Advances in Differential Equations Volume 5 (7–9) July–September 2000, pp. 1033–1090

ON THE ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE LANDAU-LIFSHITZ EQUATION

Takeshi Isobe
Department of Mathematics, Faculty of Science Tokyo Institute of Technology
Oh-okayama, Meguro-ku, Tokyo 152, Japan

(Submitted by: P.L. Lions)

Abstract. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $H \in \mathbb{R}^3$. The Landau-Lifshitz equation with external field H and boundary data $\gamma \in C^\infty(\partial\Omega; \mathbb{S}^2)$ is the following:

$$\Delta u + |\nabla u|^2 u - (H, u)u + H = 0 \quad \text{in } \Omega, \quad u = \gamma \quad \text{on } \partial\Omega.$$

Here $u \in C^\infty(\Omega; \mathbb{S}^2)$. We study the asymptotic behavior of the solutions of this equation as $H \to 0$. We show that the "large solutions" obtained by Hong and Lemaire blow up only when $\gamma \equiv$ const. and in such a case blow-up occurs only at a single point in Ω. We characterize the blow-up point as a critical point of a certain function defined in Ω. We also give the asymptotic value estimate of $\|\nabla u\|_{L^\infty(\Omega)}$ as $H \to 0$.

1. Introduction. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain, $H \in \mathbb{R}^3$ and $\gamma \in C^\infty(\Omega; \mathbb{S}^2)$. Define

$$H^1_\gamma(\Omega; \mathbb{S}^2) := \{u \in H^1(\Omega; \mathbb{R}^3) : u|_{\partial\Omega} = \gamma|_{\partial\Omega},\ |u| = 1 \text{ a.e. } \Omega\}.$$

For $u \in H^1_\gamma(\Omega; \mathbb{S}^2)$, the energy of u with the external field H is given by

$$\mathcal{E}_H(u) = \int_\Omega (|\nabla u|^2 - 2(H, u))\, dx,$$

where $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^3$.

The critical point u of $\mathcal{E}_H$ in $H^1_\gamma(\Omega; \mathbb{S}^2)$ satisfies the following *Landau-Lifshitz* equation:

$$\begin{cases} \Delta u + |\nabla u|^2 u - (H, u)u + H = 0 & \text{in } \Omega, \\ u = \gamma & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Accepted for publication July 1999.
AMS Subject Classifications: 35B40, 35J25, 35J50, 58E20.

Equation (1.1) arises in physics, for example, to describe the equilibrium states of ferromagnetism. In this case u is the spin vector and H is an external magnetic field. See [9] for the physical background of the equation (1.1). Recently, some authors studied the mathematical aspects of this equation; see [1], [9], [11], [12], [26] and [27]. Concerning the existence of the solutions to (1.1), Hong and Lemaire [12] proved that if $H \neq 0$ and γ is neither constant $\frac{H}{|H|}$ nor constant $-\frac{H}{|H|}$, then there exist at least two homotopically distinct solutions to (1.1). In the case $H = 0$, (1.1) is the harmonic map equation, and in this case Brezis and Coron [5] and Jost [16] proved the existence of at least two homotopically distinct solutions to (1.1) under the assumption $\gamma \neq$ const. Also, in this case it is known that when Ω is simply connected and $\gamma \equiv$ const., (1.1) has a unique solution $u \equiv \gamma$ (see [17]). The result of Hong and Lemaire shows that there is a strong difference between the case $H \neq 0$ and the case $H = 0$. That is, when $H \neq 0$, even for constant boundary value (not equal to $\pm\frac{H}{|H|}$), (1.1) has at least two distinct solutions. In particular, there is a nonconstant solution. (See also [27] for a similar result.) Thus from the mathematical point of view (and the physical importance of the equation (1.1)—in some applications $|H|$ is small), it seems interesting to study the behavior of the solutions to (1.1) as $H \to 0$. This is also important in understanding the structure of the solution space for equation (1.1) (see Remark 1.1 below).

Our aim in this paper is to study the asymptotic behavior of the solutions to (1.1) as $H \to 0$. Since the work of Hong and Lemaire [12] is the starting point of our work, we review their work briefly.

By direct minimization, one can prove the existence of $\underline{u}_H \in H^1_\gamma(\Omega;\mathbb{S}^2)$ such that $\mathcal{E}_H(\underline{u}_H) = \inf_{u \in H^1_\gamma(\Omega;\mathbb{S}^2)} \mathcal{E}_H(u)$. $\underline{u}_H$ satisfies (1.1) and smooth. We call $\underline{u}_H$ a *small solution*. This is the first solution.

