Advances in Differential Equations Volume 5 (4–6) April–June 2000, pp. 657–680

ON A NONLOCAL DIFFUSION EQUATION WITH DISCONTINUOUS REACTION

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(Submitted by: Michel Chipot)

Abstract. We consider the asymptotic derivation of a parabolic equation with nonlocal diffusivity and with a semi-linear nonlocal term that may also be discontinuous. From a reaction-diffusion system where the diffusivity of the second equation is arbitrarily large, using only energy estimates, we obtain a shadow system as an intermediate step for the limit equation. In particular, we obtain the existence of weak solutions and we give a rigorous derivation of a class of diffusion equations that have been used in the literature to model threshold phenomena, for instance, in porous-medium combustion or in localized patterns of excitable media.

1. Introduction. In this paper we consider the derivation and the existence of weak solutions of the parabolic equation

$$\partial_t u = a\Big(\fint_\Omega u\Big)\, \Delta u + f\Big(u, \fint_\Omega u\Big) \tag{1.1}$$

in a bounded domain $Q_T = \Omega \times]0,T[$, $\Omega \subset \mathbb{R}^N$, $T>0$, where the nonlocal term is given by

$$\fint_\Omega u = \frac{1}{\text{meas}(\Omega)} \int_\Omega u(x,t)\, dx\ . \tag{1.2}$$

Accepted for publication February 1999.
AMS Subject Classifications: 35R35, 35K50, 35K57.

The diffusion coefficient $a(\cdot)$ is a positive continuous function but the nonlinearity f is allowed to be discontinuous in u and $\fint_\Omega u$. In that case, in order to obtain an existence theory, the "filling in the jumps", i.e., the passage to a multivalued version of the problem is used, and a Neumann boundary condition and initial data are added.

We start with the reaction-diffusion system

$$\partial_t u = \nabla \cdot (a(v)\,\nabla u) + f(u,v)\ , \tag{1.3}$$

$$\tau\,\partial_t v = \sigma\,\Delta v + u - v \qquad (\sigma, \tau > 0)\ , \tag{1.4}$$

with homogeneous Neumann conditions for u and v, and in a first step, with τ fixed, we let $\sigma \to +\infty$. The so obtained intermediate system, called shadow system (see [26]), corresponds to the equation (1.3) with v replaced by $\xi = \xi(t)$, which is a function of t only solving the ordinary differential equation ($\dot{\xi} = d\xi/dt$)

$$\tau\,\dot{\xi} = \fint_\Omega u - \xi\ . \tag{1.5}$$

The equation (1.1) is then obtained from the shadow system by letting $\tau \to 0$, or from (1.3)–(1.4) in the simultaneous limit case $(\tau, \sigma) \to (0, +\infty)$. Among the many models of morphogenesis, chemical, biological and physical problems involving systems of reaction-diffusion equations, the study of stability and bifurcation of their solutions has been considered by several authors, in particular, the dynamics analysis in the case of large diffusion where almost no space dependence is expected (see, for instance, [17, 18]). Again with smooth reaction terms, the shadow system has been considered (with a heuristic derivation by Nishiura ([26]) in the steady-state case) by Hale and Sakamoto ([19]), who have conjectured that its dynamics, as $\tau \to 0$, should behave as the one of the simpler equation (1.1), and recently by [21], where it was shown that the shadow system may exhibit minimal dynamics displaying the mechanism of basic pattern formation.

On the other hand, recent experiments and numerical simulations of higher-dimensional localized patterns in excitable media have used the system (1.3)–(1.4) with the cubic type reaction $f(u,v) = u(1-u)\,(u-\vartheta) - v$ replaced by a discontinuous term involving the Heaviside step function H (see [28], for instance)

$$f(u,v) = H(u - \vartheta) - u - v\ , \tag{1.6}$$

where the parameter $\vartheta \in (0,1)$. The respective system, in particular in the degenerate case $\sigma = 0$, have been considered also as a model for the conduction of electrical impulses in the new axon, and is known as the Fitz Hugh–Nagumo equation. Following this simplification proposed by McKean [25], this system was shown to exhibit a threshold phenomenon by Terman ([34]), as it is expected due to the nonlinear discontinuity.

For the activator-inhibitor model (1.3)–(1.4) with constant diffusivity a and a nonlinear reaction of the type (1.6), a bifurcation property leading to traveling spots have been considered in [20] in the case of an excitation threshold of the type

$$\vartheta = \vartheta_0 + \vartheta_1 \fint_\Omega (u+v)\ , \tag{1.7}$$

i.e., with a threshold ϑ depending on the total activator and concentrations in the medium.

In combustion theory, reaction-diffusion models have been used with special simplifications leading also to nonlocal quantities of the type (1.2) for the temperature equation (1.1) (see, for instance, [31]). In a model of time-dependent porous-medium combustion, Norbury and Stuart ([27]) have proposed in the temperature equation, with non-constant diffusivity, a reaction rate f involving also Heaviside discontinuities similar to (1.6), in which the threshold ϑ represents a critical switching temperature. This model has motivated a considerable interest in the mathematical study of this semilinear diffusion equation with discontinuous nonlinearities. In fact, existence, uniqueness and stability properties have been shown in special cases (see [14], [12], [16], [15] and [3]).

It is interesting to remark that a similar reaction-diffusion also arises in climate models (see [11] and its references).

Nonlocal equations of the type (1.1) have been encountered in other situations (see also the recent survey [13] and its references), in particular, in mathematical biology (see [23], for instance) where nonlocal terms may also generate stable patterns. As remarked in [9] for stationary situations, interesting cases of non-uniqueness and even non-existence of solutions to nonlocal problems may arise, although, for parabolic equations a better theory may be developed, as shown recently by Chipot and Lovat [7, 8].

Our purpose in this work is to extend the rigorous derivation of the equation (1.1) as the limit as $\sigma \to \infty$ and $\tau \to 0$ of the reaction-diffusion system (1.3)–(1.4). This was done in [6] for a constant a and a smooth f

with a special structure, allowing the use of stronger estimates, in particular, L^∞ bounds on u and on v, which are not required here.

We first consider in Section 2 the case of a continuous reaction term in order to illustrate the technique based only on *a priori* energy estimates.

In Section 3 we obtain the shadow system as a limit problem when $\sigma \to +\infty$ and in Section 4 we study the nonlocal diffusion equation, which can be regarded as limit cases, as $\tau \to 0$, of the shadow system or, as $(\tau, \sigma) \to (0, +\infty)$, of the reaction-diffusion system.

Then in the Section 5, we extend the asymptotic derivation and corresponding existence results to general L^∞_{loc} discontinuities using the method of "filling in the jumps" of Rauch ([29]). It should be mentioned that general existence results for nonlinear parabolic equations (without nonlocal terms) with discontinuities have been recently obtained in [4] and [1], by the method of enclosure of solutions between upper and lower solutions, but their results do not cover our cases.

Finally we conclude, in Section 6, with two examples of application, in combustion theory and in morphogenesis and with a counterexample to the uniqueness of solution in a simple case with nonlocal discontinuous nonlinearity.

2. Existence of a solution to the reaction-diffusion system. For fixed $\tau, \sigma > 0$ ($\tau \le 1 \le \sigma$, for simplicity) we consider first weak solutions of the Cauchy problem for the reaction-diffusion system in $Q_T = \Omega \times]0, T[$:

$$\partial_t u_{\tau\sigma} = \nabla \cdot (a(v_{\tau\sigma}) \nabla u_{\tau\sigma}) + f(u_{\tau\sigma}, v_{\tau\sigma}) \quad \text{in } Q_T , \tag{2.1}$$

$$\partial_n u_{\tau\sigma} = 0 \ \text{ on } \Sigma_T , \quad u_{\tau\sigma}(0) = u_0 \ \text{ in } \Omega , \tag{2.2}$$

$$\tau\, \partial_t v_{\tau\sigma} = \sigma\, \Delta v_{\tau\sigma} + u_{\tau\sigma} - v_{\tau\sigma} \quad \text{in } Q_T , \tag{2.3}$$

$$\partial_n v_{\tau\sigma} = 0 \ \text{ on } \Sigma_T , \quad v_{\tau\sigma}(0) = v_0 \ \text{ in } \Omega , \tag{2.4}$$

where ∂_n denotes the normal derivative on $\partial\Omega$ and $\Sigma_T = \partial\Omega \times]0, T[$.

