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MONOTONICITY AND SYMMETRY RESULTS FOR P-LAPLACE EQUATIONS AND APPLICATIONS*

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Abstract. In this paper we prove monotonicity and symmetry properties of positive solutions of the equation $-div(|Du|^{p-2}Du) = f(u)$, $1 < p < 2$, in a smooth bounded domain Ω satisfying the boundary condition $u = 0$ on $\partial\Omega$. We assume f locally Lipschitz continuous only in $(0,\infty)$ and either $f \geq 0$ in $[0,\infty]$ or f satisfying a growth condition near zero. In particular we can treat the case of $f(s) = s^{\alpha} - c\,s^{q}$, $\alpha > 0$, $c \geq 0$, $q \geq p-1$. As a consequence we get an extension to the p–Laplacian case of a symmetry theorem of Serrin for an overdetermined problem in bounded domains. Finally we apply the results obtained to the problem of finding the best constants for the classical isoperimetric inequality and for some Sobolev embeddings.

1. Introduction and statement of the results. The aim of this paper is to improve some monotonicity and symmetry results that we have recently obtained in [7] as well as to give some applications to an overdetermined problem and to the search of the best constants in the isoperimetric

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and Sobolev inequalities. We consider the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1.1}$$

where Δ_p denotes the p-Laplace operator, $\Delta_p u = div\,(|Du|^{p-2}Du)$, $1 < p < 2$, Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$ and f is a real function.

We refer the reader to the introduction of [7] for more information about (1.1) as well as for a discussion of the difficulties arising in the study of that problem.

To recall our previous results we need some notations. Let ν be a direction in $\mathbb{R}^N$, i.e., $\nu \in \mathbb{R}^N$ and $|\nu| = 1$. For a real number λ we define

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\} \tag{1.2}$$

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{1.3}$$

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu, \quad x \in \mathbb{R}^N \tag{1.4}$$

(i.e., R_λ^ν is the reflection through the hyperplane T_λ^ν)

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu. \tag{1.5}$$

If $\lambda > a(\nu)$, then Ω_λ^ν is nonempty; thus we set

$$(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu). \tag{1.6}$$

If Ω is smooth and $\lambda > a(\nu)$, λ close to $a(\nu)$, then the reflected cap $(\Omega_\lambda^\nu)'$ is contained in Ω and will remain in it, at least until one of the following occurs:

(i) $(\Omega_\lambda^\nu)'$ becomes internally tangent to $\partial\Omega$ at some point not on T_λ^ν

(ii) T_λ^ν is orthogonal to $\partial\Omega$ at some point

Let $\Lambda_1(\nu)$ be the set of those $\lambda > a(\nu)$ such that for each $\mu \in (a(\nu), \lambda]$ neither of the conditions (i) and (ii) holds, and define

$$\lambda_1(\nu) = \sup \Lambda_1(\nu). \tag{1.7}$$

Note that $\lambda_1(\nu)$ is a lower-semicontinuous function whenever $\partial\Omega$ is smooth.

The main result of [7] is the following:

Theorem 1.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1.1) *where f satisfies the hypothesis*

(H1) $f : [0, +\infty) \to \mathbb{R}$ *is locally Lipschitz continuous.*

Then for any direction ν and for λ in the interval $(a(\nu), \lambda_1(\nu)]$ we have

$$u(x) \le u(x^\nu_\lambda) \qquad \forall\, x \in \Omega^\nu_\lambda. \tag{1.8}$$

Moreover,

$$\frac{\partial u}{\partial \nu}(x) > 0 \qquad \forall\, x \in \Omega^\nu_{\lambda_1(\nu)} \setminus Z, \tag{1.9}$$

where $Z = \{x \in \Omega : Du(x) = 0\}$.

From this theorem we immediately deduced the following:

Corollary 1.1. *If, for a direction ν, the domain Ω is symmetric with respect to the hyperplane $T^\nu_0 = \left\{x \in \mathbb{R}^N : x \cdot \nu = 0\right\}$ and $\lambda_1(\nu) = 0$, then u is symmetric; i.e., $u(x) = u(x^\nu_0)$ and nonincreasing in the ν–direction in Ω^ν_0 with $\frac{\partial u}{\partial \nu}(x) > 0$ in $\Omega^\nu_0 \setminus Z$.*

In particular, if Ω is a ball, then u is radially symmetric and $\frac{\partial u}{\partial r} < 0$, where $\frac{\partial u}{\partial r}$ is the derivative in the radial direction.

These results are obtained under the hypothesis (H1), i.e., requiring f to be Lipschitz-continuous near zero, but for many relevant nonlinearities this regularity does not hold, as for example in the case $f(u) = u^q$, $p - 1 < q < p^* - 1 = \frac{Np}{N-p} - 1$, q close to $p - 1$.

Here we weaken the regularity of f at zero making the hypothesis

(H2) f is locally Lipschitz-continuous in $(0, +\infty)$, and either $f \ge 0$ in $[0, +\infty)$ or there exist $s_0 > 0$ and a continuous nondecreasing function $\beta : [0, s_0] \to \mathbb{R}$ satisfying

$$\beta(0) = 0, \quad \beta(s) > 0 \text{ for } s > 0, \quad \int_0^{s_0} (\beta(s)\, s)^{-\frac{1}{p}}\, ds = \infty \tag{1.10}$$

such that $f(s) + \beta(s) \ge 0 \;\; \forall\, s \in [0, s_0]$.

Remark 1.1. A large class of nonlinearities for which (H2) holds is the one given by the functions f of the type

$$f(s) = g(s) - c\, s^q \tag{1.11}$$

with g locally Lipschitz-continuous in $(0, +\infty)$, $g(s) \geq 0$, $c \geq 0$ and $q \geq p-1$. In this case $\beta(s) = c\, s^q$, if $c > 0$.

Remark 1.2. In the case of the ball, Brock [3],[4] proved the radial symmetry of the solutions of (1.1) under quite general hypotheses on f, using the so-called continuous Steiner symmetrization.

Remark 1.3. Note that under hypothesis (H2) any nonnegative solution of the problem

$$\begin{cases} -\Delta_p u &= f(u) \qquad \text{in} \qquad \Omega \\ \quad u &= 0 \qquad\quad \text{on} \qquad \partial\Omega \end{cases}$$

is positive by the maximum principle of Vazquez [11] (see also next section). Therefore it is not restrictive to consider only the case $u > 0$.

Let us now state our main result.

Theorem 1.2. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1.1) with f satisfying (H2). Then for any direction ν and for λ in the interval $(a(\nu), \lambda_1(\nu))$ we have*

$$u(x) < u(x_\lambda^\nu) \qquad \forall\, x \in \Omega_\lambda^\nu. \tag{1.12}$$

Moreover,

$$\frac{\partial u}{\partial \nu}(x) > 0 \qquad \forall\, x \in \Omega_{\lambda_1(\nu)}^\nu \setminus Z \tag{1.13}$$

and u is strictly increasing in the ν–direction in the set $\Omega_{\lambda_1(\nu)}^\nu$.

Consequently we get:

Corollary 1.2. *If, for a direction ν, the domain Ω is symmetric with respect to the hyperplane T_0^ν and $\lambda_1(\nu) = 0$, then $u(x) = u(x_0^\nu)$ for any $x \in \Omega$. Moreover u is strictly increasing in the ν–direction in the set Ω_0^ν with $\frac{\partial u}{\partial \nu}(x) > 0$ in $\Omega_0^\nu \setminus Z$. In particular if Ω is a ball, then u is radially symmetric and $\frac{\partial u}{\partial r} < 0$.*

As compared with Theorem 1.1 and Corollary 1.1, besides substituting condition (H1) with (H2) we also get stronger information on the behaviour of the solution, namely (1.12) which implies the strict monotonicity of u in the set $\Omega^{\nu}_{\lambda_1(\nu)}$ in the ν–direction. This is a consequence of Proposition 3.1 which asserts that the solution u cannot be constant on any segment parallel to ν contained in $\Omega^{\nu}_{\lambda_1(\nu)}$.

