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# DISSIPATIVE SYSTEMS GENERATING ANY STRUCTURALLY STABLE CHAOS

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**Abstract.** Reaction-diffusion and coupled oscillator systems are considered. They possess inertial manifolds and, moreover, in a sense, their inertial dynamics can be "controlled" by some system parameters $\mathcal{P}$: the inertial dynamics can be specified to within an arbitrarily small error by adjusting of $\mathcal{P}$. Due to the classical persistence hyperbolic set theorem, this property of their inertial forms yields that any hyperbolic local attractors and invariant sets can be embedded into the global attractors of these systems. The method of the proof is based on recent results from the neural network theory and the ideas connected with the dynamics of localized modes in singular perturbed systems. This approach can be considered as a development of the method of the realization of vector fields pioneering by P. Poláčik and has a physical interpretation. It is shown, in particular, that the fundamental models of neural network type (the Hopfield systems) can generate any structurally stable (persistent) large-time behaviour.

**1. Introduction.** The reaction-diffusion systems describe a wide number of phenomena in biology, ecology, physics and chemistry (for example, [45, 23, 36, 17, 25, 47, 34, 35] among many others). An important problem about these systems is the investigation of the global attractor structure and the inertial dynamics. The concept of inertial manifold was introduced in 1985 by C. Foias, G. R. Sell and R. Temam, in the context of infinite dimensional systems [19]. In recent years, attention has been given to estimates of the global attractor dimension ([18, 3, 37, 33, 24, 8, 59]) and the inertial manifold existence ([9, 39, 38, 40, 59] and references therein). Recently also some important generalizations were introduced, for example, inertial manifolds with delay [13]. These results show the existence of finite-dimensional global attractors and inertial dynamics for wide classes of dissipative systems and in particular for reaction-diffusion systems. Detailed investigation of the global

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attractor structure is possible, however, only under essential restrictions. The attractor structure and the large-time behaviour have been well studied for the quasilinear parabolic equations, monotone and gradient-like systems ([3, 30, 16, 48, 49, 58, 59, 67, 41] among others). In these cases, the dynamics and global attractor structure are, in a sense, simple (at least, for "generic" initial data and "generic" systems).

An open important and intriguing question (which arises naturally from the work of D. Ruelle and F. Takens [54]—see also [44]) is to find an analytic proof of the strange attractor's existence and to investigate situations in which the global attractor structure and inertial dynamics are "complex."

To solve this problem, we consider here some reaction-diffusion and coupled oscillator systems of very simple forms, important for applications and having the following remarkable properties.

**A)** These systems generate global semiflows $S^t_{\mathcal{P}}$ in an ambient Hilbert or Banach phase space $H$. These semiflows depend on some parameters $\mathcal{P}$ (which could be elements of another Banach space $B$). They have inertial manifolds and thus global attractors, at least for some $\mathcal{P}$.

**B)** Inertial dynamics of $S^t_{\mathcal{P}}$ are, in a sense, "almost completely controllable." It can be described as follows. Assume the differential equations

$$\frac{dp}{dt} = Q(p), \quad Q \in C^1(\mathcal{B}^n) \tag{1.0}$$

define a dynamical system in the unit ball $\mathcal{B}^n \subset \mathbb{R}$.

For any prescribed dynamics (1.0) and any $\delta > 0$, we can choose suitable parameters $\mathcal{P} = \mathcal{P}(n, Q, \delta)$ such that

**B1)** The semiflow $S^t_{\mathcal{P}}$ has a $C^1$-smooth inertial manifold $\mathcal{M}_{\mathcal{P}}$ diffeomorphic to $\mathcal{B}^n$;

**B2)** The inertial dynamics $S^t_{\mathcal{P}}|_{\mathcal{M}_{\mathcal{P}}}$ is defined by the equations

$$\frac{dp}{dt} = \tilde{Q}(p, \mathcal{P}), \quad \tilde{Q} \in C^1(\mathcal{B}^n), \tag{1.1}$$

where the estimate $|Q - \tilde{Q}|_{C^1(\mathcal{B}^n)} < \delta$ holds. In other words, one can say that, by $\mathcal{P}$, the inertial dynamics can be specified to within an arbitrarily small error.

Thus, all robust dynamics (stable under small perturbations) can occur as inertial forms of these systems. Such structurally stable dynamics can be "chaotic." There is a rather wide variation in different definitions of "chaos." In principle, one can use here any concept of chaos. If this chaos is stable under small $C^1$-perturbations this kind of chaos occurs in the dynamics of our systems. To fix ideas, we shall use here, following classical tradition [14, 56, 57, 44, 65], such a definition. We say that finite-dimensional dynamics (1.1) is chaotic if this generates a nonquasiperiodic hyperbolic invariant set $\Gamma$. If, moreover, this set $\Gamma$ is attracting we say that $\Gamma$ is a chaotic (strange) attractor. (For definition of hyperbolic sets, see [14, 53, 46].) In this paper, we use only the following basic property of hyperbolic sets, so-called persistence ([1, 14, 56, 57, 53, 46]). This means that the hyperbolic sets are, in a sense, stable (robust): if (1.0) generates the hyperbolic set $\Gamma$ and $\delta$ is sufficiently small, then (1.1) also generates another hyperbolic set $\tilde{\Gamma}$. Dynamics (1.0) and (1.1) restricted to $\Gamma$ and $\tilde{\Gamma}$, respectively, are topologically orbitally equivalent (for the definition of this equivalence, see [14, 53] and Theorem 1.1 below).

Thus, any kind of the chaotic hyperbolic sets can occur in the dynamics of our systems, for example, the Smale horseshoes, Anosov flows, the Ruelle-Takens-Newhouse chaos, etc. [1, 14, 56, 57, 44].

The paper gives the example of a simple reaction-diffusion system with the property **B** and a constructive method that allows us, given $n$, $Q$, $\Gamma$ and $\delta$, to choose parameters $\mathcal{P} = \mathcal{P}(n, Q, \Gamma, \delta)$.

Before we start we shall recall some main definitions and facts. A semiflow (semigroup) $S^t$ in the Hilbert space $H$ is dissipative if there exists a bounded set $\mathcal{W} \subset H$ such that orbits $S^tB$ of bounded sets $B$ enter for and remain in it ($S^tB \subset \mathcal{W}$ for $t > t_0(B)$ [24, 59]). This globally attracting set $\mathcal{W}$ is the so-called absorbing set.

We say that a set $B_0$ is a (locally) attracting set if there exists an open neighborhood $V$ of $B_0$ such that $dist(S^tu, B_0) \to 0$ as $t \to \infty$ if the initial point $u \in V$.

Given a semigroup $S^t$, $t \geq 0$, an inertial manifold for this semigroup (semiflow) is a finite-dimensional Lipshitz manifold $\mathcal{M}$ having the following properties:

-$\mathcal{M}$ is locally positively invariant in the absorbing set;

-$\mathcal{M}$ attracts all the orbits with an exponential speed.

**Remark**. In many cases one can show, under some conditions, that the inertial manifold has $C^1$-smoothness; see, for example, [7]. In this paper, for some particular systems, we shall construct a $C^1$-manifold, and this smoothness will play an important role. Indeed, this property gives us the possibility of using the key results of differential dynamics on the robustness mentioned above. They hold only for $C^1$-smooth dynamical systems. Given a semigroup $S^t, t \geq 0$, a global attractor is a minimal invariant closed set $\mathcal{A}$ attracting of orbits $S^tB$ of all bounded sets $B \in H$ [24, 59] ($dist(S^tB, \mathcal{A}) \to 0$ as $t \to \infty$).

Let us describe this class of the reaction-diffusion systems with property **B**. Consider, in the two-dimensional box $\Omega = [0,1] \times [0,1]$, the following system:

$$u_t = \Delta u + w(x)u + f_0(x,y) + f_1(x,y)v, \quad u(x,y,0) = u^0(x,y) \tag{1.2}$$

$$v_t = \Delta v - a^2 v + g(x,y)\Phi(u), \quad v(x,y,0) = v^0(x,y), \quad (x,y) \in \Omega, \tag{1.3}$$

where the functions $f_i, g$ and $w$ belong to $C^2$, $a > 0$.

Let us set the Neumann boundary condition for $u$ and $v$:

$$\frac{\partial u}{\partial \nu}(x,y,t) = 0, \quad \frac{\partial v}{\partial \nu}(x,y,t) = 0 \qquad for\ (x,y) \in \partial\Omega. \tag{1.4}$$

Concerning $\Phi$ the following condition is required throughout this work:

$$\Phi'_u(u) \in S(\mathbb{R}), \quad \Phi_+ = \lim_{u \to +\infty} \Phi(u) > \lim_{u \to -\infty} \Phi(u) = \Phi_-, \tag{1.5}$$

where $S(\mathbb{R})$ denotes the Schwartz class of fast-decreasing functions (example: $\Phi(u) = \tanh u$).

Let us fix some function $\Phi$ satisfying (1.5). Denote by $\langle\ ,\ \rangle$ and $||\ ||$ the inner product and norm in $H = L_2(\Omega)$. Let $H_\alpha$ be the standard spaces of fractional powers associated with the operator $-\Delta + I$ (for a definition, see for example [29, 24, 59]). Problem (1.2)–(1.4) is well posed (which will be shown in Section 4) and defines global dissipative semiflows $S^t$ in the phase spaces $\mathcal{H}_\alpha = H_\alpha \times H_\alpha$ (with $\alpha > 0$). Let us denote $\mathcal{H} = \mathcal{H}_0 = L_2(\Omega) \times L_2(\Omega)$.

The system parameters are given by the functions $f_i, g$ and $w$ and the number $a$: $\mathcal{P} = (f_0, f_1, g, w, a)$. To embed a given hyperbolic set into the global attractor of (1.2)–(1.4), we shall adjust these parameters. This proceeds in two steps. First we shall show that the property **B** holds for some auxiliary systems of coupled oscillators.

The following system is the famous Hopfield system [31, 66]:

$$\frac{dq_i}{d\tau} = \sum_{j=1}^{m} K_{ij}\Phi(q_j) - q_i + \theta_i, \quad q \in \mathbb{R}^m, \tag{1.6}$$

where the parameters $\mathcal{P}$ of (1.6) are the matrix $\mathbf{K}$ with the elements $K_{ij}$, the vector $\theta$ and the number of neurons $m$. Notice that, under condition (1.5), system (1.6) defines a dissipative global semiflow $S_H^t$.

This system is the basic simplest model of the attractor neural network [31]. During the last decade, interest in such models has grown considerably, and the questions at hand are of great fundamental importance in brain-function modeling (see, for example, [66]).

The first main result of this paper can be formulated as follows.

**Theorem 1.1.** *Suppose equations* (1.0) *define a dynamical system* $S_Q^t$, $t \geq 0$ *in the* $n$*-dimensional unit ball* $\mathcal{B}^n$. *Assume moreover that this semiflow generates a hyperbolic set* $\Gamma$. *Then the parameters* $\mathcal{P}$ *of the Hopfield system* (1.6) *can be chosen so that*

**a)** *System* (1.6) *has the inertial manifold* $\mathcal{M}_H$ *diffeomorphic to the ball* $\mathcal{B}^n$*;*

**b)** *The inertial dynamics of this system* $S_H^t|_{\mathcal{M}_H}$ *generate a hyperbolic set* $\tilde{\Gamma}$. *The restricted dynamics* $S_Q^t|\Gamma$ *and* $S_H^t|\tilde{\Gamma}$ *are orbitally topologically equivalent;*

**c)** *If* $\Gamma$ *is an attracting set (local attractor) for semiflow* (1.0), *then* $\tilde{\Gamma}$ *also is a local attractor for* $S_H^t$.

Let us recall what topological orbital equivalence means: there exists a homeomorphism $h : \Gamma \to \tilde{\Gamma}$ mapping trajectories ${S_Q}^t|\Gamma$ onto ones ${S_H}^t|\tilde{\Gamma}$; see [14, 53].

We use this result to obtain the following Theorem 1.2. (Thus, we can observe an unexpected connection between the reaction-diffusion systems and the fundamental models of the neural network theory.)

**Theorem 1.2.** *Suppose equations* (1.0) *define a dynamical system* (*global semiflow*) $S_Q^t$, $t \geq 0$, *in the* $n$*-dimensional unit ball* $\mathcal{B}^n$ *and this dynamics has a hyperbolic set* $\Gamma$. *Then there exists a choice of parameters* $\mathcal{P}$ *of problem* (1.2)–(1.4) *such that*

**a)** *This problem is well posed and defines a dissipative global semiflow* $S_{\mathcal{P}}^t$ *in some Hilbert space. This semiflow has the inertial manifold* $\mathcal{M}_{\mathcal{P}}$, *which is diffeomorphic to the ball* $\mathcal{B}^n$*;*

**b)** *The inertial dynamics of this system* $S_{\mathcal{P}}^t|_{\mathcal{M}_{\mathcal{P}}}$ *generate a hyperbolic set* $\tilde{\Gamma}$. *The restricted dynamics* $S_Q^t|\Gamma$ *and* $S_{\mathcal{P}}^t|\tilde{\Gamma}$ *are orbitally topologically equivalent;*

**c)** *If* $\Gamma$ *is a local attractor for semiflow* (1.0), *then* $\tilde{\Gamma}$ *is a local attractor for* $S_{\mathcal{P}}^t$.

Let us notice that, in a sense, Theorem 1.2 cannot be improved: it is impossible to obtain analogous results for quasilinear parabolic equations of the second order. Indeed, it is well known that for "generic" reaction-diffusion equations the global attractor structure can be studied [3]. In this case, this attractor is a union of hyperbolic rest points together with the corresponding unstable manifolds. Thus, the appearance of complicated hyperbolic sets is impossible.

Even if the nonlinearity contains $\nabla u$ and a periodical dependence on time $t$, one obtains, under some conditions of regularity, that "almost all" trajectories converge to cycles or equilibria ([51, 52]). It can be shown by the theory of monotone flows and maps in Banach spaces (see the pioneering work [30] and review in [58]).