The diffusion coefficient is given by a measurable function $a = a(x, t, v)$:

$$a\colon Q_T \times \mathbb{R} \to \mathbb{R}^+, \ v \mapsto a(x,t,v) \ \text{ is continuous for a.e. } (x,t) \in Q_T, \tag{2.5}$$

$$0 < a_* \le a(x,t,v) \le a^* , \quad \forall\, v \in \mathbb{R}, \ \text{ a.e. } (x,t) \in Q_T . \tag{2.6}$$

The reaction function, in this section is assumed without a growth restriction, but with an "extended sign condition", i.e., $f = f(x, t, u, v)$ is such that

$$f\colon\ Q_T\times\mathbb{R}^2\to\mathbb{R},\ \ (u,v)\mapsto f(x,t,u,v)\ \text{ is continuous for a.e. } (x,t)\in Q_T \tag{2.7}$$

and there is a function $g_0\in L^1(Q_T)$, $g_0\ge 0$, and a constant $C_0>0$ satisfying

$$u\,f(x,t,u,v)\le g_0(x,t)+C_0(u^2+v^2)\,,\qquad \forall\,u,v\in\mathbb{R},\ \text{ a.e. } (x,t)\in Q_T\ . \tag{2.8}$$

Since we shall work with unbounded solutions, we also need the following assumption: for any large $M>0$, there is a function $g_M\in L^1(Q_T)$, $g_M\ge 0$, and a constant $C_M>0$, such that, for some $\delta>0$ ($\delta\le 2$)

$$\sup_{|u|\le M}|f(x,t,u,v)|\le g_M(x,t)+C_M|v|^{2-\delta},\ \ \forall\,v\in\mathbb{R},\ \text{ a.e. } (x,t)\in Q_T\ . \tag{2.9}$$

We remark that the conditions (2.8) and (2.9) are satisfied, in particular, if $f(u,v)=-u^3+c_2\,u^2+c_1\,u-v$, but we may also consider nonhomogeneous terms or any polynomial function in u of odd degree with negative higher order coefficient (see [33], for instance).

On the initial conditions it is sufficient to suppose only

$$u_0,v_0\in L^2(\Omega)\ . \tag{2.10}$$

Theorem 2.1. *Let* (2.5)–(2.10) *hold. Then there exists at least a weak solution* $(u,v)=(u_{\tau\sigma},v_{\tau\sigma})$ *of* (2.1)–(2.4), *with the following properties*

$$u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega)),$$

$$\partial_t u\in L^2(0,T;(H^1(\Omega))^*)+L^1(Q_T)\ , \tag{2.11}$$

$$v\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))\cap L^\infty(\delta,T;H^1(\Omega)), \tag{2.12}$$

$$\partial_t v\in L^2(\delta,T;L^2(\Omega))\cap L^2(0,T;(H^1(\Omega))^*)\qquad(\delta>0)\ ,$$

$$f(u,v),\ uf(u,v)\in L^1(Q_T)\ . \tag{2.13}$$

Proof. Following a standard approximation procedure, we replace f by the bounded term f_ε, for each $\varepsilon>0$, given by

$$f_\varepsilon(x,t,u,v)=\frac{f(x,t,u,v)}{1+\varepsilon|f(x,t,u,v)|}\ , \tag{2.14}$$

which still satisfies the same assumptions than f. We get the variational system

$$\partial_t u_\varepsilon=\nabla\cdot(a(v_\varepsilon)\,\nabla u_\varepsilon)+f_\varepsilon(u_\varepsilon,v_\varepsilon)\quad\text{ in }\ Q_T\ , \tag{2.15}$$

$$\tau\,\partial_t v_\varepsilon=\sigma\,\Delta v_\varepsilon+u_\varepsilon-v_\varepsilon\quad\text{ in }\ Q_T\ , \tag{2.16}$$

with the boundary and initial conditions (2.2) and (2.4). If we multiply (2.15) by u_ε and (2.16) by v_ε and integrate in $Q_t = \Omega \times]0,t[$, adding and using the assumptions (2.6) and (2.8) we easily obtain

$$\frac{1}{2}\int_\Omega u_\varepsilon^2(t) + a_* \int_{Q_t} |\nabla u_\varepsilon|^2 \leq \frac{1}{2}\int_\Omega u_0^2 + \int_{Q_t} g_0 + C_0 \int_{Q_t} (u_\varepsilon^2 + v_\varepsilon^2), \tag{2.17}$$

$$\frac{\tau}{2}\int_\Omega v_\varepsilon^2(t) + \sigma \int_{Q_t} |\nabla v_\varepsilon|^2 + \frac{1}{2}\int_{Q_t} v_\varepsilon^2 \leq \frac{\tau}{2}\int_\Omega v_0^2 + \frac{1}{2}\int_{Q_t} u_\varepsilon^2 \ . \tag{2.18}$$

Using (2.18) to estimate $\int_{Q_t} v_\varepsilon^2$ in (2.17) and the Gronwall inequality, we get first

$$\sup_{0<t<T} \int_\Omega |u_\varepsilon(t)|^2 + \int_{Q_T} |\nabla u_\varepsilon|^2 \leq C_1 \tag{2.19}$$

and, using again (2.18), also

$$\tau \sup_{0<t<T} \int_\Omega |v_\varepsilon(t)|^2 + \sigma \int_{Q_T} |\nabla v_\varepsilon|^2 + \int_{Q_T} |v_\varepsilon|^2 \leq C_2 \ , \tag{2.20}$$

where the constants C_1 and C_2 are independent of ε, τ and σ.

For each $\varepsilon > 0$, noting that $|f_\varepsilon| \leq 1/\varepsilon$, the existence of at least a solution $(u_\varepsilon, v_\varepsilon)$ follows from the estimates (2.19) and (2.20) by standard methods (see, for instance [24, 33] or [35]), for instance, by using the Schauder fixed point theorem in a large closed ball of $L^2(Q_T) \times L^2(Q_T)$.

Using (2.19) and (2.20), if we multiply (2.16) by $t\,\partial_t v_\varepsilon$, by standard computations we also obtain the estimate, for $0 < \delta < t \leq T$:

$$\tau \int_\delta^t \int_\Omega |\partial_t v_\varepsilon|^2 + \sigma \int_\Omega |\nabla v_\varepsilon(t)|^2 \leq \frac{C_\tau}{\delta} \ , \tag{2.21}$$

where C_τ is independent of ε and σ, but $C_\tau \to +\infty$ as $\tau \to 0$.

