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EXISTENCE OF MULTIDIMENSIONAL TRAVELLING FRONTS WITH A MULTISTABLE NONLINEARITY

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Abstract. This article deals with the existence of solutions of

$$\begin{cases} \Delta u - \beta(y,c)\frac{\partial u}{\partial x_1} + f(u) & = 0 \quad \text{in } \Sigma \\ \frac{\partial u}{\partial \nu} & = 0 \quad \text{on } \partial\Sigma \\ u(-\infty,\cdot) = 0,\ u(+\infty,\cdot) & = 1 \end{cases}$$

where $\Sigma = \{(x_1,y) \in \mathbb{R}\times\omega\}$ is an infinite cylinder with outward unit normal ν and whose section $\omega \subset \mathbb{R}^{n-1}$ is a bounded convex domain. The unknowns are the real parameter c and the function u (which respectively represent the speed and the profile of a travelling wave). The function β and the nonlinear term $f : [0,1] \to \mathbb{R}$ are given. We investigate the case where the function f changes sign several times. We prove that there exists a travelling front (c,u) provided that the speeds of the travelling waves for simpler problems can be compared. The proof uses the sliding method and the theory of sub- and supersolutions. This result generalizes for higher dimensions a one-dimensional result of Fife and McLeod.

1. Introduction and main results. This work is concerned with travelling wave solutions of semilinear parabolic equations in infinite cylinders $\Sigma = \mathbb{R}\times\omega = \{(x_1,y)\in\mathbb{R}^n,\ x_1\in\mathbb{R},\ y\in\omega\}$ where ω is a bounded smooth domain of $\mathbb{R}^{n-1}$. The evolution equations are of the following type:

$$\frac{\partial v}{\partial t} = \Delta v - \alpha(y)\partial_1 v + f(v). \tag{1.1}$$

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Here $t \in \mathbb{R}^+$ represents the time variable. We denote by $\partial_1 v$ the derivative $\frac{\partial v}{\partial x_1}$ and by ν the outward unit normal to $\partial\Sigma$.

The goal of this paper consists in studying the travelling front solutions of (1.1) in the case where the sign of f changes three times or more in $[0, 1]$ (more precise assumptions on f will be made later). Travelling fronts are solutions of the type $v(t, x_1, y) = u(x_1 + ct, y)$ where the real c, the speed or velocity of the front, is unknown. Renaming x_1 the variable $x_1 + ct$, these travelling wave functions u are solutions in Σ of the semilinear elliptic equation

$$\Delta u - (c + \alpha(y))\partial_1 u + f(u) = 0 \text{ in } \Sigma.$$

More generally, we look for solutions (c, u) of the equation

$$\Delta u - \beta(y, c)\partial_1 u + f(u) = 0 \text{ in } \Sigma. \tag{1.2}$$

We impose the boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Sigma \tag{1.3}$$

and the limits

$$u(-\infty, \cdot) = 0, \ \ u(+\infty, \cdot) = 1. \tag{1.4}$$

In (1.4) and throughout the paper the limits as $x_1 \to \pm\infty$ are understood to be uniform in $y \in \overline{\omega}$. The Neumann boundary condition (1.3) means that there is no flow across the walls of the tube. Further on, we assume that $\beta = \beta(y, c)$ is a given continuous function on $\overline{\omega} \times \mathbb{R}$, strictly increasing in c and such that

$$\left\{ \begin{array}{ll} \beta(y, c) \longrightarrow +\infty & \text{as } c \to +\infty \\ \beta(y, c) \longrightarrow -\infty & \text{as } c \to -\infty \end{array} \right. \quad \text{uniformly in } y \in \overline{\omega}.$$

In physical models $\beta(y, c)$ may also be of the form $c\alpha(y)$ with $\alpha > 0$ on $\overline{\omega}$. The nonlinear source term f is given in $[0, 1]$, and we systematically assume that f is Lipschitz-continuous on $[0, 1]$ and that $f(0) = f(1) = 0$.

By extension, we say that u is a travelling front over $(0, 1)$, or a connection between 0 and 1, if u satisfies the previous equations (1.2)–(1.4) and if $0 < u < 1$ in Σ. The known results for the solutions (c, u) of (1.2)–(1.4) depend mainly on the profile of f. It is a common thing to consider three types of nonlinearities f, namely the KPP or ZFK cases where $f > 0$ on $(0, 1)$, the case with an ignition temperature $\theta \in (0, 1)$ where $f \equiv 0$ on $[0, \theta]$ and $f > 0$

on $(\theta, 1)$, and lastly the "bistable" case. For the latter, it is assumed that there exists $\theta \in (0,1)$ such that

$$\begin{cases} f < 0 \text{ on } (0,\theta), \ f > 0 \text{ on } (\theta, 1) \\ f(0) = f(\theta) = f(1) = 0, \ f'(0), \ f'(1) < 0. \end{cases} \tag{1.5}$$

Both points $s = 0$ and $s = 1$ are thus stable for the simple evolution problem $\dot{X}(t) = f(X)$. This model can be found in some biological problems: population dynamics, gene developments, epidemiology (*see* Aronson, Weinberger [2], Fife [8], Fife and McLeod [9], Fisher [12] and references therein) and also in some combustion problems (Kanel' [18]).

In the one-dimensional case, problem (1.2)–(1.4) is reduced to the scalar ordinary differential equation

$$\begin{cases} \ddot{u} - c\dot{u} + f(u) = 0 \quad \text{in } \mathbb{R} \\ u(-\infty) = 0, \ u(+\infty) = 1. \end{cases} \tag{1.6}$$

In [2], [9], [18], it is proved that if f satisfies (1.5), then equation (1.6) has a unique solution (c, u), u being unique up to translation with respect to the variable x_1. Besides, an extended study including the existence and stability of solutions $U(x,t)$ of the Cauchy problem

$$U_t = U_{xx} + f(U), \quad U = U(t,x), \ (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \quad U(0,x) \text{ given} \tag{1.7}$$

was carried out in [2], [8], [9], [18].

In the course of their study, Fife and McLeod also extended to a wider class of functions f the above existence result. Namely, consider the case of a function f which has two adjacent triples of zeros, $(0, \theta_1, \theta)$ and $(\theta, \theta_2, 1)$, such that the restrictions of f to the intervals $[0,\theta]$ and $[\theta, 1]$ are of bistable type, that is to say that (see Figure 1)

$$f < 0 \text{ on } (0, \theta_1), \ f > 0 \text{ on } (\theta_1, \theta) \text{ and } f'(0), \ f'(\theta) < 0 \tag{1.8}$$

$$f < 0 \text{ on } (\theta, \theta_2), \ f > 0 \text{ on } (\theta_2, 1) \text{ and } f'(1) < 0. \tag{1.9}$$

From the results recalled above, there exist two unique couples (c_1, u_1) and (c_2, u_2) of solutions of equation (1.6), satisfying $u_1(-\infty) = 0$, $u_1(+\infty) = \theta$ and $u_2(-\infty) = \theta$, $u_2(+\infty) = 1$. If $c_1 > c_2$, Fife and Mc Leod proved that the solution U of (1.7), with suitable initial conditions, tends to split into two travelling fronts which deviate from each other. Moreover, there does

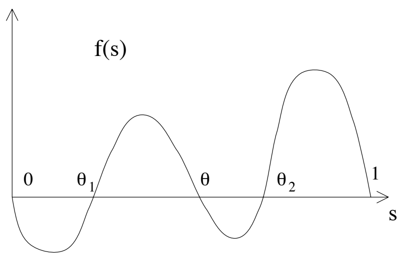

Figure 1. A function f fulfilling (1.8)–(1.9)

not exist any connection between 0 and 1. If $c_1 < c_2$, there exists a unique travelling front over $(0, 1)$ and the solution U will develop into this solution, under suitable initial conditions.

Our goal in this paper is to get similar results as those of [9], with a nonlinearity f which has a finite number of zeros between 0 and 1, for problem (1.2)–(1.4) set in infinite cylinders $\Sigma = \mathbb{R} \times \omega$.

The methods used by Fife and McLeod in [9] to prove the existence or the nonexistence of travelling waves over $(0, 1)$—according to the values of c_1 and c_2—are specific to the one-dimensional case. Indeed, equation (1.6) was studied in the phase plane of the variables $(u, \dot{u})$. With similar techniques, several results on the existence of travelling waves between two stable states were also given for systems of ordinary differential equations with nonlinearities of the bistable type or fulfilling similar monotonicity assumptions (*see e.g.* Gardner [13]; Hagan [15]; Mischaikow, Huston [21]; Reineck [24]; Terman [27]; Volpert, Volpert, Volpert [31]). In [10], [11], Fife and Peletier considered equation (1.6) with a nonlinearity f depending on x, $f(x, u)$, of the bistable type; they especially proved the existence, the uniqueness and the stability of clines with the speed $c = 0$ under various assumptions. The equivalent problems as those mentioned above, but for partial differential equations in cylinders instead of ordinary differential equations, require different tools. In particular, the shooting method, the phase plane method and the Conley index theory which can be used to look for orbits for ordinary differential equations, no longer work for partial differential equations in the cylinders Σ because of the dependence on y in the governing equations.

