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BIFURCATION FOR THE p-LAPLACIAN IN $\mathbb{R}^N$*

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1. Introduction. In this work we look for positive (weak) solutions of the following problems

$$-\Delta_p u(x) + \alpha u^{p-1}(x) = \lambda h(x) u^{q-1}(x) + g(x) u^{r-1}(x), \quad x \in \mathbb{R}^N, \qquad (P_\lambda)$$

with $1 < p < \infty$, $1 < q < p < r < p^*$, where

$$p^* := \begin{cases} \frac{Np}{N-p}, & 1 < p < N, \\ \infty, & p \geq N, \end{cases}$$

λ is a real parameter, $\alpha \geq 0$, h and g verify some integrability conditions. When $p = 2$ the problem corresponds to the classical *Laplacian* and also in this case the results are new. We will study three different cases:

(I) The case where $\alpha = 1$, $h \in L^s(\mathbb{R}^N)$, for all $s \in (1, \infty]$, and $g \in L^t(\mathbb{R}^N)$, where t is an exponent that will be made precise later. We look for solutions $u \in W^{1,p}(\mathbb{R}^N)$.

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(II) The case where $h \in L^s(\mathbb{R}^N)$, for all $s \in (1,\infty]$, $g \equiv 1$ and $\alpha = 1$. We will emphasize the semilinear case, $p = 2$.

(III) The same problem as in (I), with $\alpha = 0$. Here the natural space is $D^{1,p}(\mathbb{R}^N)$, $1 < p < N$. (See [10] for the definition of this space).

For problems (I) and (II) we will study global bifurcation properties of positive solutions and analyze the behavior of a branch (i.e., a continuum connected set) of positive solutions of the equation.

The strategy of the proof will follow an *approximation argument* as in [4]. Namely, in the first step, we define the truncated function

$$f_\varepsilon(s) = \begin{cases} s^{q-1}, & \text{if } |s| \geq \varepsilon, \\ \varepsilon^{q-p} s^{p-1}, & \text{if } |s| < \varepsilon, \end{cases}$$

and consider the associated *truncated problem*

$$-\Delta_p u + u^{p-1} = \lambda h f_\varepsilon(u) + g u^{r-1}.$$

To pass to the limit as $\varepsilon \to 0$, we will use some argument involving the *eigenvalue problem*

$$-\Delta_p u + |u|^{p-2} u = \left(\lambda h \varepsilon^{q-p}\right) |u|^{p-2} u. \tag{1.1}$$

More precisely, we study the existence of a branch of positive solutions of the equation (P_λ). The main tool will be a bifurcation result in [10], which extends to this setting the classical results by Rabinowitz in [24]. This allows us to use a method of *accumulation of branches* as in [4] by topological arguments.

In bounded domains, something more can be said about the global behavior of the branch and, as a consequence, about of the *global multiplicity* (see [3] for $p = 2$, [4] for general p and radial solutions, and also [12]).

The main difference between problems (I) and (II) are the compactness properties; in problem (II) the Palais-Smale condition can be obtained only under a critical level of energy (*locally compact* case). (See [17]). This fact implies some new phenomena in the behavior of the bifurcation branch and in the study of multiplicity of positive solutions.

Notice that in case (II), if $\lambda = 0$ the invariance under translations of problem (P_0) generates a lack of compactness, according with the results about existence and uniqueness in [16]. This fact makes difficult to study what happens when, eventually, a branch of solutions approaches $\lambda = 0$. Moreover the problem (P_0) has the trivial solution $v \equiv 1$ that is not in

$W^{1,p}$, but that can be relevant in the accumulation of branches procedure. These are some strong differences with the bounded domain case.

On the other hand, if $p = 2$ and Ω is a bounded domain in $\mathbb{R}^N$, there are several examples with *supercritical growth* in which we get a branch that blows up in L^∞ and remains bounded in $W^{1,2}$. This is the case for instance in some Emden-Fowler equations. This kind of behavior can be seen in [8] and [9]. In particular Brezis and Cabré in [8] show that the branch of positive solutions cannot be continued in $W^{1,2}_{loc}$, by proving a *complete* blow-up result.

However in our case, where the noncompactness comes from the unboundedness of the domain, we can prove that the branch starting from $(0, 0)$ is bounded in L^∞, but we are not able to prove if the branch is unbounded in $W^{1,2}$. A positive answer to the previous question should mean that we reach a dual situation of the one described in [8].

We give some additional results in the important case $p = 2$, namely the linear *Laplacian*. The behavior of problem (P_0) when $p = 2$, is well known (see [16] and [25]) Moreover the semilinear structure allows us to study the problem (P_λ) close to the manifold of solutions to problem (P_0) (defined by translations of the radial one) by using a perturbation method based in the ideas of Poincaré and Melnikov, which has been developed in this setting by Ambrosetti, Badiale and Cingolani (see [1] and [2]).

The main difficulty to adapt the abstract framework to this case comes from the hypothesis of sublinearity, $1 < q < 2$, that implies a lack of regularity in the associated energy functional. To handle this difficulty we need to prove some extra properties (i.e., L^∞ estimates) of the *perturbations*. With this approach we get multiplicity of positive solutions to problem (P_λ), for λ small enough, and show that the number of solutions obtained by perturbation, i.e., for λ small, depends on h.

The structure of the paper is the following. In the next section we study problem (I). Section 3 deals with problem (II) with a particular subsection devoted to the semilinear case and the perturbative arguments. In Section 4 we briefly discuss problem (III). Finally Section 5 contains some auxiliary results by Pohozaev needed in the proof of uniform bounds to elliptic problems and Section 6 contains the analysis of a radial problem following the ideas by Strauss in the semilinear case.

In the sequel of the paper we use the following notation $W^{1,p}(\mathbb{R}^N) =: W^{1,p}$, $W^{1,p}_{loc}(\mathbb{R}^N) =: W^{1,p}_{loc}$, $D^{1,p}(\mathbb{R}^N) =: D^{1,p}$, $L^s(\mathbb{R}^N) =: L^s$, $C^k(\mathbb{R}^N) =: C^k$.

Recall that S is the constant of the embedding $W^{1,p} \hookrightarrow L^{p^*}$, and that S_r is the constant of the embedding $W^{1,p} \hookrightarrow L^r$.

2. Problem (I), compact case. We will assume in this section that

$$h \in L^s \quad \text{for any } s > 1, \text{ and} \quad h \geq 0, \quad h \not\equiv 0 \text{ in } \mathbb{R}^N, \tag{$h1$}$$

and g is a continuous function such that

$$g \in L^t \quad \text{and} \quad g(x) > 0, \forall x \in \mathbb{R}^N, \tag{$g1$}$$

where $t = \frac{Np}{Np - r(N-p)}$, if $p < N$, $t \in (1, \infty)$, if $p \geq N$.

Note that (h_1) and (g_1) imply that the associated energy functional,

$$J_\lambda(u) = \frac{1}{p}||u||^p_{W^{1,p}} - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)u^q - \frac{1}{r}\int_{\mathbb{R}^N} g(x)u^r. \tag{2.1}$$

is well defined in $W^{1,p}$, namely, the terms hu^q, gu^r belong to L^1, for all $u \in W^{1,p}$. (The hypotheses on h are not optimal, but to avoid some technicalities we prefer to use this more restrictive setting).

In particular we can apply a result by Drabek and Huang that in our case can be read as follows.

Theorem. *Assume $h(x) \geq 0$ and $h \in L^{\frac{N}{p}}(\mathbb{R}^N)$ then the eigenvalue problem* (1.1) *has a solution $(\lambda_1(\varepsilon), \phi_1)$ such that $\lambda_1(\varepsilon)$ is a positive, isolated and simple eigenvalue and the corresponding eigenfunction verifies $\phi_1(x) > 0$ and $\phi_1 \in W^{1,p} \cap L^\infty$ (See* [10]*).*

As a consequence we have the following result.

Theorem 2.1. *Assume (h_1) and (g_1), and let $\lambda_1(\varepsilon)$ be the first positive eigenvalue of the problem* (1.1)*. Then problem*

$$-\Delta_p u + u^{p-1} = \lambda h f_\varepsilon(u) + g u^{r-1}, \quad u \in W^{1,p}, \tag{$P_{\lambda,\varepsilon}$}$$

has a branch Σ_ε of positive solutions which bifurcates from $(0, \lambda_1(\varepsilon)) \in W^{1,p} \times \mathbb{R}$.

Proof. To prove this statement it is sufficient to show that $(P_{\lambda,\varepsilon})$ approximates (1.1) if u is small enough. Namely, we need to prove

$$\lim_{||u||_{W^{1,p}} \to 0} \frac{||\lambda h(x)\varepsilon^{q-p}u^{p-1} - \lambda h(x) f_\varepsilon(u) - g(x)u^{r-1}||_{W^{-1,p'}}}{||u||^{p-1}_{W^{1,p}}} = 0,$$

uniformly for λ in a bounded set of $\mathbb{R}$. And this follows by using arguments similar to those in Lemma 4.3 in [10].

The result then follows from Theorem 4.3 in [10].

Remark. Note that $\lambda_1(\varepsilon) = \lambda_1 \varepsilon^{p-q}$, where λ_1 is the first positive eigenvalue of the equation

$$-\Delta_p u + |u|^{p-2}u = \lambda h|u|^{p-2}u.$$

In fact there holds

$$\lambda_1(\varepsilon) = \inf_{v \in W^{1,p}\setminus\{0\}} \frac{\int_{\mathbb{R}^N} (|\nabla v|^p + |v|^p)}{\varepsilon^{q-p} \int_{\mathbb{R}^N} h|v|^p} = \varepsilon^{p-q}\lambda_1.$$

Now we study the global behavior of the branch Σ_ε. According to [10] the bifurcation is on the right. Moreover either the part of the branch of positive solutions contained in the half space $\lambda \geq 0$ is unbounded, or it has to cross the hyperplane $W^{1,p} \times \{\lambda = 0\}$, because there are no other positive eigenvalues with positive eigenfunction (see [24]). The following proposition shows that the projection on the real axis of any branch of positive solutions has to be bounded on the right.

Proposition 2.2. *There exists* $\Lambda \in (0, \infty)$ *such that, if* $(u_\lambda, \lambda) \in \Sigma_\varepsilon$, *then* $\lambda \leq \Lambda$.

Proof. We argue by contradiction and suppose that there is a sequence $\{(u_k, \lambda_k)\} \subset \Sigma_\varepsilon$ such that $\lambda_k \to \infty$.