The method used in this paper is the well-known Alexandrov–Serrin moving plane method ([9]) which led to the classical result of Gidas, Ni and Nirenberg [8] about the symmetry of solutions of problem (1.1) in the case $p = 2$.

As in [7] we will use the idea of simultaneously moving hyperplanes orthogonal to directions close to a fixed direction ν_0. To ensure continuity (with respect to the directions) in this procedure we assume, as in [7], Ω smooth.

As we mentioned above, with the assumption (H2) we do not require the Lipschitz-continuity of f at zero any more. To overcome this lack of regularity we use Hopf's lemma at $\partial\Omega$, where $u = 0$, while in the interior we proceed as in [7].

When used in 1971 by Serrin, the moving plane method led to a beautiful result about the spherical symmetry of domains where certain overdetermined problems have solutions. Once we have Theorem 1.2 (or Theorem 1.1) it is easy to get the analogous result for the p–Laplacian. More precisely let us consider the following class of nonlinearities:

(H3) $f : [0,\infty) \to \mathbb{R}$ is a continuous function such that in an interval $[0, s_0]$, $s_0 > 0$, $f = f_1 + f_2$, with f_1 Lipschitz-continuous and f_2 nondecreasing

Theorem 1.3. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1.1) *with f satisfying* (H1) *or* (H2)–(H3). *If the following condition holds:*

$$-\frac{\partial u}{\partial n} = a \qquad \text{on} \quad \partial\Omega, \tag{1.14}$$

where n is the outer normal derivative and a is a positive constant, then Ω is a ball.

Let us remark that, using again the continuous Steiner symmetrization, Brock has also obtained a similar result ([5]).

Finally if we take $f(u) = u^q$, $q < p^* - 1 = \frac{Np}{N-p} - 1$, from Corollary 1.2 we get that all solutions of (1.1) in the ball are radially symmetric and strictly radially decreasing. In particular, this happens for the functions which minimize the energy functional

$$J_q^p(v) = \frac{\int_\Omega |\nabla v|^p}{\left(\int_\Omega |v|^q\right)^{\frac{p}{q}}}$$

. As explained in Section 4 this allows us to restrict our attention to radial (and radially decreasing) functions while looking for the "best" Sobolev constant for the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. In particular, letting $p \to 1$ one can find the absolute isoperimetric constant. In the existing literature the reduction to radial functions is usually made through the Schwarz symmetrization which, in turn, relies on the isoperimetric inequality.

Having proved the symmetry of the solutions of (1.1) with $f(u) = u^q$, $q < p^* - 1$, we are able to prove the isoperimetric inequality with the best constant independent of the Schwarz symmetrization, but rather as a consequence of comparison principles for elliptic operators.

The paper is organized as follows. In Section 2 we present some preliminary results, while Section 3 is devoted to the proof of Theorem 1.2. The applications are described in Section 4.

2. Preliminary results. As is well known, the moving plane method relies on comparison results. In [7] we used some weak and strong comparison theorems obtained in [6]. As originally stated they apply to the case of Lipschitz-continuous nonlinearities f. Since we will work under the hypothesis (H2) we need to modify them in a suitable way.

Let Ω be a domain in $\mathbb{R}^N$, $N \geq 2$, and u, $v \in C^1(\overline{\Omega})$ be weak solutions of

$$\begin{cases} -\Delta_p u & \leq & f(u) & \text{in} & \Omega \\ -\Delta_p v & \geq & f(v) & \text{in} & \Omega. \end{cases} \tag{2.1}$$

For any set $A \subseteq \Omega$ we define

$$M_A = M_A(u, v) = \sup_A \left(|Du| + |Dv|\right). \tag{2.2}$$

Theorem 2.1. *Suppose that Ω is bounded, $1 < p < 2$ and there exist d_1, $d_2 > 0$ such that the sets $u(\Omega^+)$, $v(\Omega^+)$ are contained in $[d_1, d_2]$, where $\Omega^+ = \{x \in \Omega : u(x) \geq v(x)\}$. If the function f satisfies*

$$f(s) = f_1(s) + c_1\, s, \quad s \in [d_1, d_2] \quad \textit{for some } c_1 \geq 0 \tag{2.3}$$

with f_1 nonincreasing[1]*, then there exist α, $M > 0$, depending on p, $|\Omega|$, M_Ω,*

[1] This is the case of any function locally Lipschitz continuous in $(0, \infty)$.

c_1, d_1, d_2 such that for any open set $\Omega' \subseteq \Omega$ with $\Omega' = A_1 \cup A_2$, $|A_1 \cap A_2| = 0$, $|A_1| < \alpha$, $M_{A_2} < M$, then $u \leq v$ on $\partial\Omega'$ implies $u \leq v$ in Ω'.

Proof. The proof is the same as that of Theorem 1.2 in [6], case $1 < p < 2$. Note only that the condition $|A_1 \cap A_2| = 0$ is not necessary, since we can always substitute A_1 with $A_1 \setminus A_2$. □

Before stating the strong comparison theorem we need to recall a "Harnack-type" inequality derived in [6].

Lemma 2.1. *Let D be an open set in $\mathbb{R}^N$, $N \geq 2$, and $u, v \in C^1(D)$ be weak solutions of*

$$-\Delta_p u + c_2\, u \leq -\Delta_p v + c_2\, v \ , \quad u \leq v \quad \text{in} \ \ D \tag{2.4}$$

with $1 < p < \infty$ and $c_2 \in \mathbb{R}$. Suppose $\overline{B(\overline{x}, 5\delta)} \subseteq D$ for some $\delta > 0$ and, if $p \neq 2$, $\inf_D (|Du| + |Dv|) > 0$. Then, for any positive number $s < \frac{N}{N-2}$ we have

$$\|v - u\|_{L^s(B(\overline{x}, 2\delta))} \leq c\, \delta^{\frac{N}{s}} \inf_{B(\overline{x}, \delta)} (v - u) \tag{2.5}$$

where c is a constant depending on N, p, s, c_2, δ and, if $p \neq 2$, also on $m = \inf_{B(\overline{x},5\delta)} (|Du| + |Dv|)$ and $M_{B(\overline{x},5\delta)}$ (defined as in (2.2)).

Theorem 2.2. *Let $u, v \in C^1(\Omega)$ be weak solutions of (2.1) with $1 < p < \infty$, $0 < u \leq v$ in Ω and f be locally Lipschitz-continuous in $(0, \infty)$. Define*

$$Z_v^u = \{x \in \Omega : |Du| = |Dv| = 0\} \quad (Z_v^u = \emptyset \text{ for } p = 2)$$

If there exists $x_0 \in \Omega \setminus Z_v^u$ such that $u(x_0) = v(x_0)$, then $u \equiv v$ in the connected component of $\Omega \setminus Z_v^u$ containing x_0.

Proof. The result follows from Lemma 2.1. Let C be the connected component of $\Omega \setminus Z_v^u$ containing x_0. It suffices to prove that the set $O = \{x \in C : u(x) = v(x)\}$ is open. Let $\overline{x}$ belong to O and consider a ball $D = B(\overline{x}, 10\delta)$ such that $\overline{D} \subset O$. If $s_1 = \inf_{\overline{D}} u$ and $s_2 = \sup_{\overline{D}} v$, by the Lipschitz-continuity of f in $[s_1, s_2]$, there exists a number $c_2 \geq 0$ such that $f_2(s) = f(s) + c_2\, s$ is nondecreasing in $[s_1, s_2]$. By (2.1), since $u \leq v$, we get

$$-\Delta_p u + c_2 u \leq -\Delta_p v + c_2 v.$$

Moreover, $\inf_{B(\overline{x},5\delta)}(|Du|+|Dv|) = m > 0$ so that we can apply Lemma 2.1, obtaining

$$\int_{B(\overline{x},2\delta)} (v-u)\,dx \leq 0.$$

This implies $u \equiv v$ in $B(\overline{x},2\delta)$, and hence $B(\overline{x},2\delta) \subseteq O$ which means that O is open. □

Next we recall the following version of the strong maximum principle and Hopf's lemma, due to Vazquez [11].

