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STRUCTURE OF PERIODIC SOLUTIONS AND ASYMPTOTIC BEHAVIOR FOR TIME-PERIODIC REACTION-DIFFUSION EQUATIONS ON R

Eduard Feireisl*
Institute of Mathematics, AVČR, Žitná 25, 115 67 Praha, Czech Republic

Peter Poláčik†
Institute of Applied Mathematics, Comenius University, Mlynská dolina
842 15 Bratislava, Slovakia

(Submitted by: Reza Aftabizadeh)

Abstract. We consider nonautonomous reaction-diffusion equations on $\mathbb{R}$,

$$u_t - u_{xx} = f(t,u),\ x \in \mathbb{R},\ t > 0,$$

and investigate solutions in $C_0(\mathbb{R})$, that is, solutions that decay to zero as $|x|$ approaches infinity. The nonlinearity is assumed to be a C^1 function which is τ-periodic in t and satisfies $f(t,0) = 0$ and $f_u(t,0) < 0 \quad (t \in \mathbb{R})$. Our two main results describe the structure of τ-periodic solutions and asymptotic behavior of general solution with compact trajectories:

(1) Each nonzero τ-periodic solution is of definite sign and is even in x about its unique peak (which is independent of t) . Moreover, up to shift in space, $C_0(\mathbb{R})$ contains at most one τ-periodic solution of a given sign.

(2) Each solution with trajectory relatively compact in $C_0(\mathbb{R})$ converges to a single τ-periodic solution.

In the proofs of these properties we make extensive use of nodal and symmetry properties of solutions. In particular, these properties are crucial ingredients in our study of the linearization at a periodic solution and in our description of center and stable manifolds of periodic

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solutions. The paper also contains a section discussing various sufficient conditions for existence of nontrivial periodic solutions.

1. Introduction and statement of the main results. In this paper we investigate solutions of the Cauchy problem

$$u_t - u_{xx} = f(t,u), \; x \in \mathbb{R}, \; t > 0, \tag{1.1}$$

$$u = u_0, \; x \in \mathbb{R}, \; t = 0. \tag{1.2}$$

The nonlinearity $f : \mathbb{R}^2 \to \mathbb{R}$ is assumed to be τ-periodic in t:

$$f(t+\tau,u) \equiv f(t,u) \;\text{ for some }\; \tau > 0,$$

and C^1 in u with locally Hölder continuous (in t and u) derivative $f_u(t,u)$. The initial datum u_0 is taken in $C_0(\mathbb{R})$, the Banach space of continuous functions on $\mathbb{R}$ vanishing at infinity (we assume the sup norm on $C_0(\mathbb{R})$). We further assume that the following hypothesis is satisfied:

(HNw) $f(t,0) = 0$ and there is a $\delta > 0$ such that

$$f_u(t,u) \le 0, \quad (t \in \mathbb{R}, |u| < \delta).$$

For some results (HNw) has to be strengthened to the following hypothesis:

(HNs) $f(t,0) = 0$ and $f_u(t,0) < 0 \quad (t \in \mathbb{R})$.

Under the above hypotheses, the problem (1.1), (1.2) is well posed on $X := C_0(\mathbb{R})$. Indeed, the operator $\varphi \mapsto -\varphi_{xx}$ with domain $\{\varphi \in X : \varphi_{xx} \in X\}$ is sectorial on X (see Lunardi [23]) and $(t,\varphi) \mapsto f(t,\varphi(\cdot)) : [0,\tau] \times X \to X$ is C^1 in φ and locally Hölder in t. Thus the existence theory in Henry [19], Lunardi [23] applies to (1.1), (1.2) yielding a unique (local) solution with the usual regularity properties.

Observe that (HNw) implies that $\phi = 0$ is a stable solution of (1.1): For any $\epsilon > 0$ there is a $\beta > 0$ such that if $\|u_0\|_X < \beta$ then the solution $u(\cdot,t;u_0)$ of (1.1), (1.2) is defined for all $t > 0$ and $\|u(\cdot,t;u_0)\|_X < \epsilon$. The stability is verified easily using comparison with spatially constant solutions in $(-\beta,\beta)$.

Our main theorem describes the time asymptotics of solutions with compact trajectories.

Theorem 1.1. *Assume (HNs). Let* $u(x,t)$ *be a solution of* (1.1), (1.2) *such that* $\{u(\cdot,t) : t \geq 0\}$ *is relatively compact in* $C_0(\mathbb{R})$. *Then there exists a solution* $p(x,t)$ *of* (1.1), (1.2), *that is,* τ*-periodic,* $p(x,t+\tau) \equiv p(x,t)$, *and such that* $\|u(\cdot,t) - p(\cdot,t)\|_X \to 0$ *as* $t \to \infty$.

Theorem 1.1 in particular implies that (1.1), (1.2) has no subharmonic solutions (that is, time-periodic solutions with minimal period $n\tau$ for some integer $n > 1$). More generally, (1.1) has no solutions $u(\cdot,t) \in X$ that are uniformly almost periodic in t, except for τ-periodic solutions.

Theorem 1.1 combined with the structure of periodic solutions, as stated in the next theorem, also gives the space asymptotics of solutions with compact trajectories.

Theorem 1.2. *Assume (HNw). Let* $p(\cdot,t) \in C_0(\mathbb{R})$ *be a solution of* (1.1), (1.2) *that is* τ*-periodic in* t. *Then either* $p \equiv 0$ *or else* $p(x,t) \neq 0$ *for any* $x,t \in \mathbb{R}$ *and there is an* $a \in \mathbb{R}$ *such that* $p(2a-x,t) \equiv p(x,t)$ *and* $p_x(x,t) \neq 0$ *for* $x \neq a$. *Moreover, if* $p_1(\cdot,t)$, $p_2(\cdot,t) \in C_0(\mathbb{R})$ *are two* τ *- periodic solutions with* $p_1(x,t)p_2(x,t) > 0$, *then there is a* $b \in \mathbb{R}$ *such that* $p_1(x+b,t) \equiv p_2(x,t)$.

Thus, up to a shift in space, $C_0(\mathbb{R})$ contains at most one τ-periodic solution of a given sign and each nonzero τ-periodic solution $p(\cdot,t) \in C_0(\mathbb{R})$ has a unique peak and is even with respect to it. These periodic solutions can be viewed as counterparts of ground state solutions of autonomous equations. For the latter, properties analogous to those in Theorem 1.2 are found easily by a phase-plane analysis.

Let us now give a sufficient condition for the problem (1.1), (1.2) to possess a solution with compact trajectory that does not converge to 0. By Theorem 1.1, such conditions in particular guarantee that nontrivial τ-periodic solutions in $C_0(\mathbb{R})$ exist. To this end, we shall assume

$$\underline{f}(t,u) \leq f(t,u) \leq \overline{f}(u) \text{ for all } t \geq 0,\ u \geq 0, \tag{1.3}$$

where the functions $\underline{f}, \overline{f}$ enjoy the same regularity as f and, moreover, satisfy the following hypotheses:

(B1) $\underline{f}$ is $\tau-$periodic in t and there is $\gamma > 0$ such that

$$\underline{f}(t,0) = \underline{f}(t,\gamma) = 0,\ \underline{f}_u(t,0) < 0,\ \underline{f}_u(t,\gamma) < 0 \text{ for all } t. \tag{1.4}$$

There is $\epsilon > 0$ such that the solution v of the problem

$$v_t - v_{xx} = \underline{f}(t, v) \qquad (1.5)$$
$$v(x, 0) = \gamma - \epsilon \text{ for } x \le 0,\ v(x, 0) = 0 \text{ for } x > 0$$

converges to the equilibrium solution $e_1 \equiv \gamma$ as $t \to \infty$ uniformly on compact subsets of $\mathbb{R}$.

(B2) One has

$$\overline{f}(u) < 0 \quad (u \in (0, \alpha)), \qquad (1.6)$$

for some $\alpha > 0$, and

$$\limsup_{u \to \infty} \frac{\overline{f}(u)}{u} \le 0. \qquad (1.7)$$

(B3) Denote $\beta \equiv \sup_{t \in [0,T]} \sup\{0 < u < \gamma \mid \underline{f}(t, u) = 0\}$ and observe that $0 < \alpha < \beta < \gamma$ as a consequence of (1.3), (1.4). We assume

$$\overline{F}(u) \equiv \int_0^u \overline{f}(s)\ ds < 0 \text{ for all } u \in (0, \beta]. \qquad (1.8)$$

Theorem 1.3. *Let* (1.3) *hold with* $\underline{f}, \overline{f}$ *satisfying the hypotheses* **(B1-B3)**. *Then there exists a solution* u *of* (1.1) *on* $\mathbb{R} \times (0, \infty)$ *such that*

$$0 < \alpha \le u(0, t),\ 0 \le u(x, t) \le \sigma(x) \text{ for all } x \in \mathbb{R},\ t > 0$$

with a certain $\sigma \in C_0(\mathbb{R})$.

Remark. The last hypothesis in (**B1**) requires, roughly speaking, the existence of a traveling wave connecting the equilibrium $e_0 \equiv 0$ with $e_1 \equiv \gamma$. Some sufficient conditions for this to occur may be found in Alikakos, Bates and Chen [1] and in the examples presented at the end of Section 6. Note that for $\underline{f}$ independent of t such a condition reads:

$$\int_0^\gamma \underline{f}(z)\ dz > 0 \qquad (1.9)$$

(see Fife-McLeod [15]).

Some specific examples of f satisfying **(B1-B3)** are given in Section 6.

In the remaining part of the introduction we take a look at our convergence theorem from a broader prospective and mention some open questions.

The problem of convergence or stabilization of solutions of semilinear reaction - diffusion equations has a long history and active presence. For autonomous equations on bounded intervals and separated boundary conditions (Dirichlet, Neumann, Robin) convergence of bounded solutions to an equilibrium has been proved as early as in 1968 by Zelenyak [34] (see also Matano [25], Hale and Raugel [17]). For nonautonomous, time-periodic counterparts of these separated boundary value problems, convergence to periodic solutions has been established by Brunovský, Poláčik and Sandstede [4] (see also Chen and Matano [7] for an earlier result on a more specific class of equations). If periodic (rather than separated) boundary conditions are assumed, bounded solutions no longer have to be convergent. In this case, Poincaré - Bendixson type theorems have been proved (see Fiedler and Mallet-Paret [13], Massatt [24], Matano [27], Fiedler and Sandstede [14], Tereščák [33] and references therein).

In higher space dimension, reaction-diffusion equations with a drift term or with periodic forcing can have very complex dynamics (see Poláčik [28], Rybakowski [30] for surveys of relevant results). Even without drift terms, the Dirichlet problem for autonomous equations of the form

$$u_t = \Delta u + f(x, u), \quad x \in \Omega, \tag{1.10}$$

can have bounded trajectories with a nontrivial continuum of limit points (see Poláčik and Rybakowski [29] for an example with Ω being a disk). On the other hand, for various specific classes of equations on bounded domains convergence has been established; see for example Simon [32], Lions [22], Hess [20], Hale and Raugel [17], Haraux and Poláčik [18], Chen and Poláčik [8] and references there.

On unbounded domains much less seems to be known on asymptotic behavior of relatively compact trajectories. Results on convergence for *autonomous* equations on $\mathbb{R}^1$ and $\mathbb{R}^N$ have recently been obtained by Fašangová [11] and Feireisl and Petzeltová [12], respectively.

The present paper is our first contribution to the study of time-periodic equations on unbounded domains. Given the result of theorem 1.1, several questions are raised naturally. For example, what is the behavior of solutions if the equation is allowed to depend on x explicitly? The method of the present paper no longer applies to such equations as it depends, at several points, on the reflection and translation symmetry of (1.1). Neither does our method apply if the solutions of (1.1) are not required to satisfy the boundary condition at infinity. Although several symmetry arguments can still be used, more insight is needed to understand the behavior of the more general solutions. Note that for a special class of such solutions, namely, solutions periodic in x, convergence to a τ-periodic solution has been proved by Chen and Matano [7].

The paper is organized as follows. An underlying linearized problem is studied in Section 2. Specifically, the properties of the nodal set of the solution are investigated in detail under very mild hypotheses concerning the data and the potential appearing in the equation. The results obtained are of fundamental importance in the proof of Theorems 1.1, 1.2.

Section 3 is devoted to the limit sets of compact solutions of the problem (1.1). Furthermore, the structure of time-periodic solutions is investigated and, finally, the proof of Theorem 1.2 is given.

In Section 4, we study the spectral properties of the period map associated with the linearization of (1.1), (1.2) along periodic solutions. We prove that, with an exception of two simple real Floquet multipliers, the entire spectrum is contained in a compact subset of the open unit disk.

In Section 5, the invariant manifolds of nonzero periodic solutions are described and their properties along with the results achieved in the preceding sections are used to prove Theorem 1.1.

Finally, some examples of functions f satisfying the hypotheses **(B1-B3)** above as well as the proof of Theorem 1.3 are given in Section 6.

2. Linear equations and nodal curves. In this section we study linear equations

$$v_t - v_{xx} = c(x,t)v, \quad x \in \mathbb{R},\ t > \tau_0, \tag{2.1}$$

where $\tau_0 \in \mathbb{R}$ and $c(x,t)$ is a continuous function on $\mathbb{R} \times [\tau_0, \infty)$. We remark that the results of this section remain valid if the right-hand side also contains the term $b(x,t)u_x$ with $b(x,t)$ continuous. The following lemma is the main result of this section.

Lemma 2.1. *Assume that there is an $M \geq 0$ such that*

(HL) $c(x,t) \leq 0$ $(|x| \geq M, t \geq \tau_0)$.