To shed light on the difficulty of emphasizing multidimensional problems and before stating our results and describing the methods used to prove them, we shall notice that, even in the case of a simple bistable nonlinearity

f fulfilling (1.5), the existence results for (1.2)–(1.4) radically differ in the multidimensional case from the one-dimensional case. Indeed, the existence and the uniqueness results established in dimension 1 for a bistable reaction term (1.5) ([9], [18]) was generalized by Berestycki and Nirenberg [6] in infinite cylinders $\mathbb{R} \times \omega$ *only for convex sections* ω:

Theorem 1.1 ([6]). *If ω is convex, if f satisfies* (1.5) *and is $C^{1,\delta}([0,1])$ for some $0 < \delta < 1$, then there exists a unique solution (c, u) of* (1.2)–(1.4), *the solution u being unique up to translation in the variable x_1. Furthermore, $\partial_1 u > 0$ in $\overline{\Sigma}$.*

The restriction that the section ω be convex cannot be omitted since Berestycki and Hamel gave some examples of nonconvex domains ω for which there does not exist any solution of (1.2)–(1.4) (*see* [3]). If the section ω is not convex, only the existence of solutions (c, u) of (1.2), (1.3) fulfilling $u(-\infty, \cdot) = 0$ and $u(+\infty, y) = \psi(y)$ holds, where ψ is a solution of the following problem in ω:

$$\begin{cases} \Delta_y \psi + f(\psi) = 0 & \text{in } \omega \\ \partial_\nu \psi = 0 & \text{on } \partial\omega. \end{cases} \tag{1.10}$$

If the section ω is convex, the following additional results about the solutions of (1.10) were proved by Casten and Holland [7], Matano [20], Berestycki and Nirenberg [6]:

Proposition 1.2 ([7], [20]). *If the domain ω is convex, if f is of class C^1 and if ψ is a nonconstant solution of* (1.10), *then ψ is unstable in the sense that the principal eigenvalue $\mu_1(\psi)$ of the linearized operator $-\Delta - f'(\psi)$ in ω with Neumann boundary conditions on $\partial\omega$, is negative.*

From this last result is derived the

Proposition 1.3 ([6]). *Under the assumptions of Proposition* 1.2, *let $\psi_- \leq \psi_+$ be two nonconstant solutions of* (1.10). *If there does not exist any zero of f between ψ_- and ψ_+, then $\psi_- \equiv \psi_+$ in $\overline{\omega}$.*

These propositions together ensure the existence of a solution (c, u) of (1.2)–(1.4) if f is bistable and if ω is convex, that is to say that the function $\psi = u(+\infty, \cdot)$ can be chosen equal to 1 in (1.10).

Papanicolaou and Xin got the same result as in [6] for equivalent problems with periodic boundary conditions instead of Neumann boundary conditions on $\partial\Sigma$ ([22], [32]). The existence of travelling waves in cylinders with Dirichlet boundary conditions was also proved by Gardner [14] and Vega [28].

In [16], Hamel proved the existence of an interval (c_-, c_+) of speeds which were solutions of some problems of the type (1.2)–(1.4) with a nonlinearity $f(x_1, u)$ nondecreasing with respect to x_1.

Main results of this paper. From now on, we emphasize problem (1.2)–(1.4) set in an infinite cylinder $\Sigma = \mathbb{R} \times \omega$. The results of this paper, which were first announced in [17], are the following lemma and theorem:

Lemma 1.4. *Let* $\Sigma = \mathbb{R}\times\omega$ *and let* f *be any function of class* $C^{1,\delta}([0,1])$ *(for some* $0 < \delta < 1$*) such that* $f(0) = f(1) = 0$*,* $f'(0) < 0$ *and* $f'(1) < 0$*. For any* $\eta, \gamma \in (0,1)$*, if there exist three travelling fronts in* Σ *solutions of* (1.2)–(1.3)*, ranging respectively over* $(0,\eta)$*,* $(0,1)$ *and* $(\gamma,1)$ *and with respective velocities* c_1*,* c *and* c_2*, then* $c_1 < c < c_2$*. Besides, the travelling front* u *over* $(0,1)$*, with velocity* c*, is unique (up to* x_1*-translations) and* $\partial_1 u > 0$ *in* $\overline{\Sigma}$*.*

Theorem 1.5. *Let* ω *be a convex domain and let* f *be a function of class* $C^{1,\delta}([0,1])$ *(for some* $0 < \delta < 1$*) satisfying* (1.8)–(1.9)*. Let* u_1 *and* u_2 *be the travelling fronts in* Σ *over* $(0,\theta)$ *and* $(\theta,1)$ *solutions of* (1.2)–(1.3) *and let* c_1 *and* c_2 *be their velocities. If* $c_1 < c_2$*, then there exists a travelling front* u *in* Σ *over* $(0,1)$ *solving* (1.2)–(1.4)*, with a velocity* c *such that* $c_1 < c < c_2$*.*

For a function f fulfilling (1.8)–(1.9), Lemma 1.4 states that the inequality $c_1 < c_2$ is a necessary condition for the problem (1.2)–(1.4) to have a solution (c, u). Theorem 1.5 states that this necessary condition is also sufficient.

Remark 1.6. Under the assumptions of Theorem 1.5, we can wonder whether we can a priori compare c_1 with c_2. From the results in [6], in order to have $c_1 < c_2$, it suffices that $\theta \le 1/2$ and that $f(t) \le\not\equiv f(t+\theta)$ on $[0,\theta]$. A symmetric condition implies $c_1 > c_2$.

The methods used in the proof of Theorem 1.5 can easily be generalized and lead to the following:

Generalization of Theorem 1.5. *Let* ω *be a convex domain and* $(\theta_i)_{i=0,\dots,2m}$ *be a finite increasing sequence such that* $0 = \theta_0 < \theta_1 < \cdots < \theta_{2m} = 1$*. Let* f *be a function of class* $C^{1,\delta}([0,1])$ *(for some* $0 < \delta < 1$*) such that*

$$\begin{cases} \forall\ 0 \le i \le m-1 & f < 0 \text{ on } (\theta_{2i}, \theta_{2i+1}) \\ \forall\ 0 \le i \le m-1 & f > 0 \text{ on } (\theta_{2i+1}, \theta_{2i+2}) \\ \forall\ 0 \le i \le 2m & f(\theta_i) = 0 \\ \forall\ 0 \le i \le m & f'(\theta_{2i}) < 0. \end{cases} \tag{1.11}$$

For any $0 \le i \le m-1$*, let* u_i *be the unique travelling front in* Σ *over*

$(\theta_{2i}, \theta_{2i+2})$ *solving* (1.2)–(1.3). *If the velocities* $(c_i)_{i=0,\ldots,m-1}$ *of these travelling fronts are strictly increasing*: $c_0 < c_1 < \cdots < c_{m-1}$, *then there exists a travelling front* (c, u) *over* $(0, 1)$ *solving* (1.2)–(1.4), *and the velocity* c *is such that* $c_0 < c < c_{m-1}$.

The main tools to prove Lemma 1.4 and Theorem 1.5 are based on the theory of sub- and super-solutions for elliptic partial differential equations and on the sliding method in cylinders (*see* Berestycki, Nirenberg [5]). One of the key points is to work out the asymptotic behaviours for the solutions u in both infinite directions $x_1 \to \pm\infty$ of the cylinder $\mathbb{R} \times \omega$, by using some general results of Agmon, Nirenberg [1]; Berestycki, Nirenberg [6]; or Pazy [23].

Lemma 1.4 is proved in Section 2. The proof of Theorem 1.5 is reached in several steps in Section 3: 1) resolution of an equivalent problem in finite cylinders, 2) construction of solutions of related problems in semi-infinite strips and 3) passage to the limit in the whole cylinder for the solution given in step 1. This process converges by comparison with the auxiliary functions given in step 2. Last, following the definitions and ideas of Fife, McLeod [9] and Roquejoffre [25], [26], Section 4 is especially devoted to the question of the stability of the travelling waves given in Theorem 1.5.

2. Comparison formulas between the speeds of different travelling waves.

2.1. Some useful preliminaries. In this subsection, we recall some results of [1], [6], [23] which are used later in the proofs. These results mainly deal with the asymptotic behaviour as $x_1 \to -\infty$ of *positive* solutions u of

$$\begin{cases} \Delta u - \beta(y)\partial_1 u + f(y,u) = 0 & \text{in } \Sigma^- = (-\infty, 0) \times \omega \\ \partial_\nu u(x_1, y) = 0 & \forall\ x_1 < 0,\ y \in \partial\omega \end{cases} \tag{2.1}$$

such that $u(x_1, y) \to 0$ as $x_1 \to -\infty$ uniformly in y. Here the function $f(y, s)$ is assumed to be of class $C^{1,\delta}$ with respect to s in a neighbourhood of $s = 0$, and $f(y, 0) = 0$ for all $y \in \overline{\omega}$. The function $\beta : \overline{\omega} \to \mathbb{R}$ is continuous. The study of the asymptotic behaviour as $x_1 \to +\infty$ systematically boils down to the previous study by changing the variables $x_1 \to -x_1$. We also mention that related results have been given by Hamel [16], Li [19] or Vega [29], [30].