Theorem 2.3. *Let Ω be a domain in $\mathbb{R}^N$ and $u \in C^1(\Omega)$, $u \geq 0$ in Ω, a weak solution of*

$$-\Delta_p u + \beta(u) = g \geq 0 \qquad \text{in } \Omega$$

where $g \in L^2_{loc}(\Omega)$ and $\beta : [0,\infty) \to \mathbb{R}$ is continuous, nondecreasing. If there exists $s_0 > 0$ such that either $\beta \equiv 0$ in $[0,s_0]$ or $\beta(s) > 0$ in $[0,s_0]$ and $\int_0^{s_0} [\beta(s)\,s]^{-\frac{1}{p}} = \infty$, then either $u \equiv 0$ in Ω or $u > 0$ in Ω.

In this latter case if the interior sphere condition is satisfied at $x_0 \in \partial\Omega$, $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$, then $\frac{\partial u}{\partial n}(x_0) > 0$ for any inward directional derivative (this means that if y approaches x_0 in a ball $B \subset \Omega$ that has x_0 on its boundary, then $\lim_{y\to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).

Let us define, using the same notations as in Section 1,

$$u^\nu_\lambda(x) = u(x^\nu_\lambda) \tag{2.6}$$

$$Z^\nu_\lambda = Z^\nu_\lambda(u) = \{x \in \Omega^\nu_\lambda : Du(x) = Du^\nu_\lambda(x) = 0\} \tag{2.7}$$

$$Z = Z(u) = \{x \in \Omega : Du(x) = 0\} \tag{2.8}$$

$$\Lambda_0(\nu) = \left\{\lambda \in (a(\nu),\lambda_1(\nu)\,] : u \leq u^\nu_\mu \text{ in } \Omega^\nu_\mu,\ \forall\, \mu \in (a(\nu),\lambda\,]\right\}$$

and, if this set is nonempty,

$$\lambda_0(\nu) = \sup \Lambda_0(\nu) \tag{2.9}$$

If ν_0 is a direction and $\delta > 0$ is a real number we set

$$I_\delta(\nu_0) = \left\{\nu \in S^{N-1} : |\nu - \nu_0| < \delta\right\},$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$.

We show now a useful property of the set Z of the critical points of a solution of (1.1).

Proposition 2.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1.1) *with f satisfying* (H2). *For any direction ν the cap $\Omega^{\nu}_{\lambda_0(\nu)}$ does not contain any subset Γ of Z on which u is constant and whose projection on the hyperplane $T^{\nu}_{\lambda_0(\nu)}$ contains an open subset of $T^{\nu}_{\lambda_0(\nu)}$ (relative to the induced topology).*

Proof. The proof is identical to the one of Proposition 3.1 of [7], where it was stated for functions f satisfying (H1). In fact, supposing the existence of such a set Γ we would have $u \equiv m > 0$ in $\overline{\Gamma}$, and f is Lipschitz continuous in $[\frac{m}{2}, 2m]$ so that the proof in [7] (which is based on Hopf's lemma in small balls touching the set Γ) goes through. □

The proof of Theorem 1.2 is based upon the following two lemmas. The first deals with the behaviour of the solution u near the boundary and exploits the Hopf's lemma, while the second gives conditions for continuing the moving plane procedure with respect to small variations of hyperplanes.

Lemma 2.2. (*i*) *There exists $\zeta_0 > 0$ such that for any direction ν we have*

$$u(x) < u^{\nu}_{\lambda}(x) \quad \forall\, x \in \Omega^{\nu}_{\lambda}\,, \quad a(\nu) < \lambda < a(\nu) + \zeta_0. \tag{2.10}$$

(*ii*) *For any direction ν_0 and any μ with $a(\nu_0) < \mu < \lambda_1(\nu_0)$, there exist $\delta_0 > 0$, $\varepsilon_0 > 0$ and a neighborhood I of $\partial\Omega$ such that*

$$u < u^{\nu}_{\lambda} \quad \text{in } \Omega^{\nu}_{\lambda} \cap I \quad \forall\, \lambda \in (\mu - \varepsilon_0, \mu + \varepsilon_0)\ ,\ \ \nu \in I_{\delta_0}(\nu_0). \tag{2.11}$$

Proof. (i) Suppose the contrary. Then there exist sequences $\{\nu_n\}$, $\{\lambda_n\}$, $\{x_n\}$ with $\{\nu_n\}$ directions, $a(\nu_n) < \lambda_n < a(\nu_n) + \frac{1}{n}$ and $x_n \in \Omega^{\nu_n}_{\lambda_n}$ such that $u(x_n) \geq u^{\nu_n}_{\lambda_n}(x_n) = u((x_n)^{\nu_n}_{\lambda_n})$. Hence for some points ξ_n in the segments connecting x_n to $(x_n)^{\nu_n}_{\lambda_n}$ we have $\frac{\partial u}{\partial \nu_n}(\xi_n) \leq 0$.

On the other hand, up to a subsequence, $\nu_n \to \nu_0$, $x_n \to x_0$, where $x_0 \in \partial\Omega$ and ν_0 is a direction with $x_0 \cdot \nu_0 = a(\nu_0)$. As a consequence $\frac{\partial u}{\partial \nu_0}(x_0) \leq 0$, which contradicts Hopf's lemma (Theorem 2.3).

(ii) As in (i) we proceed by contradiction, supposing that there exist sequences $\{\nu_n\}$, $\{\lambda_n\}$ and $\{x_n\} \subset \Omega^{\nu_n}_{\lambda_n}$ such that $\nu_n \to \nu_0$, $\lambda_n \to \mu$, $\operatorname{dist}(x_n, \partial\Omega) \to 0$ and $u(x_n) \geq u^{\nu_n}_{\lambda_n}(x_n)$. Up to a subsequence x_n converges to a point $x_0 \in \overline{\Omega^{\nu_0}_{\mu}} \cap \partial\Omega$.

If $x_0 \in \partial\Omega^{\nu_0}_{\mu} \setminus T^{\nu_0}_{\mu}$, by continuity we get $u(x_0) \geq u^{\nu_0}_{\mu}(x_0)$, which is impossible since $u(x_0) = 0$ while $u^{\nu_0}_{\mu}(x_0) > 0$ because $(x_0)^{\nu_0}_{\mu} \in \Omega$ by the condition $\mu < \lambda_1(\nu_0)$ and the definition of $\lambda_1(\nu_0)$.

If instead $x_0 \in \partial\Omega_\mu^{\nu_0} \cap T_\mu^{\nu_0}$, proceeding as in (i) we get $\frac{\partial u}{\partial \nu_0}(x_0) \le 0$, which contradicts Hopf's lemma (Theorem 2.3) since, again by the definition of $\lambda_1(\nu_0)$, the direction ν_0 is an inner direction with respect to $\partial\Omega$. □

In the sequel we will use the following notation:

$$M_\lambda^\nu(B) = \sup_B \left(|Du| + |Du_\lambda^\nu|\right).$$

Lemma 2.3. *Suppose that ν_0 is a direction and $a(\nu_0) < \mu < \lambda_1(\nu_0)$. There exist α, M, δ_0, $\varepsilon_0 > 0$, depending on μ, ν_0 (and on the other data of the problem) such that the following holds:*

Claim. *If $\nu' \in I_{\delta_0}(\nu_0)$, $\mu - \varepsilon_0 < \lambda' < \mu + \varepsilon_0$ and there exist open subsets A, B of $\Omega_{\lambda'}^{\nu'}$, such that*

$$u < u_{\lambda'}^{\nu'} \text{ in } \Omega_{\lambda'}^{\nu'} \setminus (A \cup B), \quad |A| < \frac{\alpha}{2}, \quad M_{\lambda'}^{\nu'}(B) < \frac{M}{2}, \tag{2.12}$$

then there exist in turn δ, ε, with $0 < \delta < \delta_0 - |\nu' - \nu_0|$, $0 < \varepsilon < \varepsilon_0 - |\lambda' - \lambda_0|$, such that

$$u \le u_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu \quad \text{for } |\lambda - \lambda'| < \varepsilon, \quad |\nu - \nu'| < \delta. \tag{2.13}$$