Let $v(\cdot,t) \in C_0(\mathbb{R})$ be a nontrivial solution of (2.1). *Then one of the following properties holds*

(A) *There is a sequence* $t_n \to \infty$ *and a* $k \geq 0$ *such that*

$$\min\{\sup_{x \leq -k} |v(x,t_n)|, \sup_{x \geq k} |v(x,t_n)|\} \to 0, \text{ as } n \to \infty.$$

(B) *For any* $N > 0$ *there exist a* $\theta > \tau_0$ *, an integer* m *and continuous functions* $\beta(t)$, $\gamma(t)$ *such that*

(i) $\beta(t) \leq -N < N \leq \gamma(t)$ $(t \geq \theta)$,

(ii) $v(\beta(t),t) \neq 0 \neq v(\gamma(t),t)$ $(t \geq \theta)$, *and*

(iii) *for* $t > \theta$, $v(\cdot,t)$ *has exactly* m *zeros in* $[\beta(t),\gamma(t)]$ *and all of them are simple.*

Moreover, if $\{v(\cdot,t) : t \in (\tau_0,\infty)\}$ *is relatively compact in* $C_0(\mathbb{R})$ *then the same assertion holds with the additional property that the functions* $\beta(t)$, $\gamma(t)$ *in (B) are bounded.*

To prepare the proof of this result, we investigate the nodal properties of v. Below, by a *nodal domain* of v in Q ($Q \subset \mathbb{R}^2$) we understand a connected component of the set $\{(x,t) \in Q : v(x,t) \neq 0\}$. A *nodal set* of v is a subset of $v^{-1}(0)$ and a *nodal curve* is a nodal set of the form $\Gamma = \{(\xi(t),t) : t \in J\}$, where J is an interval and ξ is a continuous function on J. If ξ is of class C^1, Γ is said to be a C^1 (or *smooth*) *nodal curve.* We say that a function $\phi(x)$ *changes sign* at x_0 if there are sequences $x_n^- < x_0 < x_n^+$, both approaching x_0, such that $\phi(x_n^-)\phi(x_n^+) < 0$.

Lemma 2.2 (Local Structure of Nodal Sets). *Let* $v \not\equiv 0$ *be a solution of* (2.1) *and* $(x_0,t_0) \in v^{-1}(0)$, $t_0 > \tau_0$. *Then there is a neighborhood* $Q = [x_0-\epsilon, x_0+\epsilon] \times [t_0-\delta, t_0+\delta]$ *of* (x_0,t_0) *such that the following properties hold:*

a) *If* $v_x(x_0,t_0) \neq 0$ *then* $Q \cap v^{-1}(0)$ *equals a single nodal curve* $\{(\xi(t),t) : t \in [t_0-\delta, t_0+\delta]\}$, *where* $\xi : [t_0-\delta, t_0+\delta] \to \mathbb{R}$ *is of class* C^1 *and* $\xi(t_0) = x_0$.

b) *If* $v_x(x_0,t_0) = 0$ *then there is an integer* $m \geq 2$ *such that* $([x_0-\epsilon, x_0+\epsilon] \times [t_0-\delta, t_0]) \cap v^{-1}(0)$ *consists of* m *nodal curves* $\{(\xi_j(t),t) : t \in [t_0-\delta, t_0]\}$, $j = 1,\ldots,m$, *where the* ξ_j *are of class* C^1 *on* $[t_0-\delta, t_0)$ *and satisfy*

$$\xi_1(t) < \xi_2(t) < \cdots < \xi_m(t), \ \ (t \in [t_0-\delta, t_0)) \text{ and } \xi_1(t_0) = \cdots = \xi_m(t_0) = x_0.$$

Furthermore, if m *is even then* $([x_0-\epsilon, x_0+\epsilon] \times [t_0, t_0+\delta]) \cap v^{-1}(0)$ *consists of the single point* (x_0,t_0), *and if* m *is odd, then* $([x_0-\epsilon, x_0+\epsilon] \times [t_0, t_0+$

$\delta]) \cap v^{-1}(0)$ *coincides with a single nodal curve* $\{(\xi(t), t) \ : t \in [t_0, t_0+\delta]\}$, *where* ξ *is of class* C^1 *on* $[t_0, t_0+\delta]$ *and* $\xi(t_0) = x_0$.

In each case, $v(\cdot, t)$ *has only simple zeros in* $[x_0 - \epsilon, x_0 + \epsilon]$ *for* $t \in [t_0 - \delta, t_0) \cup (t_0, t_0 + \delta]$.

See Chen [6] for the proof of these properties. For an earlier, less complete description of the nodal set see Angenent [2] and Chen and Poláčik [8].

The following properties are immediate consequences of Lemma 2.2.

Corollary 2.1. *Let* $v \not\equiv 0$ *be a solution of* (2.1), $x_0 \in \mathbb{R}$ *and* $t_0 > \tau_0$. *Then*

(i) *the zeros* $v(\cdot, t_0)$ *have no finite accumulation point, and*

(ii) $v(\cdot, t_0)$ *changes sign at* x_0 *if and only if there exists a* $\delta > 0$ *and a nodal curve* $\{(\xi(t), t) : t \in [t_0, t_0 + \delta)\}$ *with* $\xi(t_0) = x_0$.

Lemma 2.3 (Global Structure of Nodal Sets). *Let* $v \not\equiv 0$ *be a solution of* (2.1). *Then the following properties are satisfied.*

(i) *The singular nodal set of* v,

$$S := \{(x,t) \in \mathbb{R} \times (\tau_0, \infty) : v(x,t) = 0 = v_x(x,t)\}$$

consists of isolated points and each connected component of the nodal set of v *in* $\mathbb{R} \times (\tau_0, \infty) \setminus S$ *is a* C^1 *nodal curve. Moreover, each compact set in* $\mathbb{R} \times (\tau_0, \infty)$ *intersects at most finitely many of these connected components.*

(ii) *For any* $(x_1, t_1) \in v^{-1}(0)$, $t_1 > \ \tau_0$, *and any* $t_0 \in (\tau_0, t_1)$ *there is a nodal curve* $\{(\xi(t), t) : t \in [t_0, t_1]\}$ *with* $\xi(t_1) = x_1$; *that is, the nodal curve connects* (x_1, t_1) *to a point on the line* $\mathbb{R} \times \{t_0\}$. (*One can in addition prove that the nodal curve can be chosen with* ξ *of class* C^1 *on* $(t_0, t_1]$, *but this is not needed below.*)

(iii) *Let* $\{(\xi_j(t), t) : t \in (t_1, t_2)\}$, $j = 1, 2$, *be two nodal curves of* v. *Then* ξ_j *is piecewise smooth* (*of class* C^1). *If* $\xi_1(\vartheta) < \xi_2(\vartheta)$ *for a* $\vartheta \in (t_1, t_2)$, *then* $\xi_1(t) < \xi_2(t)$ *for any* $t \in (t_1, \vartheta]$.

Proof. Property (i) follows easily from Lemma 2.2.

We prove (ii). By Lemma 2.2, there is a nodal curve $\{(\xi(t), t) \ : t \in [t_1 - \delta, t_1]\}$ such that $\xi(t_1) = x_1$ and $\xi(t)$ is a simple zero of $v(\cdot, t)$ for $t < t_1$. Of course, $v(\cdot, t)$ changes sign at $\xi(t)$ ($t \in (t_1 - \delta, t_1)$). This and the maximum principle imply (see Matano [26]) that there are continuous functions $\xi^-(t), \xi^+(t)$ defined on $[t_0, t_1)$ such that $\xi^-(t) < \xi^+(t)$ ($t \in [t_0, t_1)$),

$\xi^-(t) < \xi(t) < \xi^+(t)$ $(t \in [t_1 - \delta, t_1))$, $v(\xi^-(t), t)v(\xi^+(t), t) < 0$ $(t \in [t_0, t_1))$ and $v(x,t) \neq 0$ $(t \in [t_1 - \delta, t_1),\, x \in [\xi^-(t), \xi(t)) \cup (\xi(t), \xi^+(t)])$. It follows that $v(\cdot, t)$ has a zero between $\xi^-(t)$ and $\xi^+(t)$ which equals $\xi(t)$ if $t \in [t_1 - \delta, t_1)$. This and Lemma 2.2 imply (ii).

The fact that the curves in (iii) are piecewise smooth follows from Lemma 2.2. The last assertion of (iii) follows from the maximum principle (see Matano [26]).

Proof of Lemma 2.1. Fix any $N \geq M$ and $t_0 > \tau_0$. We distinguish two complementary cases:

Case 1. For each $R > 0$ there exist a $t_1 > t_0$ and a nodal curve $\Gamma = \{(\xi(t), t) \ : t \in [t_0, t_1]\}$ such that

$$\xi(t) \geq M \ \ (t \in [t_0, t_1]), \quad \xi(t_1) = N \tag{2.2}$$

and $\xi(t_0) \geq R$.

Case 2. There is an R^+ such that for any nodal curve $\{(\xi(t), t) \ : t \in [t_0, t_1]\}$ satisfying (2.2) one has $\xi(t_0) < R^+$.

Fix an $R > 0$ and consider the nodal curve Γ as in case 1. We claim that $\sup_{x \geq N} |v(x, t_1)| \leq \sup_{x \geq R} |v(x, t_0)|$. To prove this, let x_1 be a point where the former supremum is assumed (recall that $v(\cdot, t) \in C_0(\mathbb{R})$). As $v \not\equiv 0$, $v(x_1, t_1) \neq 0$ (cf. Corollary 2.1). For definiteness assume $v(x_1, t_1) > 0$. Let D be the nodal domain of v in $\{(x,t) \ : t_0 \leq t \leq t_1\}$ containing (x_1, t_1). Clearly, D lies to the right of the nodal curve Γ: $\xi(t) < x$ for any $(x,t) \in D$. It follows that the parabolic boundary of D consists of two parts, one contained in $\{(x,t) \ : t = t_0\}$, the other contained in the nodal set of v. Since $\xi(t) \geq M$, we have $c(x,t) \leq 0$ in D. By the maximum principle, the maximum of v in D is assumed on $D \cap \{(x,t) \ : t = t_0\}$, at a point (x_0, t_0), say. Therefore, $v(x_1, t_1) \leq v(x_0, t_0) \leq \sup_{x \geq R} |v(x, t_0)|$. The claim is proved.

In case 1, we choose a sequence $R_n \to \infty$. Using the above claim for the corresponding nodal curves, we find a sequence $t_n > t_0$ such that $\sup_{x \geq N} |v(x, t_n)| \leq \sup_{x \geq R_n} |v(x, t_0)|$. As $v(\cdot, t_0) \in C_0(\mathbb{R})$, the right-hand side converges to 0, hence also $\sup_{x \geq N} |v(x, t_n)| \to 0$. As $v \not\equiv 0$, we necessarily have $t_n \to \infty$ (cf. Corollary 2.1), and we see that property (A) of Lemma 2.1 holds in this case.

Case 2. Similarly as in case 1, one can treat nodal curves in $(-\infty, -N] \times [t_0, \infty)$. We can thus proceed assuming, with no loss of generality, that case 2 occurs together with the analogous

Case 2'. There is an R^- such that for any nodal curve $\{(\xi(t),t)\ : t \in [t_0,t_1]\}$ satisfying

$$\xi(t) \le -M \ \ (t \in [t_0,t_1]), \quad \xi(t_1) = -N \tag{2.3}$$

one has $\xi(t_0) > R^-$. We now define two functions $\tilde\beta(t)$, $\tilde\gamma(t)$ $(t \in [t_0,\infty))$ for which all properties in (B) hold, except that they may be discontinuous and may take infinite values.

If there is a t such that $v(\cdot,t)$ changes sign at only finitely many points in $[N,\infty)$, set $\tilde\gamma(t) \equiv \infty$ $(t \ge t_0)$. Consider the other possibility: $v(\cdot,t)$ has infinitely many sign changes for any t, choose a point $\xi_0 > R^+$ at which $v(\cdot,t_0)$ changes sign. By Corollary 2.1, for some $t_1 > t_0$ there is a nodal curve $\{(\xi(t),t)\ \ : t \in [t_0,t_1)\}$ with $\xi(t_0) = \xi_0$. Moreover, by Lemma 2.2, given t_1, such a curve is uniquely defined. Assume that t_1 is the maximal number (we allow for $t_1 = \infty$) for which this nodal curve exists. As we are considering case 2 and $\xi_0 > R^+$, we necessarily have $\xi(t) > N$ $(t \in [t_0,t_1))$. Set $\tilde\gamma(t) = \xi(t)$ $(t \in (t_0,t_1)$. If $t_1 = \infty$, the definition of $\tilde\gamma(t)$ is complete. Otherwise, we proceed by proving that $\eta_1 := \lim_{t\to t_1}\xi(t)$ exists and is finite. First of all note that if $x_1 := \liminf_{t\to t_1}\xi(t) < \infty$, then $v(x_1,t_1) = 0$ and Lemma 2.2 implies that $x_1 = \lim_{t\to t_1}\xi(t)$. We thus only have to exclude the possibility

$$\lim_{t\to t_1}\xi(t) = \infty. \tag{2.4}$$

Suppose (2.4) holds. Let $x_j > N$, $j = 1,2,\ldots$, be an increasing sequence of zeros of $v(\cdot,t_1)$ (such a sequence exists for any t by the above assumption on sign changes). By Lemma 2.3, there are nodal curves $\{(\xi_j(t),t)\ : t \in [t_0,t_1]\}$ with $\xi_j(t_1) = x_j$. By the same lemma, $\xi_1(t) < \xi_2(t) < \cdots < \xi_j(t) < \xi(t)$ $(t \in [t_0,t_1))$. As $\xi_j(t_1) = x_j > -N$, the property assumed in case 2' and continuity of the ξ_j readily imply that $\xi_j(t_0) > \min\{-N,R^-\}$. Thus the zeros $\xi_1(t_0) < \xi_2(t_0) < \ldots$ of $v(\cdot,t_0)$ have a finite accumulation point, contradicting Corollary 2.1. This contradiction rules out (2.4).

Next we note that, by the maximality of t_1 and Corollary 2.1, $v(\cdot,t_1)$ does not change sign at η_1. Let $\xi_1 > \eta_1$ be the minimal point where $v(\cdot,t_1)$ changes sign. Repeating the above arguments, we now find a nodal curve $\{(\xi^1(t),t)\ : t \in [t_1,t_2)\}$ with $\xi^1(t_1) = \xi_1$, where t_2 is a maximal (possibly infinite) number for which such a nodal curve exists. By Lemma 2.3, this nodal curve can be extended so that $\xi^1(t)$ is defined for all $t \in [t_0,t_2)$ and $\xi(t) < \xi^1(t)$ $(t \in [t_0,t_1])$. In particular, $\xi^1(t_0) > \xi_0 > R^+$. As case 2 is being considered, it follows that $\xi^1(t) > N \ \ (t \in [t_0, t_2))$. Set $\tilde{\gamma}(t) = \xi^1(t) \ \ (t \in [t_1, t_2))$. Now, if $t_2 = \infty$ the definition of $\tilde{\gamma}(t)$ is complete. Otherwise we proceed repeating the previous step with t_1, ξ_1 replaced by t_2, ξ_2, respectively. By induction, we construct a function $\tilde{\gamma}(t)$ defined on $[t_0, t_\infty)$ with the following properties:

(P1) $\tilde{\gamma}(t) > N \ \ (t \in [t_0, t_\infty))$,

(P2) $v(\tilde{\gamma}(t), t) = 0 \ \ (t \in [t_0, t_\infty))$,

(P3) $\tilde{\gamma}(t)$ is continuous except at a sequence of points $t_1 < t_2 < \dots$, which is either finite and in this case $t_\infty = \infty$, or is infinite and in this case it converges to t_∞ (below we prove that $t_\infty = \infty$ anyway),

(P4) at any point t_j of discontinuity of $\tilde{\gamma}(t)$ one has $\eta_j := \lim_{t \to t_j^-} \tilde{\gamma}(t) < \xi_j := \tilde{\gamma}(t_j)$ and the function $v(\cdot, t_j)$ does not change sign at any point of $[\eta_j, \xi_j)$.

