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NONLINEAR ELLIPTIC EQUATIONS IN $\mathbb{R}^N$ WITH "ABSORBING" ZERO ORDER TERMS

Fabiana Leoni
Dipartimento di Matematica, Università di Roma "La Sapienza"
P.le A. Moro 2, 00185 Roma, Italy

(Submitted by: Juan Luis Vazquez)

0. Introduction. In [1] the semilinear elliptic problem

$$\begin{cases} -\Delta u + |u|^{s-1}u = f & \text{in } \mathcal{D}'(\mathbb{R}^N)\,, \\ \qquad u \in L^s_{loc}(\mathbb{R}^N)\,, \end{cases} \tag{0.1}$$

having merely *locally* integrable datum $f \in L^1_{loc}(\mathbb{R}^N)$, and equipped with *no prescribed behaviour at infinity* of u, has been shown to have a unique solution u whenever the exponent s satisfies $s > 1$.

This result has been subsequently generalized in [10], where the power-like nonlinearity $t \mapsto |t|^{s-1}t$ was replaced by a general function $g(t)$ satisfying the following hypotheses:

$$g : \mathbb{R} \mapsto \mathbb{R} \ \text{ is continuous, odd, increasing, } \ g(0) = 0 \tag{$G1$}$$

$$g \ \text{ is convex in } \ [0, +\infty) \tag{$G2$}'$$

$$\int^{+\infty} \frac{dt}{[tg(t)]^{\frac{1}{2}}} < +\infty\,. \tag{$G3$}'$$

Under these assumptions the problem

$$\begin{cases} -\Delta u + g(u) = f & \text{in } \mathcal{D}'(\mathbb{R}^N)\,, \\ \quad u\,, g(u) \in L^1_{loc}(\mathbb{R}^N) \end{cases} \tag{0.2}$$

has been shown to have a unique solution.

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The non linear version of equation (0.1), that is the problem

$$\begin{cases} -\text{div}(|Du|^{p-2}Du) + |u|^{s-1}u = f \qquad \text{in} \quad \mathcal{D}'(\mathbb{R}^N)\,, \\ \qquad |Du|^{p-1} \in L^1_{loc}(\mathbb{R}^N),\ u \in L^s_{loc}(\mathbb{R}^N)\,, \end{cases}$$

with $p > 1$, has been treated in [6], where a solution u belonging to the Sobolev space $W^{1,1}_{loc}(\mathbb{R}^N)$ has been obtained assuming, on the exponent s, that $s > p-1$, in the case $p > p_0 = 2 - \frac{1}{N}$, and $s > \frac{1}{p-1}$, for the case $p \in (1, p_0]$.

In this paper we are interested in the nonlinear generalization of (0.2), that is,

$$\begin{cases} -\text{div}(|Du|^{p-2}Du) + g(u) = f \qquad \text{in} \ \ \mathcal{D}'(\mathbb{R}^N)\,, \\ \qquad\qquad |Du|^{p-1},\ g(u) \in L^1_{loc}(\mathbb{R}^N)\,, \end{cases} \tag{E_g}$$

where the function $g : \mathbb{R} \mapsto \mathbb{R}$ is assumed to satisfy, besides $(G1)$, analogous hypotheses to $(G2)'\,,(G3)'$:

$$t \in (0, +\infty) \mapsto \frac{g(t)}{t^{p-1}} \ \text{ is increasing}\ , \tag{$G2$}$$

and

$$\int^{+\infty} \frac{dt}{[tg(t)]^{\frac{1}{p}}} < +\infty\,. \tag{$G3$}$$

Condition $(G1)$ in particular implies that $g(t)\,t \geq 0\ \forall t \in \mathbb{R}$, or, using the terminology introduced in [14], that the zero order term of equation (E_g) is "absorbing".

The hypotheses $(G2)$ and $(G3)$ impose growth conditions on $g(t)$ as t tends to infinity, or, in other words, conditions on the "absorbing capability" of the zero order term of equation (E_g). We notice that if g has a power-like growth, conditions $(G2)$ and $(G3)$ both require $g(t) \simeq |t|^{s-1}t$ for some $s > p-1$; however, they also include functions g such that $\frac{g(t)}{|t|^{p-2}t}$ has a logarithmic growth, that is, for example, $g(t) \simeq [\log(1+|t|)]^s |t|^{p-2}t$, with $s > p$.

Condition $(G3)$ turns out to be necessary in order to apply any natural method of approximation for solving problem (E_g), as it will be discussed in Section 4. In Section 1 we will prove that assumptions $(G1)$, $(G2)$ and $(G3)$ are sufficient in order to obtain a solution u of (E_g), with $|Du|^{p-1}$ and $g(u)$ in $L^1_{loc}(\mathbb{R}^N)$.

More precisely, we will obtain existence and regularity results concerning a class of equations, including (E_g) and modelled on (E_g), that is:

$$\begin{cases} -\operatorname{div}(a(x,Du)) + h(x,u) = f \qquad \text{in } \mathcal{D}'(\mathbb{R}^N)\,, \\ \qquad |a(x,Du)|,\ h(x,u) \in L^1_{loc}(\mathbb{R}^N)\,, \end{cases} \tag{$E1$}$$

where the functions $a(x,\xi)$ and $h(x,t)$ satisfy the following structural assumptions:

(A1) $a(x,\xi)$ is a Carathéodory function, that is, measurable with respect to $x \in \mathbb{R}^N$ for every fixed $\xi \in \mathbb{R}^N$, and continuous with respect to $\xi \in \mathbb{R}^N$ for almost every fixed $x \in \mathbb{R}^N$;

(A2) there exist constants $\lambda > 0$ and $p > 1$ such that, for every $\xi \in \mathbb{R}^N$ and for almost every $x \in \mathbb{R}^N$,

$$a(x,\xi)\cdot\xi \ge \lambda|\xi|^p\,;$$

(A3) there exists a constant $\Lambda \ge \lambda$ such that, for every $\xi \in \mathbb{R}^N$ and for almost every $x \in \mathbb{R}^N$,

$$|a(x,\xi)| \le \Lambda|\xi|^{p-1}\,;$$

(A4) for every $\xi,\ \eta \in \mathbb{R}^N$ and for almost every $x \in \mathbb{R}^N$,

$$[a(x,\xi) - a(x,\eta)]\cdot(\xi-\eta) > 0\,.$$

(H1) $h(x,t)$ is a Carathéodory function, that is, measurable with respect to $x \in \mathbb{R}^N$ for every fixed $t \in \mathbb{R}$, and continuous with respect to $t \in \mathbb{R}$ for almost every fixed $x \in \mathbb{R}^N$;

(H2) there exists a function $g(t)$, satisfying conditions $(G1)$, $(G2)$ and $(G3)$, such that, for every $t \in \mathbb{R}$ and almost every $x \in \mathbb{R}^N$,

$$h(x,t)\,\mathrm{sign}(t) \ge |g(t)|\,;$$

(H3) for all $\tau > 0$, the function $H_\tau(x) = \sup_{|t|\le\tau}|h(x,t)|$ belongs to $L^1_{loc}(\mathbb{R}^N)$.

Consider first the case $p \in (1,p_0]$. If the function g is not absorbing enough, that is if $g(t)$ has a growth close to t^{p-1}, then possible solutions of $(E1)$ are not expected to belong to the Sobolev space $W^{1,1}_{loc}(\mathbb{R}^N)$ (see [3]); nevertheless, it is still possible to give sense to a solution u of $(E1)$, and to consider its “gradient” function Du even if it has no more the usual distributional meaning.

In order to explain in more details our statements, let us introduce the truncated function

$$T_k(s) = \begin{cases} k\frac{s}{|s|} & \text{if } |s| > k \\ s & \text{if } |s| \leq k\,, \end{cases} \tag{0.3}$$

with $k > 0$ and $s \in \mathbb{R}$, and let us recall the following result (see [3]).

Lemma 1. *If $u : \mathbb{R}^N \mapsto \mathbb{R}$ is a measurable function such that its truncations $T_k(u)$ belong to $W^{1,1}_{loc}(\mathbb{R}^N)$ for all $k > 0$, then there exists a measurable function $v : \mathbb{R}^N \to \mathbb{R}^N$ such that, for all $k > 0$ and almost everywhere in $\mathbb{R}^N$,*

$$DT_k(u) = v\chi_{\{|u|<k\}},$$

where $\chi_{\{|u|<k\}}$ is the characteristic function of the set $\{|u| < k\} = \{x \in \mathbb{R}^N : |u(x)| < k\}$. Moreover, if u belongs to $W^{1,1}_{loc}(\mathbb{R}^N)$, then v coincides with the distributional gradient of u.

If u is a function as in Lemma 1, the function v will be referred to from now on as the *weak gradient* of u and it will be set $Du = v$.

We also recall the notion of Marcinkiewicz space $M^q_{loc}(\mathbb{R}^N)$, $q > 0$, which is defined as the set of all measurable functions $v : \mathbb{R}^N \mapsto \mathbb{R}$ such that, for every open, bounded set $B \subset\subset \mathbb{R}^N$ there exists a constant $c_B > 0$ satisfying, for all $\lambda > 0$

$$\text{meas}\,(\{\, x \in B \,:\, |v(x)| > \lambda \,\}) \leq \frac{c_B}{\lambda^q} \tag{0.4}$$

(here and in the following, meas(B) denotes the Lebesgue measure of the set $B \subseteq \mathbb{R}^N$).

For every $B \subset\subset \mathbb{R}^N$, the norm of v in $M^q(B)$ is defined by

$$\|v\|_{M^q(B)} = \inf\{\, c_B > 0 \,:\, (0.4) \text{ holds with } c_B \,\}\,.$$

For all $q > 1$ and $0 < \varepsilon < q - 1$, and for every subset B having finite measure, the following continuous inclusions hold:

$$L^q(B) \hookrightarrow M^q(B) \hookrightarrow L^{q-\varepsilon}(B)\,. \tag{0.5}$$

We can now state our first result.

Theorem 1. *Let* $a(x,\xi)$ *and* $h(x,t)$ *be functions satisfying assumptions* (A1)-(A4) *and* (H1)-(H3) *respectively, with some exponent* $p>1$*; then, given any function* f *in* $L^1_{loc}(\mathbb{R}^N)$*, equation* $(E1)$ *has at least one solution* u *belonging to* $M_{loc}^{\frac{N(p-1)}{N-p}}(\mathbb{R}^N)$*, such that* $T_k(u)$ *belongs to* $W^{1,p}_{loc}(\mathbb{R}^N)$ *for every* $k>0$*, and its weak gradient* Du *belongs to* $M_{loc}^{\frac{N(p-1)}{N-1}}(\mathbb{R}^N)$*. Moreover, the function* u *satisfies*

$$\int_{\mathbb{R}^N} a(x,Du)\cdot D[T_k(u-v)\theta] + \int_{\mathbb{R}^N} h(x,u)T_k(u-v)\theta \le \int_{\mathbb{R}^N} fT_k(u-v)\theta \quad (0.6)$$

for all test functions $v\in W^{1,p}_{loc}(\mathbb{R}^N)\cap L^\infty_{loc}(\mathbb{R}^N)$*,* $\theta\in\mathcal{D}(\mathbb{R}^N)$*, with* $\theta\ge 0$*, and for every* $k>0$*.*

Remark 1. Inequality (0.6) may be considered as the definition of *local entropy solution* (for the definition of entropy solution see [3]). It is stronger than the definition of distributional solution: indeed, it can be easily proved that if a function u, such that $T_k(u)$ belongs to $W^{1,p}_{loc}(\mathbb{R}^N)$ for all $k>0$, and both $|a(x,Du)|$ and $h(x,u)$ are in $L^1_{loc}(\mathbb{R}^N)$, satisfies (0.6), then u is a distributional solution of $(E1)$.

If the exponent p appearing in (A2), (A3) and (H2) is in the interval $(p_0,N]$, we obtain a stronger result.

Theorem 2. *Assume* (A1)-(A4) *and* (H1)-(H3) *are true with* p *belonging to* $(p_0,N]$*; then, for any given* f *in* $L^1_{loc}(\mathbb{R}^N)$*, there exists a solution* u *of* $(E1)$ *in* $W^{1,q}_{loc}(\mathbb{R}^N)$*, for all* q *in* $[1,\frac{N(p-1)}{N-1})$*.*

Actually, the solution found in Theorem 2 is obtained with the same method as in Theorem 1; in this sense, Theorem 2 is a regularity result for the solutions given by the previous theorem.

Observe that the regularity of u obtained in Theorem 2 is the best possible for the solutions of $-\mathrm{div}(|Du|^{p-2}Du)\in L^1_{loc}(\mathbb{R}^N)$ (see [5], and [15] for a counterexample).

In order to obtain a solution u belonging to $W^{1,\bar q}_{loc}(\mathbb{R}^N)$, where $\bar q$ is the limit exponent $\frac{N(p-1)}{N-1}$, we have to slightly strengthen the assumption on the summability of the datum f.

Theorem 3. *Under the same assumptions of Theorem 2, if the function* f *is such that* $f\,\log(1+|f|)$ *belongs to* $L^1_{loc}(\mathbb{R}^N)$*, then equation (E1) has a solution* u *in* $W^{1,\bar q}_{loc}(\mathbb{R}^N)$*, with* $\bar q=\frac{N(p-1)}{N-1}$*.*

If p is greater than N, then it is possible to obtain for u the regularity one expects from variational methods.

Theorem 4. *Suppose* (A1)-(A4) *and* (H1)-(H3) *are satisfied with an exponent* $p > N$*; then, for any given* f *in* $L^1_{loc}(\mathbb{R}^N)$*, there exists a solution* u *of (E1) belonging to* $W^{1,p}_{loc}(\mathbb{R}^N)$.

Theorems 1, 2, 3 and 4 will be proved by approximating both the domain $\mathbb{R}^N$ and the datum $f \in L^1_{loc}(\mathbb{R}^N)$ of (E1), respectively with the family of balls $B_n = \{x \in \mathbb{R}^N : |x| < n\}$, $n \in \mathbb{N}$, and with the sequence $\{f_n = T_n(f)\}$ of n-truncated functions of f; the solutions u_n of the approximating equations posed in B_n, with datum f_n, will satisfy some local *a priori* estimates appropriately depending on f, from which we will deduce the convergence of $\{u_n\}$ to a solution u of $(E1)$.

In Section 2 we will consider equations like $(E1)$ having more regular data f (that is, with greater summability), given in the Lebesgue space $L^m_{loc}(\mathbb{R}^N)$, for some $m > 1$. We will give regularity results concerning the summability of the solutions and the gradients. More precisely, we will prove that if the exponent m of the summability of the datum f increases, it is possible to obtain an accordingly increasing summability of the solutions u of $(E1)$, till they become bounded.