Fixed R, a positive real number, let $\mu_1(R)$ be the first eigenvalue of the problem

$$-\Delta_p v + |v|^{p-2}v = \lambda h|v|^{p-2}v, \quad v \in W_0^{1,p}(B_R). \tag{2.2}$$

Recalling the definition of function f_ε, and because the function g is strictly positive in B_R, by $(g1)$, we can state that there exists a λ_{k_0} sufficiently large such that it holds

$$\lambda_{k_0} h(x) f_\varepsilon(u) + g(x)u^{r-1} > (\mu_1 + \delta)h(x)u^{p-1}.$$

So u_k is a super solution of

$$-\Delta_p v + |v|^{p-2}v = (\mu_1 + \delta)h|v|^{p-2}v, \quad v \in W_0^{1,p}(B_R). \tag{2.3}$$

Now we consider $t\phi$, where ϕ is the eigenfunction of problem (2.2) and t a positive real number. This function is a subsolution of (2.3) and is less than u_{k_0}, which is strictly positive in B_R, if t is small enough . Then we can found a solution of (2.3) by usual iterative arguments for any positive δ. But this is a contradiction if δ is small, because the first eigenvalue is isolated, and then the claim is proved.

Remark. The preceding result shows also that, if $\lambda > \Lambda$, there is no positive solution of $(P_{\lambda,\varepsilon})$ in $W^{1,p}_{loc}$.

In the proof above we need the hypothesis that g is strictly positive in $\mathbb{R}^N$.

Consider a sequence of real positive numbers $\{\varepsilon_n\}$ decreasing to 0. We know (see Theorem 2.1) that there is a unbounded branch Σ_ε of positive solutions of $(P_{\lambda,\varepsilon})$, which cannot cross to the right of the hyperplane $W^{1,p} \times \{\lambda = \Lambda\}$. At this point, to pass to the limit, we will use the following classical topological result, which gives us the necessary compactness abstract setting. (See Theorem 9.1 in [26]).

Theorem 2.3. *Let $\{X_n\}$ be a sequence of connected nonempty subsets of a complete metric space X. If $\liminf X_n \neq \emptyset$ and $\cup X_n$ is precompact, then $\limsup X_n$ is a connected nonempty subset of X.*

Let Σ_n^c be the connected component of $\Sigma_{\varepsilon_n} \subset \{u \in W^{1,p} : ||u||_{W^{1,p}} \leq c\} \times [0, \Lambda]$ containing $(\lambda_1(\varepsilon_n), 0)$. It is easy to see that $\liminf \Sigma_n^c$ is nonempty, because $(0, \lambda_1(\varepsilon_n)) \in \Sigma_n^c$ tends to $(0,0)$. We have to show that the second hypothesis of Theorem 2.3 is satisfied. Namely, we need to show the precompactness of $\cup \Sigma_n^c$.

Proposition 2.4. *Let $\{(u_j, \lambda_j)\} \subset \cup_{n\in\mathbb{N}} \Sigma_n^c$ be a sequence of solutions of $(P_{\lambda_j,\varepsilon_j})$. Then there exists a convergent subsequence in $W^{1,p} \times \mathbb{R}$.*

Proof. We know that for some subsequence

$$-\Delta_p u_j + u_j^{p-1} = \lambda_j h(x) f_{\varepsilon_j}(u_j) + g(x) u_j^{r-1} \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon_j \to 0$, $\lambda_j \to \lambda$, $u_j \rightharpoonup u$ weakly in $W^{1,p}$ and $u_j \to u$ a.e. in $\mathbb{R}^N$.

We state that $\{u_j\}$ is a Palais-Smale sequence of the functional

$$J_\lambda(u) = \frac{1}{p}||u||^p_{W^{1,p}} - \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)u^q - \frac{1}{r}\int_{\mathbb{R}^N} g(x)u^r.$$

Since $(h1)$ holds, direct computation shows that

$$J'_\lambda(u_j) = h(x)\big(\lambda_j \varepsilon_j^{q-p} u_j^{p-1} - \lambda u_j^{q-1}\big)\chi_{\{u_j<\varepsilon_j\}}, \text{ then } J'_\lambda(u_j) \to 0.$$

To prove the compactness of u_j is equivalent to discard the possibility of *vanishing* and *dichotomy* in the sense of the papers by Lions, [18] and [19].

Notice that, since we are dealing with subcritical exponents $r < p^*$, *the concentration in Dirac's deltas* is impossible. A detailed argument about this point (in the case $p = 2$) can be seen in the paper by Y.Y. Li, [17].

Therefore, if we assume that the sequence is not compact then it holds that for some $\delta > 0$,

$$\int_{\{|x|>j\}} (|\nabla u_j|^p + |u_j|^p) \geq \delta > 0,$$

for a suitable subsequence. Let ρ be a radial cutoff function with $\operatorname{supp}(\rho) \subset B_{R_0}$, R_0 such that $||h||_{L^s(\mathbb{R}^N\setminus B_{R_0})}, ||g||_{L^t(\mathbb{R}^N\setminus B_{R_0})} < \varepsilon$ (this is possible by $(g1)$ and $(h1)$).

Following the arguments of [17] we can found that

$$\int_{\{|x|>j\}} (|\nabla u_j|^p + |u_j|^p) = \lambda_j \int_{\{|x|>j\}} h(x)u_j^q + \int_{\{|x|>j\}} g(x)u_j^r + o(1).$$

Using Hölder inequality and recalling the boundedness of $\{u_j\}$ and the choice of R_0, we have that the right hand side of the preceding identity is smaller than $\kappa\varepsilon + o(1)$, where κ depends on $||u_j||_{W^{1,p}}$. Taking R_0 large enough we reach a contradiction, then the sequence has to be compact. □

We summarize the result above and complete the main result in this section in the following Theorem.

Theorem 2.5. $\Sigma^c := \limsup \Sigma_n^c$ *is a nonempty and connected set of solutions of* (P_λ)*. This branch bifurcates from the trivial axis in* $(0, 0)$*. Moreover*

- (i) *There exists a constant* $c^* > 0$ *such that if* $\overline{u} \in W^{1,p}$ *is a positive solution of* (P_0)*, then* $||\overline{u}||_{W^{1,p}} \geq c^*$*. (*$\Sigma^c$ *does not collapse into* $W^{1,p} \times \{\lambda = 0\}$*.)*
- (ii) *If* $(u_0, \lambda_0) \in \Sigma$*, and* $\lambda_0 > 0$*, then* $u_0 > 0$*. (*Σ^c *does not collapse into* $\{0\} \times (0, \infty)$*.)*

Proof. The existence of a branch branching-off from $(0, 0)$ has been proved above.

To prove (i), we argue as follows: let $\overline{u} \in W^{1,p}$ be a positive solution of (P_0). Multiplying by $\overline{u}$, integrating and using Sobolev inequality we obtain

$$||\overline{u}||^p_{W^{1,p}} \leq C||g||_t ||\overline{u}||^r_{W^{1,p}}.$$

Then it is easy to see that

$$0 < C(||g||_t)^{\frac{1}{p-r}} \leq ||\overline{u}||_{W^{1,p}}.$$

Now we prove (ii). Since $(u_0, \lambda_0) \in \Sigma$, there exists a sequence $\{u_n\} \subset W^{1,p}$ such that $(u_n, \lambda_0) \in \Sigma_n^c$, with n suitable large, and $u_n \to u_0$.

Choose $R > 0$ and let be $\mu_1 = \mu_1(R)$ the first eigenvalue of the p-laplacian with weight h in B_R. There exists $c(\lambda_0) > 0$ and a small $\delta > 0$ such that

$$\lambda_0 h(x) f_{\varepsilon_n}(s) + g(x) s^{r-1} > (\mu_1 + \delta) h(x) s^{p-1},$$

for any $s \in (0, c(\lambda_0)]$. If $||u_n||_{L^\infty} < c(\lambda_0)$, arguing as in Proposition 2.2 we obtain the existence of a positive solution $v \in W^{1,p}$ of the eigenvalue problem (2.3), but this contradicts the fact that μ_1 is isolated. Then it follows that $||u_n||_{L^\infty} \geq c(\lambda_0)$, and passing to the limit on $n \to \infty$ we can prove that $||u_0||_{L^\infty} \geq c(\lambda_0)$.

Remark. i) Setting $\Sigma := \overline{\cup \Sigma^c}$ we have built an unbounded branch of solutions of (P_λ). Note that the branch is a continuum in the space $W^{1,p} \times [0, \Lambda]$.

ii) The integrability condition on h is necessary in the proof of Proposition 2.4 for proving that the sequence $\{u_n\}$ is a Palais-Smale sequence for J_λ.

iii) An open question is to know if in general the branch is bounded in some topology in the halfspace $\{\lambda > 0\}$. In particular, it is unknown if the branch crosses $\{\lambda = 0\}$ or blows up in some $\lambda \in [0, \Lambda]$. A related problem is to prove the existence of at least two solutions.

When $g \equiv 1$ something more will be said in the next section.

3. Problem (II), locally compact case.

3.1. Bifurcation from 0, multiplicity of solutions. In this section we will assume that

$$g(x) = 1, \; h \in L^s(\mathbb{R}^N), \text{ for all } s \in (1, \infty] \text{ and } h \geq 0 \text{ in } \mathbb{R}^N.$$

Observe that Theorem 2.1 and Proposition 2.2 both hold in this case. Here the difficulty is that in the branches accumulation procedure in $W^{1,p}$ we will need some condition on a norm of h to get *local compactness*, namely, we can pass to the limit in $W^{1,p}$ only for small positive levels of energy. However we can pass to the limit in the weaker topology of $W^{1,p}_{loc}$ without restrictions, using some L^∞ *a priori* estimates.

We organize this subsection in three steps.

Step 1. Local convergence in $W^{1,p}$. We study the set of solutions of (P_λ). Using the information on the continua Σ_ε we want to show that Σ (a set of solutions of the problem) has the same qualitative behavior as the branches of approximate problems $(P_{\lambda,\varepsilon})$.

Now passing to the limit on ε is not trivial and requires a precise compactness result as in [17].

Let $V_\infty = \{v \in W^{1,p} \setminus \{0\} | \langle J_0'(v), v\rangle = 0\}$ and $m_\infty = \inf_{v\in V_\infty} J_0(v)$.

Lemma 3.1. *Given $\eta > 0$, there exists a positive constant $\alpha = \alpha(\eta)$ such that, if $||h||_s \le \alpha$, $s = p^*/(p^* - q)$, and if $\{u_j\} \subset W^{1,p}$ is a Palais-Smale sequence of J_λ verifying $J_\lambda(u_j) \to c \in (-\infty, m_\infty - \eta)$, then there exists a convergent subsequence.*

Proof. We strictly follow the arguments in [17] and [27].

