Advances in Differential Equations Volume 5 (4–6) April–June 2000, pp. 623–656

RIESZ BASIS PROPERTY OF ROOT VECTORS OF NONSELFADJOINT OPERATORS GENERATED BY RADIAL DAMPED WAVE EQUATIONS

Marianna A. Shubov
Department of Mathematics, Texas Tech University, Lubbock, TX 79409

(Submitted by: J.A. Goldstein)

Abstract. We consider an infinite sequence of radial wave equations obtained by the separation of variables in the spherical coordinates from the 3-dimensional damped wave equation with spatially nonhomogeneous spherically symmetric coefficients. Our main objects of interest are the nonselfadjoint operators in the energy spaces of 2-component initial data, which are the dynamics generators for the systems governed by the aforementioned equations and nonselfadjoint boundary conditions on the sphere $r = a$. Our main result is the fact that the sets of the root vectors (generalized eigenvectors) of these operators form Riesz bases in the corresponding energy spaces. This result has several applications. The first one is the fact that the aforementioned operators are spectral in the sense of N. Dunford, and, therefore, we have a new nontrivial class of spectral operators. Another application is a precise estimate on the rate of the energy decay, which is equal to the spectral abscissa of the corresponding semigroup. Finally, we use the results of our spectral analysis to formulate the solutions of several problems in control theory of systems governed by damped wave equations (the proofs are given in another work).

1. Statement of problem. In the present paper, we study the geometric properties of the set of root vectors of a certain class of nonselfadjoint operators and related nonselfadjoint quadratic operator pencils. These operators are the dynamics generators for systems governed by radial damped wave equations with spatially nonhomogeneous coefficients defined on the interval $r \in [0, a]$ and nonselfadjoint boundary conditions at $r = a$. This paper is a continuation of our previous work [17], where we have completed a detailed asymptotic analysis of the spectrum and eigenfunctions of the

Accepted for publication January 1999.
AMS Subject Classifications: 47, 46, 45, 35, 34.

aforementioned operators and pencils. The main result of the present work is the fact that the systems of the root vectors (eigenvectors and associate vectors together) of these operators form Riesz bases in the corresponding energy spaces of 2-component initial data. As an application of the spectral results established in [17] and the present paper, we give the solutions of several controllability problems for systems governed by the aforementioned equations in our work [21]. The results of the present work are contained among the results announced in our short note [27]. The spectral results of [17] and this work can be considered as an extension of the results of the author's works [18, 19] devoted to similar problems for the equation of a nonhomogeneous damped string. The control-theoretical applications of the spectral results obtained in [18, 19] are given in [20]. In a forthcoming paper, we will use the results of [17] and this work to prove the Riesz basis property of the root vectors of the dynamics generator corresponding to the full 3-dimensional damped wave equation with spherically symmetric coefficients and nonconservative boundary conditions on the sphere $|x| = a$. (The latter fact does not follow directly from the Riesz basis property results obtained in this work, and requires an additional quite nontrivial proof.) An alternative method to prove the Riesz basis property of the root vectors for the equation of nonhomogeneous damped string has been suggested in our paper [25]. In this paper, we have used the so-called transformation operators method (for details, see [25]).

Now we recall the problem. Let Ω be an open ball of radius a in $\mathbb{R}^3$ and $\partial\Omega$ be its boundary – the sphere of radius a. In Ω, we consider the following wave equation

$$u_{tt} - \frac{1}{\rho(r)}\operatorname{div}(p(r)\nabla u) + 2d(r)u_t + q(r)u = 0, \quad r = (x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}}, \quad (1.1)$$

$t \geq 0$. All coefficients: ρ (the density of the medium), p (the elasticity coefficient), d (the viscous damping coefficient), and q (the rigidity of external harmonic force) are positive spherically symmetric functions. (Precise conditions on these functions are formulated later.)

Together with the equation, we consider the initial conditions:

$$u(x_1 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad (1.2)$$

and a one-parameter family of boundary conditions:

$$\left(\frac{\partial u}{\partial n} + hu_t\right)_{x \in \partial\Omega} = 0, \quad h \in \mathbb{C} \cup \{\infty\} \quad (1.3)$$

with $\frac{\partial u}{\partial n}$ being a normal derivative on the boundary. To $h = \infty$, we formally associate the Dirichlet condition: $u(x,t) = 0$ for $x \in \partial\Omega$.

We look for a solution of the problem (1.1)-(1.3) in the form of an expansion with respect to the spherical harmonics (see [1]), i.e.,

$$u(x,t) = \sum_{\ell,m,j} u_{\ell mj}(r,t) Y_{\ell mj}(\theta,\varphi). \tag{1.4}$$

For the initial conditions, we have

$$\begin{pmatrix} u_0(x) \\ u_1(x) \end{pmatrix} = \sum_{\ell,m,j} \begin{pmatrix} u_0^{\ell mj}(r) \\ u_1^{\ell mj}(r) \end{pmatrix} Y_{\ell mj}(\theta,\varphi). \tag{1.5}$$

Substituting (1.4), (1.5) into Eq. (1.1) and conditions (1.2), (1.3), and using the orthogonality of the spherical harmonics, we transfer 3-dimensional initial-boundary problem (1.1)-(1.3) to the infinite sequence of the following one dimensional problems:

$$\begin{aligned} (u_{\ell mj}(r,t))_{tt} = \frac{1}{\rho(r)}\Big[\frac{1}{r^2}\frac{\partial}{\partial r}\big(p(r)r^2\frac{\partial}{\partial r}u_{\ell mj}(r,t)\big) - \frac{\ell(\ell+1)}{r^2}p(r)u_{\ell mj}(r,t)\Big] \\ - 2d(r)(u_{\ell mj}(r,t)t)_t - q(r)u_{\ell mj}(r,t), \end{aligned}$$
$$\ell = 0,1,2,\ldots; \quad |m| \le \ell; \quad j = 1 \text{ if } m = 0 \text{ and } j = 1,2 \text{ if } m \neq 0, \tag{1.6}$$

with the initial conditions:

$$u_{\ell mj}(r,0) = u_0^{\ell mj}(r), \quad (u_{\ell mj})_t(r,0) = u_1^{\ell mj}(r), \tag{1.7}$$

and the boundary conditions: $u_{\ell mj}(r)$ must be bounded at $r = 0$ and satisfy the following condition at $r = a$, $\left((u_{\ell mj})_r + h(u_{\ell mj})_t\right)(a,t) = 0$. From now on, we will omit the triple index "ℓmj" and instead of $u_{\ell mj}$ simply write u. From this point, we study the following initial-boundary problem:

$$u_{tt} = L_\ell u - 2d(r)u_t, \qquad \ell = 0,1,2,\ldots \tag{1.8}$$

where L_ℓ is the Sturm-Liouville operator defined by

$$L_\ell\varphi = \frac{1}{\rho(r)}\Big[\frac{1}{r^2}\frac{d}{dr}\big(p(r)r^2\frac{d\varphi}{dr}\big) - \frac{\ell(\ell+1)}{r^2}p(r)\varphi\Big] - q(r)\varphi, \tag{1.9}$$

on any smooth function $\varphi(r)$. We impose the following boundary conditions:

$$\lim_{r\to 0} ru(r,t) = 0, \quad (u_r + hu_t)(a,t) = 0, \quad h \in \mathbb{C}\cup\{\infty\} \tag{1.10}$$

and standard initial conditions

$$u(r,0) = u_0^0(r), \quad u_t(r,0) = u_1^0(r). \tag{1.11}$$

We recall that for $h = \infty,\ 0$, we have the Dirichlet and Neumann boundary conditions respectively; for $h = 1$ and $\ell = 0$, we have the Sommerfeld radiation conditions (see [9]).

Eq. (1.8) can be represented in the form of the first order evolution equation for 2-component function $U(r,t) = \begin{pmatrix} u_0(r,t) \\ u_1(r,t) \end{pmatrix} = \begin{pmatrix} u(r,t) \\ u_t(r,t) \end{pmatrix}$:

$$U_t = i\mathfrak{L}U, \tag{1.12}$$

where $\mathfrak{L}$ is the following matrix differential expression:

$$\mathfrak{L} = -i \begin{pmatrix} 0 & 1 \\ L_\ell & -2d(r) \end{pmatrix}. \tag{1.13}$$

Under certain conditions on the coefficients which are given in Section 2, Eq. (1.12) with boundary conditions (1.10) defines a strongly continuous semigroup of transformations in a complex Hilbert space $\mathfrak{H}_\ell$ of 2-component initial data. $\mathfrak{H}_\ell$ is the closure of smooth 2-component functions $U(r) = \begin{pmatrix} u_0(r) \\ u_1(r) \end{pmatrix}$, such that $u_0(r) = 0$ in a neighborhood of $r = 0$, in the following energy norm:

$$\|U\|^2_{\mathfrak{H}_\ell} = \frac{1}{2}\int_0^a \Big[p(r)|u_0'|^2 + q(r)\rho(r)|u_0|^2 + \frac{\ell(\ell+1)}{r^2}p(r)|u_0|^2 + \rho(r)|u_1|^2\Big] r^2 dr. \tag{1.14}$$

The generator $\mathfrak{L}_\ell$ of the aforementioned semigroup is defined by (1.13), i.e., $\mathfrak{L}_\ell = \mathfrak{L}$, on the domain:

$$D(\mathfrak{L}_\ell) = \{U \in \mathfrak{H}_\ell : \mathfrak{L}_\ell U \in \mathfrak{H}_\ell,\ \lim_{r\to 0} ru_1(r) = 0,\ (u_0' + hu_1)(a) = 0\}. \tag{1.15}$$

From now on, the operator $\mathfrak{L}_\ell$ and related quadratic operator pencil $\mathcal{P}_\ell(\lambda)$ (which will be introduced below) are our main objects of interest.

To describe the pencil $\mathcal{P}_\ell(\lambda)$, let us look for a solution of problem (1.8)-(1.11) in the form

$$u(r,t) = e^{i\lambda t}v(r). \tag{1.16}$$

For $v(r)$, we have the following spectral problem:

$$\mathcal{P}_\ell(\lambda)v = 0, \qquad \lim_{r\to 0} rv(r) = 0, \quad (v' + i\lambda hv)(a) = 0, \tag{1.17}$$

where

$$\mathcal{P}_\ell(\lambda)v = L_\ell v + \lambda^2 v - 2i\lambda d(r)v \tag{1.18}$$

with L_ℓ being given in (1.9). The pencil $\mathcal{P}_\ell(\lambda)$ is defined on the domain

$$D(\mathcal{P}_\ell(\lambda)) = \big\{v \in H^2(0,a) : L_\ell v \in L^2(0,a),\ \lim_{r\to 0} rv(r) = 0, \\ (v' + i\lambda hv)(a) = 0.\big\} \tag{1.19}$$

Now we describe the relationship between the operator $\mathfrak{L}_\ell$ and pencil $\mathcal{P}_\ell(\lambda)$. We recall that $\lambda \in \mathbb{C}$ is an eigenvalue of problem (1.17), (1.18) if this problem has a nontrivial solution. This solution is called an eigenmode or eigenfunction. One can easily verify that if λ_n^ℓ is an eigenvalue of $\mathfrak{L}_\ell$, then the corresponding eigenvector $\mathcal{F}_n^\ell$ can be written in the form:

$$\mathcal{F}_n^\ell = \begin{pmatrix} \frac{1}{i\lambda_n^\ell} & F_n^\ell \\ & F_n^\ell \end{pmatrix}, \ n \in \mathbb{Z}, \tag{1.20}$$

where F_n^ℓ is an eigenvector of pencil (1.18), (1.19). (According to Theorem 2.1 below, the eigenvalues of $\mathfrak{L}_\ell$ are naturally numbered by $n \in \mathbb{Z}$, and for each eigenvalue there is only one eigenvector.) The latter fact means that the pencil $\mathcal{P}_\ell(\lambda)$ and operator $\mathfrak{L}_\ell$ have the same spectra and the eigenvectors are related through formula (1.20). Note, that relation (1.20) is valid only for the eigenvectors; associated vectors of the operator $\mathcal{L}_\ell$ and pencil $\mathcal{P}_\ell(\lambda)$ are related through more complicated formula which is not given since we do not need it for the purpose of the present paper. We would like to emphasize that neither for the operators $\mathfrak{L}_\ell$ nor for pencils $\mathcal{P}_\ell(\lambda)$ the spectral analysis has been developed before.