The complicated hyperbolic sets can appear in dynamics of very special quasilinear parabolic equations of second order. This nontrivial fact was obtained by the so-called method of "realization of vector fields" developed by the works [48, 49, 50, 11, 55]. In this case, however, the property **c)** does not hold, and thus complicated large-time behaviour is possible only for special initial data. For "almost all" initial data the large-time behavior is simple: all the trajectories are periodic or convergent. "Any invariant set with complicated dynamics is automatically unstable (due to strong monotonicity), thus, can be hardly observed" ([50, p. 26]). In [50], P. Poláčik has mentioned as well that the method of realization of vector fields gives no particular examples of physically motivated quasilinear parabolic equations with complicated dynamics.

The method of the given paper uses some ideas of the works [48, 49, 60, 61]. The results were announced in [64]; in [63] similar ones have been obtained for more complicated systems.

Here we shall study the systems (generating any hyperbolic sets) of the simple form (1.2)–(1.3) that allows us to drastically simplify an intricate proof from [63] and to find a nontrivial connection with the basic models of neural network theory. As it seems, system (1.2), (1.3) also has a connection with fundamental models of the layered pattern formation and the phase transitions (for example, such models are considered in [17, 35, 47]) and describes physical effects which actually can exist. We shall discuss this below.

Let us notice, in addition, that for quasilinear parabolic equations of the second order, Poláčik's technique is based on a very refined, nonexplicit and complex constructions of some special linear operators. Up to now, we did not know how one could essentially simplify these nontrivial constructions. In opposition to papers [48, 49, 50], here we study systems of two equations that give us a large additional freedom. This circumstance plays a crucial role in application of the approaches [48, 49, 50, 55].

Let us describe an outline of the proof. The method is constructive and gives an algorithm that allows us to embed a prescribed hyperbolic set into the global attractor of (1.2)–(1.4).

The proof of Theorem 1.1 proceeds as follows.

**Step 1**. Given system (1.0) with the hyperbolic set $\Gamma$, one can define, by the persistence hyperbolic set theorem [14, 53], a positive number $\kappa(Q, \Gamma)$.

If a $C^1$-vector field $\tilde{Q}$ satisfies

$$|Q - \tilde{Q}|_{C^1(\mathcal{B}^n)} < \delta < \kappa(Q, \Gamma) \tag{1.7}$$

then the perturbed system (1.0) with the right-hand side $\tilde{Q}$ also generates hyperbolic set $\tilde{\Gamma}$ (and the corresponding restricted dynamics are topologically orbitally equivalent).

**Step 2**. We show that, for any given $n > 0$, system (1.6) can have the $n$-dimensional inertial manifold $\mathcal{M}_H$ (diffeomorphic to the unit ball). To make it, we use the substitution $\mathbf{K} = \mathbf{AB}$ where $\mathbf{A}$ and $\mathbf{B}$ are, respectively, $m \times n$ and $n \times m$ matrices. Here the number $n$ is fixed: it defines the dimension of prescribed system (1.0). On the contrary, the number $m$ is an unknown parameter, which should be adjusted, together with $\mathbf{A}$ and $\mathbf{B}$. This substitution generalizes the well-known (in the neural network theory) one $\mathbf{K} = \mathbf{AA}^{tr}$ and allows us to find the inertial form explicitly.

Moreover, by a suitable choice of parameters $\mathcal{P} = (\mathbf{A}, \mathbf{B}, m, \theta)$, one can prove that this inertial form $\tilde{Q}(q, \mathcal{P})$ of (1.6) satisfies (1.7) with $\delta = \frac{1}{2}\kappa$. It can be shown by the special approximation lemma (which is a variant of the well-known results [32, 26, 20, 10, 4]).

Finally, this step completes the proof of Theorem 1.1. Moreover, we obtain the concrete values of parameters $m(Q, \Gamma)$, $\mathbf{K}(Q, \Gamma)$ and $\theta(Q, \Gamma)$ of system (1.6).

Let us outline essential elements of the proof of Theorem 1.2 by Theorem 1.1. The key idea in this case is to obtain weakly perturbed equations (1.6) as an $m$-dimensional inertial form of (1.2)–(1.4). We should overcome the following basic difficulties. In fact, we know [38, 40, 59, 13] that, for multidimensional general reaction-diffusion systems, the proof of the inertial manifold existence is a complicated problem, and, moreover, it is difficult to describe the inertial dynamics by explicit differential equations.

To overcome this obstacle, we shall study here systems (1.2)–(1.4) with *a special structure* defined by *a special choice of the functions $w$ and $g$*.

**Step 3**. We choose $w$ so that the spectra of the corresponding linear operators $A$ (where $Au = \Delta u + wu$) can be investigated in a simple way (by results [5, 6]). The form of the potential $w = w_m(x, \epsilon)$ depends on the number $m \geq n$ (which is obtained by steps 1, 2) and an auxiliary small parameter $\epsilon$. This potential consists of $m$ strongly localized potential wells at some points $x = x_i$. The spectrum admits a natural splitting: there exist "almost" eigenfunctions $\psi_i(x, \epsilon)$ such that $||A\psi_i + c\epsilon^{\mu}\psi_i|| < c_1 \exp(-c_2\epsilon^{-1})$ for small $\epsilon$ where $\mu > 2$ and, if $\langle u, \psi_i \rangle = 0$ for every $i$, then $\langle Au, u \rangle \leq -c||u||^2$ for small $\epsilon$.

These operators restricted to the subspace of $L_2(\Omega)$ orthogonal to all $\psi_i$ have spectral barriers depending only on $m$ and not depending on $\epsilon$ as $\epsilon \to 0$. This helps us to apply the standard results on the central and inertial manifold existence [7, 19, 29, 38, 59].

**Step 4.** Here we choose, in a special manner, the coefficients $f_i$ and $g$. Suppose $f_i$ and $g$ also depend on the same auxiliary parameter $\epsilon$. Let us set

$$f_j = \epsilon^{\mu_j} \tilde{f}_j(x, y), \quad |f_j|_{C^2} < \tilde{C}, \quad j = 1, 2 \tag{1.8}$$

and

$$g = \sum_{i=1}^{m} \exp(-\epsilon^{-2\rho}((x - x_i)^2 + (y - y_i)^2)), \quad \rho > 2, \tag{1.9}$$

where $\mu_i$ are positive numbers and $m$ is the number of potential wells. The points $x_i$ are the same ones that define the localization of these wells. The points $y_i$ can be chosen arbitrarily, but they should be different: $y_i \neq y_j$ if $i \neq j$.

Such construction allows us, for sufficiently small $\epsilon$ and appropriate $\mu_i$ and $\rho$, to demonstrate the existence of the inertial manifold. In fact, after some transformations, we obtain a standard nonlinear problem with smooth nonlinearities, where fast and slow modes are separated by a spectral barrier.

The quite straightforward a priori estimates show that the absorbing set $\mathcal{W}$ exists and inside it, the nonlinearities are very small: they have the order $O(\epsilon^s)$ where $s$ grows as a linear function of $\mu_i$ for fixed $\rho$. Thus, since the spectral barrier does not vanish as $\epsilon \to 0$, classical results ([7, 29, 59] and others) show now that the inertial manifold exists.

Here the key point is that, for special $g$ defined by (1.9), the inertial form of (1.2)–(1.4) coincides (up to small corrections) with the Hopfield equations (1.6).

**Final step**. The next key point is as follows. (Notice that all previous construction also holds for some one-dimensional variants of (1.2)–(1.4); but this last step can be done only in the two-dimensional case.)

For any given $m$ and $g$ from (1.9), and for any parameters $\mathcal{P} = (\mathbf{K}, \theta)$ in equations (1.6), these equations can be obtained as an inertial form of (1.2)–(1.4) (up to small corrections $\xi_i(q, \epsilon)$), under a suitable choice of coefficients $\tilde{f}_i(x, y)$ and the number $a$.

These small corrections $\xi_i$ to (1.6) can be evaluated (in $C^1$-norm) by $\frac{1}{4}\kappa$ inside some ball $\mathcal{B}^n(R)$, with the radius $R$ independent of $\epsilon$ as $\epsilon$ tends to zero. Thus, Theorem 1.2 follows from Theorem 1.1 and the persistence of hyperbolic sets.

To conclude this introduction, let us describe some connection of equations (1.2), (1.3) with fundamental physical systems. This also will be useful in order to explain our formula for the coefficient $w(x)$ (see (3.1) and (3.2) below). Consider the Ginzburg-Landau equation

$$u_t = \epsilon^2 \Delta u + 2(u - u^3), \tag{1.10}$$

where $u$ is the so-called "order parameter" (the coefficient 2 is introduced to simplify some notations below). This equation is of prototype character for pattern formation in dissipative nonequilibrium systems [23]. A number of works ([5, 6, 21] among others) were focused on some important special solutions of (1.10). From the physical point of view, they are chains of moving interfaces (kinks, domain walls). More precisely, for the zero Neumann conditions, these solutions define the interfaces strongly localized (as $\epsilon \to 0$) at some lines $x = x_i(t)$ where $0 < x_1(t) < x_2(t) < \cdots < x_m(t) < 1$. They have the following form [21] (up to exponentially small corrections $O(\exp(-c\epsilon^{-1}))$:

$$U_{\epsilon,m}(x, x_1, x_2, \ldots, x_m) = (-1)^{i+1} \tanh(\epsilon^{-1}(x - x_i)), \quad (x \in [\bar{x}_{i-1}, \bar{x}_i]), \tag{1.11}$$

where $i = 0, 1, \ldots, m$ and $x_0 = -x_1, x_{m+1} = 2 - x_m$, $\bar{x}_i = (x_i + x_{i+1})/2$. The time evolution of the coordinates $x_i = x_i(t)$ are defined by some ordinary differential equations [5, 6, 21].

More sophisticated models of phase transitions and of the layered pattern formation are given by systems of two equations [17, 34, 35, 47]. Such systems can describe a dissipative nonlinear medium whose state is defined by the order parameter $u$ and an additional field $v$ (for instance, the temperature). Usually the equation for the order parameter $u$ is singularly perturbed, in contrast to the equation for $v$.

Systems of such structure exhibit a number of nontrivial effects (see [34, 35]). For example, it is known that spatial inhomogeneities in such media (for example, pointed impurities or defects) give rise to the effect of the interface formation at these defects [35]. Then one can expect that such localized modes (interfaces) can oscillate periodically, quasiperiodically or even chaotically in time around these impurities (defects).

From the physical point of view, system (1.2), (1.3) can be considered as a simplified model to describe this effect. It can have the following interpretation:

1) Let us take the Ginzburg-Landau equation (1.10) for $u$ and the heat transfer equation for $v$ and let us add some terms that determine spatial

inhomogeneities. In particular, the term $g\Phi(u)$ (where $g$ is given by (1.9)) describes strongly localized defects which define a nonlinear interaction between fields $u$ and $v$.

2) Supposing that the interface-defect interaction is weak, and thus leads to small oscillations of the interface positions around $x_i$, we can linearize the term $2(u-u^3)$ about the solution $U_{\epsilon,m}$ given of (1.11). Therefore, the term $2(u-u^3)$ can be replaced by the linear term $\tilde{V}_m(x,\epsilon)u$ where the potential $\tilde{V}_m$ consists of $m$ localized potential wells:

$$\tilde{V}_m(x,\epsilon) = -4 + 6\cosh^{-2}((x-x_i)/\epsilon), \quad (x \in [\bar{x}_{i-1}, \bar{x}_i]). \tag{1.12}$$

This formula, up to exponentially small corrections, coincides with (3.1); see Section 3. The operator $\epsilon^2\Delta u + \tilde{V}_m u$ is well studied; see, for instance [5, 6].

This means that we simplify typical models of the phase transitions by linearization: instead of the movement of the nonlinear localized modes (interfaces), we shall study small oscillations of linear localized modes (which are the eigenfunctions of the operator $\epsilon^2\Delta u + \tilde{V}_m$ with exponentially small eigenvalues).

In the framework of such interpretation, our mathematical results mean the following. In the simplest model, (1.10), the interfaces interact exponentially weakly [5, 6, 21]. In the model (1.2), (1.3), the linear localized modes slowly oscillate around $x_i$ under the small perturbation $f_0 v + f_1$ in (1.2). On the other hand, for large times, the field $v$ itself is completely defined by the localized mode states. Thus, there arises a complicated nonstraight, nonlocal interaction between these localized modes through the intermediate field $v$ having the order $O(\epsilon^s)$. This small nonlocal interaction (which completely defines the state of our system for large times) is the same as the neuron interaction in the neural network.

It is interesting to note that the idea of localization plays a key role in this paper. The large-time behaviour of our system is defined by the time evolution of the localized modes, the potential $w$ consists of the well-localized wells and the coefficient $g$ also is a sum of strongly localized gaussian peaks. We shall see in Section 8 that the coefficient $f_1$ also can be chosen as a sum of finite, smooth functions with small supports. At last, in the appendix, we prove the approximation lemma by the classical famous Calderón identity, which is a basic ingredient of wavelet analysis and allows us to present a given function as a sum of the well-localized terms [2, 42, 12].

**Remark**. There exist reaction-diffusion models that are close to canonical equations from [47] and where the interfaces are nonlinear and defined by

(1.10). In this case, by the methods of the works [5, 6, 21], one can show that, as $t \to \infty$, the interface motion also can be reduced to the Hopfield equations. Recently, another approximation of any Hopfield neural dynamics by simple singular perturbed reaction diffusion systems was suggested in [15] (see also references therein). However, in these cases, the inertial manifold is not described yet. In the given paper, the main attention is focused on the inertial manifold structure, and we leave other systems (where this structure is unknown) aside.

The paper is organized as follows. Section 2 gives the proof of Theorem 1.1 (the proof of the auxiliary lemma is given in the appendix).

Section 3 contains some preliminaries (the investigation of the spectrum of linear operators $A$).

Section 4 shows that problem (1.2)–(1.4) is well posed and defines a global semiflow $S^t$.

In Section 5 we define the fast and slow modes and transform equations (1.2)–(1.4) into a standard form that allows us to use the classical results ([19, 7, 29, 39, 40, 59]).