Suppose that $\{u_j\}$ is not compact, then, as in Lemma 2.4, we have for some $\delta > 0$

$$\int_{\{|x|>j\}} (|\nabla u_j|^p + |u_j|^p) \ge \delta > 0. \tag{3.1}$$

As above, take ρ a cutoff function with support B_{R_0}, with R_0 such that $||h||_{L^s(\mathbb{R}^N\setminus B_{R_0})} < \varepsilon$.

Set $v_j = \rho u_j$ and $w_j = (1-\rho)u_j$. By the choice of R_0 we obtain that

$$\begin{aligned} &||v_j||^p_{W^{1,p}} = \lambda \int_{\mathbb{R}^N} h(x)|v_j|^q + \int_{\mathbb{R}^N} |v_j|^r + O(\varepsilon),\\ &||w_j||^p_{W^{1,p}} = \int_{\mathbb{R}^N} |w_j|^r + O(\varepsilon),\\ &J_\lambda(u_j) = J_\lambda(v_j) + J_0(w_j) + O(\varepsilon) = J_\lambda(v_j) + J_\lambda(w_j) + O(\varepsilon), \end{aligned} \tag{3.2}$$

as $j \to \infty$. Now we estimate the value of the functional J_λ on the sequences $\{v_j\}$ and $\{w_j\}$. First we have

$$\begin{aligned} J_\lambda(v_j) \ge& \big(\frac{1}{p} - \frac{1}{r}\big)||v_j||^p_{W^{1,p}} - \lambda\big(\frac{1}{q} - \frac{1}{r}\big)||h||_{L^s}||v_j||^q_{W^{1,p}} + O(\varepsilon)\\ \ge& -\kappa\lambda||h||_{L^s} + O(\varepsilon). \end{aligned}$$

By (3.1) and (3.2), if j is larger than R_0 and ε is small enough, we find

$$\int_{\mathbb{R}^N} |w_j|^r \ge \frac{\delta}{2}.$$

Set $\lambda_j := \big(\frac{||w_j||^p_{W^{1,p}}}{||w_j||^r_{L^r}}\big)^{\frac{1}{r-p}}$, and consider $\xi_j := \lambda_j w_j$. It is easy to see that $0 < \lambda_j < (1 + O(\varepsilon))$, so we can write

$$m_\infty \le J_0(\xi_j) \le \big(\frac{1}{p} - \frac{1}{r}\big) \int_{\mathbb{R}^N} |w_j|^r + O(\varepsilon) = J_\lambda(w_j) + O(\varepsilon).$$

Finally we have $c + o(1) = J_\lambda(u_j) \ge m_\infty - \kappa\lambda||h||_{L^s} + O(\varepsilon)$, and the result follows, provided $||h||_{L^s}$ is sufficiently small.

Corollary 3.2. *Assume that we have a sequence* $(u_j, \lambda_j) \in \Sigma_{\varepsilon_j}$*, with* $\varepsilon_j \to 0$*, and verifying* $||u_j||^p_{W^{1,p}} \leq (m_\infty - \eta)\frac{pr}{r-p}$*. Then there exists a strongly convergent subsequence in* $W^{1,p}$*, provided* $||h||_{L^s} \leq \alpha$*.*

Proof. As in Lemma 2.4, we have that $\{u_j\}$ is a Palais-Smale sequence for J_λ. Then we can apply the preceding result, recalling that $\{\lambda_j\} \subset [0, \Lambda]$, and the claim is proved assuming that

$$\|h\|_{L^s} \leq \alpha. \tag{$h2$}$$

We will use Theorem 2.3 again. Consider Σ^c_n the connected component of $\Sigma_{\varepsilon_n} \subset \{u \in W^{1,p} : ||u||_{W^{1,p}} \leq c\} \times [0, \Lambda]$ containing $(0, \lambda_1(\varepsilon_n))$, as above.

Theorem 3.3. *Assume* $(h1)$ *and* $(h2)$ *hold. Let* $\{\varepsilon_n\}$ *be a decreasing sequence tending to* 0 *and consider the corresponding truncated branches* Σ^c_n*, with* $c \leq (m_\infty - \eta)\frac{pr}{r-p}$*.*

Then $\Sigma := \limsup \Sigma^c_n$ *is a branch (a nontrivial connected set) of solutions of* (P_λ)*.*

Moreover

(i) *If* $(u_0, \lambda_0) \in \Sigma$*, and* $\lambda_0 > 0$*, then* $u_0 > 0$*.*
(ii) *There exists a constant* $c^* > 0$ *such that if* $\overline{u} \in W^{1,p}$ *is a positive solution of* (P_0)*, then* $||\overline{u}||_{W^{1,p}} \geq c^*$*.*

Proof. We have to prove that $\liminf \Sigma^c_n$ is a nonempty set and that $\cup_{n\in\mathbb{N}}\Sigma^c_n$ is precompact. It easy to see that $(0, \lambda_1(\varepsilon_n)) \in \Sigma^c_n$ tends to $(0, 0)$, then the first requirement is satisfied. Moreover if $\{(u_j, \lambda_j)\}$ is a sequence in $\cup\Sigma^c_n$, by the Corollary 3.2 there exists a convergent subsequence.

To prove (i) we proceed as in Theorem 2.5 (1), obtaining that the branch does not collapse in the trivial line.

On the other hand to prove (ii) we use the following argument. Let $\overline{u} \in W^{1,p}$ a positive solution of (P_0). Multiplying by $\overline{u}$, integrating and using Sobolev inequality we obtain

$$||\overline{u}||^p_{W^{1,p}} = ||\overline{u}||^r_{L^r} \leq S_r||\overline{u}||^r_{W^{1,p}}.$$

Then

$$0 < (S_r)^{\frac{1}{p-r}} \leq ||\overline{u}||_{W^{1,p}}.$$

Step 2. Uniform $L^\infty(\mathbb{R}^N)$ a priori bounds. In this case, however, we can describe more precisely the behavior of the continuum that branches-off from zero. This information is given by a result by Pohozaev, that we explain in Section 5. Its consequence is the following theorem.

Theorem 3.4. *Assume that $h \in L^\infty$ and $r < \frac{p(N-1)}{N-p}$. Then Σ_ε is bounded from above in C^0, uniformly in ε.*

Proof. Suppose that the theorem is false. Then there exists a sequence $\{(\lambda_j, u_j)\}$ such that $\lambda_j \to \lambda \in (0, \Lambda)$ and $||u_j||_{L^\infty} \to \infty$.

By the classical regularity theory (see [10]) we know that all the $u_j \in C^1$. Then there exists a sequence $\{p_j\}$ such that $u_j(p_j) = M_j/2$, where $M_j := \max_{\mathbb{R}^N} u_j \to \infty$. We use a classical blow-up argument. Consider the sequence $v_j(x) := u_j(M_j^{\frac{p-r}{p}} x + p_j)M_j^{-1}$. This sequence has unitary L^∞-norm and converges uniformly on compact sets to a nontrivial positive function $v_0 \in W^{1,p}_{loc}$ such that

$$-\Delta_p v_0 = v_0^{r-1} \qquad \text{in } \mathbb{R}^N. \tag{3.3}$$

Now the claim follows, because Theorem 5.1 below, shows that (3.3) has no positive solution in $W^{1,p}_{loc}$.

Remark. In the semilinear case the result above is true for all subcritical power r, namely $1 < r < \frac{2N}{N-2}$, (see [14]).

Remark. From the classical results in [24] we know that Σ_ε is unbounded. Theorem 3.4 and Proposition 2.2 show that the branch is forced to cross the axis $\{\lambda = 0\}$ and to continue indefinitely (in our hypotheses there is no other positive eigenvalue with positive eigenfunction).

Step 3. The global branch in $W^{1,p}_{loc}$. We have the following concrete situation: the branch Σ starts from $(0,0)$ in $W^{1,p}$ and could blow-up in such topology, but it remains bounded in the C^1 topology, if $r < p(N-1)/(N-p)$.

The problem comes from the fact that we are working in $\mathbb{R}^N$, namely in a domain which is invariant under the action of the noncompact group of the translations and since $g \equiv 1$, problems $(P_{0,\varepsilon})$ (and also (P_0)) are invariant under translations. Then in these problems there is an evident lack of compactness.

This fact is obvious taking into account the section 6, where we prove the existence of a radial positive solution of (P_0), which generates a manifold of critical points of the functional J_0.

The preceding steps show that Σ is a nontrivial branch of solutions of (P_λ), which bifurcates to the right from $(0,0) \in W^{1,p} \times \mathbb{R}$.

Consider Σ_n the untruncated branches corresponding to problem $(P_{\lambda,\varepsilon_n})$, with $\{\varepsilon_n\}$ sequence of positive real numbers decreasing to 0. Then we can prove

Proposition 3.5. $\cup\Sigma_n$ *is a relatively compact set in* $W^{1,p}_{loc} \times [0,\Lambda]$ $(r < p(N-1)/(N-p))$.

Proof. We have to prove that, if $\{(u_j,\lambda_j)\}$ is a sequence in $\cup\Sigma_n$, then $u_j \to u$ in $W^{1,p}(\Omega)$, for any $\Omega \subset \mathbb{R}^N$ compact set. But this follows easily by the regularity theory of elliptic equations, recalling that $||u_j||_{L^\infty} \leq C$, by Theorem 3.4. □

So we have proved, using Theorem 2.3, that

Theorem 3.6. $\widetilde{\Sigma} := \limsup \Sigma_n$ *is a branch of solutions of* (P_λ) *in* $W^{1,p}_{loc} \times [0,\Lambda]$. $(r < p(N-1)/(N-p))$.

Remark. In Proposition 3.5 we can consider the subsequence of $\{\varepsilon_n\}$ such that Σ_n^c converges to Σ (see Theorem 3.3), and build $\widetilde{\Sigma}$. Since the convergence in $W^{1,p}$ implies the convergence in $W^{1,p}_{loc}$, it follows that $\widetilde{\Sigma}$ is an extension of Σ.

Since $\Sigma \subset \widetilde{\Sigma}$, this branch is not trivial and does not collapse into $\{\lambda = 0\}$. Notice that the existence of $\widetilde{\Sigma}$ does not require the constant α in (h_2) to be smaller. Moreover, it is unbounded, has a bound from above and from the right; then it has a turning point and must cross the hyperplane $\{\lambda = 0\}$.