Letting $G_M = \{(x,t) \in Q_T \colon |u_\varepsilon| > M\}$ and $S_M = \{(x,t) \in Q_T \colon |u_\varepsilon| \leq M\}$, for any measurable subset $\mathcal{O} \subset Q_T$, using (2.8) and (2.9), we have

$$\begin{aligned}
\int_\mathcal{O} |f_\varepsilon| \equiv \int_\mathcal{O} |f_\varepsilon(u_\varepsilon, v_\varepsilon)| &\leq \int_{\mathcal{O}\cap S_M} |f_\varepsilon| + \frac{1}{M}\int_{\mathcal{O}\cap G_M} |u_\varepsilon\, f_\varepsilon| \\
&\leq \int_{\mathcal{O}\cap S_M} |f_\varepsilon| + \frac{1}{M}\int_{Q_T} \Big\{\widetilde{g}_\varepsilon + |u_\varepsilon\, f_\varepsilon - \widetilde{g}_\varepsilon|\Big\} \qquad (2.22) \\
&\leq \int_{\mathcal{O}\cap S_M} (g_M + |v_\varepsilon|^{2-\delta}) + \frac{2}{M}\int_{Q_T} \widetilde{g}_\varepsilon - \frac{1}{M}\int_{Q_T} u_\varepsilon f_\varepsilon \quad (\text{since } \widetilde{g}_\varepsilon - u_\varepsilon f_\varepsilon \geq 0) \\
&\leq \int_\mathcal{O} (g_M + |v_\varepsilon|^{2-\delta}) + \frac{2}{M}\int_{Q_T} \widetilde{g}_\varepsilon + \frac{1}{M}\Big|\int_{Q_T} u_\varepsilon\, f_\varepsilon\Big| \ ,
\end{aligned}$$

where we have set $\widetilde{g}_\varepsilon = g_0 + C_0(u_\varepsilon^2 + v_\varepsilon^2)$, that belongs to a bounded set of $L^1(Q_T)$ independently of ε. From (2.15), the estimate (2.19) and the assumptions (2.6) and (2.10), it follows that

$$\Big|\int_{Q_T} u_\varepsilon\, f_\varepsilon(u_\varepsilon, v_\varepsilon)\Big| \le C_3 \quad \text{(independently of } \varepsilon) \tag{2.23}$$

and since we may take M arbitrarily large, we conclude that (2.22) implies the equi-integrability of $f_\varepsilon = f_\varepsilon(u_\varepsilon, v_\varepsilon)$, that is to say the weak-L^1 compactness. On the other hand, that implies that $\partial_t u_\varepsilon$ is bounded in $L^2(0,T;(H^1(\Omega))^*)+L^1(0,T;L^1(\Omega))$. Since u_ε is bounded in $L^2(0,T;H^1(\Omega))$, a compactness result of [Si] yields that u_ε is compact in $L^2(0,T;L^2(\Omega))$. Therefore we may extract a subsequence as $\varepsilon \to 0$, such that

$$u_\varepsilon \rightharpoonup u \quad \text{in } \ L^2(0,T;H^1(\Omega))\text{-weak and } \ L^\infty(0,T;L^2(\Omega))\text{-weak*}, \tag{2.24}$$
$$u_\varepsilon \to u \quad \text{in } \ L^2(Q_T)\text{-strong and a.e. in } Q_T\ . \tag{2.25}$$

From (2.20) and (2.21), we may also suppose that

$$v_\varepsilon \rightharpoonup v \quad \text{in } \ L^2(0,T;H^1(\Omega))\text{-weak and } \ L^\infty(0,T;L^2(\Omega))\text{-weak*}, \tag{2.26}$$
$$v_\varepsilon \to v \quad \text{in } \ L^2(Q_T)\text{-strong and a.e. in } Q_T \tag{2.27}$$

which, together with (2.25), by Vitali's theorem yields

$$f_\varepsilon(u_\varepsilon, v_\varepsilon) \to f(u,v) \quad \text{strongly in } \ L^1(Q)\ . \tag{2.28}$$

Passing to the limit in (2.15) and (2.16) we conclude that (u,v) solves (2.1)–(2.4) and satisfies (2.11) and (2.12). Finally, since (2.8) as in (2.22) yields

$$|u_\varepsilon\, f_\varepsilon(u_\varepsilon, v_\varepsilon)| \le 2\,\widetilde{g}_\varepsilon - u_\varepsilon\, f_\varepsilon(u_\varepsilon, v_\varepsilon)\ , \tag{2.29}$$

by (2.23) we conclude that $uf(u,v) \in L^1(Q_T)$, completing the proof of Theorem 2.1. □

Remark 2.2. From (2.11), (2.12) and general results on vector-valued functions (see e.g. [2]), $u \in C^0([0,T];W^{-1,p}(\Omega))$, $1 \le p < N/(N-1)$ and $v \in C^0([0,T];L^2(\Omega))$. The weak solutions satisfy (2.1)–(2.4) not only in the distributional sense but also in their usual variational form obtained with integration by parts:

$$-\int_{Q_T} u\,\partial_t\psi + \int_{Q_T} a(v)\,\nabla u\cdot\nabla\psi = \int_{Q_T} f(u,v)\,\psi + \int_\Omega u_0\,\psi(0), \quad (2.30)$$
$$\forall\,\psi \in W_T \cap L^\infty(Q_T)\ ,$$
$$-\,\tau\int_{Q_T} v\,\partial_t\psi + \sigma\int_{Q_T} \nabla v\cdot\nabla\psi = \int_{Q_T} (u-v)\,\psi + \tau\int_\Omega v_0\,\psi(0), \quad (2.31)$$

$\forall\,\psi \in W_T$, where the space of test functions is defined by

$$W_T = \big\{\psi \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \colon\ \psi(T) = 0\big\}. \qquad (2.32)$$

3. The shadow system as the limit when $\sigma \to +\infty$. In this section we consider, for a fixed $\tau > 0$, the passage to the limit as $\sigma \to +\infty$ in the reaction-diffusion system (2.1)–(2.4). Its shadow system is given by the weak solutions of

$$\partial_t u_\tau = \nabla\cdot(a(\xi)\,\nabla u_\tau) + f(u_\tau,\xi) \quad \text{in } \ Q_T\ , \qquad (3.1)$$
$$\partial_n u_\tau = 0 \ \text{ on } \Sigma_T\ , \qquad u_\tau(0) = u_0 \ \text{ in } \Omega\ , \qquad (3.2)$$
$$\tau\,\dot\xi = \fint_\Omega u_\tau - \xi \quad \text{a.e. in } \ ]0,T[\ , \qquad (3.3)$$
$$\xi(0) = \fint_\Omega v_0\ . \qquad (3.4)$$

Theorem 3.1. Let (2.5)–(2.10) hold. Then there exists at least a weak solution (u_τ,ξ) of (3.1)–(3.4) with the following properties

$$u_\tau \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),$$
$$\partial_t u_\tau \in L^2(0,T;(H^1(\Omega))^*) + L^1(Q_T)\ , \qquad (3.5)$$
$$\xi \in W^{1,\infty}(0,T), \quad f(u_\tau,\xi) \text{ and } u_\tau\,f(u_\tau,\xi)\ \in\ L^1(Q_T)\ . \qquad (3.6)$$

Moreover, (u_τ,ξ) can be obtained as the limit of suitable subsequences when $\sigma \to +\infty$

$$u_{\tau\sigma} \rightharpoonup u_\tau \ \text{ in } L^2(0,T;H^1(\Omega))\text{-weak} \ \text{ and } L^\infty(0,T;L^2(\Omega))\text{-weak*}, \ (3.7)$$
$$u_{\tau\sigma} \to u_\tau \quad \text{in } \ L^2(Q_T)\text{-strong} \ \text{ and } \ \text{a.e. in } Q_T\ , \qquad (3.8)$$
$$v_{\tau\sigma} \to \xi \quad \text{in } \ L^2(0,T;H^1(\Omega))\text{-strong} \ \text{ and } \ \text{a.e. in } Q_T\ , \qquad (3.9)$$

where $(u_{\tau\sigma}, v_{\tau\sigma})$ are weak solutions of (2.1)–(2.4).

Proof. From the proof of Theorem 2.1, we have obtained solutions $u_{\tau\sigma}$ and $v_{\tau\sigma}$ of (2.1)–(2.4) that, together with their time derivatives $\partial_t u_{\tau\sigma}$ and $\partial_t v_{\tau\sigma}$, belong to the spaces indicated in (2.11) and (2.12), which are bounded independently of σ, due to the estimates (2.19)–(2.21). Repeating the argument of (2.22) now with f_ε replaced by $f_\sigma \equiv f(u_{\tau\sigma}, v_{\tau\sigma})$, we may consider a subsequence of $\sigma \to +\infty$ such that

$$u_{\tau\sigma} \to u_\tau \quad \text{and} \quad v_{\tau\sigma} \to \xi \tag{3.10}$$

converging in the sense of (2.24)–(2.25) and (2.26)–(2.27), respectively, and, arguing as in (2.28), also such that

$$f(u_{\tau\sigma}, v_{\tau\sigma}) \to f(u_\tau, \xi) \quad \text{in} \;\; L^1(Q_T)\text{-strong} \; . \tag{3.11}$$

Clearly u_τ solves (3.1)–(3.2) and a similar argument to (2.29) also yields the conclusion $u_\tau f(u_\tau, \xi) \in L^1(Q_T)$.