Proof. Choose δ_0, $\varepsilon_0 > 0$ and a closed neighborhood I of $\partial\Omega$ as in Lemma 2.2 so that (2.11) holds, i.e.,

$$u < u_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu \cap I \quad \forall\, \lambda \in (\mu - \varepsilon_0, \mu + \varepsilon_0)\,, \ \nu \in I_{\delta_0}(\nu_0),$$

and set $d_1 = \inf_{\Omega \setminus I} u$, $d_2 = \sup_\Omega u$. Since f is locally Lipschitz-continuous in $(0, \infty)$ there exists $c_1 \ge 0$ such that (2.3) holds in $[d_1, d_2]$. We take then α, $M > 0$ as in Theorem 2.1. They depend on d_1 and d_2, so they depend on I, which in turn depends on μ and ν_0, as well as on the other data of the problem. With these choices of α, M, δ_0, ε_0 the assertion is true.

In fact, suppose that $\nu' \in I_{\delta_0}(\nu_0)$, $\mu - \varepsilon_0 < \lambda' < \mu + \varepsilon_0$ and (2.12) holds. By (2.11) we have to prove only that (2.13) is satisfied in $\widetilde{\Omega_\lambda^\nu} = \Omega_\lambda^\nu \setminus I$ and we can assume that $A \cup B \subseteq \widetilde{\Omega_{\lambda'}^{\nu'}} = \Omega_{\lambda'}^{\nu'} \setminus I$.

Choose then a compact $K \subset \widetilde{\Omega_{\lambda'}^{\nu'}}$ such that $|\widetilde{\Omega_{\lambda'}^{\nu'}} \setminus K| < \frac{\alpha}{4}$. In the compact $K \setminus (A \cup B)$ we have $u_{\lambda'}^{\nu'} - u \ge m > 0$. By continuity there exist δ, ε, with

$0 < \delta < \delta_0 - |\nu' - \nu_0|$, $0 < \varepsilon < \lambda_0 - |\lambda' - \lambda_0|$ such that if $|\lambda - \lambda'| < \varepsilon$, $|\nu - \nu'| < \delta$, we have

$$u_\lambda^\nu - u \geq \frac{m}{2} \quad \text{in } K \setminus (A \cup B)\,, \quad |\widetilde{\Omega_\lambda^\nu} \setminus K| < \frac{\alpha}{2}\,, \quad M_\lambda^\nu(B) < M. \tag{2.14}$$

For these values of λ and ν, by (2.11) and (2.14) the inequality $u < u_\lambda^\nu$ holds in $\Omega_\lambda^\nu \cap I$ and in $K \setminus (A \cup B)$. So it suffices to show that $u \leq u_\lambda^\nu$ in $\widetilde{\Omega_\lambda^\nu} \setminus (K \setminus (A \cup B)) = O_\lambda^\nu$. This follows from the weak comparison principle (Theorem 2.1) because $u \leq u_\lambda^\nu$ on ∂O_λ^ν and O_λ^ν is the union of $(\widetilde{\Omega_\lambda^\nu} \setminus K) \cup (K \cap A)$, whose measure is less than α, and of $K \cap B$, where $|Du| + |Du_\lambda^\nu| < M$. □

As in [7], in order to get the full monotonicity and symmetry theorem, we first prove a preliminary result, which is an extension of Theorem 1.5 of [6].

Proposition 2.2. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1.1) *with f satisfying* (H2). *For any direction ν we have that $\Lambda_0(\nu) \neq \emptyset$ and, if $\lambda_0(\nu) < \lambda_1(\nu)$, then there exists at least one connected component C^ν of $\Omega_{\lambda_0(\nu)}^\nu \setminus Z_{\lambda_0(\nu)}^\nu$ such that $u \equiv u_{\lambda_0(\nu)}^\nu$ in C^ν. For any such component we also get*

$$Du(x) \neq 0 \quad \forall\, x \in C^\nu \tag{2.15}$$

$$Du(x) = 0 \quad \forall\, x \in \partial C^\nu \setminus T_{\lambda_0(\nu)}^\nu. \tag{2.16}$$

Moreover, for any λ with $a(\nu) < \lambda < \lambda_0(\nu)$, we have

$$u < u_\lambda^\nu \quad \text{in } \Omega_\lambda^\nu \setminus Z_\lambda^\nu \tag{2.17}$$

and finally

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \forall\, x \in \Omega_{\lambda_0(\nu)}^\nu \setminus Z. \tag{2.18}$$

Proof. By Lemma 2.2 (i) we immediately get that $\Lambda_0(\nu) \neq \emptyset$. Suppose that $\lambda_0(\nu) < \lambda_1(\nu)$ and u does not coincide with $u_{\lambda_0(\nu)}^\nu$ in any connected component of $\Omega_{\lambda_0(\nu)}^\nu \setminus Z_{\lambda_0(\nu)}^\nu$. Then by the strong comparison principle (Theorem 2.2) we have that $u < u_{\lambda_0(\nu)}^\nu$ in $\Omega_{\lambda_0(\nu)}^\nu \setminus Z_{\lambda_0(\nu)}^\nu$.

By Lemma 2.3 (with $A = \emptyset$, $\mu = \lambda_0(\nu) = \lambda'$, $\nu_0 = \nu = \nu'$, B a neighborhood of $Z_{\lambda_0(\nu)}^\nu$) there exists $\varepsilon > 0$ such that the inequality $u \leq u_\lambda^\nu$ holds in Ω_λ^ν for $\lambda \in (\lambda_0(\nu), \lambda_0(\nu) + \varepsilon)$, contradicting the definition of $\lambda_0(\nu)$.

To get (2.15) we observe that if $u \equiv u^\nu_{\lambda_0(\nu)}$, then $Du = Du^\nu_{\lambda_0(\nu)}$ in C^ν, and hence $Du \neq 0$ in C^ν, since $|Du| + |Du^\nu_{\lambda_0(\nu)}| > 0$ in C^ν by the very definition of $Z^\nu_{\lambda_0(\nu)}$. Moreover, from (ii) of Lemma 2.2 we deduce that $\partial C^\nu \cap \partial\Omega = \emptyset$, so that $\partial C^\nu \subset T^\nu_{\lambda_0(\nu)} \cup Z^\nu_{\lambda_0(\nu)}$ and (2.16) follows immediately.

Finally, the proofs of (2.17) and (2.18) are the same as those of the analogous inequalities deduced in Theorem 3.1 of [7]. □

3. Proof of Theorem 1.2.

Now we proceed to prove Theorem 1.2. Though the proof is similar to that of Theorem 1.1 of [7], in view of the hypothesis (H2) we need to make some changes. In doing that we also simplify a little the procedure contained in [7].

Let us first recall the following simple topological result, whose proof can be found in [7, Corollary 4.1].

Lemma 3.1. *Let A, B be open connected sets in a topological space and assume that $A \cap B \neq \emptyset$, $A \not\equiv B$. Then $(\partial A \cap B) \cup (\partial B \cap A) \neq \emptyset$.*

As usual let ν be a direction and define $\mathcal{F}_\nu$ as the collection of the connected components C^ν of $\Omega^\nu_{\lambda_0(\nu)} \setminus Z^\nu_{\lambda_0(\nu)}$ such that $u \equiv u^\nu_{\lambda_0(\nu)}$ in C^ν, $Du \neq 0$ in C^ν, $Du = 0$ on $\partial C^\nu \setminus (T^\nu_{\lambda_0(\nu)})$.