To obtain the second solution, decompose $H^1_\gamma(\Omega;\mathbb{S}^2)$ into countably many connected components Φ_k $(k \in \mathbb{Z})$ (see [5]):

$$H^1_\gamma(\Omega;\mathbb{S}^2) = \bigsqcup_{-\infty<k<+\infty} \Phi_k, \tag{1.2}$$

where $\Phi_k := \{u \in H^1_\gamma(\Omega;\mathbb{S}^2) : d(u) - d(\underline{u}_H) = k\}$, $d(u) = \frac{1}{4\pi}\int_\Omega (u, u_{x_1} \wedge u_{x_2})\,dx$.

Brezis and Coron showed in [5] that $d(u) - d(\underline{u}_H) \in \mathbb{Z}$ for $u \in H^1_\gamma(\Omega;\mathbb{S}^2)$ and $\Phi_k \neq \emptyset$ for any $k \in \mathbb{Z}$. In [12], modifying the argument of [5], Hong and Lemaire showed that if γ is neither constant $\frac{H}{|H|}$ nor constant $-\frac{H}{|H|}$, $\inf_{u\in\Phi_{+1}} \mathcal{E}_H(u)$ or $\inf_{u\in\Phi_{-1}} \mathcal{E}_H(u)$ is attained by $\overline{u}_H \in H^1_\gamma(\Omega;\mathbb{S}^2)$. Of course,

$\overline{u}_H$ satisfies (1.1) and is smooth. We call it a *large solution*. This is the second solution.

Our main purpose in this paper is to study the asymptotic behavior of $\overline{u}_H$ as $H \to 0$. For this we consider H of the form $H = |H|e$, where $e \in \mathbb{R}^3$ is a unit vector and $|H|$ is a positive constant.

Before stating our result, we make some remarks. First, the decomposition (1.2) is independent of H if $|H|$ is small. In fact, Theorem A (1) (see below) implies that all $\underline{u}_H$ (if $|H|$ is small) are relatively homotopic to one another. Secondly, for all small $|H|$, we may assume without loss of generality that $\overline{u}_H \in \Phi_{-1}$. In fact, from Theorem A (1) and the proof of Lemma 2.1 in [12], one can prove that there exists $|H_0| > 0$ such that if $\inf_{u\in\Phi_{\pm 1}} \mathcal{E}_{H_0}(u)$ is attained, then $\inf_{u\in\Phi_{\pm 1}} \mathcal{E}_H(u)$ is attained for all H with $|H| \leq |H_0|$. Under these remarks and conventions, we now state our main results:

Theorem A. *Let $\{H_n\} \in \mathbb{R}^n$ be such that $H_n = |H_n|e$, $|H_n| \neq 0$, $|H_n| \to 0$ (as $n \to \infty$). We write $\underline{u}_n = \underline{u}_{H_n}$ and $\overline{u}_n = \overline{u}_{H_n}$.*

(1) *There exists a subsequence of $\{\underline{u}_n\}$ (still denoted by $\{\underline{u}_n\}$) and an energy-minimizing harmonic map $\underline{u} \in C^\infty(\overline{\Omega})$ in $H^1_\gamma(\Omega;\mathbb{S}^2)$ such that $\underline{u}_n \to \underline{u}$ in $C^\infty(\overline{\Omega})$.*

(2-a) *Assume γ is not constant. There exists a subsequence of $\{\overline{u}_n\}$ (still denoted by $\{\overline{u}_n\}$) and an E_0-minimizing map $\overline{u} \in C^\infty(\overline{\Omega})$ in Φ_{-1} (that is, a "large harmonic map") such that $\overline{u}_n \to \overline{u}$ in $C^\infty(\overline{\Omega})$.*

(2-b) *Assume $\gamma \equiv$ const. and γ is not equal to $\pm\frac{H_n}{|H_n|}$ for all n (in other words, $e - (e,\gamma)\gamma \neq 0$). There exists a subsequence of $\{\overline{u}_n\}$ (still denoted by $\{\overline{u}_n\}$) and $a_\infty \in \overline{\Omega}$ such that*

(2-b-i) *$\overline{u}_n \to \gamma$ in $C^\infty_{\mathrm{loc}}(\overline{\Omega}\setminus\{a_\infty\})$.*

(2-b-ii) *$|\nabla\overline{u}_n|^2 \rightharpoonup 8\pi\delta_{a_\infty}$ in the sense of Radon measures in $\overline{\Omega}$. Here δ_{a_∞} is the Dirac measure with unit mass at a_∞.*

(2-b-iii) *There exist $\lambda_n > 0$, $\lambda_n \to 0$, $a_n \in \Omega$, $a_n \to a_\infty$ and $R_n \in SO(3)$ such that*

$$\|\nabla(\overline{u}_n - \underline{u}_n - R_n\widehat{U}_{\lambda_n,a_n})\|_{L^2(\Omega)} \to 0 \quad (n\to\infty), \tag{1.3}$$

$$\|\overline{u}_n - \underline{u}_n - R_n\widehat{U}_{\lambda_n,a_n}\|_{L^\infty(\Omega)} \to 0 \quad (n\to\infty). \tag{1.4}$$

Here, $\widehat{U}_{\lambda,a} = \frac{2\lambda}{\lambda^2+|x-a|^2}\begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ -\lambda \end{pmatrix}$ *and* $a = (a_1, a_2)$.