It remains to identify the limit problem for ξ. From (2.21), we conclude first that

$$\int_{Q_T} |\nabla v_{\tau\sigma}|^2 \le \frac{C_2}{\sigma} \tag{3.12}$$

yields $\nabla\xi$ is a.e. zero, i.e., $\xi = \xi(t)$ is a function of t only.

Integrating the equation (2.3) in Ω, for a.e. t, we have

$$\tau \int_\Omega \partial_t v_{\tau\sigma} = \int_\Omega (u_{\tau\sigma} - v_{\tau\sigma}) \; ,$$

which yields, for $0 < \delta < t \le T$:

$$\tau \int_\Omega \Big[v_{\tau\sigma}(t) - v_{\tau\sigma}(\delta) \Big] = \int_\delta^t \int_\Omega (u_{\tau\sigma} - v_{\tau\sigma}) \; . \tag{3.13}$$

Since $v_{\tau\sigma} \in C^0([0,T]; L^2(\Omega))$, letting first $\delta \to 0$ and then $\sigma \to \infty$, from (3.13) we easily conclude, using (3.10), that

$$\tau\, \xi(t) - \tau \fint_\Omega v_0 = \int_0^t \fint_\Omega u_\tau - \int_0^t \xi \; , \tag{3.14}$$

since ξ does not depend on $x \in \Omega$. Since $u_\tau \in L^\infty(0,T; L^2(\Omega))$, we conclude that $\xi \in W^{1,\infty}(0,T)$ and that ξ solves (3.3) and (3.4). From (3.12) it follows, in particular, the strong convergence (3.9), which concludes the proof of the theorem. □

The structure of the parabolic equation (3.1) in the shadow system is the same as (2.1) in the reaction-diffusion system of departure. However, the ordinary differential equation (3.3), although much simpler than (2.3), only "regularizes" the solution in the space variables, since in general we cannot expect more regularity in time than $\dot{\xi} = d\xi/dt \in L^\infty(0,T)$. Nevertheless, the following estimate holds independently of τ.

Proposition 3.2. *Let ξ_τ satisfy (3.3)–(3.4). Then for each $\tau > 0$ we have that*

$$\|\xi_\tau\|_{L^\infty(0,T)} \leq \Big| \fint_\Omega v_0 \Big| + \Big\| \fint_\Omega u_\tau \Big\|_{L^\infty(0,T)} . \tag{3.15}$$

Proof. We set $\xi = \xi_\tau$, $\zeta = \fint_\Omega u_\tau$ and $\xi_0 = \fint_\Omega v_0$ and we note that, for any $p > 2$, we have

$$|\xi|^{p-2}\,\xi\,\frac{d\xi}{dt} = \frac{1}{2}|\xi^2|^{\frac{p-2}{2}}\,\frac{d\xi^2}{dt} = \frac{1}{p}\,\frac{d}{dt}|\xi|^p .$$

Then multiplying (3.3) by $|\xi|^{p-2}\xi$ and integrating between 0 and t, we obtain

$$\frac{\tau}{p}\,|\xi|^p(t) - \frac{\tau}{p}\,|\xi_0|^p + \int_0^t |\xi|^p = \int_0^t \zeta\,|\xi|^{p-2}\,\xi \leq \int_0^t |\zeta|\,|\xi|^{p-1} .$$

Hence, applying Hölder and Young inequalities, we have

$$\begin{aligned}\int_0^T |\xi|^p &\leq \frac{\tau}{p}\,|\xi_0|^p + \Big(\int_0^T |\zeta|^p\Big)^{1/p}\Big(\int_0^T |\xi|^p\Big)^{1-1/p} \\ &\leq \frac{\tau}{p}\,|\xi_0|^p + \frac{1}{p}\int_0^T |\zeta|^p + \big(1-\frac{1}{p}\big)\int_0^T |\xi|^p .\end{aligned}$$

Whence, it follows

$$\Big(\int_0^T |\xi|^p\Big)^{1/p} \leq \Big(\tau\,|\xi_0|^p + \int_0^T |\zeta|^p\Big)^{1/p} \leq \tau^{1/p}\,|\xi_0| + \Big(\int_0^T |\zeta|^p\Big)^{1/p}$$

and we easily conclude (3.15) by letting $p \to +\infty$. □

Remark 3.3. We note that, by integrating the equation (3.1) in Ω, for a.e. $t \in [0,T]$ we obtain

$$\frac{d}{dt}\fint_\Omega u_\tau = \fint_\Omega f(u_\tau, \xi) \in L^1(0,T) , \tag{3.16}$$

by Fubini theorem, and from (3.3) the additional regularity $\dot\xi \in W^{1,1}(0,T)$ and $\xi \in C^1[0,T]$ holds.

4. The nonlocal diffusion equation. We consider now the nonlocal diffusion equation with a continuous reaction, as a limit case as $\tau \to 0$ of the shadow system (3.1)–(3.4) or as limit case as $\sigma \to +\infty$ and $\tau \to 0$ simultaneously of the reaction diffusion system (2.1)–(2.4):

$$\partial_t u = \nabla \cdot \Big(a\Big(\fint_\Omega u\Big)\nabla u\Big) + f\Big(u, \fint_\Omega u\Big) \quad \text{in } \; Q_T \;, \tag{4.1}$$

$$\partial_n u = 0 \;\; \text{on } \Sigma_T\,, \qquad u(0) = u_0 \;\; \text{in } \Omega\;. \tag{4.2}$$

Theorem 4.1. *Let* (2.5)–(2.10) *hold. Then exists at least a weak solution* u *of* (4.1)–(4.2), *satisfying*

$$\begin{aligned} &u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\\ &\partial_t u \in L^2(0,T;(H^1(\Omega))^*) + L^1(Q_T),\\ &f\big(u, \fint_\Omega u\big) \;\; \textit{and} \;\; uf\big(u, \fint_\Omega u\big) \in L^1(Q_T)\;. \end{aligned} \tag{4.3}$$

Moreover, solutions of (4.1)-(4.2) *can be obtained as limits in*

$$L^2(0,T;H^1(\Omega))\text{-weak, in } L^\infty(0,T;L^2(\Omega))\text{-weak*, in } L^2(Q_T)$$

and a.e. in Q_T, *of suitable subsequences, when* $\tau \to 0$, *of solutions* u_τ *of* (3.1)–(3.4) *or, when* $\sigma \to +\infty$ *and* $\tau \to 0$ *simultaneously, of solutions* $u_{\sigma\tau}$ *of* (2.1)–(2.4), *respectively, with*

$$\xi_\tau \underset{\tau\to 0}{\longrightarrow} \fint_\Omega u \qquad in \;\; L^p(0,T), \quad \forall\, p < \infty, \qquad and \tag{4.4}$$

$$v_{\tau\sigma} \underset{\substack{\sigma\to\infty\\ \tau\to 0}}{\longrightarrow} \fint_\Omega u \qquad in \;\; L^2(0,T;H^1(\Omega))\text{-strong}\;. \tag{4.5}$$

Proof. Let us first consider the limit $\tau \to 0$ in (3.1)-(3.4). There are weak solutions (u_τ, ξ_τ) that, since u_τ is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ independently of τ, by (3.15), ξ_τ satisfies $\|\xi_\tau\|_{L^\infty(0,T)} \le C_3$ independent of τ. Hence, from the equation (3.14), we get for some constant $C>0$, independent of τ,

$$\Big\| \fint_\Omega u_\tau - \xi_\tau \Big\|_{L^2(0,T)} \le C\sqrt{\tau}\;. \tag{4.6}$$