Consider the linearized problem of (2.1) around the function 0:

$$\begin{cases} \Delta w - \beta(y)\partial_1 w - a(y) w = 0 & \text{in } \Sigma^- \\ \partial_\nu w = 0 & \forall\ x_1 < 0,\ y \in \partial\omega \end{cases} \tag{2.2}$$

with $a(y) = -f_s(y,0)$. In various cases which are developed below, this problem has "exponential" solutions of the form $w(x_1,y) = e^{\lambda x_1}\phi(y)$ for a real $\lambda > 0$ and a function $\phi > 0$ on $\overline{\omega}$. The real λ and the function ϕ are said to be a principal eigenvalue and a principal eigenfunction. They are solutions of

$$\left\{\begin{array}{rll} -\Delta\phi + a(y)\phi & = (\lambda^2 - \lambda\beta(y))\phi & \text{in } \omega \\ \partial_\nu\phi & = 0 & \text{on } \partial\omega. \end{array}\right. \tag{2.3}$$

Generally speaking, if $a(y)$ is a bounded function on $\overline{\omega}$, we call μ_1 the first eigenvalue of the problem

$$\left\{\begin{array}{rll} (-\Delta + a(y))\sigma & = \mu_1\sigma & \text{in } \omega \\ \partial_\nu\sigma & = 0 & \text{on } \partial\omega. \end{array}\right. \tag{2.4}$$

The solutions of the elliptic equation (2.1) can be expressed in terms of the special exponential solutions of the linearized problem (2.2):

Lemma 2.1 ([6], Theorems 2.1 and 4.4). *Let u be a positive solution of* (2.1) *with $u(-\infty,\cdot) = 0$, and call μ_1 the first eigenvalue of problem* (2.4) *with $a(y) = -f_s(y,0)$.*

1) *If $\mu_1 \neq 0$, then*

$$u(x_1,y) = \alpha e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty \tag{i}$$

or $$u(x_1,y) = \alpha e^{\lambda x_1}(-x_1\phi(y) + \phi_0(y)) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty. \tag{ii}$$

In (*i*) *and* (*ii*), *α is a positive constant, $\lambda > 0$ and ϕ are respectively the principal eigenvalue and eigenfunction of* (2.3). *Furthermore, the case* (*ii*) *may only occur if $\mu_1 < 0$ and if the principal positive eigenvalue λ solving* (2.3) *is unique.*

2) *If $\mu_1 > 0$, then* (2.3) *admits exactly one positive and one negative principal eigenvalue. For each one, there exists a unique positive eigenfunction ϕ solution of* (2.3) *up to multiplication by a positive constant. Furthermore, if $\beta \leq \overline{\beta}$, $\beta \not\equiv \overline{\beta}$ then the respective principal positive eigenvalues λ and $\overline{\lambda}$ in* (2.3) *are such that $0 < \lambda < \overline{\lambda}$.*

3) *If $\mu_1 < 0$, then* (2.3) *admits* 0, 1 *or* 2 *principal eigenvalues. If two exist, they have the same sign.*

Now return to problem (1.2)–(1.3) and assume that f is $C^{1,\delta}$ in a neighbourhood of 0, $f(0) = 0$ and $f'(0) < 0$. We have the

Lemma 2.2 ([6], Lemma 4.1). *Let u and u' be positive solutions of* (1.2)–(1.3) *in Σ^- with the same c. Assume that $u \geq u'$ and that (i) is true for both u and u' with the same values of α, λ and ϕ. Then $u \equiv u'$ in Σ^-.*

2.2. Proof of Lemma 1.4. We only prove that $c_1 < c$. The other inequality $c < c_2$ holds exactly in the same way. Let u_1 and u be travelling fronts over $(0, \eta)$ and $(0, 1)$ with respective velocities c_1 and c.

First, let us suppose that $c_1 > c$. We will use the device of a sliding method as in [5] to get a contradiction. Since $f'(0) < 0$, the first eigenvalue of problem (2.4) with $a(y) \equiv -f'(0)$ is $\mu_1 = -f'(0) > 0$. Thus we can apply Lemma 2.1 to u and u_1. Let us denote by λ and λ_1 the positive principal eigenvalues involved in their asymptotic behaviour (i). Since $c_1 > c$, we have $\beta(y, c_1) > \beta(y, c)$, and then $0 < \lambda < \lambda_1$.

On the other hand, u_1 and u respectively converge to η and 1 as $x_1 \to +\infty$, and $\eta < 1$. Eventually, there exists a real R large enough such that $u(x_1 + t, y) > u_1(x_1, y)$ for all $|x_1| \geq R$ and $t \geq 0$. Furthermore, we may translate u to the left far enough so that $u > u_1$ everywhere in $\overline{\Sigma}$.

We then translate u back to the right until its graph touches the graph of u_1 (this situation necessarily happens as a result of the behaviours of u and u_1 as $x_1 \to \pm\infty$). The translation of u, that we rename u, satisfies $u \geq u_1$ with equality somewhere. Since $\beta(y, c_1) \geq \beta(y, c)$ and $\partial_1 u \geq 0$ (from Remark 2.3 below), the function $z = u - u_1 \geq 0$ satisfies a linear elliptic inequality

$$\Delta z - \beta(y, c_1)\partial_1 z + c(x_1, y)z = (\beta(y, c) - \beta(y, c_1))\partial_1 u \leq 0 \text{ in } \Sigma$$

for some bounded function c (since f is Lipschitz-continuous). Since $z = 0$ somewhere, it follows from the strong maximum principle and the Hopf Lemma that $z \equiv 0$. That is impossible because u and u_1 do not have the same limit as $x_1 \to +\infty$.

Now, assume that $c = c_1$. By Lemma 2.1 and by the uniqueness of $\lambda > 0$ and $\phi > 0$, we have

$$u(x_1, y) = \alpha e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty \tag{2.5}$$

$$u_1(x_1, y) = \alpha_1 e^{\lambda x_1}\phi(y) + o(e^{\lambda x_1}) \text{ as } x_1 \to -\infty.$$

For any real number r, the function $u^r(x_1, y) := u(x_1 + r, y)$ is a solution of (1.2)–(1.3), and it satisfies (2.5) with α replaced by $\alpha e^{\lambda r}$. With the same arguments as above, we infer that for some positive r large enough, we have $u^r > u_1$ everywhere in Σ.

Next, shift u^r back to the right until it reaches a finite value $r = s$, for which one of the following assertions first occurs: 1) $u^s = u_1$ somewhere in $\overline{\Sigma}$ or 2) $\alpha e^{\lambda s} = \alpha_1$. In case 1), we conclude, as in the case $c_1 > c$, that $u^s \equiv u_1$. This is impossible. If case 2) occurs, Lemma 2.2 yields that $u^s \equiv u_1$ in $\Sigma^- = (-\infty, 0) \times \omega$, whence $u^s \equiv u_1$ in Σ by the strong maximum principle. This completes the proof of Lemma 1.4.

Remark 2.3. The uniqueness of the solutions (c, u) of (1.2)–(1.4) and the monotonicity of u with respect to x_1 are actually consequences of the results of Berestycki and Nirenberg [6]. Only the assumptions that f is $C^{1,\delta}$ near 0 and 1, and that $f'(0)$, $f'(1) < 0$, are required in [6] to get that any solution u of (1.2)–(1.4) is increasing in x_1.

3. Proof of the existence result: Theorem 1.5. The proof is divided into three main steps: resolution of an equivalent problem in bounded domains, construction of solutions of auxiliary problems in semi-infinite cylinders and passage to the limit on the whole cylinder for the solutions constructed in the first step.

3.1. Existence of solutions in finite cylinders. In this subsection, we construct, for any $a > 0$, a couple (c_a, u_a) solution in the finite cylinder $\Sigma_a = (-a, a) \times \omega$ of the approximated equivalent problem

$$\left\{\begin{array}{rll} \Delta u_a - \beta(y, c_a)\partial_1 u_a + f(u_a) & = 0 & \text{in } \Sigma_a \\ \partial_\nu u_a & = 0 & \text{on } (-a, a) \times \partial\omega \\ u_a(-a, y) = 0 < u_a(x_1, y) < u_a(a, y) & = 1 & \forall\ (x_1, y) \in (-a, a) \times \overline{\omega}. \end{array}\right. \tag{3.1}$$

We impose the normalization condition:

$$\max_{\overline{\omega}} u_a(0, \cdot) = \theta. \tag{3.2}$$

By application of Theorem 7.1 in the paper of Berestycki and Nirenberg [5], since $\underline{u} = 0$ and $\overline{u} = 1$ are respectively sub- and supersolutions, there exists a unique solution u^c of (3.1) for any $c \in \mathbb{R}$. Set $\tilde{\Sigma_a} = (-a, a) \times \overline{\omega}$. Besides, $u^c \in W^{2,p}_{loc}(\tilde{\Sigma_a}) \cap C^0(\overline{\Sigma_a})$ for any $1 < p < \infty$. From the classical a priori estimates for elliptic operators and from the Sobolev injections, we find that the functions u^c are continuous in c.