However, also in this case, we can't say nothing on the point where $\widetilde{\Sigma}$ crosses $\{\lambda = 0\}$. This point is a function in C^1, but we do not know if it is in $W^{1,p}$. (Notice that the constant $u \equiv 1$ is a solution in $W^{1,p}_{loc}$ for $\lambda = 0$.)

This remark shows us that we have some *lack of compactness* in $\lambda = 0$, namely we don't know if the branch starting from $(0,0)$ arrives close to the manifold and if this happens we don't know how to precise the exact point of contact.

3.2. Semilinear case and the Poincaré-Melnikov method. If we assume $p = 2$, then it is well known that the problem (P_0) has an unique, up to translations, positive solution z in the Sobolev space.

The properties of this solution have been studied in [16]; if we consider the N-dimensional manifold $Z_0 = \{z_\theta(x) = z(x+\theta) | \theta \in \mathbb{R}^N\}$ its tangent space $T_{z_\theta} Z_0$ in each point is generated by $\{\frac{\partial z_\theta}{\partial x_i}\}_{i=1,\dots,N}$. Set $q_i(\theta) := \frac{\partial z_\theta}{\partial x_i} \|\frac{\partial z_\theta}{\partial x_i}\|^{-1}_{W^{1,2}}$.

We will show that with some additional hypotheses on h it is possible to apply the variational perturbative arguments in [1] and [2] (see also [6]) to get multiplicity of positive solutions to (P_λ) for λ small enough, according with the *shape* of h.

In this subsection we will assume the following extra hypothesis on h

$$\text{support}(h) \subset B_{R_0} \subset \mathbb{R}^N. \qquad (h3)$$

We choose $R_1 > 0$ large enough in such a way that

$$\sup_{|\theta|=R_1} \int_{\mathbb{R}^N} h(x) z_\theta^q(x) < \int_{\mathbb{R}^N} h(x) z_0^q(x). \tag{h4}$$

The existence of R_1 is an easy consequence of the exponential decay of z at infinity and the hypothesis $(h3)$ above.

Then we will consider the manifold Z locally defined by

$$Z = \{z_\theta(x) = z(x+\theta) | \theta \in \mathbb{R}^N,\ |\theta| \le R_1\}.$$

We try to construct $Z_\lambda = \{z_\theta(x) + w_{\theta,\lambda} | z_\theta \in Z\}$ in such a way that if we have $(J_\lambda|_{Z_\lambda})'(u) = 0$, then $J'_\lambda(u) = 0$. For this reason, according with [1], we will call Z_λ a *natural constraint*. We have the derivative

$$J'_\lambda(u) = -\Delta u + u - \lambda h(x) u^{q-1} - u^{r-1}.$$

Since z_0 is the solution corresponding to $\lambda = 0$, and z_θ is a translation, we can write

$$J'_\lambda(z_\theta + w) = J'_\lambda(z_\theta) + J''_\lambda(z_\theta) w - N_\theta(w),$$

where J'' is well defined in the manifold Z, because $|\theta| \le R_1$ and support(h) $\subset B_{R_0}$, $z_\theta > 0$ is uniformly bounded and

$$\begin{aligned} N_\theta(w) &= |z_\theta + w|^{r-1} - |z_\theta|^{r-1} - (r-1)|z_\theta|^{r-2} w \\ &\quad + \lambda h(x)(|z_\theta + w|^{q-1} - |z_\theta|^{q-1} - (q-1)|z_\theta|^{q-2} w). \end{aligned}$$

We identify $J''_\lambda(z_\theta) w \in W^{-1,2}$ with its representation in $W^{1,2}$ via Riesz theorem, namely

$$J''_\lambda(z_\theta) w = w - K(((r-1)|z_\theta|^{r-2} + \lambda h(x)(q-1)|z_\theta|^{q-2}) w),$$

where $K = (-\Delta + Id)^{-1}$. Let η be defined by

$$\eta := \inf_{\theta \in B_{R_1}, x \in B_{R_0}} z_0(x+\theta) > 0, \tag{3.4}$$

and set $\mathcal{B}_{\frac{\eta}{2}} := \{f \in L^\infty | \|f\|_{L^\infty} \le \frac{\eta}{2}\}$. For λ and θ bounded and $w \in W^{1,2} \cap \mathcal{B}_{\frac{\eta}{2}}$, we have that $N_\theta(w) = o(||w||_{W^{1,2}})$. The estimate follows from the fact that, for any $t \in (-\frac{1}{2}, \infty)$, there holds

$$(1+t)^{r-1} - 1 - (r-1)t \le C(t^{r-1} + t^2), \text{ and } (1+t)^{q-1} - 1 - (q-1)t \le Ct^2. \tag{3.5}$$

We will consider

$$\begin{aligned} &\mathcal{K}_\theta := \ker(J''_0(z_\theta)) = span\{q_i(\theta)\}_{i=1,\dots,N}, \\ &\mathcal{K}_\theta^\perp := \text{ orthogonal complement of } \mathcal{K}_\theta \text{ in } W^{1,2}, \\ &P_\theta : W^{1,2} \to \mathcal{K}_\theta^\perp, \text{the orthogonal projection} \quad \text{and} \\ &L_{\theta,\lambda} := P_\theta J''_\lambda(z_\theta). \end{aligned}$$

Proposition 3.7. *There exists a constant $C > 0$ such that $||P_\theta \phi||_\infty \leq C||\phi||_\infty$, for all $\phi \in W^{1,2} \cap L^\infty$.*

Proof. From the regularity of z_θ it is sufficient to observe that $P_\theta \phi = \phi - \sum_{i=1}^N \langle \phi, q_i \rangle q_i$.

Lemma 3.8. *There exist positive constants γ, λ_0 such that if $|\lambda| \leq \lambda_0$, and $\phi \in \mathcal{K}_\theta^\perp$, then*

$$\|L_{\theta,\lambda}\phi\|_{W^{1,2}} \geq \gamma \|\phi\|_{W^{1,2}}.$$

Proof. We follow the ideas in [11]. By contradiction, if the result is false then there exist sequences $\lambda_i \to 0, \theta_i \to \theta_0$ and $\phi_i \in K_{\theta_i}^\perp$, with $\|\phi_i\|_{W^{1,2}} = 1$, such that

$$L_{\theta_i,\lambda_i}\phi_i \to 0 \text{ in } W^{1,2}.$$

We call

$$\psi_i(x) := \phi_i(x - \theta_i),$$

then $\|\psi_i\|_{W^{1,2}} = 1$, and moreover it is orthogonal to the partial derivatives of z_0. Let ψ_∞ be the weak-$W^{1,2}$ limit, then

$$< \psi_\infty, q_k(0) >_{W^{1,2}} = 0, \;\; k = 1, \dots, N.$$

Whence $\psi_\infty \in \mathcal{K}_0^\perp$; if we prove that $\psi_\infty \in \mathcal{K}_0$, then we will have that $\psi_\infty = 0$.

So we have to prove that $J_0''(z_0)\psi_\infty = 0$. By hypothesis, we know that

$$\|L_{\theta_i,\lambda_i}\phi_i\|_{W^{1,2}} = \|L_i \psi_i\|_{W^{1,2}} \to 0,$$

where

$$L_i u = P_0(u - K(((r-1)|z_0|^{r-2} + \lambda_i h(x + \theta_i)(q-1)|z_0|^{q-2})u).$$

Notice that

$$L_i \psi_i = J_0''(z_0)\psi_i - P_0 K((\lambda_i h(x + \theta_i)(q-1)|z_0|^{q-2})\psi_i),$$

and we can pass to the limit in $W^{1,2}$, getting

$$P_0 K((\lambda_i h(x + \theta_i)(q-1)|z_0|^{q-2})\psi_i) \to 0,$$

because z_0 has exponential decay and h has compact support.

We even know that $L_i\psi_i \to 0$. So we have that

$$J_0''(z_0)\psi_\infty = 0,$$

and this means that $\psi_\infty \in \mathcal{K}_0$ and therefore $\psi_\infty = 0$. Then we have that ψ_i converges to 0 weakly in $W^{1,2}$.

On the other hand, this weak convergence, jointly with the hypotheses, $\lambda_i \to 0, \theta_i \to \theta_0$ and $L_i\psi_i \to 0$ allow us to conclude that

$$\begin{aligned} 0 &= \lim_{i\to\infty} ||P_0(\psi_i - K(((r-1)|z_0|^{r-2} + \lambda_i h(x+\theta_i)(q-1)|z_0|^{q-2})\psi_i)||_{W^{1,2}} \\ &= \lim_{i\to\infty} ||P_0(\psi_i)||_{W^{1,2}}. \end{aligned}$$

Then, since $||\psi_i||_{W^{1,2}} = 1$, this means that the projection $(I - P_0)\psi_i$ must be bounded away from zero, and, in particular, there exists some index k such that $0 < c < \langle \psi_i, q_k\rangle$. And this is a contradiction because as we have seen ψ_i converges weakly to 0. □

This lemma allows us to define the operator $L_{\theta,\lambda}^{-1} : \mathcal{K}_\theta^\perp \to \mathcal{K}_\theta^\perp$. Moreover, the following key estimate holds.

Lemma 3.9. *There exists $\gamma > 0$ such that $\forall \phi \in \mathcal{K}_\theta^\perp \cap L^\infty$*

$$||L_{\theta,\lambda}\phi||_{W^{1,2}} + ||L_{\theta,\lambda}\phi||_{L^\infty} \geq \gamma(||\phi||_{W^{1,2}} + ||\phi||_{L^\infty}).$$

Proof. By contradiction as in the preceding lemma, if the result is false then there exists sequences $\lambda_i \to 0, \theta_i \to \theta_0$ and $\phi_i \in \mathcal{K}_{\theta_i}^\perp$ with $||\phi_i||_{W^{1,2}} + \|\phi_i\|_{L^\infty} = 1$, such that $L_{\theta_i,\lambda_i}\phi_i \to 0$ in $W^{1,2}$ and L^∞. As above we call $\psi_i(x) = \phi_i(x - \theta_i)$. In particular the previous lemma shows that $\psi_i \to 0$, in $W^{1,2}$. Set L_i the translated operator defined in the previous Lemma. By hypothesis, we know that $\|L_i\psi_i\|_{L^\infty} \to 0$, Notice that

$$L_i\psi_i = J_0''(z_0)\psi_i - P_0K((\lambda_i h(x+\theta_i)(q-1)|z_0|^{q-2})\psi_i).$$

Since λ_i tends to 0, $supp(h)$ is compact and z is bounded and strictly positive, we get

$$J_0''(z_0)\psi_i \to 0 \text{ in } L^\infty.$$

In particular, this means that

$$\begin{cases} -\Delta\psi_i + (1-(r-1)z_0^{r-2})\psi_i = g_i \to 0 \text{ in } L^\infty(B_R), \\ \|\psi_i\|_{L^\infty} \to 1. \end{cases}$$

By using the elliptic estimates we obtain that $\{\psi_i\}$ is bounded in C^α, then there exists a subsequence which converges in C^0 and the limit must be 0, by the strong convergence in $W^{1,2}$. So we conclude. □

Moreover we have the following result.