As long as $m \le (p^*)' = \frac{Np}{N(p-1)+p}$, where $p^* = \frac{Np}{N-p}$ is the Sobolev imbedding exponent of $W^{1,p}$, our results hold for the solutions of (E1) constructed in the previous existence theorems.

Theorem 5. *Under the assumptions* (A1)-(A4) *and* (H1)-(H3)*, with the exponent* p *in* (p_0, N)*, if the datum* f *belongs to* $L^m_{loc}(\mathbb{R}^N)$*, with* m *in* $(1, (p^*)' = \frac{Np}{N(p-1)+p}]$*, then (E1) has a solution* u *in* $W^{1,(p-1)m^*}_{loc}(\mathbb{R}^N)$*, with* $m^* = \frac{Nm}{N-m}$*. In particular, if* f *is in* $L^{(p^*)'}_{loc}(\mathbb{R}^N)$*, then* u *is in* $W^{1,p}_{loc}(\mathbb{R}^N)$*,* $u\,h(x,u)$ *is in* $L^1_{loc}(\mathbb{R}^N)$*, and it results both*

$$\int_{\mathbb{R}^N} a(x,Du)\cdot Dv \;+\int_{\mathbb{R}^N} h(x,u)v \;=\int_{\mathbb{R}^N} fv\,, \tag{0.7}$$

for all test functions $v \in W^{1,p}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ *having compact support, and*

$$\int_{\mathbb{R}^N} a(x,Du)\cdot D(u\theta) \;+\int_{\mathbb{R}^N} h(x,u)u\theta \;=\int_{\mathbb{R}^N} fu\theta\,, \tag{0.8}$$

for every $\theta \in \mathcal{D}(\mathbb{R}^N)$.

If $m > (p^*)'$, the regularity results can be stated for *all* functions u satisfying identity (0.7).

Theorem 6. *Let $a(x,\xi)$ and $h(x,t)$ be functions satisfying* (A1)-(A3) *and* (H1)-(H3) *respectively, with p in $(1,N)$, and let u be a function belonging to $W^{1,p}_{loc}(\mathbb{R}^N)$, with $u\,h(x,u)$ in $L^1_{loc}(\mathbb{R}^N)$, and satisfying identity* (0.7) *with f in $L^m_{loc}(\mathbb{R}^N)$, m in $((p^*)', \frac{N}{p})$. Then u belongs to $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^{[(p-1)m^*]^*}_{loc}(\mathbb{R}^N)$.*

We will not deal with the limit case $m = \frac{N}{p}$; we only recall that for equations as $(E1)$ with $p = 2$, without zero order terms, and posed in bounded domains, every solutions of the corresponding homogeneous Dirichlet problem is proved to belong, if the datum f belongs to $L^{\frac{N}{2}}$, to the Orlicz space L_Φ with $\Phi(t) = \mathrm{e}^{|t|^{N/N-1}} - 1$ (see [12]).

If $m > \frac{N}{p}$, we finally obtain locally bounded solutions.

Theorem 7. *Let $a(x,\xi)$, $h(x,t)$, p and u as in Theorem 6; if the function f belongs to $L^m_{loc}(\mathbb{R}^N)$, with $m > \frac{N}{p}$, then u belongs to $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$.*

As far as the summability of the gradient Du is concerned, it is well known that for solutions of elliptic equations without lower order terms, it consistently improves with the summability of f only if the exponent m ranges from $(p^*)'$ to a certain value $\overline{m} > (p^*)'$ (this is Meyers' theorem, see [13], [4]).

We obtain a Meyers' type result for our equation assuming, for simplicity, that the absorbing term $h(x,u)$ does not depend on x and coincides with a function $g(u)$ satisfying assumptions $(G1)$ and the following one:

$$\exists\,\mu \in (0,1) \quad | \quad t \in \mathbb{R} \mapsto [G(t)]^\mu \in \mathbb{R} \quad \text{is convex}, \tag{$G4$}$$

where $G(t) = \int_0^t g(s)\,ds$ is the primitive function of g vanishing in zero. We state the result for the case $p < N$, without considering the simpler case $p \geq N$.

Theorem 8. *Let $a(x,\xi)$ and $g(t)$ be functions satisfying assumptions $(A1)-(A3)$ and $(G1)$, $(G4)$ respectively, with $p \in (1,N)$, and let u be a function in $W^{1,p}_{loc}(\mathbb{R}^N)$, with $u\,g(u)$ in $L^1_{loc}(\mathbb{R}^N)$, satisfying*

$$-\mathrm{div}\,(a(x,Du)) \;+\; g(u) \;=\; f\,, \tag{$E2$}$$

with f given in $L^m_{loc}(\mathbb{R}^N)$, $m \geq (p^*)'$, in the sense that

$$\int_{\mathbb{R}^N} a(x,Du)\cdot Dv \;+\; \int_{\mathbb{R}^N} g(u)\,v \;=\int_{\mathbb{R}^N} f\,v \tag{0.9}$$

for any v in $W^{1,p}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ having compact support. Then, there exists $\delta > 0$, depending on p, N, μ, λ, Λ, and on $\|f\|_{L^{(p^*)'}}$, such that if m is in $((p^*)', (p^*)' + \delta)$, then Du belongs to $[L^q_{loc}(\mathbb{R}^N)]^N$, for all q in $[p, p + \frac{\delta p}{(p^*)'})$.

Notice that, under the only assumptions of the theorem above, a solution u of $(E2)$ may not exist; as stated by Theorem 5, also assumptions (A4), $(G2)$, $(G3)$ and $p > p_0$ are needed in order to obtain the existence of a solution.

Examples of functions g satisfying $(G1)$ and $(G4)$ are:

a) $g(t) = \left(e^{|t|} - 1\right)\,\mathrm{sign}(t)$, which satisfies $(G4)$ for every $\mu \in \left[\frac{1}{2}, 1\right)$;

b) $g(t) = |t|^{s-1}\,t$, with $s > 0$, for which $(G4)$ holds for all μ in $[\frac{1}{s+1}, 1)$;

c) $g(t) = |t|^{s-1}\,t\,[\log(1+|t|)]^\alpha$, with $s > 0$, $\alpha > 0$, which again satisfies $(G4)$ for all μ in $[\frac{1}{s+1}, 1)$, independently on $\alpha > 0$.

Observe also that a), b) with $s > p-1$, and c) with $s \geq p-1$, and $\alpha > p$ when $s = p-1$, are examples of functions g simultaneously satisfying $(G1)-(G4)$, and for which then both Theorems 5 and 8 hold if $p > p_0$.

In Section 3 we will show how the techniques used in the previous sections may be employed to deal with equations also containing lower order terms having natural growth with respect to Du. An existence result in $W^{1,p}_{loc}(\mathbb{R}^N)\cap L^\infty_{loc}(\mathbb{R}^N)$ will be obtained, for example, for the equation

$$-\mathrm{div}\left(|Du|^{p-2}Du\right) + |u|^{s-1}u \;-|Du|^p = f \qquad \text{in } \mathbb{R}^N\,, \tag{0.10}$$

with $p > 1, f$ in $L^m_{loc}(\mathbb{R}^N)$, $m > \max\{\,1, \frac{N}{p}\,\}$, and the exponent s greater than p.

For a study of equation (0.10) in bounded domains see [7], [8] and [9].

In order to show the connection between the two equations, let us look at the model examples (E_g) and (0.10) in the semilinear case $p = 2$. Suppose we have to find a nonnegative solution u of

$$-\Delta u \;+\; u^s \;-\; |Du|^2 \;=\; f \geq 0\,;$$

if we look for a solution of the form $u = \log(1+v)$, for some function $v \geq 0$, then v is a positive solution of

$$-\Delta v \;+\; [\log(1+v)]^s(1+v) \;=\; f(1+v) \;= \overline{f}\,.$$

From the previous theorems we know that it is possible to find such a function v if $s > 2$.

More in general, we can handle equations as

$$-\text{div}\,(a(x,Du)) \;+h(x,u) \;+\; F(x,Du) \;= f\,, \qquad (E3)$$

where $h(x,t)$, besides (H1) and (H3), satisfies:

(H4) there exists a continuous, odd and increasing function $j:\mathbb{R}\mapsto\mathbb{R}$, with $j(0)=0$, satisfying

$$\int^{+\infty}\frac{dt}{[j(t)]^{\frac{1}{p}}}<+\infty\,,$$

such that, for every $t\in\mathbb{R}$ and for almost every $x\in\mathbb{R}^N$

$$h(x,t)\,\text{sign}(t)\geq|j(t)|;$$

and the function $F(x,\xi)$ satisfies:

(F1) $F(x,\xi)$ is a Carathéodory function, that is, measurable with respect to $x\in\mathbb{R}^N$ for every fixed $\xi\in\mathbb{R}^N$, and continuous with respect to $\xi\in\mathbb{R}^N$ for almost every fixed $x\in\mathbb{R}^N$;

(F2) there exists a positive constant γ such that, for every $\xi\in\mathbb{R}^N$ and for almost every $x\in\mathbb{R}^N$,

$$|F(x,\xi)|\leq\gamma|\xi|^p\,.$$

Theorem 9. *Let $a(x,\xi)$, $h(x,t)$ and $F(x,\xi)$ be functions satisfying assumptions* (A1)-(A4), (H1), (H3), (H4), *and* (F1)-(F2) *respectively, for some $p>1$, then for any given f in $L^m_{loc}(\mathbb{R}^N)$, with $m>\max\{1,\frac{N}{p}\}$, there exists a solution u of $(E3)$ in $W^{1,p}_{loc}(\mathbb{R}^N)\cap L^\infty_{loc}(\mathbb{R}^N)$.*

1. Equations with data in L^1_{loc}. In this section we will give the proofs of Theorems 1, 2, 3, and 4.

As in [6], solutions of $(E1)$ will be obtained by approximating both the domain $\mathbb{R}^N$ and the datum $f\in L^1_{loc}(\mathbb{R}^N)$.

Throughout this Section assumptions (A1)-(A4) and (H1)-(H3) will be always supposed to hold.

We recall the well known fact that, for any given $p>1$, $R>0$, f in $L^\infty(B_R)$, where $B_R=\{\,x\in\mathbb{R}^N\,:\,|x|<R\,\}$ is the ball of $\mathbb{R}^N$ with radius

R, and for any boundary datum $\psi \in W^{1-\frac{1}{p},p}(\partial B_R) \cap L^\infty(\partial B_R)$, there exists a weak solution u of the Dirichlet problem

$$\begin{cases} -\mathrm{div}\,(a(x,Du)) \ + \ h(x,u) = f & \text{in} \quad B_R \\ \hfill u = \psi & \text{on} \quad \partial B_R \,, \end{cases}$$

that is a function u belonging to $W^{1,p}(B_R) \cap L^\infty(B_R)$ satisfying

$$\int_{B_R} a(x,Du) \cdot Dv \ + \ \int_{B_R} h(x,u)\, v \ = \ \int_{B_R} f\, v \tag{1.1}$$

for any test function $v \in W_0^{1,p}(B_R)$, and such that $u - \tilde{\psi}$ belongs to $W_0^{1,p}(B_R)$, where $\tilde{\psi}$ is a function of $W^{1,p}(B_R)$ which coincides with ψ on ∂B_R (see for instance [2]).

We need some local estimates on the solution u, depending only on $\|f\|_{L^1(B_R)}$, which will be deduced from the following lemmas.

The first one summarizes all the absorbing properties of the function g introduced in (H2). It is an extension to our case of Lemma 3 in [10].

Lemma 1.1. *Let $g : [0,+\infty) \mapsto [0,+\infty)$ be a continuous and increasing function, with $g(0) = 0$, satisfying $(G2)$ and $(G3)$ for some $p > 1$. Then for any $C > 0$ and $\alpha \geq 0$, there exists a smooth function $\varphi : [0,1] \mapsto [0,1]$, with $\varphi(0) = \varphi'(0) = 0, \varphi(1) = 1$, depending only on g, C, p and α, satisfying*

$$t^{\alpha+p-1} \, \frac{\varphi'(\sigma)^p}{\varphi(\sigma)^{p-1}} \leq \frac{1}{C}\, t^\alpha\, g(t)\, \varphi(\sigma) \ + \ 1 \tag{1.2}$$

for any $\sigma \in [0,1]$, and $t \geq 0$.

Proof. Given $\alpha \geq 0$, setting $g_\alpha(t) = t^\alpha g(t)$ we obtain an increasing, hence invertible, function on $[0,+\infty)$.

We then define the continuous function

$$k_\alpha(y) \ = \begin{cases} \frac{1}{g_\alpha^{-1}\left(\frac{C}{y}\right)} & \text{if } y > 0 \\ 0 & \text{if } y = 0 \,, \end{cases} \tag{1.3}$$

where C is the constant which appears in the statement.

We claim that the Cauchy problem

$$\begin{cases} \varphi'(\sigma) = \left[\varphi(\sigma)^{p-1}\, k_\alpha(\varphi(\sigma))^{\alpha+p-1}\right]^{\frac{1}{p}} & \text{in } (0,\sigma_0), \ \sigma_0 > 0 \\ \varphi(0) = 0 \end{cases}$$

has the nontrivial solution (φ, σ_0) given by

$$\begin{cases} \displaystyle\int_0^{\varphi(\sigma)} \frac{dt}{\left[k_\alpha(t)^{\alpha+p-1}\, t^{p-1}\right]^{\frac{1}{p}}} = \sigma \qquad \text{for } \sigma \in [0, \sigma_0] \\ \sigma_0 \;=\; \min\left\{ 1, \displaystyle\int_0^1 \frac{dt}{\left[k_\alpha(t)^{\alpha+p-1}\, t^{p-1}\right]^{\frac{1}{p}}} \right\}. \end{cases} \tag{1.4}$$

Indeed we will prove that

$$\int^{+\infty} \frac{dt}{\left[t\, g(t)\right]^{\frac{1}{p}}} < +\infty \;\Leftrightarrow\; \int_0 \frac{dt}{\left[k_\alpha(t)^{\alpha+p-1}\, t^{p-1}\right]^{\frac{1}{p}}} < +\infty \tag{1.5}$$

and this implies that φ is well defined by (1.4).

Assume for a moment that (1.5) holds. By (1.3), one then has

$$\frac{1}{C} \frac{g\left(\frac{1}{k_\alpha(y)}\right)}{\left(\frac{1}{k_\alpha(y)}\right)^{p-1}} \;=\; \frac{1}{C} \frac{g_\alpha\left(\frac{1}{k_\alpha(y)}\right)}{\left(\frac{1}{k_\alpha(y)}\right)^{p-1+\alpha}} \;=\; \frac{[k_\alpha(y)]^{p-1+\alpha}}{y}$$

for all $y > 0$, and, since $t \mapsto \frac{g(t)}{t^{p-1}}$ is increasing by $(G2)$, we get

$$\frac{1}{C} \frac{g(t)}{t^{p-1}} \geq \frac{[k_\alpha(y)]^{p-1+\alpha}}{y}$$

for any $t, y > 0$ with $t\, k_\alpha(y) \geq 1$. Thus,

$$[t\, k_\alpha(y)]^{p-1+\alpha} \leq \frac{1}{C}\, t^\alpha\, g(t)\, y \;+\; 1$$

for all $t,\ y \geq 0$. Setting $y = \varphi(\sigma)$, we obtain (1.2) in $[0, \sigma_0]$.