In this paper, we call the point a_∞ in Theorem A (2-b) the *blow-up point* of $\overline{u}_n$. The next theorem gives the complete characterization of the location of the blow-up point. Since $\{\overline{u}_n\}$ blows up only when $\|\nabla\overline{u}_n\|_{L^\infty(\Omega)} \to +\infty$ (as $n \to \infty$), it is also interesting to know the asymptotic value of $\|\nabla\overline{u}_n\|_{L^\infty(\Omega)}$ as $n \to \infty$. This is also given in the next theorem.

Theorem B. *Assume $\gamma \equiv$ const. and $e' := e - (e,\gamma)\gamma \neq 0$. Let a_∞ be the blow-up point of $\{\overline{u}_n\}$, where $\{\overline{u}_n\}$ and $\{H_n\}$ are as in Theorem* A (2-b). *We then have*

(1) *Let E be a function in Ω defined by the solution of*

$$\Delta E = -1 \text{ in } \Omega \ , \qquad E = 0 \text{ on } \partial\Omega \ . \tag{1.5}$$

For $a \in \Omega$, we also define functions h_a^i ($i = 1, 2$) by the solutions of

$$\Delta h_a^i = 0 \text{ in } \Omega \ , \qquad h_a^i = \frac{2(x_i - a_i)}{|x-a|^2} \text{ on } \partial\Omega \ . \tag{1.6}$$

Define the function $\Psi(a)$ ($a \in \Omega$) by

$$\Psi(a) = \frac{|\nabla E(a)|^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}. \tag{1.7}$$

Then $a_\infty \in \Omega$ (interior point), and a_∞ maximizes the function Ψ in Ω; that is, $\Psi(a_\infty) = \sup_{a\in\Omega}\Psi(a)$.

(2) *We have*

$$\lim_{n\to\infty} |H_n|\|\nabla\overline{u}_n\|_{L^\infty(\Omega)} = \frac{2\sqrt{2}}{|e'|} \cdot \frac{\frac{\partial h_{a_\infty}^1}{\partial x_1}(a_\infty) + \frac{\partial h_{a_\infty}^2}{\partial x_2}(a_\infty)}{|\nabla E(a_\infty)|}.$$

Remark 1.1. (a) The assumption $e' \neq 0$ in Theorem A (2-b) and Theorem B is necessary, since, in general, if $e' = 0$, the large solution does not exist; see [12]. (b) The function $\Omega \ni a \to \frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)$ is positive and goes to $+\infty$ if a approaches to $\partial\Omega$; see Lemma 3.8. Thus Ψ takes the maximum value at the interior point of Ω. (c) In the above theorems, we only consider the case $H = |H|e$ for some fixed unit vector e. However, our proof is easily extended to the general case $H = |H|e_H$, where the unit vector e_H varies with H and satisfies $e_H - (e_H,\gamma)e_H \neq 0$. We leave the details for this case to the reader. (d) In [27], Shen and Yan studied the asymptotic behavior of the axially symmetric solutions of (1.1) when $\Omega =$ unit ball in $\mathbb{R}^2$ and $H = |H|\gamma$. Using an ODE argument, they showed that axially symmetric solutions constructed in [27] blow up at the origin. On the other hand, when Ω = unit ball and $e-(e,\gamma)\gamma \neq 0$, Theorem B (1) implies that the large solutions blow up at the point a_∞ with $|a_\infty| = \frac{1}{\sqrt{3}}$. (In this case, $\Psi(a) = \frac{1}{16}|a|^2(1-|a|^2)^2$.) (e) Related blow-up problems arise in other areas of variational problems. For example, Rey studied in [22] the asymptotic behavior of the solutions to the equation $-\Delta u + \lambda u + u^{\frac{N+2}{N-2}} = 0$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$ when $\lambda \downarrow 0$. Here, $N =$ dim$\Omega \geq 3$. Bethuel et al. [2] and others ([18], [31], ... , etc.) studied the asymptotic behavior of the solutions to the Ginzburg-Landau equation $-\Delta u + \frac{1}{\epsilon^2}(|u|^2-1)u = 0$ in Ω, $u = g$ on $\partial\Omega$ as $\epsilon \to 0$, where dim $\Omega = 2$, $u \in H^1(\Omega;\mathbb{C})$ and $g \in C^1(\partial\Omega;\mathbb{S}^1)$. See [21], [24] for other related results. All these results suggest the following conjecture:

Conjecture: Let γ be a constant unit vector and $e-(e,\gamma)\gamma \neq 0$. Let a be a nondegenerate local maximum point (or more generally, nondegenerate critical point) of Ψ. Then, there exists $|H_0| > 0$ such that for all $H = |H|e$ with $0 < |H| \leq |H_0|$, there exists a solution $\overline{u}_H$ to (1.1) with $|\nabla \overline{u}_H|^2 \rightharpoonup 8\pi\delta_a$ as $H \to 0$.

