}{|\nabla\Phi|} \qquad \text{for all } x \in \Gamma(t). \tag{3.4}$$

Note that, for each $k = 1, \cdots, N$ and $x \in \Gamma(t)$, we have

$$\Phi_{x_k} = \frac{x_k}{|x|} - \eta e^{\mu t} \sum_{j=1}^{N} \widetilde{\rho}_{y_j}\Big(\tfrac{x}{|x|}\Big)\Big(\tfrac{\delta_{kj}}{|x|} - \tfrac{x_k x_j}{|x|^3}\Big) + O(\eta^2), \quad x \in \Gamma(t).$$

Hence, $|\nabla\Phi(x)| = 1 + O(\eta^2)$ for all $x \in \Gamma(t)$. Consequently, $V = \mu\eta e^{\mu t}\widetilde{\rho}\big(\tfrac{x}{|x|}\big) + O(\eta^2)$, $x \in \Gamma(t)$, and

$$\begin{aligned}(N-1)H &= \Delta\Phi + O(\eta^2) = (N-1)/|x| - \eta e^{\mu t}\big\{\Sigma_{i,j=1}^{N}\widetilde{\rho}_{y_i y_j}\big(\tfrac{x}{|x|}\big)\big[\tfrac{\delta_{ij}}{|x|^2} - \tfrac{x_i x_j}{|x|^4}\big] \\ &\quad + \Sigma_{i=1}^{N}\widetilde{\rho}_{y_i}\big(\tfrac{x}{|x|}\big)\tfrac{(1-N)x_i}{|x|^3}\big\} + O(\eta^2) \\ &= \frac{N-1}{R_0} - \frac{\eta e^{\mu t}}{R_0^2}\big\{(N-1)\rho(\xi) + \Delta_{S^{N-1}}\rho(\xi)\big\}\big|_{\xi = x/|x|} + O(\eta^2),\end{aligned}$$

for $x \in \Gamma(t)$. Here $\Delta_{S^{N-1}}$ is the Laplace-Beltrami operator defined for all (smooth) functions $\rho : S^{N-1} \to \mathbf{R}$. It can be calculated as follows: Given $\rho(\xi) : S^{N-1} \to \mathbf{R}$,

$$\Delta_{S^{N-1}}\rho(\xi) = \sum_{i,j=1}^{N}(\delta_{i,j} - \xi_i\xi_j)\widetilde{\rho}_{y_i y_j}(y) - (N-1)\sum_{k=1}^{N}\xi_k\widetilde{\rho}_{y_k}(y)\Big|_{y=\xi}$$

for $\xi \in S^{N-1} \subset \mathbf{R}^N$. Here $\widetilde{\rho}(y) : \mathbf{R}^N \to \mathbf{R}$ is any smooth extension of ρ. One can directly verify that the expression of $\Delta_{S^{N-1}}\rho$ is independent of the extension $\widetilde{\rho}$.

Now substituting the expression of $c(v)$, H and V in (1.1), dividing both side by $\eta e^{\mu t}$ and sending η to 0, we obtain

$$\begin{aligned}\tau\mu\rho(\xi) &= c'(v_0(R_0))[w(R_0\xi) + v_{0,r}(R_0)\rho(\xi)] \\ &\quad + \varepsilon R_0^{-2}\Big(\Delta_{S^{N-1}}\rho(\xi) + (N-1)\rho(\xi)\Big).\end{aligned} \tag{3.5}$$

for all $\xi \in S^{N-1}$. Therefore, (1.1)–(1.2) has a solution (Γ, v) of the form (3.2) if (μ, ρ, w) satisfies (3.3) and (3.5). To simplify equations for (μ, ρ, w), we shall decompose ρ into spherical harmonics, defined as follows:

Definition 1. Let $H_n(x)$ be a homogeneous polynomial of degree n with $\Delta H_n = 0$. Then $H_n(\xi) = H_n|_{S^{N-1}}$ is called a spherically harmonic function of degree n.

For readers' convenience, we recall a few properties of spherical harmonics and the Laplace-Beltrami operator $\Delta_{S^{N-1}}$. Every spherically harmonic function $\Phi(\xi)$ of degree n satisfies

$$-\Delta_{S^{N-1}}\Phi = \kappa_n\Phi, \qquad \kappa_n = n(n+N-2),$$

where $n = 0, 1, \ldots$. The dimension of the eigenspace of $\Delta_{S^{N-1}}$ associated with κ_n is $\frac{(2n+N-2)\Gamma(n+N-2)}{\Gamma(n+1)\Gamma(N-1)}$, which equals the dimension of all spherical harmonics of degree n. The system of all orthonormal spherical harmonics, say, $\{\Phi_j\}_{j=1}^{\infty}$, is complete for the continuous functions on S^{N-1}. In the polar coordinate (r,ξ), $x = r\xi$, the Laplace operator $\Delta = \sum_{i=1}^{N}\partial_{x_i x_i}$ has the expression

$$\Delta = \Delta_r + r^{-2}\Delta_{S^{N-1}} \quad \text{where} \quad \Delta_r \stackrel{\text{def}}{=} \partial_{rr} + (N-1)r^{-1}\partial_r.$$

For more details, we refer interested readers to [4, 5].

Now we go back to (3.3) and (3.5). Let Φ_j be any spherically harmonic function of degree n. Multiplying these two equations by $\Phi_j(x/|x|)$ and defining $\rho_j \in \mathbf{R}$, $w_j : [0,\infty) \to \mathbf{R}$ by

$$\rho_j = \int_{S^{N-1}} \rho(\xi)\Phi_j(\xi)\,d\xi, \qquad w_j(r) = \int_{S^{N-1}} w(r\xi)\Phi_j(\xi)\,d\xi,$$

we obtain

$$\begin{aligned} &\Big\{ -\Delta_r + \kappa_n r^{-2} - [G_v^+(v_0)\chi_{\{r<R_0\}} + G_v^-(v_0)\chi_{\{r>R_0\}}] + \mu\Big\} w_j \\ &\quad = [G^+(v_0(R_0)) - G^-(v_0(R_0))]\,\rho_j\,\delta(r-R_0), \\ \tau\mu\rho_j &= c'(v_0(R_0))[v_{0,r}(R_0)\rho_j + w_j(R_0)] + \varepsilon R_0^{-2}(N-1-\kappa_n)\rho_j. \end{aligned}$$