If $\sigma_0 < 1$, it then results $\varphi(\sigma_0) = 1$ and we can extend φ to $[0, 1]$ by setting

$$\varphi(\sigma) \;= \varphi(\sigma_0) + (\sigma - \sigma_0)\, \varphi'(\sigma_0) \;= 1 + (\sigma - \sigma_0)\, [k_\alpha(1)]^{\frac{\alpha+p-1}{p}}$$

for $\sigma \in [\sigma_0, 1]$. Then (1.2) holds in the whole interval $[0, 1]$ and, by replacing $\varphi(\zeta)$ with $\frac{\varphi(\zeta)}{\varphi(1)}$, we still have $0 \leq \varphi(\zeta) \leq 1$, $\varphi(1) = 1$.

It remains to prove the equivalence (1.5). We first observe that

$$\int_0 \frac{dt}{[k_\alpha(t)^{\alpha+p-1}\, t^{p-1}]^{\frac{1}{p}}} < +\infty \Leftrightarrow \int_0 \frac{dt}{[k_\alpha(Ct)^{\alpha+p-1}\, t^{p-1}]^{\frac{1}{p}}} < +\infty$$
$$\Leftrightarrow \int_0 \frac{[g_\alpha^{-1}(\frac{1}{t})]^{\frac{\alpha+p-1}{p}}}{t^{\frac{p-1}{p}}}\, dt < +\infty\,.$$

By a density argument, we can assume $g \in C^1([0,+\infty))$; then, with the substitution $t = \frac{1}{g_\alpha(u)}$, we get

$$\int_0 \frac{[g_\alpha^{-1}(\frac{1}{t})]^{\frac{\alpha+p-1}{p}}}{t^{\frac{p-1}{p}}}\, dt < +\infty$$
$$\Leftrightarrow \alpha \int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} + \int^{+\infty} \frac{u^{\frac{p-1}{p}}}{g(u)^{1+\frac{1}{p}}}\, g'(u)\, du < +\infty$$
$$\Leftrightarrow \alpha \int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} - p \int^{+\infty} u^{\frac{p-1}{p}} \, (\frac{1}{g(u)^{\frac{1}{p}}})'\, du < +\infty$$
$$\Leftrightarrow \alpha \int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} - p \frac{u^{\frac{p-1}{p}}}{g(u)^{\frac{1}{p}}}\Big|^{+\infty} + (p-1) \int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} < +\infty\,.$$

Since $u\, g(u)$ is increasing and $\int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} < +\infty$, then $\frac{1}{[u\, g(u)]^{\frac{1}{p}}} = o\left(\frac{1}{u}\right)$ for $u \to +\infty$, i.e., $\frac{u^{\frac{p-1}{p}}}{g(u)^{\frac{1}{p}}} = o(1)$ as $u \to +\infty$. Thus,

$$(p-1+\alpha) \int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} - p \frac{u^{\frac{p-1}{p}}}{g(u)^{\frac{1}{p}}}\Big|^{+\infty} < +\infty \Leftrightarrow \int^{+\infty} \frac{du}{[u\, g(u)]^{\frac{1}{p}}} < +\infty\,.$$

As an example, if $g(t) = t^s$, with $s > p-1$, then assuming $C \geq [\frac{s-(p-1)}{p(\alpha+s)}]^{\frac{p(\alpha+s)}{\alpha+p-1}}$, the function φ provided by Lemma 1.1 is $\varphi(\sigma) = \sigma^{\frac{p(\alpha+s)}{s-(p-1)}}$.

We are now able to state the announced local estimates.

Lemma 1.2. *Let f be a given function in $L^\infty(B_R)$ and let u in $W^{1,p}(B_R) \cap L^\infty(B_R)$ be such that (1.1) holds. Then, for all $r \in (0,R)$, there exists* *a constant* $c > 0$ *depending only on* p, r, R, g, λ, Λ, *and on the norm* $\|f\|_{L^1(B_R)}$, *such that, for every* $k > 0$

$$\int_{B_R} |D(T_k(u\varphi(\zeta)))|^p \leq c\,(1+k)\,, \tag{1.6}$$

where $\zeta \in \mathcal{D}(B_R)$ *is a cut-off function, with* $0 \leq \zeta \leq 1$ *and* $\zeta \equiv 1$ *in* B_r, $\varphi \in C^1([0,1])$ *is a function as in Lemma* 1.1, *and* $T_k(s)$ *is the truncated function defined in* (0.3).

Proof. Fixed r in $(0,R)$, let ζ be as in the statement and let φ be the function constructed in Lemma 1.1 in correspondence of the function g given in $(H2)$, of $\alpha = 1$, and of a constant $C > 0$ which will be chosen later. We set $\eta = \varphi(\zeta)$, and observe that also η is a cut-off function of class $C^1(B_R)$, with $0 \leq \eta \leq 1$ in B_R and $\eta \equiv 1$ in B_r.

Choosing $v = T_k(u\,\eta)$ in (1.1), we have

$$\int_{B_R} a(x,Du)\cdot DT_k(u\,\eta) + \int_{B_R} h(x,u)\,T_k(u\,\eta) = \int_{B_R} f\,T_k(u\,\eta) \leq k\|f\|_{L^1(B_R)} = c_1 k,$$

using assumptions (A2), (A3) and (H2), and observing that $DT_k(u\eta) \equiv 0$ in $\{x \in B_R : |u(x)|\eta(x) \geq k\}$, $DT_k(u\eta) = D(u\eta)$ in $\{x \in B_R : |u(x)|\eta(x) < k\} = \{|u|\eta < k\}$, this implies

$$\lambda\int_{\{\,|u|\,\eta<k\,\}} |Du|^p\eta + \int_{B_R} g(u)T_k(u\eta) \leq c_1 k + \Lambda\int_{\{\,|u|\,\eta<k\,\}} |Du|^{p-1}|D\zeta|\varphi'(\zeta)|u|.$$

Using the Young inequality, the last integral may be estimated as follows

$$\Lambda\int_{\{\,|u|\,\eta<k\,\}} |Du|^{p-1}\,|D\zeta|\varphi'(\zeta)\,|u| \leq c_2\int_{\{\,|u|\,\eta<k\,\}} |Du|^{p-1}\,\varphi'(\zeta)\,|u|$$
$$\leq \varepsilon\int_{\{\,|u|\,\eta<k\,\}} |Du|^p\,\eta + \frac{c_3}{\varepsilon^{p-1}}\int_{\{\,|u|\,\eta<k\,\}} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,|u|^p\,,$$

where $\varepsilon > 0$ may be arbitrarily chosen.

Combining this inequality with the previous one, and choosing $\varepsilon = \frac{\lambda}{2}$, we get

$$\frac{\lambda}{2}\int_{\{\,|u|\,\eta<k\,\}} |Du|^p\,\eta + \int_{B_R} g(u)\,T_k(u\,\eta) \leq c_1\,k + c_4\int_{\{\,|u|\,\eta<k\,\}} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,|u|^p\,.$$

Since $\eta = \varphi(\zeta) \le 1$, we have

$$|D(u\,\eta)|^p \le 2^{p-1}\left(|Du|^p\eta^p + |D\zeta|^p\varphi'(\zeta)^p|u|^p\right) \le c_5\big(|Du|^p\eta + |u|^p\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\big), \tag{1.7}$$

from which it follows

$$\frac{\lambda}{2}\int_{\{\,|u|\,\eta<k\,\}}|Du|^p\,\eta \ge \frac{\lambda}{2\,c_5}\int_{\{\,|u|\,\eta<k\,\}}|D(u\,\eta)|^p - \frac{\lambda}{2}\int_{\{\,|u|\,\eta<k\,\}}|u|^p\,\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,.$$

We then deduce

$$\int_{\{\,|u|\,\eta<k\,\}}|D(u\,\eta)|^p + \int_{B_R} g(u)\,T_k(u\,\eta) \le c_6\,k + c_7\int_{\{\,|u|\,\eta<k\,\}}|u|^p\,\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,.$$

We use now Lemma 1.1. Applying (1.2) with $t=|u|$, $\alpha=1$ and $C=\frac{1}{c_7}$, and remembering that $g(t)\,t\ge 0$, it results

$$\begin{aligned} c_7\int_{\{\,|u|\,\eta<k\,\}}|u|^p\,\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} &\le \int_{\{\,|u|\,\eta<k\,\}} g(|u|)\,|u|\,\varphi(\zeta) + \int_{\{\,|u|\,\eta<k\,\}} c_7 \\ &\le \int_{\{\,|u|\,\eta<k\,\}} g(u)u\eta + c_8 \le \int_{B_R} g(u)T_k(u\,\eta) + c_8, \end{aligned}$$

and so

$$\int_{\{\,|u|\,\eta<k\,\}}|D(u\,\eta)|^p + \int_{B_R} g(u)\,T_k(u\,\eta) \le c_9\,(k+1) + \int_{B_R} g(u)\,T_k(u\,\eta)\,,$$

from which the assertion follows. □

We can now give the proof of Theorem 1, which is essentially based on the proof of Theorem 6.1 of [3]; the only differences are due to the local nature of equation $(E1)$.

Proof of Theorem 1. Let us define a sequence of homogeneous boundary value problems approximating $(E1)$: set $B_n = \{x \in \mathbb{R}^N : |x| < n\}$ and $f_n = T_n(f)$, let u_n be a weak solution of

$$\begin{cases} -\mathrm{div}\,(a(x,Du_n)) + h(x,u_n) = f_n & \text{in } B_n \\ u_n = 0 & \text{on } \partial B_n \end{cases} \tag{$(E1)_n$}$$

that is a function u_n in $W_0^{1,p}(B_n) \cap L^\infty(B_n)$ satisfying

$$\int_{B_n} a(x, Du_n) \cdot Dv + \int_{B_n} h(x, u_n)\, v = \int_{B_n} f_n\, v\,, \tag{1.8}$$

for any v in $W_0^{1,p}(B_n)$.

For every fixed $\varrho > 0$ and for every $n \geq 3\varrho$, u_n satisfies in $B_{3\varrho}$ the assumptions of Lemma 1.2, with $f = f_n$ and $R = 3\varrho$. Applying (1.6) with $r = 2\varrho$, for all $k > 0$ and $n \geq 3\varrho$, we thus have

$$\int_{B_{3\varrho} \cap \{\, |u_n|\, \eta < k\,\}} |D(u_n\, \eta)|^p \leq c_1\, (1 + k)\,, \tag{1.9}$$

where c_1 is a positive constant depending neither on n nor on k.

From (1.9), as proved in Lemmas 4.1 and 4.2 in [3], it follows that the sequences $\{\, u_n\,\}_{n \geq 3\varrho}$ and $\{\, |Du_n|\,\}_{n \geq 3\varrho}$ are bounded in the Marcinkiewicz spaces $M^{\frac{N(p-1)}{N-p}}(B_{2\varrho})$ and $M^{\frac{N(p-1)}{N-1}}(B_{2\varrho})$ respectively, that is, there exist positive constants c_2 and c_3, not depending on n, such that, for every $\lambda > 0$ and for all $n \geq 3\varrho$,

$$\text{meas}\,(\{\, x \in B_{2\varrho} : |u_n(x)| > \lambda\,\}) \leq \frac{c_2}{\lambda^{\frac{N(p-1)}{N-p}}}\,, \tag{1.10}$$

and

$$\text{meas}\,(\{\, x \in B_{2\varrho}\, :\, |Du_n(x)| > \lambda\,\}) \leq \frac{c_3}{\lambda^{\frac{N(p-1)}{N-1}}}\,. \tag{1.11}$$

By (0.5) and (1.11) we have that the sequence $\{\, |Du_n|^{p-1}\,\}$ is bounded in $L^q(B_{2\varrho})$, for any $q \in [1, \frac{N}{N-1})$. This and (1.8) imply that $\{h(x, u_n)\}_{n \geq 3\varrho}$ is bounded in $L^1(B_\varrho)$; indeed, if ζ is a cut-off function having compact support in $B_{2\varrho}$, with $\zeta \equiv 1$ in B_ϱ, and $T : \mathbb{R} \to \mathbb{R}$ is any bounded, continuous, increasing function of class $C^1(\mathbb{R})$, with $|T(s)| \leq 1$ and $T(s)\, s \geq 0$ for all $s \in \mathbb{R}$, then, applying (1.8) written for all $n \geq 3\varrho$ with test function $v = T(u_n)\zeta$, and using assumptions (A2) and (A3), and the $L^1(B_{2\varrho})$ bound on $\{|Du_n|^{p-1}\}$, it results

$$\int_{B_{2\varrho}} h(x, u_n)\, T(u_n)\, \zeta \leq \|f_n\|_{L^1(B_{2\varrho})} + c_4 \int_{B_{2\varrho}} |Du_n|^{p-1} \leq c_5\,;$$

on the other hand, by (H3), we have

$$\begin{aligned}\int_{B_{2\varrho}} h(x,u_n)T(u_n)\zeta &\geq \int_{B_\varrho} h(x,u_n)T(u_n) \geq T(1)\int_{B_\varrho\cap\{|u_n|>1\}} |h(x,u_n)| \\ &= T(1)\Big[\int_{B_\varrho} |h(x,u_n)| - \int_{B_\varrho\cap\{|u_n|\leq 1\}} |h(x,u_n)|\Big] \\ &\geq T(1)\Big[\int_{B_\varrho} |h(x,u_n)| - \int_{B_\varrho} H_1(x)\Big] \\ &\geq T(1)\Big[\int_{B_\varrho} |h(x,u_n)| - c_6\Big],\end{aligned}$$

from which we get

$$\int_{B_\varrho} |h(x,u_n)| \leq c_7\,. \tag{1.12}$$

By a diagonal process we can now extract from $\{u_n\}$ a subsequence which converges to a solution of $(E1)$. Indeed, at the first step, using (1.9) and (1.10) written for $\varrho = 1$ as shown in Theorem 6.1 of [3], we obtain the existence of a subsequence $\{u_{n^{(1)}}\}$, extracted from $\{u_n\}_{n\geq 3}$, and of a function u_1 belonging to $M^{\frac{N(p-1)}{N-p}}(B_2)$, whose truncations $T_k(u_1)$ are in $W^{1,p}(B_2)$ for all $k>0$, such that $u_{n^{(1)}} \to u_1$ almost everywhere in B_2, and, for every fixed $k>0$,

$$T_k(u_{n^{(1)}}) \rightharpoonup T_k(u_1) \quad \text{weakly in } W^{1,p}(B_2)\,,$$

so that, by Lemma 1, there exists the weak gradient Du_1 of u_1 in B_2.