We claim that the following properties also hold true:

(P5) If $\{(\xi(t), t) \ : t \in [t_0, \vartheta]\}$ is a nodal curve of v such that $\vartheta \in (t_0, t_\infty) \setminus \{t_j : j = 1, 2, \dots\}$ and $\xi(\vartheta) < \tilde{\gamma}(\vartheta)$, then

$$\xi(t) < \tilde{\gamma}(t) \quad \text{for any } t \in [t_0, \vartheta]. \tag{2.5}$$

(P6) $t_\infty = \infty$.

We prove (P5). By continuity of $\xi(t)$ and $\tilde{\gamma}(t)$ at ϑ, (2.5) holds for any t sufficiently close to ϑ. Let $t_j < \vartheta$ be the discontinuity point of $\tilde{\gamma}(t)$ nearest to ϑ. By definition of $\tilde{\gamma}$, $\{(\tilde{\gamma}(t), t) \ : t \in [t_j, \vartheta]\}$ coincides with a nodal curve. Thus, by Lemma 2.3, (2.5) holds for $t \in [t_j, \vartheta]$. Moreover, $\xi(t_j) < \eta_j = \lim_{t \to t_j^-} \tilde{\gamma}(t)$. This follows from (P4) and the fact that $v(\cdot, t_j)$ changes sign at $\xi(t_j)$ (cf. Corollary 2.1). Arguing as above one shows that (2.5) holds for $t \in [t_{j-1}, t_j]$. An induction argument then proves that it holds for any $t \in [t_0, \vartheta]$.

In order to prove (P6), we only need to consider the case when the sequence $\{t_j\}$ is infinite. Suppose $t_\infty < \infty$. Note that $x_0 := \liminf_{t \to t_\infty} \tilde{\gamma}(t) < \infty$ is impossible in such a case. Indeed, if it was true, x_0 would be a zero of $v(\cdot, t_\infty)$. The definition of $\tilde{\gamma}(t)$ and Lemma 2.2 would then imply that $\tilde{\gamma}(t)$ is continuous near t_∞, which contradicts $t_j \to t_\infty$. We may therefore assume

$\lim_{t\to t_\infty}\tilde\gamma(t)=\infty$. But this leads to a contradiction, too, as one can show by the arguments that were used above to exclude (2.4). This contradiction proves that $t_\infty=\infty$.

We have completed the definition of $\tilde\gamma$. In an analogous way one defines a function $\tilde\beta(t)<-N$. Specifically, $\tilde\beta(t)\equiv-\infty\ (t\ge t_0)$ if there is a t such that $v(x,t)$ changes sign at only finitely many points in $(-\infty,-N]$. If there is no such t then $\tilde\beta(t)$ is finite and properties (P1)-(P6) hold with $\tilde\gamma$ replaced by $\tilde\beta$, N replaced by $-N$ and the inequality signs reversed.

As the next step, we show that there is a $\hat t\ge t_0$ such that $v(\cdot,t)$ has only finitely many zeros in $(\tilde\beta(t),\tilde\gamma(t))\ (t>\hat t)$. This is obvious if $\tilde\beta(t)$ and $\tilde\gamma(t)$ are both finite, as the zeros are isolated. Suppose $\tilde\beta(t)\equiv-\infty$, that is, there is a $\hat t$ such that $v(\cdot,\hat t)$ changes sign at only finitely many points of $(-\infty,-N]$. Assume that $\tilde\gamma(t)$ is finite. If for some $t>\hat t$ there are infinitely many points (x_j,t) with $x_j\le\tilde\gamma(t)$ and $v(x_j,t)=0$ then nodal curves as in Lemma 2.3 connect them to mutually distinct points $(\hat x_j,\hat t)$ with $\hat x_j<\tilde\gamma(t)$ (the latter follows from (P5)). By Corollary 2.1, $v(\cdot,\hat t)$ changes sign at each of the points $\hat x_j$. By the choice of $\hat t$, infinitely many of these points must be contained in $(-N,\tilde\gamma(\hat t))$ and hence they have a finite accumulation point - a contradiction. This shows that $v(\cdot,t)$ can have only finitely many zeros in $(-\infty,\tilde\gamma(t))\ (t\ge\hat t)$. The case $\tilde\gamma(t)=\infty$ is treated similarly.

For $t>\hat t$ let $z(t)$ denote the number of zeros of $v(\cdot,t)$ in $(\tilde\beta(t),\tilde\gamma(t))$. By (P5) and Lemma 2.3 (ii), $t\to z(t)$ is monotone nonincreasing, hence there is an m such that $z(t)\equiv m$ for t sufficiently large, $t>\theta$ say. In combination with Lemma 2.2, this implies that for $t>\theta$ all zeros of $v(\cdot,t)$ in $(\tilde\beta(t),\tilde\gamma(t))$ are simple. Therefore, these zeros are C^1 functions of t; we denote them by $\xi_1(t)<\cdots<\xi_m(t)$. Choosing any continuous functions $\beta(t)$, $\gamma(t)$ with

$$\tilde\beta(t)<\beta(t)<\min\{-N,\xi_1(t)\},\ \ \max\{N,\xi_m(t)\}<\gamma(t)<\tilde\gamma(t)\ \ (t\ge\theta)$$

one checks easily that properties (B)(i)-(B)(iii) of Lemma 2.1 are satisfied.

To complete the proof of Lemma 2.1 assume that $\{v(\cdot,t)\ :t>\tau_0\}$ is relatively compact in $C_0(\mathbb{R})$. We show that either (A) holds or the function $\gamma(t)$ in (B) can be chosen bounded (for $\beta(t)$ the proof is analogous). This is obvious if the function $\xi_m(t)$ is bounded above. If it is not, we let j denote the minimal of the indices $1,\dots,m$ such that $\xi_j(t)$ is unbounded above. If there is a $\overline\theta$ such that $\xi_j(t)>N\ (t\ge\overline\theta)$, then there is a continuous bounded function $\gamma(t)$, $t\ge\overline\theta$, such that $\gamma(t)$ lies between $\xi_{j-1}(t)$ and $\xi_j(t)$ (or between N and $\xi_j(t)$ if $j=1$). For this function properties (B)(i)-(B)(iii) still hold (with θ replaced by $\overline\theta$). Let us consider the opposite case, that is, there are sequences $s_k < t_k$ approaching ∞ such that $\xi_j(s_k) \to \infty$, $\xi_j(t) > N \;\; (t \in (s_k, t_k))$, $\xi_j(t_k) = N \;\; (k = 1, 2, \dots)$. We show that this implies property (A). Indeed, using (HL) and the maximum principle as in the above proof (see the claim in case 1), one shows that

$$\sup_{x \geq N} |v(x, t_k)| \leq \sup_{x \geq \xi_j(s_k)} |v(x, s_k)|. \tag{2.6}$$

As $\{v(\cdot, s_k) : k = 1, 2 \dots\}$ is relatively compact in $C_0(\mathbb{R})$ and $\xi_j(s_k) \to \infty$, the right-hand side of (2.6), and the left-hand side along, converges to zero. Thus (A) holds as claimed. The proof of Lemma 2.1 is complete. □

Remark. The main difficulties in the above proof stem from the fact that $v(\cdot, t)$ can have infinitely many zeros (which we in general cannot rule out in applications below). If $v(x, t)$ has only finitely many zeros then Lemma 2.1 is a consequence of Lemma 2.2, 2.3. One can prove that a sufficient condition for $v(\cdot, t)$ to have only finitely many zeros for any $t > 0$ is that $v(x, 0)$ has support contained in an interval (a, b) and $v(x, 0) > 0$ for $x \in (a, b)$ near the boundary points.

The last result of this section deals with periodic solutions of (2.1).

Lemma 2.4. *Assume that $c(x, t)$ is a continuous function on $\mathbb{R}^2$ and (HL) is satisfied for some $M \geq 0$. Let $v(\cdot, t) \in C_0(\mathbb{R})$, $t \in \mathbb{R}$, be a nontrivial solution of* (2.1) *such that $v(x, t + \tau) \equiv v(x, t)$ for some $\tau > 0$. If $v^{-1}(0)$ is not empty then it is the union of finitely many nodal curves*

$$\{(\xi_j(t), t) \;\; : t \in (-\infty, \infty)\}, \;\; j = 1, \dots, m, \tag{2.7}$$

where the ξ_j are C^1 τ-periodic functions satisfying $\xi_1(t) < \xi_2(t) < \cdots < \xi_m(t)$, $(t \in \mathbb{R})$. Moreover, each of the nodal curves (2.7) *divides the set $\{(x, t) \;\; : c(x, t) > 0\}$, that is,*

$$\{(x, t) : c(x, t) > 0 \;\;\text{ and } x < \xi_j(t)\} \neq \emptyset \neq \{(x, t) : c(x, t) > 0 \text{ and } x > \xi_j(t)\} \tag{2.8}$$

Proof. Periodicity of v implies that property (A) of Lemma 2.1 does not apply to v (otherwise $v \equiv 0$ by Corollary 2.1). Thus property (B) holds, which, combined with periodicity, implies that $v(\cdot, t)$ has only simple zeros for any t. By Lemma 2.3, $v^{-1}(0)$ is the union of nonintersecting nodal curves. Let us consider any nodal curve $\{(\xi(t), t) : t \in (t_1, t_2)\}$ which is a connected component of $v^{-1}(0)$. Then, by Lemma 2.3 (ii), $t_1 = -\infty$. Since $\{v(\cdot,t)\ : t \in \mathbb{R}\}$ is compact in $C_0(\mathbb{R})$, Lemma 2.1 implies that $t_2 = \infty$ and ξ is bounded on $[0,\infty)$. We next prove that $\xi(t)$ is τ-periodic.

Let K be the (discrete) set of all zeros of $v(\cdot,\tau) \equiv v(\cdot,0)$. For any $\zeta \in K$ there is a point $\Phi(\zeta) \in K$ such that $(\zeta,0)$ and $(\Phi(\zeta),\tau)$ lie on the same nodal curve. As nodal sets do not intersect, $\Phi(\zeta)$ is uniquely determined. We claim that Φ equals the identity:

$$\Phi(\zeta) = \zeta \ \ (\zeta \in K). \tag{2.9}$$

Once this is proved, we obtain as a consequence that for any nodal curve $\{(\xi(t),t)\ : t \in (-\infty,\infty)\}$ the function $\xi(t)$ satisfies $\xi(\tau) = \xi(0)$. In view of periodicity of v, this readily implies that $\xi(t)$ is τ-periodic.

We prove (2.9) by contradiction. Suppose $\Phi(\zeta) \neq \zeta$ for some $\zeta \in K$. For definiteness let $\Phi(\zeta) > \zeta$. We claim that this implies that $\Phi^n(\zeta)$ is an increasing sequence. Indeed, otherwise there would be an n such that

$$\Phi^{n-1}(\zeta) < \Phi^n(\zeta) \geq \Phi^{n+1}(\zeta) \tag{2.10}$$

By definition of Φ, there are nodal curves $\{(\xi_i(t),t) : t \in [0,\tau]\}$, $i = 1,2$ such that $\xi_1(0) = \Phi^{n-1}(\zeta)$, $\xi_1(\tau) = \Phi^n(\zeta)$, $\xi_2(0) = \Phi^n(\zeta)$, $\xi_2(\tau) = \Phi^{n+1}(\zeta)$. In view of (2.10), these nodal curves would have to intersect, which is impossible. Thus $\Phi^n(\zeta)$ is indeed increasing, therefore unbounded because K is discrete. On the other hand, it is easily seen, by periodicity of v, that all the points $(\Phi^n(\zeta), n\tau)$ lie on one nodal curve $\{(\xi(t),t) : t \in (-\infty,\infty)\}$, contradicting boundedness of $\xi(t)$ established above. This contradictions proves (2.9).

We next prove (2.8) for each nodal curve of v. Suppose that (2.8) fails. For definiteness, assume that $\{(x,t) : c(x,t) > 0 \ \text{ and } x < \xi_j(t)\} = \emptyset$. Then c is nonpositive in $Q := \{(x,t) : t \geq 0,\ x \leq \xi_j(t)\}$. Using periodicity of v and ξ_j and the fact that $v(x,t) = 0$, at $x = \xi_j(t), x = -\infty$, we obtain that v assumes its positive maximum at (infinitely many) interior points of Q. The strong maximum principle therefore implies that $v \equiv const$ hence $v \equiv 0$ in Q. This is a contradiction, which proves (2.8).

Finally, we claim that there are only finitely many nodal curves $\{(\xi(t),t) : t \in (-\infty,\infty)\}$ of v. Indeed, if it was not true, then by property (B) of Lemma 2.1, infinitely many of these curves would be contained in the set $\{(x,t)\ : t \in \mathbb{R},\ |x| > M\}$, with M as in (HL). However, for such nodal curves (2.8) would clearly be violated. The proof of Lemma 2.4 is complete. ☐

3. Limit sets and periodic solutions. In this section we first prove that the limit sets of solutions of (1.1) contain time-periodic solutions that are even with respect to some point (see Corollary 3.1). Then we investigate the structure of τ-periodic solutions and prove Theorem 1.2.

Let $u(\cdot, t; u_0)$ denote the solution of (1.1), (1.2). We use the following notation for the limit sets of solutions defined for all $t \geq 0$:

$\tilde{\omega}(u_0) = \{\phi \in C_0(\mathbb{R}) \ : \phi = \lim_{n\to\infty} u(\cdot, t_n)$ for some sequence $t_n \in \mathbb{R}$ with $t_n \to \infty\}$,
$\omega(u_0) = \{\phi \in C_0(\mathbb{R}) \ : \phi = \lim_{j\to\infty} u(\cdot, n_j\tau)$ for some sequence $n_j \in \mathbb{N}$ with $n_j \to \infty\}$.