In the sequel, we will make several uses of the following comparison principle stated in Corollary 5.1 in [6]:

Lemma 3.1 ([6]). *Let u and u' be solutions of*

$$\begin{cases} \Delta u - \beta(y)\partial_1 u + f(u) = \Delta u' - \beta'(y)\partial_1 u' + f(u') = 0 & \text{in } \Sigma_a \\ \partial_\nu u = \partial_\nu u' = 0 & \text{on } (-a,a)\times\partial\omega. \end{cases}$$

If $\beta' \le \beta$, $\beta' \not\equiv \beta$ in $\overline{\omega}$ and if $u \le u'$ on $\{\pm a\}\times\overline{\omega}$, then $u < u'$ in Σ_a.

Now, if $c < c'$, then $\beta(y,c) < \beta(y,c')$ in $\overline{\omega}$ (from the assumption made in the introduction), whence $u^c > u^{c'}$ in Σ_a by Lemma 3.1.

On the other hand, for any real k, let v_k be the solution of the one-dimensional problem (which can for instance be solved with the same tools)

$$\begin{cases} v_k'' - kv_k' + f(v_k) & = 0 \text{ in } (-a,a) \\ v_k(-a) = 0,\ v_k(+a) & = 1. \end{cases}$$

Direct computations, using the boundedness of f and comparisons with exponential solutions, lead to the limits $\lim_{k\to-\infty} v_k(0) = 1$, $\lim_{k\to+\infty} v_k(0) = 0$. Since the real $v_k(0)$'s are strictly decreasing in k, we then infer that there exists a unique $k^* \in \mathbb{R}$ such that $v_{k^*}(0) = \theta$.

Let c_0 be such that $\beta(y,c_0) < k^*$ for all $y \in \overline{\omega}$. Lemma 3.1 then yields that $\max_{\overline{\omega}} u^{c_0}(0,.) > \theta$. Similarly, if c_1 is such that $\beta(y,c_1) > k^*$ for all $y \in \overline{\omega}$, then $\max_{\overline{\omega}} u^{c_1}(0,\cdot) < \theta$. We eventually conclude that there exists a unique c_a such that (c_a, u_a) is a solution of (3.1) with the normalization (3.2).

The next step consists in proving that the real c_a's are bounded.

Lemma 3.2. *There exists a constant $K > 0$ such that, for all $a \ge 1$, $|c_a| \le K$.*

Proof. Note first that $0 < \theta_1 < \theta < \theta_2 < 1$ are the 5 zeros of f in $[0,1]$. In order to get an upper bound for the speeds c_a, we first call (c_a', u_a') the unique couple which is a solution of

$$\begin{cases} \Delta u_a' - \beta(y,c_a')\partial_1 u_a' + f(u_a') = 0 & \text{in } \Sigma_a \\ \partial_\nu u_a' = 0 & \text{on } (-a,+a)\times\partial\omega \\ u_a'(-a,y) = 0,\ u_a'(+a,y) = 1 & \text{for } y \in \overline{\omega} \end{cases}$$

with the normalization condition

$$\max_{\overline{\omega}} u_a'(0,\cdot) = \theta_1. \tag{3.3}$$

We infer that $c_a \le c_a'$. Indeed, if $c_a > c_a'$, then Lemma 3.1 yields that $u_a < u_a'$ in Σ_a; this is in contradiction with the normalization conditions (3.2) and (3.3) on $\{0\}\times\overline{\omega}$ (indeed, $0 < \theta_1 < \theta$).

Now, to get an upper bound for c'_a, consider the unique pair (k_a, v_a) solving

$$\begin{cases} v''_a - k_a v'_a + f(v_a) = 0 \text{ in } (-a,a) \\ v_a(-a) = 0,\ v_a(a) = 1, \quad v_a(0) = \theta_1. \end{cases}$$

We claim that

$$\min_{\overline{\omega}} \beta(\cdot, c'_a) \leq k_a. \tag{3.4}$$

Otherwise, we would have $\min_{\overline{\omega}} \beta(\cdot, c'_a) > k_a$, and Lemma 3.1 would yield that $u'_a < v$ in $(-a,a) \times \overline{\omega}$. This contradicts the normalization condition (3.3) and $v_a(0) = \theta_1$.

Now, to get an upper bound for k_a, we observe that $v''_a - k_a v'_a \geq -M\chi_a$, where $M = \max|f|$ and χ_a is the characteristic function of $(0,a)$. We now construct a C^1 function z on $[-a,a]$ such that

$$\begin{cases} z'' - k_a z' &= -M\chi_a \text{ on } (-a,a) \\ z(-a) = 0,\ z(a) &= 1. \end{cases}$$

Set $z(0) = \tau$ and suppose that $k_a > 0$. Thus,

$$\begin{cases} z(x_1) = \tau \frac{e^{k_a x_1} - e^{-k_a a}}{1 - e^{-k_a a}} & \text{for } x_1 < 0 \\ z(x_1) = \frac{M x_1}{k_a} + \tau + \alpha(e^{k_a x_1} - 1) & \text{for } x_1 > 0 \end{cases}$$

where the real α is determined by $z(a) = 1$, namely $\frac{Ma}{k_a} + \tau + \alpha(e^{k_a a} - 1) = 1$. Furthermore, since z is C^1 at $x = 0$, we have $z'(0) = \frac{\tau k_a}{1 - e^{-k_a a}} = \frac{M}{k_a} + \alpha k_a$ and

$$\frac{\tau}{1 - e^{-k_a a}}(e^{k_a a} - 1) = \frac{M}{k_a^2}(e^{k_a a} - 1) + \alpha(e^{k_a a} - 1) = \frac{M}{k_a^2}(e^{k_a a} - 1) + 1 - \frac{Ma}{k_a} - \tau.$$

The maximum principle and the Hopf Lemma yield that $v \leq z$ on $[-a,a]$ and then $\theta_1 \leq \tau$. Hence, $\theta_1 \leq \tau \leq \frac{M}{k_a^2} + \frac{1}{e^{k_a} - 1}$ if $a \geq 1$. This eventually implies that

$$k_a \leq \max(K_1, 0) = K_2, \quad \forall a \geq 1 \tag{3.5}$$

where K_1 (and then K_2) is independent of $a \geq 1$. Inequalities (3.4) and (3.5) then give that $\min_{\overline{\omega}} \beta(y, c'_a) \leq K_2$, $\forall a \geq 1$. We then conclude that there exists a real K such that $c_a \leq c'_a \leq K$, $\forall a \geq 1$.

The lower bound $c_a \geq K'$ could be obtained similarly, by considering the unique pair (c''_a, u''_a) solving

$$\begin{cases} \Delta u''_a - \beta(y, c''_a)\partial_1 u''_a + f(u''_a) = 0 & \text{in } \Sigma_a \\ \partial_\nu u''_a = 0 & \text{on } (-a, +a) \times \partial\omega \\ u''_a(-a, y) = 0, \ u''_a(+a, y) = 1 & \forall \ y \in \overline{\omega} \end{cases}$$

with the new normalization condition $\min_{\overline{\omega}} u''_a(0, \cdot) = \theta_2$. Since $\theta < \theta_2$, we can then get that there exists a constant K' such that $c_a \geq c''_a \geq K'$ for any $a \geq 1$.

3.2. Construction of some auxiliary solutions.

Lemma 3.3. *Let f be a function satisfying* (1.8)–(1.9). *For any fixed $c > c_1$, there exists a solution v_c, in the half-cylinder $\Sigma^- = \mathbb{R}^*_- \times \omega$, of the following problem:*

$$\begin{cases} \Delta v_c - \beta(y, c)\partial_1 v_c + f(v_c) = 0 & \text{in } \Sigma^- \\ \partial_\nu v_c = 0 & \text{on } \mathbb{R}^*_- \times \partial\omega \\ v_c(-\infty, y) = 0, \ v_c(0, y) = \theta & \text{for all } y \in \overline{\omega}, \end{cases} \tag{3.6}$$

and for any fixed $c < c_2$, there exists a solution w_c, in the half-cylinder $\Sigma^+ = \mathbb{R}^*_+ \times \omega$, of the following problem:

$$\begin{cases} \Delta w_c - \beta(y, c)\partial_1 w_c + f(w_c) = 0 & \text{in } \Sigma^+ \\ \partial_\nu w_c = 0 & \text{on } \mathbb{R}^*_+ \times \partial\omega \\ w_c(0, y) = \theta, \ w_c(+\infty, y) = 1 & \text{for all } y \in \overline{\omega}. \end{cases} \tag{3.7}$$

In [9], Fife and McLeod solved the same problem in dimension 1 by using the device of the phase plane. Then, by a continuity argument, they proved the existence of a real $c \in (c_1, c_2)$ such that the function u defined by $u = u_c$ in $\mathbb{R}_-$ and $u = v_c$ in $\mathbb{R}_+$ is a solution of (1.6) with the speed c. To do that, it is sufficient that $v'_c(0) = w'_c(0)$. Unfortunately, in the multidimensional case, these arguments no longer work.