Moreover, by (1.12) with $n = n^{(1)}$ and $\varrho = 1$, reasoning as in the proof of Theorem 1 in [6], we obtain that

$$h(x,u_{n^{(1)}}) \to h(x,u_1) \quad \text{strongly in } L^1(B_1)\,.$$

From this and (1.11) written for $n = n^{(1)}$ and $\varrho = 1$, exactly following [3], only paying attention to the fact that we have to localize the test functions used in [3] by means of a cut-off function $\zeta \in \mathcal{D}(B_2)$, we obtain that

$$Du_{n^{(1)}} \to Du_1 \quad \text{almost everywhere in } B_1\,.$$

This implies that $|Du_1|$ belongs to $M^{\frac{N(p-1)}{N-1}}(B_1)$, and that

$$|Du_{n^{(1)}}|^{p-1} \to |Du_1|^{p-1} \quad \text{strongly in } L^q(B_1)\,,\ \forall\, q\in[1,\frac{N}{N-1})\,,$$

and then by (A1), (A2) and the dominated convergence theorem,

$$a(x, Du_{n^{(1)}}) \to a(x, Du_1) \quad \text{strongly} \quad \text{in } L^1(B_1)\,.$$

Going on in a similar manner, at the j-th step we find a subsequence $\{\, u_{n^{(j)}} \,\}$, extracted from $\{\, u_{n^{(j-1)}} \,\}_{n^{(j-1)} \geq j+2}$, and a function u_j in $M^{\frac{N(p-1)}{N-p}}(B_{j+1})$, with $T_k(u_j)$ in $W^{1,p}(B_{j+1})$ for all $k > 0$ and $|Du_j|$ in $M^{\frac{N(p-1)}{N-1}}(B_j)$, such that

$$\begin{aligned} u_{n^{(j)}} \to u_j \quad & \text{almost everywhere in } B_{j+1}\,, \\ T_k(u_{n^{(j)}}) \rightharpoonup T_k(u_j) \quad & \text{weakly in } W^{1,p}(B_{j+1})\,, \\ h(x, u_{n^{(j)}}) \to h(x, u_j) \quad & \text{strongly in } L^1(B_j)\,, \\ Du_{n^{(j)}} \to Du_j \quad & \text{almost everywhere in } B_j\,, \\ |Du_{n^{(j)}}|^{p-1} \to |Du_j|^{p-1} \quad & \text{strongly in } L^q(B_j)\,,\ \forall\, q \in [1, \frac{N}{N-1})\,, \\ a(x, Du_{n^{(j)}}) \to a(x, Du_j) \quad & \text{strongly in } L^1(B_j)\,, \end{aligned}$$

and $u_j = u_{j-1}$ almost everywhere in B_j.

If we define $u = u_j$ in B_j, then u is a globally defined function belonging to $M_{loc}^{\frac{N(p-1)}{N-p}}(\mathbb{R}^N)$, whose truncations $T_k(u)$ belong to $W^{1,p}_{loc}(\mathbb{R}^N)$ for all $k > 0$, with $|Du|$ in $M_{loc}^{\frac{N(p-1)}{N-1}}(\mathbb{R}^N)$, and such that the diagonal sequence $\{\, u_{j^*} = u_{j^{(j)}} \,\}$ in particular satisfies:

$$u_{j^*} \to u \quad \text{almost everywhere in } \mathbb{R}^N\,, \tag{1.13}$$

$$h(x, u_{j^*}) \to h(x, u) \quad \text{strongly in } L^1_{loc}(\mathbb{R}^N)\,, \tag{1.14}$$

$$Du_{j^*} \to Du \quad \text{almost everywhere in } \mathbb{R}^N\,, \tag{1.15}$$

$$a(x, Du_{j^*}) \to a(x, Du) \quad \text{strongly in } L^1_{loc}(\mathbb{R}^N)\,. \tag{1.16}$$

The convergences (1.14) and (1.16) allow us to pass to the limit in identity (1.8), written for $n = j^*$, for every fixed v in $\mathcal{D}(\mathbb{R}^N)$, and to conclude that the function u is a distributional solution of $(E1)$.

It remains to prove inequality (0.6), that is, that u is a local entropy solution of $(E1)$.

Given θ in $\mathcal{D}(\mathbb{R}^N)$, with $\theta \geq 0$, and v in $C^\infty(\mathbb{R}^N)$, we can apply (1.8), written for $n = j^*$, choosing as test function $T_k(u_{j^*} - v)\,\theta$ for every j^* so large that $\operatorname{supp}\theta \subset B_{j^*}$; we obtain

$$\int_{B_{j^*}} a(x, Du_{j^*}) \cdot DT_k(u_{j^*} - v)\,\theta = \int_{B_{j^*}} f_{j^*} T_k(u_{j^*} - v)\theta$$
$$- \int_{B_{j^*}} h(x, u_{j^*}) T_k(u_{j^*} - v)\,\theta - \int_{B_{j^*}} a(x, Du_{j^*}) \cdot D\theta T_k(u_{j^*} - v).$$

The left hand side term, thanks to (1.13), (1.14) and (1.16), is easily seen to converge to

$$\int_{\mathbb{R}^N} f T_k(u - v)\,\theta - \int_{\mathbb{R}^N} h(x, u) T_k(u - v)\,\theta - \int_{\mathbb{R}^N} a(x, Du) \cdot D\theta T_k(u - v)\,.$$

As the first integral is concerned, it suffices to observe that

$$\int_{B_{j^*}} a(x, Du_{j^*}) \cdot DT_k(u_{j^*} - v)\,\theta = \int_{\{\,|u_{j^*} - v| < k\,\}} a(x, Du_{j^*}) \cdot Du_{j^*}\,\theta$$
$$- \int_{\{\,|u_{j^*} - v| < k\,\}} a(x, Du_{j^*}) \cdot Dv\,\theta\,;$$

being $\theta \geq 0$, assumption (A2) and Fatou Lemma imply

$$\liminf_{j^* \to +\infty} \int_{\{\,|u_{j^*} - v| < k\,\}} a(x, Du_{j^*}) \cdot Du_{j^*}\,\theta \geq \int_{\{\,|u - v| < k\,\}} a(x, Du) \cdot Du\,\theta\,,$$

while, by (1.16), it is clear that

$$\lim_{j^* \to +\infty} \int_{\{\,|u_{j^*} - v| < k\,\}} a(x, Du_{j^*}) \cdot Dv\,\theta = \int_{\{\,|u - v| < k\,\}} a(x, Du) \cdot Dv\,\theta\,.$$

Combining the limits obtained, we deduce (0.6) for all nonnegative $\theta \in \mathcal{D}(\mathbb{R}^N)$ and $v \in C^\infty(\mathbb{R}^N)$. By a density argument we then obtain (0.6) for all v in $W^{1,p}_{loc}(\mathbb{R}^N) \cap L^\infty_{loc}(\mathbb{R}^N)$. $\square$

Proof of Theorem 2. Let u_n be in $W^{1,p}_0(B_n) \cap L^\infty(B_n)$ defined as in Theorem 1; using (0.5) and the assumption $p > p_0$, from (1.10) and (1.11) we immediately obtain that, for every fixed $\varrho > 0$, the sequence $\{\,u_n\,\}_{n \geq 3\varrho}$ is uniformly bounded in $W^{1,q}(B_{2\varrho})$ for all $q \in [1, \frac{N(p-1)}{N-1})$, from which, reasoning as in Theorem 1, the conclusion follows. $\square$

Proof of Theorem 3. Assume that $f\,\log(1+|f|)$ belongs to $L^1_{loc}(\mathbb{R}^N)$ and let u_n be as in the proof of Theorem 1; the only thing which has to be shown in this case is that, for every fixed $\varrho > 0$, the sequence $\{\,Du_n\,\}$ is uniformly bounded in $L^{\bar q}(B_\varrho)$.

Fixed $\varrho > 0$, chosen ζ in $\mathcal{D}(B_{2\varrho})$, with $0 \le \zeta \le 1$ and $\zeta \equiv 1$ in B_ϱ, and set $\eta = \varphi(\zeta)$ as above, let us apply (1.8) with test function $v = \log(1 + |u_n|\,\eta)\,\mathrm{sign}(u_n)$, for all $n \ge 2\varrho$. We get:

$$\int_{B_{2\varrho}} \frac{a(x, Du_n)\cdot D(u_n\,\eta)}{(1+|u_n|\,\eta)} + \int_{B_{2\varrho}} h(x,u_n)\,\log(1+|u_n|\,\eta)\,\mathrm{sign}(u_n) = \int_{B_{2\varrho}} f_n\,\log(1+|u_n|\,\eta)\,\mathrm{sign}(u_n)\,.$$

From (A2), (A3) and (H2) it follows

$$\lambda \int_{B_{2\varrho}} \frac{|Du_n|^p\,\eta}{(1+|u_n|\,\eta)} \le \int_{B_{2\varrho}} |f|\,\log(1+|u_n|\,\eta) + c_1 \int_{B_{2\varrho}} \frac{|Du_n|^{p-1}}{(1+|u_n|\,\eta)}|u_n|\varphi'(\zeta)\,. \tag{1.17}$$

Recalling that $ab \le a\log(1+a) + e^b$ for all $a,\ b \ge 0$, and using the $L^1(B_{2\varrho})$ bound for u_n obtained in the proof of Theorem 1, we have, for all $n \ge 3\varrho$,

$$\int_{B_{2\varrho}} |f|\,\log(1+|u_n|\,\eta) \le \int_{B_{2\varrho}} |f|\log(1+|f|) + \int_{B_{2\varrho}} (1+|u_n|\,\eta) \le c_2\,. \tag{1.18}$$

Moreover, from the Young inequality it follows

$$\begin{aligned} &c_1 \int_{B_{2\varrho}} \frac{|Du_n|^{p-1}}{(1+|u_n|\,\eta)}|u_n|\varphi'(\zeta) \\ &\le \frac{\lambda}{2}\int_{B_{2\varrho}} \frac{|Du_n|^p\eta}{(1+|u_n|\,\eta)} + c_3 \int_{B_{2\varrho}} \frac{|u_n|^p}{(1+|u_n|\,\eta)}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}. \end{aligned} \tag{1.19}$$

Inequalities (1.18) and (1.19), used in (1.17), give

$$\int_{B_{2\varrho}} \frac{|Du_n|^p\,\eta}{(1+|u_n|\,\eta)} \le c_4 \int_{B_{2\varrho}} \frac{|u_n|^p}{(1+|u_n|\,\eta)}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} + c_5\,.$$

Furthermore, using estimate (1.7), we easily obtain

$$\int_{B_{2\varrho}} \frac{|D(u_n\,\eta)|^p}{(1+|u_n|\,\eta)} \le c_6 \int_{B_{2\varrho}} \frac{|u_n|^p}{(1+|u_n|\,\eta)}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} + c_7\,.$$

Lemma 1.1 applied with $t=|u_n|,\ \alpha=1$, and $C=c_7$, assumption (H2) and the bound (1.12) with ϱ replaced by 2ϱ, then imply, for all $n\geq 6\varrho$,

$$c_6\int_{B_{2\varrho}}\frac{|u_n|^p}{(1+|u_n|\,\eta)}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\leq\int_{B_{2\varrho}}\frac{|u_n|\,\eta}{(1+|u_n|\,\eta)}g(|u_n|)+\int_{B_{2\varrho}}\frac{c_7}{(1+|u_n|\,\eta)}$$
$$\leq\int_{B_{2\varrho}}g(|u_n|)+c_6|B_{2\varrho}|\leq\int_{B_{2\varrho}}|h(x,u_n)|+c_6|B_{2\varrho}|\leq c_8\,,$$

so that

$$\int_{B_{2\varrho}}\frac{|D(u_n\,\eta)|^p}{(1+|u_n|\,\eta)}\leq c_9\,.$$

From this and the Hölder inequality with exponents $(\frac{p}{\bar q},\frac{p}{p-\bar q})$, we deduce

$$\begin{aligned}\int_{B_{2\varrho}}|D(u_n\,\eta)|^{\bar q}&=\int_{B_{2\varrho}}\frac{|D(u_n\,\eta)|^{\bar q}}{(1+|u_n|\,\eta)^{\frac{\bar q}{p}}}\,(1+|u_n|\,\eta)^{\frac{\bar q}{p}}\\&\leq\Big(\int_{B_{2\varrho}}\frac{|D(u_n\,\eta)|^p}{(1+|u_n|\,\eta)}\Big)^{\frac{\bar q}{p}}\Big(\int_{B_{2\varrho}}(1+|u_n|\,\eta)^{\frac{\bar q}{p-\bar q}}\Big)^{\frac{p-\bar q}{p}}\\&\leq c_{10}\Big[1+\Big(\int_{B_{2\varrho}}(|u_n|\,\eta)^{\frac{\bar q}{p-\bar q}}\Big)^{\frac{p-\bar q}{p}}\Big]\,.\end{aligned}$$

Observing that $\bar q^*=\frac{\bar q}{p-\bar q}=\frac{N(p-1)}{N-p}$ and $\frac{p-\bar q}{p}=\frac{\bar q}{p\,\bar q^*}=\frac{N-p}{p(N-1)}$, the previous inequality can be rewritten as

$$\int_{B_{2\varrho}}|D(u_n\,\eta)|^{\bar q}\leq c_{10}\Big[1+\Big(\int_{B_{2\varrho}}(|u_n|\,\eta)^{\bar q^*}\Big)^{\frac{\bar q}{p\,\bar q^*}}\Big]\,.\qquad(1.20)$$

Applying to the left hand side integral the Sobolev inequality, and to the right hand side one the Young inequality, we have

$$\begin{aligned}\Big(\int_{B_{2\varrho}}(|u_n|\eta)^{\bar q^*}\Big)^{\frac{\bar q}{\bar q^*}}&\leq c_{11}\int_{B_{2\varrho}}|D(u_n\eta)|^{\bar q}\\&\leq c_{12}\Big[1+\frac{\varepsilon}{p}\Big(\int_{B_{2\varrho}}(|u_n|\eta)^{\bar q^*}\Big)^{\frac{\bar q}{\bar q^*}}+\frac{1}{p'\varepsilon^{\frac{1}{p-1}}}\Big],\end{aligned}$$

for any $\varepsilon>0$. Choosing $\varepsilon=\frac{p}{2c_{12}}$, we deduce from the above inequality that

$$\Big(\int_{B_{2\varrho}}(|u_n|\,\eta)^{\bar q^*}\Big)^{\frac{\bar q}{\bar q^*}}\leq c_{13}\,,$$

and then

$$\int_{B_{2\varrho}} |D(u_n\,\eta)|^{\bar{q}} \leq c_{14}\,,$$

which is the desired estimate. □

In order to give the proof of Theorem 4 we need the following preliminary result (which can be compared with Lemma 2.1 in [6]).