Indeed writing (3.14) in the form $\tau\,\dot\xi_\tau = \fint_\Omega u_\tau - \xi_\tau$, we obtain

$$\begin{aligned}&\int_0^T \Big(\fint_\Omega u_\tau - \xi_\tau\Big)^2\,dt = \tau\int_0^T \dot\xi_\tau\Big(\fint_\Omega u_\tau - \xi_\tau\Big)\,dt\\ &\le \tau\, C_{0,T} + \tau\Big\|\frac{d}{dt}\fint_\Omega u_\tau\Big\|_{L^1(0,T)}\,\|\xi_\tau\|_{L^\infty(0,T)} \le C^2\,\tau\ ,\end{aligned}\tag{4.7}$$

where we have integrated by parts in t, used the estimates (3.15) and (3.16) and set

$$C_{0,T} = \frac{1}{2}\Big(\Big|\fint_\Omega v_0\Big|^2 - |\xi_\tau(T)|^2\Big) + \xi_\tau(T)\fint_\Omega u_\tau(T) - \fint_\Omega v_0 \fint_\Omega u_0\ ,$$

which is bounded independently of τ.

By compactness, we may consider a subsequence of $\tau\to 0$ such that

$$u_\tau \rightharpoonup u \ \text{ in } L^2(0,T;H^1(\Omega))\text{-weak}\ \text{ and in } L^\infty(0,T;L^2(\Omega))\text{-weak*},\tag{4.8}$$

$$u_\tau \to u \ \text{ in } L^2(Q_T) \text{ and a.e. in } Q_T,\tag{4.9}$$

$$\xi_\tau \rightharpoonup \zeta \ \text{ in } L^\infty(0,\infty)\text{-weak*}.\tag{4.10}$$

We have

$$\int_0^T\Big(\fint_\Omega u_\tau - \fint_\Omega u\Big)^2 \le C\int_{Q_T}|u_\tau - u|^2\tag{4.11}$$

and using (4.6) and (4.9), we conclude that $\xi_\tau \to \fint_\Omega u$ in $L^2(0,T)$ and therefore $\zeta = \fint_\Omega u$. Since ξ_τ and $\fint_\Omega u_\tau$ are bounded in $L^\infty(0,T)$ independently of τ, we may conclude (4.4), the remaining properties being obtained as in the proof of Theorem 3.1.

In the second case, we recall the estimates (2.19) and (2.20) for $u_{\tau\sigma}$ and $v_{\tau\sigma}$, respectively, and by compactness, we may also consider a subsequence $(\tau,\sigma)\to(0,+\infty)$, such that

$$u_{\tau\sigma}\to u \quad \text{in the sense of (4.8) and (4.9)},\tag{4.12}$$

$$v_{\tau\sigma}\rightharpoonup v \ \text{ in } L^2(0,T;H^1(\Omega))\text{-weak}.\tag{4.13}$$

In particular, (3.12) implies that $|\nabla v_{\tau\sigma}|\to 0$ strongly in $L^2(Q_T)$ and $v = v(t)$ depends only on t. On the other hand, integrating (2.3) in Ω, for each $t\in[0,T]$, we obtain

$$\tau\int_\Omega v_{\tau\sigma}(t) - \tau\int_\Omega v_0 = \int_0^t\int_\Omega (u_{\tau\sigma} - v_{\tau\sigma}).\tag{4.14}$$

As in Proposition 3.2, we have that $\xi_{\tau\sigma} = \fint_\Omega v_{\tau\sigma}$ solves the problem (3.3)–(3.4) with the non-homogeneous term given now by $\zeta = \fint_\Omega u_{\tau\sigma}$, which belongs to a bounded set of $L^\infty(0,T)$ independently of τ and σ, and therefore satisfies

$$\Big\| \fint_\Omega v_{\tau\sigma} \Big\|_{L^\infty(0,T)} \leq C \quad \text{(independently of } \tau, \sigma). \tag{4.15}$$

Thus, we deduce from (4.14) exactly as in (4.7) that for some $C' > 0$,

$$\Big\| \fint_\Omega u_{\tau\sigma} - \fint_\Omega v_{\tau\sigma} \Big\|_{L^2(0,T)} \leq C' \sqrt{\tau}$$

and from (3.12), which by (4.11) implies $\fint_\Omega u_{\tau\sigma} \to \fint_\Omega u$ in $L^2(0,T)$-strong we deduce that $v = \fint_\Omega u$ and the conclusion (4.5) follows by Poincaré inequality.

Remark 4.1. Since, in general, the problem (4.1)–(4.2) may have more than one solution, we cannot "a priori" guarantee that the limits obtained in Theorem 4.1 from the shadow system when $\tau \to 0$ or from the initial reaction-diffusion system when $(\tau, \sigma) \to (0, +\infty)$, are the same solution.

Remark 4.2. Using the approximation defined by (2.14) and similar a priori estimates as in the proof of Theorem 2.1 (now with v replaced directly by $\fint_\Omega u$) it is possible to give a direct proof of existence of a solution of the problem (4.1)–(4.2), for instance, by using the Schauder fixed point theorem in a ball of $L^2(Q_T) \times L^2(0,T)$ and then letting $\varepsilon \to 0$.

For completeness, we shall include additional conditions under which we may guarantee that (4.1)–(4.2) have a unique solution in the class (4.3). Assume

$$\big|a(x,t,v) - a(x,t,w)\big| \leq a'(|v|,|w|)\,|v-w|, \tag{4.16}$$
$$\big|f(x,t,0,v)\big| \leq g_1(v)\,g_2(x,t), \tag{4.17}$$
$$\big|f(x,t,u,v) - f(x,t,z,v)\big| \leq C_1|u-z|, \tag{4.18}$$
$$\big|f(x,t,u,v) - f(x,t,u,w)\big| \leq \big(g_2(x,t) + C_1|u|\big)\,|v-w|, \tag{4.19}$$

for a.e. $(x,t) \in Q_T$, $u, v, w, z \in \mathbb{R}$, where $C_1 > 0$ is a constant, a' and g_1 are nonnegative L^∞_{loc} functions and $g_2 \in L^2(Q_T)$, $g_2 \geq 0$.

Proposition 4.3. *Under the additional assumptions* (4.16)–(4.19), *there exists at most one solution u to* (4.1)–(4.2) *in the class* (4.3).

Proof. First we observe that, since $\fint_\Omega u \in L^\infty(0,T)$, the conditions (4.17) and (4.18) actually imply $f(u, \fint_\Omega u) \in L^2(Q_T)$. Then multiplying the difference of the two equations (4.1) for two different solutions u_1 and u_2 in

$L^2(0,T;(H^1(\Omega))^*)$:

$$\partial_t\overline{u} - \nabla\cdot\Big(a\Big(\fint_\Omega u_1\Big)\,\nabla\overline{u}\Big) = f\Big(u_1,\,\fint_\Omega u_1\Big) - f\Big(u_2,\,\fint_\Omega u_2\Big) - \nabla\cdot\Big(\Big(a\Big(\fint_\Omega u_1\Big) - a\Big(\fint_\Omega u_2\Big)\Big)\nabla u_2\Big),$$

by their difference $\overline{u} = u_1 - u_2$, after standard calculations using the assumptions (4.16), (4.18) and (4.19), we obtain, for $t \in [0,T]$

$$\int_\Omega |\overline{u}|^2(t) + a_*\int_0^t\!\!\int_\Omega |\nabla\overline{u}|^2 \le 2\,C_1\int_0^t\!\!\int_\Omega |\overline{u}|^2 + C_2\int_0^t\Big|\fint_\Omega \overline{u}\Big|^2 \le C'\int_0^t\!\!\int_\Omega |\overline{u}|^2, \tag{4.20}$$

where the constant C_2 involves the Lipschitz constant in (4.16), the L^2-norms of u_1, $|\nabla u_2|$, g_2, the constant a_* from (2.6) and the Poincaré constant. From (4.20) the uniqueness follows by Gronwall inequality.