If $\lambda_0(\nu) < \lambda_1(\nu)$, then $\mathcal{F}_\nu \neq \emptyset$ by Proposition 2.2. If this is the case and $C^\nu \in \mathcal{F}_\nu$ there are two alternatives: either $Du(x) = 0$ for all $x \in \partial C^\nu$, in which case we define $\widetilde{C}^\nu = C$, or there are points $x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)}$ such that $Du(x) \neq 0$. In this latter case we define $\widetilde{C}^\nu = C^\nu \cup C^\nu_1 \cup C^\nu_2$ where $C^\nu_1 = R^\nu_{\lambda_0(\nu)}(C^\nu)$, $C^\nu_2 = \{x \in \partial C^\nu \cap T^\nu_{\lambda_0(\nu)} : Du(x) \neq 0\}$. It is easy to check that $\widetilde{C}^\nu$ is open and connected, with $Du \neq 0$ in $\widetilde{C}^\nu$, $Du = 0$ on $\partial\widetilde{C}^\nu$.

Let us finally define the collection $\widetilde{\mathcal{F}}_\nu = \{\widetilde{C}^\nu : C^\nu \in \mathcal{F}_\nu\}$.

Remark 3.1. An important remark for the sequel is the following: if ν_1, ν_2 are directions and $C^{\nu_1} \in \mathcal{F}_{\nu_1}$, $C^{\nu_2} \in \mathcal{F}_{\nu_2}$, then either $\widetilde{C}^{\nu_1} \cap \widetilde{C}^{\nu_2} = \emptyset$ or $\widetilde{C}^{\nu_1} \equiv \widetilde{C}^{\nu_2}$. In fact if $\widetilde{C}^{\nu_1} \cap \widetilde{C}^{\nu_2} \neq \emptyset$ and $\widetilde{C}^{\nu_1} \not\equiv \widetilde{C}^{\nu_2}$, then by Lemma 3.1 either $\partial\widetilde{C}^{\nu_1} \cap \widetilde{C}^{\nu_2}$ or $\partial\widetilde{C}^{\nu_2} \cap \widetilde{C}^{\nu_1}$ is nonempty, and this is not possible since $Du \neq 0$ in $\widetilde{C}^{\nu_i}$, $Du = 0$ on $\partial\widetilde{C}^{\nu_i}$, $i = 1, 2$.

Proof of Theorem 1.2. As in [7] the idea of the proof is to get a contradiction by showing that if for a direction ν_0 we have $\lambda_0(\nu_0) < \lambda_1(\nu_0)$, then it is possible to construct a set Γ as in Proposition 2.1.

Suppose now that $\lambda_0(\nu_0) < \lambda_1(\nu_0)$ for a direction ν_0. By Proposition 2.2 $\mathcal{F}_{\nu_0} \neq \emptyset$, and obviously it contains at most countably many components that we denote by $\mathcal{F}_{\nu_0} = \{C_i^{\nu_0}\ ,\ i \in \mathcal{I} \subseteq \mathbb{N}\}$.

We fix then α, M, ε_0, δ_0 as in Lemma 2.3 (with $\mu = \lambda_0(\nu_0)$), a compact $K \subset \Omega^{\nu_0}_{\lambda_0(\nu_0)}$ and an open neighborhood B of $Z^{\nu_0}_{\lambda_0(\nu_0)}$ such that

$$|\Omega^{\nu_0}_{\lambda_0(\nu_0)} \setminus K| < \frac{\alpha}{4}\ , \qquad M^{\nu_0}_{\lambda_0(\nu_0)}(B) < \frac{M}{4}.$$

Let us define the compact set $S_0 = ((K \setminus B) \setminus \cup_{i\in\mathcal{I}} C_i^{\nu_0})$ and observe that

$$u^{\nu_0}_{\lambda_0(\nu_0)} - u \geq m > 0 \qquad \text{in } S_0.$$

By continuity, taking eventually ε_0 and δ_0 smaller, we have that for $\lambda \in (\lambda_0(\nu_0) - \varepsilon_0, \lambda_0(\nu_0) + \varepsilon_0)\ ,\ \ \nu \in I_{\delta_0}(\nu_0)$:

$$K \subset \Omega^{\nu}_{\lambda} \tag{3.1}$$

$$|\Omega^{\nu}_{\lambda} \setminus K| < \frac{\alpha}{2} \tag{3.2}$$

$$M^{\nu}_{\lambda}(B) < \frac{M}{2} \tag{3.3}$$

$$u^{\nu}_{\lambda} - u \geq \frac{m}{2} > 0 \quad \text{ in } S_0, \tag{3.4}$$

and since the function $\lambda_1(\nu)$ is lower semicontinuous with respect to ν also

$$\lambda_0(\nu_0) + \varepsilon_0 < \lambda_1(\nu). \tag{3.5}$$

We now proceed in several steps in order to show that there exist $i \in \mathcal{I}$ and a direction $\nu_1 \in I_{\delta_0}(\nu_0)$ such that $\widetilde{C}_i^{\nu_0} \in \widetilde{\mathcal{F}}_{\nu}$ for any direction in a suitable neighbourhood $I_\delta(\nu_1)$ of ν_1, and $\partial C_i^{\nu_0}$ contains a set Γ as in Proposition 3.1 (with respect to the direction ν_1).

In what follows we implicitly assume that $\varepsilon > 0$ means $0 < \varepsilon \leq \varepsilon_0$, $\delta > 0$ means $0 < \delta \leq \delta_0$.

Step 1. *The function $\lambda_0(\nu)$ is continuous; i.e., for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\nu \in I_\delta(\nu_0)$, then*

$$\lambda_0(\nu_0) - \varepsilon < \lambda_0(\nu) < \lambda_0(\nu_0) + \varepsilon. \tag{3.6}$$

Proof of Step 1. Let $\varepsilon > 0$ be fixed. By the definition of $\lambda_0(\nu_0)$ there exist $\lambda \in (\lambda_0(\nu_0), \lambda_0(\nu_0) + \varepsilon)$ and $x \in \Omega_\lambda^{\nu_0}$ such that $u(x) > u_\lambda^{\nu_0}(x)$. By continuity there exists $\delta_1 > 0$ such that, for every $\nu \in I_{\delta_1}(\nu_0)$, x belongs to Ω_λ^ν and $u(x) > u_\lambda^\nu(x)$. This implies that for all $\nu \in I_{\delta_1}(\nu_0)$, we have

$$\lambda_0(\nu) < \lambda < \lambda_0(\nu_0) + \varepsilon.$$

Next we show that there exists $\delta_2 > 0$ such that $\lambda_0(\nu) > \lambda_0(\nu_0) - \varepsilon$ for any $\nu \in I_{\delta_2}(\nu_0)$. Suppose the contrary; then there exists a sequence $\{\nu_n\}$ of directions such that $\nu_n \to \nu_0$ and $\lambda_0(\nu_n) \leq \lambda_0(\nu_0) - \varepsilon$. By Lemma 2.2 we have that $\lambda_0(\nu_n) \geq a(\nu_n) + \zeta_0$. Up to a subsequence we have that $\lambda_0(\nu_n)$ converges to a number $\lambda' \in [a(\nu_0) + \zeta_0, \lambda_0(\nu_0) - \varepsilon]$.

By (2.17), we have that

$$u < u_{\lambda'}^{\nu_0} \quad \text{in } \Omega_{\lambda'}^{\nu_0} \setminus Z_{\lambda'}^{\nu_0},$$

so by Lemma 2.3 (with $\mu = \lambda'$, $\nu_0 = \nu'$, $A = \emptyset$, B a neighborhood of $Z_{\lambda'}^{\nu_0}$) there exist δ, $\varepsilon > 0$ such that the inequality $u \leq u_\lambda^\nu$ holds in Ω_λ^ν for $|\lambda - \lambda'| < \varepsilon$, $|\nu - \nu_0| < \delta$. This in particular holds for $\nu = \nu_n$, $\lambda = \lambda_0(\nu) + \eta$ for n sufficiently large and η sufficiently small, contradicting the definition of $\lambda_0(\nu_n)$.