Section 6 investigates the dissipative properties of this transformed system. It will be shown that the dynamics (1.2)–(1.4) is dissipative.

This dynamics is studied in detail in Sections 7 and 8, which describe the last steps in the proof of Theorem 1.2.

**2. Proof of Theorem 1.1. A. Preliminaries.** Let us define a family of smooth vector fields $\Psi(p, \mathbf{A}, \mathbf{B}, \theta)$ in $\mathbb{R}^n$ depending on the following parameters: the number $m > n$, an $m \times n$ matrix $\mathbf{A}$, $n \times m$ matrix $\mathbf{B}$ and the vector $\theta \in \mathbb{R}^m$. Let us set

$$\Psi_i(p, \mathbf{A}, \mathbf{B}, m, \theta) = \sum_{k=1}^{m} B_{ik} \Phi(\sum_{j=1}^{n} A_{kj} p_j + \theta_k) - p_i \tag{2.1}$$

where $p = (p_1, p_2, \dots, p_n) \in \mathbb{R}^n$ and $i = 1, 2, \dots, n$.

Denote by $\mathcal{B}^n(R)$ the ball in $\mathbb{R}^n$ of radius $R$:

$$\mathcal{B}^n(R) = \{p \in \mathbb{R}^n : |p| \le R\}. \tag{2.2}$$

(For brevity, let us denote $\mathcal{B}^n = \mathcal{B}^n(R)$ if $R = 1$.)

**Approximation Lemma 2.1.** *If $\Phi$ satisfies condition* (1.5), *then for any $C^1$-vector field $Q(p)$ (defined in the ball $\mathcal{B}^n$ and directed inside this ball at its boundary $\partial\mathcal{B}^n$) and any positive number $\delta$, there exist a number $m \ge n$, matrices $\mathbf{A}, \mathbf{B}$ and an $m$-vector $\theta$ such that*

$$|Q(\cdot) - \Psi(\cdot, \mathbf{A}, \mathbf{B}, m, \theta)|_{C^1(\mathcal{B}^n)} < \delta, \tag{2.3}$$

*where the conditions*

$$A_{ij} = \delta_{ij}, \quad \theta_i = 0 \quad (i, j \leq n) \tag{2.4}$$

$$p \cdot \Psi = \sum_{l=1}^{n} p_l \Psi_l(p, \mathbf{A}, \mathbf{B}, m, \theta) < -\delta_1 < 0, \quad \textit{for any } p \textit{ such that } |p| > 1 \tag{2.5}$$

*hold.*

**Remark 1.** Condition (2.5) yields that the vector $\Psi(p)$ is directed towards the ball $\mathcal{B}^n$ if $p$ lies outside of this ball. Thus, all the $p$-trajectories of the system $dp/dt = \Psi(p)$ attain this ball. In fact, (2.5) entails $\frac{1}{2}\frac{d|p|^2}{dt} = -p \cdot \Psi < -\delta_1$ if $|p| > 1$.

The proof of this quite-technical lemma can be found in the appendix.

**B.** Let us now turn to the proof of Theorem 1.1. The first step is a substitution for $\mathbf{K}$. Suppose $m > n$. Let us set

$$\mathbf{K} = \mathbf{AB}, \tag{2.6}$$

where $\mathbf{A}$ and $\mathbf{B}$ are from Lemma 2.1. Now system (1.6) takes the form

$$\frac{dq_i}{dt} = \sum_{k=1}^{n} A_{ik} R_k(q) - q_i + \theta_i, \tag{2.7}$$

where

$$R_k(q) = \sum_{j=1}^{m} B_{kj} \Phi(q_j). \tag{2.8}$$

The next part of the proof is based on the following two key points. The first point is that system (2.7) has an inertial manifold $\mathcal{M}_H$ attracting all the trajectories with an exponential rate. The second point is that, in a sense, we can control the corresponding inertial form.

Let us introduce new variables $z_i$ and $p_k$ by

$$z_i = q_i - \sum_{k=1}^{n} A_{ik} p_k - \theta_i, \quad i = n+1, \ldots, m, \quad p_k = q_k, \quad k = 1, \ldots, n. \tag{2.9}$$

Simple calculations show (see also [62, 63]) that, under conditions (2.4), equations (1.6) can be rewritten as

$$\frac{dz_i}{dt} = -z_i, \quad i > n, \tag{2.10}$$

$$\frac{dp_j}{dt} = R_j(\bar{q}(p, z)) - p_j, \quad j = 1, \ldots, n, \tag{2.11}$$

where $\bar{q}_i(p, z) = \sum_{k=1}^{n} A_{ik} p_k + \gamma_i(z)$, the functions $R_j$ are defined by (2.8), $\gamma_i(z) = 0$ for $i \le n$ and $\gamma_i(z) = \theta_i + z_i$ for $i > n$.

Equations (2.10) lead to the estimate $|z_i(t)| < C \exp(-t)$, $t > 0$. Therefore, taking into account Remark 1, we observe that any set defined by the relations $|z| \le r$, with any $r > 0$, $|p| \le 1$ is absorbing. Thus, the relations $z_i = 0$, $|p| \le 1$ define the inertial manifold. Using conditions (2.4) we notice that equations

$$\frac{dp_j}{dt} = \Psi_j(p, \mathbf{A}, \mathbf{B}, \theta, m) \quad j = 1, \ldots, n \tag{2.12}$$

define the inertial form of (1.6). Due to the approximation lemma, any prescribed $C^1$-vector field can be approximated in $\mathcal{B}^n$ (with any given precision in $C^1$-norm and under a suitable choice of the parameters $\mathcal{P}$) by the field $\Psi$ from (2.12). This completes the proof.

Let us turn now to the proof of Theorem 1.2. It is based on Theorem 1.1 and proceeds in some steps.

**3. Preliminaries: construction of $w$ and some properties of linear operators.** Let us begin with the construction of $w$. Consider the interval $I = [0, 1]$ and an arbitrary subset $X_m$ consisting of $m$ different points $x_i$ such that $x_i < x_{i+1}$. Let us denote by $d_m$ the minimum of distances between these points and boundaries of $I$: $d_m = \min\{x_{i+1} - x_i,\ x_1, 1 - x_m\}$, where $i = 1, 2, \ldots, m-1$. Let us set $\delta_m = d_m/4$.

Let us define a special potential $V_m(x, \epsilon)$ consisting of $m$ identical potential wells at the points $x_i$ and depending on an auxiliary parameter $\epsilon$:

$$V_m(x, \epsilon) = -4 + 6 \sum_{i=1}^{m} \cosh^{-2}((x - x_i)/\epsilon). \tag{3.1}$$

Let us define the operator

$$A_\epsilon u = \Delta u + w_m(x, \epsilon) u, \quad w_m = \epsilon^{-2} V_m - \epsilon^{\mu}, \quad \mu > 2 \tag{3.2}$$

with a dense domain $Dom\ A_\epsilon = \{u \in W^{2,2}(\Omega) : \partial u/\partial n = 0,\ (x, y) \in \partial\Omega\}$.

With the above standing assumptions this operator has the following "almost" eigenfunctions:

$$\psi_i(x, \epsilon) = \cosh^{-2}((x - x_i)/\epsilon)\xi((x - x_i)/\delta_m), \tag{3.3}$$

where $\xi$ is a $C^\infty$ cut-off function such that $0 \le \xi \le 1$, $\xi(z) \equiv 1$ for $|z| < 1/2$, *supp* $\xi = [-1, 1]$. Simple calculations show

$$\sup |(A_\epsilon + \epsilon^\mu)\psi_i| < h_\epsilon \tag{3.4}$$

$$||\psi_i|| = \frac{2}{\sqrt{3}}\sqrt{\epsilon} + h(\epsilon), \tag{3.5}$$

where $|h(\epsilon)| < C\exp(-c/\epsilon)$ for some constants $c = c(X_m)$ and $C = C(X_m)$.

Below, to simplify notations, we use the following conventions. The exponentially small quantities (which are less than $C\exp(-c\epsilon^{-\delta})$ as $\epsilon \to 0$ for some $\delta > 0$) are denoted by $h(\epsilon)$ or $h_i(\epsilon)$.

Moreover, one denotes by $C_i$ and $c_i$ sufficiently large positive constants which do not depend on small parameters $\epsilon$ as $\epsilon \to 0$. We shall often omit the index $i$. These constants can vary from line to line. Small positive constants (which do not depend on $\epsilon$) are denoted by $\delta_j$. Define the projection operators

$$\mathbf{P}_i u = \psi_i \langle u, \psi_i \rangle ||\psi_i||^{-2}, \quad \mathbf{P} = \sum \mathbf{P}_i, \quad \mathbf{Q} = \mathbf{I} - \mathbf{P}. \tag{3.6}$$

Then $\mathbf{Q}H$ is a subspace in $H = L_2(\Omega)$ consisting of functions $u$ orthogonal to all $\psi_i$:

$$\mathbf{Q}H = \{u \in H : \ \ \langle u, \psi_i \rangle = 0\}. \tag{3.7}$$

Let us formulate now the key assertion of this section.

**Proposition 3.1.** *Suppose positive $\epsilon$ is small enough: $\epsilon < \epsilon_1(X_m, \delta_m)$. Then, if $u \in \mathbf{Q}H \bigcap DomA_\epsilon$, one has*

$$\langle A_\epsilon u, u \rangle \ \le \ -c||u||^2, \tag{3.8}$$

*and, for arbitrary $u \in DomA_\epsilon$, one has*

$$\langle A_\epsilon u, u \rangle \ \le \ -c\epsilon^\mu ||u||^2. \tag{3.9}$$

**Proof**. Using the decomposition $u = \bar{u}(x) + \tilde{u}(x, y)$, where $\bar{u} = \int_0^1 u(x, y)dy$, one has

$$\langle A_\epsilon u, u \rangle = \langle A_\epsilon \bar{u}, \bar{u} \rangle + \langle A_\epsilon \tilde{u}, \tilde{u} \rangle \tag{3.10}$$

(the contribution $\langle A\bar{u}, \tilde{u} \rangle$ vanishes since $\int_0^1 a(x)\tilde{u}(x, y)dy \equiv 0$ for any function $a(x)$).

To evaluate these quadratic forms, we use the well-known result [5, 6]

$$J_\epsilon(v) = \int_0^1 [(\frac{\partial v}{\partial x})^2 + \epsilon^{-2} V_m(x,\epsilon) v^2(x)]\, dx \ \leq -c\epsilon^{-2} \int_0^1 v^2\, dx, \qquad (3.11)$$

if $\langle \psi_i, v \rangle = 0$ for all $i$ and

$$J_\epsilon(v) \ \leq h_\epsilon \int_0^1 v^2\, dx \qquad (3.12)$$

for any $u \in Dom A_\epsilon$. Moreover, from the Poincaré inequality one has

$$c_2 \int_\Omega \tilde{u}_y^2\, dx\, dy \ \geq \ \int_\Omega \tilde{u}^2\, dx\, dy. \qquad (3.13)$$

By integrating (3.11) with $v = \bar{u}$ over $y$, one obtains $\langle A_\epsilon \bar{u}, \bar{u} \rangle \leq -c\epsilon^{-2} ||\bar{u}||^2$. In a similar way, using (3.12) and (3.13) one finds $\langle A_\epsilon \tilde{u}, \tilde{u} \rangle \leq -c||\tilde{u}||^2$, completing the proof of (3.8). Similar arguments prove (3.9).

Estimate (3.8) means that the operator $A$ restricted to corresponding subspaces $\mathbf{Q}H$ has spectrum in the negative half-plane. This spectrum is separated from the imaginary axis by some "spectral barrier." The existence of such a barrier (which does not vanish as $\epsilon \to 0$) is a crucial factor in proving the inertial manifold existence. The second key point is a special form of functions $\psi_i$ which are localized at points $x_i$. This localization property will be used below to obtain the simple inertial form.

**4. Global existence and uniqueness of solutions.** To prove global existence and uniqueness, we use the well-known approach of [29]. Let us define the following scale of the Hilbert spaces (spaces of fractional powers):

$$\mathcal{H}_\alpha = \{w = (u,v) \in L_2(\Omega) \times L_2(\Omega) : ||(-A_\epsilon + I)^\alpha u|| < \infty,\ ||(-\tilde{A}_a)^\alpha v|| < \infty\}, \qquad (4.1)$$

where $0 \leq \alpha < 1$ and $\mathcal{H} = L_2(\Omega) \times L_2(\Omega)$ and where $\tilde{A}_a$ denotes the operator $\Delta - a^2$ with the domain $\{v : v \in W^{2,2} : \frac{\partial v}{\partial \nu} = 0 \ \ at\ \partial\Omega\}$. For brevity, we denote $\tilde{A} = \tilde{A}_1$ and, moreover, we shall suppose $a \geq 1$.

The corresponding norms $||\ ||_{\alpha,\epsilon}$ are defined by

$$||w||_{\alpha,\epsilon}{}^2 = ||u||^2_{\alpha,\epsilon} + ||v||^2_\alpha, \quad ||u||^2_{\alpha,\epsilon} = ||A^\alpha_\epsilon u||^2, \quad ||v||^2_\alpha = ||\tilde{A}^\alpha v||^2. \quad (4.2)$$

Let us define the map

$$F : (u,v)^t \to (F_1(u,v), F_2(u,v))^t, \qquad (4.3)$$

where $F_1 = f_0(x,y) + f_1(x,y)v$ and $F_2 = g(x,y)\Phi(u)$. Let us recall that $f_i$ and $g$ lie in $C^2$ and thus are a priori bounded: $|f_i(\cdot,\cdot)|$, $|g(\cdot,\cdot)| < C$ (where $|\ |$ denotes sup-norm). Let us show following [29] that $F \in C^{1+\omega}(\mathcal{H}_\alpha, \mathcal{H})$ (where $\omega \in [0,1)$) if $\alpha > 3/4$ and, moreover, the corresponding Fréchet derivative is a priori uniformly bounded. Indeed, let us consider, for instance, the second component. We obtain, due to the Sobolev embedding theorem [29], that $||F_2(u+\tilde{u}) - F_2'(u)\tilde{u} - F_2(u)|| < C_0(g,\Phi)||\tilde{u}|| \sup_\Omega |\tilde{u}| < C(\epsilon)||\tilde{u}||^2_{\alpha,\epsilon}$ , where the Fréchet derivative $F_2'$ is the following linear operator: $\tilde{u} \to g(x,y)\Phi'(u)\tilde{u}$. According to the converse Taylor theorem (see [29]), this yields $F_2 \in C^{1+\omega}(\mathcal{H}_\alpha, H)$ .