Here κ_n is the eigenvalue of $-\Delta_{S^{N-1}}$ associated with Φ_j. Since $\kappa_n \geq 0$, $G_v^{\pm} < 0$, and $\mathrm{Re}(\mu) \geq 0$, one can verify that if $\rho_j = 0$, then $w_j \equiv 0$. Hence, if (ρ_j, w_j) is non-trivial, then $\rho_j \neq 0$, and we can always scale the solution with

$$\rho_j = \rho^0 \stackrel{\text{def}}{=} \frac{1}{R_0[G^+(v_0(R_0)) - G^-(v_0(R_0))]}. \tag{3.6}$$

Now assume that (ρ_j, w_j) is non-trivial. We take ρ_i as above and define

$$\begin{cases} W(s) = w_j(R_0 s), \qquad s \geq 0, \\ \lambda = \mu R_0^2, \\ g(s) = -R_0^2[G_v^+(v_0(R_0 s))\chi_{\{s<1\}} + G_v^-(v_0(R_0 s))\chi_{\{s>1\}}] > 0\,, \\ \sigma = c'(v_0(R_0))\, v_{0,r}(R_0) R_0^2 > 0 \\ \nu = -c'(v_0(R_0))[G^+(v_0(R_0)) - G^-(v_0(R_0))]R_0^3 > 0\,. \end{cases} \tag{3.7}$$

Then (λ, W) solves the following eigenvalue problem:

$$\begin{cases} \left(-\Delta_s + \kappa s^{-2} + g(s) + \lambda\right) W(s) = \delta, \quad s \in (0, \infty), \\ \sigma + \varepsilon(N - 1 - \kappa) - \nu W(1) = \tau\lambda \end{cases} \tag{3.8}$$

where $\kappa = \kappa_n$ and δ is the Dirac delta function concentrated at $s = 1$. Since ρ and w can be written as $\rho = \sum_{j=1}^{\infty} \rho_j \Phi_j(\xi)$ and $w = \sum_{j=1}^{\infty} w_j(r)\Phi_j(x/|x|)$, we see that to find solutions of (3.3) and (3.5), it suffices to solve the linear system (3.8) for all $\kappa = \kappa_n \stackrel{\text{def}}{=} n(n + N - 2)$, $n = 0, 1, \cdots$. In the sequel, we call λ an eigenvalue and W an eigenfunction if (λ, W) solves (3.8) and W does not vanish identically.

The following is the main theorem of this paper.

Theorem 3.1. *Consider the eigenvalue problem* (3.8). *There exist positive constants $\tau_0 > 0$ and ε_0 such that for every $\tau > \tau_0$ and every $\varepsilon \in (0, \varepsilon_0]$, the following assertions hold:*

1. *If λ is an eigenvalue with non-negative real part, then λ is real.*
2. *There exists $\kappa^* = [\sigma + O(\sqrt{\varepsilon})]\varepsilon^{-1}$ such that if $\kappa \in [0, N-1) \cup (\kappa^*, \infty)$, every eigenvalue has a negative real part, and if $\kappa \in [N - 1, \kappa^*]$, there exists exactly one eigenvalue having a non-negative real part.*
3. *For $\kappa \in [N - 1, \kappa^*]$, if $\Lambda(\kappa)$ stands for the eigenvalue with a non-negative real part, then $\Lambda(\kappa)$ is real with $\Lambda(N - 1) = \Lambda(\kappa^*) = 0$ and the curve $\lambda = \Lambda(\kappa)$ is concave, i.e. $\Lambda''(\kappa) < 0$ for all $\kappa \in [N - 1, \kappa^*]$.*
4. *The largest eigenvalue $\lambda^{\max}$ is equal to $\sigma/\tau + O(\varepsilon^{\frac{1}{3}})$. It is obtained at a unique $\kappa = \kappa^{\max}$ that has the estimate $\kappa^{\max} = [(\nu/4)^{\frac{2}{3}} + O(\varepsilon^{\frac{1}{3}})]\varepsilon^{-\frac{2}{3}}$ as $\varepsilon \to 0$.*

Using this theorem, the transformation (3.7), and Theorem 2.1, we then obtain the following corollary concerning the stability of the stationary solution $(v^{\mathrm{L}}, R^{\mathrm{L}})$ of (1.1), (1.2).

Corollary 3.2. *Assume that* $\tau > \tau_0$ *and* $\varepsilon \in (0, \varepsilon_0]$ *and let* (v_0, R_0) *be the stationary solution of* (1.1), (1.2) *given by* (v^L, R^L) *in Theorem* 2.1. *Then,* (R_0, v_0) *is unstable and the set of all eigenvalues with non-negative real part is given by* $\{\mu : \mu = R_0^{-2}\Lambda(\kappa_n), n = 1, 2, \cdots, n^*\}$ *where* n^* *is the largest integer such that* $n^*(n^* + N - 2) \leq \kappa^*$ *holds true. The associated eigenspace with respect to* $\mu = R_0^{-2}\Lambda(\kappa_n)$, $n = 1, 2, \cdots, n^*$, *is given by*

$$\operatorname{span}\left\{(\rho^0\Phi, W(r/R_0)\Phi) \;\middle|\; \begin{array}{l} (\lambda = \Lambda(\kappa_n), W) \textit{ solves } (3.8) \textit{ with } \kappa = \kappa_n, \textit{ where} \\ \Phi \textit{ is any spherically harmonic function of order } n' \\ \textit{so that either } n' = n \textit{ or } \Lambda(\kappa_{n'}) = \Lambda(\kappa_n) \textit{ holds} \end{array}\right\}$$

where ρ^0 *is defined in* (3.6). *Moreover, every eigenvalue with a non-negative real part is real and the largest eigenvalue* $\mu^{\max}(\varepsilon)$ *satisfies*

$$\lim_{\varepsilon \to 0} \mu^{\max}(\varepsilon) = \tau^{-1} c'(v^*)\, V_r(R^*) \;\; > 0$$

where V *is the unique solution to* (2.2). *The associated orders of spherical harmonics for the eigenfunctions with eigenvalue* $\mu^{\max}$ *consists of one positive integer or two successive positive integers that satisfy*

$$\big[\big(\frac{1}{4}c'(v^*)[G^+(v^*) - G^-(v^*)]\big)^{\frac{1}{3}} R^* + o(1)\big]\varepsilon^{-\frac{1}{3}}.$$

as ε *goes to zero. The maximum order* $n^*(\varepsilon)$ *of spherical harmonics that give unstable modes satisfies*

$$\lim_{\varepsilon \to 0} \big(\varepsilon^{\frac{1}{2}}\, n^*(\varepsilon)\big) = \big(c'(v^*)\, V_r(R^*)\big)^{\frac{1}{2}} R^*.$$

In the next section, we shall study the eigenvalue problem (3.8) and in particular prove Theorem 3.1.