Remark 4.4. This uniqueness property is essentially the remark of [10]. Analogously it can be also shown for the systems (2.1)–(2.4) and (3.1)–(3.4).

For the special case $a = a(v)$ and $f \equiv 0$, under the only assumption of continuity and positivity of a, an existence and uniqueness result has been given recently, in [7]. For that case, with $f = f(x,t)$ see also [8] for stability results.

5. The case of a discontinuous reaction term. In this Section we keep the assumptions (2.8) and (2.9) but, following the approach of [29], we allow the reaction term $f = f(x,t,u,v)$ to be discontinuous in the variables u and v, i.e., we replace the assumption (2.7) by

$$f\colon\ Q_T\times\mathbb{R}^2\to\mathbb{R},\ (u,v)\mapsto f(x,t,u,v)\ \text{ in } L^\infty_{\mathrm{loc}}(\mathbb{R}^2) \text{ for a.e. } (x,t)\in Q_T. \tag{5.1}$$

For fixed $(x,t)\in Q_T$ and for any $\delta>0$, we let

$$\overline{\varphi}_\delta(x,t,u,v) = \operatorname*{ess\,sup}_{|z-u|+|w-v|\le\delta} f(x,t,z,w) \quad \text{and}$$

$$\underline{\varphi}_\delta(x,t,u,v) = \operatorname*{ess\,inf}_{|z-u|+|w-v|\le\delta} f(x,t,z,w).$$

It is easy to see that $\overline{\varphi}_\delta$ is a decreasing function as $\delta\to 0$ and $\underline{\varphi}_\delta$ is increasing as $\delta\to0$. Furthermore defining

$$\overline{f}(x,t,u,v) = \lim_{\delta\to0}\overline{\varphi}_\delta(x,t,u,v) \quad\text{ and }\quad \underline{f}(x,t,u,v) = \lim_{\delta\to0}\underline{\varphi}_\delta(x,t,u,v)$$

we have that $\overline{f}$ is an upper semicontinuous function in (u, v) and $\underline{f}$ is lower semicontinuous in (u, v). We define, for a.e. $(x, t) \in Q_T$ the multivalued function $\widehat{f}$ by

$$(x, t, u, v) \mapsto \widehat{f}(x, t, u, v) = \big[\underline{f}(x, t, u, v), \overline{f}(x, t, u, v)\big], \tag{5.2}$$

and we assume that both $\underline{f}$ and $\overline{f}$ are $(N+1)$-measurable in the sense of [5] so that their compositions with measurable functions $u = u(x, t)$ and $v = v(x, t)$ are still measurable in Q_T.

The Neumann problem (4.2) for the nonlocal diffusion equation (4.1) with discontinuous reaction is now given by

$$\partial_t u = \nabla \cdot \Big(a\Big(\fint_\Omega u \Big) \nabla u \Big) + \varphi \quad \text{in } Q_T, \tag{5.3}$$

$$\varphi \in L^1(Q_T) \quad \text{and} \quad \varphi \in \widehat{f}\Big(u, \fint_\Omega u\Big) \quad \text{a.e. } (x,t) \in Q_T, \tag{5.4}$$

$$\partial_n u = 0 \quad \text{on } \Sigma_T \quad \text{and} \quad u(0) = u_0 \quad \text{in } \Omega. \tag{5.5}$$

Theorem 5.1. *Under the assumptions* (2.5)–(2.6), (5.1) *and* (2.8)–(2.10), *there exists at least a weak solution of* (5.3)–(5.5) *satisfying*

$$u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \partial_t u \in L^2(0, T; (H^1(\Omega))^*) + L^1(Q_T). \tag{5.6}$$

Proof. We use the natural approximation of discontinuities by mollification in the variable (u, v), i.e., for $\rho \in C_0^\infty((-1, 1) \times (-1, 1)), \rho \geq 0$ and $\int_{-1}^1 \int_{-1}^1 \rho(s)\, ds = 1$, we set $\rho_\varepsilon(s) = \frac{1}{\varepsilon^2} \rho(\frac{s}{\varepsilon})$ with $s = (s_1, s_2)$ and $f_\varepsilon(u, v) = [\rho_\varepsilon * f](u, v)$ for $\varepsilon > 0$. The function f_ε is then continuous in (u, v) for a.e. $(x, t) \in Q_T$ and satisfies also for $\varepsilon > 0$ the assumptions (2.8) and (2.9).

By Theorem 4.1, we know that there exists at least a solution u_ε, in the class (5.6), to the problem (4.1)–(4.2) with f replaced by f_ε, for each $\varepsilon > 0$. In addition, u_ε satisfies the estimate (2.19), uniformly in ε, and the same argument used in (2.22), now with $f_\varepsilon = f_\varepsilon(u_\varepsilon, \fint_\Omega u_\varepsilon)$ and $v_\varepsilon = \fint_\Omega u_\varepsilon$, yields the equi-integrability of f_ε. Consequently, as $\varepsilon \to 0$, there is a $\varphi \in L^1(Q_T)$ and a subsequence such that

$$f_\varepsilon\Big(u_\varepsilon, \fint_\Omega u_\varepsilon\Big) \rightharpoonup \varphi \quad \text{in } L^1(Q_T)\text{-weak}. \tag{5.7}$$

Then, $\partial_t u_\varepsilon$ being also bounded in $L^2(0,T;(H^1(\Omega))^*) + L^1(Q_T)$, it follows the compactness of u_ε in $L^2(Q_T)$, and we may still suppose

$$u_\varepsilon \to u \text{ in } L^2(0,T;H^1(\Omega))\text{-weak, } L^2(Q_T)\text{-strong and a.e. in } Q_T, \tag{5.8}$$

$$⨍_\Omega u_\varepsilon \to ⨍_\Omega u \text{ in } L^2(0,T) \text{ and a.e. } t \in [0,T]. \tag{5.9}$$

The passage to the limit in the equation being as in the previous cases, it remains to show that $\varphi \in \widehat{f}(u, ⨍_\Omega u)$ a.e. in Q_T, or equivalently, that

$$\underline{f}\Big(x,t,u(x,t), ⨍_\Omega u(t)\Big) \le \varphi(x,t) \le \overline{f}\Big(x,t,u(x,t), ⨍_\Omega u(t)\Big) \text{ a.e. } (x,t) \in Q_T. \tag{5.10}$$

This can be done by adapting the argument of [29]. From (5.8) and (5.9) for any $\eta > 0$, there exist measurable subsets $\mathcal{O} \subset Q_T$ and $E \subset]0,T[$ such that $\text{meas}(\mathcal{O}) < \eta$ and $\text{meas}(E) < \eta$ and $u_\varepsilon \to u$ uniformly in $Q_T \backslash \mathcal{O}$ and $v_\varepsilon = ⨍_\Omega u_\varepsilon \to ⨍_\Omega u = v$ uniformly in $J \equiv]0,T[\backslash E$. So, for any $\delta > 0$, there is $\varepsilon_0 < \delta/4$, such that for $\varepsilon < \varepsilon_0$, we have

$$|u_\varepsilon(x,t) - u(x,t)| + |v_\varepsilon(t) - v(t)| \le \frac{\delta}{4} \text{ for all } (x,t) \in Q_T \backslash \mathcal{O} \text{ such that } t \in J.$$