The standard proof (see also [29]) gives then the local existence and uniqueness of solutions within some bounded time interval in the spaces $\mathcal{H}_\alpha$. The global existence is a consequence of the uniform boundedness of the Fréchet derivatives of maps $F_i$.

Thus problem (1.2)–(1.4) defines a global semiflow $S^t$, $t \geq 0$.

**5. Transformation of system (1.2)–(1.3). Fast and slow modes.** In this section we introduce fast and slow modes. The slow modes capture an essential part of the dynamics and define the inertial form of the system. Let us notice that formally we can make such separation of our dynamics into slow and fast modes for coefficients $f_i$ and $g$ of a very general form. We use the specific form (1.9) of $g$ only beginning with Section 6, where this helps us to obtain a priori estimates.

Let us define the functions $u_0$, $u_1$ and $q = (q_1, q_2, \dots, q_m) \in \mathbb{R}^m$ by the relations

$$u_0(x,q,\epsilon) = \sum_{i=1}^{m} q_i \psi_i(x,\epsilon) \tag{5.1}$$

and

$$u = u_0(x,q,\epsilon) + u_1, \quad \langle \psi_i, u_1 \rangle \equiv 0. \tag{5.2}$$

Then

$$q_i = \gamma_i \langle u, \psi_i \rangle, \quad \gamma_i = ||\psi_i||^{-2} = \frac{3}{4}\epsilon^{-1} + h_i(\epsilon). \tag{5.3}$$

Let us define $v_0(x,y,q,\epsilon)$ as a solution of the following Neumann problem:

$$\Delta v_0 - a^2 v_0 + g(x,y)\Phi(u_0(x,q,\epsilon)) = 0, \quad \frac{\partial v_0}{\partial \nu}(x,y) = 0 \ \ (x,y) \in \partial\Omega. \tag{5.4}$$

One sets

$$v_1 = v - v_0. \tag{5.5}$$

Let us substitute (5.1)–(5.5) into (1.2) and (1.3). Then, applying the projection operators $\mathbf{P}_i$ and $\mathbf{Q}$ to (1.2), one obtains the following system:

$$\frac{dq_i}{dt} = -\epsilon^{\mu} q_i + \tilde{R}_i(q, v_1, \epsilon) = R_i(q, v_1, \epsilon), \tag{5.6}$$

$$\frac{\partial u_1}{\partial t} = A_\epsilon u_1 + F_1(u_1, v_1, q, \epsilon), \tag{5.7}$$

$$\frac{\partial v_1}{\partial t} = \Delta v_1 - a^2 v_1 + F_2(u_1, v_1, q, \epsilon) \tag{5.8}$$

where

$$\tilde{R}_i = \gamma_i \langle f_0 + f_1(v_0 + v_1), \psi_i \rangle + \eta_i(u_1, q, \epsilon) \tag{5.9}$$

$$F_1 = \mathbf{Q}(f_0 + f_1(v_0 + v_1)) + \eta(u_1, q, \epsilon), \tag{5.10a}$$

$$F_2 = g(x, y)[\Phi(u_0 + u_1) - \Phi(u_0)] - \sum_{j=1}^{m} \frac{\partial v_0}{\partial q_j} R_j, \tag{5.10b}$$

and where $\eta_i$ and $\eta$ are exponentially small corrections. It can be checked, by (3.3) and (3.4), that they are linear functions of $u_1$ and $q_i$ and satisfy

$$|\eta_i| \ \le \ ch_i(\epsilon)(||u_1|| + |q|), \quad ||\eta|| \ \le \ ch(\epsilon)(||u_1|| + |q|), \tag{5.11}$$

where $h_i$ and $h$ are exponentially small as $\epsilon \to 0$.

Let us investigate equations (5.6)–(5.8) in the phase space $X = \mathbf{Q}\mathcal{H}_\alpha \times E$ where $E$ is the Euclidean space $E = \mathbb{R}^m$ with the usual norm $|q|^2 = \sum q_i^2$, $\mathcal{H}_\alpha = H_\alpha \times H_\alpha$ and $\mathbf{Q}\mathcal{H}_\alpha$ denotes $\mathbf{Q}H_\alpha \times H_\alpha$.

Due to the results of previous section, the right-hand sides $R$ and $F$ are from $C^{1+\omega}$-Holder class as maps from $\mathbf{Q}\mathcal{H}_\alpha \times E$ to $E$ and $X$ respectively where $\omega > 0$.

The existence of the spectral barrier for $A_\epsilon$ (see Proposition 3.1) means that $u_1$ and $v_1$ are fast modes whereas $q_i$ are slow ones. Moreover, we notice that, for elements $u_1$ from $\mathbf{Q}H$, the norm $||u_1||_{\alpha,\epsilon}$ is equivalent to the standard norm $|||u_1|||_\alpha = ||(-\Delta + I)^\alpha u_1||$.

To keep the notations simple, we, using this fact, do not make explicit this dependence of the norms of the functions $w = (v_1, u_1)$ and $u_1$ on $\epsilon$ in the rest of the paper.

**6. Preliminaries: choice of coefficients $f_i$ and $g$ and auxiliary calculations.** In this section we describe a special choice of coefficients $f_i$ and $g$. Such a choice allows to us to attain two basic goals. The first one is to prove the existence of the inertial dynamics. Second, we would like to obtain a simple representation for $v_0$, in order to simplify equations (5.6)–(5.8) and reduce these to neural equations (1.6).

Suppose the coefficients $g(x, y)$ lie in the following class of functions:

$$\mathcal{G}(\rho, \epsilon, X_m) = \{g : g = \sum_{i=1}^{m} \exp(-\epsilon^{-2\rho}[(x - x_i)^2 + (y - y_i)^2])\}. \tag{6.1}$$

Here $\rho > 2$ and the set of points $X_m = \{x_1, \ldots, x_m\}$ has been defined in Section 3; the points $\{y_1, \ldots, y_m\} \in (0, 1)$ can be chosen arbitrarily (but they should be different, $y_i \neq y_j$ for $i \neq j$).

In the sequel, throughout we shall assume that this set $X_m$ and points $y_i$ are fixed. Denote $Z_m = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}$.

Let us define the following class of the pairs of functions $(\tilde{f}_0, \tilde{f}_1)$:

$$\mathcal{F}(\tilde{C}, Z_m, \tilde{\delta}) = \{(\tilde{f}_0, \tilde{f}_1) : |\tilde{f}_i(\cdot, \cdot)|_{C^2(\Omega)} < \tilde{C},\ dist(supp\ \tilde{f}_1, Z_m) > \tilde{\delta}\}. \tag{6.2}$$

This means in particular that the support of $\tilde{f}_1$ does not intersect the set $Z_m$. This class $\mathcal{F}$ depends on the parameters $\tilde{C}, \tilde{\delta}$ and the set $Z_m$. We choose positive $\tilde{\delta}$ so that $\delta < \frac{1}{2} \min_{i \neq j} |x_i - x_j|$.

To simplify notations, we shall sometimes denote these classes of functions $\mathcal{G}$ and $\mathcal{F}$ omitting the dependence on $\rho, \epsilon, X_m$ and $\tilde{C}, Z_m, \tilde{\delta}$.

Assume the numbers $\mu_i$ satisfy

$$\mu_0 = \mu, \quad \mu_1 = \mu - 2\rho, \quad \mu > 2\rho. \tag{6.3}$$

More precisely, the numbers $\mu_i$ and $\rho > 2$ will be defined below. In the sequel, one supposes that $f_0$ and $f_1$ are defined by

$$f_i = \epsilon^{\mu_i} \tilde{f}_i, \quad (\tilde{f}_0, \tilde{f}_1) \in \mathcal{F}. \tag{6.4}$$

The choice $g \in \mathcal{G}$ allows us to obtain, for small $\epsilon$, the solution $v_0$ of problem (5.4) in a simple form. To show it, let us use the integral representation

$$v_0(x, y, q, \epsilon) = \int_{\Omega} G_a(x, y, x', y') g(x', y', \epsilon) \Phi(u_0(x', q, \epsilon))\, dx'\, dy' \tag{6.5}$$

where $G_a$ is the Green's function of problem (5.4). Taking into account $\rho > 2$, we observe that the functions $g \in \mathcal{G}$ are "well localized" at the points $x_i$ and $y_i$. This localization is stronger than the one of the functions $\psi_i$ at

$x_i$. Thus, for small $\epsilon$ and points $(x, y)$ such that $dist\{(x, y), Z_m\} > \tilde{\delta}$, we can calculate $v_0(x, y, q, \epsilon)$, up to small corrections. To do this, let us fix an arbitrary index $i$ and consider the following integral:

$$J_{i,\epsilon,\phi} = \int_\Omega \phi(x', y', \epsilon) \exp(-\epsilon^{-2\rho}[(x' - x_i)^2 + (y' - y_i)^2])\, dx'\, dy'.$$

Suppose $\phi(x', y', \epsilon)$ is a smooth, bounded function in the domain $\Omega_{i,\epsilon} = \{(x', y') : dist\{(x_i, y_i), (x', y')\} < 2\epsilon^{\rho-1/2}\}$ and, moreover, $\phi \in L_1(\Omega)$.

By the substitution $x' = x_i + \epsilon^\rho r \cos\zeta$, $y' = y_i + \epsilon^\rho r \sin\zeta$, we obtain

$$|J_{i,\epsilon,\phi} - b\epsilon^{2\rho}\phi(x_i, y_i, \epsilon)| \leq h_\epsilon ||\phi||_{L_1(\Omega)} + \epsilon^{2\rho}\int_0^{2\pi}\int_0^{\epsilon^{-1/2}} r \exp(-r^2)\tilde{\phi}_i(r, \zeta, \epsilon)\, dr\, d\zeta,$$

where $h_\epsilon$ is exponentially small, $b = 2\pi\int_0^\infty r\exp(-r^2)dr = \pi$ and $\tilde{\phi}_i(r, \zeta, \epsilon) = \phi(x_i + \epsilon^\rho r\cos\zeta, y_i + \epsilon^\rho r \sin\zeta, \epsilon) - \phi(x_i, y_i, \epsilon)$. We apply this estimate of $J_{i,\epsilon,\phi}$ for

$$\phi(x', y', \epsilon, q, x, y) = G_a(x, y, x', y')\Phi(\sum_{j=1}^m q_j \cosh^{-2}((x' - x_j)/\epsilon)),$$

where $\epsilon, q, x, y$ are parameters.

Assuming $|q| < c\epsilon^{-2\rho}$ we notice that for any point $(x', y')$ from $\Omega_{i,\epsilon}$ and small $\epsilon$, this expression for $\phi$ can be simplified:

$$\phi(x', y', \epsilon, q, x, y) = G_a(x, y, x', y')\Phi(q_i \cosh^{-2}((x' - x_i)/\epsilon)) + h_\epsilon,$$

where $h_\epsilon$ is an exponentially small error. One observes then that

$$|\tilde{\phi}_i| < C_1\epsilon^{\rho-1/2}\sup_{t\in\mathbb{R}}|\Phi(t)|\Big(\sup_{(x',y')\in\Omega_{i,\epsilon}}|\frac{\partial G_a}{\partial x'}(x, y, x', y')|$$

$$+ \sup_{(x',y')\in\Omega_{i,\epsilon}}|\frac{\partial G_a}{\partial y'}(x, y, x', y')|\Big) + C_2\epsilon^{2\rho-3}\sup_{t\in\mathbb{R}}|t\Phi'(t)| \sup_{(x',y')\in\Omega_{i,\epsilon}}|G_a(x, y, x', y')|$$

$$< C(\epsilon^{\rho-1/2} + \epsilon^{2\rho-3}) < C\epsilon.$$

Moreover, we have

$$||\phi(\cdot, \cdot, \epsilon, q, x, y)||_{L_1(\Omega)} < \sup|\Phi| \int_\Omega G_a(x, y, x', y')\, dx'\, dy' < \sup|\Phi| a^{-2} < C,$$

uniformly with respect to parameters $\epsilon, q, x, y$. Therefore, under such choice of $\phi$ and $x, y, q$, we have (for sufficiently small $\epsilon$) that

$$|J_{i,\epsilon,\phi} - \pi\epsilon^{2\rho} G_a(x, y, x_i, y_i)\Phi(q_i)| < C\epsilon^{2\rho+1}.$$

Analogous estimates hold for the first derivatives of the integral $J_{i,\epsilon,\phi}$ with respect to $q_j$. For arbitrary $(x, y)$ and $|q|$ one can obtain a rough estimate.

We can evaluate $J_{i,\epsilon,\phi}$ by

$$|J_{i,\epsilon,\phi}| < h_\epsilon ||G_a||_{L_1(\Omega)} + \sup|\Phi| \int_{\Omega_{i,\epsilon}} g_i(x', y', \epsilon) G_a(x, y, x', y')\, dx'\, dy',$$

where $g_i = \exp(-\epsilon^{-2\rho}[(x' - x_i)^2 + (y' - y_i)^2])$. Here $h_\epsilon$ is exponentially small and evaluates the quantity $\sup|\Phi| \sup_{\Omega\setminus\Omega_{i,\epsilon}} g_i$. We apply to the second term of the right-hand side the Hölder inequality with exponents $p_1$ and $p_2$, $p_1^{-1} + p_2^{-1} = 1$. Then the integral $\int_{\Omega_{i,\epsilon}} |G_a|^{p_1}$ is bounded. In fact, the function $G_a$ is positive, smooth outside of the point $x' = x, y' = y$ and due to condition (6.2) has a logarithmic singularity at this point. Choosing $p_2$ as $p_2 = 1 + \delta_2$, where $\delta_2 = \delta_3(1 - \delta_3)^{-1} > 0$, we obtain

$$(\int_{\Omega_{i,\epsilon}} g_i^{p_2}(x, y, \epsilon)\, dx\, dy)^{\frac{1}{p_2}} < C(\delta_3)\epsilon^{2\rho-\delta_3},$$

with any $\delta_3 > 0$.