Here the limits are understood in $C_0(\mathbb{R})$. By standard regularity results, for trajectories relatively compact in $C_0(\mathbb{R})$ the limit sets do not change if the convergence is understood in the C^2 sup norm. Note that $\omega(u_0)$ coincides with the ω-limit set of u_0 with respect to the discrete dynamical system generated on $C_0(\mathbb{R})$ by the period map of (1.1).

Let us recall the following standard properties of the limit sets (see Hale [16] for the proof).

Lemma 3.1. *If* $\{u(\cdot, t; u_0) : t > \tau_0\}$ *is relatively compact in* $C_0(\mathbb{R})$ *then* $\tilde{\omega}(u_0)$ *is nonempty, compact and invariant for* (1.1) *in the following sense: if* $\phi \in \tilde{\omega}(u_0)$*, then there is a solution* $u(\cdot, t)$ *of* (1.1) *defined for all* $t \in \mathbb{R}$ *such that* $u(\cdot, t) \in \tilde{\omega}(u_0)$ $(t \in \mathbb{R})$ *and* $u(\cdot, t_0) = \phi$ *for some* t_0.

In the remaining part of this section we assume that (HNw) of Section 1 and the following standing hypothesis are satisfied.

(SH) $u_0 \in C_0(\mathbb{R})$ is such that the solution $u(\cdot, t; u_0)$ is defined for all $t \geq 0$ and $\{u(\cdot, t; u_0) : t > 0\}$ is relatively compact in $C_0(\mathbb{R})$.

As u_0 is fixed, we drop it from the notation of limit sets: $\tilde{\omega} = \tilde{\omega}(u_0)$, $\omega = \omega(u_0)$. For a function $\phi \in C_0(\mathbb{R})$, let $V_a\phi(x) = \phi(x) - \phi(2a - x)$, $(x, a \in \mathbb{R})$. We say ϕ is even (or symmetric) with respect to a if $V_a\phi \equiv 0$. The symmetry plays in this section a similar role as in Chen and Matano [7].

Lemma 3.2. *Let* $a \in \mathbb{R}$*. Then either* $\tilde{\omega}$ *contains a function symmetric with respect to* a *or else* $u_x(a, t; u_0) \neq 0$ *for all sufficiently large* t.

Proof. Let $v(x, t) = V_a u(x, t; u_0)$. Then v satisfies linear equation (2.1) with

$$c(x, t) = \int_0^1 f_u(t, su(x, t; u_0) + (1 - s)u(2a - x, t; u_0))ds.$$

By hypotheses (HNw) and (SH), $c(x,t)$ satisfies (HL) of Lemma 2.1. Note that $v(a,t)=0$ $(t>0)$ thus $v_x(a,t)=0$ means that a is a multiple zero of $v(\cdot,t)$. An application of Lemma 2.1 shows that either $v_x(a,t)\neq 0$ for all sufficiently large t or else $\tilde{\omega}$ contains a point ϕ for which $V_a\phi$ vanishes on an interval. Now, by invariance of $\tilde{\omega}$, the solution $u(\cdot,t,\phi)$ is defined for all $t\in\mathbb{R}$. As $V_a u(x,t;\phi)$ solves a linear equation (2.1), it follows from Corollary 2.1 that $V_a u(x,t;\phi)\equiv 0$. In particular, ϕ is symmetric, as desired. □

Lemma 3.3. *There exists an $a\in\mathbb{R}$ such that $\tilde{\omega}$ contains a function symmetric with respect to a.*

Proof. By Lemma 3.2, if there is an x such that $\operatorname{sgn} u_x(x,t;u_0)$ does not stabilize as $t\to\infty$ then $\tilde{\omega}$ contains a symmetric point. Assume that $\operatorname{sgn} u_x(x,t;u_0)$ stabilizes to a value $s(x)\in\{-1,1\}$ for any $x\in\mathbb{R}$. If $s(x)$ is the same for each x then every $\phi\in\tilde{\omega}$ is monotone in x, hence $\phi=0$ as $\tilde{\omega}\subset C_0(\mathbb{R})$. In the opposite case, there is an a such that $\phi_x(a)=0$ for any $\phi\in\tilde{\omega}$. Then, by invariance of $\tilde{\omega}$, $u_x(a,t;\phi)=0$ $(t\in\mathbb{R})$. Applying Lemma 3.2 with u_0 replaced by ϕ, we obtain that $\tilde{\omega}(\phi)\subset\tilde{\omega}$ contains a function symmetric with respect to a. □

Lemma 3.4. *Assume u_0 is symmetric with respect to an $a\in\mathbb{R}$. Then either $\tilde{\omega}=\{0\}$ or else for each $x\neq a$ $\operatorname{sgn} u_x(x,t;u_0)$ stabilizes as $t\to\infty$.*

Proof. If the latter is not true then, by Lemma 3.2, $\tilde{\omega}$ contains a function ϕ symmetric with respect to a point $b\neq a$. Clearly such a ϕ also inherits the symmetry of u_0, thus it is symmetric with respect to a and b. It follows that it is periodic with period $2a-2b$ and therefore, being an element of $C_0(\mathbb{R})$, it necessarily equals 0. As $\phi=0$ is stable (cf. Section 1), $0\in\tilde{\omega}$ implies $\tilde{\omega}=\{0\}$. □

Lemma 3.5. *Let u_0 be symmetric with respect to an $a\in\mathbb{R}$. Then there is a $\phi\in\tilde{\omega}$ such that $u(\cdot,t;\phi)$ is τ-periodic in t.*

Proof. Denote $v(x,t)=u(x,t+\tau;u_0)-u(x,t;u_0)$ and observe that v solves a linear equation (2.1) with the coefficient $c(x,t)$ satisfying (HL) for some M (the latter follows by (SH) and (HNw)). Thus Lemma 2.1 applies. Assume first that property (A) of that lemma holds: for a sequence $t_n\to\infty$ one has $v(x,t_n)\to 0$ uniformly on an interval J. Replacing t_n by a subsequence, if necessary, we have $u(\cdot,t_n;u_0)\to\phi\in\tilde{\omega}$ and $u(\cdot,t_n+\tau;u_0)\to\psi\in\tilde{\omega}$. Then, obviously, $\psi=u(\cdot,\tau;\phi)$ and $\phi-\psi$ vanishes on J. Set $w(x,t):=$

$u(x, t+\tau; \phi) - u(x, t; \phi)$. Then $w(x, t)$ is defined on $\mathbb{R}^2$ and solves there a linear equation (2.1). As $w(x, 0) = \psi(x) - \phi(x) = 0$ $(x \in J)$, Corollary 2.1 gives $w(x, t) \equiv 0$. Thus $u(\cdot, t, \phi)$ is the sought τ-periodic solution.

Now assume that property (B) of Lemma 2.1 applies to v. As $v(\cdot, t)$ is symmetric with respect to a for any t, we have $v_x(a, t) \equiv 0$. Therefore, in accordance with (B), $v(a, t) \neq 0$ for all sufficiently large t. For such t, $\{u(a, t+n\tau; u_0)\}_n$ is a monotone bounded sequence. Let $\eta(t)$ be its limit. Clearly

$$\eta(t+\tau) = \eta(t) \ \ (t > \theta). \tag{3.1}$$

We now prove that $w(x, t) := u(\cdot, t+\tau; \phi) - u(\cdot, t; \phi) \equiv 0$ for any $\phi \in \omega \subset \tilde{\omega}$. To see this, first observe that $\phi(a) = \eta(0)$ and $u(a, t; \phi) = \eta(t)$. Next, as ϕ inherits the symmetry of $u(\cdot, t; u_0)$, $u_x(a, t; \phi) = 0$. This and (3.1) yield $w(a, t) = w_x(a, t) \equiv 0$. By Lemma 2.2, this is possible only if $w(x, t) \equiv 0$. This completes the proof. □

Corollary 3.1. *For any u_0 satisfying (SH), $\tilde{\omega}(u_0)$ contains a function ϕ that is symmetric with respect to some $a \in \mathbb{R}$ and such that the solution $u(x, t; \phi)$ is τ-periodic in time.*

Proof. By Lemma 3.3, $\tilde{\omega}$ contains a symmetric function ψ. Applying Lemma 3.5 with $u_0 = \psi$ and using the invariance of $\tilde{\omega}$ we obtain the conclusion of Corollary 3.1. □

We can now give a proof of Theorem 1.2. We do it in the following propositions.

Proposition 3.1. *Let $p(\cdot, t) \in C_0(\mathbb{R})$ be a τ-periodic solution of* (1.1). *Then either $p(x, t) \equiv 0$ or else there is an $a \in \mathbb{R}$ such that $p(\cdot, t)$ is symmetric with respect to a and $p_x(x, t) \neq 0$ for any $t \in \mathbb{R}$, $x \neq a$ (in particular, $p(x, t) \neq 0$ for all $x, t \in \mathbb{R}$ in this case).*

Proof. As $\tilde{\omega}(p(\cdot, 0)) = \{p(\cdot, t) : t \in [0, \tau]\}$, Corollary 3.1 implies that $p(\cdot, t)$ is symmetric with respect to an $a \in \mathbb{R}$ (this is true for some t and consequently for any t). If for some t and $x \neq a$ one has $0 = p_x(x, t) = p_x(x, t+n\tau)$ then, by Lemma 3.4, $p \equiv 0$. Finally note that $p_x(x, t) \neq 0$ for $x \neq a$ and $p(\cdot, t) \in C_0(\mathbb{R})$ imply $p(x, t) \neq 0$ for any $x, t \in \mathbb{R}$. □

Proposition 3.2. *Let $p^j(\cdot, t) \in C_0(\mathbb{R})$, j=1,2, be two nonzero τ-periodic solutions of* (1.1) *that are of the same sign. Then there is a $b \in \mathbb{R}$ such that $p^1(x, t) \equiv p^2(x - b, t)$.*

Proof. Without loss of generality assume that these solutions are both positive. For $\lambda \in \mathbb{R}$, let $v^\lambda(x,t) = v(x,t,\lambda) = p^1(x,t) - p^2(x-\lambda,t)$ $(x,t \in \mathbb{R})$. Suppose $v^\lambda(x,t) \not\equiv 0$ for any λ. We find a contradiction. Observe that for any λ, $v^\lambda(\cdot,t)$ is a τ-periodic solution of the linear equation (2.1), where

$$c(x,t) = \int_0^1 f_u(t, sp^1(x,t;u_0) + (1-s)p^2(x-\lambda,t;u_0)ds. \tag{3.2}$$

This coefficient satisfies condition (HL) (for some M depending on λ). Therefore, by Lemma 2.4, $v^\lambda(\cdot,t)$ has for each t the same finite number of zeros, each of which is simple. Let a_i be the point where $p^i(\cdot,t)$ assumes its maximum (cf.Proposition 3.1). If $p^1(a_1,t) = p^2(a_2,t)$ for some t then there is a λ such that $v^\lambda(\cdot,t)$ has a multiple zero which is impossible. We may therefore assume, interchanging the roles of p^1, p^2 if necessary, that $p^2(a_2,t) > p^1(a_1,t)$ $(t \in \mathbb{R})$. Then of course

$$p^2(a_2,t) > p^1(x,t) \ \ (x,t \in \mathbb{R}). \tag{3.3}$$

Taking $\lambda = \lambda_0$, where λ_0 is large negative, we see that $v^\lambda(\cdot,0)$ has a zero to the right of $a_2+\lambda$. Let η^λ be the minimum of all such zeros. We now start to increase λ. As all the zeros of $v^\lambda(\cdot,0)$ remain simple and

$$v^\lambda(a_2+\lambda,t) > 0, \tag{3.4}$$

(cf. (3.3)), there exists a $\lambda_m \in (\lambda_0,\infty]$ such that the minimal zero $\eta^\lambda > a_2+\lambda$ is defined for $\lambda \in [\lambda_0,\lambda_m)$ and

$$\eta^\lambda \to \infty \ \text{ as } \lambda \to \lambda_m. \tag{3.5}$$

For $\lambda \in [\lambda_0,\lambda_m)$ let $\{(\xi^\lambda(t),t) : t \in [0,\tau]\}$ be the nodal curve of v^λ with $\xi^\lambda(0) = \eta^\lambda$. Then

$$\xi^\lambda(t) \to \infty \ \text{ as } \ \lambda \to \lambda_m \quad \text{uniformly for } \ t \in [0,\tau]. \tag{3.6}$$

Indeed, if $\lambda_m = \infty$ then this follows trivially from (3.3). Otherwise it follows from (3.5) and the structure of the nodal curves of v^λ for $\lambda = \lambda_m$ (cf. Lemma 2.4). As $p^1(\cdot,t) \in C_0(\mathbb{R})$, (3.6) implies that for λ sufficiently close to λ_m one has

$$p^1(x,t) < \delta \ \ (t \in \mathbb{R}, x \geq \xi^\lambda(t)\}, \tag{3.7}$$

where δ is as in hypotheses (HNw). Further, as $p^2(x-\lambda,t)$ is decreasing for $x > a_2+\lambda$, we see that for λ as in (3.7) and $x > \xi^\lambda(t)$ one has

$$0 < p^2(x-\lambda,t) < p^2(\xi^\lambda(t)-\lambda,t) = p^1(\xi^\lambda(t),t) < \delta.$$

Using this, together with (3.7) and hypothesis (HNw), we obtain that the function $c(x,t)$ in (3.2) satisfies $c(x,t) \le 0$ $(t \in \mathbb{R}, x \ge \xi^\lambda(t))$. This contradicts the last statement of Lemma 2.4. □

4. Linearization at periodic solutions. In this section we consider eigenvalue problems associated with linearization along nonzero τ-periodic solutions $p(\cdot,t) \in C_0(\mathbb{R})$ of (1.1). Recall that for any such solution $p(x,t)$ is different from zero everywhere and there is a $b \in \mathbb{R}$ such that $p(\cdot,t)$ is symmetric (even) with respect to b and $p_x(x,t) \neq 0$ for $x \neq b$.

Theorem 4.1. *Assume (HNs). Let $p(\cdot,t) \in C_0(\mathbb{R})$ be a nonzero τ-periodic solution of* (1.1)*. Let $\Pi : C_0(\mathbb{R}) \to C_0(\mathbb{R})$ be the period τ map of the linear equation*

$$v_t = v_{xx} + f_u(t,p(x,t))v \tag{4.1}$$

and $\sigma(\Pi)$ its spectrum. Then $\sigma(\Pi) = \{\lambda_1, 1\} \cup \sigma_2$, where σ_2 is a compact subset of $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$, $\lambda_1 > 1$ and both λ_1 and 1 are (algebraically) simple eigenvalues of Π. Furthermore, λ_1 has an everywhere positive eigenfunction φ that is symmetric with respect to the same point as $p(\cdot,t)$ and the eigenvalue 1 has the eigenfunction $p_x(\cdot,0)$.