Therefore, for a.e. $(x,t) \in Q_T \backslash \mathcal{O}$ such that $t \in J$, we have

$$\begin{aligned}
& f_\varepsilon\Big(x,t,u_\varepsilon(x,t), v_\varepsilon(t)\Big) \\
&= \int_{u_\varepsilon(x,t)-\varepsilon}^{u_\varepsilon(x,t)+\varepsilon} \int_{v_\varepsilon(t)-\varepsilon}^{v_\varepsilon(t)+\varepsilon} \rho_\varepsilon\Big(u_\varepsilon(x,t) - s_1, v_\varepsilon(t) - s_2\Big) f(x,t,s_1,s_2)\, ds_1\, ds_2 \\
&\ge \operatorname*{ess\,inf}_{\substack{s_1 \in [u_\varepsilon(x,t)-\varepsilon,\, u_\varepsilon(x,t)+\varepsilon] \\ s_2 \in [v_\varepsilon(t)-\varepsilon,\, v_\varepsilon(t)+\varepsilon]}} f(x,t,s_1,s_2) \quad \text{ and since } \varepsilon < \delta/4 \\
&\ge \operatorname*{ess\,inf}_{\substack{s_1 \in [u(x,t)-\delta,\, u(x,t)+\delta] \\ s_2 \in [v(t)-\delta,\, v(t)+\delta]}} f(x,t,s_1,s_2) = \underline{\varphi}_\delta\Big(x,t,u(x,t),v(t)\Big).
\end{aligned}$$

Taking any $g \in L^\infty(Q_T)$, $g \ge 0$ a.e. in Q_T, for all $\varepsilon < \varepsilon_0$, we have in $D_\eta = Q_T \backslash \mathcal{O} \cap (\Omega \times J)$

$$\int_{D_\eta} f_\varepsilon(u_\varepsilon, v_\varepsilon)\, g \ge \int_{D_\eta} \underline{\varphi}_\delta(u,v)\, g,$$

and, recalling (5.7), letting $\varepsilon \to 0$, we obtain

$$\int_{D_\eta} \varphi\, g \geq \int_{D_\eta} \underline{\varphi}_\delta\Big(u, \fint_\Omega u\Big)\, g.$$

Since $\underline{\varphi}_\delta$ is monotone increasing as $\delta \searrow 0$, it follows that

$$\int_{D_\eta} \varphi\, g \geq \int_{D_\eta} \underline{f}\Big(u, \fint_\Omega u\Big)\, g\,, \qquad \forall\, g \in L^\infty(Q_T), \quad g \geq 0.$$

Hence, we deduce

$$\varphi(x,t) \geq \underline{f}\Big(x, t, u(x,t), \fint_\Omega u(t)\Big) \ \text{ a.e. } (x,t) \in D_\eta \tag{5.11}$$

and since we may choose η arbitrarily small, the measure of $Q_T \backslash D_\eta$ is also as small as we wish and we have in fact (5.11) a.e. in Q_T. The upper inequality of (5.10) can be shown in a similar way and the proof of Theorem 5.1 is complete.

Remark 5.2. Of course, the supremum in the assumption (2.9) is now understood as the essential supremum. In this sense, it is easy to see that the construction of $\underline{f}$ and $\overline{f}$ in the beginning of this section is such that the properties on f in assumptions (2.8) still hold for those two functions.

In a similar way, we may use the results of Sections 2 and 3, in order to extend the existence results for the reaction-diffusion and shadow systems, respectively, (2.1)–(2.4) and (3.1)–(3.4), with discontinuous reaction terms. The precise formulations consist in replacing (2.1) by

$$\partial_t u_{\tau\sigma} = \nabla \cdot \big(a(v_{\tau\sigma})\, \nabla u_{\tau\sigma}\big) + \varphi_{\tau\sigma} \ \text{ in } Q_T, \tag{5.12}$$

$$\varphi_{\tau\sigma} \in L^1(Q_T) \ \text{ and } \ \varphi_{\tau\sigma} \in \widehat{f}(u_{\tau\sigma}, v_{\tau\sigma}) \ \text{ a.e. } (x,t) \in Q_T, \tag{5.13}$$

keeping (2.2)–(2.4), and replacing (3.1) by

$$\partial_t u_\tau = \nabla \cdot (a(\xi_\tau)\, \nabla u_\tau) + \varphi_\tau \ \text{ in } Q_T, \tag{5.14}$$

$$\varphi_\tau \in L^1(Q_T) \ \text{ and } \ \varphi_\tau \in \widehat{f}(u_\tau, \xi_\tau) \ \text{ a.e. } (x,t) \in Q_T, \tag{5.15}$$

with (3.2)–(3.4), respectively. An analogous of the asymptotic convergences (3.7)–(3.9) for $u_{\tau\sigma} \to u_\tau$, $v_{\tau\sigma} \to \xi_\tau$ as $\sigma \to +\infty$, and (4.7)–(4.8), (4.4) for $u_\tau \to u$, $\xi_\tau \to \fint_\Omega u$, as $\tau \to 0$, or (4.11), (4.5) for $u_{\tau\sigma} \to u$, $v_{\tau\sigma} \to \fint_\Omega u$

as $(\tau,\sigma) \to (0,+\infty)$, still hold for subsequences, but now an additional argument is required, since we cannot expect more than weak convergences in $L^1(Q_T)$ of $\varphi_\tau \to \varphi$ and $\varphi_{\tau\sigma} \to \varphi$, respectively. We limit ourselves to prove the following result, the case corresponding to Theorem 3.1 being very similar.

Theorem 5.3. *Under the previous assumptions, we can obtain solutions u to the problem* (5.3)–(5.5) *as limits when $\tau \to 0$*

$$u_\tau \to u \quad \text{in } L^2(0,T;H^1(\Omega))\text{-weak}, \quad \text{in } L^2(Q_T) \quad \text{and a.e. in } Q_T, \tag{5.16}$$

$$\xi_\tau \to \fint_\Omega u \quad \text{in } L^p(0,T), \quad \forall\, p<\infty \quad \text{and} \quad L^\infty(0,T)\text{-weak*}, \tag{5.17}$$

where u_τ, ξ_τ are weak solutions of (5.14)–(5.15), (3.2)–(3.4), *or as limits when $(\tau,\sigma) \to (0,+\infty)$*

$$u_{\tau\sigma} \to u \quad \text{in } L^2(0,T;H^1(\Omega))\text{-weak}, \quad \text{in } L^2(Q_T) \quad \text{and a.e. in } Q_T, \tag{5.18}$$

$$v_{\tau\sigma} \to \fint_\Omega u \quad \text{in } L^2(0,T;H^1(\Omega))\text{-strong}, \tag{5.19}$$

where $u_{\tau\sigma}$, $v_{\tau\sigma}$ are weak solutions of (5.12)–(5.13), (2.2)–(2.4).

Proof. We may repeat the proof of Theorem 4.1, provided we have proved the weak-L^1 compactness of φ_τ and $\varphi_{\tau\sigma}$ and we have identified their accumulation points, respectively, as $\tau \to 0$ and $(\tau,\sigma) \to (0,+\infty)$ as functions satisfying (5.4).

The first case being similar, let us concentrate in the second one, for which we consider solutions $u_{\tau\sigma}$, $v_{\tau\sigma}$ obtained by regularization of the discontinuities, as in the proof of Theorem 5.1. In particular, we may consider $\varphi_{\tau\sigma}$ as a weak limit in $L^1(Q_T)$ of certain functions $f_{\tau\sigma\varepsilon} = f_\varepsilon(u_{\tau\sigma\varepsilon}, v_{\tau\sigma\varepsilon})$, given in terms of solutions $u_{\tau\sigma\varepsilon}$, $v_{\tau\sigma\varepsilon}$ obtained in Theorem 2.1. Here $\varepsilon > 0$ denotes, as in the proof of Theorem 5.1 the regularization parameter.