Finally, as a consequence of these estimates, we have the following important result:

**Proposition 6.1.** *Suppose that* $dist\{(x, y), Z_m\} > \tilde{\delta}$ *and* $|q| < c\epsilon^{-2\rho}$. *Then, for small* $\epsilon$, *the function* $v_0$ *takes the form*

$$v_0(x, y, q, \epsilon) = \pi\epsilon^{2\rho}[\sum_i G_a(x, y, x_i, y_i)\Phi(q_i) + \sigma(q, \epsilon) + h(\epsilon)], \tag{6.6}$$

*where a* $C^1$*-smooth correction* $\sigma$ *satisfies the estimates* $|\sigma(q, \epsilon)|$, $|\sigma'_q(q, \epsilon)| < c\epsilon$. *For arbitrary* $(x, y)$ *and* $|q|$ *we have the estimate*

$$|v_0(x, y, q, \epsilon)|, |\frac{\partial v_0(x, y, q, \epsilon)}{\partial q_j}| < c\epsilon^{2\rho-\delta_3}, \tag{6.6a}$$

*for any* $\delta_3 > 0$ *and for sufficiently small* $\epsilon$.

Using the localization properties of $\psi_i$, one obtains (if $(\tilde{f}_0, \tilde{f}_1) \in \mathcal{F}$)

$$\langle f_0, \psi_i\rangle = c\epsilon^{\mu_0+1}(\tilde{\theta}_i(\tilde{f}_0) + \xi_i(\epsilon)), \tag{6.7}$$

where

$$\tilde{\theta}_i = \int_0^1 \tilde{f}_0(x_i, y)\, dy, \tag{6.8}$$

and smooth corrections $\xi_i$ satisfy $|\xi_k(\epsilon)| < C_1\epsilon^{s_1}$ as $\epsilon \to 0$ (with $s_1 > 0$).

Let us define the matrix $\mathbf{K}$ by

$$K_{ij} = \int_0^1 \tilde{f}_1(x_i, y) G_a(x_i, y, x_j, y_j) dy. \tag{6.9}$$

By (6.5) and (6.6) we obtain that, under condition $|q| < C\epsilon^{-2\rho}$ and for small $\epsilon$, the following relation,

$$\langle f_1 v_0, \psi_i \rangle = c\epsilon^{\mu_1+2\rho+1}(\sum_{j=1}^{m} K_{ij}\Phi(q_j) + \bar{\xi}_i(q, \epsilon)), \tag{6.10}$$

holds where the $\bar{\xi}_i$ satisfy $|\bar{\xi}_k(\cdot, \epsilon)|_{C^1} < C_2\epsilon^{s_2}$ as $\epsilon \to 0$ (with $s_2 > 0$).

(Notice that relation (6.10) holds due to the assumption about the support $f_1$ that allows us to use (6.6).)

These results will be used in the following section, where we establish a priori estimates and prove the absorbing set existence.

**7. Absorbing set.** In this section we suppose that the numbers $m, \rho > 2$, the set $X_m$ and the points $y_j$ are fixed, $(\tilde{f}_0, \tilde{f}_1) \in \mathcal{F}$. Then, for each $\epsilon > 0$, class $\mathcal{G}$ contains a single function $g$ which is defined by

$$g = \sum_{i=1}^{m} \exp(-\epsilon^{-2\rho}[(x - x_i)^2 + (y - y_i)^2]), \tag{7.0}$$

with $(x_i, y_i) \in Z_m$. We shall take such positive $\epsilon < \epsilon_0(m, X_m)$ that Proposition 3.1 holds. Let us recall some standard facts about the linear operators [29]. The operators $A_\epsilon$ are self-adjoint, with a dense domain and negatively defined in $\mathbf{Q}H$ due to Proposition 3.1. Thus,

$$||(-A_\epsilon)^\alpha \exp(A_\epsilon t)u|| \ \le C||(-A_\epsilon)^\alpha u|| \exp(-\beta t), \quad u \in \mathbf{Q}H \tag{7.1}$$

$$||(-A_\epsilon)^\alpha \exp(A_\epsilon t)u|| \le C b_\alpha(t)||u|| \exp(-\beta t), \quad u \in \mathbf{Q}H \tag{7.2}$$

where $C$ and $\beta$ do not depend on $\epsilon$ for small $\epsilon$ and $b_\alpha = \max\{1, t^{-\alpha}\}$. The same estimates hold for the operator $\tilde{A}_a = \Delta - a^2$, with $a \ge 1$.

In the proof of dissipativity of equations (5.6)–(5.8), our main tools are simple $L_2$-estimates and (7.1)–(7.2).

**Lemma 7.1.** *For $1 > \alpha > \frac{3}{4}$, the following a priori estimates hold:*

$$||v(t)||_\alpha \leq C_1 \exp(-c_1 t) + \bar{C}_1 \epsilon^\rho, \tag{7.3}$$

$$|q(t)| < C_2 \exp(-c_2 \epsilon^\mu t) + \bar{C}_2 \epsilon^{-\rho-1/2}, \tag{7.4}$$

$$||u_1(t)||_\alpha \leq C_3 \exp(-c_3 t) + \bar{C}_3 \epsilon^{\mu-\rho}, \tag{7.5}$$

*where constants $C_i$ and $\bar{C}_i$ do not depend on the small parameter $\epsilon$ as $\epsilon \to 0$ (but they, naturally, can depend on $\alpha$). Moreover, the $\bar{C}_i$ do not depend on the initial data $(u^0, v^0)$.*

**Proof**. Multiplying (1.3) by $v$ one finds

$$\frac{d||v||^2}{dt} \leq -c_0 ||v||^2 + c_1 ||g|| ||v||. \tag{7.6}$$

Notice that $||g|| < c\epsilon^\rho$. Thus

$$||v(t)|| < C_1 \exp(-a^2 t) + \bar{C}_1 \epsilon^\rho. \tag{7.7}$$

Using the condition $(\tilde{f}_0, \tilde{f}_1) \in \mathcal{F}$ and (3.9), in a similar way one obtains from (1.2)

$$||u(t)|| < c_1 \exp(-c_0 \epsilon^\mu t) + \bar{c} \epsilon^{-\rho}, \tag{7.8}$$

where the constant $\bar{c}$ does not depend on initial data. This important preliminary inequality shows that, in (5.9) and (5.10a), the terms $\eta_i$ and $\eta$ can be a priori bounded by the exponentially small quantities $h(\epsilon)$. Therefore, these contributions can be removed in the following estimates.

Let us now turn to (5.7). Multiplying it by $u_1$ and using (7.7) and (3.8), one has

$$||u_1(t)|| < C_3 \exp(-c_1 t) + \bar{C}_3 \epsilon^{\mu_1 + \rho}. \tag{7.9}$$

Using (6.2)–(6.4) and (3.5), one finds

$$|\langle f_1 v + f_0,\ \psi_i \rangle| \leq (|f_1|||v|| + ||f_0||) ||\psi_i|| < c\epsilon^{\mu+1/2} (\epsilon^{-2\rho} ||v|| + 1).$$

Now (7.7) gives

$$|\tilde{R}_i| < c\epsilon^{\mu-1/2} [\epsilon^{-2\rho} (C_1 \exp(-c_0 t) + \bar{C}_1 \epsilon^\rho) \ + 1]. \tag{7.10}$$

Multiplying the $i$-th equation (5.6) by $q_i$ and using (7.10), one has

$$|q(t)| < C_2 \exp(-c_2 \epsilon^\mu t) + \bar{C}_2 \epsilon^{\mu_1 + \rho - 1/2 - \mu} = C_2 \exp(-c_2 \epsilon^\mu t) + \bar{C}_2 \epsilon^{-\rho-1/2}. \tag{7.11}$$

Estimate (7.4) is proved. Estimates (7.3) and (7.5) are obtained now for $\alpha = 0$. Now we can derive these estimates for $\alpha \in (3/4, 1)$ in a standard way by (7.1), (7.2) and inequalities (7.7) and (7.9) [29]. Rewrite (5.7) and (5.8) in integral forms using evolution operators $\exp(At)$ and $\exp(\tilde{A}t)$. Then, for example, for $u_1$ one has

$$||u_1(t)||_\alpha \leq c\exp(-\beta t)||u_1(0)||_\alpha + \int_0^t b_\alpha(t-t_1)\exp(-\beta(t-t_1))||F_1(u(t_1), v(t_1), q(t_1))||\,dt_1. \tag{7.12}$$

Using this relation, one concludes that (7.5) holds, similarly for $v_1$. The proof is complete.

Inequalities (7.3)–(7.5) show that the global semiflow $S^t$ (generated by (1.2)–(1.4)) is dissipative.

These preliminary estimates (7.3)–(7.5) are sufficiently rough and their proof does not use Proposition 6.1. With these preparations, and using estimate (7.4) of $|q|$, we can obtain rather exact estimates based on Proposition 6.1 and (6.6a) and describe an absorbing set of the system.

**Proposition 7.2.** *Suppose $\mu > 2\rho+1, \rho > 2$ and $\epsilon$ is a small enough. Then, for sufficiently large $t > T_0(\epsilon)$, solutions of equations (5.6)–(5.8) satisfy*

$$||v_1(t)||_\alpha, \quad ||u_1(t)||_\alpha \leq \bar{C}\epsilon^\mu, \tag{7.13}$$

$$|q(t)| < C_a = K_0(\sup_{x,y\in\Omega} |\tilde{f}_0(x,y)| + \sup_{x,y\in\Omega} |\tilde{f}_1(x,y)|) \leq \tilde{C}K_0, \tag{7.14}$$

*where $K_0$ and $\bar{C}$ are some constants that do not depend on $\epsilon$ as $\epsilon \to 0$ and the initial data. Moreover, $K_0$ also does not depend on $|\tilde{f}_i|_{C^2}$.*

**Proof**. First let us establish a preliminary estimate for $||v_1||$. Notice that (7.5) entails

$$||g(\Phi(u_0+u_1) - \Phi(u_0))|| \leq C||g||||u_1|| \leq c\epsilon^\rho||u_1|| < c\epsilon^\mu$$

for large $t$. From (5.9), (7.4) and (7.10) it results that $|R_i| < C\epsilon^{\mu-\rho-1/2}$ $(t > T_0(\epsilon, u^0, v^0, Z_m, \dots))$. Now (6.6a) with $\delta_3 = 1/2$ gives

$$|\frac{\partial v_0}{\partial q_j}|\ |R_j| < c|\frac{\partial v_0}{\partial q_j}|\epsilon^{\mu-\rho-1/2} < c_1\epsilon^{\mu+\rho-1},$$

for large $t$. Thus, one obtains

$$||F_2|| \leq ||g[\Phi(u_0+u_1)-\Phi(u_0)]||+\sum_{j=1}^{m}|\frac{\partial v_0}{\partial q_j}||R_j| < c\epsilon^{\mu},\ (t > T_0(\epsilon, u^0, v^0, \dots)),$$

which yields estimate (7.13) for $v_1$. Now, using this estimate of $v_1$ in equation (5.6), we see that equation (5.6) takes the following form:

$$\frac{dq_i}{dt} = \epsilon^{\mu}(-q_i + \mathcal{Q}_i(q, v_1, u_1, \epsilon)) \tag{7.15}$$

where, due to estimate (7.13) of $||v_1||$ and (6.10), one has

$$|\mathcal{Q}_i| < C, \tag{7.16}$$

for large $t$. (Notice that (6.10) holds since Proposition 6.1 is valid.)

Equation (7.15) and estimate (7.16) yield now the inequality

$$|q(t)| < C_a \quad (t > T_1(\epsilon, u^0, v^0, \dots)). \tag{7.17}$$

We observe that the constant $C_a$ depends on $\sup|\tilde{f}_i|$ and (7.14) holds. It completes the proof of (7.14) and (7.13) for $v_1$ (in the case $\alpha = 0$).

Let us prove estimate (7.13) of $u_1$. Since $\mu > 2\rho + 1$, one notices

$$||f_1 v + f_0|| < ||f_1|| \sup_{x,y \in supp f_1} |v_0(x, y, q, \epsilon)| + |f_1|||v_1|| + ||f_0||,$$

where the sup is taken over the support of $f_1$. Using now (6.5) and estimate $v_1$ from (7.13) we have

$$||f_1 v + f_0|| < c\epsilon^{\mu}, \quad t > T_2(\epsilon, u^0, v^0, \dots), \tag{7.18}$$

which proves estimate (7.13) of $u_1$ for $\alpha = 0$. The estimates $||u_1||_{\alpha}$ and $||v_1||_{\alpha}$ can be obtained by (7.1) and (7.2) (see the end of the proof of Lemma 7.1).

Thus, under conditions standing above, there exists an absorbing set $\mathcal{W}_{\epsilon}$ defined by

$$\mathcal{W}_{\epsilon} = \{(q, u_1, v_1) : |q| \leq C_a, ||u_1||_{\alpha}, ||v_1||_{\alpha} \leq C\epsilon^{\mu}\}. \tag{7.19}$$

Notice that without any loss of generality one can suppose $C_a > 1$.

**8. Existence of the inertial manifold $\mathcal{M}_1$. Inertial form.** Our aim now is to investigate (5.6)–(5.8) inside this absorbing set $\mathcal{W}_\epsilon$. Suppose the pair $(\tilde{f}_0, \tilde{f}_1) \in \mathcal{F}$ and the coefficient $g \in \mathcal{G}$; the exponent $\rho$ from (6.1) and the set $Z_m$ are fixed.