4. The eigenvalue problem. In this section, we study the eigenvalue problem (3.8); namely, we prove Theorem 3.1. In the sequel, we shall use $\overline{u}$ to denote the complex conjugate of u. Also we use $(\,\cdot\,,\,\cdot\,)$ to denote the $L^2(\mathbf{R}^N)$ inner product and $\|\cdot\|$ the $L^2(\mathbf{R}^N)$ norm. We put

$$(u, v) = \textstyle\int_0^\infty s^{N-1} u(s)\overline{v(s)}\, ds, \qquad \|u\|^2 = (u, u).$$

Since the $H^1(\mathbf{R}^N)$ norm of a function $u(|x|)\Phi(x/|x|)$ is equivalent to $\big[\int_0^\infty s^{N-1}(u_s^2 + u^2 + \kappa_n s^{-2}u^2)ds\big]^{1/2}$ if Φ is a spherically harmonic function of degree n, we introduce

$$H^1(\kappa) \stackrel{\text{def}}{=} \Big\{ w \in L^2(0,\infty) : \|u\|^2_{H^1(\kappa)} \stackrel{\text{def}}{=} \textstyle\int_0^\infty [s^{N-1}(|u_s|^2 + |u|^2) + s^{N-3}\kappa|u|^2] ds < \infty \Big\}.$$

In the sequel, if there is no confusion, we shall write $H^1(\kappa)$ simply as H^1. We use $(H^1)'$ to denote the dual of H^1 consisting of all bounded linear functional over H^1. Also, if $u \in H^1$ and $f \in (H^1)'$, we use $\langle u, f\rangle$ to denote the value of the functional f acted on u. In particular, if $v \in L^2$, then the induced functional $(\,\cdot\,, v)$ will also be denoted by v so that we have

$$\langle u, v\rangle = (u, v) = \int_0^\infty s^{N-1} u\overline{v}\, ds, \qquad \text{for all } u \in H^1, v \in L^2.$$

We introduce an operator $\mathcal{L}$ defined by

$$\mathcal{L} \stackrel{\text{def}}{=} -\frac{1}{s^{N-1}}\frac{d}{ds}\big(s^{N-1}\frac{d}{ds}\big) + g(s) + \frac{1}{s^2}\kappa + \lambda \tag{4.1}$$

For any $v \in H^1$, the functional $\mathcal{L}[v] \in (H^1)'$ is defined by

$$\langle u, \mathcal{L}[v]\rangle = B(u, v) \stackrel{\text{def}}{=} \int_0^\infty \Big[s^{N-1} u_s \overline{v}_s + s^{N-3}\kappa u\overline{v} + s^{N-1}(g + \overline{\lambda}) u\overline{v} \Big] ds.$$

One can directly verify, if u, v are smooth and in H^1, then

$$(\overline{\mathcal{L}}[u], v) = (u, \mathcal{L}[v]) = \langle u, \mathcal{L}[v]\rangle = \langle \overline{\mathcal{L}}[u], v\rangle;$$

namely, the definition of $\mathcal{L}[u]$ is an extension of the differential operator $\mathcal{L}$ over C^2 functions. Finally, we introduce

$$g_m \stackrel{\text{def}}{=} \inf\nolimits_{s\in[0,\infty)} g(s), \qquad g_M \stackrel{\text{def}}{=} \sup\nolimits_{s\in[0,\infty)} g(s).$$

Recalling the definition of g, we have $0 < g_m \le g_M < \infty$.

We begin with solving the first equation in (3.8) for every $\kappa \ge 0$ and λ with $\mathrm{Re}(\lambda) > -g_m$.

Lemma 4.1. *For every $\kappa \ge 0$ and complex number λ with* $\mathrm{Re}(\lambda) > -g_m$*, there exists a unique $w(s, \kappa, \lambda) \in H^1(\kappa)$ solving*

$$\mathcal{L}w = \delta \qquad \text{in } \ (H^1)'. \tag{4.2}$$

Consequently, (λ, W) *with* $\mathrm{Re}(\lambda) > -g_m$ *is a solution to* (3.8) *for some* $\kappa \geq 0$ *if and only if* $W = w(\,\cdot\,, \kappa, \lambda)$ *and* λ *solves*

$$F(\kappa, \lambda) \stackrel{\text{def}}{=} \sigma + \varepsilon(N - 1 - \kappa) - \tau\lambda - \nu w(1, \kappa, \lambda) = 0. \tag{4.3}$$

Proof. Uniqueness of solutions to (4.2) follows from the fact that $\mathcal{L}$ is linear and the real part of $B(u, u)$ is strictly positive for any non-trivial $u \in H^1(\kappa)$.