Lemma 3.10. *There exists a constant* λ_0 *such that if* $0 < |\lambda| < \lambda_0$, *then*

$$F_{\theta,\lambda}(w) := L_{\theta,\lambda}^{-1}(P_\theta(\lambda h(x) z_\theta^{q-1} + N_\theta(w)))$$

is a contraction in $W^{1,2} \cap L^\infty$.

Proof. First, notice that if $w \in \mathcal{B}_\delta \subset L^\infty$, with $\delta \le \frac{\eta}{2}$, then

$$\|N_\theta(w)\|_{W^{1,2}\cap L^\infty} = o(\|w\|_{W^{1,2}\cap L^\infty}).$$

Moreover,

$$\begin{aligned} &\|N_\theta(w_1) - N_\theta(w_2)\|_{W^{1,2}\cap L^\infty} \\ &\quad \le (c_1 \max(\|w_i\|_{W^{1,2}\cap L^\infty}) + c_2(\eta)\lambda)\|w_1 - w_2\|_{W^{1,2}\cap L^\infty} \end{aligned} \tag{3.6}$$

by (3.5). Also, for $|\lambda|$ small enough, there holds

$$\|P_{\theta,\lambda} \lambda h(x) z_\theta^{q-1}\|_{W^{1,2}\cap L^\infty(B_R)} \le \delta', \tag{3.7}$$

by Proposition 3.7. As a conclusion we can prove that $F_{\theta,\lambda}$ maps $B_\delta^\perp := B_\delta(z_0) \cap \mathcal{K}_\theta^\perp$ into itself.

Let $\phi \in B_\delta^\perp$. Then, taking into account the lower bound for $L_{\theta,\lambda}$, we obtain

$$||F_{\theta,\lambda}(\phi)||_{W^{1,2}\cap L^\infty} \le \frac{1}{\gamma}(||P_\theta(\lambda h(x) z_\theta^{q-2})||_{W^{1,2}\cap L^\infty} + ||P_\theta N_\theta(\phi)||_{W^{1,2}\cap L^\infty})$$

and this is bounded by δ, if λ and δ are small, by using the estimates (3.6) and (3.7) for $P_\theta(\lambda h(x) z_\theta^{q-2})$ and N_θ respectively.

Next, we have to prove that in fact $F_{\theta,\lambda}$ is a contraction in $B_\delta^\perp$. By using again the estimates for $L_{\theta,\lambda}^{-1}$ and N_θ, we can write

$$\begin{aligned} \|F_{\theta,\lambda}(w_1) - F_{\theta,\lambda}(w_2)\|_{W^{1,2}\cap L^\infty} &\le \frac{1}{\gamma}(c_1 \max(\|w_i\|_{W^{1,2}\cap L^\infty}) \\ &+ c_2(\eta)|\lambda|)\|w_1 - w_2\|_{W^{1,2}\cap L^\infty} \le \frac{1}{2}||w_1 - w_2||_{W^{1,2}\cap L^\infty}, \end{aligned}$$

if δ and $|\lambda|$ are small enough. We point out that the constant γ (see Lemma 3.9) doesn't depend on λ, so the claim follows. □

The previous lemma shows that there exists a fixed point $w_{\theta,\lambda} \in \mathcal{K}_\theta^\perp \cap L^\infty$, for which we have the following estimate:

Lemma 3.11. *$||w_{\theta,\lambda}||_{W^{1,2}} \le C||J'_\lambda(z_\theta)||_{W^{1,2}}$, namely $||w_{\theta,\lambda}||_{W^{1,2}} \to 0$, as $\lambda \to 0$.*

Proof. First we identify (in $W^{1,2}$) $J'_\lambda(z_\theta) = \lambda h(x) z_\theta^{q-1}$. Now, taking into account that z_θ is a solution of $J'_0(z_\theta) = 0$, we obtain the following estimate

$$\|J'_\lambda(z_\theta)\|_{W^{1,2}} = \|J'_\lambda(z_\theta) - J'_0(z_\theta)\|_{W^{1,2}} = |\lambda| \|h(x) z_\theta^{q-1}\|_{W^{1,2}} \to 0,$$

as $\lambda \to 0$. Since $w_{\theta,\lambda}$ is a fixed point of $F_{\theta,\lambda}$, we have that

$$w_{\theta,\lambda} = \lim_{k\to\infty} F^{(k)}_{\theta,\lambda}(0).$$

Moreover, $\|F_{\theta,\lambda}(0)\|_{W^{1,2}} = \|L^{-1}_{\theta,\lambda}(P_\theta J'_\lambda(z_\theta))\|_{W^{1,2}} \le \frac{1}{\gamma}\|J'_\lambda(z_\theta)\|_{W^{1,2}}$, and then by recurrence

$$||F^{(k)}_{\theta,\lambda}(0) - F^{(k-1)}_{\theta,\lambda}(0)\|_{W^{1,2}} \le 2^{1-k}\|F_{\theta,\lambda}(0)\|_{W^{1,2}},$$

hence it follows that

$$\|w_{\theta,\lambda}\|_{W^{1,2}} \le \frac{2}{\gamma}\|J'_\lambda(z_\theta)\|_{W^{1,2}}.$$

Remark. Since $w_{\theta,\lambda}$ is a fixed point of $F_{\theta,\lambda}$, we have that

$$w_{\theta,\lambda} = -L^{-1}_{\theta,\lambda}(P_\theta(J'_\lambda(z_\theta)) + P_\theta(N_\theta(w_{\theta,\lambda}))),$$

then

$$P_\theta J''_\lambda(z_\theta)(w_{\theta,\lambda}) = L_{\theta,\lambda} w_{\theta,\lambda} = -(P_\theta(J'_\lambda(z_\theta)) + P_\theta(N_\theta(w_{\theta,\lambda}))).$$

Since

$$J'_\lambda(z_\theta + w_{\theta,\lambda}) = J'_\lambda(z_\theta) + J''_\lambda(z_\theta) w_{\theta,\lambda} + N_\theta(w_{\theta,\lambda}),$$

we conclude that $P_\theta J'_\lambda(z_\theta + w_{\theta,\lambda}) = 0$. In other words

$$\nabla J_\lambda(z_\theta + w_{\theta,\lambda}) = J'_\lambda(z_\theta + w_{\theta,\lambda}) \in \mathcal{K}_\theta = \ker(J''_0(z_\theta)).$$

Remark. Note that we have found a fixed point w such that $(z_\theta + w) > \frac{\eta}{2}$ a.e. in B_R. The crucial point is that with this bound, we can deal with negative powers (namely $(q-2)$) of $(z_\theta + w)$ in B_R (the support of h).

Consider

$$Z_\lambda = \{u \in W^{1,2} | u = z_\theta + w_{\theta,\lambda}, z_\theta \in Z\}. \tag{3.8}$$

To show that Z_λ is diffeomorphic to the starting manifold, and also that it is a *natural constraint* we have to show the regularity of $w_{\theta,\lambda}$ with respect to θ. To this end, we fix $|\lambda| \le \lambda_0$ and we consider $w = w_{\theta,\lambda}$ defined for $\theta \in B := \{|\theta| \le R_1\}$.

Then we consider the map

$$H : B \times (W^{1,2} \cap L^\infty) \times \mathbb{R}^N \to (W^{1,2} \cap L^\infty) \times \mathbb{R}^N$$

defined in the following way: $H := (H_1, H_2)$ where

$$H_1(\theta, w, \alpha) := z_\theta + w - K((|z_\theta + w|^{r-1} + \lambda h(x)|z_\theta + w|^{q-1})) - \sum_i \alpha_i q_i(\theta),$$

and

$$H_2(\theta, w, \alpha) := (\langle w, q_1\rangle_{W^{1,2}}, \dots, \langle w, q_n\rangle_{W^{1,2}}).$$

Notice that in fact H depends on the value of λ fixed at the beginning; for simplicity, we avoid the use of subscripts in the notation.

In particular, $H(\theta, w, \alpha) = 0$ implies that $P_\theta J'_\lambda(z_\theta + w) = 0$. The dependence on θ comes from z_θ and then it is easily seen that H is C^1 with respect to θ . Moreover, if $x \in \text{support}(h)$ and $||v||_\infty$ is small, then $z_\theta(x) + v(x) > 0$. Therefore H is C^1 in a small neighborhood in $W^{1,2} \cap L^\infty$ of $w_{\theta,\lambda}$.

Next, given the fixed point $(w_{\theta,\lambda})$ found in the previous results, there exists $\alpha \in \mathbb{R}^N$ such that $H(\theta, w, \alpha) = 0$, and the idea is to apply the Implicit Function theorem in a neighborhood of this point. Therefore, we have to compute the derivatives with respect to w and α, to show that $D_{w,\alpha}H$ is invertible. By a direct computation we have

$$\begin{aligned}\frac{\partial H_1}{\partial(w,\alpha)}(v,b) =& v - K(((r-1)|z_\theta + w|^{r-2} + \lambda h(x)(q-1)|z_\theta + w|^{q-2})v) \\ &- \sum_i b_i q_i(\theta) =: T_1,\\ \frac{\partial H_2}{\partial(w,\alpha)}(v,b) =& (\langle v, q_1(\theta)\rangle_{W^{1,2}}, \dots, \langle v, q_n(\theta)\rangle_{W^{1,2}}) =: T_2.\end{aligned}$$

We will write $T := (T_1, T_2) = A + R$ where

$$A(v,b) = (v - K[((r-1)|z_\theta|^{r-2} + \lambda h(x)(q-1)|z_\theta|^{q-2})v] - \sum_i b_i q_i(\theta),$$

$$(\langle v, q_1(\theta)\rangle_{W^{1,2}}, \dots, \langle v, q_n(\theta)\rangle_{W^{1,2}}))$$

and

$$R(v,b) = (T-A)(v,b) = (-K(((r-1)(|z_\theta + w|^{r-2} - |z_\theta|^{r-2}) + \lambda h(x)(q-1)(|z_\theta + w|^{q-2}) - |z_\theta|^{q-2}))), 0)(v,b).$$

The result is the following:

Lemma 3.12. 1. *There exists a constant* $C > 0$ *such that*

$$\|A(v,b)\|_{(W^{1,2}\cap L^\infty)\times\mathbb{R}^N} > C\|(v,b)\|_{(W^{1,2}\cap L^\infty)\times\mathbb{R}^N},$$

for any $\lambda \in (0, \lambda_0]$ *and for any* $(v,b) \in W^{1,2} \times \mathbb{R}^N$. *In particular* A *is invertible.*

2. $|||R||| \to 0$, *as* $\lambda \to 0$ (*where* $|||\cdot|||$ *means the norm as a linear operator*).