Proof of Lemma 3.3. We only prove the existence of the functions v_c (the existence of the w_c is completely similar). Fix a real $c > c_1$. For any $a > 0$, let v^c_a be the unique solution of the following problem:

$$\begin{cases} \Delta v^c_a - \beta(y, c)\partial_1 v^c_a + f(v^c_a) = 0 & \text{in } \Sigma'_a = (-2a, 0) \times \omega \\ \partial_\nu v^c_a = 0 & \text{on } (-2a, 0) \times \partial\omega \\ v^c_a(-2a, \cdot) = 0, \ v^c_a(0, \cdot) = \theta & . \end{cases} \tag{3.8}$$

This solution exists and is unique since both $\overline{u} = \theta$ and $\underline{u} = 0$ are super- and sub-solutions for this problem (see [5]). We also have $\partial_1 v_a^c \geq 0$ in Σ'_a.

Using standard elliptic estimates up to the boundary, we can see that for a subsequence of a, $a_j \to +\infty$, the functions $v_{a_j}^c$ converge to some function v_c uniformly on compact sets of $\mathbb{R}_- \times \overline{\omega}$. The limit function v_c is in $W^{2,p}_{loc}(\Sigma^-)$ for any $p < \infty$, is nondecreasing in x_1 in $\mathbb{R}_- \times \overline{\omega}$ and satisfies

$$\begin{cases} \Delta v_c - \beta(y,c)\partial_1 v_c + f(v_c) = 0 & \text{in } \Sigma^- \\ \partial_\nu v_c = 0 & \text{on } \mathbb{R}^*_- \times \partial\omega \\ v_c(-\infty, y) = \psi_1(y),\ v_c(0,y) = \theta & \text{for all } y \in \omega. \end{cases}$$

By the standard elliptic estimates and since $\partial_1 v_c \geq 0$, it follows that $v_c(x_1 - n, y) \to \psi_1(y)$ in $W^{2,p}_{loc}(\mathbb{R} \times \overline{\omega})$ as $n \to +\infty$. Hence, $\psi_1(y)$ is in $W^{2,p}_{loc}(\overline{\omega})$ and is a solution of

$$\begin{cases} \Delta\psi + f(\psi) = 0 & \text{in } \omega \\ \partial_\nu \psi = 0 & \text{on } \partial\omega \end{cases} \tag{3.9}$$

In addition, $0 \leq \psi_1 \leq \theta$.

Let us now prove that $\psi_1 \equiv 0$, by *reductio ad absurdum*. Suppose that $\psi_1 \not\equiv 0$. Since $\psi_1 \geq 0$, it follows from the maximum principle and the Hopf Lemma that $\psi_1 > 0$ in $\overline{\omega}$. Fix a real number $d > 0$ such that $0 < d < \min(\min_{\overline{\omega}} \psi_1, \theta_1)$. Since $\partial_1 v_a^c > 0$ in Σ'_a, there is a unique $\tau_a \in (0, 2a)$ such that $\max_{\overline{\omega}} v_a^c(-\tau_a, \cdot) = d$. Since $v_a^c \to v_c$ locally and $v_c \geq \psi_1$, we see that $\tau_a \to +\infty$ as $a \to +\infty$. Let us now shift the origin to $x_1 = \tau_a$ by setting $w_a^c(x_1, y) := v_a^c(x_1 - \tau_a, y)$. This function w_a^c is defined on $[-2a + \tau_a, \tau_a] \times \overline{\omega}$. For a sequence of $a_j \to +\infty$, we have $-2a_j + \tau_{a_j} \to b \in [-\infty, 0]$ as $a_j \to +\infty$, and the functions $w_{a_j}^c$ converge to a function w_c locally in $C^{1,\mu}([b, +\infty[\times\overline{\omega})$ for any $0 < \mu < 1$. This function w_c is a solution in $(b, +\infty) \times \omega$ of the same equation as v_c. Moreover $\max_{\overline{\omega}} w_c(0, \cdot) = d$ and $w_c(+\infty, y) = \psi_2(y)$ where ψ_2 is a solution of (3.9) satisfying $0 \leq \psi_2 \leq \theta$.

We will consider both cases $b > -\infty$ and $b = -\infty$ to get a contradiction. If b is a finite number, then w_c satisfies

$$\begin{cases} \Delta w_c - \beta(y,c)\partial_1 w_c + f(w_c) = 0 & \text{in } (b, +\infty) \\ \partial_\nu w_c = 0 & \text{on } (b, +\infty) \times \partial\omega \\ w_c(b, y) = 0,\ w_c(+\infty, y) = \psi_2(y) & \text{for all } y \in \overline{\omega}. \end{cases}$$

Let us now compare w_c with the travelling wave u_1, which is a connection between 0 and θ with the speed c_1. Clearly $u_1 > w_c$ if $x_1 = b$. Two cases occur:

– If $\psi_2 \not\equiv \theta$, then $\psi_2 < \theta$ by the strong maximum principle. Thus $u_1 > w_c$ if x_1 is large. Since $c > c_1$, by using a sliding method as in the proof of Lemma 1.4, we would get a contradiction. Hence, $\psi_1 \equiv 0$ and v_c satisfies (3.6).

– If $\psi_2 \equiv \theta$, then we can study the asymptotic behaviour of w_c near $+\infty$. Since $\partial_1 w_c \geq 0$, then $w_c \leq \theta$ and we even have $w_c < \theta$ in $(b, +\infty) \times \overline{\omega}$ from the strong maximum principle and the Hopf Lemma. Thus the function $\xi(x_1, y) := \theta - w_c(-x_1, y)$ is positive, goes to 0 as $x_1 \to -\infty$ and satisfies

$$\begin{cases} \Delta\xi - (-\beta(y,c))\partial_1\xi + g(\xi) = 0 & \text{in } (-\infty, -b) \times \omega \\ \partial_\nu \xi = 0 & \text{on } (-\infty, -b) \times \partial\omega \end{cases}$$

where $g(\xi) = -f(\theta - \xi)$. We have $g(0) = 0$ and $a(y) := -g'(0) = -f'(\theta) > 0$. With the notations in Lemma 2.1, the first eigenvalue μ_1 of problem (2.4) is positive. From Lemma 2.1, it follows that

$$\xi(-x_1, y) = \theta - w_c(x_1, y) = \alpha e^{-\tau x_1}\phi(y) + o(e^{-\tau x_1}) \text{ as } x_1 \to +\infty,$$

where $\tau > 0$, $\alpha > 0$ and $\phi(y) > 0$. Similarly, we can write the asymptotic behaviour of u_1 near $+\infty$:

$$\theta - u_1(x_1, y) = \alpha_1 e^{-\tau_1 x_1}\phi_1(y) + o(e^{-\tau_1 x_1}) \text{ as } x_1 \to +\infty,$$

where $\tau_1 > 0$, $\alpha_1 > 0$ and $\phi_1(y) > 0$. Since $c > c_1$, we have $-\beta(y, c_1) > -\beta(y, c)$ for any $y \in \overline{\omega}$. Lemma 2.1 then yields that $\tau_1 > \tau > 0$. Thus, $u_1 > w_c$ for x_1 large and we get a contradiction by arguing as in Section 2.

If $b = -\infty$, then the function w_c satisfies

$$\begin{cases} \Delta w_c - \beta(y,c)\partial_1 w_c + f(w_c) = 0 & \text{in } \Sigma \\ \partial_\nu w_c = 0 & \text{on } \partial\Sigma \\ w_c(-\infty, y) = \psi_1'(y),\ w_c(+\infty, y) = \psi_2(y) & \text{for all } y \in \overline{\omega}, \end{cases}$$

where ψ_1' is a solution of (3.9). Since $\max_{\overline{\omega}} w_c(0, .) = d < \theta_1$ and $\partial_1 w_c \geq 0$, it follows that $\psi_1' < \theta_1$. By integration (3.9) and by using the fact that $f < 0$ on $(0, \theta_1)$, it follows that $\psi_1' \equiv 0$. Since $c > c_1$, Lemma 2.1 implies that u_1 and w_c have different exponential behaviours as $x_1 \to -\infty$ and that $u_1 > w_c$ for $-x_1$ large. By examining the behaviours of u_1 and w_c as $x_1 \to +\infty$, the same arguments as in the case where b is finite, eventually lead to a contradiction.

Thus, both cases $b > -\infty$ and $b = -\infty$ lead to a contradiction. We conclude that $\psi_1 \equiv 0$. This achieves the proof of the existence of a solution v_c of (3.6), for $c > c_1$.

3.3. Proof of Theorem 1.5. Remember first that u_1 and u_2 are the travelling wave solutions of (1.2)–(1.3), with respective speeds c_1 and c_2 and that u_1 and u_2 are respectively connections between 0 and θ, and between θ and 1.