The existence of a solution to (4.2) follows from a classical theory of linear second order ordinary differential equations. For reader's convenience and later applications, we sketch the proof here. First, from the Liouville-Green approximation technique [9, chapter 6], there exists a unique $J(s, \kappa, \lambda)$ smooth in $s \in (0, \infty)$ such that $\mathcal{L}[J] = 0$ is valid in $(0, \infty)$ with

$$J(s, \kappa, \lambda) = s^{(1-N)/2} e^{-s\sqrt{g(\infty)+\lambda}}\, (1 + O(1/s)) \qquad \text{as} \quad s \to \infty.$$

Here $g(\infty) = -R_0^2 G_v^-(0) > 0$, and $-\sqrt{g(\infty) + \lambda}$ is the root of $g(\infty) + \lambda$ with a positive real part. Also, there is a unique solution $K(s, \kappa, \lambda)$ smooth in $s \in [0, \infty)$ such that $\mathcal{L}[K] = 0$ is valid with

$$K(s, \kappa, \lambda) = s^{m(\kappa)}[1 + O(s)] \qquad \text{as} \quad s \searrow 0$$

where

$$m(\kappa) \stackrel{\text{def}}{=} -\tfrac{N-2}{2} + \sqrt{\left(\tfrac{N-2}{2}\right)^2 + \kappa}\,, \qquad \kappa \geq 0\,.$$

Note that J and K are linearly independent because otherwise we obtain a solution of $\mathcal{L}[w] = 0$ in $H^1(\kappa)$ that must be identically 0.

Now defining $w(s, \kappa, \lambda) = c_1(\kappa, \lambda)J$ for all $s \geq 1$ and $w(s, \kappa, \lambda) = c_2(\kappa, \lambda)K$ for all $s \in [0, 1]$ where c_1 and c_2 satisfies

$$c_1 J(1, \kappa, \lambda) - c_2 K(1, \kappa, \lambda) = 0, \qquad c_1 J_s(1, \kappa, \lambda) - c_2 K_s(1, \kappa, \lambda) = 1\,,$$

we obtain a solution to (4.2). □

Next we study properties of $w(\,\cdot\,, \kappa, \lambda)$ for all $\kappa \geq 0$ and complex λ with $\mathrm{Re}(\lambda) > -g_m$.

Lemma 4.2. *Let* $\kappa \geq 0$, $\mathrm{Re}(\lambda) > -g_m$, $w(\,\cdot\,, \kappa, \lambda) \in H^1(\kappa)$ *be the unique solution to* (4.2). *Then the following assertions hold:*

(1) There exist non-zero (complex) constants $c_1(\kappa, \lambda)$ *and* $c_2(\kappa, \lambda)$ *with*

$$w(s, \kappa, \lambda) = c_1(\kappa, \lambda) s^{m(\kappa)}[1 + O(s)] \quad \text{as} \;\; s \searrow 0,$$
$$w(s, \kappa, \lambda) = c_2(\kappa, \lambda) s^{(1-N)/2} e^{-s\sqrt{g(\infty)+\lambda}}[1 + O(1/s)] \quad \text{as} \;\; s \to \infty.$$

(2) *$w(\cdot,\kappa,\lambda)$ is smooth (actually analytic) in (κ,λ) in the domain $\{(\kappa,\lambda) : \kappa > 0, \mathrm{Re}(\lambda) > -g_m\}$ with*

$$\mathcal{L}[w_\lambda] = -w, \qquad \mathcal{L}[w_\kappa] = -s^{-2}w,$$
$$\mathcal{L}[w_{\lambda\lambda}] = -2w_\lambda, \quad \mathcal{L}[w_{\kappa\lambda}] = -w_\kappa - s^{-2}w_\lambda, \quad \mathcal{L}[w_{\kappa\kappa}] = -2s^{-2}w_\kappa.$$

Here w_λ, w_κ, $w_{\kappa\lambda}$, $w_{\kappa\kappa}$ and $w_{\lambda\lambda}$ belong to $H^1(\kappa)$.

(3) *As $\kappa \searrow 0$, $w(s,\kappa,\lambda) \to w(s,0,\lambda)$ uniformly in $s \in [\eta,\infty)$ for any $\eta > 0$. In addition, the function $w(s,0,\lambda)$ is smooth in λ for all $s \in [0,\infty)$ and $\mathrm{Re}(\lambda) > -g_m$.*

Proof. (1) The first assertion follows from the proof of the previous lemma. (2) The second assertion follows from the continuous dependence of K and J in κ and λ. We only prove $\mathcal{L}[w_\kappa] = -s^{-2}w$ for instance. Let $\mathcal{L}$ and w be as in (4.1) and (4.2) be denoted by $\mathcal{L}(\kappa)$ and $w(\,\cdot\,,\kappa)$. Let κ, $\kappa' \in (0,\infty)$. From (4.2) for κ and κ', we have

$$\mathcal{L}(\kappa)\left[(w(s,\kappa') - w(s,\kappa))/(\kappa' - \kappa)\right] = -s^{-2}w(s,\kappa').$$

Because the norms of $H^1(\kappa)$ and $H^1(\kappa')$ are equivalent, and $s^{-2}w(s,\kappa) \in \left(H^1(\kappa)\right)'$, the Lax-Milgram theorem implies that

$$(w(s,\kappa') - w(s,\kappa))/(\kappa' - \kappa) = -\mathcal{L}(\kappa)^{-1}[s^{-2}w(s,\kappa')].$$

Letting $\kappa' \to \kappa$, we obtain $w_\kappa = -\mathcal{L}(\kappa)^{-1}[s^{-2}w(s,\kappa)]$, and hence $w_\kappa \in H^1(\kappa)$.

(3) The proof of the latter part is omitted. To prove the former part of the assertion, first we consider the case $N \geq 3$. In such case, $H^1(\kappa) = H^1(0)$ for all $\kappa \geq 0$. In fact, for any real function u in $H^1(0)$,

$$\int_0^\infty s^{N-3}u^2\,ds = -\int_0^\infty s^{N-3}\left(\int_s^\infty 2uu_s ds\right)ds = -\tfrac{2}{N-2}\int_0^\infty s^{N-2}uu_s\,ds$$
$$\leq \tfrac{2}{N-2}\Big(\int_0^\infty s^{N-3}u^2 ds \int_0^\infty s^{N-1}u_s^2 ds\Big)^{1/2}$$

so that $\int_0^\infty s^{N-3}u^2\,ds \leq \frac{4}{(N-2)^2}\int_0^\infty s^{N-1}u_s^2\,ds$ holds true. Therefore, the norm for $H^1(0)$ and $H^1(\kappa)$ with $\kappa > 0$ are equivalent. With this fact, one can show that $w(\,\cdot\,,\kappa,\lambda)$ is continuous in $H^1(0)$ in $\kappa \in [0,\infty)$. From the Sobolev embedding theorem, $w(s,\kappa,\lambda)$ goes to $w(s,0,\lambda)$ as $\kappa \searrow 0$ uniformly in s in any compact subset of $(0,\infty)$. From the assertion (1) and the continuous dependence of K and J for κ and λ, $|w(s,\kappa,\lambda)|$ goes to zero as $s\to\infty$ uniformly in κ. Thus $w(s,\kappa,\lambda)$ goes to $w(s,0,\lambda)$ as $\kappa\searrow 0$ uniformly in $s\in[\eta,\infty)$. Here we omit the details.