In the remaining part of this section we assume the solution $p(x,t)$ to be fixed. Shifting the origin in $\mathbb{R}$, we may assume that the point of symmetry of p is at 0: $p(-x,t) \equiv p(x,t)$ and $p_x(x,t) \neq 0$ for $x \neq 0$. Denote $c(x,t) := f_u(t,p(x,t))$. By (HNs), there are positive constants δ and M such that

$$c(x,t) < -\delta \ \ (|x| \ge M, t \in \mathbb{R}). \tag{4.2}$$

The proof of Theorem 4.1 is split in several lemmas. The first one deals with the essential spectrum $\sigma_{ess}(\Pi)$ of Π. By definition, $\sigma_{ess}(\Pi)$ consists of those $\lambda \in \mathbb{C}$ for which one of the following properties is satisfied:

(i) λ is a limit point of $\sigma(\Pi)$,

(ii) the range of $\lambda - \Pi_c$ is not closed (Π_c is the complexification of Π),

(iii) the space $\cup_{k\ge1} \ker(\lambda - \Pi)^k$ is infinite dimensional.

Note that any element of $\sigma(\Pi) \setminus \sigma_{ess}(\Pi)$ is a pole of the resolvent of Π, and, in particular, it is an eigenvalue of Π (see e.g. Kato [21]).

Lemma 4.1. *$\sigma_{ess}(\Pi) \subset \{\lambda \in \mathbb{C} : |\lambda| < e^{-\delta\tau}\}$, with δ as in (4.2).*

The proof can be found in Daners and Koch [10] (see Proposition 6.1 of [10], where higher-dimensional space variables are treated as well).

Lemma 4.2. *$\lambda = 1$ is an eigenvalue of Π with the eigenfunction $p_x(x, 0)$.*

Proof. This is obtained by differentiation of (1.1) with respect to x. Note that the differentiation is justified by standard regularity results and $u_x(\cdot, t) \in C_0(\mathbb{R})$ because $u(\cdot, t) \in \{\varphi \in C^2(\mathbb{R}) : \varphi, \varphi_{xx} \in C_0(\mathbb{R})\}$ which is the domain of the $C_0(\mathbb{R})$ realization of $\varphi \to -\varphi_{xx}$ (cf. Lunardi [23]). □

Lemma 4.3. *The spectral radius $r(\Pi)$ of Π is a simple eigenvalue of Π and it has a positive even eigenfunction. In particular, $r(\Pi) > 1$.*

Proof. We apply the following theorem on positive operators.
(TPO) *Let Y be a Banach lattice with the positive cone Y_+. Let $T \in \mathcal{L}(X)$ be an irreducible positive operator on Y. Then $r(T) \in \sigma(T)$. Moreover, if $r(T)$ is a pole of the resolvent of T then it is a simple eigenvalue of T, it has an eigenvector in Y_+, and no other eigenvalue has eigenvector in Y_+.*

Recall that an operator on Y is positive if $TY_+ \subset Y_+$. Such an operator is called irreducible if there is no nontrivial proper closed subspace Y_1 of Y that is T-invariant and solid (that is, $x \in Y$, $y \in Y_1$ and $|x| \le |y|$ together imply $x \in Y_1$; here $|\cdot|$ is the modulus associated with the lattice structure). The proof of (TPO) can be found in Schaeffer [31] (Proposition V.4.1 of [31] gives $r(T) \in \sigma(T)$ and Theorem V.5.2 and its corollary yield the remaining properties).

To apply this theorem, note that $X := C_0(\mathbb{R})$ is a Banach lattice with the pointwise ordering and, by the maximum principle, Π is a positive operator on X. We verify that Π is irreducible. The modulus $|u|$ of a $u \in X$ is obviously given by $|u|(x) = |u(x)|$. Therefore, if a solid subspace $X_1 \subset X$ contains an everywhere positive function φ (and with it any function $\alpha\varphi$, $\alpha > 0$) then it contains all functions with compact support. Thus if X_1 is closed in X then it coincides with X. Now, any solid space contains a nonnegative function ψ and, if it is Π invariant, it contains an everywhere positive function $\Pi(\psi)$. From these properties it follows that Π is irreducible.

By (TPO), $r(\Pi) \in \sigma(\Pi)$. As $\lambda = 1$ is an eigenvalue of Π, we have $r(\Pi) \ge 1$. In particular, by Lemma 4.1, $r(\Pi) \in \sigma(\Pi) \setminus \sigma_{ess}(\Pi)$ and therefore $r(\Pi)$ is a pole of the resolvent. The conclusion of Lemma 4.3 now becomes a direct consequence of (TPO). We have $r(\Pi) > 1$, for otherwise $r(\Pi) = 1$

would not be a simple eigenvalue (in addition to the positive eigenfunction, it would have the eigenfunction $p_x(\cdot, 0)$.) □

In the proof of the next lemma, X_e and X_o stand for the subspaces of $X = C_0(\mathbb{R})$ consisting of all even and odd functions, respectively. As $p(x,t)$ is even in x, both X_e and X_o are invariant under Π. Denote $\Pi_e := \Pi|_{X_e}$, $\Pi_o := \Pi|_{X_o}$. Note that, since $X = X_e \oplus X_o$, each eigenvalue of Π is an eigenvalue of one of the operators Π_e, Π_o.

Lemma 4.4. *Let $\lambda \in \sigma(\Pi) \cap \{\lambda : |\lambda| \geq 1\}$. Then λ is a positive real eigenvalue of* Π.

Proof. By Lemma 4.1, λ is an eigenvalue. We prove that it is real. Suppose to the contrary that

$$\lambda = Re^{i\omega},\ R \geq 1,\ \omega \in \mathbb{R} \setminus 2\pi\mathbb{Z}. \tag{4.3}$$

Then there is a two-dimensional Π-invariant (real) subspace $X_1 \subset X$ such that in an appropriate basis $\Pi|_{X_1}$ is represented by the matrix

$$R \begin{bmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{bmatrix}$$

As remarked above, $\lambda \in \sigma(\Pi_e) \cup \sigma(\Pi_o)$, hence $X_1 \subset X_e$ or $X_1 \subset X_o$.

We claim that any nonzero function in X_1 has only simple zeros. As $\Pi X_1 = X_1$, this is equivalent to

$$\Pi(\varphi) \text{ has only simple zeros for any } \varphi \in X_1. \tag{4.4}$$

To demonstrate (4.4), we use the following two consequences of the fact that $R^{-1}\Pi$ acts on X_1 as a rotation.

(P1) For any $\varphi \in X_1$ the sequence $R^{-n}\Pi^n\varphi$ is contained in a compact set in $X_1 \setminus \{0\}$.

(P2) For any $\varphi \in X_1$ there is an increasing sequence n_k such that $R^{-n_k}\Pi^{n_k}\varphi \to \varphi$.

Let $v(\cdot, t, v_0)$ denote the solution of (4.1) with $v(\cdot, 0) = v_0$. Set $\rho := \frac{\log R}{\tau} \geq 0$. We now show that the alternative (A) of Lemma 2.1 does not apply to $v(\cdot, t, \varphi)$. Let t_k be any sequence with $t_k \to \infty$ and $t_k = m_k\tau + \theta_k$, $m_k \in \mathbb{N}$,

$\theta_k \in [0, \tau)$. Passing to a subsequence, we may assume that θ_k converges to a $\theta_0 \in [0, \tau]$ and, at the same time,

$$R^{-m_k}\Pi^{m_k}\varphi \to \psi, \quad R^{-m_k-1}\Pi^{m_k+1}\varphi \to \psi_1 \tag{4.5}$$

for some $\psi, \psi_1 \in X_1$. Observe that $w = e^{-\rho t}v$ solves the equation

$$w_t = w_{xx} + (c(x,t) - \rho)w.$$

Using periodicity of $c(x,t)$ and continuous dependence on initial conditions, one checks easily that (4.5) implies $\psi_1 = Rv(\cdot, \tau, \psi)$ and

$$[e^{-\rho t_k}v(\cdot, t_k, \varphi) \to e^{-\rho\theta_0}v(\cdot, \theta_0, \psi_1) = e^{-\rho(\tau+\theta_0)}v(\cdot, \tau + \theta_0, \psi) \tag{4.6}$$

(the convergence is in the norm of X). As $v(\cdot, \tau + \theta, \psi)$ has only isolated zeros (cf. Corollary 2.1) and $\rho \geq 0$, (4.6) clearly rules out alternative (A) of Lemma 2.1. We can thus use Lemma 2.1(B) to conclude that, given any $N > 0$, $v(\cdot, t, \varphi)$ has only simple zeros in $[-N, N]$ for all sufficiently large t. Using this conclusion we now derive (4.4).

Suppose that $\Pi(\varphi) = v(\cdot, \tau, \varphi)$ has a multiple zero x_0. We find a contradiction. Let n_k be as in (P2). Then, by continuous dependence on initial conditions and parabolic regularity, $e^{-\rho(n_k\tau+\theta)}v(\cdot, n_k\tau + \theta, \varphi) \to e^{-\rho\theta}v(\cdot, \theta, \varphi)$, as $k \to \infty$, uniformly for $\theta \in [0, 2\tau]$, where the convergence is in the C^1 norm. Using the description of the structure of the nodal set of $v(x, t, \varphi)$ near (x_0, τ), as given Lemma 2.2, one checks easily that for all sufficiently large k, $v(\cdot, n_k\tau + \theta, \varphi)$ has a multiple zero near x_0 for some $\theta \in [0, 2\tau]$ - a contradiction to the above conclusion.

We have thus demonstrated that $\lambda \notin \mathbb{R}$ implies that any nonzero function in the two-dimensional eigenspace X_1 has only simple zeros. However, if φ, ψ is a basis of X_1 then these functions are both even or both odd, hence some nontrivial linear combination of these functions has a multiple zero. This contradiction shows that λ must be real.

To see that $\lambda > 0$, we choose an eigenfunction φ corresponding to λ: $v(\cdot, \tau, \varphi) = \lambda\varphi$. As $e^{\rho t}v(\cdot, t, \varphi)$ is 2τ-periodic for $\rho = -\frac{\log|\lambda|}{\tau}$, $v(\cdot, t, \varphi)$ has only simple zeros for any t. Therefore, $v(0, t, \varphi) \neq 0$ $(t \in \mathbb{R})$ in case φ is even and $v_x(0, t, \varphi) \neq 0$ $(t \in \mathbb{R})$ in case φ is odd. In either case, $v(0, \tau, \varphi) = \lambda\varphi(0)$ or $v_x(0, \tau, \varphi) = \lambda\varphi_x(0)$ implies $\lambda > 0$. □

Lemma 4.5. $\lambda = 1$ *is a simple eigenvalue of* Π_o (*with the eigenfunction* $p_x(\cdot, 0)$) *and*

$$\sigma(\Pi_o) \setminus \{1\} \subset \{\lambda \ : |\lambda| < \varrho\} \tag{4.7}$$

for some $\varrho < 1$.

Proof. Note that X_o has the structure of a Banach lattice with ordering given by the cone $X_o^+ = \{\varphi \in X : \varphi(x) \geq 0 \text{ for any } x \geq 0)\}$ and Π_o is a positive operator on X_o. Applying (TPO) similarly as in the proof of Lemma 4.3, one obtains that

(i) $r(\Pi_o) = 1$ (because 1 has the eigenfunction $p_x(\cdot, 0) \in X_o^+$), and

(ii) $\lambda = 1$ is a simple eigenvalue of Π_o.

By Lemma 4.4, there is no eigenvalue $\lambda \neq 1$ of Π_o with $|\lambda| = 1$. This and Lemma 4.1 imply (4.7). □

Lemma 4.6. $\sigma(\Pi_e) \cap \{\lambda \ : |\lambda| \geq 1 \ \ \textit{and}\ \lambda \neq r(\Pi)\} = \emptyset$.

Proof. Suppose the intersection is not empty and let μ be its element. By Lemma 4.4, μ is an eigenvalue and it is positive, thus $\mu \geq 1$. Let $\varphi \in X_e$ be an eigenfunction of μ: $v(\cdot, \tau, \varphi) = \mu\varphi$. We first derive a contradiction for $\mu = 1$. In this case both $v(x, t, \varphi)$ and $p_x(x, t)$ are τ-periodic solutions of (4.1) and so is any their linear combination. As these functions are linearly independent (one is odd the other even) any linear combination is not identical to zero. Therefore, by Lemma 2.4, for each $\beta \in \mathbb{R}$, $p_x(\cdot, t) - \beta v(\cdot, t, \varphi)$ has only simple zeros that depend periodically on t. As $p_x(x, t) \neq 0$ $(x > 0)$, using a continuation argument as in the proof of Proposition 3.2, we find a β such that $p_x(\cdot, t) - \beta v(\cdot, t, \varphi)$ has a nodal curve contained entirely in the set $\{(x, t) : x \geq M\}$. This contradicts the last assertion of Lemma 2.4.

In case $\mu = 1$ we are done. We proceed assuming $\mu > 1$.

We use a comparison of zero sets of eigenfunctions of Π that is analogous to the classical Sturm comparison theorem for second order ODE. We remark that similar comparison arguments were used in a different context in Chen and Poláčik [8]. Denote $\nu := log\mu > 0$ and $w(x, t) := e^{-\nu t} v(x, t, \varphi)$. Then w is a τ-periodic solution of $w_t = w_x x + (c(x, t) - \nu)w$, where $c(x, t) - \nu < 0$ for $|x| \geq M$. By Lemma 2.4, the zeros $\xi_1(t) < \cdots < \xi_m(t)$ of $w(\cdot, t)$ are simple, finite in number, and τ-periodically depending on t. This of course remains valid for v. In particular, as $v(x, t)$ is even in x, $v(0, t) \neq 0$ $(t \in \mathbb{R})$. Replacing φ by $-\varphi$, if necessary, we may assume that $v(0, t)$ has the sign opposite to the sign of $p_x(x, t)$ for $x > 0$. For definiteness assume $p_x(x, t) > 0$ $(x > 0)$ (the other case is analogous). Thus

$$v(0, t) < 0 < p_x(x, t) \ \ (x > 0, t \in \mathbb{R}). \tag{4.8}$$

By (TPO), φ must have a zero (it corresponds to an eigenvalue different from $r(\Pi)$) and, as it is even and $\varphi(0) \neq 0$, it has a positive zero. We distinguish two cases:

(i) φ has at least two positive zeros.