This problem will be treated in a future work, [13].

Here we outline the proofs of Theorem A and Theorem B. The proof of Theorem A follows from a standard blow-up argument; see [23], [4]. This is carried out in Section 2.

The main idea of the proof of Theorem B is the following: First, we prove the following lower bound of $\mathcal{E}_{H_n}(\overline{u}_n)$:

$$\mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - 4\pi\Psi(a_n)|e'|^2|H_n|^2 + o(|H_n|^2) \tag{1.8}$$

as $n \to \infty$. Next, for a given $a \in \Omega$, we construct $\varphi_n \in \Phi_{-1}$ such that

$$\mathcal{E}_{H_n}(\varphi_n) = 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - 4\pi\Psi(a)|e'|^2|H_n|^2 + o(|H_n|^2) \tag{1.9}$$

as $n \to \infty$. (1.8), (1.9) and the minimizing property of $\overline{u}_n$ imply Theorem B (1).

The proof of (1.8) is based on the following expansion of $\overline{u}_n$: $\overline{u}_n = \underline{u}_n + R_n U_{\lambda_n,a_n} - R_n h_n + v_n$, where R_n, λ_n and a_n are as in Theorem A (2-b-iii), h_n is the harmonic extension of $\widehat{U}_{\lambda_n,a_n}|_{\partial\Omega}$ to Ω and $\|\nabla v_n\|_{L^2(\Omega)} = o(|H_n|)$. The proofs of it and (1.8) are given in Section 3. φ_n in (1.9) is constructed as a solution of the problem: $\inf_{\varphi \in H^1_\gamma(\Omega;\mathbb{S}^2)} \|\nabla(\varphi - \underline{u}_n - RU_{\lambda_n,a})\|_{L^2(\Omega)}$ for some appropriate $R \in SO(3)$ and $\lambda_n > 0$. The proof of (1.9) is given in Section 4.

Theorem B (2) is a corollary of the proofs of (1.8) and (1.9), and its proof is given in Section 4. Thus the main ingredient in this paper is the proof of (1.8) and (1.9).

A related asymptotic expansion like (1.8) is obtained for the minimizing solutions of the Ginzburg-Landau functional $\int_\Omega |\nabla u|^2 + \frac{1}{\epsilon^2}(|u|^2-1)^2\,dx$ as $\epsilon \to 0$. See [2]. However, in the Ginzburg-Landau case, the nature of the problem and difficulties are completely different from the ones in our problem. In fact, in the Ginzburg-Landau case, the difficulties come from the existence of the topological obstruction and the existence of the infinite energy. However, these are not in our problem. In our problem, difficulty mainly comes from the noncompactness (in the sense that $H^1(\Omega;\mathbb{S}^2)$ is not compactly embedded in $C^0(\Omega;\mathbb{S}^2)$) of our variational problem.

In Section 5, some technical results are proved.

The following conventions and notations are used throughout this paper: $C>0$ will denote various constants independent of H and n. $O(\rho)$ denotes a quantity X such that $|X| \le C\rho$, where C is independent of n and $|H|$. $o(\rho)$ denotes a quantity Y such that $\rho^{-1}|Y| \to 0$ as $n \to \infty$ or $H \to 0$. When $\gamma \equiv$ const., we always assume, by rotating the solution of (1.1) if necessary, $\gamma = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

2. Proof of Theorem A.

Proof of Theorem A (1). We first prove $\underline{u}_n \to \underline{u}$ strongly in $H^1(\Omega)$ for some energy-minimizing harmonic map $\underline{u}$. Let $\underline{u}_0$ be an energy-minimizing harmonic map in $H^1_\gamma(\Omega;\mathbb{S}^2)$. Define $I^0_0 := \inf_{u\in H^1_\gamma(\Omega;\mathbb{S}^2)} \int_\Omega |\nabla u|^2\,dx$. Then $I^0_0 = \int_\Omega |\nabla \underline{u}_0|^2\,dx$. Since $\mathcal{E}_{H_n}(\underline{u}_n) = \inf_{u\in H^1_\gamma(\Omega;\mathbb{S}^2)} \mathcal{E}_{H_n}(u)$, we have

$$\mathcal{E}_{H_n}(\underline{u}_n) \le \mathcal{E}_{H_n}(\underline{u}_0) = I^0_0 - 2\int_\Omega (H_n, \underline{u}_0)\,dx.$$