Next we consider the case $N=2$. In such a case $H^1(\kappa)$ with $\kappa>0$ is a proper subset of $H^1(0)$. In fact, since $w(s,0,\lambda)=c_1(1+O(s))$ for small s with $c_1\neq 0$, $w(\,\cdot\,,0,\lambda)\notin H^1(\kappa)$ for any $\kappa>0$. Here we prove the assertion of the lemma as follows.

Using $\mathcal{L}[w]=\delta$ and taking w as a test function, we obtain $B(w,w)=w(1,\kappa,\lambda)$. As $|w(1,\kappa,\lambda)|\le\eta\int_{1/2}^{2}(|w_s|^2+|w|^2)ds+C_\eta$ for any $\eta>0$, we find from the real part of $B(w,w)$ that $\|w(\,\cdot\,,\kappa,\lambda)\|_{H^1(0)}$ is bounded uniformly in $\kappa\ge 0$, with a bound depending only on the positive lower bound of $\mathrm{Re}(\lambda)+g_m$. Hence, any sequence $\{w(\,\cdot\,,\kappa_j,\lambda_j)\}$ with $\kappa_j\searrow 0$ and $\lambda_j\to\lambda$ $(\mathrm{Re}(\lambda)>-g_m)$ has a subsequence converging in C^0 to a function $\tilde{w}\in H^1(0)$ in any compact set of $(0,\infty)$. In addition, one can show that $\mathcal{L}[\tilde{w}]=\delta$ with $\tilde{w}=w(\,\cdot\,,0,\lambda)$. Since $w(\,\cdot\,,0,\lambda)$ is unique, we then conclude that $w(\,\cdot\,,\kappa,\tilde{\lambda})$ converges in C^0 to $w(\,\cdot\,,0,\lambda)$ in any compact subset of $(0,\infty)$ as $\kappa\searrow 0$ and $\tilde{\lambda}\to\lambda$. Because $|w(s,\kappa,\lambda)|$ goes to zero as $s\to\infty$ uniformly in κ, one can also show that the convergence is uniform in $s\in[\eta,\infty)$ for any $\eta>0$. This completes the proof. □

Now we study Equation (4.3). First we show that if τ is suitably large, then all eigenvalues with non-negative real parts are real.

Lemma 4.3. *Assume that τ satisfies*

$$\tau>\tau_0\stackrel{\text{def}}{=}\nu\,\|w(\,\cdot\,,0,0)\|^2. \tag{4.4}$$

Then any eigenvalue λ of (3.8) *with a non-negative real part must be real. In addition, $F_\lambda(\kappa,\lambda)<0$ for all $\kappa\ge 0$ and real $\lambda\ge 0$.*

Proof. Assume that $\lambda=\lambda_R+\mathbf{i}\lambda_I$ is an eigenvalue of (3.8) with $\lambda_R\ge 0$. We write $w(s,\kappa,\lambda)=w_R+\mathbf{i}w_I$ where both w^R and w^I are real.

We can calculate $w(1,s,\lambda)$ by

$$w(1,\kappa,\lambda)=\langle w,\delta\rangle=\langle w,\mathcal{L}[w]\rangle=B(w,w). \tag{4.5}$$

Taking the imaginary part, we obtain $w_I(1,\kappa,\lambda)=-\lambda_I\|w\|^2$. Since (4.3) implies that $\tau\lambda_I=\nu w^I(1,\kappa,\lambda)$, we have the identity

$$\lambda_I\big[\tau-\nu\|w(\,\cdot\,,\kappa,\lambda)\|^2\big]=0. \tag{4.6}$$

We now estimate $\|w(\,\cdot\,,\kappa,\lambda)\|^2$. First, we show that as a function of λ_I, the function $\|w(\,\cdot\,,\kappa,\lambda_R+\mathbf{i}\lambda_I)\|^2$ obtains its maximum at $\lambda_I=0$. In fact, since $\mathcal{L}w_{\lambda_I}=-\mathbf{i}w$, we have

$$\begin{aligned}\frac{\partial}{\partial\lambda_I}\|w(\,\cdot\,,\kappa,\lambda_R+\mathbf{i}\lambda_I)\|^2 &= 2\mathrm{Re}(w,w_{\lambda_I}) = 2\ \mathrm{Re}\langle w,w_{\lambda_I}\rangle\\ &= 2\mathrm{Re}\langle \mathbf{i}\mathcal{L}[w_{\lambda_I}],w_{\lambda_I}\rangle = 2\mathrm{Re}\Big\{\mathbf{i}\overline{B(\bar w_{\lambda_I},\bar w_{\lambda_I})}\Big\} = -2\lambda_I\|w_{\lambda_I}\|^2.\end{aligned}$$

Therefore, $\|w(\,\cdot\,,\kappa,\lambda_R+\mathbf{i}\lambda_I)\|^2<\|w(\,\cdot\,,\kappa,\lambda_R)\|^2$ for all $\lambda_I\neq 0$.