Observe that, since we implicitly assume that $\varepsilon \leq \varepsilon_0$ and $\delta \leq \delta_0$, by (3.5) and (3.6) we have that $\lambda_0(\nu) < \lambda_1(\nu)$ $\forall\, \nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, and (3.1)–(3.4) hold for $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, $\lambda = \lambda_0(\nu)$.

Step 2. *There exists a direction ν_1 near ν_0, $\delta_1 > 0$ and $i_1 \in \mathcal{I}$ such that the set $\widetilde{C}_{i_1}^{\nu_0} \in \widetilde{\mathcal{F}}_\nu$ for any ν in $I_{\delta_1}(\nu_1)$.*

Proof of Step 2. Let us recall that for every $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$ we have

$$u_{\lambda_0(\nu)}^\nu - u \geq \frac{m}{2} > 0 \quad \text{in } S_0 = ((K \setminus B) \setminus \cup_{i \in I} C_i^{\nu_0}). \tag{3.7}$$

Let us set, for $i \in \mathcal{I}$, $S_i = (K \setminus B) \cap C_i^{\nu_0}$ and observe that since $\partial C_i^{\nu_0} \setminus T_{\lambda_0(\nu_0)}^{\nu_0} \subset B$, each S_i is a compact subset of $C_i^{\nu_0}$ and we have

$$K \setminus B = S_0 \cup (\cup_{i \in \mathcal{I}} S_i). \tag{3.8}$$

Let us take $\nu' \in I_{\delta(\varepsilon_0)}(\nu_0)$ and $i \in \mathcal{I}$. If there exists $x \in S_i$ such that $u(x) = u(x_{\lambda_0(\nu')}^{\nu'})$, since $|Du(x)| + |Du_{\lambda_0(\nu')}^{\nu'}(x)| > 0$, by the strong comparison principle (Theorem 2.2) we get that $u \equiv u^{\nu'}_{\lambda_0(\nu')}$ in some component $C^{\nu'}$ of $\Omega^{\nu'}_{\lambda_0(\nu')} \setminus Z^{\nu}_{\lambda_0(\nu)}$, and since $x \in C^{\nu'} \cap C^{\nu_0}_i$, by Remark 3.1 we get

$$\widetilde{C}^{\nu'} = \widetilde{C}^{\nu_0}_i \in \widetilde{\mathcal{F}}_{\nu_0} \cap \widetilde{\mathcal{F}}_{\nu'},$$

and in particular $u \equiv u^{\nu'}_{\lambda_0(\nu')}$ in S_i. The other possibility is that $u < u^{\nu'}_{\lambda_0(\nu')}$ in S_i, in which case, by the compactness of S_i, we have that for every ν in a neighbourhood of ν' we have the same inequality $u < u^{\nu}_{\lambda_0(\nu)}$ in S_i.

Take now $i = 1$. If $u \equiv u^{\nu}_{\lambda_0(\nu)}$ in S_1 for all $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, then as remarked $\widetilde{C}^{\nu_0}_1 \in \widetilde{\mathcal{F}}_{\nu}$ for all $\nu \in I_{\delta(\varepsilon_0)}(\nu_0)$, and the assertion is proved with $\nu_1 = \nu_0$.

If not, there exists $\nu_1 \in I_{\delta(\varepsilon_0)}(\nu_0)$ such that $u < u^{\nu_1}_{\lambda_0(\nu_1)}$ in S_1 and $C^{\nu_0}_1 \notin \mathcal{F}_{\nu_1}$; i.e., $u > u^{\nu_1}_{\lambda_0(\nu_1)}$ in S_i. By the continuity with respect to ν, $u < u^{\nu}_{\lambda_0(\nu)}$ in S_1 and $\widetilde{C}^{\nu_0}_1 \notin \widetilde{\mathcal{F}}_{\nu}$ for all $\nu \in I_{\delta_1}(\nu_1)$ for some $\delta_1 < \delta(\varepsilon_0) - |\nu_1 - \nu_0|$. Now if $u \equiv u^{\nu}_{\lambda_0(\nu)}$ in S_2, i.e., $\widetilde{C}^{\nu_0}_2 \in \mathcal{F}_{\nu}$, for all $\nu \in I_{\delta_1}(\nu_1)$ the assertion is proved.

If not, we find a direction $\nu_2 \in I_{\delta_1}(\nu_1)$ and a neighbourhood $I_{\delta_2}(\nu_2)$ with $\delta_2 < \delta_1 - |\nu_2 - \nu_1|$, such that $u <^{\nu}_{\lambda_0(\nu)}$ in S_2 and $C^{\nu_0}_2 \notin \mathcal{F}_{\nu}$ for all $\nu \in I_{\delta_2}(\nu_2)$. Proceeding in this way,

(i) either we stop after k iterations, i.e., $\widetilde{C}^{\nu_0}_k \in \widetilde{\mathcal{F}}_{\nu}\ \forall\, \nu \in I_{\delta_k}(\nu_k)$ and Step 2 is proved,

(ii) or the iterations are infinite.

Now we claim that (ii) cannot arise. In case (ii), we construct a sequence of compact sets $\{\overline{I_{\delta_i}(\nu_i)}\}_{i\in\mathcal{I}}$ which have the finite intersection property (since they are nested). Then by Cantor's intersection theorem $\cap_{i\in\mathcal{I}}\overline{I_{\delta_i}(\nu_i)}$ is nonempty and contains a direction ν' for which $C^{\nu_0}_i \notin \mathcal{F}_{\nu'}$ and $u < u^{\nu'}_{\lambda_0(\nu')}$ in S_i for all $i \in \mathcal{I}$. Thus by (3.7) and (3.8) we get $u < u^{\nu'}_{\lambda_0(\nu')}$ in $K \setminus B$. But then using Lemma 2.3 (with $A = \Omega^{\nu'}_{\lambda_0(\nu')} \setminus K$) we get that the inequality $u < u^{\nu'}_{\lambda}$ holds in $\Omega^{\nu'}_{\lambda}$ for some $\lambda > \lambda_0(\nu')$, contradicting the maximality of $\lambda_0(\nu')$. Hence only i) is possible and Step 2 is proved.

Step 3. *Let ν_1, i_1, δ_1 be as in Step 2 and set $C = C^{\nu_0}_{i_1}$. The set $\partial C \cap \Omega^{\nu_1}_{\lambda_0(\nu_1)}$ contains a subset Γ on which u is constant and whose projection on the hyperplane $T^{\nu_1}_{\lambda_0(\nu_1)}$ contains an open subset of the hyperplane. Since $Du = 0$ on $\partial C \cap \Omega^{\nu_1}_{\lambda_0(\nu_1)}$ this gives a contradiction with Proposition* 2.1 *and shows that $\lambda_0(\nu) = \lambda_1(\nu)$ for every direction ν.*

Proof of Step 3. It is exactly the same as in [7].

Having proved that $\lambda_0(\nu) = \lambda_1(\nu)$ for every direction ν, (1.13) follows from (2.18).

To prove the strict inequality (1.12) let us observe that if it does not hold, then the cap $\Omega^{\nu}_{\lambda_0(\nu)}$ must contain a segment parallel to the direction ν on which u is constant. In fact if there exists a direction ν' such that $u(\overline{x}) = u((\overline{x})^{\nu'}_{\lambda'}) = m$ for some $\overline{x} \in \Omega^{\nu'}_{\lambda'}$ and $\lambda' < \lambda_1(\nu')$, then, taking any $\lambda \in (\lambda', \lambda_1(\nu'))$ we would have that $(\overline{x})^{\nu'}_{\lambda'} = (\tilde{x})^{\nu'}_{\lambda}$ for some $\tilde{x} \in \Omega^{\nu'}_{\lambda}$. Hence by the monotonicity of u in the set $\Omega^{\nu'}_{\lambda}$ we would deduce $m = u(\overline{x}) \leq u(\tilde{x}) \leq u((\tilde{x})^{\nu'}_{\lambda}) = u((\overline{x})^{\nu'}_{\lambda'}) = m$ so that $u \equiv m$ on the segment connecting $\overline{x}$ with $\tilde{x}$. This is not possible as Proposition 3.1 below shows.