Thus $\int_\Omega |\nabla \underline{u}_n|^2\,dx \le I^0_0 + 4|H_n||\Omega|$ ($|\Omega| =$ Lebesgue measure of Ω) and

$$\varlimsup_{n\to\infty} \int_\Omega |\nabla \underline{u}_n|^2\,dx \le I^0_0. \tag{2.1}$$

Obviously, we have $\varliminf_{n\to\infty} \int_\Omega |\nabla \underline{u}_n|^2\,dx \ge I^0_0$. By this and (2.1), there exists an energy-minimizing harmonic map $\underline{u}$ such that (up to extracting a subsequence) $\underline{u}_n \to \underline{u}$ strongly in $H^1(\Omega)$. $C^\infty(\overline{\Omega})$-convergence follows from the following Regularity Theorem and the strong H^1-convergence:

Regularity Theorem. *Let $u_H \in H^1_\gamma(\Omega;\mathbb{S}^2)$ be a weak solution of* (1.1). *Then $u_H \in C^\infty(\overline{\Omega})$. Moreover, assume $|H| \le 1$. Then for any $k \in \mathbb{N}$, there exist $\epsilon > 0$ and $C_k > 0$ (ϵ and C_k are independent of H) such that if $\int_{\mathbb{B}_R(a)\cap\Omega} |\nabla u_H|^2\,dx \le \epsilon^2$, then $\|\nabla u_H\|_{C^k(\mathbb{B}_{R/2}(a)\cap\overline{\Omega})} \le C_k\epsilon$. Here $\mathbb{B}_r(a) := \{x \in \mathbb{R}^2 : |x-a| < r\}$.*

Now the proof of this theorem is standard; see [10], [12], [23] and [25]. □

Proof of Theorem A (2-a). Assume that γ is not identically equal to a constant. Recall that we have assumed that $\inf_{u\in\Phi_{-1}} \mathcal{E}_{H_n}$ is attained for all n. By Brezis and Coron [5] and Jost [14], $\inf_{u\in\Phi_{-1}} \mathcal{E}_0(u)$ or $\inf_{u\in\Phi_{+1}} \mathcal{E}_0(u)$ is attained. We may assume that $I_0^{-1} := \inf_{u\in\Phi_{-1}} \mathcal{E}_0(u)$ is attained. In fact, by the proof of Lemma 2.1 in [12] and Theorem A (1), if $\inf_{u\in\Phi_{\pm1}} \mathcal{E}_0(u)$ is attained, then $\inf_{u\in\Phi_{\pm1}} \mathcal{E}_H$ is attained for small $|H|$. Note that I_0^{-1} and I_0^0 are related by the inequality (see [5])

$$I_0^{-1} < I_0^0 + 8\pi. \tag{2.2}$$

Define $I_n^{-1} := \inf_{u\in\Phi_{-1}} \mathcal{E}_{H_n}(u)$. One can prove, by an argument similar to the proof of Theorem A (1),

$$\lim_{n\to\infty} I_n^{-1} = I_0^{-1}. \tag{2.3}$$

Since $\{\overline{u}_n\}$ is bounded in $H^1(\Omega;\mathbb{S}^2)$, there exist a subsequence of $\{\overline{u}_n\}$ (still denoted by $\{\overline{u}_n\}$) and $\overline{u}_0 \in H^1_\gamma(\Omega;\mathbb{S}^2)$ such that

$$\overline{u}_n \rightharpoonup \overline{u}_0 \quad \text{weakly in } H^1(\Omega). \tag{2.4}$$

We show that $\overline{u}_0 \in \Phi_{-1}$. In fact, assume $\overline{u}_0 \notin \Phi_{-1}$. Then $|d(\overline{u}_n) - d(\overline{u}_0)| \ge 1$. We may assume without loss of generality $d(\overline{u}_n) \ge d(\overline{u}_0) + 1$. By Brezis and Coron [5], the functional $\mathcal{E}_0 - 8\pi d$ is H^1-weakly lower semicontinuous. Thus

$$\begin{aligned}\mathcal{E}_0(\overline{u}_0) - 8\pi d(\overline{u}_0) &\le \varliminf_{n\to\infty} \left(\mathcal{E}_{H_n}(\overline{u}_n) - 8\pi d(\overline{u}_n)\right)\\ &\le \varliminf_{n\to\infty} \left(\mathcal{E}_{H_n}(\overline{u}_n) - 8\pi d(\overline{u}_0) - 8\pi\right).\end{aligned}$$

Therefore we have

$$\varliminf_{n\to\infty} \mathcal{E}_{H_n}(\overline{u}_n) \ge \mathcal{E}_0(\overline{u}_0) + 8\pi \ge I_0^0 + 8\pi. \tag{2.5}$$