(ii) φ has a unique positive zero.

In both cases we derive a contradiction.

Consider **(i)**. Let $\zeta_{01} < \zeta_{02}$ be the first two positive zeros of $\varphi = v(\cdot, 0, \varphi)$ and let $\zeta_1(t)$, $\zeta_2(t)$ be the zeros of $v(\cdot, t, \varphi)$ such that $\zeta_i(0) = \zeta_{0i}$, $i = 1, 2$ (the $\zeta_i(t)$ are assumed continuous in t). By (4.8), we have $v(x,t) > 0$, $(x \in (\zeta_1(t), \zeta_2(t)),\ t \in \mathbb{R})$. Clearly, there is $\beta > 0$ such that

$$0 < \beta\varphi(x) < p_x(x,0) \ \ (x \in (\zeta_{01}, \zeta_{02})) \tag{4.9}$$

Furthermore,

$$p_x(x,t) > 0 \ \ (x \geq 0, t \geq 0) \ \text{ and } v(\zeta_i(t), t) = 0, \ \ (t \geq 0, i = 1, 2). \tag{4.10}$$

By the maximum principle, (4.10) and (4.9) together imply that $\beta v(x,t) < p_x(x,t)$, $(x \in [\zeta_1(t), \zeta_2(t)],\ t > 0)$. This is absurd, however, because $p_x(\cdot, t)$ is τ-periodic, whereas $v(x, n\tau, \varphi) = \mu^n \varphi(x) \to \infty$, $(x \in (\zeta_{10}, \zeta_{02}))$. Now consider **(ii)**. Let $\zeta(t)$ be the unique positive zero of $v(\cdot, t)$. Then

$$\begin{aligned} v(x,t) < 0 \ \ (x \in [0, \zeta(t))), \\ v(x,t) > 0 \ \ (x \in (\zeta(t), \infty)). \end{aligned} \tag{4.11}$$

As $\zeta(t)$ is periodic, there is an x_0 satisfying $x_0 \geq \max\{M, \zeta(t)\}$ (for any t), where M is as in (4.2). By (4.11), there is a $\beta_1 > 0$ such that

$$\beta v(x, 0, \varphi) = \beta\varphi(x) < p_x(x, 0) \ \ (x \in [0, x_0], \beta \in (0, \beta_1).$$

Since we also have $v(0, t, \varphi) < 0 = p_x(0, t)$, the maximum principle implies that

$$\beta v(x, t, \varphi) < p_x(x, t) \ \ (x \in [0, x_0)) \tag{4.12}$$

as long as

$$\beta v(x_0, t, \varphi) < p_x(x_0, t). \tag{4.13}$$

Since $v(x_0, n\tau) = \mu^n \varphi(x_0) \to \infty$, (4.13) fails for some t. Let $t_m = t_m(\beta)$ be the minimal t with this property. Then (x_0, t_m) is a zero of $\beta v(\cdot, \cdot, \varphi) - p_x(\cdot, t)$

and, by Lemma 2.4, it is connected by a nodal curve $\{(\xi(t), t) : t \in [0, t_m]\}$ to a point $(\xi_0, 0)$. (Of course, ξ_0 depends on β.) In view of (4.12), this nodal curve is contained entirely in the set $\{(x, t) : x \geq x_0\}$ where $c(x, t) \leq 0$ (because $x_0 \geq M$). The maximum principle therefore yields (cf. the claim at the beginning of the proof of Lemma 2.1).

$$\sup_{x \geq x_0} |\beta v(x, t_m, \varphi) - p_x(x, t_m)| \leq \sup_{x \geq \xi_0} |\beta v(x, 0, \varphi) - p_x(x, 0)| = \sup_{x \geq \xi_0} |\beta \varphi(x) - p_x(x, 0)|. \tag{4.14}$$

Being a zero of $\beta\varphi(\cdot) - p_x(\cdot, 0)$, $\xi_0 = \xi_0(\beta)$ clearly approaches ∞ as $\beta \to 0+$. From (4.14) we therefore infer that

$$\sup_{x \geq x_0} |\beta v(x, t_m(\beta), \varphi) - p_x(x, t_m(\beta))| \to 0 \quad \text{as } \beta \to 0+. \tag{4.15}$$

Choose a sequence $\beta_j \to 0+$ such that $t_m(\beta_j) = n_j\tau + \theta_j$, where $n_j \in \mathbb{N}$ and $\theta_j \to \theta_0 \in [0, \tau]$. Then $v(x, t_m(\beta_j), \varphi) = \mu^{n_j} v(x, \theta_j, \varphi)$ and (4.15) implies

$$\sup_{x \geq x_0} |\beta_j \mu^{n_j} v(x, \theta_j, \varphi) - p_x(x, \theta_j)| \to 0 \quad \text{as } j \to \infty. \tag{4.16}$$

If $\beta_j \mu^{n_j} \to \infty$ then (4.16) yields $v(x, \theta_0, \varphi) = 0$, $(x \geq x_0)$, contradicting the fact that $v(\cdot, \theta_0, \varphi)$ has only simple zeros. On the other hand, if $\beta_j \mu^{n_j} \to d < \infty$, at least for a subsequence, then (4.16) yields

$$dv(x, \theta_0, \varphi) - p_x(x, \theta_0) = 0 \quad (x \geq x_0).$$

This is a contradiction to Corollary 2.1 for $dv - p_x$ is a nontrivial solution of (4.1). Lemma 4.6 is proved. □

Proof of Theorem 4.1. The conclusion of the theorem follows from Lemmas 4.2, 4.3, 4.5, 4.6. □

5. Invariant manifolds and convergence. In this section we prove Theorem 1.1. We start by a description of the stable and center manifolds of nonzero periodic solutions of (1.1).

Let F be the period τ map of (1.1), that is, $F(u_0) = u(\tau, \cdot, u_0)$, where $u_0 \in X = C_0(\mathbb{R})$ and $u(t, \cdot, u_0)$ is the solution of (1.1), (1.2). F is a C^1 X-valued map defined on an open set in X consisting of the initial conditions for which the solution exists up to time τ. Obviously, ϕ is a fixed point of F if and only if $p(\cdot, t) = u(\cdot, t, \phi)$ is a τ-periodic solution of (1.1). At any such fixed point,

$F'(\phi) = \Pi$, where Π is the period map of the linear variational equation (4.1). If $\phi \neq 0$ then by Theorem 4.1, $\lambda = 1$ is a simple eigenvalue of Π with eigenfunction $\psi := \phi_x$ and there is another simple eigenvalue $\lambda_1 > 1$ with eigenfunction φ symmetric with respect to the same point as ϕ and positive; the remaining part of $\sigma(\Pi)$ is contained in a compact subset of $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$. This form of the spectrum implies that there exist one-dimensional local center and unstable manifolds of ϕ. More specifically, there exist a neighborhood U of ϕ in X and two one-dimensional Lipschitz submanifolds of U, W^c_{loc}, W^u_{loc}, such that the following properties are satisfied:

(M1) Both W^c_{loc} and W^u_{loc} contain ϕ and W^u_{loc} is a submanifold of class C^1 that is tangent at ϕ to $\text{span}\{\varphi\}$.

(M2) For $z \in U$ one has $z \in W^u_{loc}$ if and only if z has a negative trajectory (that is a sequence $\{z_{-n}\}_{n=0,1,\dots}$, satisfying $F(z_{-n-1}) = z_{-n}$ and $z_0 = z$) which is contained in U and converges to ϕ exponentially:

$$\|z_{-n} - \phi\|_X \leq C\varrho^n, \quad n = 0, 1, 2, \dots,$$

for some $C > 0$ and $\varrho < 1$.

(M3) $\text{Fix}(F) \cap U \subset W^c_{loc}$.

(M4) Let S be any of the sets W^c_{loc}, W^u_{loc}. Then S is locally invariant in the following sense: if $z \in S$ and $F(z) \in U$, then $F(z) \in S$ and if $z \in U$ and $F(z) \in S$, then $z \in S$.

These are standard properties of the local center manifold and (strong) local unstable manifold of maps (see for example Chen, Chen and Hale [5, Appendix] or Henry [19], Lunardi [23]). One can also show, applying abstract results, that W^c_{loc} is of class C^1 and is tangent at ϕ to ψ. This would require some additional work (modification of the nonlinearity f near the periodic solution $p(x,t)$) which is not necessary here. We shall see in the proof of the following theorem that in the present situation W^c_{loc} is actually a C^1 curve of fixed points.

Theorem 5.1. *Assume (HNs). Let $\phi \in \text{Fix}(F) \setminus \{0\}$. Then there is neighborhood V of ϕ in X such that the following properties are satisfied:*

(i) $W^c_{loc} \cap V \subset \text{Fix}(F)$.

(ii) $W^u_{loc} \cap V$ *is simply ordered and consists entirely of symmetric sub and supersolutions with no zero point. More specifically, for any* $z_1, z_2 \in W^u_{loc}$, $z_1 \neq z_2$, *one has*

$$\textit{either} \;\; z_1(x) < z_2(x) \;(x \in \mathbb{R}) \quad \textit{or} \;\; z_2(x) < z_1(x) \;(x \in \mathbb{R}), \tag{5.1}$$

and if $z \in W^u_{loc} \cap V \setminus \{\phi\}$ *then* z *is symmetric with respect to the same point as* ϕ, *it has no zero and* (5.1) *holds with* $z_1 = z$, $z_2 = F(z)$.

Proof. (i) By spatial homogeneity of (1.1), $\phi(\cdot + b) \in \mathrm{Fix}(F)$ for any $b \in \mathbb{R}$. Thus, ϕ lies on a curve of fixed points. As W^c_{loc} is also a curve and contains all fixed points in U (cf. (M3)), these curves must locally coincide.

(ii) Shifting the origin in $\mathbb{R}$ we may assume that ϕ is even. As λ_1 has an even eigenfunction, ϕ has a one-dimensional unstable manifold relative to $F|_{X_e}$, the restriction of F to the space of even functions. By (M2), this manifold coincides with W^u_{loc}, hence $W^u_{loc} \subset X_e$.

By (M1), near ϕ the manifold W^u_{loc} coincides with a curve

$$\{\phi + s\varphi + g(s) \;\; : s \in (-\epsilon, \epsilon)\}, \tag{5.2}$$

where $g \;: (-\epsilon, \epsilon) \to X$ is a C^1 function with

$$g(0) = 0 \;\; \text{and} \; g'(0) = 0. \tag{5.3}$$

We assume that the eigenfunction φ is normalized in X: $\|\varphi\|_X = 1$.

We next prove (5.1). Fix any two distinct points $z_1, z_2 \in W^u_{loc}$. There are solutions $u_1(\cdot, t), u_2(\cdot, t) \in X$ of $u_t = u_{xx} + f(t, u)$ on $(-\infty, 0]$ such that $u_i(\cdot, 0) = z_i$, and

$$\|u_i(\cdot, t) - p(\cdot, t)\| \to 0 \;\; \text{as } t \to -\infty \;\; (i = 1, 2), \tag{5.4}$$

where $p(x, t)$ is the τ-periodic solution of (1.1) associated with ϕ. Indeed, if $\{z^i_{-n}\} \in U$ is the negative trajectory of z_i then u_i is obtained by defining $u_i(\cdot, -n\tau + s) = u(\cdot, s, z^i_{-n})$ for $s \in [0, \tau]$. Clearly, these solutions are even in x. Set $v(\cdot, t) = u_2(\cdot, t) - u_1(\cdot, t)$. Observe that v solves a linear equation (2.1) with

$$c(x, t) = \int_0^1 f_u(t, su_2(x, t) - (1 - s)u_1(x, t))ds. \tag{5.5}$$

By (5.4), $u_i(\cdot, t)$, $t \leq 0, i = 1, 2$, are contained in a compact set in X. Hence, by (HNs), there is an M such that $c(x, t) \leq 0$, $(|x| \geq M, t \in \mathbb{R})$. For each

$t \leq 0$ let R_t be the supremum of $R \geq 0$ such that $v(\cdot, t)$ has no zero in the interval $(-R, R)$. If $R_{t_0} = \infty$ for some t_0 then, by the maximum principle, $v(\cdot, t)$ has no zero for any $t \geq t_0$. In particular, $v(\cdot, 0)$ has no zero which is equivalent to (5.1). We next show that $R_t < \infty$ for all $t \leq 0$ is impossible. Suppose it holds. We claim that $R_t \to \infty$ as $t \to -\infty$. To show this, first note that

$$\frac{(z_{-n}^2 - z_{-n}^1)}{\|z_{-n}^2 - z_{-n}^1\|_X} \to \pm\varphi \tag{5.6}$$

This follows from (5.2), (5.3) and the fact that $z_{-n}^i \to \phi$ $(i = 1, 2)$. Moreover, as $\lambda_1 > 1$, F preserves the ordering of W_{loc}^u given by the parameterization in (5.2). It follows that the sequence in (5.6) actually converges (to one of the functions φ, $-\varphi$). Exchanging the role of z_1, z_2, we may assume that it converges to φ. By (5.4), (5.5),

$$\sup_{x \in \mathbb{R}} |c(x, t) - f_u(t, p(x, t))| \to 0 \quad \text{as } t \to -\infty. \tag{5.7}$$

Using (5.7), (5.6) and continuous dependence of solutions on the initial conditions and the coefficient in the equation, we obtain that

$$v(x, -n\tau + t)\|z_{-n}^2 - z_{-n}^1\|_X{}^{-1}$$

converges, uniformly for $x \in \mathbb{R}$, $t \in [0, \tau]$, to the solution of

$$\tilde{v}_t = \tilde{v}_{xx} + f_u(t, p(x, t)), \qquad \tilde{v}(\cdot, 0) = \varphi.$$

As $v(x, t)$ is everywhere positive (because φ is) our claim follows.

Choose a t_0 such that $R_t \geq M$ $(t \leq t_0)$. By definition of R_t and since $v(\cdot, t)$ is even, $x_0 = R_{t_0}$ is a zero of $v(\cdot, t_0)$. By Lemma 2.3, there is a nodal curve $\{(\xi(t), t) \; : t \leq t_0\}$ of v such that $\xi(t_0) = x_0$. Obviously, $\xi(t) \geq R_t \geq M$. Thus $c(x, t) \leq 0$ for any $x \geq \xi(t)$, $t \leq t_0$. Therefore, by the maximum principle,

$$\sup_{x \geq x_0} |v(x, t_0)| \leq \sup_{x \geq \xi(t)} |v(x, t)| \;\; (t \leq t_0).$$

As $v(\cdot, t)$ is contained in a compact set in X and $\xi(t) \geq R_t \to 0$, the right-hand side of this inequality approaches 0 as $t \to -\infty$. Thus $v(\cdot, t_0)$ vanishes on $[x_0, \infty)$, in contradiction to Corollary 2.1. This contradiction completes the proof of (5.1).