The main result of the present paper is the fact that the set of root vectors (eigenvectors and associated vectors together) of the operator $\mathfrak{L}_\ell$ forms a Riesz basis in the energy space $\mathfrak{H}_\ell$, and the set of root vectors of the pencil $\mathcal{P}_\ell(\lambda)$ forms a Riesz basis in an appropriate weighted L^2-space. These results are stated precisely in Section 2. In Section 3, we formulate our control-theoretical results. As was already mentioned in the abstract, we do not provide any proofs of the statements from Section 3 (these proofs are given in our work [21]). However, we decided to present control-theoretical results here just to demonstrate that the spectral results are not only important for pure mathematics, but also useful for applied sciences. Sections 4-6 are devoted to the proof of our main spectral result - Theorem 2.2. This theorem deals with the Riesz basis property of the root vectors of the operator $\mathfrak{L}_\ell$. We prove this result in several steps. On each step, we consider an auxiliary problem and obtain certain partial results which might be of interest in themselves. The Riesz basis property results for the pencils $\mathcal{P}_\ell(\lambda)$ are formulated without proofs (Theorem 2.3). We do not provide any proofs of the pencil results by two reasons: first, we do not need them for control-theoretical applications, and, secondly, the proofs can be reconstructed based on the approach developed in our paper [18], where the case $\ell = 0$ was studied. We formulate the pencil results because of their importance: as we already mentioned, the pencil $\mathcal{P}_\ell(\lambda)$ does not belong to any class of nonselfadjoint operator pencils for which the spectral analysis has been developed (see [8] and references therein).

2. Statement of main spectral results. We begin with the properties of the coefficients ρ, d, q and p. (These properties are given in our paper [17]; to keep this paper self contained, we repeat them here).

We assume that the density ρ satisfies the conditions:

$$\rho \in H^2[0,a], \quad \rho(r) > 0 \text{ for } \; r \in [0,a]. \tag{2.1}$$

In the case of real h, we need an additional restriction

$$\rho(a)/p(a) \neq h^2. \tag{2.2}$$

The discussion on condition (2.2) is presented in our paper [20].

About the damping, rigidity and elasticity coefficients we assume:

$$d \in H^1(0,a), \qquad d(r) \geq 0 \text{ for } r \in [0,a]; \tag{2.3}$$

$$q \in L^1(0,a), \qquad q(r) \geq 0 \text{ for } r \in [0,a]; \tag{2.4}$$

$$\begin{aligned} p \in C^2[0,a], \qquad & p(r) > 0 \text{ for } r \in [0,a] \text{ and} \\ p(r) \quad = \quad & p_0 + p_1 r^2 + O(r^3) \text{ when } r \to 0, \\ & (p_0, p_1 \text{ are two positive constants}). \end{aligned} \tag{2.5}$$

The condition on the behavior of p at the vicinity of zero has been imposed only to simplify the asymptotic analysis in [17]. However, this restriction can be eliminated if necessary. We introduce the quantities $\mathcal{M}$ and $\mathcal{N}$ which are important in the following:

$$\mathcal{M} = \int_0^a \sqrt{\rho(t)/p(t)}dt, \quad \mathcal{M} < \infty, \tag{2.6}$$

$$\mathcal{N} = \int_0^a d(t)\sqrt{\rho(t)/p(t)}dt > 0, \quad \mathcal{N} < \infty. \tag{2.7}$$

We mention that, in fact, the asymptotic behavior of the eigenvalues depends on the behavior of the function $\tau = \rho/p$ only at the vicinity of the endpoint $r = a$. So, we can admit singularities and zeros somewhere in the inner points of the interval $[0, a]$, even at the endpoint $r = 0$ (see [17] and [19] for the case of $d = 0$). The leading term of the asymptotics will be the same. We do not allow this behavior only to simplify the asymptotic analysis. If we allowed singularities or zeros of τ on $[0, a]$, then (2.6), (2.7) would be additional restrictions on τ.

Our main theorem (Theorem 2.2 below) describes the geometry of the set of root vectors of the operator $\mathfrak{L}_\ell$. Before we formulate it, we recall several definitions related to the notion of Riesz basis.

As is well known, the convergence of expansions with respect to any complete orthonormal system $\{\varphi_n\}$ in a Hilbert space H is unconditional, i.e., the corresponding Fourier series converges to the same sum after any permutation of its terms. The latter fact becomes true for any system $\{\psi_n\}$ obtained from an orthonormal basis by means of a bounded and boundedly invertible transformation of H.

Definition 2.1. Any complete system $\{\psi_n\}_{n\in\mathbb{Z}}$ in a Hilbert space H is called a Riesz basis (R-basis) if there exist an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{Z}}$ and bounded, boundedly invertible operator A such that $\varphi_n = A\psi_n$. The operator A is called an orthogonalizer of the system $\{\psi_n\}_{n\in\mathbb{Z}}$. (Note that the system $\{\psi_n\}$ is almost normalized, i.e., there exist positive constants C_1 and C_2 such that $0 < C_1 \le \|\psi_n\| \le C_2 < \infty$).

Definition 2.2. A family of nonzero vectors $\{\psi_n\}_{n\in\mathbb{Z}}$ in a Hilbert space H is called an unconditional basis in H if a) this family spans H (i.e., the set of all finite linear combinations of $\{\psi_n\}$ is dense in H); b) there exist positive constants C_3 and C_4 such that for every finite set of complex numbers $\{a_n\}$ the following inequalities hold:

$$C_3 \sum_n |a_n|^2 \|\psi_n\|^2 \le \left\|\sum_n a_n\psi_n\right\|^2 \le C_4 \sum_n |a_n|^2 \|\psi_n\|^2. \tag{2.8}$$

This definition requires some explanation. According to the standard definition of an unconditional basis $\{\psi_n\}_{n\in\mathbb{Z}}$, every element f of the space can be uniquely expanded in an unconditionally convergent series $f = \sum_{n\in\mathbb{Z}} a_n\psi_n$. The latter definition is equivalent to Definition 2.2 due to the following property of unconditional bases (see [4, 5]): a complete system $\{\psi_n\}_{n\in\mathbb{Z}}$ forms an unconditional basis if and only if for any $f \in H$ the "approximate Parceval identity" holds, i.e., there exist $C_5, C_6 > 0$ such that $C_5 \sum_{n\in\mathbb{Z}} |a_n|^2 \le \|f\|^2 \le C_6 \sum_{n\in\mathbb{Z}} |a_n|^2$.

As follows from Definitions 2.1 and 2.2, every R-basis is unconditional and every unconditional and almost normalized basis is an R-basis.

To formulate the main result of this paper, we need the description of the spectrum obtained in our paper [17]. We reproduce the necessary information from [17] in the form of the following theorem.

Theorem 2.1. a) *The operator $\mathfrak{L}_\ell$ has a countable set of complex eigenvalues, which are located in a strip parallel to the real axis, and has only two points of accumulation: $+\infty$ and $-\infty$ in the sense that Re $\lambda_n^\ell \to \pm\infty$ as $n \to \pm\infty$ and Im $\lambda_n^\ell \to$ const as $n \to \pm\infty$ (see* (2.10) *below). For this reason, the spectrum can be represented in the form $\{\lambda_n^\ell, \quad n \in \mathbb{Z}\}$, where Re $\lambda_n^\ell \le$ Re λ_{n+1}^ℓ and Re $\lambda_n^\ell \to \pm\infty$ as $n \to \pm\infty$. If we denote*

$$\alpha_\pm = \sqrt{\rho(a)/p(a)} \pm h, \tag{2.9}$$

then for any $h \in \mathbb{C}$, the following asymptotic formula for the eigenvalues of problem (1.17) *holds*

$$\begin{aligned} &\lambda_n^\ell = \Lambda_n^\ell + O(\ell n|n|/|n|), \ \text{ as } |n| \to \infty, \\ &\Lambda_n^\ell = \mathcal{M}^{-1}\big(n + \frac{\ell}{2} + \frac{1}{2} sgn\ n\big)\pi + i\mathcal{M}^{-1}\big(\frac{1}{2}\ell n(\alpha_+\alpha_-^{-1}) + \mathcal{N}\big), \end{aligned} \tag{2.10}$$

where $\mathcal{M}$ and $\mathcal{N}$ are given by (2.6), (2.7), *and under ℓn we understand the principal value of the logarithm. In* (2.10), *the enumeration is asymptotical but not absolute. It is possible to pass to the limit $|h| \to \infty$. In the case $|h| = \infty$, which corresponds to the Dirichlet boundary condition $u(a) = 0$, the expression under the sign of logarithm in* (2.10) *should be replaced with* (-1), *so that $\ell n(-1) = i\pi$.*

b) *All eigenvalues of the operator $\mathfrak{L}_\ell$ have geometric multiplicities equal to 1, i.e., for each λ_n^ℓ there exists only one linearly independent eigenvector $\mathcal{F}_n^\ell$. However, a finite number of eigenvalues $\{\lambda_n^\ell, n \in R_\ell \subset \mathbb{Z}\}$ may have* *finite algebraic multiplicities* m_n^ℓ, *i.e., for such* λ_n^ℓ *there exists a finite chain of associate vectors* $\{\mathcal{F}_{n,j}^\ell\}_{j=1}^{m_n^\ell-1}$:

$$(\mathfrak{L}_\ell - \lambda_n^\ell I)\mathcal{F}_{n,j}^\ell = \mathcal{F}_{n,j-1}^\ell, \quad \mathcal{F}_{n,0}^\ell = \mathcal{F}_n^\ell, \quad \mathcal{F}_{n,-1}^\ell = 0. \tag{2.11}$$

Corollary 2.1. *The eigenvalues are asymptotically equidistant:*

$$\lim_{|n|\to\infty} \lambda_n^\ell / n = \pi \mathcal{M}^{-1}.$$

Remark 2.1. In Theorem 2.1, we have collected the necessary information from Theorems 2.1 and 2.2 of work [17]. Statement a) of the above theorem coincides with Statement a) of Theorem 2.2 [17]. Statement b) is a corollary of Theorem 2.1 [17]. The latter theorem describes the properties of the so-called Jost solution of Eq. (1.17), and does not mention the eigenvectors and associated vectors explicitly. However, Statement b) of the above theorem is an immediate result of the following properties of the Jost function $J_\ell(\lambda)$ (see [17]). $J_\ell(\lambda)$ is an entire function with an infinite set of roots (which are precisely the eigenvalues of $\mathfrak{L}_\ell$). All roots, except for maybe a finite number of them, are simple.

Now, we are in a position to formulate the main spectral result of the paper.

Theorem 2.2. *Assume that the linearly independent eigenvectors* $\{\mathcal{F}_n^\ell, \; n \in \mathbb{Z}\}$ *are selected in such a way that they are almost normalized* (*see Definition* 2.1). *Then the whole set of the root vectors (eigenvectors and associate vectors together) of* $\mathfrak{L}_\ell$ *forms a Riesz basis in the energy space* $\mathfrak{H}_\ell$.

An important corollary of Theorem 2.2 is the fact that the operators $\mathfrak{L}_\ell$ provide a class of nontrivial examples of spectral operators. While an abstract theory of spectral operators has been developed long ago [2], there is still a problem of finding specific examples of such operators.

To formulate the aforementioned corollary precisely, we recall the definition of a spectral operator. The definition we give below is a particular case of the general definition. However, it is sufficient for our purpose.

Definition 2.3. a) Let $\{\psi_n\}_{n=1}^\infty$ be a Riesz basis in a complex Hilbert space H. Denote by $\{\psi_n^*\}_{n=1}^\infty$ the unique biorthogonal basis defined by the relations: $(\psi_n, \psi_m^*) = \delta_{nm}$. Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of complex numbers. Define an operator S in H by:

$$S\varphi = \sum_{n=1}^\infty \lambda_n (\varphi, \psi_n^*)\psi_n, \; D(S) = \Big\{\varphi \in H : \sum_{n=1}^\infty |\lambda_n|^2 (\varphi, \psi_n^*)|^2 < \infty\Big\}. \tag{2.12}$$

The operators of the type (2.12) are called scalar operators.

b) An operator $\mathfrak{L}$ in H is called a spectral operator if it can be represented in the form:

$$\mathfrak{L} = S + N \tag{2.13}$$

where S is a scalar operator and N is a bounded finite rank nilpotent operator (i.e., there exists a positive integer k such that $N^k = 0$), which commutes with S.

(In a more general definition of a spectral operator [2], the spectral representation (2.12) may be continuous, i.e., it involves integration with respect to a spectral measure, and N may be a quasinilpotent operator, i.e., it is bounded and its spectrum $\mathfrak{S}(N) = \{0\}$).

Corollary 2.2. *All operators $\mathfrak{L}_\ell$ are spectral in the sense of Definition 2.3. The operators S_ℓ and N_ℓ from Definition* 2.3 (*we equipped them with a subindex* ℓ) *can be written in the form*

$$S_\ell \varphi = \sum_{n\in\mathbb{Z}\setminus R_\ell} \lambda_n^\ell (\varphi, \mathcal{F}_n^{\ell *})\mathcal{F}_n^\ell + \sum_{n\in R} \lambda_n^\ell \sum_{j=0}^{m_n^\ell - 1} (\varphi, \mathcal{F}_{n,j}^{\ell *})\mathcal{F}_{n,j}^\ell,$$

$$N_\ell \varphi = \sum_{n\in R_\ell} \sum_{j=1}^{m_{n-1}^\ell} (\varphi, \mathcal{F}_{n,j}^{\ell *})\mathcal{F}_{n,j-1}^\ell, \quad \varphi \in \mathcal{D}(\mathfrak{L}_\ell), \tag{2.14}$$

where $\mathcal{F}_{n,j}^\ell$ is an associate vector corresponding to the eigenvalue λ_n^ℓ and the asterisk denotes the corresponding vector from the biorthogonal basis.