(2.3) and (2.5) imply $I_0^{-1} \geq I_0^0 + 8\pi$. But this contradicts (2.2). Thus $\overline{u}_0 \in \Phi_{-1}$. Since $\overline{u}_0 \in \Phi_{-1}$, we have

$$I_0^{-1} \leq \int_\Omega |\nabla \overline{u}_0|^2 \, dx \leq \varliminf_{n\to\infty} \mathcal{E}_{H_n}(\overline{u}_n) = I_0^{-1}.$$

Therefore, $\overline{u}_n \to \overline{u}_0$ strongly in $H^1(\Omega)$ and $\overline{u}_0$ minimizes $\mathcal{E}_0$ in Φ_{-1}. $C^\infty(\overline{\Omega})$-convergence follows from the H^1-convergence and the Regularity Theorem.

Proof of Theorem A (2-b). The following argument is related to the work of P.L. Lions' concentration compactness principle [19], the work of Brezis-Coron's H-systems [4], the works of Struwe's elliptic equations involving the critical Sobolev exponent and H-systems [29], [30] and the work of Bethuel et al.'s Ginzburg-Landau systems [2].

Assume $\gamma =$ const. and $e - (e, \gamma)\gamma \neq 0$. We first note that $I_0^{-1} = 8\pi$. In fact, it is true that $8\pi \leq I_0^{-1} \leq I_0^0 + 8\pi$ if $\gamma \equiv$ const. (See [5].) When $\gamma =$ const., $I_0^0 = 0$. Thus $I_0^{-1} = 8\pi$. So we have

$$\lim_{n\to\infty} \int_\Omega |\nabla \overline{u}_n|^2 \, dx = 8\pi. \tag{2.6}$$

Let $\epsilon > 0$ be a constant in the Regularity Theorem. Define the set

$$\Sigma := \bigcap_{r>0} \Big\{ a \in \overline{\Omega} : \varlimsup_{n\to\infty} \int_{\mathbb{B}_r(a)\cap\Omega} |\nabla \overline{u}_n|^2 \, dx \geq \epsilon \Big\}.$$

One can easily verify $\#\Sigma < +\infty$, and by the Regularity Theorem we have

$$\overline{u}_n \to v \quad \text{in } C^\infty_{\text{loc}}(\overline{\Omega} \setminus \Sigma) \tag{2.7}$$

for some $v \in H^1_\gamma(\Omega; \mathbb{S}^2)$.

We claim $v \equiv \gamma$. In fact, assume $v \in \Phi_k$ for some $k \in \mathbb{Z}$. Then $d(v) - d(\overline{u}_n) = k + 1$ and

$$\begin{aligned} \int_\Omega |\nabla v|^2 \, dx + 8\pi d(v) &\leq \varliminf_{n\to\infty} \Big(\int_\Omega |\nabla \overline{u}_n|^2 \, dx + 8\pi d(\overline{u}_n) \Big) \\ &\leq I_0^{-1} + 8\pi d(v) - 8\pi(k+1). \end{aligned}$$

Thus

$$I_0^k \leq \int_\Omega |\nabla v|^2 \, dx \leq I_0^{-1} - 8\pi(k+1) = -8\pi k \leq I_0^k,$$

where $I_0^k := \inf_{u\in\Phi_k} \mathcal{E}_0(u) = 8\pi|k|$. Therefore, $\int_\Omega |\nabla v|^2\,dx = I_0^k = 8\pi d(v)$. But in such a case, it is well known that v is holomorphic (see, for example, [5]). Then by the unique continuation theorem, we have $v \equiv \gamma$.

Thus $\Sigma \neq \emptyset$. Set $\Sigma = \{a_1, \dots, a_p\}$ ($p \geq 1$). Assume $p \geq 2$. Since $\{|\nabla \overline{u}_n|^2\}$ is $L^1(\Omega)$-bounded, there exist a subsequence of $\{\overline{u}_n\}$ (still denoted by $\{\overline{u}_n\}$) and a Radon measure μ in $\overline{\Omega}$ such that $|\nabla \overline{u}_n|^2 \rightharpoonup \mu$in the sense of measures. Define the numbers α_i ($1 \leq i \leq p$) by

$$\alpha_i = \varliminf_{r\downarrow 0} \varliminf_{n\to\infty} \int_{\mathbb{B}_r(a_i)\cap\Omega} |\nabla \overline{u}_n|^2\,dx.$$

Then $\alpha_i > 0$ ($1 \leq i \leq p$) and

$$\mu \geq \alpha_1 \delta_{a_1} + \cdots + \alpha_p \delta_{a_p}. \tag{2.8}$$

On the other hand, by Brezis et al. [6] and the fact that $v \equiv \gamma$, there exist $b_1, \dots, b_q \in \overline{\Omega}$, $d_1, \dots, d_p \in \mathbb{Z} \setminus \{0\}$ ($q \geq 0$) such that

$$\frac{1}{4\pi}(\overline{u}_n, (\overline{u}_n)_{x_1} \wedge (\overline{u}_n)_{x_2}) \rightharpoonup d_1\delta_{b_1} + \cdots + d_q\delta_{b_q} \tag{2.9}$$

in the sense of measures.