Next we use (5.1) in conjunction with the facts that for $z \in W^u_{loc}$ sufficiently close to ϕ one has $F(z) \in W^u_{loc}$ and $F(z) \neq z$ if $z \neq \phi$. This implies that (5.1) holds for $z_1 = z$ and $z_2 = F(z)$.

Finally, we prove that $z \in W^u_{loc}$ has no zeros. This is obvious if $z \geq \phi > 0$ or $z \leq \phi < 0$. Consider the case $\phi > 0$ and $z < \phi$. Let $v(\cdot, t)$, $t \leq 0$ be the solution of

$$v_t = v_{xx} + f(t, v)$$

such that $v(\cdot, 0) = z$ and $\|v(\cdot, t) - p(\cdot, t)\|_X \to 0$ as $t \to -\infty$. Observe that

$$f(t, v(x,t)) = c(x,t)v(x,t) \quad \text{with } c(x,t) = \int_0^1 f_u(t, sv(x,t))ds \leq 0$$

for large $|x|$. Thus, v solves a linear equation as in Section 2. Let again R_t be the supremum of $R \geq 0$ such that $v(\cdot, t)$ has no zero in $(-R, R)$. The arguments similar to those above give that $R_t = \infty$ for some $t \neq 0$ and thus z has no zero in $\mathbb{R}$. □

Remark. Using the fact that F preserves the pointwise ordering, one can show that property (ii) of Theorem 5.1 holds for the global unstable manifold of ϕ, $W^u(\phi)$, that is the set of all $z \in X$ that have a negative trajectory converging to ϕ. In a recent publication (see [9]), Dancer has proved and employed a similar property of unstable manifolds in a more general context.

Corollary 5.1. *If $\phi > 0$ then $F^n(u_0) \to 0$ for any $u_0 \in W^u_{loc}$ with $u_0 < \phi$, and $F^n(u_0)$ is not precompact in X for any $u_0 \in W^u_{loc}$ with $u_0 > \phi$. An analogous statement holds for $\phi < 0$.*

Proof. By Theorem 5.1(ii), the sequence $F^n(u_0)$ is monotone and consists of positive functions. Therefore if it is precompact, it converges to a nonnegative fixed point $\overline{\phi}$ of F. This cannot be the case if $z > \phi$ for $\overline{\phi}$ would clearly satisfy $\overline{\phi} > \phi$ which violates Proposition 3.2. If $z < \phi$ then $\phi > \overline{\phi} \geq 0$. By Proposition 3.2, this implies $\overline{\phi} = 0$. □

We are now ready to give a **proof of Theorem 1.1**. The conclusion of the theorem follows, provided we prove that the trajectory of u_0 relative to F, $F^n(u_0)$, $n = 0, 1, 2, \ldots$, converges to a fixed point of F. Let $\omega(u_0)$ be the set of all limit points of this trajectory. If $0 \in \omega(u_0)$ then $\omega(u_0) = \{0\}$, because 0 is stable for (1.1). In this case $F^n(u_0) \to 0$. We thus only have to consider the possibility $0 \notin \omega(u_0)$. We apply the following result of Brunovský and Poláčik [3]. This result is an extension of an earlier convergence theorem of

Hale and Raugel [17] which does not apply here as we do not know a priori if the limit set consists of fixed points.

(TC) *Let Y be a Banach space, U an open set in X and $F\ :U\to X$ a C^1 map. Suppose that for a $u_0\in U$ the trajectory $u_0, F(u_0), F^2(u_0),\dots$ is contained in a compact subset of U and its ω-limit set $\omega(u_0)$ contains a fixed point ϕ of F satisfying the following properties:*

(a) *$\sigma(F'(z)) = \sigma_s\cup\sigma_c\cup\sigma_u$, where σ_s, σ_c and σ_u are closed subsets of $\{\lambda\ :|\lambda|<1\}$, $\{\lambda\ :|\lambda|=1\}$ and $\{\lambda\ :|\lambda|>1\}$, respectively.*

(b) *ϕ is stable relative to $F|_{W^c_{loc}}$, where W^c_{loc} is a local center manifold of ϕ.*

Then either $\omega(u_0)=\{\phi\}$, that is $F^n(u_0)\to\phi$ or else $\omega(u_0)$ contains a point of $W^u\setminus\{\phi\}$ where W^u is the (strong) unstable manifold of ϕ.

Let us verify the hypotheses of (TC) in the present setting. By Proposition 3.1, $\omega(u_0)$ contains a $\phi\in \mathrm{Fix}(F)$ which is nonzero by the above assumption. By Theorem 4.1, hypothesis (a) is satisfied. By Theorem 5.1 (i), W^c_{loc} consists of fixed points of F thus (b) is trivially satisfied.

Using (TC), we see that u_0 has the desired converge property if $\omega(u_0)$ contains no point of $W^u(\phi)\setminus\{\phi\}$. We show that the latter is indeed the case. By standard results, $\omega(u_0)$ is compact and invariant under F (cf. Lemma 3.1). If there is a point in $\omega(u_0)\cap W^u(\phi)\setminus\{\phi\}$ then its trajectory is contained in $\omega(u_0)$. By Corollary 5.1 this trajectory converges to 0, thus $0\in\omega(u_0)$ - a contradiction. This completes the proof of Theorem 1.1. □

6. On the existence of compact trajectories. The aim of this section is to prove Theorem 1.3 as well as to give some concrete examples of the functions f satisfying the hypotheses (**B1-B3**).

We start with some simple observations. The first one is an immediate consequence of (1.6) and the comparison principle.

Lemma 6.1. *Assume that $f(t,0)\equiv 0$ for all $t\geq 0$ and that the relation* (1.3) *holds with the upper bound $\overline{f}$ satisfying* (1.6). *Then the equilibrium solution $e_0\equiv 0$ of* (1.1) *is locally asymptotically stable in $C_0(\mathbb{R})$, more specifically, any solution u of* (1.1) *satisfying*

$$0\leq u(x,t_0)<\alpha \text{ for all } x\in\mathbb{R} \text{ and a certain } t_0\geq 0$$

vanishes for large times, i.e.,

$$\sup_{x\in\mathbb{R}}|u(x,t)|\to 0 \text{ as } t\to\infty.$$

To continue our analysis, we introduce a new topology of the phase space. Consider the space $C_{loc}(\mathbb{R})$ of continuous functions on $\mathbb{R}$ endowed with the (metrizable) topology of uniform convergence on compact subsets of $\mathbb{R}$.

Let γ be as in **(B1)**. Take $\mathcal{B}_\gamma \subset C_{loc}(\mathbb{R})$ - the set of all continuous functions on $\mathbb{R}$ with range in the closed interval $[0,\gamma]$. As $\mathcal{B}_\gamma$ is *bounded,* the topology of C_{loc} restricted to $\mathcal{B}_\gamma$ may be given by a norm

$$\|v\|_{\mathcal{B}_\gamma} \equiv sup_{x\in\mathbb{R}}\rho(x)|v(x)| \tag{6.1}$$

where $\rho : \mathbb{R} \to (0,1]$, $\lim_{|x|\to\infty}\rho(x) = 0$. Observe that different choices of ρ give equivalent norms on $\mathcal{B}_\gamma$.

By (1.4) and the comparison principle, the solution operator corresponding to (1.5) maps $\mathcal{B}_\gamma$ into itself and, moreover, it is continuous :

Lemma 6.2. *Let u, v be two solutions of the equation* (1.5) *on a time interval $[t_0, t]$ such that $u(t_0), v(t_0) \in \mathcal{B}_\gamma$. Then*

$$\|u(t) - v(t)\|_{\mathcal{B}_\gamma} \le L(t - t_0)\|u(t_0) - v(t_0)\|_{\mathcal{B}_\gamma}$$

where L is bounded for bounded values of $t - t_0$.

Proof. We take $\rho(x) = \frac{1}{(1+x^2)}$ in (6.1). Subtracting the equations for u, v and multiplying the result by ρ we get

$$w_t - w_{xx} + [\frac{\rho_{xx}}{\rho} - 2\frac{\rho_x^2}{\rho^2}]w + 2\frac{\rho_x}{\rho}w_x = \rho(\underline{f}(t,u) - \underline{f}(t,v))$$

where we have denoted $w \equiv \rho(u - v)$.

Thus, the desired result follows from the standard linear estimates as $\underline{f}$ is Lipschitz for arguments belonging to the compact interval $[0,\gamma]$ and the coefficients in the equation are bounded. □

Next, we show the following assertion.

Lemma 6.3. *Let $\underline{f}$ satisfy the hypothesis* **(B1)**. *Then the equilibrium $e_1 \equiv \gamma$ is locally asymptotically stable in the topology of $\mathcal{B}_\gamma$ for solutions of* (1.5).

Remark. Observe that e_1 being locally asymptotically stable in $\mathcal{B}_\gamma$ means that it is stable in the topology of $\mathcal{B}_\gamma$ and there are constants $\lambda, \delta > 0$ such that any solution v of (1.5) with the initial datum $v(0)$,

$$v(0) \in [0,\gamma],\ v_0(x) > \gamma - \delta \text{ on an interval } I \text{ of length } l(I) > \lambda,$$

satisfies $v(t) \to \gamma$ as $t \to \infty$ uniformly on compacts in $\mathbb{R}$.

Proof. Let ϵ be as in **(B1)**. With a $\delta \in (0, \epsilon)$, consider the solution v of (1.5) satisfying $v(x, 0) = \gamma - \delta$ for $x \leq 0$, $v(x, 0) = 0$ for $x > 0$. By **(B1)**, there is an integer $n > 0$ such that $v(2, t) > \gamma - \delta$ for all $t \geq T \equiv n\tau$. Consequently, as v is nonincreasing in x, $v(x, t) > \gamma - \delta$ for all $x \in (-\infty, 2]$, $t \geq T$. By Lemma 6.2 , the solution operator of (1.5) is continuous with respect to the topology of uniform convergence on compacts, i.e., there is $M_0 = M_0(T, \delta) > 0$ such that the solution w of (1.5) starting from the initial datum $w_0 = 0$ on $(-\infty, -2M) \cup (0, \infty)$, $w_0 = \gamma - \delta$ on $[-2M, 0]$ for arbitrary $M \geq M_0$ satisfies $w(2, t) > \gamma - \delta$, $t \in [T, 2T]$. On the other hand, the solution w is even and decreasing with respect to the center $-M$, i.e., $w(x, t) > \gamma - \delta$ for all $x \in [-2(M + 1), 2]$, $t \in [T, 2T]$. Using the results just obtained and the comparison principle we get the following conclusion:

Given $\delta \in (0, \epsilon)$ there exist M and $T = n\tau$ such that for any data $v(0)$ satisfying $v(0) \in [0, \gamma]$, $v(0) > \gamma - \delta$ on an interval $[a, b]$, $b - a > 2M$. the corresponding solution v of (1.5) has the property $v(x, t) > \gamma - \delta$ for all $x \in [a - 1, b + 1]$, $t \in [T, 2T]$. By induction, for $k = 1, 2, \dots,$

$$v(x, t) > \gamma - \delta \text{ for all } x \in [a - k, b + k],\ t \in [kT, (k + 1)T], \tag{6.2}$$

and, in particular,

$$\liminf_{t\to\infty} v(t) \geq \gamma - \delta \text{ uniformly on compacts in } \mathbb{R}^1. \tag{6.3}$$

Note that for any $\delta \in (0, \epsilon)$ condition (6.2) is true provided

$$v(0) > \gamma - \delta \text{ on an interval } I \text{ of length } l(I) > 2M,\ (\text{with } M = M(\delta)). \tag{6.4}$$

Property (6.2) and Lemma 6.2 readily imply the local stability of e_1.

To prove the asymptotic stability, we fix a $\delta_0 \in (0, \epsilon)$ such that the (spatially constant) solution z of (1.5) with the initial data $z(0) \equiv \gamma - \delta_0$ satisfies $z(t) \to \gamma$ as $t \to \infty$. Such choice is possible since $e_1 \equiv \gamma$ is asymptotically stable as a solution of the the corresponding ODE ((1.5) without the diffusion term). Next take a $0 < \delta_1 < \delta_0$ and let $M_1 = M(\delta_1)$ be the constant corresponding to δ_1, as in (6.4). Let v be any solution of (1.5) satisfying (6.4) with with $\delta = \delta_1$ and $M = M_1$, so that by (6.3) we have

$$\liminf_{t\to\infty} v(t) \geq \gamma - \delta_1 \text{ uniformly on compacts in } \mathbb{R}^1. \tag{6.5}$$

We claim that given any $\delta \in (0, \epsilon)$ there is an integer m such that (6.4) is satisfied with $v(0)$ replaced by $v(m\tau)$.

If this is true then (6.3) holds for any $\delta \in (0, \epsilon)$. Hence

$$\lim_{t \to \infty} v(t) \geq \gamma \text{ uniformly on compacts in } \mathbb{R}^1,$$

as needed for the asymptotic stability.

The proof will be thus completed if we prove the claim. For this end, fix an arbitrary $\delta \in (\gamma - \epsilon, \gamma)$. Consider the solution y of (1.5) corresponding to the initial data

$$y(0) = (\gamma - \delta_0) 1_I \tag{6.6}$$

where 1_I denotes the characteristic function of interval I. We choose I of large length, so that $y(0)$ is sufficiently close to the constant $z(0) = \gamma - \delta_0$ in the norm of $\mathcal{B}_\gamma$. Applying Lemma 6.2 to the solutions $z(t)$, $y(t)$ we obtain that there is an integer p such that

$$y(x, p\tau) \geq \gamma - \delta \;\text{ on an interval } [a, b] \;\text{ with }\; b - a > 2M(\delta). \tag{6.7}$$

Now, using (6.5), (6.6) and the comparison principle, we find another integer m such that $v(x, m\tau) > y(x, p\tau)$ $(x \in \mathbb{R})$. The last two estimates combined prove the claim. □

As an immediate corollary we obtain the following result which generalizes a theorem of Fife-McLeod [15] to the time-periodic case.