In our next statement, we give the formulation (without proof) of the result on the R-basis property of the set of root vectors of the pencil $\mathcal{P}_\ell(\lambda)$. Let us extend all coefficients of the wave equation to the interval $(a, 2a]$ by the rule: $p(r) = \rho(r) = 1$, $q(r) = d(r) = 0$, $r \in (a, 2a]$. The following statement holds.

Theorem 2.3. *The set of root vectors of the pencil $\mathcal{P}_\ell(\lambda)$ forms an R-basis in the weighted space $L^2_\nu(0, 2a)$, $\nu(r) = r^2\rho(r)/p(r)$.*

The details of the proof of Theorem 2.3 for the case $\ell = 0$ can be found in [18]. Based on [18], the proof for the case $\ell > 0$ can be reconstructed.

Below we give the formulation of the results on nonharmonic exponentials from our paper [17] in the form convenient for the present paper.

These results will be used in Section 3 where we give the formulations of our controllability results.

Theorem 2.4. *Assume that the operator $\mathfrak{L}_\ell$ has a simple spectrum $\{\lambda_n^\ell\}_{n\in\mathbb{Z}}$, i.e., there are no associated vectors. Denote by $\mathcal{E}_h^\ell$ the closed linear span of the set of exponentials $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$ in the space $H = L^2(0, 2\mathcal{M})$. Then the set of nonharmonic exponentials $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$ forms a Riesz basis in $\mathcal{E}_h^\ell$. Moreover,*

$$\dim H(mod\ \mathcal{E}_h^\ell) = \begin{cases} \ell, & if\ \ |h| \neq \infty \quad and \quad h \neq 0, \\ \ell + 1, & if\ \ |h| = \infty \quad or \ \ h = 0. \end{cases} \tag{2.15}$$

In the conclusion of this section, we formulate a certain result from [3] which will be used in the proof of Theorem 2.2 given in Sections 4-6. Below, we formulate this result in the form convenient for this present work.

Theorem 2.5. (see [3]) *Let a dissipative operator B in a complex separable Hilbert space H (B is closed and Re $(Bf, f)_H \geq 0$ for all $f \in D(B) \subset H$) satisfy the following conditions:* a) *the resolvent of B is a meromorphic function on H all poles of which are simple except, maybe, for a finite number of them;* b) *B is a "full" operator, i.e., the whole space H is spanned by the root vectors of B (finite linear combinations of these vectors are dense in H);* c) *the set of eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ satisfies the Carleson condition* ([10, 11]):

$$\inf_{m\in\mathbb{Z}} \prod_{\substack{n\in\mathbb{Z}\\ n\neq m}} \left|\frac{\lambda_n - \bar{\lambda}_m}{\lambda_n - \lambda_m}\right| > 0. \tag{2.16}$$

Then the set of the root vectors of B forms an R-basis in H.

3. Exact controllability results.

In this section, we describe a series of control problems and then formulate their solutions in Theorems 3.1-3.3. These problems can be solved by the method of the spectral decomposition. To apply this method, we need all spectral results represented in Section 2. The goal of this section is to demonstrate immediate and effective applications of the Riesz spectral property of the dynamics generator $\mathfrak{L}_\ell$.

Let us consider the following nonhomogeneous radial damped wave equation

$$u_{tt} - L_\ell u + 2d(r)u_t = g(r)f(t) \tag{3.1}$$

with the boundary conditions

$$\lim_{r\to 0} ru(r,t) = 0, \quad (u_r + hu_t)(a,t) = \varphi(t), \quad h \in \mathbb{C} \text{ or } u(a,t) = \varphi(t). \quad (3.2)$$

In Eq. (3.1), L_ℓ is given in (1.9) and u obeys standard initial conditions (1.11). $g(r)$ is called a force profile function; the functions $f(t)$ and $\varphi(t)$ are called distributed and boundary controls,respectively.

We consider the following control problems.

A) *Zero controllability problem.* Let initial conditions (1.11) and $T > 0$ be given. Do there exist a distributed control $f \in L^2(0,T)$ and a boundary control $\varphi \in L^2(0,T)$ such that the solution of problem (3.1), (3.2) satisfies also an additional condition at $t = T$: $u(r,T) = u_t(r,T) = 0, \quad r \in [0,a]$?

B) *Closed loop control problem.* Do there exist controls f and φ such that the terminal state of the system at the moment $t = T$ is a multiple of its initial state (δ is a given number): $u(r,T) = \delta u(r,0), \quad u_t(r,T) = \delta u_t(r,T)$?

C) *Periodic control problem.* Let $T_1 > 0$ be given and let both control functions f and φ be switched on at the moment $t = T_1$. Do there exist f and φ such that the state of the system at $t = T_1 + T$ is equal to the initial state $u(r, T_1 + T) = u(r,0), \quad u_t(r, T_1 + T) = u_t(r,0)$?

We say that for a given initial state $U_0 = \begin{pmatrix} u_0^0 \\ u_1^0 \end{pmatrix}$, the system is controllable in time T if the desired control exists. If the control is also unique, we say that the system is uniquely controllable.

From this moment, we assume that the dynamics generator $\mathfrak{L}_\ell$ of our system has a simple spectrum, i.e., it does not have associate vectors. Similar assumptions occur in many control papers (see, i.e., [20, 14-16]). In the presence of associate vectors, the formulas for controls become significantly more complicated. For the sake of brevity in this work, we restrict ourselves to the case of a simple spectrum. However, the aforementioned assumption has been eliminated in [21], where we give complete proofs of all controllability results.

To answer questions A)-C), we study evolution problem (3.1), (3.2) in the energy space $\mathfrak{H}_\ell$ with the energy norm given by (1.14). Let us represent our initial boundary value problem (3.1), (3.2) in the form of the following operator equation in $\mathfrak{H}_\ell$ for the function $U = \begin{pmatrix} u \\ u_t \end{pmatrix} \equiv \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$:

$$U_t = i\mathfrak{L}_\ell U + \hat{G}, \quad U|_{t=0} = U_0 \in \mathfrak{H}_\ell, \quad (3.3)$$

where $\mathfrak{L}_\ell$ is given by (1.13), (1.15), and

$$U_0(r) = \begin{pmatrix} u_0^0(r) \\ u_1^0(r) \end{pmatrix}, \quad \hat{G}(r,t) = f(t)G(r), \quad G(r) = \begin{pmatrix} 0 \\ g(r) \end{pmatrix}. \tag{3.4}$$

The crucial fact about the dynamics generator $\mathfrak{L}_\ell$ (Theorem 2.2) is that $\mathfrak{L}_\ell$ is a Riesz spectral operator. We also need Theorem 2.4, (case a)) which says that the set of nonharmonic exponentials $\{e^{i\lambda_n^\ell t}\}_{n\in\mathbb{Z}}$ forms an R-basis in $\mathcal{E}_h^\ell \subset L^2(0, 2\mathcal{M})$, where $\mathcal{M}$ is given in (2.6). Having these results at our disposal, we are in a position to formulate the controllability results.

Assume that $U_0, G \in \mathfrak{H}_\ell$ have the following expansions with respect to R-basis (1.20):

$$U_0(r) = \sum_{n\in\mathbb{Z}} u_n^0 \mathcal{F}_n^\ell(r), \quad G(r) = \sum_{n\in\mathbb{Z}} g_n \mathcal{F}_n^\ell(r). \tag{3.5}$$

Theorem 3.1 (Problem A)**.** *Assume that the spectrum of the operator $\mathfrak{L}_\ell$ is simple (no associate functions).*

(i) *If $f = 0$, then the following statements hold.*

a) *Problem* (3.1), (3.2) *is exactly controllable on the time interval* $[0, 2\mathcal{M}]$ *through the boundary control φ for any initial state $U_0 \in \mathfrak{H}_\ell$. Let $\{w_n^\ell(t)\}_{n\in\mathbb{Z}}$ be the Riesz basis in $\mathcal{E}_h^\ell$, biorthogonal* [4] *to the basis $\{e^{i\bar{\lambda}_n^\ell t}\}_{n\in\mathbb{Z}}$, i.e., $\int_0^{2\mathcal{M}} e^{-i\lambda_m^\ell t} w_n^\ell(t)dt = \delta_{m,n}$. The desired boundary control function, which brings the system to zero state on the time interval* $[0, 2\mathcal{M}]$, *is uniquely defined by the formula*

$$\varphi(t) = \frac{2}{p(a)} \sum_{n\in\mathbb{Z}} u_n^0 w_n^\ell(t). \tag{3.6}$$

b) *If $T < 2\mathcal{M}$, then the system is not controllable in time T for an arbitrary initial condition $U_0 \in \mathfrak{H}_\ell$.*

c) *If $T > 2\mathcal{M}$, then the system is controllable in time T and our control problem has infinitely many solutions $\varphi \in L^2(0,T)$.*

(ii) *If $\varphi = 0$, then assume*

$$g_n \neq 0 \text{ for all } n \in \mathbb{Z}. \tag{3.7}$$

The following statements hold:

a) *For a given* $g \in L^2(0,a)$, *the system* (3.1), (3.2) *is controllable on the time interval* $[0, 2\mathcal{M}]$ *if and only if the initial state* $U_0 \in \mathfrak{H}_\ell^g$, *where* $\mathfrak{H}_\ell^g$ *is the dense subspace of* $\mathfrak{H}_\ell$ *defined by the condition*

$$\{\gamma_n = u_n^0 / g_n\}_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}), \text{ i.e., } \sum_{n\in\mathbb{Z}} |\gamma_n|^2 < \infty. \tag{3.8}$$

The desired control function $f(t)$ *is defined by the formula*

$$f(t) = -\sum_{n\in\mathbb{Z}} \gamma_n w_n^\ell(t), \tag{3.9}$$

where $\{\gamma_n\}_{n\in\mathbb{Z}}$ *is given by* (3.8) *and* $\{w_n^\ell(t)\}_{n\in\mathbb{Z}}$ *is the Riesz basis in* $\mathcal{E}_h^\ell$ *defined in* (i). *Claims* b) *and* c) *which follow* (3.6) *are valid in this case as well.*

Theorem 3.2 (Problem B**).** *If the terminal state of system* (3.1), (3.2), *is*

$$U(x,T) = \delta U_0(x), \tag{3.10}$$

then the solution of the closed loop problem is given by formulas similar to (3.6) *and* (3.9) *with some additional factors under the summation sign. Namely,* (i) *If* $f = 0$ *then the boundary control is given by*

$$\varphi(t) = \frac{2}{p(a)} \sum_{n\in\mathbb{Z}} (1 - \delta e^{-i\lambda_n^\ell T}) u_n^0 w_n^\ell(t), \quad T = 2\mathcal{M}. \tag{3.11}$$

(ii) *If* $\varphi = 0$ *and* (3.8) *is satisfied, then the bilinear control function is given by*

$$f(t) = -\sum_{n\in\mathbb{Z}} (1 - \delta e^{-i\lambda_n^\ell t} T) \gamma_n w_n^\ell(t), \quad T = 2\mathcal{M}. \tag{3.12}$$

We consider the periodic control problem only for the case $\varphi = 0$. The case of a boundary control can be done in a similar manner.

Theorem 3.3 (Problem C**).** *If the initial state* U_0 *satisfies* (3.8), *then for any* $T_1 > 0$ *there exists a control* f *which returns the system to the initial state on the time interval* $[0, T_1 + T]$ *with* $T = 2\mathcal{M}$; f *is defined by the formula*

$$f(T_1 + t) = -\sum_{n\in\mathbb{Z}} \gamma_n (e^{i\lambda_n^\ell T_1} - e^{-i\lambda_n^\ell T}) w_n^\ell(t), \quad t \geq 0. \tag{3.13}$$

(*Note that formula* (3.13) *makes sense for any* $T_1 \geq 0$. *Moreover, when* $T_1 = 0$ *we obtain* (3.12) *with* $\delta = 1$).

For the case when both $f \neq 0$ and $\varphi \neq 0$, we refer to our work [26] in which the damped string equation is considered. It turns out that in this case, the control time can be reduced to $T = \mathcal{M}$.

4. Auxiliary problem. Completeness of root vectors in energy space. As was already mentioned, Sections 4-6 are devoted to the proof of our main spectral result– Theorem 2.2. We split this proof into several steps. In Sections 4 and 5, we study an auxiliary problem. In Section 4, we prove the completeness for the set of root vectors of the dynamics generator corresponding to this auxiliary problem. In Section 5, we prove the Riesz basis property of this set of the root vectors. In Section 6, we prove Theorem 2.2 using the results on the auxiliary problem. To introduce this new problem, we make a certain change of variable and reduce problem (1.8)-(1.11) to some standard form.