Since $d(\overline{u}_n) = 1$, $q \geq 1$ and $b_i \in \Sigma$ ($1 \leq i \leq q$). We may assume, without loss of generality, $a_1 = b_1$. Then by (2.6) and (2.8)

$$8\pi = \lim_{n\to\infty} \int_\Omega |\nabla \overline{u}_n|^2\,dx \geq 8\pi + \alpha_2 + \cdots + \alpha_p > 8\pi.$$

This is a contradiction. Thus $\#\Sigma = 1$ and $|\nabla \overline{u}_n|^2 \rightharpoonup 8\pi\delta_{a_\infty}$ for some $a_\infty \in \overline{\Omega}$. This proves Theorem A (2-b-i) and (2-b-ii).

Next, we prove Theorem A (2-b-iii). As in [19], [4], define the concentration function Q_n as $Q_n(t) := \sup_{x\in\Omega} \int_{\mathbb{B}_t(x)} |\nabla \overline{u}_n|^2\,dx$. Here, $\overline{u}_n$ is extended to $\mathbb{R}^2 \setminus \overline{\Omega}$ by γ.

Let $\epsilon > 0$ be a constant in the Regularity Theorem. There exist $\lambda_n > 0$ and $a_n \in \overline{\Omega}$ such that $Q_n(\lambda_n) = \int_{\mathbb{B}_{\lambda_n}(a_n)\cap\Omega} |\nabla \overline{u}_n|^2\,dx = \frac{\epsilon}{2}$. Since $\Sigma = \{a_\infty\}$, one can verify $\lambda_n \to 0$ and $a_n \to a_\infty$. Define $\tilde{u}_n(x) := \overline{u}_n(\lambda_n x + a_n)$. $\tilde{u}_n$ satisfies

$$\Delta \tilde{u}_n + |\nabla \tilde{u}_n|^2 \tilde{u}_n - \lambda_n^2 (H_n, \tilde{u}_n)\tilde{u}_n + \lambda_n^2 H_n = 0 \quad \text{in } \Omega_n \tag{2.10}$$

and

$$\int_{\mathbb{B}_1(y)\cap\Omega_n} |\nabla \tilde{u}_n|^2\,dx = \int_{\mathbb{B}_{\lambda_n}(a_n+\lambda_n y)\cap\Omega} |\nabla \overline{u}_n|^2\,dx \le Q_n(\lambda_n) = \frac{\epsilon}{2} \tag{2.11}$$

for any $y \in \Omega_n$. Here $\Omega_n = \{\lambda_n^{-1}(x-a_n) : x \in \Omega\}$.

Let Ω_∞ be a limiting set of Ω_n. Then $\Omega_\infty = \mathbb{R}^2$ if $\lambda_n^{-1} d_n \to \infty$; otherwise $\Omega_\infty =$ half-plane in $\mathbb{R}^2$. Here $d_n = \operatorname{dist}(a_n, \partial\Omega)$. By the Regularity Theorem and (2.11), there exists $U \in H^1(\Omega_\infty; \mathbb{S}^2)$ such that $\tilde{u}_n \to U$ in $C^\infty_{\mathrm{loc}}(\overline{\Omega_\infty})$. Moreover, by (2.10), U is a harmonic map; that is,

$$\Delta U + |\nabla U|^2 U = 0 \quad \text{in } \Omega_\infty.$$

Since

$$\int_{\mathbb{B}_1(0)\cap\Omega_\infty} |\nabla U|^2\,dx = \lim_{n\to\infty} \int_{\mathbb{B}_1(0)\cap\Omega_n} |\nabla \tilde{u}_n|^2\,dx = \lim_{n\to\infty} \int_{\mathbb{B}_{\lambda_n}(a_n)} |\nabla \tilde{u}_n|^2\,dx = \frac{\epsilon}{2},$$

U is not constant.

We claim $\lambda_n^{-1} d_n \to \infty$ and $\Omega_\infty = \mathbb{R}^2$. In fact, assume $\lambda_n^{-1} d_n$ is bounded. Then for some subsequence (still denoted by the same symbol), $\lambda_n^{-1} d_n \to$ finite value, and then $\Omega_\infty =$ half plane in $\mathbb{R}^2$. Then U satisfies

$$\begin{cases} \Delta U + |\nabla U|^2 U = 0 & \text{in } \Omega_\infty \\ U = \text{const.} & \text{on } \partial\Omega_\infty. \end{cases}$$

By the nonexistence result of Lemaire [17], we then have $U \equiv$ const. This is a contradiction. Thus $\lambda_n^{-1} d_n \to \infty$ and $\Omega_\infty = \mathbb{R}^2$.