Lemma 1.3. *Let R, f, u, r, ζ, and φ be as in Lemma 1.2; then there exists a constant $c > 0$, depending on $p, r, R, g, \lambda, \Lambda$, and on the norm $\|f\|_{L^1(B_R)}$, such that*

$$\int_{B_R} \frac{|D(\varphi(\zeta)\,u)|^p}{[(1+\varphi(\zeta)\,|u|)\,g(1+\varphi(\zeta)\,|u|)]^{\frac{1}{p}}} \leq c\,. \tag{1.21}$$

Proof. We define the function

$$\Phi(s) \;=\begin{cases} \int_0^s \frac{dt}{[(1+t)\,g(1+t)]^{\frac{1}{p}}} & \text{if } s \geq 0,\\ -\Phi(-s) & \text{if } s < 0, \end{cases}$$

and observe that, under assumption (H2), $\Phi_\infty \;= \sup_{\mathbb{R}} |\Phi(s)| < +\infty$.

Using $v = \Phi(u\,\eta)$, where $\eta = \varphi(\zeta)$, as test function in (1.1), we have

$$\int_{B_R} \frac{a(x,Du)\cdot D(u\eta)}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}} + \int_{B_R} h(x,u)\,\Phi(u\,\eta) = \int_{B_R} f\,\Phi(u\,\eta)\,,$$

from which

$$\lambda \int_{B_R} \frac{|Du|^p\,\eta}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}} + \int_{B_R} g(|u|)\,\Phi(|u|\,\eta)$$
$$\leq \Phi_\infty \|f\|_{L^1(B_R)} + c_1 \int_{B_R} \frac{|Du|^{p-1}\,|u|\,\varphi'(\zeta)}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}}\,.$$

We can now proceed as in Lemma 1.2: first, use Young inequality to estimate the last integral, and obtain

$$\int_{B_R} \frac{|Du|^p\,\eta}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}} + \int_{B_R} g(|u|)\,\Phi(|u|\,\eta)$$
$$\leq c_2 + c_3 \int_{B_R} \frac{|u|^p \varphi'(\zeta)^p}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}\,\varphi(\zeta)^{p-1}}\,;$$

second, remember (1.8) and deduce

$$\begin{aligned}&\int_{B_R} \frac{|D(u\,\eta)|^p}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}} + \int_{B_R} g(|u|)\,\Phi(|u|\,\eta)\\ &\le c_4 + c_5 \int_{B_R} \frac{|u|^p \varphi'(\zeta)^p}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}\,\varphi(\zeta)^{p-1}}\,;\end{aligned}$$

third, use (1.2) of Lemma 1.1 applied to the last integral and get

$$\begin{aligned}&\int_{B_R} \frac{|D(u\,\eta)|^p}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}} + \int_{B_R} g(|u|)\,\Phi(|u|\,\eta)\\ &\le c_4 + \int_{B_R} \frac{|u|\,g(|u|)\,\eta}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}} + c_5 \int_{B_R} \frac{1}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}}\\ &\le c_6 + \int_{B_R} \frac{|u|\,g(|u|)\,\eta}{[(1+|u|\,\eta)\,g(1+|u|\,\eta)]^{\frac{1}{p}}}\,.\end{aligned}$$

Finally, observe that $\Phi(s) \ge \frac{s}{[(1+s)\,g(1+s)]^{\frac{1}{p}}}$ for all $s \ge 0$ to conclude. □

Remark 2. Estimate (1.21) has been obtained assuming on g only $(G1)$, $(G2)$ and $(G3)$. Nevertheless, it becomes useful only if the function g is not so big that the denominator of the integral (1.21) weighs too hard. In other words, such estimate works only for lower order terms $g(t)$ having growth "close" to t^{p-1}. On the other hand, the other cases may be treated as in [6].

Proof of Theorem 4. Assume $p > N$ and let $u_n \in W_0^{1,p}(B_n) \cap L^\infty(B_n)$ be as in the proof of Theorem 1. In order to achieve the conclusion, the only thing which has to be proved is an estimate for $\{\, u_n \,\}$ in $W^{1,p}(B_\varrho)$, for every fixed $\varrho > 0$.

Inequality (1.21) applied to u_n in $B_{2\varrho}$, with $r = \varrho$ and for all $n \ge 2\varrho$, gives

$$\int_{B_{2\varrho}} |D\Psi(u_n\,\eta)|^p \le c\,, \tag{1.22}$$

where

$$\Psi(s) \;=\begin{cases} \int_0^s \frac{dt}{[(1+t)\,g(1+t)]^{\frac{1}{p^2}}} & \text{if } s \ge 0,\\ -\Psi(-s) & \text{if } s < 0.\end{cases}$$

By Remark 2 we may assume that $\Psi(s)$ diverges to $+\infty$ as s tends to $+\infty$ (otherwise g would grow as a power function with exponent greater than $p-1$, and we can use the result of [6]).

By (1.22), $\{\Psi(u_n\,\eta)\}$ is uniformly bounded in $W_0^{1,p}(B_{2\varrho})$ and then, being $p > N$, in $L^\infty(B_{2\varrho})$. Therefore, $\|u_n\|_{L^\infty(B_\varrho)} \le c_1$, and, using again (1.22), we get

$$\int_{B_\varrho} |Du_n|^p \le c[(1+c_1)\,g(1+c_1)]^{\frac{1}{p}} = c_2.$$

Reasoning as in the proof of Theorem 1, it follows that there exists a subsequence $\{u_{j^*}\}$ of $\{u_n\}$ such that $u_{j^*} \rightharpoonup u$ weakly in $W^{1,p}_{loc}(\mathbb{R}^N)$, where u is a solution of $(E1)$. Observe that in this case the solution u satisfies

$$\int_{\mathbb{R}^N} a(x,Du)\cdot Dv + \int_{\mathbb{R}^N} h(x,u)\,v = \int_{\mathbb{R}^N} f\,v$$

for every test function v belonging to $W^{1,p}(\mathbb{R}^N)$ and with compact support.

2. Equations with data in $L^m_{loc}, m > 1$. In this Section we are concerned with equation $(E1)$ having not a merely locally integrable datum: we will suppose that f belongs to $L^m_{loc}(\mathbb{R}^N)$, for some $m > 1$.

We will first prove Theorem 5, which is a regularity result, analogous to Theorem 3 of [5], for the solutions u constructed in Theorem 2.

We recall that these solutions may be obtained as limits of the sequence $\{\,u_n\,\}$, where the functions $u_n \in W_0^{1,p}(B_n)\cap L^\infty(B_n)$ satisfy

$$\int_{B_n} a(x,Du_n)\cdot Dv + \int_{B_n} h(x,u_n)\,v = \int_{B_n} f_n\,v\,, \tag{2.1}$$

for all v in $W_0^{1,p}(B_n)$, with $f_n = T_n(f)$.

Proof of Theorem 5. Set

$$\Psi_\theta(s) = \frac{1}{\theta}\,\left[(1+|s|)^\theta - 1\right]\,\mathrm{sign}(s)\,, \quad s\in\mathbb{R}\,,$$

where θ is a parameter in $(0,1]$ to be fixed in the sequel, and let us take as test function in (2.1) $v = \Psi_\theta(u_n\,\varphi(\zeta))$, where ζ is a cut-off function belonging to $\mathcal{D}(B_{2\varrho})$, $\varrho > 0$ being arbitrarily fixed, with $\zeta\equiv 1$ in B_ϱ, and φ is a function as in Lemma 1.1, constructed in correspondence of g, $\alpha = 1$ and an arbitrary constant $C > 0$. Writing η instead of $\varphi(\zeta)$, we get

$$\int_{B_{2\varrho}} \frac{a(x,Du_n)\cdot Du_n\,\eta}{(1+|u_n|\,\eta)^{1-\theta}} + \int_{B_{2\varrho}} h(x,u_n)\,\Psi_\theta(u_n\,\eta) = \int_{B_{2\varrho}} f_n\Psi_\theta(u_n\eta) - \int_{B_{2\varrho}} \frac{a(x,Du_n)\cdot D\zeta\varphi'(\zeta)\,u_n}{(1+|u_n|\,\eta)^{1-\theta}},$$

from which we obtain

$$\lambda \int_{B_{2\varrho}} \frac{|Du_n|^p \, \eta}{(1+|u_n|\,\eta)^{1-\theta}} + \int_{B_{2\varrho}} h(x,u_n)\, \Psi_\theta(u_n\, \eta) \leq \int_{B_{2\varrho}} |f_n| (|u_n|\,\eta)^\theta + c_1 \int_{B_{2\varrho}} \frac{|Du_n|^{p-1}\, \varphi'(\zeta)\, |u_n|}{(1+|u_n|\,\eta)^{1-\theta}} .$$

From this estimate, using first Young inequality and then estimate (1.8), we obtain

$$\int_{B_{2\varrho}} \frac{|D(u_n\, \eta)|^p}{(1+|u_n|\,\eta)^{1-\theta}} + \int_{B_{2\varrho}} h(x,u_n)\, \Psi_\theta(u_n\, \eta) \leq c_2 \int_{B_{2\varrho}} |f_n| (|u_n|\eta)^\theta + c_3 \int_{B_{2\varrho}} \frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}} \frac{|u_n|^p}{(1+|u_n|\,\eta)^{1-\theta}}.$$

Applying now (1.2), with $t = |u_n|$, $\alpha = 1$ and $C = 2c_3$, using assumption (H2) and observing that $\frac{s}{(1+s)^{1-\theta}} \leq \Psi_\theta(s)$ for all $s \geq 0$ and $\theta \in (0,1]$, we have

$$\int_{B_{2\varrho}} \frac{|D(u_n\, \eta)|^p}{(1+|u_n|\,\eta)^{1-\theta}} + \int_{B_{2\varrho}} h(x,u_n) \Psi_\theta(u_n \eta) \leq c_4 \int_{B_{2\varrho}} |f_n| (|u_n|\eta)^\theta + c_5 .$$

Furthermore, estimating the last integral by means of the Hölder inequality with exponents $(m, m' = \frac{m}{m-1})$, it follows

$$\begin{aligned} &\int_{B_{2\varrho}} \frac{|D(u_n\, \eta)|^p}{(1+|u_n|\,\eta)^{1-\theta}} + \int_{B_{2\varrho}} h(x,u_n)\, \Psi_\theta(u_n\, \eta) \\ &\leq c_4 \Big(\int_{B_{2\varrho}} |f_n|^m \Big)^{\frac{1}{m}} \Big(\int_{B_{2\varrho}} (|u_n|\,\eta)^{\theta\, m'} \Big)^{\frac{1}{m'}} + c_5 \\ &\leq c_4 \|f\|_{L^m(B_{2\varrho})} \Big(\int_{B_{2\varrho}} (|u_n|\,\eta)^{\theta\, m'} \Big)^{\frac{1}{m'}} + c_5 = c_6 \Big(\int_{B_{2\varrho}} (|u_n|\eta)^{\theta m'} \Big)^{\frac{1}{m'}} + c_5. \end{aligned} \tag{2.2}$$

If $m = (p^*)' = \frac{Np}{N(p-1)+p}$, then, choosing $\theta = 1$, (2.2) becomes

$$\int_{B_{2\varrho}} |D(u_n\, \eta)|^p + \int_{B_{2\varrho}} |h(x,u_n)\, u_n|\, \eta \leq c_6 \Big(\int_{B_{2\varrho}} (|u_n|\,\eta)^{p^*} \Big)^{\frac{1}{p^*}} + c_5 .$$

The Sobolev and Young inequalities then imply both

$$\int_{B_\varrho} |Du_n|^p \le c_7 \quad \text{and} \quad \int_{B_\varrho} |h(x,u_n)\, u_n| \le c_8,$$

from which it follows that the limit function u, obtained as in Theorem 1, belongs to $W^{1,p}_{loc}(\mathbb{R}^N)$, is such that $h(x,u)\,u$ belongs to $L^1_{loc}(\mathbb{R}^N)$, and satisfies (0.7) for any test function $v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ having compact support. Moreover, being $h(x,u)\,u$ in $L^1_{loc}(\mathbb{R}^N)$, by a density argument we can also deduce (0.8).

If m belongs to $(1,(p^*)')$, let us go back to (2.2) with $\theta \in (0,1)$, which in particular implies

$$\int_{B_{2\varrho}} \frac{|D(u_n\,\eta)|^p}{(1+|u_n|\,\eta)^{1-\theta}} \le c_6 \Big(\int_{B_{2\varrho}} (|u_n|\,\eta)^{\theta\, m'} \Big)^{\frac{1}{m'}} + c_5\,.$$

Reasoning as in the proof of Theorem 3, we get by the Hölder inequality with exponents $(\frac{p}{q}, \frac{p}{p-q})$, for any $q \in [1,p)$,

$$\begin{aligned}
\int_{B_{2\varrho}} |D(u_n\eta)|^q &= \int_{B_{2\varrho}} \frac{|D(u_n\eta)|^q}{(1+|u_n|\,\eta)^{(1-\theta)\frac{q}{p}}} (1+|u_n|\,\eta)^{(1-\theta)\frac{q}{p}} \\
&\le \Big(\int_{B_{2\varrho}} \frac{|D(u_n\eta)|^p}{(1+|u_n|\,\eta)^{1-\theta}} \Big)^{\frac{q}{p}} \Big(\int_{B_{2\varrho}} (1+|u_n|\,\eta)^{(1-\theta)\frac{q}{p-q}} \Big)^{\frac{p-q}{p}} \\
&\le \Big[c_6 \Big(\int_{B_{2\varrho}} (|u_n|\eta)^{\theta m'} \Big)^{\frac{1}{m'}} + c_5 \Big]^{\frac{q}{p}} \Big(\int_{B_{2\varrho}} (1+|u_n|\,\eta)^{(1-\theta)\frac{q}{p-q}} \Big)^{\frac{p-q}{p}} \\
&\le c_9 \Big[1 + \Big(\int_{B_{2\varrho}} (|u_n|\eta)^{\theta m'} \Big)^{\frac{q}{m' q}} \Big] \Big[1 + \Big(\int_{B_{2\varrho}} (|u_n|\eta)^{(1-\theta)\frac{q}{p-q}} \Big)^{\frac{p-q}{p}} \Big].
\end{aligned}$$

We now choose θ in $(0,1)$ and q in $[1,p)$ such that

$$\theta\, m' = \frac{q\,(1-\theta)}{p-q} = q^* = \frac{N\,q}{N-q}\,;$$

observe that such a choice is possible if and only if p is in (p_0, N) and m in $(1,(p^*)')$, and yields:

$$\theta = \frac{N(p-1)\,(m-1)}{N-p\,m}, \quad q = (p-1)m^* = (p-1)\frac{N\,m}{N-m}.$$

For these values of θ and q, we thus obtain

$$\int_{B_{2\varrho}} |D(u_n\,\eta)|^q \le c_{10}\Big[1 + \Big(\int_{B_{2\varrho}} (|u_n|\eta)^{q^*}\Big)^{\frac{q}{q^* p}}\Big],$$

which is inequality (1.20) of the proof of Theorem 3, with $\bar{q}$ replaced by q. We have already shown that this inequality implies the bound

$$\int_{B_{2\varrho}} |D(u_n\,\eta)|^q \le c_{11}\,.$$

Thus, for any m in $(1,(p^*)']$, we have obtained a $W^{1,q}(B_\varrho)$ bound on the sequence $\{\,u_n\,\}_{n\ge 2\varrho}$, with $q=(p-1)\,m^*$; reasoning as in the case $m=(p^*)'$, we conclude. $\square$

Theorem 5 implies in particular that, if f is in $L^m_{loc}(\mathbb{R}^N)$ and m is in $(1,(p^*)']$, then the solution u constructed in Theorem 2 is in $L^{((m-1)p^*)^*}_{loc}(\mathbb{R}^N)$. This summability still holds if m belongs to $((p^*)', \frac{N}{p})$ for *every* function u in $W^{1,p}_{loc}(\mathbb{R}^N)$ satisfying identity (0.7).