Let us rewrite (1.8)-(1.11) for the function $w(r,t)$:

$$u(r,t) = \frac{1}{r\sqrt{p(r)}} w(r,t). \tag{4.1}$$

Instead of Eq. (1.8), we have the following equation for w:

$$w_{tt} = \frac{1}{\tau(r)}\big(w_{rr} - \frac{\ell(\ell+1)}{r^2} w\big) - 2d(r)w_t - \psi(r)w, \tag{4.2}$$

with

$$\tau(r) = \rho(r)/p(r), \tag{4.3}$$

$$\psi(r) \quad = \quad q(r) + \frac{p''(r)}{2r\sqrt{p(r)}} - \frac{(p'(r))^2}{4rp^{3/2}(r)} + \frac{p'(r)}{r^2\sqrt{p(r)}}. \tag{4.4}$$

$w(r,t)$ satisfies the following initial conditions:

$$w(r,0) = r\sqrt{p(r)}u_0(r) \equiv w_0^0(r), \quad w_t(r,0) = r\sqrt{p(r)}u_1(r) \equiv w_1^0(r), \tag{4.5}$$

and the boundary conditions

$$w(0,t) = 0, \tag{4.6}$$

$$(w_r + hw_t + \Gamma w)(a,t) = 0, \quad h \in \mathbb{C} \cup \{\infty\}, \tag{4.7}$$

$$\Gamma \equiv \Big[r\sqrt{p(r)}(d/dr)(r^{-1}p^{-\frac{1}{2}}(r))\Big]_{r=a}. \tag{4.8}$$

The problem defined by (4.2)–(4.8) is exactly the auxiliary problem which will be studied in Sections 4 and 5.

At this moment, we introduce an important assumption about $\psi(r)$. Namely, assume that

$$\psi(r) \geq 0. \tag{4.9}$$

Restriction (4.9) will be eliminated at the end of Section 6.

Initial boundary value problem (4.2) - (4.8) can be written in the form of the first order in time evolution system in the energy space $\mathcal{H}_\ell$, which is a Hilbert space of two-component Cauchy data

$$W(r,t) = \begin{pmatrix} w_0(r,t) \\ w_1(r,t) \end{pmatrix} = \begin{pmatrix} w(r,t) \\ w_t(r,t) \end{pmatrix}$$

with the following energy norm:

$$\|W\|^2_{\mathcal{H}_\ell} = \frac{1}{2}\int_0^a \left(|w_0'|^2 + \frac{\ell(\ell+1)}{r^2}|w_0|^2 + \psi(r)|w_0|^2 + \tau(r)|w_1|^2 \right) dr. \tag{4.10}$$

We note that due to the Hardy inequality $\int_0^a r^{-2}|w_0(r)|^2 dr \leq 4\int_0^a |w_0'(r)|^2 dr$, $w_0 \in H^1(0,a)$, $w_0(0) = 0$, all spaces $\mathcal{H}_\ell$ are metrically equivalent to $\mathcal{H}_0$. Problem (4.2) - (4.8) can be written as

$$W_t = i\mathfrak{L}^\ell_{h,\Gamma} W, \tag{4.11}$$

$$W|_{t=0} = \begin{pmatrix} w_0^0(r) \\ w_1^0(r) \end{pmatrix} \equiv W_0(r), \tag{4.12}$$

where the dynamics generator $\mathfrak{L}^\ell_{h,\Gamma}$ is the matrix differential operator

$$\mathfrak{L}^\ell_{h,\Gamma} = -i \begin{pmatrix} 0 & 1 \\ \mathcal{L}_\ell & -2d(r) \end{pmatrix}, \tag{4.13}$$

and $\mathcal{L}_\ell$ is the Sturm-Liouville operator defined on a smooth function φ by the formula

$$\mathcal{L}_\ell \varphi = \frac{1}{\tau(r)}\big(\varphi'' - \frac{\ell(\ell+1)}{r^2}\varphi\big) - \psi(r)\varphi. \tag{4.14}$$

The domain of the operator $\mathfrak{L}^\ell_{h,\Gamma}$ is given by:

$$D(\mathfrak{L}^\ell_{h,\Gamma}) = \big\{ W \in \mathcal{H}_\ell : w_0 \in H^2(0,a), \quad w_1 \in H^1(0,a), \tag{4.15}$$
$$w_1(0) = 0, \quad (w_0' + \Gamma w_0 + h w_1)(a) = 0 \big\}.$$

The operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ is our main object of interest in Sections 4 and 5. Our goal is to prove that $\mathfrak{L}^{\ell}_{h,\Gamma}$ is a Riesz spectral operator for any complex h, real Γ and any real function ψ. In Sections 4 and 5, this goal will be achieved only for nonnegative ψ. We remove this restriction in Section 6.

To formulate and prove the main result of this section (Theorem 4.2 below), we represent the statement which describes the spectrum of the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ with any $\Gamma \in \mathbb{C}$ and $h \in \mathbb{C}$. In the particular case, when Γ is given by (4.8), the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ is obtained from $\mathfrak{L}_{\ell}$ by the change of the dependent variable (4.1). Therefore, in this case, $\mathfrak{L}^{\ell}_{h,\Gamma}$ has the same spectrum as $\mathfrak{L}_{\ell}$, and all of the statements of Theorem 2.1 hold for this operator.

Theorem 4.1. *For any $\Gamma \in \mathbb{C}$, the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ has a countable set of complex eigenvalues $\{\lambda_n^{h,\Gamma}\}_{n\in\mathbb{Z}}$ which is located in a strip parallel to the real axis, and has only two points of accumulation: $+\infty$ and $-\infty$ in the sense that $Re\ \lambda_n^{\ell} \to \pm\infty$ as $n \to \pm\infty$ and $Im\ \lambda_n^{\ell} \to const$ as $n \to \pm\infty$ (see* (2.10)*). The asymptotics of the spectrum is given by the same formulas* (2.9), (2.10) *as for the operator $\mathfrak{L}_{\ell}$. (This means that both the leading term and the order of the remainder term in the asymptotics of $\lambda_n^{h,\Gamma}$ are the same as in* (2.9), (2.10). *However, the remainder term depends on Γ.) The asymptotic estimate of the remainder term $O(|n|^{-1}\ell n|n|)$ is uniform with respect to Γ if Γ runs over a compact subset of $\mathbb{C}$. Statement b) of Theorem* 2.1 *holds for the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ as well.*

We postpone the proof of this theorem to the end of the section. The proof will only be outlined, since it is similar to the proof of Theorem 2.1 given in [17].

Remark 4.1. In fact, we will need only the particular case of Theorem 4.1 corresponding to $Im\ \Gamma = 0$. However, the proof is the same for any $\Gamma \in \mathbb{C}$. So, it is natural and convenient to state and prove this theorem in full generality.

Theorem 4.2. *The set of root vectors of the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ is complete in $\mathcal{H}_{\ell}$ for any $h \in \mathbb{C} \cup \{\infty\}$ and $\Gamma \in \mathbb{R}$.*

Proof. First of all, let us split the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ into two parts: $\mathfrak{L}^{\ell}_{h,\Gamma} = \mathcal{A}_{h,\Gamma} + \mathcal{D}$, where

$$\mathcal{A}_{h,\Gamma} = -i \begin{pmatrix} 0 & 1 \\ \mathcal{L}_{\ell} & 0 \end{pmatrix}, \tag{4.16}$$

and $\mathcal{L}_\ell$ defined in (4.14). The domain of $\mathcal{A}_{h,\Gamma}$ coincides with $D(\mathfrak{L}^\ell_{h,\Gamma})$. $\mathcal{D}$ is a bounded operator in $\mathcal{H}_\ell$ given by

$$\mathcal{D} = -i \begin{pmatrix} 0 & 0 \\ 0 & -2d(r) \end{pmatrix}. \tag{4.17}$$

The operator $\mathcal{A}_{h,\Gamma}$ has been studied in our papers [22, 23]. (To be more precise, in [23] we obtain the results for the case $\ell = 0$. However, the extension to the case $\ell \geq 1$ can be done simply by repeating all of the steps completed for $\ell = 0$.) We need some results on this operator. To keep this paper self contained, we collect these results in the next proposition.

Theorem 4.3. *For any $h \in \mathbb{C} \cup \{\infty\}$ and $\Gamma \in \mathbb{C}$ the following statements hold.*

a) *$\mathcal{A}_{h,\Gamma}$ has purely discrete spectrum. Its resolvent is a meromorphic operator-valued function of λ, all poles of which are simple except, maybe, for a finite number of them of a finite algebraic multiplicity each.*

b) *Let λ be a regular point, i.e., $(\mathcal{A}_{h,\Gamma} - \lambda I)^{-1}$ exists and is bounded. Then $(\mathcal{A}_{h,\Gamma} - \lambda I)^{-1}$ is a compact operator of the class $\mathfrak{S}_2$. (We recall that $K \in \mathfrak{S}_2$ if the sequence of eigenvalues of the operator $(KK^*)^{1/2}$ belongs to ℓ^2).*

c) *The set of all root vectors of $\mathcal{A}_{h,\Gamma}$ forms a Riesz basis in $\mathcal{H}_\ell$.*

d) *For $h = \Gamma = 0$, the operator $\mathcal{A}^{-1}_{0,0}$ exists and belongs to $\mathfrak{S}_2$. If $h \neq 0$ and $\Gamma \neq 0$, then the operator $\mathcal{A}^{-1}_{h,\Gamma}$ is a rank two perturbation of $\mathcal{A}^{-1}_{0,0}$, i.e.,*

$$\mathcal{A}^{-1}_{h,\Gamma} = \mathcal{A}^{-1}_{0,0} + \mathcal{B}_{h,\Gamma}, \tag{4.18}$$

where $\mathcal{B}_{h,\Gamma}$ is a rank - two operator in $\mathcal{H}_\ell$. If one of the parameters h or Γ equal to zero, then the corresponding operator ($\mathcal{A}^{-1}_{0,\Gamma}$ or $\mathcal{A}^{-1}_{h,0}$) is a rank-one perturbation of $\mathcal{A}^{-1}_{0,0}$.

Now we can proceed with the proof of Theorem 4.2. First, let us check that zero does not belong to the spectrum of $\mathfrak{L}^\ell_{h,\Gamma}$. (This means that $\lambda = 0$ is not an eigenvalue since the spectrum is discrete.) Indeed, using contradiction argument, let us assume that there exists a nontrivial vector $G = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \in \mathcal{H}_\ell$ such that $\mathfrak{L}^\ell_{h,\Gamma} G = 0$. It is easy to see that the latter equation implies:

$$\frac{1}{\tau(r)}\big(g_0'' - \frac{\ell(\ell+1)}{r^2} g_0\big) - \psi(r) g_0 - 2d(r) g_1 = 0, \tag{4.19}$$

$$g_1(r) = 0, \quad g_0(0) = 0, \quad g_0'(a) + \Gamma g_0(a) + h g_1(a) = 0. \tag{4.20}$$

If we rewrite (4.19), (4.20) for the function $u_0 = \dfrac{1}{r\sqrt{p}} g_0$, then we arrive at the problem:

$$\frac{1}{p(r)}\Big(\frac{1}{r^2}(p(r)r^2u_0')' - \frac{\ell(\ell+1)}{r^2}u_0p(r)\Big) - q(r)u_0 = 0, \tag{4.21}$$

$$u_0(0) = 0, \qquad u_0'(a) = 0. \tag{4.22}$$

Multiplying (4.21) by $\bar{u}_0$, integrating by parts and using (4.22), we obtain:

$$\int_0^a \big[p(r)r^2|u_0|^2 + \ell(\ell+1)p(r)|u_0|^2 + q(r)\rho(r)r^2|u_0|^2\big]^2\,dr = 0.$$

From the latter equation, we immediately get $u_0 = 0$ and, thus, $g_0 = g_1 = 0$.