Proof. 1). We argue by contradiction. Assume that there exists a sequence $\{(w_k, b_k)\}$ such that $||(w_k, b_k)||_{(W^{1,2}\cap L^\infty)\times\mathbb{R}^N} = 1$ and $|A_{\lambda_k}(w_k, b_k)| \to 0$, with $\lambda_k \to 0$. Then by definition of $A_{\lambda_k}(w_k, b_k)$ we have

$$w_k - K(((r-1)|z_\theta|^{r-2} + \lambda_k h(x)(q-1)|z_\theta|^{q-2})w_k) - \sum_{i=1}^{N} b_k^i q_i \to 0,$$

and $\langle w_k, q_i\rangle \to 0$, $i = 1, \ldots, N$. Consider the projections P_θ and $(I - P_\theta)$. Then $w_k^T := (I - P_\theta)w_k \to 0$, as $\lambda_k \to 0$. But now we have

$$L_{\theta,\lambda_k} w_k = P_\theta A_{\lambda_k}(w_k, b_k) \to 0, \text{ and } L_{\theta,\lambda_k} w_k^T \to 0,$$

namely,

$$L_{\theta,\lambda_k} w_k^\perp \to 0,$$

where $w_k^\perp := P_\theta w_k$. Now the estimates in Lemma 3.8 give us

$$\|L_{\theta,\lambda_k} w_k^\perp\|_{W^{1,2}\cap L^\infty} \geq \gamma \|w_k^\perp\|_{W^{1,2}\cap L^\infty}$$

because $w_k^\perp \in \mathcal{K}_\theta^\perp$. Hence we conclude that $w_k \to 0$, and therefore $b_k \to 0$. And this is a contradiction with the fact that $\|(w_k, b_k)\|_{(W^{1,2}\cap L^\infty)\times\mathbb{R}^N} = 1$.

2). Taking into account that r is subcritical, then by Hölder inequality, if $||u|| = ||v|| = 1$:

$$\langle K(|z_\theta + w|^{r-2} - |z_\theta|^{r-2})v, u\rangle_{W^{1,2}} \leq C\||z_\theta + w|^{r-2} - |z_\theta|^{r-2}\|_{L^{\frac{N}{2}}}.$$

Since $w \to 0$ in $W^{1,2}$, as $\lambda \to 0$, we conclude that the first part of R goes to zero, as $\lambda \to 0$, that is,

$$(-K((r-1)(|z_\theta + w|^{r-2} - |z_\theta|^{r-2})), 0)(v, b) \to 0,$$

because $r > 2$.

The second part uses essentially the regularity of w, namely the fact that $w \in W^{1,2} \cap L^\infty$; hence in the ball B_{R+R_1} we have that if $\|u\|_{W^{1,2}\cap L^\infty} = \|v\|_{W^{1,2}\cap L^\infty} = 1$,

$$\lambda(q-1)\langle K(h(x)(|z_\theta + w|^{q-2} - |z_\theta|^{q-2}))v, u\rangle_{W^{1,2}} \leq C\lambda,$$

taking into account that

$$\int_{B_R} \big(h(x)(|z_\theta + w|^{q-2} - |z_\theta|^{q-2})\big)^{N/2} \leq C,$$

because $\|w\|_{L^\infty(B_R)} \leq \frac{\eta}{2}$ and then $(|z_\theta + w|^{q-2} - |z_\theta|^{q-2}) \in L^\infty(B_R)$. So we conclude that $|||R||| \to 0$, as $\lambda \to 0$.

Remark. The proof above uses the regularity of the fixed point $w_{\theta,\lambda}$ in order to show the properties of A and R as operators in $W^{1,2} \cap L^\infty$.

Therefore we can apply the *Implicit Function Theorem* to obtain the next result.

Corollary 3.13. *Fixed* $\lambda \in (0, \lambda_0)$*,* $w_{\theta,\lambda}$ *is* C^1 *with respect to* θ*.*

Moreover, we can show a more precise estimate, in the $W^{1,2}$ topology:

Lemma 3.14. $D_{\theta_i} w_{\theta,\lambda} \to 0$ *in* $W^{1,2}$*, as* $\lambda \to 0$*.*

Proof. First recall that

$$z_\theta + w - K((z_\theta + w)^{r-1} + \lambda h(x)(z_\theta + w)^{q-1}) - \sum_i \alpha_i q_i(\theta) = 0$$

with $\alpha = \alpha(\theta, \lambda)$, and $(\langle w, q_1\rangle_{W^{1,2}}, \ldots, \langle w, q_n\rangle_{W^{1,2}}) = 0$. It is easy to see that $\alpha_j(\theta, \lambda) \to 0$, as $\lambda \to 0$: in fact, $0 = \langle z_\theta, q_j\rangle_{W^{1,2}}$ (because $\|z_\theta\|_{W^{1,2}}$ doesn't depend on θ), and $\langle w, q_j\rangle_{W^{1,2}} = 0$, because $w \in \mathcal{K}_\theta^\perp$. Moreover, as $\lambda \to 0$, we obtain

$$\langle K(z_\theta + w)^{r-1}, q_j\rangle_{W^{1,2}} = \int_{\mathbb{R}^N} (z_\theta + w)^{r-1} q_j \to \int_{\mathbb{R}^N} (z_\theta)^{r-1} q_j,$$

and, since $(z_\theta)^{r-1} = z_\theta - \Delta z_\theta$, the last integral is $\langle z_\theta, q_j\rangle_{W^{1,2}} = 0$. Finally, recalling the hypotheses on h, θ and the properties of z_θ and w, it is easy to see that

$$\lambda\langle Kh(x)(z_\theta + w)^{q-1}, q_j\rangle_{W^{1,2}} \to 0.$$

Collecting these estimates we get that $\alpha \to 0$, as $\lambda \to 0$.

Next, we derive with respect to θ_j. We denote $D_j := \frac{\partial}{\partial\theta_j}$, getting:

$$\begin{aligned} D_j z_\theta + D_j w - K((r-1)(z_\theta + w)^{r-2}(D_j z_\theta + D_j w) \\ +\lambda(q-1)h(x)(z_\theta + w)^{q-2}(D_j z_\theta + D_j w)) \\ = \sum(D_j\alpha_k)q_k + \sum \alpha_k(D_j q_k). \end{aligned}$$

And, on the other hand,

$$< D_j w, q_k >_{W^{1,2}} + < w, D_j q_k >_{W^{1,2}} = 0.$$

Since $w \to 0$ in $W^{1,2}$, as $\lambda \to 0$, we have that $< w, D_j q_k >_{W^{1,2}} \to 0$, and therefore, $< D_j w, q_k >_{W^{1,2}} \to 0$. Now we write the first equation in the following way:

$$\begin{aligned} D_j w - K((r-1)(z_\theta)^{r-2} + \lambda(q-1)h(x)(z_\theta)^{q-2} D_j w) \\ - K((r-1)((z_\theta + w)^{r-2} - z_\theta^{r-2}) \\ + \lambda(q-1)h(x)((z_\theta + w)^{q-2} - z_\theta^{q-2}))D_j w \\ = -\sum(D_j\alpha_k)q_k - \sum \alpha_k(D_j q_k) - D_j z_\theta. \end{aligned}$$

Applying the projection P_θ to this equation, and taking into account that $\alpha_k \to 0$ and that $< D_j w, q_k >_{W^{1,2}} \to 0$, we get

$$\begin{aligned} L_{\theta,\lambda}(D_j w)^\perp - P_\theta K((r-1)((z_\theta + w)^{r-2} - z_\theta^{r-2}) \\ + \lambda(q-1)h(x)((z_\theta + w)^{q-2} - z_\theta^{q-2}))(D_j w)^\perp = \sigma_\lambda, \end{aligned}$$

where $(D_j w)^\perp = P_\theta D_j w$ and $\sigma_\lambda \to 0$ in $W^{1,2}$, as $\lambda \to 0$.

Finally, choosing λ small enough, we get:

$$\begin{aligned} \|\sigma_\lambda\|_{W^{1,2}} \geq (\|L_{\theta,\lambda}(D_j w)^\perp\|_{W^{1,2}}) - \|P_\theta K((r-1)((z_\theta + w)^{r-2} - z_\theta^{r-2}) \\ + \lambda(q-1)h(x)((z_\theta + w)^{q-2} - z_\theta^{q-2}))(D_j w)^\perp\|_{W^{1,2}} \\ \geq \gamma\|(D_j w)^\perp\|_{W^{1,2}} - \frac{\gamma}{2}\|(D_j w)^\perp\|_{W^{1,2}}, \end{aligned}$$

which implies that $(D_j w)^\perp \to 0$, and, since $< D_j w, q_k >_{W^{1,2}} \to 0$, $D_j w \to 0$ in $W^{1,2}$, as $\lambda \to 0$.

Theorem 3.15. *The manifold* Z_λ *defined by* (3.8) *is diffeomorphic to the starting manifold* Z*, and moreover it is a natural constraint.*

Proof. The previous Lemmas imply that $Z_\lambda = \{z_\theta + w_{\theta,\lambda}\}$ is a manifold locally diffeomorphic to Z. If $u = z_\theta + w \in Z_\lambda$ is a constrained critical point, we have that $(J_\lambda|_{Z_\lambda})'(u) = 0$. In particular, $J'_\lambda(u) \perp T_u Z_\lambda$.

On the other hand, by construction of Z_λ, this means that $J'_\lambda(u) \in K_\theta = ker J''_0(z_\theta) \equiv T_{z_\theta} Z$. But by Lemma 3.14, we have that $T_{z_\theta} Z$ is close to $T_u Z_\lambda$, for λ small. Then $J'_\lambda(u) = 0$.