Proof of Theorem 6. Let u and f be as in the statement. Let further ζ and φ be respectively a cut-off function belonging to $\mathcal{D}(B_{2r})$, for a fixed $r>0$, with $\zeta\equiv 1$ in B_r, and the function given by Lemma 1.1 in correspondence of g and some $\alpha,\ C>0$ which will be chosen later.

The function $v=|T_k(u\,\eta)|^{p\gamma}T_k(u\eta)$, with $\eta=\varphi(\zeta)$ and $k,\,\gamma>0$, may be used as test function in (0.7), setting $B(k,2r)=B_{2r}\cap\{\,|u|\,\eta<k\,\}$, provides

$$(p\gamma+1)\int_{B(k,2r)} a(x,Du)\cdot D(u\eta)|T_k(u\,\eta)|^{p\gamma} + \int_{B_{2r}} h(x,u)T_k(u\,\eta)|T_k(u\,\eta)|^{p\gamma} = \int_{B_{2r}} fT_k(u\,\eta)|T_k(u\,\eta)|^{p\gamma},$$

and then

$$\int_{B(k,2r)} |Du|^p|T_k(u\,\eta)|^{p\gamma}\eta + \int_{B_{2r}} g(|u|)|T_k(u\,\eta)|^{p\gamma+1} \le c_1\int_{B_{2r}} |f||T_k(u\,\eta)|^{p\gamma+1} + c_2\int_{B(k,2r)} |Du|^{p-1}\varphi'(\zeta)|T_k(u\,\eta)|^{p\gamma}|u|.$$

We can deal with the last integral using the Young inequality and obtain

$$\int_{B(k,2r)}|Du|^p|T_k(u\,\eta)|^{p\gamma}\eta+\int_{B_{2r}}g(|u|)|T_k(u\,\eta)|^{p\gamma+1}$$
$$\le c_3\int_{B_{2r}}|f|\,|T_k(u\,\eta)|^{p\gamma+1}+c_4\int_{B(k,2r)}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}|T_k(u\,\eta)|^{p\gamma}|u|^p\,.$$

Moreover, observing that in $B(k,2r)$ we have

$$|D(|T_k(u\,\eta)|^\gamma T_k(u\,\eta))|^p\le c_5|T_k(u\,\eta)|^{p\gamma}\left(|Du|^p\eta^p+\varphi'(\zeta)^p|u|^p\right)$$
$$\le c_5|T_k(u\,\eta)|^{p\gamma}\Big(|Du|^p\eta+\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}|u|^p\Big)\,,$$

it results

$$\int_{B(k,2r)}|D(|T_k(u\,\eta)|^\gamma T_k(u\,\eta))|^p+\int_{B_{2r}}g(|u|)|T_k(u\,\eta)|^{p\gamma+1}$$
$$\le c_6\int_{B_{2r}}|f|\,|T_k(u\,\eta)|^{p\gamma+1}+c_7\int_{B(k,2r)}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}|T_k(u\,\eta)|^{p\gamma}|u|^p.$$

From inequality (1.2) of Lemma 1.1, applied with $t=|u|,\ \alpha=1$ and $C=c_7$, we get

$$\int_{B(k,2r)}|D(|T_k(u\,\eta)|^\gamma T_k(u\,\eta))|^p\le c_6\int_{B_{2r}}|f||T_k(u\,\eta)|^{p\gamma+1}+c_7\int_{B(k,2r)}|T_k(u\,\eta)|^{p\gamma}$$
$$\le c_8\Big[\int_{B_{2r}}(|f|+1)|T_k(u\,\eta)|^{p\gamma+1}+1\Big].$$

The right hand side term may be estimated by means of the Hölder inequality with exponents $(m,m'=\frac{m}{m-1})$, while the Sobolev inequality may be applied to the right hand side integral, this leads to

$$\Big(\int_{B_{2r}}|T_k(u\,\eta)|^{(\gamma+1)p^*}\Big)^{\frac{p}{p^*}}\le c_9\int_{B(k,2r)}|D(|T_k(u\,\eta)|^\gamma T_k(u\,\eta))|^p$$
$$\le c_{10}\Big[\int_{B_{2r}}(|f|+1)\,|T_k(u\,\eta)|^{p\gamma+1}+1\Big]$$
$$\le c_{10}\Big[\Big(\int_{B_{2r}}(|f|+1)^m\Big)^{\frac{1}{m}}\Big(\int_{B_{2r}}|T_k(u\,\eta)|^{(p\gamma+1)m'}\Big)^{\frac{1}{m'}}+1\Big]$$
$$\le c_{11}\Big[\Big(\int_{B_{2r}}|T_k(u\,\eta)|^{(p\gamma+1)m'}\Big)^{\frac{1}{m'}}+1\Big].$$

The choice

$$\gamma = \frac{m\,(N(p-1)+p) - Np}{p\,(N-pm)}\,,$$

which is possible since m belongs to $((p^*)', \frac{N}{p})$, yields

$$m'(p\gamma+1) = p^*(\gamma+1) = ((p-1)m^*)^*\,,$$

so that we have obtained

$$\Big(\int_{B_{2r}} |T_k(u\,\eta)|^{((p-1)m^*)^*}\Big)^{\frac{p}{p^*}} \le c_{11}\Big[\Big(\int_{B_{2r}} |T_k(u\,\eta)|^{((p-1)m^*)^*}\Big)^{\frac{1}{m'}} + 1\Big].$$

The inequality $m < \frac{N}{p}$ implies $\frac{1}{m'} < \frac{p}{p^*}$, and thus, from the above and the Young inequality, it follows

$$\int_{B_{2r}} |T_k(u\,\eta)|^{((p-1)m^*)^*} \le c_{12}\,.$$

In particular, we have

$$\int_{B_r} |T_k(u)|^{((p-1)m^*)^*} \le c_{12}\,,$$

for every $k > 0$, with c_{12} not depending on k. Passing to the limit as k tends to infinity yields the conclusion. □

The next result is concerned with locally bounded solutions of $(E1)$.

Proof of Theorem 7. Fixed $0 \le r < R \le 1$, let ζ be a cut-off function belonging to $\mathcal{D}(B_R)$, with $\zeta \equiv 1$ in B_r, and such that

$$|D\zeta| \le \frac{c_0}{R-r}\,. \tag{2.3}$$

For every $k > 0$, and for all $s \in \mathbb{R}$, we set

$$G_k(s) = s - T_k(s) = \begin{cases} 0 & \text{if } |s| \le k, \\ s-k & \text{if } s > k, \\ s+k & \text{if } s < -k. \end{cases} \tag{2.4}$$

Since $u \in W^{1,p}_{loc}(\mathbb{R}^N)$ satisfies (0.7), and $u\,h(x,u)$ belongs to $L^1_{loc}(\mathbb{R}^N)$, the function $v = G_k(u)\,\eta$ may be used in (0.7), where $\eta = \varphi(\zeta)$, with φ given by

Lemma 1.1 in correspondence of g, and some $\alpha,\ C>0$ to be fixed in the sequel. Observing that supp (v) is contained in the set $A(k,R)=B_R\cap\{|u|\geq k\}$, it results

$$\int_{A(k,R)} a(x,Du)\cdot DG_k(u)\eta+\int_{A(k,R)} h(x,u)\,G_k(u)\eta \\ =\int_{A(k,R)} f\,G_k(u)\eta-\int_{A(k,R)} a(x,Du)\cdot D\zeta\varphi'(\zeta)G_k(u)\,,$$

from which, being $Du\equiv DG_k(u)$ in $A(k,R)$ and using the bound (2.3),

$$\lambda\int_{A(k,R)}|DG_k(u)|^p\eta+\int_{A(k,R)} g(|u|)|G_k(u)|\eta \\ \leq\int_{A(k,R)}|f|\,|G_k(u)|\eta+\frac{c_0\Lambda}{R-r}\int_{A(k,R)}|DG_k(u)|^{p-1}|G_k(u)|\varphi'(\zeta)\,.$$

Applying the Young inequality to the last integral, we obtain

$$\int_{A(k,R)}|DG_k(u)|^p\eta+\int_{A(k,R)} g(|u|)|G_k(u)|\eta \\ \leq c_1\int_{A(k,R)}|f|\,|G_k(u)|\eta+\frac{c_2}{(R-r)^p}\int_{A(k,R)}|G_k(u)|^p\,\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,,$$

where c_1 and c_2 are positive constants not depending on $r,\ R$. Observing further that

$$|D(g_k(u)\eta)|^p\leq c_3\big(|DG_k(u)|^p\eta^p+\frac{1}{(R-r)^p}\varphi'(\zeta)^p|G_k(u)|^p\big) \\ \leq c_3\big(|DG_k(u)|^p\eta+\frac{1}{(R-r)^p}\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}|G_k(u)|^p\big)\,,$$

we deduce

$$\int_{A(k,R)}|D(G_k(u)\eta)|^p+\int_{A(k,R)} g(|u|)|G_k(u)|\eta \\ \leq c_4\int_{A(k,R)}|f|\,|G_k(u)|\eta+\frac{c_5}{(R-r)^p}\int_{A(k,R)}|G_k(u)|^p\,\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,.$$

From Lemma 1.1, applied with $t = |G_k(u)|$, $C = \frac{c_5}{(R-r)^p}$ and $\alpha = 1$, and from the monotonicity of g, we obtain

$$\int_{A(k,R)} |D(G_k(u)\eta)|^p \leq c_4 \int_{A(k,R)} |f||G_k(u)|\eta + \frac{c_5}{(R-r)^p} \text{meas}\,(A(k,R)) \ .$$

By the Hölder inequality with exponents $((p^*)', p^*)$, the Sobolev inequality and the Young inequality, the second integral may be estimated by:

$$\begin{aligned}
\int_{A(k,R)} |f||G_k(u)|\eta \leq & \Big(\int_{A(k,R)} |f|^{(p^*)'} \Big)^{\frac{1}{(p^*)'}} \Big(\int_{A(k,R)} (|G_k(u)|\eta)^{p^*} \Big)^{\frac{1}{p^*}} \\
\leq & c_6 \Big(\int_{A(k,R)} |f|^{(p^*)'} \Big)^{\frac{1}{(p^*)'}} \Big(\int_{A(k,R)} |D(G_k(u)\eta)|^p \Big)^{\frac{1}{p}} \\
\leq & \frac{c_6^{p'}}{\varepsilon^{\frac{1}{1-p}}} \Big(\int_{A(k,R)} |f|^{(p^*)'} \Big)^{\frac{p'}{(p^*)'}} + \varepsilon \int_{A(k,R)} |D(G_k(u)\eta)|^p \, ,
\end{aligned}$$

for any $\varepsilon > 0$; choosing $\varepsilon = \frac{1}{2c_4}$, we then get

$$\int_{A(k,R)} |D(G_k(u)\eta)|^p \leq c_7 \Big(\int_{A(k,R)} |f|^{(p^*)'} \Big)^{\frac{p'}{(p^*)'}} + \frac{c_8}{(R-r)^p} \text{meas}\,(A(k,R)) \ .$$

A further use of the Hölder inequality with exponents $(\frac{m}{(p^*)'}, \frac{m}{m-(p^*)'})$ gives

$$\begin{aligned}
& \int_{A(k,R)} |D(G_k(u)\eta)|^p \\
& \leq c_7 \Big(\int_{A(k,R)} |f|^m \Big)^{\frac{p'}{m}} \text{meas}(A(k,R))^{\frac{p'}{(p^*)'} - \frac{p'}{m}} + \frac{c_8 \text{meas}\,(A(k,R))}{(R-r)^p} \\
& \leq c_7 \|f\|^{p'}_{L^m(B_1)} \text{meas}\,(A(k,R))^{\frac{p'}{(p^*)'} - \frac{p'}{m}} + \frac{c_8 \text{meas}\,(A(k,R))}{(R-r)^p} \\
& = c_9 \text{meas}(A(k,R))^{\frac{p'}{(p^*)'} - \frac{p'}{m}} + \frac{c_8 \text{meas}(A(k,R))}{(R-r)^p} \, .
\end{aligned}$$

We remark that, for all $k > 0$ and $0 \leq r < R \leq 1$, $\text{meas}\,(A(k,R)) \leq \text{meas}\,(B_1)$ and $\frac{1}{(R-r)^p} \geq 1$; it then follows

$$\int_{A(k,R)} |D(G_k(u)\eta)|^p \leq \frac{c_{10}}{(R-r)^p} \text{meas}\,(A(k,R))^{\nu} \ , \tag{2.5}$$

where $\nu = \min\{\frac{p'}{(p^*)'} - \frac{p'}{m}, 1\}$.

On the other hand, using again the Sobolev inequality, and remembering that $\eta \equiv 1$ in B_r, it results for every $h > k$:

$$\begin{aligned}\int_{A(k,R)} |D(G_k(u)\eta)|^p \geq & \Big(\int_{A(k,R)} |G_k(u)\eta|^{p^*} \Big)^{\frac{p}{p^*}} \geq c_{11} \Big(\int_{A(k,r)} |G_k(u)|^{p^*} \Big)^{\frac{p}{p^*}} \\ \geq & c_{11} \Big(\int_{A(h,r)} |G_k(u)|^{p^*} \Big)^{\frac{p}{p^*}} \geq c_{11}(h-k)^p \text{meas}(A(h,r))^{\frac{p}{p^*}} .\end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) we obtain

$$\text{meas}\,(A(h,r)) \leq \frac{c_{12} \text{meas}\,(A(k,R))^{\frac{\nu p^*}{p}}}{(R-r)^{p^*}(h-k)^{p^*}} ,$$

for all $h > k \geq 0$ and $0 \leq r < R \leq 1$.