Now we can use Statement b) of Theorem 4.3 and have:

$$\mathfrak{L}^{\ell}_{h,\Gamma} = (I + \mathcal{D}\mathcal{A}^{-1}_{h,\Gamma})\mathcal{A}_{h,\Gamma}, \quad \mathcal{D}\mathcal{A}^{-1}_{h,\Gamma} \in \mathfrak{S}_2(\mathcal{H}_\ell).$$

Due to the fact that zero is not an eigenvalue of $\mathfrak{L}^{\ell}_{h,\Gamma}$, we obtain that (-1) is not an eigenvalue of $\mathcal{D}\mathcal{A}_{h,\Gamma}{}^{-1}$. Therefore, we have

$$(\mathfrak{L}^{\ell}_{h,\Gamma})^{-1} = \mathcal{A}^{-1}_{h,\Gamma}(I + S), \qquad S = -V(I + V)^{-1}$$

with $V = \mathcal{D}\mathcal{A}_{h,\Gamma}{}^{-1}$, $S \in \mathfrak{S}_2(\mathcal{H}_\ell)$. Based on Statement d) of Theorem 4.3, we have

$$(\mathfrak{L}^{\ell}_{h,\Gamma})^{-1} = (\mathcal{A}^{-1}_{0,0} + \mathcal{B}_{h,\Gamma})(I + S) = (I + \mathcal{B}_{h,\Gamma}\mathcal{A}_{0,0})\mathcal{A}^{-1}_{0,0}(I + S).$$

Due to the fact that $\mathcal{A}^{-1}_{0,0}$ is selfadjoint, we can apply the following version of the Keldysh Theorem (see [8, 4]).

Theorem 4.4. *Let A be an operator in a Hilbert space H such that $A = (I + S_1)B(I + S_2)$, where $B = B^* \in \mathfrak{S}_2(H)$ and S_1, S_2 are compact. If the operator A vanishes only on zero vector, then the system of root vectors of A is complete in H.*

If we identify $(\mathfrak{L}^{\ell}_{h,\Gamma})^{-1}$, $\mathcal{A}^{-1}_{0,0}$, $\mathcal{B}_{h,\Gamma}\ \mathcal{A}_{0,0}$, S with A, B, S_1 and S_2 respectively, then from Theorem 4.4, we obtain the desired completeness of root vectors of the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$. Theorem 4.2 is shown.

Proof of Theorem 4.1. 1) First, we need some information from [17] about the asymptotic properties of the solutions of the initial value problem (4.2)-(4.7) in which Γ is an arbitrary complex number. Let us recall the corresponding spectral problem. We look for nontrivial solutions of the following boundary problem:

$$w'' - \frac{\ell(\ell+1)}{r^2}w + \lambda^2\tau(r)w - 2i\lambda d(r)\tau(r)w - \psi(r)\tau(r)w = 0, \tag{4.23}$$

$$w(0) = 0, \quad (w' + i\lambda hw + \Gamma w)(a) = 0. \tag{4.24}$$

As was shown in [17], Eq. (4.23) has two linearly independent solutions denoted by $F_\ell^j(\lambda, r), j = 1, 2$. These solutions are asymptotically close to the Hankel functions of the first and second kinds respectively. More precisely, by Theorem 3.2 from [17], each solution is an analytic function of λ for every $r \in (0, a]$ with one possible pole of order ℓ at $\lambda = 0$. In addition to two Hankel-like solutions, Eq. (4.23) has a solution $\Phi_\ell(\lambda, r)$ (Bessel-like solution) which is an entire function of λ for each $r \in [0, a]$; when $r \to 0$, this solution behaves as $(\lambda r)^{\ell+1}$. For sufficiently large $|\lambda|$, the following estimates are valid:

$$\begin{aligned} &\left|[F_\ell^1(\lambda, \omega(\lambda, r)) - \sqrt{i\omega(\lambda, r)}H^1_{\ell+1/2}(i\omega(\lambda, r))]e^{\omega(\lambda,r)}\right| \\ &\qquad \le C_\ell \Delta_\ell(\omega(\lambda, r))|\lambda|^{-1}\ell n|\lambda|, \end{aligned} \tag{4.25}$$

$$\begin{aligned} &\left|[F_\ell^2(\lambda, \omega(\lambda, r)) - \sqrt{i\omega(\lambda, r)}H^2_{\ell+1/2}(i\omega(\lambda, r))]e^{-\omega(\lambda,r)}\right| \\ &\qquad \le C_\ell \Delta_\ell(\omega(\lambda, r))|\lambda|^{-1}\ell n|\lambda|, \end{aligned} \tag{4.26}$$

$$\begin{aligned} &\left|[\Phi_\ell(\lambda, \omega(\lambda, r)) - \sqrt{i\omega(\lambda, r)}J_{\ell+1/2}(i\omega(\lambda, r))]e^{\pm\omega(\lambda,r)}\right| \\ &\qquad \le C_\ell (\Delta_{\ell+1}(\omega(\lambda, r)))^{-1}|\lambda|^{-1}\ell n|\lambda|, \end{aligned} \tag{4.27}$$

where $\omega(\lambda, r) = i\lambda\int_0^r \sqrt{\rho(x)/p(x)}dx - \int_0^r d(x)\sqrt{\rho(x)/p(x)}dx$; $\Delta_\nu(z) = (1 + |z|^\nu)|z|^{-\nu}$. In (4.27), "+" is taken for $Im\ \lambda \le 0$ and "−" for $Im\ \lambda > 0$. In (4.25) and (4.26), $H^j_{\ell+1/2}(z)$, $j = 1, 2$, are the Hankel functions of the first and second kinds respectively; in (4.27), $J_{\ell+1/2}(z)$ is the Bessel function.

We call the solution of Eq. (4.23), satisfying only one boundary condition at $r = a$, the generalized Jost solution and denote it by $\mathcal{J}^\ell_{h,\Gamma}(\lambda, r)$. (Recall that for $h = 0$, $\ell = 0$ and $\Gamma = 0$, this Jost solution is well known in Quantum Mechanics [9]). Let us look for the generalized Jost solution in the form:

$$\mathcal{J}^\ell_{h,\Gamma}(\lambda, r) = \mathcal{A}^\ell_{h,\Gamma}(\lambda)F_\ell^1(\lambda, r) + \mathcal{B}^\ell_{h,\Gamma}(\lambda)F_\ell^2(\lambda, r). \tag{4.28}$$

Assuming that $h \ne 0$ and normalizing this function by the condition:

$$\mathcal{J}^\ell_{h,\Gamma}(\lambda, a) = h^{-1}, \tag{4.29}$$

we arrive at the following linear system for the coefficients:

$$\mathcal{A}^{\ell}_{h,\Gamma}(\lambda)F^1_{\ell}(\lambda,a)+\mathcal{B}^{\ell}_{h,\Gamma}(\lambda)F^2_{\ell}(\lambda,a)=h^{-1}, \tag{4.30}$$

$$\begin{aligned}[\mathcal{A}^{\ell}_{h,\Gamma}(\lambda)(d/dr)F^1_{\ell}(\lambda,r)+\mathcal{B}^{\ell}_{h,\Gamma}(\lambda)(d/dr)F^2_{\ell}(\lambda,r)]_{r=a}&\\ =-i\lambda+\Gamma h^{-1}=-i\lambda(1+O(|\lambda|^{-1})).&\end{aligned} \tag{4.31}$$

Direct comparison (4.30), (4.31) with system (4.6), (4.7) from [17] shows that the only difference is the fact that in the right of Eq. (4.31), there is $(-i\lambda(1+O(|\lambda|^{-1})))$ instead of simply $(-i\lambda)$. Note that if Γ belongs to a bounded region of the complex plane, the estimate $O(|\lambda|^{-1})$ is uniform with respect to Γ. Arguing exactly as we did in the proof of Theorem 4.1 of [17], we arrive at the following expressions for the coefficients: if $h\neq 0$, then

$$\begin{aligned}\mathcal{A}^{\ell}_{h,\Gamma}(\lambda)&=K[h^{-1}\sqrt{\rho(a)/p(a)}-1]H^2_{\ell+1/2}(\omega(\lambda,a))(1+O(|\lambda|^{-1}\ell n|\lambda|)),\\ \mathcal{B}^{\ell}_{h,\Gamma}(\lambda)&=K[h^{-1}\sqrt{\rho(a)/p(a)}+1]H^1_{\ell+1/2}(\omega(\lambda,a))(1+O(|\lambda|^{-1}\ell n|\lambda|)).\end{aligned} \tag{4.32}$$

In both formulas (4.32), K is an absolute constant the exact value of which is immaterial for us; $\omega(\lambda,a)=i\lambda\int_0^a\sqrt{\rho(x)/p(x)}dx-\int_0^a d(x)dx$. If $h=0$, then $\mathcal{J}^{\ell}_{0,\Gamma}(\lambda,r)=\frac{d}{dr}\Phi_{\ell}(\lambda,r)$. If $h=\infty$, then $\mathcal{J}^{\ell}_{\infty,\Gamma}(\lambda,r)=\Phi_{\ell}(\lambda,r)$.

All asymptotical estimates in system (4.30), (4.31) are uniform with respect to Γ when Γ runs over a compact subset of $\mathbb{C}$. Therefore, formulas (4.32) for the coefficients $\mathcal{A}^{\ell}_{h,\Gamma}$ and $\mathcal{B}^{\ell}_{h,\Gamma}$ have the estimates $(O(|\lambda|^{-1}\ell n|\lambda|)$ which are uniform with respect to Γ. The latter fact means that the dependence on Γ would appear in the higher order terms of asymptotical representations, but we do not need them for the purpose of the present paper.

2) Arguing precisely as we did in the proof of Theorem 2.1 from [17], we obtain the following result. The generalized Jost solution $\mathcal{J}^{\ell}_{h,\Gamma}(\lambda,r)$ is an entire function of the parameter λ for each $r\in(0,a]$. It is clear that the same is valid for the generalized Jost function $\mathcal{J}^{\ell}_{h,\Gamma}(\lambda)$:

$$\mathcal{J}^{\ell}_{h,\Gamma}(\lambda)=\lim_{r\to 0}r^{\ell}\mathcal{J}^{\ell}_{h,\Gamma}(\lambda,r). \tag{4.33}$$

The equation for eigenvalues: $\mathcal{J}^{\ell}_{h,\Gamma}(\lambda)=0$ has a countable set of roots $\{\lambda^{\ell,\Gamma}_n\}_{n\in\mathbb{Z}}$ whose asymptotics coincides with (2.10). The dependence on Γ is hidden in the lower order terms having the asymptotics $O(|n|^{-1}\ell n|n|)$. The latter estimates are uniform with respect to Γ as long as Γ belongs to a compact set of the complex plane. The proof is complete.

5. Auxiliary problem. Riesz basis property of root vectors. In this section, we prove the Riesz basis property of root vectors of the operator $\mathfrak{L}^\ell_{h,\Gamma}$ under condition (4.9). We do this in two steps: first, we show that each operator from the one-parameter family $\mathfrak{L}^\ell_{h,0}(\Gamma = 0)$ has a set of root vectors which is a Riesz basis in $\mathcal{H}_\ell$, and then we extend this result to all $\Gamma \in \mathbb{R}$.

Theorem 5.1. *For any $h \in \mathbb{C} \cup \{\infty\}$ and $\Gamma = 0$, the set of root vectors of the operator $\mathfrak{L}^\ell_{h,0}$ forms a Riesz basis in $\mathcal{H}_\ell$ if condition* (4.9) *is satisfied.*

Proof. The main tool in the proof of this theorem is Theorem 2.5. According to the latter theorem and Theorem 4.2, one has to verify two facts: first, the set of eigenvalues satisfies the Carleson condition (2.16) and, secondly, the operator $\mathfrak{L}^\ell_{h,0}$ is dissipative.

To verify the first fact, we note (see Corollary 2.1) that, the eigenvalues of $\mathfrak{L}^\ell_{h,0}$ are separated and such that $|\mathcal{I}m\lambda^\ell_n| \le C < \infty$. For this type of sets of complex points, condition (2.16) can be verified by a straightforward calculation (see [17] and, also, [10, Lecture IX, n.2] or [11]). So, the only fact, remaining to show, is the dissipativity of $\mathfrak{L}^\ell_{h,0}$. To do this, we consider two cases: $Re\ h \ge 0$ and $Re\ h < 0$.

a) $Re\ h \ge 0$. To claim that $\mathfrak{L}^\ell_{h,0}$ is dissipative, we must show that $Im(\mathfrak{L}^\ell_{h,0}F, F)_{\mathcal{H}_\ell} \ge 0$ for $F \in D(\mathfrak{L}^\ell_{h,0})$. If $F = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}$, then

$$\mathcal{I}m(\mathfrak{L}^\ell_{h,0}F, F)_{\mathcal{H}_\ell} = \frac{1}{2}\mathcal{I}m \int_0^a \Big(- i f_1' \bar{f}_0' - i\frac{\ell(\ell+1)}{r^2} f_1 \bar{f}_0 - i\psi(r) f_1 \bar{f}_0 \tag{5.1}$$
$$- i(\mathcal{L}_\ell f_0)\bar{f}_1 \tau(r) + 2id(r)|f_1(r)|^2\Big) dr.$$

Taking into account expression (4.14) for $\mathcal{L}_\ell$ and the boundary condition at $r = a$, we arrive at the desired result:

$$\mathcal{I}m(\mathfrak{L}^\ell_{h,0}F, F)_{\mathcal{H}_\ell} = \tfrac{1}{2}\mathcal{I}m\, i\int_0^a \Big\{ - (\bar{f}_0' f_1' + \bar{f}_1' f_0') + (\psi(r) + \frac{\ell(\ell+1)}{r^2})(f_0\bar{f}_1 - \bar{f}_0 f_1)$$
$$+ 2d(r)|f_1(r)|^2\Big\} dr + Re\ h|f_1(a)|^2 = 2\int_0^a d(r)|f_1(r)|^2 dr + Re\ h|f_1(a)|^2 \ge 0. \tag{5.2}$$

b) $Re\ h < 0$. Let us consider the following "shifted" operator

$$\mathcal{T}_\ell = \mathfrak{L}^\ell_{h,0} - C_0 I, \quad \text{with} \ \ C_0 = 2 \max_{r\in[0,a]} d(r), \tag{5.3}$$

where I is the identity operator. Clearly, the sets of root vectors of $\mathfrak{L}^{\ell}_{h,0}$ and $\mathcal{T}_\ell$ coincide. We have

$$\begin{aligned}\mathcal{I}m(\mathcal{T}_\ell F, F)_{\mathcal{H}_\ell} &= 2\int_0^a d(r)|f_1(r)|^2 dr + Re\ h|f_1(a)|^2 - C_0\|F\|^2_{\mathcal{H}_\ell}\\ &\le Re\ h|f_1(a)|^2 \le 0.\end{aligned}$$

Therefore, the set of root vectors of the operator $(-\mathcal{T}_\ell)$ (as well as $\mathfrak{L}^{\ell}_{h,0}$) is a Riesz basis in $\mathcal{H}_\ell$ for $Re\ h < 0$. Theorem is completely shown.