Therefore, we have that Z_λ is a *natural constraint* for our problem, and we conclude the proof. □

Now we define and study the properties of the Poincare-Melnikov function associated to our variational problem. In first place we prove the following Lemma:

Lemma 3.16. *If* $\beta < 1$*, then* $\lambda^{-\beta} w_{\theta,\lambda} \to 0$ *in* $W^{1,2}$*, as* $\lambda \to 0$.

Proof. Taking into account that $w = -L^{-1}_{\theta,\lambda}(P_\theta J'_\lambda(z_\theta) + P_\theta N_\theta(w))$ we can write

$$\lambda^{-\beta} w = -L^{-1}_{\theta,\lambda}(\lambda^{-\beta} P_\theta J'_\lambda(z_\theta) + \lambda^{-\beta} P_\theta N_\theta(w)).$$

Since we know that $N_\theta(w) = o(||w||_{W^{1,2}})$ and $w(\theta,\lambda) \to 0$, as $\lambda \to 0$, we have that $N_\theta(w) = wR(\lambda)$, with $R(\lambda) \to 0$. Moreover, $0 = J'_0(z_\theta)$, and then

$$w_\lambda = -L^{-1}_{\theta,\lambda}(P_\theta \psi_\lambda + w_\lambda R(\lambda)),$$

where $w_\lambda = \lambda^{-\beta} w$ and $\psi_\lambda = -\lambda^{-\beta} K(\lambda h(x)|z_\theta|^{q-1})$. By using our hypotheses on h and the exponential decay of z_θ we can conclude that if $\beta < 1$, then $\psi_\lambda \to 0$ in $W^{1,2}$, and finally the continuity of $L^{-1}_{\theta,\lambda}$ gives the result. □

This lemma allows us to estimate the energy functional.

Lemma 3.17. *Let* $u(\theta,\lambda) = z_\theta + w_{\theta,\lambda}$*. Then we have the following estimate*

$$\Phi_\lambda(\theta) := J_\lambda(u(\theta,\lambda)) = J_0(z_\theta) + \frac{\lambda}{q}\int_{\mathbb{R}^N} (h(x) z^q_\theta) + o(\lambda).$$

Proof. According to our definition, we have

$$\Phi_\lambda(\theta) = \frac{\|z_\theta\|^2_{W^{1,2}}}{2} + < z_\theta, w_{\theta,\lambda} >_{W^{1,2}} + \frac{\|w_{\theta,\lambda}\|^2_{W^{1,2}}}{2}$$
$$+ \int_{\mathbb{R}^N} \frac{\lambda h(x)(z_\theta + w_{\theta,\lambda})^q}{q} - \int_{\mathbb{R}^N} \frac{|z_\theta + w_{\theta,\lambda}|^r}{r}.$$

Since $z_\theta - \Delta z_\theta = z_\theta^{r-1}$, it follows that $< z_\theta, w_{\theta,\lambda} >_{W^{1,2}} = \int_{\mathbb{R}^N} z_\theta^{r-1} w_{\theta,\lambda}$. Using the estimate of w that we got in the previous lemma, and since

$$\frac{\lambda}{q}\int_{\mathbb{R}^N} h(x)(z_\theta + w_{\theta,\lambda})^q = \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x) z_\theta^q + o(\lambda),$$

it follows that

$$\Phi_\lambda(\theta) = J_0(z_\theta) + \frac{\lambda}{q}\int_{\mathbb{R}^N} h(x) z_\theta^q + o(\lambda).$$

Then, if we define the function

$$\Gamma(\theta) = \int_{\mathbb{R}^N} h(x) z_\theta^q dx,$$

following Theorem 2 in [1] we have in our case

Theorem 3.18. *Assume* $(h1)$, $(h3)$, *and* $(h4)$ *hold. Suppose also that* Γ *has a (possibly degenerated) maximum or minimum in* θ_0 *(i.e., there is* $r > 0$ *such that* $\Gamma(\theta_0) > \sup_{\partial B(\theta_0,r)} \Gamma(\theta)$ *or* $\Gamma(\theta_0) < \inf_{\partial B(\theta_0,r)} \Gamma(\theta)$*).*

Then, the functional J_λ *has a critical point* $u(\lambda)$, $\forall \lambda \in [0, \lambda_0]$ *(*λ_0 *small enough). Moreover, we have that* $u(\lambda) = z_{\theta_\lambda} + w_\lambda$, *with* $\theta_\lambda \to \theta \in B(\theta_0, r)$ *and* $w_\lambda \to 0$, *as* $\lambda \to 0$.

Remarks. i) Note that our assumption on R_1 (see $(h4)$) gives us that Γ has at least one maximum point and then the problem (P_λ) has at least one solution near Z, for λ small.

ii) Roughly speaking, the previous result states that the local maxima and minima of Γ select the points of Z_λ where the branch Σ, possibly, crosses the hyperplane $\{\lambda = 0\}$.

If we have more precise hypotheses on h we can prove the multiplicity of the solutions of (P_λ) close to Z. A result in this direction is the following, which can be proved in a similar way to the results in [20].

Proposition 3.19. *Assume that* $(h1)$, $(h2)$, $(h3)$ *and* $(h4)$ *hold and moreover that there exist positive constants* β, δ, r *and points* p_i, $i = 1, \dots, k$, *such that*

$$\begin{cases} h(x) \geq \beta \quad a.e. \quad x \in B(p_i, r), \\ (\beta - \delta) \leq h(x) \leq \beta \quad a.e. \quad x \in B(p_i, 2r) \setminus B(p_i, r), \\ h(x) \leq (\beta - \delta) \quad a.e. \quad x \in B(p_i, 4r) \setminus B(p_i, 2r). \end{cases} \tag{h5}$$

Then, if r *is sufficiently large,* Γ *has* k *maxima, i.e.,* (P_λ) *possesses* k *distinct solutions for* λ *small.*

Corollary 3.20. *Problem* (P_λ) *has* $k+1$ *solutions for* λ *small, where* k *is the number of maxima and minima of* Γ (*in the sense of Theorem* 3.18).

Remark. The positivity of the solutions obtained by the Poincaré-Melnikov method can be proved independently of the sign of h. See [5] for details.

4. Problem (III). In this section we will briefly consider the problem

$$-\Delta_p u(x) = \lambda h(x) u^{q-1}(x) + g(x) u^{r-1}(x), \quad x \in \mathbb{R}^N, \qquad (P_\lambda)$$

where $1 < p < N$, $1 < q < p < r < p^*$, h, g are positive functions such that $h \in L^s$ for all $s \in [1, \infty]$ and $g \in L^t$, where $t = \frac{Np}{Np - r(N-p)}$. In this problem the natural space to use a variational framework is $D^{1,p}(\mathbb{R}^N)$, defined as the closure of $C_0^\infty(\mathbb{R}^N)$ in the norm $||u||_{D^{1,p}}^p = \int |\nabla u|^p dx$.

In particular, it is impossible to use a variational setting if $g \equiv 1$ and $p < r < p^*$; therefore the arguments in Sections 3 and 4 do not apply in this case.

The result in this section is the following.

Theorem 4.1. *Problem* (P_λ) *has at least two positive solutions, if* λ *is small enough.*

Proof. Note that it is easy to prove the Palais-Smale condition following the arguments in Section 2, since $g \in L^t$. If we write the functional

$$J(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p - \frac{\lambda}{q} \int_{\mathbb{R}^N} h(x) u^q - \frac{1}{r} \int_{\mathbb{R}^N} g(x) u^r,$$

under the general hypotheses, it is easy to see that for λ small we have the geometry that permits to prove the existence of at least two solutions, namely, a local minimum and a *mountain-pass* critical point. (See [12] for the details to study the geometry of the functional in a similar problem.) $\square$

A more detailed study of this case and some other related problems will be the subject of a forthcoming paper.

5. Nonexistence result: A Pohozaev energy estimate. We explain in this appendix, for the reader convenience, the result by Pohozaev, [22]. See also the paper by Bidaut-Veron, [7], where the result follows by a different argument.

We will consider C^1 solutions to the problem

$$\begin{cases} -\Delta_p u = u^q & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N. \end{cases} \qquad (5.1)$$

In this section we will show the following result

Theorem 5.1. *Assume that* $p-1 < q < \frac{N(p-1)}{N-p}$, *(*$p-1 < q < +\infty$, *if* $p \geq N$*), then problem* (5.1) *has no positive solution* $u \in W^{1,p}_{loc}$.

Proof. Let $u > 0$ be a solution to (5.1). Then we can consider as a multiplier the function ρu^α, where ρ is a convenient cutoff function of compact support, that we call Ω, and $\alpha < 0$. Then by integration we obtain the identity

$$\int_{\mathbb{R}^N} \rho u^{\alpha+q} = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\langle \nabla u, \nabla \rho\rangle u^\alpha + \alpha \int_{\mathbb{R}^N} |\nabla u|^p \rho u^{\alpha-1},$$

that we can write as follows

$$\begin{aligned} \int_{\mathbb{R}^N} \rho u^{\alpha+q} - \alpha \int_{\mathbb{R}^N} |\nabla u|^p \rho u^{\alpha-1} &= \int_{\mathbb{R}^N} |\nabla u|^{p-2}\langle \nabla u, \nabla \rho\rangle u^\alpha \\ &\leq \int_{\mathbb{R}^N} |\nabla u|^{p-1} u^{\alpha-\lambda} u^\lambda \frac{|\nabla \rho|}{\rho^{\mu_1+\mu_2}} \rho^{\mu_1+\mu_2}. \end{aligned} \tag{5.2}$$

By application of the Hölder inequality we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla u|^{p-1} u^{\alpha-\lambda} u^\lambda \frac{|\nabla \rho|}{\rho^{\mu_1+\mu_2}} \rho^{\mu_1+\mu_2} \\ &\leq \left(\int_{\mathbb{R}^N} |\nabla u|^p u^{\frac{p\lambda}{p-1}} \rho^{\mu_1 \frac{p}{p-1}}\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} u^{(\alpha-\lambda)\gamma} \rho^{\mu_2\gamma}\right)^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}^N} \frac{|\nabla \rho|^\eta}{\rho^{\mu_1+\mu_2}\eta}\right)^{\frac{1}{\eta}}, \end{aligned} \tag{5.3}$$

where the exponents will be chosen in such a way that we get again a term like the second term in the left hand side in 5.2, namely,

i) $\frac{p-1}{p} + \frac{1}{\gamma} + \frac{1}{\eta} = 1$,
ii) $\lambda = \frac{(\alpha-1)(p-1)}{p} < 0$,
iii) $\mu_1 \frac{p}{p-1} = 1$, $\mu_2\gamma = 1$,
iv) $(\alpha-\lambda)\gamma = q + \alpha$, namely $\gamma = \frac{p(q+\alpha)}{p\alpha-(p-1)(\alpha-1)}$.