Proposition 6.1. *Let the function* $\underline{f}$ *satisfy the hypothesis* **(B1)**. *Let* z *denote the maximal (unstable) time-periodic solution of the problem*

$$\frac{dz}{dt} = \underline{f}(t, z)$$

lying between the stable equilibria $e_0 \equiv 0$, $e_1 \equiv \gamma$. *Then for any* $\delta > 0$ *there exists* $\lambda > 0$ *such that any solution* v *of* (1.5) *with the initial data*

$$v(0) \in [0, \gamma], v_0(x) \geq z(0) + \delta \textit{ for all } x \in I,\; l(I) > \lambda$$

satisfies $v(t) \to \gamma$ *as* $t \to \infty$ *uniformly on compacts in* $\mathbb{R}$.

Proof. In view of the translation invariance, it suffices to show the conclusion for the initial datum $v(0) = 0$ on $(-\infty, 0) \cup (\lambda, \infty)$, $v_0 = z(0) + \delta$ on

$[0,\lambda]$. By Lemma 6.3 , the solution e_1 is locally asymptotically stable in $\mathcal{B}_\gamma$, in particular, the set of all solutions converging to e_1 in $\mathcal{B}_\gamma$ is open. On the other hand, the solution w of (1.5) with the data $w_0 \equiv z(0)+\delta$ obviously converges to e_1 in $\mathcal{B}_\gamma$ and, consequently, so does any solution with the data u_0 sufficiently close to w_0 *in* $\mathcal{B}_\gamma$. □

Lemma 6.4. *Let* (1.3) *hold where the functions* $\underline{f}, \overline{f}$ *satisfy the hypotheses* **(B1-B3)**. *Then for any* $\delta > 0$ *there is* $\lambda > 0$ *such that any solution* u *of* (1.1) *has the following property*:

$$u(.,t_0) \geq 0,\ u(x,t_0) > \beta+\delta \textit{ for all } x \in I,\ \textit{length}(I) \geq \lambda \ \textit{ for a certain } t_0 \tag{6.8}$$

implies

$$\liminf_{t\to\infty} u(x,t) \geq \gamma \textit{ uniformly in } x \textit{ on compact subsets of } \mathbb{R}. \tag{6.9}$$

Proof. By (1.3), the solutions of equation (1.5) are subsolutions for (1.1).

Now, it follows from Proposition 6.1 that for any $\delta > 0$ there is $\lambda > 0$ such that for any initial data $v(0)$ satisfying (6.8), the corresponding solution v of (1.5) satisfies $\lim_{t\to\infty} v(x,t) = \gamma$ uniformly in x on compact sets in $\mathbb{R}$ which implies (6.9). □

At this stage, we are ready to prove Theorem 1.3. In accordance with (1.8), we can fix $\delta > 0$ so that

$$\overline{F}(z) < 0 \text{ for all } z \in (0, \beta+\delta]. \tag{6.10}$$

Let us take a function $U \in C^2(\mathbb{R})$ having the following properties: $U(x) \geq 0$, $U(-x) = U(x)$ for all $x \in \mathbb{R}$, U is nonincreasing on $[0,\infty)$, $U(\frac{\lambda}{2}) > 0$, where $\lambda = \lambda(\delta)$ appears in Lemma 6.4,

$$U \equiv 0 \text{ on } (-\infty,-\lambda] \cup [\lambda,\infty). \tag{6.11}$$

Suppose that (1.3) holds along with the hypotheses **(B1-B3)** and denote by u_μ the (unique global) solution of the equation (1.1) emanating from the initial value $u_\mu(x,0) = \mu U(x)$, $x \in \mathbb{R}$, $\mu \geq 0$. Consider two sets:

$$\mathcal{M}_0 = \{\mu \geq 0 \mid \sup_{x\in\mathbb{R}} u_\mu(x,t_0) < \alpha \text{ for a certain } t_0 \geq 0\}$$

$$\mathcal{M}_\infty = \{\mu \geq 0 \mid u_\mu(\frac{\lambda}{2},t_0) > \beta+\delta \text{ for a certain } t_0 \geq 0\}.$$

Obviously, both sets $\mathcal{M}_0, \mathcal{M}_\infty$ are open and nonvoid subsets of $[0,\infty)$. More importantly, by Lemmas 6.1, 6.4 they *do not* intersect (notice that u_μ is even and decreasing in $(0,\infty)$). Consequently, as the interval $[0,\infty)$ is connected, there exists μ_c which belongs to the complement of both $\mathcal{M}_0$ and $\mathcal{M}_\infty$.

We examine the solution $u \equiv u_{\mu_c}$ of (1.1) with the initial data $u(0) = \mu_c U$. Since $\mu_c \notin \mathcal{M}_0$, we immediately get $u(0,t) \geq \alpha > 0$ for all $t \geq 0$. On the other hand, as $\mu_c \notin \mathcal{M}_\infty$ and u is nonincreasing for $x \geq 0$ we obtain

$$0 \leq u(\lambda,t) \leq \beta + \delta \text{ for all } t \geq 0 \tag{6.12}$$

Now, there is a (stationary) solution w of the problem $-w_{xx} = \overline{f}(w)$ for $x \in (\lambda,\infty)$ satisfying the boundary conditions $w(\lambda) = \beta+\delta$, $\lim_{x\to\infty} w(x) = 0$. This solution can be found by explicit integration of $w_x = -\sqrt{-2\overline{F}}$. Note that $\overline{F}$ is strictly negative as stated in (6.10). Using (6.11),(6.12) and the comparison principle we obtain $0 \leq u(x,t) \leq w(x)$ for all $x,\ |x| \geq \lambda,\ t \geq 0$. Finally, the value of u on the interval $[-\lambda,\lambda]$ is estimated from above by the maximum of the solution v of the initial boundary value problem:

$$\begin{aligned}&v_t - v_{xx} = \overline{f}(v) \text{ for } x \in (-\lambda,\lambda),\ v(.,0) = \lambda_c U(0),\\ &v(-\lambda,t) = v(\lambda,t) = \lambda_c U(0) \text{ for } t \geq 0.\end{aligned}$$

By (1.7), there is a constant K such that $v \leq K$ for all $x \in [-\lambda,\lambda]$ and $t \geq 0$.

Summing up the previous results the conclusion of Theorem 1.3 follows.

Let us conclude this Section by presenting a few examples where the hypotheses of Theorem 1.3 are satisfied.

Example 1. Assume that $f(t,v) = r(t)g(v)$, where

$$0 < r_1 \leq r(t) \leq r_2 < \infty \text{ for all } t \geq 0 \tag{6.13}$$

and there is $\beta > 0$ such that

$$g(0) = g(\beta) = 0,\ g(z) < 0 \text{ for all } z \in (0,\beta), \tag{6.14}$$

$g(z) > 0$ for $z > \beta$, $\limsup_{z\to\infty} \frac{g(z)}{z} = 0$ and

$$\int_\beta^\infty g(s)\ ds = \infty. \tag{6.15}$$

By (6.15), there is a function $\underline{g}$ and a constant γ such that $g(z) \geq \underline{g}(z)$ for all $z \geq 0$, $\underline{g}(0) = \underline{g}(\beta) = \underline{g}(\gamma) = 0$, $\underline{g}'(0), \underline{g}'(\gamma) < 0$ and

$$r_1 \int_\beta^\gamma \underline{g}(z)\ dz > -r_2 \int_0^\beta \underline{g}(z)\ dz.$$

Now, the hypotheses **(B1-B3)** are satisfied with

$$\underline{f}(z) = r_2 \underline{g}(z),\ \ z \in [0,\beta],\ \ \underline{f}(z) = r_1 \underline{g}(z),\ \ z \in (\beta,\infty) \tag{6.16}$$

$$\overline{f}(z) = r_1 g(z),\ \ z \in [0,\beta],\ \ \overline{f}(z) = r_2 g(z),\ \ z \in (\beta,\infty). \tag{6.17}$$

Note that the last hypothesis in **(B1)** follows from (1.9) as $\underline{f}$ is independent of t (cf. Fife-McLeod [15]).

Example 2. The second example is the so called bistable nonlinearity, specifically, $f(t,u) = r(t)g(u)$, where r is as in (6.13). The function g satisfies (6.14) and there is $\gamma > \beta$ such that $g(z) > 0$ for $z \in (\beta,\gamma)$, and $g(\gamma) = 0$. It is easy to see that a function $\underline{g}$ can be found so that the hypotheses **(B1-B3)** may be satisfied for $\underline{f}, \overline{f}$ defined as in (6.16),(6.17) provided that

$$r_2 \int_0^\beta g(z)\ dz + r_1 \int_\beta^\gamma g(z)\ dz > 0.$$

Example 3. Up to now, all the nonlinearities have possessed fixed zeros only. Let us examine the case where $f(t,z) = rz(\gamma - z)(z - b(t))$, $r > 0$ with $b(t+T) = b(t)$ for all t, $0 < \alpha = \inf_{t\in[0,T]} b(t) \leq \sup_{t\in[0,T]} b(t) = \beta < \gamma$. It follows from the results of Alikakos, Bates and Chen [1] that **(B1)** is satisfied with $\underline{f} = f$ if and only if

$$\int_0^T b(t)\ dt > \frac{\gamma}{2}. \tag{6.18}$$

Indeed the condition (6.18) guarantees the existence of a traveling wave with *positive speed* connecting the equilibria $0, \gamma$ (see [1]). On the other hand, we can take $\overline{f}(z) = rz(\gamma - z)(z - \alpha)$ for $z \in [0,\gamma]$, $\overline{f} \equiv 0$ otherwise satisfying clearly **(B2)** while **(B3)** holds if $3\beta^2 - 4\beta(\alpha+\gamma) + 6\alpha\gamma > 0$.

Example 4 . Asymmetric nonlinearity. Let $f = f(t,z)$ satisfy $f(t+T,z) = f(t,z)$ for all $t, z \geq 0$,

$$f(t,0) = f(t,\beta) = 0,\ \ f_z(t,0) < 0,\ \ f_z(t,\beta) > 0 \text{ uniformly for all } t, \tag{6.19}$$

$$\sup_{t\in[0,T]} f(t,z) \le K < \infty \text{ for all } z \ge 0,$$
$$\sup_{t\in[0,T]} f(t,z) < 0 \text{ for all } z \in (0,\beta) \tag{6.20}$$

and there is $a \ge \beta$ such that $f(t,z) > 0$ for all t and $z \in (\beta, a)$ (if $a > \beta$), $f(t, a+y) > -f(t, \beta - y)$ for all $y \in (0,\beta]$ and uniformly in t. At this stage, it is easy to see that the function $\overline{f}(z) = \sup_{t\in[0,T]} f(t,z)$ satisfies **(B2-B3)** where the function $\underline{f}$ is taken

$$\underline{f}(t,z) = f(t,z) \text{ for } z \in [0, a+\beta], \tag{6.21}$$

and prolonged outside of $[0, a+\beta]$ in such a way that

$$\underline{f}(t,\gamma) = 0 \text{ for all } t \text{ and a certain } \gamma > a+\beta. \tag{6.22}$$

Consequently, it suffices to show that $\underline{f}$ defined by (6.21), (6.22) satisfies **(B1)**. To this end, we find a decomposition

$$\underline{f}(t,z) \ge h(t,z) + m \tag{6.23}$$

where h is odd with respect to a point $c \in \mathbb{R}$, i.e., $-h(t, c-y) = h(t, c+y)$ for all $y \ge 0,\ t \ge 0$ and $m > 0$ is a constant.

First of all, observe that we can always assume $a > \beta$. Indeed if $a = \beta$, we have

$$\underline{f}(t, \beta+y) > -\underline{f}(t, \beta-y) \text{ for all } y \in (0,\beta] \text{ and uniformly in } t. \tag{6.24}$$

We look for $\delta > 0$ such that

$$\underline{f}(t, \beta+\delta+y) > -\underline{f}(t, \beta-y),\ y \in [0,\beta]. \tag{6.25}$$

Arguing by contradiction, we obtain a sequence $\delta_n \to 0$, $t_n \to t \in [0,T]$ and $y_n \to y \in [0,\beta]$ such that

$$\underline{f}(t_n, \beta+\delta_n+y_n) \le -f(t_n, \beta-y_n) \tag{6.26}$$

Observe that $y > 0$ since, if not, the relation (6.19) would imply

$$\underline{f}(t_n, \beta+\delta_n+y_n) \ge \underline{f}(t_n, \beta+y_n) > -\underline{f}(t_n, \beta-y_n)$$

where the last inequality follows from (6.24). Thus passing to the limit for $n \to \infty$ in (6.26) we obtain $\underline{f}(t, \beta + y) \le -\underline{f}(t, \beta - y)$ for a certain $y \in (0, \beta]$ in contrast with (6.24).

Taking $a = \beta + \delta$ and making use of (6.25) we can find a constant $q > 0$ such that $\underline{f}(t, a + y) > -\underline{f}(t, \beta - y) + q$ for all $y \in [0, \beta]$, $t \in [0, T]$ and, obviously, $\underline{f}(t, y) = \underline{f}(t, y) - q + q$, $y \in [0, \beta]$. Taking $h(t, z) = \underline{f}(t, z) - q$ for $z \in [0, \beta]$, $h(t, z) = -(\underline{f}(t, z) - q)$ for $z \in [a, a + \beta]$ and prolonging h suitably outside these intervals we obtain the desired decomposition (6.23) with a certain $m > 0$ and $c = \frac{\beta + a}{2}$ where $h(t, -\frac{a-\beta}{2}) = h(t, c) = h(t, a + \beta) = 0$, $h_z(t, -\frac{a-\beta}{2}), h_z(t, a + \beta) < 0$, $a + \beta \le \gamma$. Thus, the problem (1.5) may be written in the form

$$v_t - v_{xx} = h(t, v) + m, \;\; h(t, c + y) = -h(t, c - y), \;\; y \ge 0. \tag{6.27}$$

By the results of Alikakos, Bates and Chen [1] and the symmetry properties of h, the related "symmetric" equation $w_t - w_{xx} = h(t, w)$ possesses a time periodic solution w with the following properties $w(x, t + T) = w(x, t)$ for all x, t, $w(-x, t) - c = -w(x, t)$ for all $x \ge 0$, t, $\lim_{x \to -\infty} w(x, t) = a + \beta$, $\lim_{x \to \infty} w(x, t) = -\frac{a-\beta}{2}$ uniformly in t and $0 \ge w_x(x, t) \ge -const$ for all x, t. Now, the function $z(x, t) = w(x - pt, t)$ solves the equation $z_t - z_{xx} = h(t, z) - p w_x(x - pt, t)$. Thus, for small *positive constants* p, z is a subsolution to (6.27). It follows that problem (6.27) possesses a traveling wave with a positive speed connecting 0 and γ. By the results of [1], the hypothesis **(B1)** holds.

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