Remark 5.1. The above proof does not work if $\Gamma \neq 0$. The reason is the fact that for the operator $\mathcal{L}^{\ell}_{h,\Gamma}(\Gamma \neq 0)$, we cannot show the dissipativity. Indeed, in this case in formula (5.2), we have an additional term $\Gamma f_0(a)\overline{f_1(a)}$ which destroys the fact that the imaginary part of the corresponding quadratic form is nonnegative.

To extend the results of Theorem 5.1 to arbitrary $\Gamma \in \mathbb{R}$, we need Lemma 5.1 below. In fact, this lemma is a corollary of Theorem 4.1. Now, we give only its formulation; the proof is given after the proof of Theorem 5.2.

Lemma 5.1. *Let $\Gamma \in \Omega \subset \mathbb{R}$ and Ω be a compact subset of $\mathbb{R}$ containing zero. Let $\{\mathcal{F}^{\ell,\Gamma}_n\}_{n\in\mathbb{Z}}$ be the set of root vectors of the operator $\mathcal{L}^{\ell}_{h,\Gamma}$. There exists an absolute constant $\mathfrak{C}$ such that for all $n \in \mathbb{Z}$ and $\Gamma \in \Omega$, the following relation holds:*

$$\|\mathcal{F}^{\ell,\Gamma}_n - \mathcal{F}^{\ell,0}_n\|_{\mathcal{H}_\ell} \le \mathfrak{C}\frac{\ell n(|n|+2)}{|n|+1}, \tag{5.4}$$

where $\mathfrak{C} = \mathfrak{C}(\Gamma)$. (In fact, $\mathfrak{C}(\Gamma)$ is a continuous function of $\Gamma \in \mathbb{C}$.)

Now we are in a position to give our main result of this section.

Theorem 5.2. *The set of the root vectors of the operator $\mathfrak{L}^{\ell}_{h,\Gamma}$ in the space $\mathcal{H}_\ell$ forms a Riesz basis for any real Γ.*

Proof. In the proof of this theorem, we use Lemma 5.1 and the following well known theorem by N.K Bari [4].

Theorem (N.K. Bari). *Any ω-linearly independent sequence of vectors in a Hilbert space quadratically close to an R-basis is an R-basis itself.*

We recall that a system of vectors is ω-linearly independent if the equality $\sum_{n\in\mathbb{Z}} c_n\psi_n = 0$ is impossible when

$$0 < \sum_{n\in\mathbb{Z}} |c_n|^2\|\psi_n\|^2 < \infty. \tag{5.5}$$

We say that two systems of vectors are quadrically close if the following relation takes place:

$$\sum_{n\in\mathbb{Z}} \|f_n - \psi_n\|^2 < \infty.$$

The fact that the systems of root vectors $\{\mathcal{F}_n^{\ell,\Gamma}\}_{n\in\mathbb{Z}}$ and $\{\mathcal{F}_n^{\ell,0}\}_{n\in\mathbb{Z}}$ are quadratically close is guaranteed by Lemma 5.1 (it follows immediately from (5.4)). Therefore, the only fact, remaining to show, is the ω-linear independence of the sequence $\{\mathcal{F}_n^{\ell,\Gamma}\}_{n\in\mathbb{Z}}$. Let us do this.

Having Lemma 5.1 at our disposal, we prove that there exists a compact (actually Hilbert-Schmidt) operator V such that the operator $(I+V)$ has inverse and both systems of root vectors are related to each other by the rule:

$$\mathcal{F}_n^{\ell,\Gamma} = (I+V)\mathcal{F}_n^{\ell,0}, \quad n \in \mathbb{Z}. \tag{5.6}$$

The latter fact together with the Riesz basis property of the vectors $\{\mathcal{F}_n^{\ell,0}\}_{n\in\mathbb{Z}}$ leads to the desired result of the theorem.

Let us consider the following linear mapping defined on finite linear combinations of the elements of the Riesz basis $\{\mathcal{F}_n^{\ell,0}\}_{n\in\mathbb{Z}}$ by the rule:

$$\sum a_n \mathcal{F}_n^{\ell,0} \quad \stackrel{V}{\longrightarrow} \quad \sum a_n(\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}). \tag{5.7}$$

We have:

$$\|V\sum a_n\mathcal{F}_n^{\ell,0}\|_{\mathcal{H}_\ell} = \|\sum a_n(\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0})\|_{\mathcal{H}_\ell} \leq$$

$$\sqrt{\sum |a_n|^2}\sqrt{\sum \|\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}\|^2_{\mathcal{H}_\ell}} \leq \tilde{K}\|\sum a_n\mathcal{F}_n^{\ell,0}\|_{\mathcal{H}_\ell}, \tag{5.8}$$

where $\tilde{K}$ is an absolute constant.

On the last step, we have used the facts that $\{\mathcal{F}_n^{\ell,0}\}_{n\in\mathbb{Z}}$ is a Riesz basis in $\mathcal{H}_\ell$ and $\{\|\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}\|_{\mathcal{H}_\ell}\}_{n\in\mathbb{Z}} \in \ell^2$ due to (5.4). (5.8) obviously means that V is bounded. We extend this operator by continuity to the whole space $\mathcal{H}_\ell$. Now, we prove that $V \in \mathfrak{S}_2$ ($\mathfrak{S}_2$-class of Hilbert-Schmidt operators). Let $\left\{G_s = \sum_{n\in\mathbb{Z}} a_n^s \mathcal{F}_n^{\ell,0}\right\}_{s\in\mathbb{N}}$ be a sequence of vectors in $\mathcal{H}_\ell$ weakly converging to zero:

$$G_s \rightharpoonup 0 \quad \text{when} \quad s \longrightarrow \infty. \tag{5.9}$$

We show that the sequence $\{VG_s\}_{s\in\mathbb{N}}$ converges to zero strongly:

$$VG_s \longrightarrow 0 \quad \text{when} \quad s \longrightarrow \infty. \tag{5.10}$$

From (5.9), we have the following properties of the coefficients $\{a_n^s\}_{n\in\mathbb{Z}}$:

a) $\sum_{n\in\mathbb{Z}}|a_n^s|^2 \le K_1$, K_1 does not depend on s;

b) $a_n^s \to 0$ as $s \to \infty$ for each $n \in \mathbb{Z}$.

Now we prove (5.10). We have

$$\|VG_s\|_{\mathcal{H}_\ell} \le \Big\|\sum_{|n|\le N_0} a_n^s(\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0})\Big\|_{\mathcal{H}_\ell} + \Big\|\sum_{|n|\ge N_0+1} a_n^s(\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0})\Big\|_{\mathcal{H}_\ell}. \tag{5.11}$$

Let us fix $\epsilon > 0$. Since $\{\|\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}\|_{\mathcal{H}_\ell}\}_{n\in\mathbb{Z}} \in \ell^2$, there exists a number M such that

$$\sum_{|n|\ge M} \|\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}\|^2_{\mathcal{H}_\ell} \le \frac{\epsilon^2}{4K_1}. \tag{5.12}$$

Due to property a), the second sum in (5.11) with $N_0 \ge M$ can be estimated as

$$\Big\|\sum_{|n|\ge N_0+1} a_n^s(\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0})\Big\|_{\mathcal{H}_\ell} \le \sqrt{\sum_{n\in\mathbb{Z}}|a_n^s|^2}\sqrt{\sum_{|n|\ge N_0+1}\|\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}\|^2_{\mathcal{H}_\ell}} \le \frac{\epsilon}{2}. \tag{5.13}$$

The first sum in (5.11) contains $(2N_0+1)$ elements. Due to property b), there exists a number S such that the following estimate holds:

$$|a_n^s| \le \frac{\epsilon}{2\sqrt{2N_0+1}\;\;K_2}, \quad s \ge S, \tag{5.14}$$

where $K_2 = \sqrt{\sum_{n\in\mathbb{Z}}\|\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0}\|^2_{\mathcal{H}_\ell}}$. Therefore, for all $s \ge S$, we have

$$\Big\|\sum_{|n|\le N_0} a_n^s(\mathcal{F}_n^{\ell,\Gamma} - \mathcal{F}_n^{\ell,0})\Big\|_{\mathcal{H}_\ell} \le K_2\sqrt{\sum_{|n|\le N_0}|a_n^s|^2} \le \frac{\epsilon}{2}. \tag{5.15}$$

Collecting together (5.13) and (5.15), we obtain (5.10), which means that V is compact. To show that $V \in \mathfrak{S}_2$, it suffices to verify that $\sum_{n\in\mathbb{Z}}\|V\mathcal{F}_n^{\ell,0}\|^2_{\mathcal{H}_\ell} < \infty$. The latter fact is obviously true.

Now, the fact that the system $\{\mathcal{F}_n^{\ell,\Gamma}\}_{n\in\mathbb{Z}}$ is ω-linearly independent can be shown without any difficulties. Assume that there exists a sequence of complex numbers $\{c_n\}$ such that

$$\sum c_n \mathcal{F}_n^{\ell,\Gamma} = 0. \tag{5.16}$$

We will show that $c_n = 0, n \in \mathbb{Z}$. From (5.16), it follows that the vector $\sum c_n \mathcal{F}_n^{\ell,\Gamma}$ must belong to $Ker(I+V)$. We prove now that $Ker(I+V) = \{0\}$. Indeed, from the fact that the set $\{\mathcal{F}_n^{\ell,\Gamma}\}_{n\in\mathbb{Z}}$ is complete in $\mathcal{H}_\ell$ and V is compact, we have: $Range\ (I+V) = \mathcal{H}_\ell$. The latter fact gives: $Ker(I+V^*) = \{0\}$, which means that (-1) is not an eigenvalue of V^*. Thus, (-1) is not an eigenvalue of V as well, which shows that $(I+V)^{-1}$ exists. The latter fact immediately implies that $c_n = 0$, $n \in \mathbb{Z}$. This completes the proof of Theorem 5.2.

Proof of Lemma 5.1. Using (4.32) and (4.33) from Theorem 4.1 of the present paper and (4.27) and (4.28) from [17], we easily show that the eigenvalue equation can be represented in the following asymptotical form:

$$e^{2(i\lambda\mathcal{M}+\mathcal{N})} = (-1)^\ell \quad \frac{h^{-1}\sqrt{\rho(a)/p(a)} - 1}{h^{-1}\sqrt{\rho(a)/p(a)} + 1}\left(1 + O\left(\frac{\ell n|\lambda|}{|\lambda|}\right)\right), \tag{5.17}$$

where the asymptotical estimate for the remainder term does not depend on Γ, when Γ runs over a compact subset of $\mathbb{C}$; $\mathcal{M}$ and $\mathcal{N}$ are given in (2.6) and (2.7) respectively. We note that a very detailed derivation of formula (5.17) can be found in [17].