In order to get an $L^\infty(B_r)$ bound for u we need the following result due to G. Stampacchia, a proof of which can be found in [16].

Lemma 2.1. *Let $\omega(h,r)$ be a nondecreasing in r, and nonincreasing in h, function defined in $[0,+\infty) \times [0,1]$; suppose that there exist constants $k_0 \geq 0$, D, α, $\gamma > 0$ and $\beta > 1$ such that*

$$\omega(h,r) \leq \frac{D\,\omega(k,R)^\beta}{(h-k)^\alpha (R-r)^\gamma} ,$$

for all $h > k \geq k_0$ and $0 \leq r < R \leq 1$. Then, for every ρ in $(0,1)$, there exists $d > 0$, given by

$$d^\alpha = \frac{D\, 2^{\frac{\beta(\alpha+\gamma)}{\beta-1}}\, \omega(k_0,1)^{\beta-1}}{(1-\rho)^\gamma} ,$$

such that $\omega(d,\rho) = 0$.

Observing that $\frac{\nu p^*}{p} > 1$ if and only if $m > \frac{N}{p}$, under the assumptions of Theorem 7 the previous Lemma can be used with $\omega(h,r) = \text{meas}\,(A(h,r))$ and $k_0 = 0$, and it provides, for $\rho = \frac{1}{2}$, the existence of a positive constant d depending on p, N, λ, Λ, and on $\|f\|_{L^m(B_1)}$, such that

$$\text{meas}(A(d, \frac{1}{2})) = 0 ,$$

which implies $\|u\|_{L^\infty(B_{\frac{1}{2}})} \le d$.

By a standard covering argument, we then deduce $u \in L^\infty_{loc}(\mathbb{R}^N)$. $\square$

Let us now turn to investigate the summability of the gradient Du of the solutions of $(E2)$.

Proof of Theorem 8. Let u be a solution of $(E2)$ in the sense of (0.9), with u belonging to $W^{1,p}_{loc}(\mathbb{R}^N)$ and $u\,g(u)$ belonging to $L^1_{loc}(\mathbb{R}^N)$.

Fixed $R_0 > 0$ and $0 < r \le \frac{R_0}{2}$, we choose a cut-off function ζ in $\mathcal{D}(B_{2r})$, with $\zeta \equiv 1$ in B_r, and $|D\zeta| \le \frac{c_0}{r}$, and we define u_{2r} as the integral mean of u over B_{2r}, that is

$$u_{2r} = \fint_{B_{2r}} u = \frac{1}{\text{meas}\,(B_{2r})} \int_{B_{2r}} u \,.$$

The use of $v = \zeta^p\,(u - u_{2r}) \in W^{1,p}_0(B_{2r})$ as test function in (0.9), which is possible since $u\,g(u)$ belongs to $L^1_{loc}(\mathbb{R}^N)$, provides

$$\begin{aligned}\int_{B_{2r}} a(x,Du)\cdot Du\,\zeta^p + \int_{B_{2r}} g(u)\,(u-u_{2r})\,\zeta^p \\ = \int_{B_{2r}} f(u-u_{2r})\,\zeta^p - p\int_{B_{2r}} a(x,Du)\cdot D\zeta\,\zeta^{p-1}(u-u_{2r})\,,\end{aligned}$$

from which

$$\begin{aligned}\lambda \int_{B_{2r}} |Du|^p\zeta^p + \int_{B_{2r}} g(u)\,(u-u_{2r})\,\zeta^p \\ \le \int_{B_{2r}} |f||u-u_{2r}|\zeta^p + \frac{c_0\Lambda p}{r}\int_{B_{2r}} |Du|^{p-1}\zeta^{p-1}|u-u_{2r}|\,.\end{aligned}$$

From this estimate, reasoning as in the proof of Theorem 7, we easily obtain

$$\begin{aligned}\int_{B_{2r}} |D((u-u_{2r})\zeta)|^p + \int_{B_{2r}} g(u)\,(u-u_{2r})\,\zeta^p \\ \le c_1\int_{B_{2r}} |f||u-u_{2r}|\zeta^p + \frac{c_2}{r^p}\int_{B_{2r}} |u-u_{2r}|^p\,.\end{aligned}$$

The Hölder, Sobolev and Young inequalities applied to the first integral of the right hand side term gives

$$\begin{aligned}\int_{B_{2r}}|f||u-u_{2r}|\zeta^p &\le \Big(\int_{B_{2r}}(|u-u_{2r}|\zeta)^{p^*}\Big)^{\frac{1}{p^*}}\Big(\int_{B_{2r}}(|f|\zeta^{p-1})^{(p^*)'}\Big)^{\frac{1}{(p^*)'}}\\ &\le c_3\Big(\int_{B_{2r}}|D((u-u_{2r})\zeta)|^p\Big)^{\frac{1}{p}}\Big(\int_{B_{2r}}|f|^{(p^*)'}\Big)^{\frac{1}{(p^*)'}}\\ &\le \varepsilon\int_{B_{2r}}|D((u-u_{2r})\zeta)|^p+\frac{c_3^{p'}}{\varepsilon^{\frac{1}{p-1}}}\Big(\int_{B_{2r}}|f|^{(p^*)'}\Big)^{\frac{p'}{(p^*)'}}\\ &\le \varepsilon\int_{B_{2r}}|D((u-u_{2r})\zeta)|^p+\frac{c_3^{p'}}{\varepsilon^{\frac{1}{p-1}}}\Big(\int_{B_{R_0}}|f|^{(p^*)'}\Big)^{\frac{p'}{N}}\int_{B_{2r}}|f|^{(p^*)'}\\ &= \varepsilon\int_{B_{2r}}|D((u-u_{2r})\zeta)|^p+\frac{c_4}{\varepsilon^{\frac{1}{p-1}}}\int_{B_{2r}}|f|^{(p^*)'},\end{aligned}$$

for any $\varepsilon > 0$; choosing $\varepsilon = \frac{1}{2c_1}$, we deduce

$$\int_{B_{2r}}|D((u-u_{2r})\zeta)|^p+\int_{B_{2r}}g(u)(u-u_{2r})\zeta^p \le c_5\int_{B_{2r}}|f|^{(p^*)'}+\frac{c_6}{r^p}\int_{B_{2r}}|u-u_{2r}|^p.$$

Let us define $G(s) = \int_0^s g(t)\,dt$ as in (G4), and notice that G is a C^1, convex, nonnegative function on $\mathbb{R}$, then for all $a, b \in \mathbb{R}$, it results

$$G(a) - G(b) \le G'(a)\,(a-b) = g(a)\,(a-b)\,.$$

In particular, we have

$$G(u) - G(u_{2r}) \le g(u)\,(u-u_{2r}) \qquad \text{almost everywhere in } B_{2r}\,,$$

so that

$$\begin{aligned}&\int_{B_{2r}}|D((u-u_{2r})\zeta)|^p+\int_{B_{2r}}G(u)\zeta^p\\ &\quad\le c_5\int_{B_{2r}}|f|^{(p^*)'}+\frac{c_6}{r^p}\int_{B_{2r}}|u-u_{2r}|^p+G(u_{2r})\int_{B_{2r}}\zeta^p\\ &\quad\le c_5\int_{B_{2r}}|f|^{(p^*)'}+\frac{c_6}{r^p}\int_{B_{2r}}|u-u_{2r}|^p+G(u_{2r})\operatorname{meas}(B_{2r})\,.\end{aligned}$$

Using that $\zeta \equiv 1$ in B_r, we further have

$$\int_{B_r} |Du|^p + \int_{B_r} G(u) \le c_5 \int_{B_{2r}} |f|^{(p^*)'} + \frac{c_6}{r^p} \int_{B_{2r}} |u - u_{2r}|^p + G(u_{2r}) \text{meas}\,(B_{2r})\,.$$

By means of the Poincaré-Sobolev inequality (see [13])

$$\int_{B_{2r}} |u - u_{2r}|^p \le c_7 r^{p+N-N/\nu} \Big(\int_{B_{2r}} |Du|^{p\nu} \Big)^{\frac{1}{\nu}},$$

which holds for all ν such that $\max\{\frac{1}{p}, \frac{N}{N+p}\} \le \nu \le 1$, we deduce

$$\begin{aligned} &\int_{B_r} |Du|^p + \int_{B_r} G(u) \\ &\le c_5 \int_{B_{2r}} |f|^{(p^*)'} + c_8 r^{N-N/\nu} \Big(\int_{B_{2r}} |Du|^{p\nu} \Big)^{\frac{1}{\nu}} + G(u_{2r}) \text{meas}\,(B_{2r})\,. \end{aligned}$$

Dividing both terms of the previous inequality by meas (B_r), it results

$$⨍_{B_r} (|Du|^p + G(u)) \le c_9 ⨍_{B_{2r}} |f|^{(p^*)'} + c_{10} \Big(⨍_{B_{2r}} |Du|^{p\nu} \Big)^{\frac{1}{\nu}} + c_{11} G(u_{2r}).$$

Using now the assumption (G4) and the Jensen inequality, we have

$$[G(u_{2r})]^\mu = \Big[G\Big(⨍_{B_{2r}} u \Big) \Big]^\mu \le ⨍_{B_{2r}} [G(u)]^\mu\,,$$

and thus

$$⨍_{B_r} (|Du|^p + G(u)) \le c_9 ⨍_{B_{2r}} |f|^{(p^*)'} + c_{10} \Big(⨍_{B_{2r}} |Du|^{p\nu} \Big)^{\frac{1}{\nu}} + c_{11} \Big(⨍_{B_{2r}} [G(u)]^\mu \Big)^{\frac{1}{\mu}}.$$

Another consequence of the Jensen inequality is the increasing monotonicity of the map

$$\tau \in (0,1] \mapsto \Big(⨍_B |\psi|^\tau \Big)^{\frac{1}{\tau}},$$

for every given function ψ in $L^1(B)$, for any ball B. Applying this property with $\psi = |Du|^p \in L^1(B_{2r})$ and $\psi = G(u) \in L^1(B_{2r})$, we get

$$\Big(⨍_{B_{2r}} |Du|^{p\nu} \Big)^{\frac{1}{\nu}} \le \Big(⨍_{B_{2r}} |Du|^{p\tau} \Big)^{\frac{1}{\tau}} \quad \text{and} \quad \Big(⨍_{B_{2r}} [G(u)]^\mu \Big)^{\frac{1}{\mu}} \le \Big(⨍_{B_{2r}} [G(u)]^\tau \Big)^{\frac{1}{\tau}},$$

where $\tau = \max\{\mu, \nu\} < 1$. It then results:

$$\fint_{B_r} (|Du|^p + G(u)) \le c_{12}\Big[\fint_{B_{2r}} |f|^{p^{*\prime}} + \Big(\fint_{B_{2r}} \Big(|Du|^p + G(u)\Big)^{\tau}\Big)^{\frac{1}{\tau}}\Big],$$

which is a reverse Hölder inequality for the function $|Du|^p + G(u)$. Applying the M. Giaquinta and G. Modica Theorem (see [11]), the proof of Theorem 8 is thus complete. $\square$

Remark 3. The previous Theorem also gives a greater summability for the function $G(u)$, and then for $u\, g(u)$, which results to belong to $L^{\sigma}_{loc}(\mathbb{R}^N)$, for some $\sigma > 1$ which can be calculated in terms of δ.

3. The equation $(E3)$**.** In this Section, using both the approximation technique of [8] and the "absorbing" properties proved in Section 1, we will obtain an existence result for the equation

$$-\mathrm{div}\,(a(x, Du)) \;+\; h(x, u) \;+\; F(x, Du) \;= f\,. \qquad (E3)$$

For all $n \ge 1$, and for every ξ in $\mathbb{R}^N$ and almost every x in $\mathbb{R}^N$, let us set

$$F_n(x, \xi) \;=\; \frac{F(x, \xi)}{1 + \frac{1}{n}\,|F(x, \xi)|}\,,$$

and observe that F_n is a Carathéodory function satisfying assumption (F2), being

$$|F_n(x, \xi)| \le |F(x, \xi)|\,.$$

Let u_n be a solution of the following boundary value problem

$$\begin{cases} -\mathrm{div}\,(a(x, Du_n)) + h(x, u_n) + F_n(x, Du_n) \;= f \quad \text{in } B_n\,, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad u_n = 0 \quad \text{on } \partial B_n\,, \end{cases} \qquad (E3)_n$$

where B_n is, as usual, the ball of $\mathbb{R}^N$ with radius n.

We recall that for every fixed n, and for any given function f in $L^m_{loc}(\mathbb{R}^N)$, with $m > \max\{1, \frac{N}{p}\}$, there exists at least one solution u_n belonging to $W^{1,p}_0(B_n) \cap L^\infty(B_n)$ of problem $(E3)_n$, in the sense that

$$\int_{B_n} a(x, Du_n) \cdot Dv + \int_{B_n} h(x, u_n)\, v + \int_{B_n} F_n(x, Du_n)\, v = \int_{B_n} f\, v\,, \quad (3.1)$$

for every test function v in $W_0^{1,p}(B_n) \cap L^\infty(B_n)$ (see [7] and [8]).

Following [8] and [9], the proof of the existence of a solution of $(E3)$ is essentially reduced to obtaining an $L^\infty(B_r)$ uniform bound on the sequence $\{u_n\}$, for every fixed $r > 0$. Such an estimate is achieved in the following proof, which is closely related to that of Theorem 7.

Proof of Theorem 9. Fixed $0 \le r < R \le 1$, let ζ be a cut-off function belonging to $\mathcal{D}(B_R)$, with $\zeta \equiv 1$ in B_r and $|D\zeta| \le \frac{c_0}{R-r}$.

Let then $\Psi_\beta(t)$ be the function

$$\Psi_\beta(t) \,=\frac{1}{\beta}\,(\mathrm{e}^{\beta|t|}-1)\,\mathrm{sign}(t)\,, \qquad t \in \mathbb{R}\,,$$

where β is a positive constant to be fixed later. Set

$$g(t) \,= |t|^{p-2}t\,j(\frac{1}{\beta}\log(\beta|t|^{p-1}+1))\,, \qquad t \in \mathbb{R},$$

and observe that, thanks to (H4), g satisfies assumptions $(G1)-(G4)$ for every $\beta > 0$; finally, let φ be the function provided by Lemma 1.1, in correspondence of g, of $\alpha = 0$, and of an arbitrary positive constant C.

We apply identity (3.1) with test function $v = \Psi_\beta[G_k(u_n)]\,\eta$, where G_k is defined by (2.4) and $\eta = \varphi(\zeta)$; notice that as β tends to zero, v becomes the same test function used in the proof of Theorem 7.