In subsection 3.1, for any $a > 0$, we proved the existence and uniqueness of a solution (c_a, u_a) in the finite cylinder $\Sigma_a = (-a, a) \times \omega$, of

$$\begin{cases} \Delta u_a - \beta(y, c_a)\partial_1 u_a + f(u_a) = 0 & \text{in } \Sigma_a \\ \partial_\nu u_a = 0 & \text{on } (-a, a) \times \partial\omega \\ u_a(-a, \cdot) = 0,\ u_a(a, \cdot) = 1 & \end{cases}$$

with the normalization condition $\max_{y\in\overline{\omega}} u_a(0, y) = \theta$. Lemma 3.2 states that the real numbers c_a are bounded independently of $a \geq 1$. Hence, from the standard elliptic estimates up to the boundary, there exists a subsequence $a_j \to +\infty$ such that $c_{a_j} \to c$ and the functions u_{a_j} converge locally in $C^{1,\mu}$ to a function u solving

$$\begin{cases} \Delta u - \beta(y, c)\partial_1 u + f(u) = 0 & \text{in } \Sigma \\ \partial_\nu u = 0 & \text{on } \partial\Sigma \\ u(-\infty, y) = \psi_1(y),\ u(+\infty, y) = \psi_2(y) & \text{for all } y \in \overline{\omega} \end{cases}$$

where ψ_1 and ψ_2 are solutions of (3.9). Moreover, $\partial_1 u \geq 0$ and $\max_{\overline{\omega}} u(0, \cdot) = \theta$. In order to complete the proof of Theorem 1.5, it is sufficient to prove that $\psi_1 \equiv 0$ and $\psi_2 \equiv 1$.

Step 1. Let us first prove that $c_1 < c < c_2$ and that $\psi_1 \equiv 0$. Assume first that $c \geq c_2$. Since $c_2 > c_1$, there exists a real c' such that $c_a > c' > c_1$ for a large enough. Let $v_{a/2}^{c'}$ be the auxiliary function solving (3.8) for c' and $a/2$, namely

$$\begin{cases} \Delta v_{a/2}^{c'} - \beta(y, c')\partial_1 v_{a/2}^{c'} + f(v_{a/2}^{c'}) = 0 & \text{in } (-a, 0) \times \omega \\ \partial_\nu v_{a/2}^{c'} = 0 & \text{on } (-a, 0) \times \partial\omega \\ v_{a/2}^{c'}(-a, \cdot) = 0,\ v_{a/2}^{c'}(0, \cdot) = \theta. & \end{cases}$$

Since $c_a > c'$ and $u_a \leq v_{a/2}^{c'}$ on $\{0, -a\} \times \overline{\omega}$, Lemma 3.1 (applied on $(-a, 0) \times \omega$) yields that $u_a \leq v_{a/2}^{c'}$ in $[-a, 0] \times \overline{\omega}$. Then, as $a_j \to +\infty$, it follows that

$u \le v_{c'}$ in $\mathbb{R}_- \times \overline{\omega}$, where $v_{c'}$ is a solution of (3.6). Hence u is a connection between 0 and the function $\psi_2(y)$. Since $\max_{\overline{\omega}} u(0,\cdot) = \theta$, it follows from the strong maximum principle and the Hopf Lemma that

$$\max_{\overline{\omega}} \psi_2 > \theta. \tag{3.10}$$

By sliding u with respect to u_2, with the same tools as in the proof of Lemma 1.4, we could see that the hypothesis $c \ge c_2$ would lead to a contradiction. Actually, the case $\psi_2 \equiv 1$ was explicitly covered in Section 2 by writing the exponential behaviours of u and u_2 as $x_1 \to +\infty$. The other case $\psi_2 < 1$ is actually easier to deal with. Notice that it is necessary to have $\max_{\overline{\omega}} \psi_2 > \theta$ so that the graphs of u and u_2 touch at a common point, up to translation. Hence, we conclude that $c < c_2$.

Assume now that $c \le c_1$. Let (c'_a, u'_a) be the unique couple which is the solution of

$$\begin{cases} \Delta u'_a - \beta(y, c'_a)\partial_1 u'_a + f(u'_a) = 0 & \text{in } \Sigma_a \\ \partial_\nu u'_a = 0 & \text{on } (-a,a) \times \partial\omega \\ u'_a(-a,\cdot) = 0,\ u'_a(a,\cdot) = 1 & \end{cases}$$

fulfilling this time the normalization condition $\min_{\overline{\omega}} u'_a(0,\cdot) = \theta$ instead of the max as for (c_a, u_a). As in Section 3.1, we infer that the real numbers c'_a are bounded. Hence, for some subsequence $\varphi(a_j) \to +\infty$, which we rename a_j, we get that $c'_{a_j} \to c'$ and $u'_{a_j} \to u'$ uniformly on compact sets. The function u' satisfies

$$\begin{cases} \Delta u' - \beta(y, c')\partial_1 u' + f(u') = 0 & \text{in } \Sigma \\ \partial_\nu u' = 0 & \text{on } \partial\Sigma \\ u'(-\infty, y) = \psi'_1(y),\ u'(+\infty, y) = \psi'_2(y) & \text{for all } y \in \overline{\omega} \end{cases}$$

where the functions ψ'_1 and ψ'_2 are solutions of (3.9).

We claim that $c'_a \le c_a$. Otherwise, if $c_a < c'_a$, then $u'_a < u_a$ by Lemma 3.1, and this contradicts the normalization conditions on $\{0\} \times \overline{\omega}$. Thus, by passing to the limit $a_j \to +\infty$, we get that $c' \le c \le c_1 < c_2$. In addition, there exists $c'' < c_2$ such that $c'_a < c'' < c_2$ for a large enough. Since $c'' < c_2$, by Lemma 3.3, there exists a function $w_{c''}$ solving (3.7), namely

$$\begin{cases} \Delta w_{c''} - \beta(y, c'')\partial_1 w_{c''} + f(w_{c''}) = 0 & \text{in } \Sigma^+ \\ \partial_\nu w_{c''} = 0 & \text{on } \mathbb{R}^*_+ \times \partial\omega \\ w_{c''}(0,\cdot) = \theta,\ w_{c''}(+\infty,\cdot) = 1. & \end{cases}$$

As we did earlier for the functions u and $v_{c'}$, we can compare u' and $w_{c''}$ and get that $u' \geq w_{c''}$ in Σ^+. Hence u' is a connection between the function $\psi_1'(y)$ and 1. The strong maximum principle then yields that $\min_{\overline{\omega}} \psi_1' < \theta$. Since we have supposed that $c \leq c_1$ and since $c' \leq c$, we get that $c' \leq c_1$. We could then slide u' with respect to the travelling front u_1 with the same tools as in Section 2. This would lead to a contradiction.

Finally, we conclude that $c_1 < c < c_2$. In particular, the first part of the proof of this step 1 implies then that $\psi_1 \equiv 0$.

Step 2. To complete the proof of Theorem 1.5, only the equality $\psi_2 \equiv 1$ remains to be shown. The proof is rather similar to [6], but we give it here for the sake of completeness.

Suppose that $\psi_2 \not\equiv 1$. Since $\psi_2 \leq 1$ is a solution of (3.9), it follows from the strong maximum principle and the Hopf Lemma that $\psi_2 < 1$ in $\overline{\omega}$. Note that $\theta < \max_{\overline{\omega}} \psi_2$ by (3.10). Fix a real d such that

$$\theta < \max_{\overline{\omega}} \psi_2 < d < 1. \tag{3.11}$$

Since $\partial_1 u_a > 0$ in $\overline{\Sigma}_a$, there exists a unique $t_a \in (-a, a)$ such that

$$\max_{y \in \overline{\omega}} u_a(t_a, y) = d.$$

Since $u_a \to u$ locally and $u \leq \psi_2$, it follows that $t_a \to +\infty$ as $a \to +\infty$. We then shift the functions u_a by setting $v_a(x_1, y) := u_a(x_1 + t_a, y)$. The functions v_a are defined on $[-a - t_a, a - t_a] \times \overline{\omega}$ and, for some subsequence $a_j \to +\infty$, we have $a_j - t_{a_j} \to b \in [0, +\infty]$. Since the functions v_{a_j} are bounded locally in $W^{2,p}$, we can assume that $v_{a_j} \longrightarrow v$ locally in $C^{1,\mu}$ as $a_j \to +\infty$ (up to extraction of some subsequence). Thus v is a solution of (1.2)–(1.3) in $\Sigma_b = (-\infty, b) \times \omega$ for the same c as for u. Furthermore $\partial_1 v \geq 0$ in this domain, and

$$\max_{\overline{\omega}} \ v(0, \cdot) = d. \tag{3.12}$$

The same arguments as above show that v has a limit as $x_1 \to -\infty$: $v(-\infty, y) = \psi_1'(y)$ where ψ_1' is a solution of (3.9). For any $y \in \overline{\omega}$, $x_1 \in \mathbb{R}$ and any $A > 0$, we have $x_1 + t_a > A$ for a large, whence $v_a(x_1, y) > u_a(A, y)$ for $y \in \omega$. The limit $a_j \to +\infty$ gives $v(x_1, y) \geq u(A, y)$ for any $A > 0$, and therefore $v(x_1, y) \geq \psi_2(y)$. Thus $\psi_1'(y) \geq \psi_2(y)$ and, by condition (3.12), it follows that

$$\psi_2 \leq \psi_1' \leq d < 1. \tag{3.13}$$

If ψ_1' is a constant, then it is a zero of f and (3.11) and (3.13) imply that $\psi_1' = \theta_2$. With the same kind of arguments as in section 3.2, by successively considering the cases $b \in \mathbb{R}$ and $b = +\infty$ and by comparing v with the travelling front u_2, the inequality $c < c_2$ eventually leads to a contradiction. Hence, ψ_1' is not a constant. Now, if ψ_2 is a constant, then (3.11) and (3.13) imply that $\psi_2 = \theta_2$, and then $f(\psi_1') \geq 0$ in ω. By integration of (3.9) over ω with $\psi = \psi_1'$, it follows that $\int_\omega f(\psi_1') = 0$ and then that $\psi_1' \equiv \theta_2$ (remember that $\psi_1' < 1$). Thus, ψ_1' and ψ_2 are both nonconstant solutions of (3.9), $\psi_2 \leq \psi_1'$ and there cannot exist any zero of f between ψ_2 and ψ_1. From Proposition 1.3, it then follows that $\psi_1' \equiv \psi_2$ in $\overline{\omega}$.