When $\lambda_I=0$, we have $w(1,\kappa,\lambda_R)>0$ from (4.5). Since $w(0,\kappa,\lambda_R)=0$ and $w(\infty,\kappa,\lambda_R)=0$ are valid, the maximum principle implies $w(s,\kappa,\lambda_R)>0$ for all $s>0$. From $\mathcal{L}w_\lambda=w<0$, $\mathcal{L}w_\kappa=-s^{-2}w<0$, $0=w_\lambda(s,\kappa,\lambda_R)|_{s=\infty}$ and $0=w_\kappa(s,\kappa,\lambda_R)|_{s=\infty}$ as in Lemma 4.2, we can see that $w_\lambda(s,\kappa,\lambda_R)<0$ and $w_\kappa(s,\kappa,\lambda_R)<0$ for all $s>0$ using the maximum principle. Thus from $\frac{\partial}{\partial\lambda}\|w\|^2=2(w,w_\lambda)<0$ and $\frac{\partial}{\partial\kappa}\|w\|^2=2(w,w_\kappa)<0$, one can show that

$$\|w(\,\cdot\,,\kappa,\lambda_R)\|^2\le\|w(s,0,0)\|^2$$

holds true. In conclusion, $\|w(\,\cdot\,,\kappa,\lambda)\|^2\le\|w(\,\cdot\,,0,0)\|^2$ for all $\kappa\ge 0$ and $\mathrm{Re}(\lambda)\ge 0$. The identity (4.6) and the assumption on τ then yield $\lambda_I=0$.

Finally, we show $F_\lambda<0$ when $\kappa\ge 0$ and λ is real. Since $\mathcal{L}[w_\lambda]=-w$,

$$w_\lambda(1,\kappa,\lambda) \;=\; \langle w_\lambda,\delta\rangle=\langle w_\lambda,\mathcal{L}[w]\rangle=\langle\mathcal{L}[w_\lambda],w\rangle=-\langle w,w\rangle=-\|w\|^2.$$

It then follows that

$$F_\lambda=-\tau-\nu w_\lambda(1,\kappa,\lambda)=-\tau+\nu\|w(\cdot,\kappa,\lambda)\|^2\le-\tau+\nu\|w(\cdot,0,0)\|^2<0.$$

This completes the proof of the lemma. □

Lemma 4.4. *For every $\kappa>0$ and real $\lambda\ge 0$, $F_{\lambda\lambda}<0$, $F_{\kappa\kappa}<0$, $F_{\lambda\lambda}F_{\kappa\kappa}-F_{\kappa\lambda}^2>0$.*

Proof. Using $\mathcal{L}[w_{\lambda\lambda}]=-2w_\lambda$ and $w=-\mathcal{L}[w_\lambda]$, we can calculate

$$\begin{aligned}w_{\lambda\lambda}(1,\kappa,\lambda)&=\langle w_{\lambda\lambda},\delta\rangle=\langle w_{\lambda\lambda},\mathcal{L}[w]\rangle=\langle\mathcal{L}[w_{\lambda\lambda}],w\rangle\\ &=-2\langle w_\lambda,w\rangle=-2\langle w_\lambda,-\mathcal{L}[w_\lambda]\rangle=2B(w_\lambda,w_\lambda)>0.\end{aligned}$$

Similarly, since $\mathcal{L}[w_{\kappa\kappa}]=-s^{-2}w_\kappa$ and $s^{-2}w=\mathcal{L}[w_\kappa]$,

$$\begin{aligned}w_{\kappa\kappa}(1,\kappa,\lambda)&=\langle w_{\kappa\kappa},\delta\rangle=\langle w_{\kappa\kappa},\mathcal{L}[w]\rangle=\langle\mathcal{L}[w_{\kappa\kappa}],w\rangle\\ &=-2\langle w_\kappa,s^{-2}w\rangle=2\langle w_\kappa,\mathcal{L}[w_\kappa]\rangle=2B(w_\kappa,w_\kappa)>0.\end{aligned}$$

As $\mathcal{L}[w_{\kappa\lambda}] = -w_\kappa - s^{-2}w_\lambda$ and $w_\kappa \not\equiv w_\lambda$, we obtain

$$\begin{aligned} w_{\kappa\lambda}(1,\kappa,\lambda) &= \langle w_{\kappa\lambda}, \delta\rangle = \langle w_{\kappa\lambda}, \mathcal{L}[w]\rangle = \langle \mathcal{L}[w_{\kappa\lambda}], w\rangle \\ &= -\langle w_\kappa, w\rangle - \langle w_\lambda, s^{-2}w\rangle = \langle w_\kappa, \mathcal{L}[w_\lambda]\rangle + \langle w_\lambda, \mathcal{L}[w_\kappa]\rangle \\ &= 2B(w_\kappa, w_\lambda) < 2\sqrt{B(w_\kappa,w_\kappa)B(w_\lambda,w_\lambda)} = \sqrt{w_{\lambda\lambda}w_{\kappa\kappa}}. \end{aligned}$$

The assertion of the lemma thus from the definition of F. □

Now we are ready to study the equation (4.3).

Lemma 4.5. *Assume that τ satisfies* (4.4). *Then there exist a positive constant $\kappa^* \in (N-1, N-1+\sigma\varepsilon^{-1})$ and a smooth function $\Lambda(\kappa)$ defined on $[N-1,\kappa^*]$ such that the following assertions hold:*

1. *$\Lambda(N-1) = \Lambda(\kappa^*) = 0$, $\Lambda(\kappa) \in (0,\sigma/\tau)$ for all $\kappa \in (N-1,\kappa^*)$, and $F(\kappa,\Lambda(\kappa)) = 0$ for all $\kappa \in [N-1,\kappa^*]$;*
2. *the curve $\lambda = \Lambda(\kappa)$ is strictly concave, i.e., $\Lambda''(\kappa) < 0$ for all $\kappa \in [N-1,\kappa^*]$;*
3. *λ solves* (4.3) *with a non-negative real part and with respect to some $\kappa \ge 0$ if and only if $\kappa \in [N-1,\kappa^*]$ and $\lambda = \Lambda(\kappa)$.*