First we prove the existence of the inertial manifold $\mathcal{M}_1$ for these equations. The main tool used to prove this assertion is the standard results about the central manifolds [7, 29, 59]. Choosing sufficiently large $\mu > \mu_0$ and small $\epsilon < \epsilon_0(m, Z_m, \tilde{\delta})$ we obtain the following.

If the number $\mu_0$ is sufficiently large and $(u, v)$ is inside the absorbing set $\mathcal{W}_\epsilon$, the nonlinearities in equations (5.6) and (5.8) are small (for small $\epsilon$). As a result, the existence of the spectral barrier independent of $\epsilon$ (see Proposition 3.1) allows us to prove the following:

**Proposition 8.1.** *Suppose the numbers $m$, $\rho > 2$ and the set $X_m$ and the points $y_i$ are fixed. Let the family of the potentials $w_m(x, \epsilon)$ be defined by* (3.2), *the function $g(x, y, \epsilon)$ be defined by* (7.0), *the coefficients $f_i$ be defined the relation $f_i = \epsilon^{\mu_i} \tilde{f}_i$ and suppose the pair $(\tilde{f}_0, \tilde{f}_1)$ lies in the class $\mathcal{F}$ where the constants $\tilde{C}$ and $\tilde{\delta}$ are fixed. Then we can choose such a number $\mu_*(\rho)$ that*

**I**. *If $\mu > \mu_*$, $\epsilon$ is sufficiently small, then there exists a manifold $\mathcal{M}_1$ locally invariant in the absorbing set $\mathcal{W}_\epsilon$ and globally attracting;*

**II**. *Under condition* **I**, *this manifold is at least $C^1$-class and defined by equations*

$$u_1 = U_1(q, \epsilon), \quad v_1 = V_1(q, \epsilon) \quad (q \in \mathcal{B}^n(R_0)) \tag{8.1}$$

*where the radius $R_0 = C_a$ and the maps $U_1$ and $V_1$ satisfy the estimates*

$$||U_1||, ||V_1||, ||U'_{1q}||, ||V'_{1q}|| \leq c\epsilon^{\mu - \rho_0}, \tag{8.2}$$

*where $\rho_0$ depends only on $\rho$;*

**III.** *In addition, one can choose $\tilde{f}_i$ so that the inertial dynamics of equations* (5.6)–(5.8) *takes the following form:*

$$\frac{dq_i}{dt} = c_0\epsilon^\mu[-q_i + \sum_j K_{ij}\Phi(q_j) + \theta_i + \phi_i(q, \epsilon)], \quad q \in \mathcal{B}^n(R_0), \tag{8.3}$$

*where $c_0$ is some constant and where corrections $\phi_i$ satisfy*

$$|\phi_i(\cdot, \epsilon)|_{C^1(\mathcal{B}^n(R_0))} < c\epsilon^s, \quad s > 0. \tag{8.4}$$

*The radius $R_0 > 1$ depends only on the numbers $m, \tilde{C}, \delta$ and the set $Z_m$, the matrix $\mathbf{K}$ is defined by relation* (6.9).

**Proof**. The assertions **I** and **II** can be proved by standard estimates and methods [19, 7, 29, 13]. If we denote $w = (u_1, v_1)$ and suppose that $w \in \mathcal{W}_\epsilon$, then equations (5.6) and (5.8) can be rewritten as follows:

$$w_t = A_\epsilon w + \epsilon^s W(w, q, \epsilon), \quad q_t = \epsilon^s P(w, q, \epsilon),$$

where the maps $W$ and $P$ lie in the Hölder classes $C^{1+\omega}(\mathcal{H}_\alpha, H)$ and $C^{1+\omega}$ $(\mathcal{H}_\alpha, \mathbb{R}^m)$ respectively, and the operator $A_\epsilon$ has the spectral barrier which does not depend on $\epsilon$ as $\epsilon \to 0$. Without any loss of generality, we can assume that these maps are uniformly Hölder bounded, for any $q$ and $w$, by a quantity $C$ independent of $\epsilon$; the exponent $s$ depends on $\mu_i$ and $\rho$: $s = s(\mu, \rho)$ (in fact, outside of $\mathcal{W}_\epsilon$, one can always make the standard cut-off procedure; see for example [13]). It is clear that if the numbers $\mu_i$ are chosen sufficiently large then the exponent $s$ is positive: $s > 0$. (This assertion is a simple consequence of the explicit formulae (5.9), (5.10) and a priori estimates of Section 7: we observe that $s = s(\mu_1, \mu_0, \rho) > \mu - \rho_1(\rho)$, and thus $s > 0$ for $\mu > \rho_1$.)

Thus, the existence of the $C^1$-manifold, locally invariant in $\mathcal{W}_\epsilon$, is a trivial corollary of well-known results; see, for example [29, Chapter 6] or [7]. Moreover, by Theorem 6.1.7 of [29, Chapter 6], one can check that this manifold is attracting in the set $\mathcal{W}_\epsilon$. In fact, we can derive the following relation:

$$(w(t) - W(q(t)))_t = A_\epsilon(w - W) + S(q, w, \epsilon)(w - W),$$

where $W = (U_1, V_1)^{tr}$ and, inside $\mathcal{W}_\epsilon$, the operator $Sw \to w$ is bounded ($S \in C(\mathcal{H}, \mathcal{H})$) by the quantity $c\epsilon^{s_3}$, $s_3 > 0$. Due to the existence of (nonvanishing as $\epsilon \to 0$) the spectral barrier for $A_\epsilon$, we immediately obtain the attracting property.

Assertion (8.3) and estimates (8.4) can be obtained as follows. Consider (5.6). It is clear that the inertial dynamics has the form

$$\frac{dq_i}{dt} = \epsilon^\mu(-q_i + \bar{Q}_i(q, \epsilon)),$$

where $\bar{Q}_i = \epsilon^{-\mu}\gamma_i\langle f_0 + f_1(v_0 + V_1(q, \epsilon)), \psi_i\rangle + \eta_i(U_1(q, \epsilon), q, \epsilon)$ and $\gamma_i$ is defined by (5.3).

Then, using relations (6.9) and (6.10), we find that the scalar product $\gamma_i\langle f_1 v_0 + f_0, \psi_i\rangle$ gives, up to small corrections, the terms $\epsilon^\mu[-q_i +$

$\sum_j K_{ij}\Phi(q_j) + \theta_i]$ in (8.3). Thus, to prove our assertions (8.3) and (8.4), it is sufficient to evaluate the contributions $\eta_i$ and $\langle f_1 V_1, \psi_i\rangle$ which give corrections $\phi_i$ in (8.3). The first one has the exponentially small order $h_\epsilon$ in $C^1$-norm; see (5.11) and (8.2). The second term can be evaluated as $c||f_1||||V_1||$. Since $||f_1|| < c\epsilon^{\mu-2\rho}$ (see Section 6, relation (6.3) and the definition of $f_i$) and $||V_1||$ can be estimated by (8.2), we see that this contribution is less than $c\epsilon^{2\mu-2\rho-\rho_0} < c\epsilon^{s+\mu}, \quad s > 0$, for large $\mu$. In a similar way, we can estimate the derivatives $\partial\langle f_1 V_1, \psi_i\rangle/\partial q_j$. Thus, the $\phi_i$ satisfy (8.4). This completes the proof.

The last auxiliary assertions that allow to derive Theorem 1.2 from Theorem 1.1 are the following lemmas.

**Lemma 8.1.** *For any given vector $\bar\theta \in \mathbb{R}^m$, there exists a smooth function $\tilde f_0$ such that*

$$\bar\theta_i = \theta_i(\tilde f_0), \tag{8.5a}$$

*where the vector $\theta$ in the right-hand side is defined by relation* (6.8).

The trivial proof of this lemma is omitted.

To simplify the proof of the following lemma, one suppose that all the points $(x_i, y_j)$ lies in the ball of diameter $< \frac{1}{6}$ centered at $(1/2, 1/2)$ :

$$(x_i - 1/2)^2 + (y_j - 1/2)^2 < 6^{-2}, \quad i, j = 1, \dots, m. \tag{8.6}$$

This assumption implies that the distances between any point $(x_i, y_j)$ and the boundary $\partial\Omega$ is larger than the reciprocal distances between these points.

**Lemma 8.2.** *Let the sets $X_m$, the different points $y_i$ (and thus the set $Z_m$) be fixed and suppose* (8.6) *holds. Then, if $a > a_0(m, Z_m)$ is large enough and $\tilde\delta$ is sufficiently small, for any given $m \times m$ matrix $\bar{\mathbf{K}}$ there exists a smooth function $\tilde f_1$ satisfying the condition dist(supp $\tilde f_1, Z_m) > \tilde\delta$ and such that*

$$\bar K_{ij} = K_{ij}(\tilde f_1), \tag{8.5b}$$

*where the matrix* $\mathbf{K}$ *in the right-hand side is defined by relation* (6.9).

**Remark**. In contrast to the approach used in [63] which was not explicit and not completely justified, the proof given below is constructive and gives us the required $\tilde f_1$ in an explicit form.

**Proof of Lemma 8.2**. We shall reduce (6.9) to a linear system of algebraic equations and shall demonstrate that this system has a solution for $a > a_0 > 1$ if $a_0$ is large enough.

We seek $f_1$ in a special form,

$$f_1(x,y) = \sum_{l=1}^{m} \sum_{k=1}^{m} b_{kl}(x,y) X_l^k, \tag{8.7}$$

where the $X_l^k$ are unknown coefficients (here $k$ denotes an index) and the $b_{kl}$ are smooth cut-off functions satisfying the conditions

**a**) the functions $b_{kl}$ are nonnegative and such that $\int_0^1 b_{kl}(x_k, y)\, dy = 1$;

**b**) the supports of $b_{kl}$ are balls $\Omega_{kl}$ such that $\Omega_{kl}$ lies in a $2r_0$-neighborhood of the point $(x_k, y_l)$ and does not intersect an $r_0$-neighborhood of this point.

From **b**) it follows that the function $\tilde{f}_1$ defined by (8.7) satisfies condition 6.2:

$$\mathrm{dist}(\mathrm{supp}\, \tilde{f}_1, Z_m) > \tilde{\delta}, \text{ for sufficiently small } r_0 \text{ and some } \tilde{\delta} > 0.$$

Let us choose a sufficiently small number $r_0 > 0$. Then such choice of $\Omega_{kl}$ yields, under our assumptions on the location of the points $x_i, y_j$, the following important facts: the supports $\Omega_{kl}$ do not intersect and

$$if\ (x_i, y) \in \Omega_{kl} \ \ then \ \ k = i, \tag{8.8$a$}$$

and, moreover,

$$\sup_{(x,y)\in\Omega_{ij}} dist\{(x,y), \Omega_{jj}\} < \inf_{(x,y)\in\Omega_{ik}} dist\{(x,y), \Omega_{jj}\} \ for\ any\ k \neq j. \tag{8.8$b$}$$

The last condition means, in other words, that all points of $\Omega_{ij}$ are closer to $\Omega_{jj}$ than any point from $\Omega_{ik}$ with $k \neq j$. This holds, for small $r_0$, due to **b**) and since $(x_i, y_j)$ is closer to $(x_j, y_j)$ than all $(x_i, y_k)$, $k \neq j$.

Using (8.8a), we find that relation (8.5b) takes the form

$$\sum_{l=1}^{m} T_{jl}^i X_l^i = K_{ij}, \tag{8.9}$$

where the $i$-th matrix $\mathbf{T}^i$ is defined by the relation

$$T_{jl}^i = \int_{I_{il}} G_a(x_i, y, x_j, y_j) b_{il}(x_i, y)\, dy \tag{8.10}$$

with the set $I_{il} \subset (0,1)$ consisting of such $y \in (0,1)$ that $(x_i, y) \in \Omega_{il}$.

Let us notice now, for a fixed index $i$, that relation (8.9) is a linear system of $m$ equations relatively $m$ unknown numbers $X_l^i, \quad l = 1, 2, \ldots, m$.

Let us show that all these systems are solvable if the parameter $a$ is sufficiently large, $a > a_0$. To make this, we shall prove that the matrices $\mathbf{T}^i$ satisfy the classical Gershgorin condition [22]. Namely, in each row, sums of the modules of its off-diagonal elements is less than the corresponding diagonal element. Otherwise, since all elements are positive, this means that

$$T_{jj}^i > \sum_{l \neq j} T_{jl}^i. \tag{8.11}$$

Then these matrices are invertible and each solution $X_l^i$ can be found by iterations [22]. This assertion is a simple consequence of assumption (8.6), relations (8.8) and the well-known properties of the Green's function $G_a$. In fact, for large $a$, the Green's function is exponentially decreasing. We shall show now that, as a result, matrices $T_{jk}^i$ are almost diagonal for such $a$. Indeed, we can represent $G_a$ as $G_a(x, y, x_j, y_j) = G_a^0(x, y, x_j, y_j) + \tilde{G}_a^{(j)}(x, y)$, where $G_a^0(x, y, x_j, y_j)$ is the fundamental solution for the operator $\Delta - a^2$ in all $\mathbb{R}^2$, $\tilde{G}_a^{(j)}$ is the solution of the following Neumann problem:

$$(\Delta - a^2)\tilde{G}_a^{(j)} = 0, \quad \frac{\partial \tilde{G}_a^{(j)}(x, y)}{\partial \nu} = \beta_j(x, y), \quad ((x, y) \in \partial\Omega), \tag{8.12}$$

where $\beta_j$ is the trace of the normal derivative of the fundamental solution on the boundary: $\beta_j = -\partial G_a^0(x, y, x_j, y_j)/\partial \nu$ for $(x, y) \in \partial\Omega$.