Based on the Rouchet Theorem, we claim that the following result is valid. There exists a number N_0 such that for each Γ from the aforementioned region, part of the spectrum $\{\lambda_n^{h,\Gamma}\}_{|n|\geq N_0}$ is simple and located in a small vicinity of the roots $\{\lambda_n^0\}_{n\geq N_0}$ of the equation

$$e^{2(i\lambda\mathcal{M}+\mathcal{N})} = (-1)^\ell \left[h^{-1}\sqrt{\rho(a)/p(a)} - 1\right]\left[h^{-1}\sqrt{\rho(a)/p(a)} + 1\right]^{-1}. \tag{5.18}$$

Taking into account the estimates for the remainder term in (5.17), we obtain:

$$|\lambda_n^{h,\Gamma} - \lambda_n^0| \leq \mathfrak{C} \quad \frac{\ell n|n|}{|n|}, \quad |n| \geq N_0, \quad \mathfrak{C}\text{-absolute constant.} \tag{5.19}$$

Therefore, from (5.19), we have

$$|\lambda_n^{h,\Gamma} - \lambda_n^{h,0}| \leq \mathfrak{C} \quad \frac{\ell n|n|}{|n|}, \quad |n| \geq N_0. \tag{5.20}$$

(5.20) means that for $|n| \geq N_0$, there exists the one-to-one correspondence between the sets $\{\lambda_n^{h,\Gamma}\}_{|n|\geq N_0}$ and $\{\lambda_n^{h,0}\}_{|n|\geq N_0}$.

Now, we discuss the correspondence between finite sets $\{\lambda_n^{h,\Gamma}\}_{|n|\leq N_0-1}$ and $\{\lambda_n^{h,0}\}_{|n|\leq N_0-1}$. We know that the whole spectrum $\{\lambda_n^{h,\Gamma}\}_{n\in\mathbb{Z}}$ lives in the strip parallel to the real axis. As follows from [17], the generalized Jost function $\mathcal{J}_{h,\Gamma}^{\ell}(\lambda)$ (see (4.33)) for sufficiently large $|\lambda|$ can be represented in the form

$$\mathcal{J}_{h,\Gamma}^{\ell}(\lambda) = \tag{5.21}$$
$$C_1 \sum_{j=0}^{1} [h^{-1}\sqrt{\rho(a)/p(a)} - (-1)^j](-1)^j H_{\ell+1/2}^{2-j}(\lambda\mathcal{M} - i\mathcal{N})\big(1 + O\big(\frac{\ell n|\lambda|}{|\lambda|}\big)\big),$$

where C_1 is an absolute constant and $H_{\ell+1/2}^{j}(z),\ j = 1, 2$, are the Hankel functions. From (5.21), it follows that we can find two horizontal lines L_1 and L_2 such that

$$|\mathcal{J}_{h,\Gamma}^{\ell}(\lambda) - \mathcal{J}_{h,0}^{\ell}(\lambda)| \leq \mathfrak{C}\frac{\ell n|\lambda|}{|\lambda|} \text{ and } |\mathcal{J}_{h,\Gamma}^{\ell}(\lambda)| \geq C_2 > 0,\ \lambda \in L_j,\ j = 1, 2. \tag{5.22}$$

C_2 is an absolute constant. In the second estimate (5.22), we have used the properties of the Hankel functions.

Let $V_j, j = 1, 2$, be two vertical lines on the complex λ-plane given by the equations

$$Re\ \lambda = 1/2Re\ (\lambda_{N_0} + \lambda_{N_0+1}),\quad Re\ \lambda = 1/2Re\ (\lambda_{-N_0} + \lambda_{-N_0-1}). \tag{5.23}$$

It can be easily verified that the estimates similar to (5.22) are valid on $V_j,\ j = 1, 2$, as well. If R is a rectangle bounded by the lines L_j and $V_j, j = 1, 2$, then the Rouchet Theorem can be applied to the generalized Jost functions $\mathcal{J}_{h,\Gamma}^{\ell}(\lambda)$ and $\mathcal{J}_{h,0}^{\ell}(\lambda)$ when λ changes on R. The result of this application is the following: both functions $\mathcal{J}_{h,\Gamma}^{\ell}(\lambda)$ and $\mathcal{J}_{h,0}^{\ell}(\lambda)$ have in R the same number of roots counting their multiplicities. The latter exactly means that both operators $\mathcal{L}_{h,\Gamma}^{\ell}$ and $\mathcal{L}_{h,0}^{\ell}$ have the same number of root vectors corresponding to those eigenvalues (simple and multiple) $\{\lambda_n^{h,\Gamma}\}$ and $\{\lambda_n^{h,0}\}$ that occur in the rectangle R. Therefore, the one-to-one correspondence between these finite dimensional root subspaces can be established without difficulties.

Collecting together all of the above, we obtain that the following relation holds:

$$|\lambda_n^{h,\Gamma} - \lambda_n^{h,0}| \le \mathfrak{C} \quad \frac{\ell n(|n|+2)}{|n|+1}, \quad n \in \mathbb{Z}. \tag{5.24}$$

The enumeration in (5.24) is absolute, but not asymptotical.

Now, we are in a position to prove formula (5.4). In fact, this result is an immediate corollary of Theorem 4.1 and (5.24), i.e., formulas (4.25), (4.26) and (4.32). Since all of the remainder terms in these formulas are estimated uniformly with respect to Γ (recall Γ is from a compact subset of $\mathbb{C}$), and the one-to-one correspondence follows from (5.24), the result of Lemma 5.1 is now obvious. Lemma is shown.

6. Riesz basis property of root vectors of main operator. Based on Theorem 5.2 on the auxiliary operator $\mathfrak{L}_{h,\Gamma}^{\ell}$, we prove our main result, Theorem 2.2. This result does not follow from Theorem 5.2 immediately. Recall that auxiliary problem (4.2), (4.6)-(4.8) was obtained from the original problem (1.9)-(1.12) by change of variables (4.1). However, the new operator $\mathcal{L}_{h,\Gamma}^{\ell}$ is considered in the space $\mathcal{H}_\ell$ with metric (4.10) which is totally different from original metric (1.15). (4.10) is not even positively definite unless condition (4.9) is satisfied. Furthermore, if this condition holds, as was assumed in Section 5, the result still requires some proof. Again, we will use a two-step procedure: first we prove the result for the case $\psi(r) \ge 0$ and then extend it to the function $\psi(x)$ of any sign.

Proof of Theorem 2.2. a) Case $\psi(r) \ge 0$. Our main tool in this proof is Theorem 2.5. We start with the verification of the fact that the set of root vectors of $\mathfrak{L}_\ell$ is complete in $\mathfrak{H}_\ell$. Using the contradiction argument, let us assume that there exists a nontrivial vector $G = \begin{pmatrix} g_0 \\ g_1 \end{pmatrix} \in \mathfrak{H}_\ell$ which is orthogonal to $\{\mathcal{F}_n^\ell\}_{n\in\mathbb{Z}}$, where $\{\mathcal{F}_n^\ell\}_{n\in\mathbb{Z}}$ is the set of root vectors of $\mathfrak{L}_\ell$. Using (1.14) and(1.20), let us consider the following scalar product in the energy space $\mathfrak{H}_\ell$:

$$2i\lambda_n^\ell(\mathcal{F}_n^\ell, G)_{\mathfrak{H}_\ell} = \int_0^a \Big[p(r)(F_n^\ell(r))'\bar{g}_0' + q(r)\rho(r)F_n^\ell(r)\bar{g}_0(r) \\ +\ell(\ell+1)r^{-2}p(r)F_n^\ell(r)\bar{g}_0(r) + i\lambda_n^\ell\rho(r)F_n^\ell(r)\bar{g}_1\Big]\, r^2 dr. \tag{6.1}$$

Taking into account that $F_n^\ell(r)$ is a solution of problem (1.17) and integrating by parts, we can rewrite (6.1) as follows:

$$\begin{aligned}2i\lambda_n^\ell(\mathcal{F}_n^\ell, G)_{\mathfrak{H}_\ell} &= -i\lambda_n^\ell h a^2 F_n^\ell(a)\bar{g}_0(a)\\ &+ \int_0^a \Big[-2i\lambda_n^\ell d(r)(r\sqrt{p(r)}F_n^\ell(r))(r\sqrt{p(r)}\bar{g}_0(r))\\ &+ (\lambda_n^\ell)^2(r\sqrt{p(r)}F_n^\ell(r))(r\sqrt{p(r)}\bar{g}_0(r))\\ &+ i\lambda_n^\ell(r\sqrt{p(r)}F_n^\ell(r))(r\sqrt{p(r)}\bar{g}_1(r))\Big]\,\tau(r)dr.\end{aligned} \tag{6.2}$$

Setting

$$\begin{aligned}\tilde{\mathcal{F}}_n^\ell &= r\sqrt{p(r)}\mathcal{F}_n^\ell = \begin{pmatrix} \frac{1}{i\lambda_n^\ell} r\sqrt{p(r)}F_n^\ell(r) \\ r\sqrt{p(r)}F_n^\ell(r)\end{pmatrix} \equiv \begin{pmatrix} \frac{1}{i\lambda_n^\ell}\tilde{F}_n^\ell(r) \\ \tilde{F}_n^\ell(r)\end{pmatrix},\\ \tilde{G} &= r\sqrt{p(r)}G = \begin{pmatrix} r\sqrt{p(r)}g_0(r) \\ r\sqrt{p(r)}g_1(r)\end{pmatrix} \equiv \begin{pmatrix} k_0(r) \\ k_1(r)\end{pmatrix},\end{aligned} \tag{6.3}$$

we can rewrite (6.1) in the form

$$\begin{aligned}&2i\lambda_n^\ell(\mathcal{F}_n^\ell, G)_{\mathfrak{H}_\ell} = -i\lambda_n^\ell h\tilde{F}_n^\ell(a)\bar{k}_0(a)\\ &+ \int_0^a \Big[-2i\lambda_n^\ell d(r)\tilde{F}_n^\ell(r)\bar{k}_0(r) + (\lambda_n^\ell)^2\tilde{F}_n^\ell(r)\bar{k}_0(r) + i\lambda_n^\ell\tilde{F}_n^\ell(r)\bar{k}_1(r)\Big]\,\tau(r)dr.\end{aligned} \tag{6.4}$$

From our construction, we know that $\tilde{F}_n^\ell$ is an eigenfunction of pencil (1.19), (1.20) with the eigenvalue λ_n^ℓ. Using (4.10), let us consider the following scalar product:

$$\begin{aligned}2i\lambda_n^\ell(\tilde{\mathcal{F}}_n^\ell, \tilde{G})_{\mathcal{H}_\ell} = \int_0^a &\Big[(\tilde{F}_n^\ell(r))'\bar{k}_0'(r) + \frac{\ell(\ell+1)}{r^2}\tilde{F}_n^\ell(r)\bar{k}_0(r)\\ &+ \psi(r)\tilde{F}_n^\ell(r)\bar{k_0}(r) + i\lambda_n^\ell\tau(r)\tilde{F}_n^\ell(r)\bar{k}_1(r)\Big]dr.\end{aligned} \tag{6.5}$$

Integrating by parts in (6.5) and taking into account that $\tilde{F}_n^\ell$ satisfies (4.24), we obtain:

$$\begin{aligned}&2i\lambda_n^\ell(\tilde{\mathcal{F}}_n^\ell, \tilde{G})_{\mathcal{H}_\ell} = -i\lambda_n^\ell h\tilde{F}_n^\ell(a)\bar{k}_0(a) - i\lambda_n^\ell\Gamma\tilde{F}_n^\ell(a)\bar{k}_0(a)\\ &+ i\lambda_n^\ell\int_0^a\Big(-2d(r)\tilde{F}_n^\ell(r)\bar{k}_0(r) - i\lambda_n^\ell\tilde{F}_n^\ell(r)\bar{k}_0(r) + \tilde{F}_n^\ell(r)\bar{k}_1(r)\Big)\,\tau(r)dr.\end{aligned} \tag{6.6}$$

Comparing (6.4) and (6.6), we see that

$$(\mathcal{F}_n^\ell, G)_{\mathfrak{H}_\ell} = (\tilde{\mathcal{F}}_n^\ell, \tilde{G})_{\mathcal{H}_\ell} - 1/2\ \ \Gamma\tilde{F}_n^\ell(a)\bar{k}_0(a). \tag{6.7}$$

By assumption $(\mathcal{F}_n^\ell, G)_{\mathfrak{H}_\ell} = 0, n \in \mathbb{Z}$. Therefore, from Eq. (6.7), we obtain

$$(\tilde{\mathcal{F}}_n^\ell, \tilde{G})_{\mathcal{H}_\ell} = 1/2 \;\; \Gamma \tilde{F}_n^\ell(a)\bar{k}_0(a). \tag{6.8}$$

Now we show that the only possibility for (6.8) is $g_0 \equiv 0$. Let us rewrite (6.7) using the definition (6.3) of $\tilde{F}_n^\ell(r)$ and the normalization condition $F_n^\ell(a) = h^{-1}$ (which follows from (5.19) after the substitution $\lambda = \lambda_n^\ell$). We have

$$(\tilde{\mathcal{F}}_n^\ell, \tilde{G})_{\mathcal{H}_\ell} = 1/2 \;\; \Gamma a\sqrt{p(a)}h^{-1}\bar{k}_0(a) = \text{Const.} \tag{6.9}$$

Due to the fact that $\{\tilde{\mathcal{F}}_n^\ell\}_{n\in\mathbb{Z}}$ is an R-basis in $\mathcal{H}_\ell$, the sequence of Fourier coefficients (6.9) must be from $\ell^2(\mathbb{Z})$. Therefore, (6.8) is not valid unless $g_0(a) = 0$. But if $g_0(a) = 0$, then from (6.8), we obtain $\tilde{G} \equiv 0$. Therefore, the set of root vectors of $\mathfrak{L}_\ell$ is complete in $\mathfrak{H}_\ell$.