With this choices we get

$$\begin{aligned} &\int_{\mathbb{R}^N} \rho u^{\alpha+q} - \alpha \int_{\mathbb{R}^N} |\nabla u|^p \rho u^{\alpha-1} = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\langle \nabla u, \nabla \rho\rangle u^\alpha \\ &\leq \left(\int_{\mathbb{R}^N} |\nabla u|^p u^{\alpha-1}\rho\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} u^{(\alpha-\lambda)\gamma} \rho^{\mu_2\gamma}\right)^{\frac{1}{\gamma}} \left(\int_{\mathbb{R}^N} \frac{|\nabla \rho|^\eta}{\rho^{\mu_1+\mu_2}\eta}\right)^{\frac{1}{\eta}}, \end{aligned} \tag{5.4}$$

moreover in iv) we must assume $\gamma > 1$ because we use the Hölder inequality, but

$$\gamma = \frac{p(q+\alpha)}{p\alpha - (p-1)(\alpha-1)} \to \frac{pq}{p-1}, \quad \alpha \to 0^-.$$

Hence for $|\alpha|$ small enough we have indeed $\gamma > 1$ (we remark that we need only that $q > \frac{p-1}{p}$). Hereafter we assume that

$$-\varepsilon < \alpha < 0, \text{ where } \varepsilon = \varepsilon(\gamma), \text{ is small enough to have } \gamma > 1.$$

Now we will fix the test function ρ and the exponents η and α.

We start with a smooth function ϕ_0 defined in the positive halfline, such that $\phi_0(0) = 1$, $\phi_0(x) \geq 0$, $\phi_0(x) = 0$ if $x > 1$, and

$$\lim_{x \to 1^-} \frac{\phi_0'(x)}{\phi_0^{\mu_1+\mu_2}(x)} = -c < 0.$$

(For instance $\phi_0(x) \sim (1-x)^d$, for $x \sim 1$, with $\mu_1 + \mu_2 < 1$ and d large). Fix $R > 0$ we define $\phi_1(r) = \phi_0(\frac{r}{R})$, $r = |x|$, and finally for R_1 fixed

$$\rho(r) = \begin{cases} 1 & \text{if } r \leq R_1, \\ \phi_1(r - R_1) & \text{if } r \geq R_1, \end{cases} \tag{5.5}$$

we call $R_2 = R_1 + R$. We need to evaluate

$$I = \int_{\mathbb{R}^N} \frac{|\nabla \rho|^\eta}{\rho^{(\mu_1+\mu_2)\eta}},$$

by the definition of ρ and writing the above integral in polar coordinates we obtain

$$I = \omega_N (R_2 - R_1)^{N-\eta} \int_0^1 \frac{|\phi_0'(s)|^\eta}{\phi_0^{(\mu_1+\mu_2)\eta}(s)} \left(s + \frac{R_1}{R_2 - R_1} \right)^{N-1} ds.$$

To simplify we choose $R_2 = 2R_1$ and then we have

$$I = \omega_N (R_1)^{N-\eta} \int_0^1 \left(\frac{|\phi_0'(s)|}{\phi_0^{(\mu_1+\mu_2)}(s)} \right)^\eta (s+1)^{N-1} ds.$$

By the hypotheses on ϕ_0 we conclude

$$I \leq C R_1^{N-\eta}, \tag{5.6}$$

and we have that, if $\eta > N$, then $I(R_1) \to 0$, if $R_1 \to \infty$. We collect all the conditions on the exponents

A) $\frac{p-1}{p} + \frac{1}{\gamma} + \frac{1}{\eta} = 1$, $\eta > N$. As a consequence, $\gamma < p^*$ (if $p \geq N$ $p^* = \infty$, and we can go ahead).

B) $\mu_1 = \frac{p-1}{p}$, $\quad \mu_2 = \frac{1}{\gamma}$ and $\mu_1 + \mu_2 < 1$. This implies $\gamma > p$.

Moreover, according with v), we have $\gamma = \frac{p(q+\alpha)}{p\alpha-(p-1)(\alpha-1)}$ and by A) and B): $p < \gamma < p^*$ with $\alpha < 0$ small enough in absolute value. So the condition, taking limits for $\alpha \to 0^-$, is

$$p < \frac{pq}{p-1} < p^* = \frac{Np}{N-p} \quad \text{or equivalently} \quad p-1 < q < \frac{N(p-1)}{N-p},$$

that is the hypothesis on q (obviously, if $p \geq N$, we don't have the upper bound on q).

Now we can conclude the proof of the theorem. From (5.2) and (5.6) we have, by Young inequality, that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \rho u^{\alpha+q} - \alpha \int_{\mathbb{R}^N} |\nabla u|^p \rho u^{a-1} \\
&\leq CR^{-k} \left(\int_{\mathbb{R}^N} |\nabla u|^p u^{\alpha-1} \rho \right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^N} u^{(\alpha-\lambda)\gamma} \rho^{\mu_2 \gamma} \right)^{\frac{1}{\gamma}} \\
&\leq C(\alpha) R^{-k(\alpha)} + \frac{|\alpha|}{2} \int_{\mathbb{R}^N} |\nabla u|^p u^{\alpha-1} \rho + \frac{1}{2} \int_{\mathbb{R}^N} u^{(q+\alpha)} \rho,
\end{aligned} \tag{5.7}$$

where $k(\alpha)$ is positive, namely

$$\frac{1}{2} \int_{\mathbb{R}^N} \rho u^{\alpha+q} + \frac{|\alpha|}{2} \int_{\mathbb{R}^N} |\nabla u|^p \rho u^{a-1} \leq C(\alpha) R^{-k(\alpha)}.$$

If $u > 0$ and $R > \frac{1}{2}$, then

$$\int_{B_{2R}} u^{q+\alpha} \rho > \int_{B_R} u^{q+\alpha} > c_0,$$

but this is a contradiction because $c_0 \leq C(\alpha) R^{-k(\alpha)} \to 0$, as $R \to \infty$.

Remarks. i) The theorem in this section was obtained, with a different method, by Bidaut-Veron in [7]. ii) It is very important to observe that the proof doesn't need any asymptotic behavior on the solutions at infinity. In this sense this result can be seen also as an extension of the result by Gidas and Spruck [14] to the quasilinear case, but with the restriction in the range of q.

6. Existence of a radial solution. In order to show the lack of compactness of (P_0) (see Section 3), now we prove the existence of a radial positive solution in $W^{1,p}$ of the equation

$$-\Delta_p u + u^{p-1} = u^{r-1} \quad \text{in } \mathbb{R}^N. \tag{6.1}$$

So, by the invariance of the problem, also for the quasilinear case there exists a critical manifold parametrized by $\mathbb{R}^N$, however we are not able to prove that the radial solution is, up to translations, the unique solution of (6.1). Pucci and Serrin have recently proved that in the class of radial solutions there is only a solution (see [23]). In our case there is no symmetry result like in the case $p = 2$, where the uniqueness result by Kwong (see [16]) and the classical result by Gidas, Ni and Nirenberg (see [13]) solves the uniqueness problem. In this section we follow the arguments in [25].

First a result on the properties of the space $W^{1,p}_{rad} := \{u \in W^{1,p} | u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^N\}$.

Lemma 6.1. *Let $v \in W^{1,p}_{rad}$. Then there exists $V(s) \in C^0(\mathbb{R}^+)$ such that $v(x) = V(|x|)$ a.e. and*

$$V(s) \le \kappa ||v||^p_{W^{1,p}} s^{\frac{1-N}{p}},$$

for $s > 1$, where κ depends on N and p only.

Proof. Set $m := \frac{N-1}{p}$. Then we can write

$$\begin{aligned}
&((s^m V)^p)_s = p|s^m V|^{p-2}(s^m V)(s^m V)_s \le (p-1)(s^m V)^p + ((s^m V)_s)^p \\
&(p-1)(s^m V)^p + \left(s^m V_s + m s^{m-1} V\right)^p \\
&\le (p-1)(s^m V)^p + 2^p \left((s^m V_s)^p + (m s^{m-1} V)^p\right) \\
&\le 2^p s^{pm} V_s^p + ((p-1) + 2^p m^p)\left(s^{pm} V^p\right) \le \kappa(p,N)\left(s^{pm}(V^p + V_s^p)\right).
\end{aligned}$$

Integrating from 1 to s we obtain

$$s^{pm} V^p(s) - V^p(1) \le \kappa(p,N) \int_1^s \rho^{N-1}\left(V^p(\rho) + V^p_\rho(\rho)\right) \le \kappa(p,N) \frac{||v||^p_{W^{1,p}}}{\omega_N},$$

where ω_N is the measure of surface of the unit ball in $\mathbb{R}^N$.

Since $V \in W^{1,p}(1, +\infty)$ we have that $V(1) \le \|V\|_{L^\infty} \le C\|v\|_{W^{1,p}}$, so it follows

$$V(s) \le \tilde{\kappa} \|v\|_{W^{1,p}} s^{\frac{1-N}{p}}.$$

This lemma allows us to handle problem (6.1) and to prove the existence of a positive radial solution, because now we have the necessary compactness.

Theorem 6.2. *There exists a positive radial function $u \in W^{1,p}$ solution of* (6.1).

Proof. Set $M := \{u \in W^{1,p}_{rad} | \frac{1}{r}\int_{\mathbb{R}^N} |u|^r = 1\}$ and consider the functional $\frac{1}{p}\|u\|^p_{W^{1,p}}$ on this manifold. Obviously the functional is bounded from below and coercive, then there exists a bounded minimizing sequence $\{u_j\} \subset W^{1,p}_{rad} \cap M$. Then there exists $u \in W^{1,p}_{rad}$ such that $u_j \rightharpoonup u$ in $W^{1,p}_{rad}$ and a.e. in $\mathbb{R}^N$.

Using Lemma 2 in [25] (see also the proof of Lemma 3), we have that $u_j \to u$ in L^r. Because the norm is weakly lower semicontinuous we obtain then u attains the minimum of $\frac{1}{p}\| \cdot \|^p_{W^{1,p}}$, then there exists $\lambda \in \mathbb{R}^+$ such that

$$-\Delta_p u + u^{p-1} = \lambda u^{r-1} \quad \text{in } \mathbb{R}^N,$$

rescaling the function we obtain a positive radial solution of (6.1), i.e. (P_0).

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