Eventually, for the same value of c, there is a connection u from 0 to ψ_2 and a solution v of (1.2)–(1.3) in Σ_b with $v(-\infty, y) = \psi_2(y)$. By analyzing the asymptotic behaviour of these solutions, we will now show that this is impossible. Indeed, since the function $\psi_2 \equiv \psi_1'$ is a nonconstant solution of (3.9), Proposition 1.2 states that the first eigenvalue $\mu_1(\psi_2)$ of the linearized operator $-\Delta - f'(\psi_2)$ in ω with Neumann boundary conditions on $\partial\omega$, is negative (we here use the convexity of ω).

Consider first the behaviour of u as $x_1 \to +\infty$. The function $w(x_1, y) = \psi_2(y) - u(-x_1, y)$ is positive, goes to 0 as $x_1 \to -\infty$ and satisfies the equation

$$\Delta w + \beta(y, c)\partial_1 w + g(y, w) = 0 \text{ in } \Sigma,$$

where $g(y, w) = f(\psi_2(y)) - f(\psi_2(y) - w)$. We have $g(y, 0) = 0$, $g_w(y, 0) = f'(\psi_2(y))$ and the first eigenvalue of $-\Delta_y - g_w(y, 0)$ with Neumann boundary conditions, namely $\mu_1(\psi_2)$, is negative. By Lemma 2.1, there exist a positive principal eigenvalue $\lambda > 0$ and an eigenfunction $\phi(y) > 0$ in $\overline{\omega}$, which are solutions of the problem

$$\begin{cases} -\Delta_y \phi - f'(\psi_2)\phi &= (\lambda^2 + \lambda\beta(y, c))\phi \quad \text{in } \omega \\ \partial_\nu \phi &= 0 \quad \text{on } \partial\omega. \end{cases} \tag{3.14}$$

The behaviour of u as $x_1 \to +\infty$ is given by

$$u(x_1, y) = \psi_2(y) - \alpha e^{-\lambda x_1}\phi(y) + o(e^{-\lambda x_1}) \text{ as } x_1 \to +\infty, \quad \text{or}$$

$$u(x_1, y) = \psi_2(y) - \alpha e^{-\lambda x_1}(x_1\phi(y) + \phi_0(y)) + o(e^{-\lambda x_1}) \text{ as } x_1 \to +\infty,$$

where α and λ are positive and ϕ is a positive function in $\overline{\omega}$ satisfying (3.14).

Let us now emphasize the behaviour of v as $x_1 \to -\infty$. By applying again Lemma 2.1, it follows that

$$v(x_1, y) = \psi_2(y) + \alpha' e^{\lambda' x_1}\phi'(y) + o(e^{\lambda' x_1}) \text{ as } x_1 \to -\infty, \quad \text{or}$$

$$v(x_1,y)=\psi_2(y)+\alpha' e^{\lambda' x_1}(-x_1\phi'(y)+\phi_0'(y))+o(e^{\lambda' x_1}) \text{ as } x_1\to-\infty,$$

where α', λ' are positive and ϕ' is a positive function in $\overline{\omega}$ solving

$$\begin{cases} -\Delta_y\phi'-f'(\psi_2)\phi' &= ({\lambda'}^2-\lambda'\beta(y,c))\phi \quad \text{in } \omega\\ \partial_\nu\phi' &= 0 \quad \text{on } \partial\omega.\end{cases}$$

Therefore, the same problem (3.14) admits one positive principal eigenvalue, λ, and one negative principal eigenvalue, $-\lambda'$. Since $\mu_1(\psi_2)<0$, Lemma 2.1 asserts that the principal eigenvalues of (3.14) necessarily have the same sign. We then have a contradiction. This proves that $\psi_2\equiv 1$ and that u is necessarily a connection between 0 and 1. This completes the proof of Theorem 1.5.

4. Remarks on the stability of these travelling waves. In this section, we assume that $\beta(y,c)=c+\alpha(y)$, where α is a given and continuous function in $\overline{\omega}$. Let us now study the Cauchy problem

$$\begin{cases} \partial_t v=\Delta v-\alpha(y)\partial_1 v+f(v) & \text{for } t\in\mathbb{R}_+, (x_1,y)\in\overline{\Sigma}\\ \partial_\nu v=0 & \text{on } \partial\Sigma\\ v(0,x_1,y)=v_0(x_1,y) & \text{given function in } \overline{\Sigma}.\end{cases} \tag{4.1}$$

For a function f fulfilling (1.8)-(1.9), the solutions (c,u) given in Theorem 1.5 are travelling fronts $u(x_1+ct,y)$ for this evolution problem (4.1). A natural question consists in investigating the stability of these waves u.

More generally speaking, following the definitions and ideas of Fife, McLeod ([9]) and Roquejoffre ([25], [26]), we state in this section various results dealing with the behaviour for large time of the solutions of (4.1). If problem (1.2)–(1.4) has a solution of the travelling wave type, that is to say if $c_1<c_2$, we will speak about the asymptotic or global stability of this wave and about extension phenomena. If such waves do not exist, we will mention some results of the splitting type.

4.1. Asymptotic stability. If there exists a travelling front (c,u) for (4.1), namely if $c_1<c_2$, we say that this front is asymptotically stable if the solutions of the Cauchy problem (4.1) converge, as $t\to+\infty$, to a shift of this front in the frame which moves with the speed c to the left, provided that the initial condition be close enough to the travelling front.

In recent works, Berestycki, Larrouturou and Roquejoffre ([4], [25], [26]) established results on the stability of travelling fronts in the multidimensional case for a bistable nonlinearity f. Consider a function f of class $C^3([0,1])$, satisfying (1.5), and assume that there exists a travelling wave (c,ϕ) solving

$$\begin{cases} \Delta\phi - (c+\alpha(y))\partial_1\phi + f(\phi) = 0 & \text{in } \Sigma \\ \partial_\nu \phi = 0 & \text{on } \partial\Sigma \\ \phi(-\infty,\cdot) = 0,\ \phi(+\infty,\cdot) = 1 \end{cases} \tag{4.2}$$

(the existence of travelling fronts is guaranteed if ω is convex). Let X be the space of the bounded and uniformly continuous functions on $\overline{\Sigma}$. In [25], [26], Roquejoffre proved that there exist constants δ, K, $\omega > 0$ and a function τ of class C^1 in the ball $B_X(0,\delta)$ such that $\tau(0) = 0$, and if $v_0 = \phi + \tilde{v_0}$ with $||\tilde{v_0}||_\infty < \delta$, then

$$|v(t,x_1,y) - \phi(x_1 + ct - \tau(\tilde{v_0}), y)| \le Ke^{-\omega t}, \quad \forall (x_1,y) \in \overline{\Sigma},\ \forall t \ge 0. \tag{4.3}$$

This result actually works for a wider class of functions f even if it means changing the definition of X (see [25], [26] for more details).

Similarly, it is clear that we can extend this result (4.3) to the multiple crossing case, i.e., for a function f satisfying (1.8)–(1.9), or (1.11) in the general case, provided that there is a travelling front solution of (1.2)–(1.4).

Let us explain in a few words how this extension works. The proof in [25] is divided into two main steps: first, a precise study of the linearized operator, $L = -\Delta + (c+\alpha(y))\partial_1 - f'(\phi)$ and second, the application of the implicit function theorem. The result given in [25] only requires the facts that u is increasing in x_1 and that $f'(0)$, $f'(1)$ are negative. Since these properties are true in the problems we emphasize, we conclude that the asymptotic stability (4.3) works for the travelling fronts given in Theorem 1.5 or in its generalization for multistable functions f.

4.2. Global stability. Global stability is a stronger notion than asymptotic stability, in the sense that the initial condition can be more general. In [26], Roquejoffre established that if the nonlinear term f satisfies (1.5) and if the initial condition v_0 is such that

$$\limsup_{x_1 \to -\infty} v_0(x_1,\cdot) < \theta, \quad \liminf_{x_1 \to +\infty} v_0(x_1,\cdot) > \theta$$

(front-like data), then there exist $x_0 \in \mathbb{R}$ and constants K, $\omega > 0$ such that

$$|v(t,x_1,y) - \phi(x_1 + ct - x_0, y)| \le Ke^{-\omega t}, \quad \forall (x_1,y) \in \overline{\Sigma},\ \forall t \ge 0, \tag{4.4}$$

where ϕ is the unique travelling wave solution of (4.2). This result actually generalizes a former result by Fife and McLeod in the one-dimensional case [9]. Notice that similar results were also obtained by Hagan [15] for fully nonlinear one-dimensional equations.