It is well known that the fundamental solution $G_a^0(x, y, x_j, y_j)$ is exponentially decreasing. Indeed, as $a \to \infty$, one has the exponentially decreasing asymptotic (see, for example, [43])

$$G_a^0(x, y, x_j, y_j) \approx c r^{-1/2} \exp(-a\ r), \tag{8.13}$$

where $r = dist\{(x, y), (x_j, y_j)\}$ and $r > \delta > 0$, similarly for $x$ and $y$- derivatives of $G_a^0$. The estimate of type (8.13) for derivatives shows, by (8.6), that $|\beta_j| < c \exp(-a/6)$. Thus, from (8.12) one obtains

$$|\tilde{G}_a^{(j)}(x, y)| < c \exp(-a/6). \tag{8.14}$$

Then, for large $a$, we find

$$T_{jl}^i = (1 + O(\exp(-c\ a))) \int_{I_{il}} G_a^0(x_i, y, x_j, y_j) b_{il}(x_i, y)\, dy \tag{8.15}$$

with some $c > 0$. Using now again the asymptotic (8.13) of the fundamental solution $G_a^0$, (8.14) and property (8.8b), we obtain

$$\sum_{l \neq j} T_{jl}^i < C \exp(-c_1 a) T_{jj}^i \tag{8.16}$$

for large $a$ and some positive $C, c_1$. In the $i$-th equation (8.9), let us divide the $j$-th row on the corresponding diagonal element. We obtain then linear systems of the following form:

$$(\mathbf{I} + \tilde{\mathbf{T}}^i)\mathbf{X}^i = \mathbf{B}^i,$$

where $\mathbf{I}$ is the identity matrix, elements of the matrix $\tilde{\mathbf{T}}^i$ are exponentially small ($< C \exp(-ca)$) and $\mathbf{X}^i = (X_1^i, \ldots, X_m^i)$. Clearly such systems have solutions for large $a$, and they can be obtained by iterations.

The proof is complete. Let us remark, in addition, that for large $a$ we can find, by this constructive method, a very good first approximation for $X_l^i$ and $\tilde{f}_1$ which satisfies (8.5b) with an exponential small error $O(\exp(-ca))$. In fact, one can take $X_l^i = (T_{ll}^i)^{-1} K_{il}$.

Let us describe, as a whole, the embedding procedure of a prescribed hyperbolic set into the global attractor of (1.2)–(1.4). This proceeds in three steps.

The first step is the following. Consider system (1.0) possessing a hyperbolic set $\Gamma$. Given $\Gamma$, we can define the constant of "structural stability" $\kappa(\Gamma, Q)$ of this set. Then, according to Theorem 1.1, one can choose such $m(\Gamma, Q)$, $\bar{\mathbf{K}} = \mathbf{K}(\Gamma, Q)$ and $\bar{\theta}_i = \theta_i(\Gamma, Q)$ that system (1.6) has an inertial form close to (1.0) and satisfying (1.7) with $\delta = \frac{1}{2}\kappa$.

The second step is as follows. Given $m = m(\Gamma, Q)$, let us fix some points $x_i$ and $y_j$ from $(0, 1)$ such that $i \neq j$ implies $x_i \neq x_j$ and $y_i \neq y_j$ where $i, j = 1, 2, \ldots, m$. Moreover, these points should satisfy (8.6).

Let us define, by (3.1) and (3.2), the potential $w_m(x, \epsilon, X_m)$ depending on the auxiliary parameter $\epsilon$ and consisting of $m$ potential wells at $x_i$. Let us fix a number $\rho > 2$.

At the third step, first we choose a sufficiently large $a$ and small $\tilde{\delta}$ (so that Lemma 8.2 holds). This choice depends naturally on the number $m$ and $Z_m$, i.e., on the location of points $x_i, y_j$.

After this, we find the functions $\tilde{f}_i$ satisfying (8.5a) and (8.5b) for $\theta$ and $\mathbf{K}$ obtained in the first step.

Furthermore, we take a sufficiently large $\mu$ and define $\mu_0$ and $\mu_1$ according to relations (6.3). One sets $f_i = \epsilon^{\mu_i} \tilde{f}_i$.

Let us define now $\tilde{C} = \max\{|\tilde{f}_0|_{C^2}, |\tilde{f}_1|_{C^2}\}$. Moreover, for each $\epsilon$ we define $g$ by (7.0). Then, for any sufficiently small $\epsilon < \epsilon_0(Z_m, m, \rho, \tilde{C}, a)$ Proposition 8.1 holds. Moreover, one notices that, if necessary, the quantity $c\epsilon^s$ in the right-hand side of (8.4) can always be done (by diminishing $\epsilon_0$) less than $\frac{1}{2}\kappa(\Gamma)$.

This remark completes the proof and the construction of the embedding algorithm. Required coefficients $f_i(x, y), g(x, y)$ and $w(x)$ can be defined by the relations $w = w_m(x, \epsilon, X_m)$ and $f = f_i(x, y, \epsilon)$, $g = g(x, y, \epsilon)$ and the formulae of Sections 3, 6 and 7, with any fixed $\epsilon < \epsilon_0$.

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**Appendix.** This appendix contains the proof of the approximation Lemma 2.1. Let us recall its formulation. To simplify denotations, we use the notation $X \in \mathbb{R}^m$ instead of $p$ for points of the space $\in \mathbb{R}^m$.

Let us consider the family of vector fields $\Psi(X)$ defined by

$$\Psi_i(X, \mathbf{A}, \mathbf{B}, m, \theta) = \sum_{p=1}^{m} B_{ip}\Phi(\sum_{j=1}^{n} A_{pj}X_j + \theta_p), \quad (i = 1, 2, \ldots, n) \tag{A1}$$

depending on parameters $\mathbf{P} = (m, \mathbf{A}, \mathbf{B}, \theta)$.

**Approximation Lemma.** *Assume that $\Phi$ satisfies (1.5). Let $G$ be an arbitrary $C^1$-vector field defined in the unit ball $\mathcal{B}^n$ and directed inside $\mathcal{B}^n$ at the boundary $\partial\mathcal{B}^n$. Let $\mathcal{H}_i(X) = G_i(X) + X_i$. Then for any such $G$ and any positive number $\delta$ there exist a number $m$, matrices $\mathbf{A}, \mathbf{B}$ and $m$-vector $\theta$ such that*

$$|\mathcal{H} - \Psi(\cdot, \mathbf{A}, \mathbf{B}, m, \theta)|_{C^1(\mathcal{B}^n)} < \delta, \tag{A2}$$

*where, in addition, the equalities*

$$A_{ij} = \delta_{ij}, \quad \theta_i = 0 \quad (i, j \le n) \tag{A3}$$

*hold, and for any $X$ such that $|X| > 1$ the inequality*

$$X \cdot \Psi = \sum_{l=1}^{n} X_l \Psi_l(X, \mathbf{A}, \mathbf{B}, m, \theta) < -\delta_1 + |X|^2, \quad \delta_1 > 0, \tag{A4}$$

*is valid.*

Let us note that this lemma is a somewhat standard, to a certain extent. Namely, more simple variants of this lemma are well known. For example, see [10, 20, 32, 4]. The proof given below uses some ideas from these works and allows us to determine $\mathbf{A}, \mathbf{B}, m, \theta$ in a constructive way. The main instrument is the so-called Calderón identity (see below).

We note also that without loss of generality one can suppose that $\Phi(0) \neq 0$.

**Proof of lemma. Step 1. Extension of a vector field**. Let us continue the vector field $\mathcal{H}$ defined in the ball $\mathcal{B}^n$ to the whole space so that the continuation $\tilde{\mathcal{H}}$ has the following properties:

**a**) $\tilde{\mathcal{H}} \in C^1(\mathbb{R}^n)$ and the vector $\tilde{\mathcal{H}}(X) - X$ is directed inside the unit ball $\mathcal{B}^n$ if $X$ lies in the set $V = \{X : 1 < |X| < 1 + \delta_2\}$; that is, $\tilde{\mathcal{H}}(X) \cdot X < -\delta_1 + |X|^2, X \in V$.

**b**) $\tilde{\mathcal{H}} \equiv \mathcal{H}$ for any $X \in \mathcal{B}^n$ and $\tilde{\mathcal{H}} = 0$ for any $X \in \mathbb{R}^n \setminus (\mathcal{B}^n \cup V)$.

**Step 2. Approximation of the smooth field by averaging**. Given $\mathcal{H}$, let us construct a $C^\infty(\mathbb{R}^n)$ vector field $\mathcal{H}_{av}$ from the L. Schwartz class $S(\mathbb{R}^n)$ such that $|\mathcal{H} - \mathcal{H}_{av}|_{C^1(\mathbb{R}^n)} < \delta/10$. This approximation can be obtained by an averaging procedure. For example, one can set

$$\mathcal{H}_{av}^r(X) = r^{-n} \int_{\mathbb{R}^n} \mathcal{H}(X - \tilde{X})\omega(\tilde{X}/r)d^n\tilde{X},$$

where $\omega(x)$ is a standard cut-off function such that $\int \omega(x)d^nx = 1$. Let us set $\mathcal{H}_{av} = \mathcal{H}_{av}^r$ for a small $r$.

Now our aim is to solve the approximation problem for $\mathcal{H}_{av}$ from $S(\mathbb{R}^n)$.

**Step 3. Representation $\mathcal{H}_{av}$ as a sum of "plane waves" and reformulation of the approximation problem**. Here we follow the paper [4]. It is easy to show that each component $(\mathcal{H}_{av})_j$ can be approximated by a sum of "plane waves" $\eta_{jl} = \eta_{jl}((X, s^{jl}))$

$$\bar{\mathcal{H}}_j(X) = L_j^{-1} \sum_{l=1}^{L_j} \eta_{jl}((X, \mathbf{s}^{jl})), \quad (X, \mathbf{s}) = \sum_{i=1}^{n} X_i s_i, \qquad (A5)$$

where the vectors $\mathbf{s}^{jl}$ are distributed over the unit sphere and the functions $\eta_{jl}(z)$ are from the Schwartz class $S(\mathbb{R})$. This decomposition can reduce a problem for $\mathbb{R}^n$ to a scalar problem in $\mathbb{R}$. In fact, the functions $\eta_{jl}$ depend on one-dimensional arguments $z = (X, \mathbf{s}^{jl})$, and their surface levels are hyperplanes (that explains our terminology).

Let us fix an index $j$ and below, for brevity, let us omit temporarily this index in notations $\mathbf{s}^{jl}$ and $L_j$.

To obtain approximation (A5), we can use the so-called Radon transformation ([27], Chapter 1, or [28], Chapter 1, Theorems 2.4 and 2.13). Indeed, any function $f(X) \in S(\mathbb{R}^n)$ can be represented as an integral over the unit sphere $S^{n-1}$, namely

$$f(X) = \int_{S^{n-1}} \phi_f(\mathbf{s}, (X, \mathbf{s}))\, d\mu(\mathbf{s}),$$

where $\phi_f(\mathbf{s}, z) \in C^\infty$ and lies in the Schwartz class as a function of the argument $z = (X, \mathbf{s})$ and $d\mu$ is the usual uniform measure on the unit sphere (see [27, 28]).

In order to obtain (A5), one can approximate the integral over the unit sphere by a finite Riemannian sum.

In fact, let us introduce the spherical coordinates and decompose all regions of integration into $L$ small boxes $\Pi_k$ of diameter $\kappa(L)$. The number $L$ of such boxes has then the order $C\kappa^{-n}$. We choose $\mathbf{s}^l$ as a center of $l$-th box, and let us set $\eta_{jl}(\mathbf{s},(X,\mathbf{s})) = \phi_f(\mathbf{s},(X,\mathbf{s}))$, $f = (\mathcal{H}_{av})_j$. Let us estimate now the difference between $(\mathcal{H}_{av})_j$ and $\bar{\mathcal{H}}_j$ from (A5). We notice that the first function is the above integral over the unit sphere and the second one is the corresponding Riemannian sum for this integral.

Let us define, for such $X$, the subset $U_X \subset \mathbb{S}^{n-1}$ of the sphere by $U_X = \{\mathbf{s} : |(\mathbf{s},X)| < |X|^{1/2}\}$. The measure of this set is $o(|X|)$ as $|X| \to \infty$ and its contribution into the integral over the sphere also vanishes as $|X| \to \infty$. It is clear that the contribution of $\mathbf{s}^l \in U_X$ in the discrete sum for $\bar{\mathcal{H}}$ also tends to zero as $|X| \to \infty$. On the other hand, for $\mathbf{s}$ that lie outside of $U_X$, the integrand $\phi(\mathbf{s},(X,\mathbf{s}))$ is smooth and fast decreasing in $X$. Therefore, for any $j$, since the field $\mathcal{H}_{av} \in S(\mathbb{R}^n)$, we can choose the radius $R_0$ so that the estimate $|(\mathcal{H}_{av})_j - \bar{\mathcal{H}}_j| < \delta/10$ holds for $|X| > R_0$, together with the same estimate for the $X$-derivatives of this difference.

On the other hand, for $|X| < R_0$ we have

$$|\phi(\mathbf{s},(X,\mathbf{s})) - \phi(\mathbf{s}^l,(X,\mathbf{s}^l))| \le C|\mathbf{s}^l - \mathbf{s}| \le C\kappa(L),$$

where the second inequality holds if the points $\mathbf{s}^l$ and $\mathbf{s}$ are from the same box. Thus, the integral of $|\phi(\mathbf{s},(X,\mathbf{s})) - \phi(\mathbf{s}^l,(X,\mathbf{s}^l))|$ over each small box has the order $O(\kappa^{n+1})$. This implies the estimate of $|(\mathcal{H}_{av})_j - \bar{\mathcal{H}}_j|$ for $|X| < R_0$ and large $L$.

Finally, for sufficiently large $L_j$ and appropriate $\eta_{jl}$, we obtain

$$|\mathcal{H}_{av} - \bar{\mathcal{H}}|_{C^1(\mathbb{R}^n)} < \delta/10. \qquad (A6)$$

Let us construct an approximation $\Psi(X,\mathbf{A},\mathbf{B},m,\theta)$ of $\bar{\mathcal{H}}$ such that

$$|\bar{\mathcal{H}}(X) - \Psi(X,\mathbf{A},\mathbf{B},m,\theta)| < \frac{\delta}{3}(1+|X|), \quad (X \in \mathbb{R}^n) \qquad (A7)$$

$$|\frac{\partial\bar{\mathcal{H}}}{\partial X_j}(X) - \frac{\partial\Psi}{\partial X_j}(X,\mathbf{A},\mathbf{B},m,\theta)| < \frac{1}{3}\delta, \quad (X \in \mathbb{R}^n). \qquad (A8)$$

It is obvious that (A7) and (A8) yield (A2) and (A4); thus it is sufficient to ensure (A7) and (A8).