To use Theorem 2.5, we have to verify that the operator $\mathfrak{L}_\ell$ is dissipative. For $F \in D(\mathfrak{L}_\ell)$, we have

$$\begin{aligned} 2\mathcal{I}m\,(\mathfrak{L}_\ell F, F)_{\mathfrak{H}_\ell} &= 2\mathcal{I}m\left(\begin{pmatrix} -if_1 \\ -iL_\ell f_0 + 2idf_1 \end{pmatrix}, \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}\right)_{\mathfrak{H}_\ell} \\ &= a^2 p(a) Re\; h|f_1(a)|^2. \end{aligned} \tag{6.10}$$

For $Re\; h \geq 0$, we can immediately apply Theorem 2.5 and have the result on the R-basis property of root vectors shown.

Let us discuss the case $Re\; h < 0$. In this case, we consider the following "shifted" operator

$$\mathfrak{L}_\ell^\beta = \mathfrak{L}_\ell - i\beta I, \quad \beta \geq 2 \max_{r\in[0,a]} d(r). \tag{6.11}$$

This operator has the same set of root vectors as $\mathfrak{L}_\ell$. Since the set of root vectors of $\mathfrak{L}_\ell$ is complete, the same is valid for the root vectors of $\mathfrak{L}_\ell^\beta$. Now we verify the dissipativity of $(-\mathfrak{L}_\ell^\beta)$. For $F \in D(\mathfrak{L}_\ell)$, we have:

$$\begin{aligned} &\mathcal{I}m(-\mathfrak{L}_\ell^\beta F, F)_{\mathfrak{H}_\ell} = -\mathcal{I}m(\mathfrak{L}_\ell F, F)_{\mathfrak{H}_\ell} + \mathcal{I}m \;\; i\beta\|F\|^2_{\mathfrak{H}_\ell} \\ &= -a^2 p(a)(f_1(a)|^2 Re\; h + \beta\|F\|^2_{\mathfrak{H}_\ell} - 2\int_0^a d(r)|f_1(r)|^2\rho(r)r^2 dr \\ &\geq a^2 p(a)|f_1(a)|^2|Re\; h| + \int_0^a (\beta - 2d(r))|f_1(r)|^2\rho(r)r^2 dr \geq 0. \end{aligned}$$

From Theorem 2.5, we obtain that the operator $(-\mathfrak{L}_\ell^\beta)$ is Riesz spectral. So, the same is valid for $\mathfrak{L}_\ell^\beta$ and, therefore, for $\mathfrak{L}_\ell$. Case a) is shown.

b) Our final step in the proof of the R-basis property of the root vectors of the operator $\mathfrak{L}_\ell$ is an elimination of assumption (4.9) ($\psi(r) \geq 0$). Assume now that ψ is not necessarily nonnegative. In this case, we consider the following operator:

$$\mathfrak{L}_\ell^\alpha = \mathfrak{L}_\ell + i\alpha I, \quad \alpha \geq 2 \max_{r\in[0,a]} |\psi(r)| \max_{r\in[0,a]} d(r). \tag{6.12}$$

Let us write the corresponding spectral problem for $\mathfrak{L}_\ell^\alpha$:

$$\mathfrak{L}_\ell^\alpha U = \lambda U, \quad U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in D(\mathfrak{L}_\ell) \subset \mathfrak{H}_\ell. \tag{6.13}$$

For the components of U, we have the system

$$\begin{aligned} &i\alpha u_0 - iu_1 = \lambda u_0, \\ &\frac{1}{\rho(r)}\Big(\frac{1}{r^2}(r^2 p(r) u_0')' - \frac{\ell(\ell+1)}{r^2} u_0\Big) - q(r)u_0 - (2d(r)+\alpha)u_1 = i\lambda u_1. \end{aligned} \tag{6.14}$$

Substituting $u_1 = (\alpha + i\lambda)u_0$ from the first equation in (6.14) into the second one, we obtain the following equation for u_0:

$$\begin{aligned} &\frac{1}{\rho(r)}\Big(\frac{1}{r^2}(r^2 p(r) u_0')' - \frac{\ell(\ell+1)}{r^2} u_0\Big) - q u_0 \\ &- 2i\lambda(d(r)+\alpha)u_0 - \alpha(2d(r)+\alpha)u_0 + \lambda^2 u_0 = 0. \end{aligned} \tag{6.15}$$

Rewriting Eq. (6.15) for the function $w_0 = r\sqrt{p(r)}u_0$, we arrive at the following equation for w_0:

$$\frac{1}{\tau(r)}\Big(w_0'' - \frac{\ell(\ell+1)}{r^2} w_0\Big) - 2i\lambda(d(r)+\alpha)w_0 + \lambda^2 w_0 = \tilde{\psi}(r) w_0, \tag{6.16}$$

where

$$\tilde{\psi}(r) = \psi(r) 2d(r)\alpha + \alpha^2, \tag{6.17}$$

with ψ being given in (4.4). Clearly, $\tilde{\psi} \geq 0$ due to the choice of α in (6.12). Eq. (6.16) is the eigenvalue equation for the following quadratic operator pencil:

$$\mathcal{P}_\ell^\alpha(\lambda)\varphi = \frac{1}{\tau(r)}\Big(\varphi'' - \frac{\ell(\ell+1)}{r^2}\varphi\Big) - 2i\lambda(d(r)+\alpha)\varphi + \lambda^2\varphi - \tilde{\psi}(r)\varphi, \tag{6.18}$$

defined on the domain $D(\mathcal{P}_\ell^\alpha(\lambda)) = \{\varphi \in H^2(0,a) : \varphi(0) = 0,\ (\varphi' + i\lambda h\varphi + \Gamma\varphi)(a) = 0,\ h \in \mathbb{C} \cup \{\infty\},\ \Gamma \in \mathbb{R}\}$. It can be verified in a straightforward manner, that the spectrum of the pencil $\mathcal{P}_\ell^\alpha(\lambda)$ coincides with the spectrum of the following nonselfadjoint operator:

$$\mathfrak{L}_{h,\Gamma}^{\ell,\alpha} = -i \begin{pmatrix} 0 & 1 \\ \frac{1}{\tau(r)}\left(\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} - \tilde{\psi}(r)\right) & -2(d(r)+\alpha) \end{pmatrix} \tag{6.19}$$

defined on the domain

$$D(\mathfrak{L}_{h,\Gamma}^{\ell,\alpha}) = \left\{ \begin{array}{c} U \in \mathfrak{H}_\ell{}^\alpha : u_0 \in H^2(0,a), u_1 \in H^1(0,a), u_1(0) = 0, \\ (u_0' + ihu_1 + \Gamma u_1)(a) = 0 \end{array} \right\} \tag{6.20}$$

in the energy space $\mathfrak{H}_\ell{}^\alpha$ with the norm given by formula (4.10) in which $\psi(r)$ has been replaced with $\tilde{\psi}(r)$ from (6.17) ($\tilde{\psi}(r) \geq 0$). As follows from the first part of the proof (Case a)), for any complex h and Γ, the operator $\mathfrak{L}_{h,\Gamma}^{\ell,\alpha}$ has the set of root vectors which is a Riesz basis in $\mathfrak{H}_\ell{}^\alpha$. Therefore, the same is valid for the set of root vectors of the operator (6.12) in the energy space $\mathfrak{H}_\ell$. The latter obviously means that the operator $\mathfrak{L}_\ell$ is Riesz spectral. Theorem is shown.

Acknowledgment Partial support by the National Science Foundation Grant DMS #9706882, Advanced Research Program-95 of Texas Grant #0036-44-124, and Advanced Research Program-97 of Texas Grant #0036-44-045 is highly appreciated.

REFERENCES

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions*, Dover Publ., Inc., New York, 1972.

[2] N. Dunford, J.T. Schwartz, *Linear Operators, Part III: Spectral Operators.* New York – London, Toronto, (1971).

[3] S.A. Ivanov and B.S. Pavlov, Carleson series of resonances and the Regge problem, *Math. USSR Izv.*, **12**(1), (1978).

[4] I.C. Gohberg, M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators.* Transl. of Math. Monographs, Vol. 18, AMS, Providence, RI, (1969).

[5] S.V. Hruscev, N.K. Nikolśkii, B.S. Pavlov, Unconditional bases of exponentials and reproducing kernels. *Complex Analysis and Spectral Theory, Lect. Notes Math.*, Vol. 864, Springer-Verlag, (1981) p. 215-335.

[6] J.-L. Lions, *Controlabilite Exacte, Perturbations et Stabilisation de Systemes Distribues, T.1: Controlabilite Exacte*, Masson, RMA8, 1988; *T.2, Perturbations*, Masson, RMA9, 1988.

[7] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, (1971).

[8] A.S. Marcus, *Introduction to the Spectral Theory of Polynomial Pencils*. Transl. of Math. Monographs, Vol. 71, AMS, Providence, RI, (1988).

[9] R.G. Newton, *Scattering Theory of Waves and Particles*, 2nd Ed., Springer-Verlag, New York, 1982.

[10] N.K. Nikolśkii, *Treatise on the Shift Operator*, Springer-Verlag, Berlin, (1986).

[11] B.S. Pavlov, Spectral analysis of a differential operator with "blurred" boundary condition. *Problemy Matematicheskoy Fiziki (Problems of Mathematical Physics)*, **6**, p. 101-119, (1973), Leningrad University Press.

[12] M.A. Pekker (M.A. Shubov), Resonances in the scattering of acoustic waves by a spherical inhomogeneity of the density, *Amer. Math. Soc. Transl.* (2), 115 (1980), 143-163.

[13] M.A. Pekker (M.A. Shubov), The nonphysical sheet for the string equation, *J. Soviet Math* **10** (1978).

[14] D.L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, *Studies in Appl. Math.*, LII, (1973), p. 189-211.

[15] D.L. Russell, On boundary-value controllability of linear symmetric hyperbolic system, *Math. Theory of Control*, Academic Press, New York, (1967), p. 312-321.

[16] D.L. Russell, Nonharmonic Fourier series in the control theory of distributed parameter systems, *J. Math. Anal. Appl.*, **18**, (1967), p. 542-559.

[17] Marianna A. Shubov, Asymptotics of spectrum and eigenfunctions for nonselfadjoint operators generated by radial nonhomogeneous damped wave equations, *Asymptotic Analysis*, **16**, (1998), p.245-272.

[18] Marianna A. Shubov, Basis property of eigenfunctions of nonselfadjoint operator pencils generated by the equation of nonhomogeneous damped string. *Int. Eq. and Oper. Theory*, **25**, (1996), p. 289-328.

[19] Marianna A. Shubov, Asymptotics of resonances and eigenvalues for nonhomogeneous damped string. *Asympt. Anal.*, **13**, (1996), p. 1-48.

[20] M.A. Shubov, C.F. Martin, J.P. Dauer, B.P. Belinskiy, Unique controllability of the damped wave equation, *SIAM J. Contr. Opt.*, Vol. **35**, No 5, (1997), p. 1773-1789.

[21] Marianna A. Shubov, Exact boundary and distributed controllability of radial damped wave equation, *J. de Mathematiques Pures et Appliquees*, **77**, (1998), p.415-437.

[22] Marianna A. Shubov, Asymptotics of resonances and geometry of resonance states in the problem of scattering of acoustical waves by a spherically symmetric inhomogeneity of the density, *Dif. Int. Eq.*, **8**, (5), (1995), p. 1073-1115.

[23] Marianna A. Shubov, Nonselfadjoint operators generated by equation of nonhomogeneous damped string, *Transactions of American Mathematical Society*, **349**, (1997), p.4481-4499.

[24] Marianna A. Shubov, Certain class of unconditional bases in Hilbert space and its applications to functional model and scattering theory, *Integral Equations and Operator Theory*, Vol. **13** (1990), p. 750-770.

[25] Marianna A. Shubov, *Riesz basis property of the system of root vectors for the equation of nonhomogeneous damped string,* Methods and Applications of Analysis, to appear.

[26] Marianna A. Shubov, Spectral decomposition method for controlled damped string. Reduction of control time,*Applicable Analysis*, **68**, (1998), p.241-259.

[27] Marianna A. Shubov, Spectral operators generated by damped hyperbolic equations, *Integral Equations and Operator Theory*, **28** (1997), p. 358-372.

[28] M. Zwaan, Moment Problems in Hilbert Space with Applications to Magnetic Resonance Imaging, Centrum voor Wiskunde en Informatica, CWI TRACT, (1991).