The proof of [26] in the bistable case is based on the construction of suitable sub- and supersolutions for problem (4.1), which are close, exponentially in time, to a shift of the front ϕ. These comparisons yield the compactness of the orbits of the family of functions $\{V(t) = v(t, \cdot - ct, \cdot),\ t > 0\}$. This implies the existence of an increasing function in the ω-limit set, and this eventually leads to the convergence of the whole family $v(t, \cdot - ct, \cdot)$ to a shift of the front ϕ, exponentially in time.

This proof can be adapted word for word to the multistable case. Hence, with the notations of Theorem 1.5, if $c_1 < c_2$, if ϕ is the unique travelling front solution of (4.2) over $(0,1)$ and if $\limsup_{x_1 \to -\infty} v_0(x_1, \cdot) < \theta_1$, $\liminf_{x_1 \to +\infty} v_0(x_1, \cdot) > \theta_2$, then (4.4) is true.

4.3. Splitting phenomenon. If the two speeds c_1 and c_2 of the travelling fronts u_1 and u_2 over $(0, \theta)$ and $(\theta, 1)$ are ordered in such a way that there does not exist any travelling front over $(0,1)$ (i.e., if $c_1 \geq c_2$), then a splitting phenomenon happens: under suitable initial conditions, the solutions of the Cauchy problem (4.1) develop into two different fronts with the speeds c_1 and c_2. More precisely, if we assume that $c_1 > c_2$ and if $\limsup_{x_1 \to -\infty} v_0(x_1, \cdot) < \theta_1$ and $\liminf_{x_1 \to +\infty} v_0(x_1, \cdot) > \theta_2$, then we claim that there exist two reals x_0 and x_0' and two constants K, $\omega > 0$ such that $\forall (x_1, y) \in \overline{\Sigma}$, $\forall t \geq 0$,

$$|v(t, x_1, y) - u_1(x_1 + c_1 t - x_0, y) + \theta - u_2(x_1 + c_2 t - x_0', y)| \leq K e^{-\omega t}. \quad (4.5)$$

As a consequence, according to the value of the speed c, the functions $(x_1, y) \mapsto v(t, x_1 + ct, y)$ converge to the travelling front u_1 if $c = c_1$, to the travelling front u_2 if $c = c_2$, or to the stationary states 0 if $c > c_1$, θ if $c_2 < c < c_1$ or 1 if $c < c_2$ (see also Hagan [15] for similar results).

Inequality (4.5) was proved in [9] for a function f fulfilling (1.8)–(1.9) in the one-dimensional case. It can be extended to the multidimensional case, by arguing in two main steps: the first one consists in comparing, with exponential decay in time, the functions $(x_1, y) \mapsto v(t, x_1 - c_1 t, y)$ with u_1 over $(0, \theta)$, and the functions $(x_1, y) \mapsto v(t, x_1 - c_2 t, y)$ with u_2 over $(\theta, 1)$ (with sub- and supersolutions); in the second step, the same tools as for the global stability in [26] lead to (4.5).

Remark 4.1. Following the definition of Fife and McLeod given in [9], if there exists a sequence $\theta_0 = 0 < \theta_1 < \cdots < \theta_{2m} = 1$ and if f satisfies (1.11), we infer that there exists a unique minimal decomposition of the interval $[0,1]$ of the form $\theta_{i_0} = 0 < \theta_{i_1} < \cdots < \theta_{i_k} = 1$, in the following sense: there exist travelling fronts $\tilde{u}_j$ over the intervals $(\theta_{i_j}, \theta_{i_{j+1}})$ with corresponding speeds $\tilde{c}_j$ $(j = 0, \cdots, k-1)$ which are such that $\tilde{c}_0 \geq \cdots \geq \tilde{c}_{k-1}$. We can strengthen the splitting result (4.5) if we now assume that $\tilde{c}_0 > \cdots > \tilde{c}_{k-1}$. More precisely, the result is that there exist k reals $\tilde{x}_0, \cdots, \tilde{x}_{k-1}$ and 2 constants K, $\omega > 0$ such that, if the initial datum v_0 satisfies $\text{limsup}_{x_1 \to -\infty}\, v_0(x_1, \cdot) < \theta_1$ and $\text{liminf}_{x_1 \to +\infty}\, v_0(x_1, \cdot) > \theta_{2m-1}$, then

$$\begin{aligned}&|v(t,x_1,y) - \tilde{u}_0(x_1 + \tilde{c}_0 t - \tilde{x}_0, y) + \theta_{i_1} - \tilde{u}_1(x_1 + \tilde{c}_1 t - \tilde{x}_1, y) + \cdots \\ &\quad + \theta_{i_{k-1}} - \tilde{u}_{k-1}(x_1 + \tilde{c}_{k-1} t - \tilde{x}_{k-1}, y)| \leq K e^{-\omega t}, \quad \forall (x_1, y) \in \overline{\Sigma},\ \forall t \geq 0.\end{aligned}$$

4.4. Extinction and extension phenomena. Other types of situations may happen if the initial condition v_0 is close to 0 as $x_1 \to \pm\infty$. We speak about extinction when the solution of the evolution problem collapses to 0, and about extension when it develops into two fronts moving in opposite directions.

Let f be of the bistable type (1.5). Consider an initial datum v_0 such that

$$\operatorname*{limsup}_{|x_1| \to \infty} v_0(x_1, \cdot) < \theta \tag{4.6}$$

(pulse-like datum). Let c and $\tilde{c}$ be the speeds of the fronts u and $\tilde{u}$ over $(0,1)$, solving (1.2)–(1.3) and respectively increasing and decreasing in x_1. Roquejoffre proved in [26] that if $\tilde{c} < c$, if $v_0 \geq \theta + \eta$ in $[-L, L] \times \overline{\omega}$ and if $v_0 \leq \theta - \eta$ in $((-\infty, -L-\delta] \cup [L+\delta, +\infty)) \times \overline{\omega}$ for some η, L and $\delta > 0$ with δ and L small enough, then there exist 2 constants K and $\omega > 0$ such that

$$|v(t, x_1, y)| \leq K e^{-\omega t}, \quad \forall (x_1, y) \in \overline{\Sigma},\ \forall t \geq 0.$$

There is an extinction of the front. This extinction phenomenon immediately holds good for a function f fulfilling (1.8)–(1.9) if θ is replaced with θ_1 in (4.6). On the other hand, with the same notations, if L is large enough, the solution $v(t, \cdot, \cdot)$ splits into the fronts u and $\tilde{u}$ (which move in opposite directions if $\tilde{c} < 0 < c$): $\forall\ (x_1, y) \in \overline{\Sigma}$, $\forall\ t \geq 0$,

$$|v(t, x_1, y) - u(x_1 + ct - x_0, y) + 1 - \tilde{u}(x_1 + \tilde{c}t - \tilde{x}_0, y)| \leq K e^{-\omega t}. \tag{4.7}$$

We speak about extension. This last result had already been established by Fife and McLeod for the problem of multiple crossings in dimension 1, for functions v_0 such that $v_0 \geq \theta_2 + \eta$ in $[-L, L]$.

The generalization of these results in the multidimensional case and for a multistable nonlinearity can be done but requires additional assumptions. Indeed, for problem (1.2)–(1.3), if $\beta(y, c)$ is not uniform in y, we cannot compare the speeds c_1 and c_2 of the increasing fronts over $(0, \theta)$ and $(\theta, 1)$ with the speeds $\tilde{c_1}$ and $\tilde{c_2}$ corresponding to the unique decreasing fronts over $(0, \theta)$ and $(\theta, 1)$. We have to consider several cases.

If $c_1 < c_2$, we proved in Theorem 1.5 that there exists a unique increasing travelling front over $(0, 1)$, with a speed $c \in (c_1, c_2)$. If $\tilde{c_1} > \tilde{c_2}$, there exists similarly a decreasing front over $(0, 1)$ with a speed $\tilde{c}$ in $(\tilde{c_2}, \tilde{c_1})$. If $c > \tilde{c}$, if v_0 is such that $v_0 \geq \theta_2 + \eta$ in $[-L, L] \times \overline{\omega}$, $v_0 \leq \theta_1 - \eta$ in $((-\infty, -L - \delta] \cup [L + \delta, +\infty)) \times \overline{\omega}$ (for some L, δ, $\eta > 0$) and if L is large enough, then the estimate (4.7) is true.

Furthermore, if $\tilde{c_j} < c_i$ for all $i, j \in \{1, 2\}$, then, by combining the results of the previous subsections, we find that the function $v(t, \cdot, \cdot)$ develops into 2, 3 or 4 travelling fronts over $(0, 1)$, $(0, \theta)$ or $(\theta, 1)$ according to the relative positions of c_1 and c_2 and of $\tilde{c_1}$ and $\tilde{c_2}$.

Last, let us notice that if the initial condition satisfies (4.6), then the exhaustive study of all possible behaviours, especially those which are intermediate between the extinction and the extension, is still an open question.

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