Finally the strict monotonicity of u in the ν–direction in the cap $\Omega^{\nu}_{\lambda_0(\nu)}$ is an immediate consequence of (1.12). $\square$

Proof of Corollary 1.2. The first part is an obvious consequence of Theorem 1.2, while the inequality $\frac{\partial u}{\partial r} < 0$ when Ω is a ball follows from Hopf's lemma as in [7]. $\square$

Let us now prove the property of the solutions of (1.1) that was used in the proof of Theorem 1.2 to get (1.12).

Proposition 3.1. *Let $u \in C^1(\overline{\Omega})$ be a weak solution of* (1.1) *and let ν_0 be a direction. There cannot exist in the cap $\Omega^{\nu_0}_{\lambda_1(\nu_0)}$ any segment parallel to the ν–direction on which u is constant.*

Proof. For simplicity we assume that $\nu_0 = e_1 = (1, \dots, 0)$ and omit the dependence on ν_0 in all notations (e.g. $\Omega^{\nu_0}_{\lambda_1(\nu_0)}$ will be denoted simply by Ω_{λ_1}).

Arguing by contradiction we suppose that there exists a point $x' \in \mathbb{R}^{N-1}$ such that the segment $I = \{(t, x') : t \in (a, b)\}$, $a < b < \lambda_1(e_1)$, is contained in Ω_{λ_1} and $u \equiv m > 0$ on I.

Let us first consider the case $f(m) \leq 0$.

Since $\lambda_1(\nu)$ is lower semicontinuous with respect to ν there exists $\delta > 0$ such that for any direction ν with $|\nu - e_1| < \delta$ the point $P = (b, x')$ lies in $\Omega^{\nu}_{\lambda_1(\nu)}$. Let K be the open cone made up of the segments parallel to these directions connecting P with $\partial\Omega$, and let O be the part of that cone where $u > \frac{m}{2}$. Since there are points on ∂O where $u = \frac{m}{2}$, it cannot be that $u \equiv m$ in O, and by the monotonicity of the solutions in the caps $\Omega^{\nu}_{\lambda_1(\nu)}$ we have that $u \leq m$ in O. On the other hand since f is locally Lipschitz-continuous and $f(m) \leq 0$ we have

$$\begin{aligned} -\Delta_p(u-m) + \Lambda(u-m) &= f(u) + \Lambda u - \Lambda m \\ &\leq f(u) + \Lambda u - (f(m) + \Lambda m) \leq 0 \quad \text{ in } O, \end{aligned} \tag{3.9}$$

where $\Lambda \geq 0$ is such that the function $f(s) + \Lambda s$ is nondecreasing in $\left[\frac{m}{2}, m\right]$. Hence by the strong maximum principle (Theorem 2.3) we get $u < m$ in O which is a contradiction since the segment I is contained in O and $u \equiv m$ on I.

In the case $f(m) > 0$ we argue in the same way and consider the open cone O' made up of segments parallel to the directions ν, for ν in a neighborhood of e_1, and connecting $Q = (a, x')$ with $T^{\nu}_{\lambda_1(\nu)}$. In this case m is the minimum of u in O'. If Λ' is a Lipschitz constant of f in $[m, \|u\|_\infty]$ we get

$$-\Delta_p(u-m) + \Lambda'(u-m) > 0 \qquad \text{in } O',$$

and u is not constant in the cone because $f(m) > 0$. So we reach again a contradiction using the strong maximum principle, because $u \equiv m$ on $I \subset O'$. □

4. Applications. We start by proving Theorem 1.3.

Proof of Theorem 1.3. The procedure is the same as in [9]. Under hypothesis (H1) Theorem 1.1 holds while under (H2) Theorem 1.2 is true. In both situations we have that, for any direction ν,

$$u(x) \leq u^{\nu}_{\lambda_1(\nu)}(x) = u(x^{\nu}_{\lambda_1(\nu)}) \quad \forall\, x \in \Omega^{\nu}_{\lambda_1(\nu)}, \tag{4.1}$$

$\lambda_1(\nu)$ being defined in (1.7). Thus, by the strong comparison principle (Theorem 2.2) in any connected component of $\Omega^{\nu}_{\lambda_1(\nu)} \setminus Z^{\nu}_{\lambda_1(\nu)}$ either $u < u^{\nu}_{\lambda_1(\nu)}$ or $u \equiv u^{\nu}_{\lambda_1(\nu)}$. Since, by (1.14) $|Du| \neq 0$ in a connected neighborhood I of $\partial\Omega$, one of the previous two alternatives must hold in $I \cap \Omega^{\nu}_{\lambda_1(\nu)}$.

It is obvious that if

$$u \equiv u^{\nu}_{\lambda_1(\nu)} \qquad \text{in } I \cap \Omega^{\nu}_{\lambda_1(\nu)}, \tag{4.2}$$

then Ω must be symmetric about the hyperplane $T^{\nu}_{\lambda_1(\nu)}$, so that changing arbitrarily the direction ν, we get that Ω is a ball.

To prove (4.2) we assume, for the sake of contradiction, that

$$u < u^{\nu}_{\lambda_1(\nu)} \qquad \text{in } I \cap \Omega^{\nu}_{\lambda_1(\nu)}. \tag{4.3}$$

Since $Du(x) \neq 0$ for any $x \in I \cap \Omega$, by standard results we have that $u \in C^2(I \cap \Omega)$ and the difference $u^\nu_{\lambda_1(\nu)} - u$ satisfies in $\Omega^\nu_{\lambda_1(\nu)} \cap I$ either

$$L(u^\nu_{\lambda_1(\nu)} - u) = 0 \tag{4.4}$$

in the hypothesis (H1), or

$$L(u^\nu_{\lambda_1(\nu)} - u) = f_2(u^\nu_{\lambda_1(\nu)}) - f_2(u) \geq 0 \tag{4.5}$$

in the hypothesis (H3), where L is some linear uniformly elliptic operator and f_2 is nondecreasing (see [9]). Moreover, by definition, at $\lambda_1(\nu)$ either condition (i) or (ii) of Section 1 holds.

In the first case $(\Omega^\nu_{\lambda_1(\nu)})'$ becomes internally tangent to $\partial\Omega$ at some point P' not on $T^\nu_{\lambda_1(\nu)}$. Thus, at the reflected point $P \in \partial\Omega \cap \overline{\Omega}^\nu_{\lambda_1(\nu)}$ we have $u^\nu_{\lambda_1(\nu)}(P) - u(P) = 0$, and hence, by (4.3) and (4.4) or (4.5), applying the usual form of Hopf's lemma we obtain

$$\frac{\partial u}{\partial n}(P) > \frac{\partial}{\partial n}\, u^\nu_{\lambda_1(\nu)}(P),$$

getting a contradiction with (1.14).

In the second case, i.e., when $T^\nu_{\lambda_1(\nu)}$ is orthogonal to $\partial\Omega$ at some point, a contradiction arises using the version of the Hopf's lemma at "corners" due to Serrin. Since, using (4.4) or (4.5) there are not changes with respect to the proof of [9], we refer the reader to this paper for the details. □

Now we would like to show how, as a consequence of the symmetry of the solutions of (1.1), one can find the exact value S^1 of the absolute isoperimetric constant in $\mathbb{R}^N$.

It is well known that S^1 is actually the best constant for the Sobolev embedding of the space $W^{1,1}(\mathbb{R}^N)$ in the space $L^{\frac{N}{N-1}}(\mathbb{R}^N)$, and it is achieved in the space $BV(\mathbb{R}^N)$ (which is also embedded in the space in $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ with the same best constant) by the characteristic functions of balls.