Proof. First we show that when $\kappa = N-1$, $\lambda = 0$ is an eigenvalue. In fact, since the problem (1.1), (1.2) is space independent, a translation (in space) of the solution (R_0, v_0) also gives a solution. In other words, for every unit vector $\mathbf{e}$ and $h \in \mathbf{R}$, $\Gamma(t) \stackrel{\text{def}}{=} h\mathbf{e} + \partial B(R_0)$, together with $v = v_0(h\mathbf{e} + \cdot\,)$ is a stationary solution of (1.1) and (1.2). Hence, $\frac{d}{dh}v_0(h\mathbf{e}+\cdot\,)|_{h=0} = v_{0,r}\Phi_1$, where $\Phi(\xi) \stackrel{\text{def}}{=} \mathbf{e}\cdot\xi$ is a spherically harmonic function of degree 1, is an eigenfunction. In terms of (3.8), we can verify this as follows. Differentiating the equation for $v_0(r)$ with respect to r, one can check that the function $\rho^0 v_r(sR_0, R_0)$ (ρ^0 is as in (3.6)) solves the first equation in (3.8). The other condition in (3.8) follows from the definition of the constants σ and ν. In conclusion, we have $F(N-1,0) = 0$. Note that for all $\kappa > 0$ and real $\lambda > -g_m$,

$$w_\kappa(1,\kappa,\lambda) = \langle w_\kappa, \delta\rangle = \langle w_\kappa, \mathcal{L}[w]\rangle = \langle \mathcal{L}[w_\kappa], w\rangle = -\langle s^{-2}w, w\rangle = -\|s^{-1}w\|^2.$$

Using this, we have

$$F_\kappa(N-1,0) = -\varepsilon - \nu w_\kappa(1,N-1,0) = -\varepsilon + 2\nu\|r^{-1}w(\cdot,N-1,0)\|^2 > 0,$$

because ε is small and $w(\,\cdot\,, N-1, 0)$ is of order 1. Hence, $F(\kappa, 0) > 0$ for all κ close to $N-1$ from the right.

Observe that $F(N-1+\sigma\varepsilon^{-1}, 0) = -\nu w(1, N-1+\sigma\varepsilon^{-1}, 0) < 0$, we conclude that there exists a $\kappa^* \in (N-1, N-1+\sigma\varepsilon^{-1})$ with $F(\kappa^*, 0) = 0$. As $F_{\kappa\kappa}(\kappa, 0) < 0$ for all $\kappa > 0$, and $F(0,0) = \lim_{\kappa\searrow 0} F(\kappa, 0)$, we then conclude that κ^* is unique, and $F(\kappa, 0) < 0$ in $[0, N-1) \cup (\kappa^*, \infty)$, $F(\kappa, 0) > 0$ in $(N-1, \kappa^*)$.

In the interval $\kappa \in [0, N-1) \cup (\kappa^*, \infty)$, $F_\lambda < 0$ so that $F(\kappa, \lambda) < 0$ for all $\kappa \in [0, N-1) \cup (\kappa^*, \infty)$ and all $\lambda \geq 0$.

In the interval $\kappa \in [N-1, \kappa^*]$, $F(\kappa, \lambda) < 0$ when $\lambda \geq \sigma/\tau$. Hence, as $F_\lambda < 0$, for every $\kappa \in [N-1, \kappa^*]$, there exists a unique $\lambda = \Lambda(\kappa) \in (0, \sigma/\tau)$ with $F(\kappa, \Lambda(\kappa)) = 0$. In addition, by the implicit function theorem,

$$\Lambda'(\kappa) = -F_\kappa(\kappa, \Lambda) F_\lambda(\kappa, \lambda)^{-1} \qquad \text{for all } \kappa \in [N-1, \kappa^*].$$

Furthermore,

$$\Lambda''(\kappa) = -\left(F_{\lambda\lambda}(\Lambda')^2 + 2F_{\kappa\lambda}\Lambda' + F_{\kappa\kappa}\right) F_\lambda^{-1} < 0, \qquad \kappa \in [N-1, \kappa^*],$$

because $F_{\lambda\lambda} < 0$ and $F_{\lambda\lambda}F_{\kappa\kappa} - F_{\kappa\lambda}^2 > 0$. This completes the proof of the lemma. □

Finally, we estimate κ^*, as well as the point $\kappa^{\max}$ where Λ obtains the unique maximum $\lambda^{\max}$. To do this, we need to estimate $w(1, \kappa, \lambda)$.

Lemma 4.6. *Assume that* $\lambda \in [0, \sigma/\tau]$, $\kappa \geq 0$, *and* $(N-2)^2 + \kappa \geq 4$. *Define*

$$\theta \overset{\text{def}}{=} \tfrac{N-2}{2} + \sqrt{(\tfrac{N-2}{2})^2 + \kappa}, \tag{4.7}$$

$$z(s, \beta, m) \overset{\text{def}}{=} \begin{cases} s^m/(\theta + \beta + m) & \text{if } s \in [0,1], \\ s^{-\theta} e^{-\beta(s-1)}/(\theta + \beta + m) & \text{if } s \in (1, \infty). \end{cases} \tag{4.8}$$

Then

$$z(s, \beta^-, m^-) \leq w(s, \kappa, \lambda) \leq z(s, \beta^+, m^+) \tag{4.9}$$

holds true for all $s \in (0, \infty)$, *where*

$$\beta^- = \sqrt{g_M + \lambda}, \quad \beta^+ = -\sqrt{\left(\tfrac{N-2}{2}\right)^2 + \kappa} + \sqrt{\left(\tfrac{N-2}{2}\right)^2 + \kappa + g_m + \lambda},$$
$$m^- = -\tfrac{N-2}{2} + \sqrt{\left(\tfrac{N-2}{2}\right)^2 + \kappa + g_M + \lambda}, \quad m^+ = -\tfrac{N-2}{2} + \sqrt{\left(\tfrac{N-2}{2}\right)^2 + \kappa}.$$

Consequently, when $\kappa \to \infty$, the following estimates

$$w(1,\kappa,\lambda) = \frac{1}{2}\kappa^{-\frac{1}{2}}[(1+O(\kappa^{-\frac{1}{2}})] \tag{4.10}$$

$$\|s^{-1}w(s,\kappa,\lambda)\|^2 = \frac{1}{4}\kappa^{-\frac{3}{2}}[1+O(\kappa^{-\frac{1}{2}})] \tag{4.11}$$

hold true. The convergence of $O(\kappa^{-\frac{1}{2}})$ is uniform in $\lambda \in [0,\sigma/\tau]$.