In fact, for small $\delta$ outside of the set $V$ defined in step 1 this results from condition **a)** of step 1 that inequalities $|\Psi \cdot X| < -\delta_1 + |X|^2$, and thus (A4) holds. On the other hand, in the layer $V$ one obtains $|\Psi \cdot X - (G+X)\cdot X)| < 2\delta$ that entails $|\Psi \cdot X| < |(G+X)\cdot X)| + 2\delta$, and since $G \cdot X < -\delta_3 < 0$, one finds that again (A4) holds.

The following step gives an approximation of the plane waves $\eta_{jl}(z)$ that allows us to satisfy (A7) and (A8).

**Step 4. Approximation of functions $\eta_j(z)$.** Given numbers $a_i, c_i \in \mathbb{R}$ and points $Z_i \in \mathbb{R}$ where $i = 0,1,\ldots,M$ consider

$$\xi(z,a,c,Z,M) = \sum_{i=0}^{M} c_i \Phi(a_i(z - Z_i)), \qquad (A9)$$

where $c = (c_0, \dots, c_M), a = (a_0, \dots, a_M)$, $Z = (Z_0, \dots, Z_M)$. Let us check that for any $\varepsilon > 0$ there exist $a, c, Z$ such that

$$|\nu(z) - \xi'_z(z, a, c, Z, M)| < \varepsilon, \quad (z \in \mathbb{R}), \quad \nu = \eta'_z. \tag{A10}$$

Notice that without loss of generality we can assume $\nu(0) = \xi(0, a, c, z, M)$. In fact, if this property does not hold, we can add to the sum (A9) the constant term $c_{M+1}\Phi(a_{M+1}\cdot(z - Z_{M+1}))$ with $c_{M+1} = \nu(0)/\Phi(0), \quad a_{M+1} = 0$. (Let us recall that we suppose $\Phi(0) \neq 0$; see above.)

As a result, integrating (A10) over the interval $(0, z)$, we get the existence of $\xi$ of the form (A9) such that

$$|\eta - \xi(z, a, c, Z)| < \varepsilon(1 + |z|), \quad (z \in \mathbb{R}). \tag{A11}$$

Below (see the final step) we shall show that approximations satisfying (A10) and (A11) give a solution to the approximation problem (A3) and (A4).

Let us focus our attention on (A10), since it yields (A11). To satisfy this inequality, one uses a "wavelet-type" approach.

Let $\gamma_k, z_k$, $k = 1, 2, 3$, be real numbers such that $z_1 < z_2 < z_3$ and

$$\gamma_1 + \gamma_2 + \gamma_3 = 0, \quad \gamma_1 z_1 + \gamma_2 z_2 + \gamma_3 z_3 = 0. \tag{A12}$$

Then the function

$$\rho(z) = \sum_{i=1}^{3} \gamma_i \Phi'(z - z_i) \tag{A13}$$

is a "wavelet-type" function such that

$$\int_{-\infty}^{\infty} \rho(z)\, dz = 0, \quad \int_{-\infty}^{\infty} z\rho(z)\, dz = 0. \tag{A14}$$

Assume (without any loss of generality) that for $\nu(z) = \eta'(z)$

$$\int_{-\infty}^{\infty} \nu(z)\, dz = 0, \quad \int_{-\infty}^{\infty} z\nu(z)\, dz = 0 \tag{A15}$$

hold. In fact, the first relation always holds. If the second one is invalid, then one sets

$$\tilde{\nu} = \nu + \tilde{c}(\Phi'(z) - \Phi'(z - z_0)) \tag{A16}$$

with

$$\tilde{c} = z_0^{-1} \, (\int_{-\infty}^{\infty} x\Phi'\, dx)^{-1} \int_{-\infty}^{\infty} x\nu dx$$

and $z_0 \neq 0$. Then $\tilde{\nu}$ satisfies both conditions (A15). (This substitution was suggested by the anonymous referee of the paper.)

To solve the approximation problem, let us use the well-known Calderón formula,

$$\nu(z) = C_\rho \int_0^{\infty} \int_{-\infty}^{\infty} \lambda^{-3} C(\lambda, X) \rho(\lambda^{-1}(z - X))\, dX\, d\lambda, \tag{A17}$$

where

$$C(\lambda, X) = \int_{-\infty}^{\infty} \nu(z)\rho(\lambda^{-1}(z - X))\, dz, \quad X \in \mathbb{R} \tag{A18}$$

and $C_\rho$ is a constant depending on the function $\rho$. Notice that relations (A17) and (A18) hold due to the fact that $\Phi$ satisfies (1.5), and thus $\rho \in S(\mathbb{R})$.

This identity is a fact long known to harmonic analysts and plays also an important role in quantum and mathematical physics. The Aslaksen-Klauder construction of so-called $(ax+b)$-coherent states can be seen as an independent derivation of this formula [2]. Also, this is one of the basic ingredients of the wavelet theory. This can be considered as an example of a continuous wavelet transform which allows us to break up a complicated phenomenon into many simple pieces and study each of the pieces separately. See, for example, [12, 42]. At last, let us notice that this formula can be easily proved by the Fourier transformation.

The next step is an approximation of (A17) by a discrete sum. To make this, let us notice the following. Suppose the integral (A17) can be approximated by

$$I(r, R_1, R, z) = C_\rho \int_r^{R_1} (\int_{-R}^{R} \lambda^{-3} C(\lambda, X)\rho(\lambda^{-1}(z - X))\, dX)\, d\lambda \tag{A19}$$

(where $R_1, R$ and $r$ are positive) with any precision $\delta_1$. Then the approximation problem (A10) is solved. In fact, for any $\varepsilon > 0$ one can write, using Riemannian sums, that

$$\tilde{I}_{res}(r, R_1, R, z) = \sup_z |I(r, R_1, R, z) - \sum_{i=1}^{M_1}\sum_{j=1}^{M_1} \lambda_i{}^{-3} C(\lambda_i, X_j)\rho(\lambda_i^{-1}(z - X_j))| < \varepsilon \tag{A20}$$

for some $\lambda_i$ and $X_j$. One can take, for example, $\lambda_i = r + i(R_1 - r)M_1^{-1}$ and $X_i = (2i - 1)RM_1^{-1} - R$. Then, due to (A13), the sum in (A20) can be rewritten as $\sum_{i=1}^{M} c_i \Phi'(a_i(x - Z_i))$, where $Z_i = X_i$, $M = M_1^2$. Let us prove that estimate (A20) holds uniformly in $z$. Let us denote $\kappa = M_1^{-1}\max(2R, R_1 - r)$. Then, since $\rho \in S(\mathbb{R})$, using the same arguments as above in the derivation of (A6), we have

$$\sup_{z\in\mathbb{R}} \tilde{I}_{res}(r, R_1, R, z) < C\kappa \sup_{z\in\mathbb{R}, \lambda\in[r,R_1], X\in[-R,R]} (|\rho'_\lambda(\lambda^{-1}(z - X))| + |\rho'_X(\lambda^{-1}(z - X))|)$$

$$< C_1(r, R_1, R)\kappa,$$

which immediately implies (A20) for sufficiently large $M_1$ (and, respectively, small $\kappa$).

Finally, to solve the approximation problem, it is sufficient to check that the integrals

$$J_1 = \int_0^r (\int_{-\infty}^{\infty} \lambda^{-3} C(\lambda, X)\rho(\lambda^{-1}(z - X))\, dX)\, d\lambda,$$

$$J_2 = \int_r^{R_1} (\int_R^{\infty} \lambda^{-3} C(\lambda, X)\rho(\lambda^{-1}(z - X))\, dX)\, d\lambda,$$

$$J_3 = \int_{R_1}^{\infty} (\int_{-\infty}^{\infty} \lambda^{-3} C(\lambda, X)\rho(\lambda^{-1}(z - X))\, dX)\, d\lambda,$$

satisfy

$$|J_1| < \kappa_1(r), \quad |J_2| < \kappa_2(r, R, R_1), \quad |J_3| < \kappa_3(R_1), \tag{A21}$$

where $\kappa_i \to 0$ as $R, R_1 \to \infty$ and $r \to 0$. In order to investigate $J_k$, let us evaluate $C(\lambda, X)$. First let us notice that, due to condition (1.5), we have $\rho \in S(\mathbb{R})$; thus, after the substitution $z = X + \lambda y$ relation (A14) yields

$$C(\lambda, X) = \lambda \int_{-\infty}^{\infty} (\nu(X + \lambda y) - \nu'(X)\lambda y - \nu(X))\rho(y)\, dy, \quad X \in \mathbb{R},$$

and now the Taylor formula gives

$$|C(\lambda, X)| < \lambda^3 | \int_R y^2 \nu''(X + \lambda s y)\rho(y)\, dy\ |, \tag{A22}$$

for $0 < \lambda < 1$ and some $s = s(\lambda, y) \in [0, 1]$. Since $\nu \in S(\mathbb{R})$, one obtains

$$|C(\lambda, X)| < c_N \lambda^3 (1 + |X|)^{-N}, \tag{A23}$$

and, therefore, $|J_1| < cr$. Thus, estimate (A21) holds for $J_1$.

Consider $J_2$. One observes from (A18) that $|C(\lambda, X)| < c_N |1 + \lambda^{-1} X|^{-N}$ for any $N > 0$ and $\lambda > 1$. In fact,

$$(1 + \lambda^{-1}|z - X|)(1 + |z|) \geq 1 + \lambda^{-1}|z| + \lambda^{-1}|z - X| \geq 1 + \lambda^{-1}|X|,$$

and hence, since $\rho$ lies in the Schwartz class,

$$|C(X, \lambda)| \leq c_0 \int_0^{\infty} [(1 + \lambda^{-1}|z - X|)(1 + |z|)]^{-2N} dz$$

$$\leq c \max[(1 + \lambda^{-1}|z - X|)(1 + |z|)]^{-N} \int_0^{\infty} (1 + |z|)^{-N} dz,$$

which gives, by the previous inequality, the estimate of $C(X, \lambda)$.

To obtain the estimate of $J_2$, let us split this integral in two parts: the integral $J_2^1$ over $[1, \infty]$ and $J_2^0$ over $[r, 1]$. Then

$$J_2^1 \leq C_N \int_1^{\infty} \lambda^{-3} d\lambda \int_R^{\infty} (1 + \lambda^{-1} X)^{-N} dX < \int_1^{\infty} \lambda^{-2}\, d\lambda\, (1 + \lambda^{-1} R)^{-N+1} < cR^{-1}.$$

The part of the integral $J_2$ over $\lambda \in [r, 1]$ can be easily evaluated by (A23). In fact, one finds

$$|J_2^0| = | \int_r^1 (\int_R^{\infty} \lambda^{-3} C(\lambda, X) \rho(\lambda^{-1}(z - X))\, dX)\, d\lambda|$$

$$\leq C_N \max|\rho| \int_R^{\infty} (1 + |X|)^{-N} dX < c_2 R^{-N+1}.$$

As a result we get the second inequality in (A21).

In a similar way

$$|J_3| \leq C_N \int_{R_1}^{\infty} \lambda^{-3} d\lambda \int_{-\infty}^{\infty} (1 + \lambda^{-1}|X|)^{-N} dX \leq C_0 \int_{R_1}^{\infty} \lambda^{-2} d\lambda \leq C_1 {R_1}^{-1}.$$

Hence (A2) holds. This means that the approximation problem for $\eta_j$ is solved. Namely it is shown that, for any $\varepsilon$, there exist $a_i, c_i$ and $Z_i$ such that (A10) and (A11) hold.

**Final step**. This step is a combination of the previous step and the "plane wave" representation. It results in

$$\Psi_i(X, \tilde{\mathbf{A}}, \mathbf{C}, X, m_1, L) = \sum_{l=1}^{L} \sum_{k=1}^{m_1} B_{ikl} \Phi(\tilde{A}_{ik}^{l}((X, \mathbf{s}^{il}) + Z_{ikl})). \qquad (A24)$$

Notice that sums in (A24) can be rewritten in the form (A1). It can be done by simple algebraic transformations.

Let us show it. First we observe that, by a renumeration, the double index $(l, k)$ can be replaced by a single index $p$. Then (A24) takes the form

$$\Psi_i(X, Z, m_1, L) = \sum_{p=1}^{M} B_{ip} \Phi(\sum_{j=1}^{n} \bar{A}_{ipj} X_j + Z_{ip}),$$

where $\bar{A}$ is some matrix. This sum almost coincides with sum (A1), but $\bar{A}$ and $Z$ depend on the component index $i$.

We can however transform this last sum to form (A1) in a very similar way: simply adding new terms with zero coefficients. For example, if

$$\Psi_1(X) = \sum_{p=1}^{M} B_{1p} \Phi(\sum_{j=1}^{n} \bar{A}_{1pj} X_j + Z_{1p}),$$

$$\Psi_2(X) = \sum_{p=1}^{M} B_{2p} \Phi(\sum_{j=1}^{n} \bar{A}_{2pj} X_j + Z_{2p}),$$

then, introducing a $2M$-vector $\theta$ and a $2M \times n$ matrix $A$ instead of the $M \times n$ matrices $\bar{A}_1$, $\bar{A}_2$ and the $M$-vectors $Z_1$ and $Z_2$, we notice, for example, that the sum for $\Psi_1$ can be rewritten as

$$\Psi_1(X) = \sum_{p=1}^{M} B_{1p} \Phi(\sum_{j=1}^{n} A_{pj} X_j + \theta_p) + \sum_{p=M+1}^{2M} 0 \cdot \Phi(\sum_{j=1}^{n} A_{pj} X_j + \theta_p).$$

Finally, we have obtained representation (A1) satisfying (A2) and (A4). It remains to satisfy (A3). If restriction (A3) does not hold, we can add, in a similar way, new terms with zero coefficients $B_p$ in (A1) and satisfy condition (A3). The proof is complete.

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