Arguing as in (2.22), for any measurable subset $\mathcal{O} \subset Q_T$ and any $M > 0$, we arrive to the estimate

$$\int_{\mathcal{O}} |\varphi_{\tau\sigma}| \le \liminf_{\varepsilon\to 0} \int_{\mathcal{O}} |f_{\tau\sigma\varepsilon}| \le \int_{\mathcal{O}} \widetilde{\widetilde{g}}_M + \frac{3C}{M} \tag{5.20}$$

where, by the "a priori" estimates on $u_{\tau\sigma\varepsilon}$ and $v_{\tau\sigma\varepsilon}$ in $L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ independently of τ, σ and ε, the constant $C > 0$ and the integrable function $\widetilde{\widetilde{g}}_M \geq 0$ are fixed. Then, (5.20) meaning the equi-integrability of $\varphi_{\tau\sigma}$, we may conclude that

$$\varphi_{\tau\sigma} \rightharpoonup \varphi \quad \text{in } L^1(Q_T)\text{-weak}, \tag{5.21}$$

when $(\tau,\sigma) \to (0,+\infty)$, as well as the convergences (5.18)–(5.19) by a similar argument to the proof of Theorem 4.1. It remains to prove that $\varphi \in \widehat{f}(u, ⨍_\Omega u)$ a.e. in Q_T, i.e., (5.10). By (5.13) we know that

$$\begin{aligned} &\underline{f}(x,t,u_{\tau\sigma}(x,t),v_{\tau\sigma}(x,t)) \leq \varphi_{\tau\sigma}(x,t) \\ &\quad \leq \overline{f}(x,t,u_{\tau\sigma}(x,t),v_{\tau\sigma}(x,t)), \quad \text{a.e. } (x,t) \in Q_T. \end{aligned} \tag{5.22}$$

For any $\eta > 0$, there exists a measurable subset $\mathcal{O} \subset Q_T$ such that meas$(\mathcal{O})$ $< \eta$ and

$$u_{\tau\sigma} \to u \;\text{ and }\; v_{\tau\sigma} \to ⨍_\Omega u \;\text{ uniformly in } Q_T \backslash \mathcal{O}. \tag{5.23}$$

Therefore, for any $g \in L^\infty(Q_T)$, $g \geq 0$ a.e., by the lower semi-continuity of $\underline{f}$ in the last two variables we obtain

$$\begin{aligned} \int_{Q_T\backslash\mathcal{O}} g\,\varphi &= \lim_{\substack{\tau\to 0\\ \sigma\to+\infty}} \int_{Q_T\backslash\mathcal{O}} g\,\varphi_{\tau\sigma} \geq \liminf_{\substack{\tau\to 0\\ \sigma\to+\infty}} \int_{Q_T\backslash\mathcal{O}} g\,\underline{f}(u_{\tau\sigma},v_{\tau\sigma}) \\ &\geq \int_{Q_T\backslash\mathcal{O}} g \liminf_{\substack{\tau\to 0\\ \sigma\to+\infty}} \underline{f}(u_{\tau\sigma},v_{\tau\sigma}) \geq \int_{Q_T\backslash\mathcal{O}} g\,\underline{f}(u, ⨍_\Omega u). \end{aligned} \tag{5.24}$$

Here we have used the Fatou lemma, since $u_{\tau\sigma}$ and $v_{\tau\sigma}$ are bounded in $Q_T\backslash\mathcal{O}$ and by (2.9) $\underline{f}(u_{\tau\sigma}, v_{\tau\sigma})$ is integrable in this set. Analogously, by the upper semi-continuity of $\overline{f}$ we get

$$\int_{Q_T\backslash\mathcal{O}} g\,\varphi \leq \int_{Q_T\backslash\mathcal{O}} g\,\overline{f}\Big(u, ⨍_\Omega u\Big), \qquad \forall\, g \in L^\infty(Q_T), \quad g \geq 0 \text{ a.e.} \tag{5.25}$$

Since the measure of $\mathcal{O}$ is arbitrarily small, from (5.24) and (5.25) we conclude (5.10) a.e. in Q_T, that completes the proof of Theorem 5.3.

6. Examples of application and a counterexample to uniqueness.

6.1. A model for porous-medium combustion. A simplified version of a model proposed in [27] and considered by several authors with different assumptions leads to a parabolic equation of the type (see, for instance [12] or [15])

$$\partial_t u = \Delta u + H(u - \vartheta)\, f(u) \quad \text{in } Q_T, \tag{6.1}$$

where f is a smooth reaction term with no precise sign and H denotes the maximal monotone graph associated with the Heaviside function, i.e.,

$$H(z) = \begin{cases} 0 & \text{for } z < 0, \\ [0,1] & \text{for } z = 0, \\ 1 & \text{for for } z > 0. \end{cases}$$

Here $u = u(x,t)$ represents the temperature field and ϑ the ignition temperature or a critical switching temperature. Although ϑ is often taken to be a constant, we may suppose that it depends, through a continuous function

$$\vartheta = \vartheta(v), \tag{6.2}$$

on some other physical quantity $v = v(x,t)$, like some concentration of water (see, for instance, [31]). Assuming that v satisfies also a diffusion problem of the type (2.3)–(2.4), by letting $(\tau, \sigma) \to (0, +\infty)$, we obtain then a problem of the type (6.1), with

$$\vartheta = \vartheta\Big(\fint_\Omega u \Big), \tag{6.3}$$

being the global term $\fint_\Omega u$ related to the total energy of the system.

Under the assumptions of Section 2 on the behaviour of f, since H is "a priori" bounded and has a unique jump discontinuity at $u = \vartheta$, we may apply the results of Section 5 in order to ensure the existence of a weak solution, and the derivation of the law (6.3) from (6.2) with an auxiliary system of the form (2.3)–(2.4) for v.

6.2. An activator-inhibitor with nonlocal threshold. Morphogenesis in excitable media also leads to consider systems of the type (see [28] and its references)

$$\partial_t u = \Delta u - u + H(u - \vartheta) - v, \tag{6.4}$$

$$\tau\, \partial_t v = \sigma\, \Delta v - v + u, \tag{6.5}$$

where u and v denote the activator and inhibitor concentrations in the medium. The excitation threshold ϑ may depend on the total values of

u and v. For instance, in [20] it has been chosen a particular case of

$$\vartheta = \theta\Big(\fint_\Omega (u+v)\Big), \tag{6.6}$$

where θ is a given continuous function (see (1.7)).

Clearly the system (6.4)–(6.5), with appropriate boundary and initial conditions can be handled with the techniques of the preceding sections. For instance, with (6.6) when $(\tau,\sigma) \to (0,+\infty)$ we would obtain the following equation

$$\partial_t u = \Delta u - u + H\Big(u - \vartheta\Big(2\fint_\Omega u\Big)\Big) - \fint_\Omega u \tag{6.7}$$

which, certainly, is still in the type considered in this paper.

6.3. An example of nonuniqueness of solution. As it was already observed in [12] and in [16], the presence of a discontinuous reaction term, such as the Heaviside function, may easily produce a multiplicity of solutions even in the evolutionary case.

With the above notations, consider the nonlocal problem

$$\partial_t u - \Delta u + u \;\in\; H\Big(u - \fint_\Omega u\Big) \;\text{ in } Q_T, \tag{6.8}$$

$$\partial_n u = 0 \;\text{ on } \Sigma_T\,, \qquad u(0) = \gamma \;\text{ in } \Omega, \tag{6.9}$$

where $\gamma \in [0,1]$ is a given constant. For any other constant $\eta \in [0,1]$, we easily see that the one-parameter family of functions

$$u_\eta = u_\eta(t) = \gamma\, e^{-t} + \eta(1-e^{-t}), \tag{6.10}$$

taking values between γ and $\eta = \lim_{t\to\infty} u_\eta(t)$, constitutes a continuum of solutions to (6.8), (6.9). We also observe that their limit values η also constitute a one-parameter family of solutions to the corresponding time independent nonlocal Neumann problem.

Acknowledgments. The authors wish to thank J. Hulshof for a kind suggestion that led us to the counterexample to the uniqueness of Section 6.3. This work was partially supported in the framework of a grant of the Fundação de Ciências e Tecnologia (Project Praxis/2/2.1/Mat/125/94).

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