Proof. First notice that $(N-2)^2 + 4\kappa \geq 4$ is valid from the assumption. Note that $z(\,\cdot\,,\beta,m)$ is continuous in $[0,\infty)$ and $z_s|_{s=1-0} - z_s|_{s=1+0} = 1$. Also, when $s \in (0,1]$, $\mathcal{L}[z] = s^{-2}z[\kappa - m(m+N-2) + (g+\lambda)s^2]$, so that we obtain $\pm\mathcal{L}[z(s,\beta^{\pm},m^{\pm}) \geq 0$ holds true. When $s \in (1,\infty)$,

$$\mathcal{L}[z] = s^{-1}z\left\{\beta\left[1-\sqrt{(N-2)^2+4\kappa}\right] + (g+\lambda-\beta^2)s\right\}$$

so that $\pm\mathcal{L}[z(s,\beta^{\pm},m^{\pm})] \geq 0$ holds true.

In summary, we obtain $\mathcal{L}[z(\,\cdot\,,\beta^+,m^+)] \geq 0$ and $\mathcal{L}[z(\,\cdot\,,\beta^-,m^-)] \leq 0$ with $-\left[z_s(s,\beta^{\pm},m^{\pm})\right]_{s=1-0}^{s=1+0} = 1$. Hence, by a comparison principle, (4.9) holds. The estimates (4.10) and (4.11) follow from (4.9). □

With these estimates, we can now conclude the following:

Lemma 4.7. *Assume that τ satisfies* (4.4). *Let $\Lambda(\,\cdot\,)$ and κ^* be as in Lemma* 4.5. *Also let $\kappa^{\max} \in (N-1,\kappa^*)$ and $\lambda^{\max}$ be the constants such that $\lambda^{\max} = \Lambda(\kappa^{\max}) = \max_{\kappa\in[N-1,\kappa^*]}\Lambda(\kappa)$ holds true. Then*

$$(\kappa^{\max})^{\frac{1}{2}} = \left(\frac{\nu}{4\varepsilon}\right)^{\frac{1}{3}} + O(1), \tag{4.12}$$

$$\lambda^{\max} = \frac{\sigma}{\tau} - \frac{3(2\nu)^{\frac{2}{3}}}{4\tau}\varepsilon^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}), \tag{4.13}$$

$$(\kappa^*)^{\frac{1}{2}} = \sigma^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}} - \frac{\nu}{4\sigma} + O(\varepsilon^{\frac{1}{2}}) \tag{4.14}$$

hold true as $\varepsilon \to 0$.

Proof. At $\kappa^{\max}$, $\Lambda' = 0$, so that $F_\kappa(\lambda^{\max},\kappa^{\max}) = 0$. Recall that $F_\kappa = -\varepsilon - \nu w_\kappa(1,\kappa,\lambda)$ and $w_\kappa = -\|s^{-1}w\|^2$. We have $\nu\|s^{-1}w(\,\cdot\,,\kappa^{\max},\lambda^{\max})\|^2 = \varepsilon$. As $\lambda^{\max} < \frac{\sigma}{\tau}$, we can use (4.11) to conclude (4.12). Consequently, solving $F(\lambda^{\max},\kappa^{\max}) = 0$, we obtain

$$\tau\lambda^{\max} = \sigma + \varepsilon(N-1-\kappa^{\max}) - \nu w(1,\kappa^{\max},\lambda^{\max}) = \sigma - \frac{3}{4}2^{\frac{2}{3}}\nu^{\frac{2}{3}}\varepsilon^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}).$$

The estimate (4.13) thus follows. Finally, solving $F(\kappa^*, 0) = 0$, that is,

$$\begin{aligned} 0 &= \sigma + \varepsilon(N - 1 - \kappa^*) - \nu w(1, \kappa^*, 0) \\ &= \sigma + \varepsilon(N - 1 - \kappa^*) - \frac{\nu}{2}(\kappa^*)^{-\frac{1}{2}} + O((\kappa^*)^{-1}), \end{aligned}$$

we obtain (4.14). This completes the proof of the lemma. □

Clearly, Theorem 3.1 follows from Lemmas 4.3, 4.5 and 4.7.

REFERENCES

[1] X. Chen, *Generation and propagation of interfaces in reaction-diffusion systems*, Trans. Amer. Math. Soc. **334**, No. 2 (1992), pp 877–913

[2] X-Y. Chen, *Dynamics of interfaces in reaction diffusion systems*, Hiroshima Math. J. **21** (1991), pp 47–83

[3] P. C. Fife, "Dynamics of internal layers and diffusive interfaces", CBMS-NSF Regional Conference Series in Applied Mathematics 53, SIAM, Philadelphia, 1988

[4] R. Courant and D. Hilbert, "Methods of Mathematical Physics", A Wiley-intersciences, New York, Chichester, 1953

[5] C. Müller, *Spherical Harmonics*, Lecture Notes in Mathematics 17, Springer-Verlag, Berlin, Heidelberg, New York, 1966

[6] Y. Giga, S .Goto and H. Ishii, *Global existence of weak solutions for interface equations coupled with diffusion equations*, SIAM J. Math. Anal. **23** (1992), pp 821–835

[7] D. Hilhorst, Y. Nishiura and M. Mimura, *A free boundary problem arising in some reaction-diffusion systems*, Proc. Roy. Soc. Edinburgh **118A** (1991), pp 355–378

[8] T. Ohta, M. Mimura and R. Kobayashi, *Higher-dimensional localized patterns in excitable media*, Physica D **34** (1989), pp 115–144

[9] F. W. J. Olver, "Asymptotics and Special Functions", Academic Press, New York, London, 1974

[10] M. Protter and H. Weinberger, "Maximum principles in Differential Equations", Springer-Verlag, New York, Berlin, 1984

[11] P. Soravia and P. E. Souganidis, *Phase-field theory for FitzHugh-Nagumo-type systems*, SIAM J. Math. Anal. **27** (1996), No. 5, pp 1341–1359

[12] M. Taniguchi, *Multiple existence and linear stability of equilibrium balls in a nonlinear free boundary problem*, Quart. Appl. Math. (to appear)

[13] M. Taniguchi and Y. Nishiura, *Instability of planar interfaces in reaction-diffusion systems*, SIAM J. Math. Anal. **25** (1994), No. 1, pp 99–134

[14] T. Tsujikawa and M. Mimura and S. Yotsutani, unpublished