Let us also consider the best constants S^p for the embedding of $W^{1,p}(\mathbb{R}^N)$ into $L^{p^*}(\mathbb{R}^N)$, where $p^* = \frac{Np}{N-p}$, $1 < p < N$. Another well-known fact is that S^p is also the best constant for the embedding of $W^{1,p}_0(\Omega)$ into $L^{p^*}(\Omega)$ where Ω is any bounded domain, and hence its value does not depend on Ω. Let us then fix the ball B in $\mathbb{R}^N$ with $|B| = 1$. Hence,

$$S^p = \inf\left\{ \frac{\int_B |\nabla u|^p}{\left(\int_B |u|^{p^*}\right)^{\frac{p}{p^*}}} \ : u \in W^{1,p}_0(B)\, , u \not\equiv 0 \right\}. \tag{4.6}$$

Let us recall that the above infimum is never achieved because of the lack of compactness of the previous embedding.

Now we consider the problems

$$\begin{cases} -\Delta_p u = u^{q^*-\frac{1}{n}} & \text{in} \quad B \\ u > 0 & \text{in} \quad B \\ u = 0 & \text{on} \quad \partial B \end{cases} \tag{4.7}$$

$n \in \mathbb{N}$, $1 < p < 2$, $q^* = p^* - 1 = \frac{Np}{N-p} - 1$, $N \geq 2$. By virtue of Theorem 1.2 all solutions of (4.7) are radial and strictly radially decreasing. In particular this is true for the functions v_n^p which minimize the functional

$$J_n^p(u) = \frac{\int_B |\nabla u|^p}{\left(\int_B |u|^{p^*-\frac{1}{n}}\right)^{\frac{p}{p^*-\frac{1}{n}}}}. \tag{4.8}$$

Note that such solutions exist because the embedding of $W_0^{1,p}(B)$ into $L^{p^*-\frac{1}{n}}(B)$ is compact. Let us set

$$S_n^p = \inf_{W_0^{1,p}(B)\setminus\{0\}} J_n^p(u) = J_n^p(v_n^p). \tag{4.9}$$

We have:

Lemma 4.1. *For n sufficiently large we have that $S_n^p \geq S_{n+1}^p$ for every $p \in [1,2]$.*

Proof. For any $p \in [1,2]$, p^* is greater than 1, so that for n sufficiently large $p_n = p^* - \frac{1}{n} > 1$. If $1 < r < s < p^*$, using Hölder's inequality we get

$$\left(\int_B |u|^r\right)^{\frac{p}{r}} \leq |B|^{\frac{p}{r}-\frac{p}{s}} \left(\int_B |u|^s\right)^{\frac{p}{s}}$$

for any $u \in W_0^{1,p}(B)$. Since $|B| = 1$, from the previous inequality we get the assertion, taking $r = p_n$, $s = p_{n+1}$. □

Lemma 4.2.

$$\lim_{n\to\infty} S_n^p = S^p \tag{4.10}$$

Proof. Fixing $\varepsilon > 0$ we have that there exists $v_\varepsilon \in W_0^{1,p}(B)$ such that

$$J_{p^*}^p(v_\varepsilon) = \frac{\int_B |\nabla v_\varepsilon|^p}{\left(\int_B |v_\varepsilon|^{p^*}\right)^{\frac{p}{p^*}}} < S^p + \varepsilon.$$

Since $J_n^p(v_\varepsilon) \to J_{p^*}^p(v_\varepsilon)$ as $n \to \infty$, we get $S_n^p < S^p + 2\varepsilon$ for n sufficiently large. This, together with Lemma 4.1, gives (4.10). $\square$

In the same way it is easy to show that S_n^p and S^p are continuous functions of p. Hence if we consider a sequence $p_n \to 1$ we get that $S_n^{p_n} \to S^1$, and the functions $v_n^{p_n} \in C^1(\overline{B})$ give a minimizing sequence for S^1, i.e.,

$$\frac{\int_B |\nabla v_n^{p_n}|}{\left(\int_B |v_n^{p_n}|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}}} \to S^1 \quad \text{as } n \to \infty. \tag{4.11}$$

As remarked before the functions $v_n^{p_n}$ are radial and strictly radially decreasing because they solve (4.7) for $p = p_n$. Thus (4.11) allows us to restrict our attention to radial functions to determine the value of S^1. But for radial functions an elementary argument gives the best constant for the imbedding of $W^{1,1}(\mathbb{R}^N)$ into $L^{\frac{N}{N-1}}(\mathbb{R}^N)$ as shown in the following proposition.

Proposition 4.1. *For any $u \in W^{1,1}(\mathbb{R}^N)$, with $u(x) = U(|x|)$, we have*

$$\left(\int_{\mathbb{R}^N} |u(x)|^{\frac{N}{N-1}}\,dx\right)^{\frac{N-1}{N}} \le \frac{1}{N\omega_N^{\frac{1}{N}}} \int_{\mathbb{R}^N} |Du(x)|\,dx, \tag{4.12}$$

where ω_N is $|B|$.

From (4.11), (4.12) we get $S^1 \ge N\omega_N^{\frac{1}{N}}$, and using the characteristic function of B, which belongs to $BV(\mathbb{R}^N)$, we deduce that $S^1 = N\omega_N^{\frac{1}{N}}$.

Proof of Proposition 4.1. It suffices to show (4.12) for radial functions $u \in C_c^1(\mathbb{R}^N)$. Let us set $q = \frac{N}{N-1}$ so that $q' = N$ ($\frac{1}{q} + \frac{1}{q'} = 1$). We have, for $\rho \ge 0$, $U(\rho) = -\int_0^\infty U'(\rho + r)\,dr = -\int_\rho^\infty U'(s)\,ds$; hence $|U(\rho)| \le \int_\rho^\infty |U'(s)|\,ds$. Integrating with respect to the measure $\rho^{N-1}d\rho$ and using

Hölder's inequality and Fubini's Theorem we get

$$\int_0^\infty |U(\rho)|^q \rho^{N-1}\, d\rho \le \int_0^\infty \rho^{N-1} \left(\int_\rho^\infty |U'(s)|\, ds \right)^q d\rho =$$
$$\int_0^\infty \rho^{N-1} \left(\int_\rho^\infty \left[|U'(s)|^{\frac{1}{q}} s^{\frac{1-N}{q'}} \right] \left[|U'(s)|^{\frac{1}{q'}} s^{\frac{N-1}{q'}} \right] ds \right)^q d\rho \le$$
$$\int_0^\infty \rho^{N-1} \left(\int_\rho^\infty |U'(s)| s^{(1-N)(q-1)}\, ds \right) \left(\int_0^\infty |U'(s)| s^{N-1}\, ds \right)^{q-1} d\rho =$$
$$\left(\int_0^\infty |U'(s)| s^{N-1}\, ds \right)^{q-1} \left(\int_0^\infty |U'(s)| s^{(1-N)(q-1)} \left[\int_0^s \rho^{N-1} d\rho \right] ds \right) =$$
$$\frac{1}{N} \left(\int_0^\infty |U'(s)| s^{N-1}\, ds \right)^{q-1} \left(\int_0^\infty |U'(s)| s^{N-(N-1)(q-1)}\, ds \right).$$

Since $q = \frac{N}{N-1}$ we have that $q-1 = \frac{1}{N-1}$ and $N-(N-1)(q-1) = N-1$, so that we have

$$\int_0^\infty |U(\rho)|^q \rho^{N-1}\, d\rho \le \frac{1}{N} \left(\int_0^\infty |U'(s)|\, s^{N-1}\, ds \right)^q, \tag{4.13}$$

i.e.,

$$\frac{1}{N\omega_N} \int_{\mathbb{R}^N} |u(x)|^q\, dx \le \frac{1}{N} \left(\frac{1}{N\omega_N} \right)^q \left(\int_{\mathbb{R}^N} |Du(x)|\, dx \right)^q, \tag{4.14}$$

which is the same as (4.12) because $1 - \frac{1}{q} = \frac{1}{N}$. □

The previous procedure allows us to assert that to determine also the other best constants S^p $(1 < p < 2)$ we can consider only radial and strictly radially decreasing functions. In fact, the sequence $\{v_n^p\}$ is a minimizing sequence for S^p. Then to find the exact value of S^p one can use an inequality due to Bliss [2], as done in [1] or [10].

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