Observing that $\mathrm{supp}\,(v)$ is contained in the set $A_n(k,R) = B_R \cap \{|u_n| \ge k\}$, we get, for every $n \ge R$,

$$\begin{aligned}
&\int_{A_n(k,R)} a(x,Du_n)\cdot DG_k(u_n)\mathrm{e}^{\beta|G_k(u_n)|}\eta + \int_{A_n(k,R)} h(x,u_n)\Psi_\beta[G_k(u_n)]\eta \\
&+\int_{A_n(k,R)} F_n(x,Du_n)\Psi_\beta[G_k(u_n)]\eta \\
&=\int_{A_n(k,R)} f\Psi_\beta[G_k(u_n)]\eta - \int_{A_n(k,R)} a(x,Du_n)\cdot D\zeta\varphi'(\zeta)\Psi_\beta[G_k(u_n)].
\end{aligned}$$

Then, observing that $DG_k(u_n) \equiv Du_n$ in $A_n(k,R)$ and using assumptions (A2), (A3), (H4) and (F2),

$$\begin{aligned}
&(\lambda-\frac{\gamma}{\beta})\int_{A_n(k,R)} |DG_k(u_n)|^p\mathrm{e}^{\beta|G_k(u_n)|}\eta + \int_{A_n(k,R)} j(u_n)\Psi_\beta[G_k(u_n)]\eta \\
&\le \int_{A_n(k,R)} |f||\Psi_\beta[G_k(u_n)]|\eta + \frac{c_0\Lambda}{R-r}\int_{A_n(k,R)} |DG_k(u_n)|^{p-1}|\Psi_\beta[G_k(u_n)]||\varphi'(\zeta).
\end{aligned}$$

Applying Young inequality to the last integral, and subtracting from the left hand side term, we obtain

$$\begin{aligned}&(\lambda-\frac{\gamma+1}{\beta})\int_{A_n(k,R)}|DG_k(u_n)|^p \mathrm{e}^{\beta|G_k(u_n)|}\eta+\int_{A_n(k,R)} j(u_n)\Psi_\beta[G_k(u_n)]\eta\\&\le\int_{A_n(k,R)}|f||\Psi_\beta[G_k(u_n)]|\eta+\frac{c_0{}^p\Lambda^p}{(R-r)^p}\int_{A_n(k,R)}|\Psi_\beta[G_k(u_n)]|\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}.\end{aligned}$$

Choosing $\beta=2(\gamma+1)/\lambda$ and observing that

$$\begin{aligned}&\left|D\left\{\Psi_\beta[G_k(u_n)/p]\eta^{\frac{1}{p}}\right\}\right|^p\\&\le c_1|DG_k(u_n)|^p\mathrm{e}^{\beta|G_k(u_n)|}\eta+\frac{c_2}{(R-r)^p}|\Psi_\beta[G_k(u_n)/p]|^p\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\\&\le c_1|DG_k(u_n)|^p\mathrm{e}^{\beta|G_k(u_n)|}\eta+\frac{c_2}{(R-r)^p}|\Psi_\beta[G_k(u_n)]|\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,,\end{aligned}$$

we obtain

$$\begin{aligned}&\int_{A_n(k,R)}|D\{\Psi_\beta[G_k(u_n)/p]\eta^{1/p}\}|^p+\int_{A_n(k,R)} j(u_n)\Psi_\beta[G_k(u_n)]\eta\\&\le c_3\int_{A_n(k,R)}|f||\Psi_\beta[G_k(u_n)]|\eta+\frac{c_4}{(R-r)^p}\int_{A_n(k,R)}|\Psi_\beta[G_k(u_n)]|\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\,.\end{aligned}$$

If we now set $t^{p-1}=|\Psi_\beta[G_k(u_n)]|$, and apply (1.2) of Lemma 1.1 to φ, with $\alpha=0$, and $C=\frac{(R-r)^p}{2c_4}$, remembering the definition of g and using the increasing monotonicity of j, it then results

$$\begin{aligned}&\frac{c_4}{(R-r)^p}|\Psi_\beta[G_k(u_n)]|\frac{\varphi'(\zeta)^p}{\varphi(\zeta)^{p-1}}\le\frac{1}{2}g(t)\eta+\frac{c_4}{(R-r)^p}\\&=\frac{1}{2}|\Psi_\beta[G_k(u_n)]|j[|G_k(u_n)|]+\frac{c_4}{(R-r)^p}\le\frac{1}{2}\Psi_\beta[G_k(u_n)]j(u_n)+\frac{c_4}{(R-r)^p},\end{aligned}$$

almost everywhere in $A_n(k,R)$. Thus,

$$\begin{aligned}&\int_{A_n(k,R)}|D\{\Psi_\beta[G_k(u_n)/p]\eta^{1/p}\}|^p+\int_{A_n(k,R)} j(u_n)\Psi_\beta[G_k(u_n)]\eta\\&\le c_5\int_{A_n(k,R)}|f||\Psi_\beta[G_k(u_n)]|\eta+\frac{c_6\,\mathrm{meas}\,(A_n(k,R))}{(R-r)^p}.\end{aligned}$$

Using again the monotonicity of j, we further have

$$j(u_n)\,\mathrm{sign}(u_n) \geq j(k) \qquad \text{almost everywhere in } A_n(k,R)\,,$$

so that

$$\int_{A_n(k,R)} |D\{\Psi_\beta[G_k(u_n)/p]\eta^{1/p}\}|^p + j(k)\int_{A_n(k,R)} |\Psi_\beta[G_k(u_n)]|\eta$$
$$\leq c_5 \int_{A_n(k,R)} |f||\Psi_\beta[G_k(u_n)]|\eta + \frac{c_6\,\mathrm{meas}\,(A_n(k,R))}{(R-r)^p}\,.$$

From this estimate, using first Hölder inequality, then Sobolev inequality, and finally applying Lemma 2.1, as done in [9], we obtain an $L^\infty(B_r)$ bound for $\{\, u_n \,\}_{n\geq R}$.

Once the $L^\infty(B_r)$ estimate has been obtained, reasoning as in [8] and [9], it follows that $\{\, u_n \,\}$ converges, up to a subsequence, in the strong topology of $W^{1,p}_{loc}(\mathbb{R}^N)$, to a locally bounded solution of $(E3)$.

4. On the necessity of the condition $(G3)$**.** In this Section we will show, through a very simple example, that the growth condition $(G3)$ on the zero order term of the equation (E_g) is necessary, as stated in the introduction, in order to solve (E_g) (or, more in general, $(E1)$) by approximation.

First of all, we observe that, following exactly the proof of Proposition 5.1 of [6], it can be proved that if g is a function satisfying assumptions $(G1)$ and $(G2)$, but not $(G3)$, that is such that

$$\int^{+\infty} \frac{dt}{(t\,g(t))^{\frac{1}{p}}} = +\infty\,, \tag{4.1}$$

or, equivalently, such that

$$\int^{+\infty} \frac{dt}{(G(t))^{\frac{1}{p}}} = +\infty\,, \tag{4.2}$$

where $G(\tau) = \int_0^\tau g(t)\,dt$, and if $f \in L^1_{loc}(\mathbb{R}^N)$ is a nonnegative radially symmetric function such that there exists a nonnegative radially symmetric solution u of problem (E_g), then there exist positive constants c_1, c_2 and c_3 such that

$$\frac{1}{r^N}\int_{B_r} f(x)\,dx \leq c_1 g\left(c_2 + L^{-1}(c_3(r-1))\right) \qquad \text{for all } \; r \geq 1\,, \tag{4.3}$$

where $L(\tau)$ is the nonnegative, continuous, increasing and unbounded function defined as

$$L(\tau) \; = \int_0^\tau \frac{dt}{(G(c_2+t)+1)^{\frac{1}{p}}} \,.$$

Observe that if a solution u of (E_g), with f as above, is obtained as limit of a sequence of solutions of equations posed in bounded, radially symmetric domains and approximating (E_g), then u is a nonnegative radially symmetric function and thus the growth condition (4.3) has to hold true.

The necessity of the growth condition (4.3) on the datum f clearly appears in the simplest case $N = 1$ and $p = 2$, arguing as follows.

Let g satisfy $(G1)$, $(G2)$ and (4.1) (or (4.2)), and let further $f \in C^1(\mathbb{R})$ be a positive even function, with $f(0) = 0$. Given an arbitrary positive number $a > 0$, consider the second order initial value problem

$$-u'' + g(u) = f(x)\,,\; u(0) = a\,,\; u'(0) = 0\,, \tag{4.4}$$

which has a unique, even, maximal solution u in $C^2(-\mu,\mu)$, for some positive $\mu = \mu(a) \le +\infty$.

We will show that there exists a positive value $\mu^* = \mu^*(a) < \mu$ such that $u(x) > 0$ for $x \in (-\mu^*, \mu^*)$ and $u(\mu^*) = u(-\mu^*) = 0$; moreover, it will be proved that $\mu^*(a)$ is a continuous and increasing function of a, such that $\lim_{a\to+\infty} \mu^*(a) = +\infty$.

First of all, observe that, by assumption (4.2), the homogeneous problem

$$-u_0'' + g(u_0) = 0\,,\; u_0(0) = a\,,\; u_0'(0) = 0$$

has a unique globally defined solution u_0, implicitly given by

$$\frac{1}{\sqrt{2}} \int_a^{u_0(x)} \frac{dt}{\sqrt{G(t) - G(a)}} = x\,.$$

If we set, for all $\tau \ge a$,

$$L_a(\tau) \; = \frac{1}{\sqrt{2}} \int_a^\tau \frac{dt}{\sqrt{G(t)-G(a)}}\,,$$

then we have $u_0(x) = L_a^{-1}(x)$, $\forall\, x \in \mathbb{R}$. Let us now look at the solution u; of course, it suffices to consider $u(x)$ for $x \ge 0$. Observing that $u''(0) = g(a) >$

0, by continuity there will be a $\xi = \xi(a) \in (0, \mu]$ such that $u'' > 0$ in $[0, \xi)$. Thus, it results $u' > 0$ in $(0, \xi]$ and, being $f > 0$, we have

$$\frac{1}{2}\left[(u')^2\right]' = u'u'' = [g(u) - f(x)]\, u' < g(u)u' = [G(u)]' \quad \text{in } (0, \xi]\,;$$

integrating twice between 0 and x, we deduce

$$u(x) < L_a^{-1}(x) = u_0(x) \tag{4.5}$$

for all x in $(0, \xi]$.

Assume now that the estimate (4.3) fails for f, and, more than that, suppose that f satisfies

$$\lim_{x \to +\infty} \frac{f(x)}{g\left(L_a^{-1}(x)\right)} = +\infty \tag{4.6}$$

for every fixed $a > 0$. In this case, there exists a $\eta = \eta(a) > 0$ such that $f(x) > g(L_a^{-1}(x))$ for all $x > \eta$. This implies that the real number

$$\mu_1 = \mu_1(a) \; = \sup\{\, x \in [0, \mu) \; : g(u(x)) \geq f(x)\,\}$$

is strictly lesser than μ. For, if $g(u(x)) \geq f(x)$ for all $x \in [0, \mu)$, then u is a convex increasing function lying, by (4.5), below u_0; therefore, $\mu = +\infty$ and, for all $x > \xi$, it results

$$f(x) \leq g(u(x)) \leq g\left(L_a^{-1}(x)\right) < f(x)\,,$$

a contradiction.

Thus, we have $u''(\mu_1) = 0$, $u'(\mu_1) > 0$ and $u''(x) < 0$ for all $x \in (\mu_1, \mu)$. At least in a small neighborhood of μ_1 the function u' decreases and remains positive. If it never vanishes in $[0, \mu)$, then u remains increasing and $\mu = +\infty$; furthermore, there exist the limits of u, u' and u'' as $x \to +\infty$ and, since u' converges to a finite value, u'' must converge to zero. On the other hand

$$u''(x) = g(u(x)) - f(x) \leq g\left(L_a^{-1}(x)\right) - f(x)$$

and the right hand side diverges to $-\infty$ as $x \to +\infty$: a contradiction. Thus, there exists a point $\mu_2 = \mu_2(a)$ in (μ_1, μ) such that $u'(\mu_2) = 0$ and $u'(x) < 0$ in (μ_2, μ) (μ_2 is thus the maximum point of the function u); moreover, being

u still a concave function in (μ_2, μ), there exists $\mu^* = \mu^*(a) \in (\mu_2, \mu)$ such that $u(\mu^*) = 0$. As an example, Figure 1 shows the graph of the function u obtained for $g(u) = u$ and $f(x) = x\,\mathrm{e}^x$.

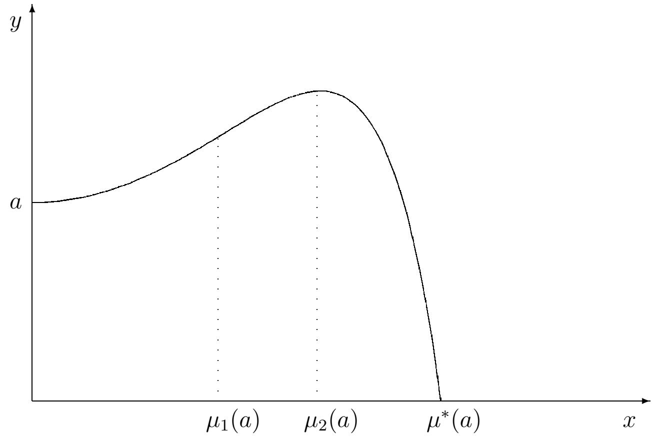

Figure 1

From the uniqueness of the solution of problem (4.4), and from its continuous dependence on the initial value a, it follows that $\mu^*(a)$ is a continuous increasing function defined in $(0, +\infty)$.

Moreover, remembering that $\mu^*(a) > \mu_1(a)$, where $\mu_1(a)$ is the flex point of u, and observing that $f(\mu_1(a)) = g(u(\mu_1(a))) > g(a)$, we deduce that both $\mu_1(a)$ and $\mu^*(a)$ diverge to $+\infty$ as $a \to +\infty$, since, by $(G2)$ with $p = 2$, it results $g(t) \geq g(1)\,t$ for all $t \geq 1$ and therefore $g(t)$ tends to $+\infty$ as $t \to +\infty$.

The previous discussion implies that the equation

$$-u'' + g(u) = f \qquad \text{in } \mathbb{R}$$

cannot be generally solved by approximation with boundary value problems posed in bounded domains. Indeed, if the datum f is as above, and if u_n denotes, for every natural number $n \geq 1$, the solution of (4.4) having initial value $a = a_n = (\mu^*)^{-1}(n)$, then u_n is the unique solution of the boundary value problem

$$\begin{cases} -u_n'' + g(u_n) = f & \text{in } (-n, n)\,, \\ u_n(n) = u_n(-n) = 0\,; \end{cases}$$

using the above notations, it results that $u_n \geq a_n$ in $[-\mu_2(a_n), \mu_2(a_n)]$, and, being $a_n, \mu_2(a_n) \to +\infty$ as $n \to +\infty$, this implies that the sequence $\{u_n\}$ identically diverges to $+\infty$ as $n \to +\infty$.

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