Then by the removable singularity theorem of Sacks and Uhlenbeck [23], U can be extended to $\mathbb{S}^2$ as a harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^2$, and we have

$$\begin{aligned}
&\int_\Omega \Big|\nabla\Big(\overline{u}_n - U\Big(\frac{\cdot - a_n}{\lambda_n}\Big)\Big)\Big|^2\,dx \\
&= \int_\Omega |\nabla \overline{u}_n|^2\,dx - 2\int_\Omega \nabla \overline{u}_n \cdot \nabla\Big(U\Big(\frac{\cdot - a_n}{\lambda_n}\Big)\Big)\,dx + \int_\Omega \Big|\nabla\Big(U\Big(\frac{\cdot - a_n}{\lambda_n}\Big)\Big)\Big|^2\,dx \\
&= \int_\Omega |\nabla \overline{u}_n|^2\,dx - 2\int_{\Omega_n} \nabla \tilde{u}_n \cdot \nabla U + \int_{\Omega_n} |\nabla U|^2\,dx \\
&= 8\pi - \int_{\mathbb{R}^2} |\nabla U|^2\,dx + o(1) \quad \text{as } n \to \infty.
\end{aligned} \tag{2.12}$$

Since U is a nonconstant harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^2$, $\int_{\mathbb{R}^2} |\nabla U|^2\,dx = 8\pi d$ for some positive $d \in \mathbb{Z}$, and by (2.12) we have $\int_{\mathbb{R}^2} |\nabla U|^2\,dx = 8\pi$. Then by (2.12) and Theorem A (1), we have

$$\Big\|\nabla\Big(\overline{u}_n - \underline{u}_n - U\Big(\frac{\cdot - a_n}{\lambda_n}\Big)\Big)\Big\|_{L^2(\Omega)} \to 0 \quad \text{as } n \to \infty. \tag{2.13}$$

(2.13) implies, since $d(\overline{u}_n) = -1$ for all n, $\frac{1}{4\pi}\int_{\mathbb{R}^2}(U, U_{x_1}\wedge U_{x_2})\,dx = -1$. That is, the degree of U is -1. All such harmonic maps are known. Using complex notation (by identifying $\mathbb{S}^2$ with $\mathbb{C}\cup\{\infty\}$ by the stereographic projection from the north pole of $\mathbb{S}^2$), $U(z) = \frac{a\bar{z}+b}{c\bar{z}+d}$ $(a,b,c,d \in \mathbb{C},\ ad-bc \neq 0)$. Or equivalently, $U(x) = RU_{\lambda,a}(x)$ for some $R \in SO(3)$, $\lambda > 0$ and $a \in \mathbb{R}^2$. Here $U_{\lambda,a} = U_0\big(\frac{\cdot - a}{\lambda}\big)$ and U_0 is the inverse of the stereographic projection from the north pole of $\mathbb{S}^2$:

$$U_0(x) = \frac{2}{1+|x|^2}\begin{pmatrix} x_1 \\ x_2 \\ -1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x = (x_1, x_2).$$

For $U = RU_{\lambda,a}$, $U\big(\frac{\cdot - a_n}{\lambda_n}\big) = U_{\lambda\lambda_n, a_n+\lambda_n a}$, and (1.3) follows by replacing $\lambda\lambda_n$ by λ_n and $a_n + \lambda_n a$ by a_n.

Next, we prove (1.4). Now R_n, λ_n and a_n are as in (1.3). Let h_n be the solution of

$$\begin{cases} \Delta h_n = 0 & \text{in } \Omega \\ h_n = \widehat{U}_{\lambda_n,a_n} & \text{on } \partial\Omega. \end{cases} \tag{2.14}$$

Since $\widehat{U}_{\lambda_n,a_n} = O\big(\frac{\lambda_n}{d_n}\big)$ on $\partial\Omega$ (as $n \to \infty$), by the maximum principle we have

$$\|h_n\|_{L^\infty(\Omega)} = O\Big(\frac{\lambda_n}{d_n}\Big) = o(1) \quad \text{as } n \to \infty. \tag{2.15}$$

(Recall $\lambda_n^{-1} d_n \to \infty$.) Set $v_n := \overline{u}_n - R_n\widehat{U}_{\lambda_n,a_n} - \gamma + R_n h_n$. By (2.15) and Theorem A (1), we need only to show that $\|v_n\|_{L^\infty(\Omega)} = o(1)$ as $n \to \infty$. In the following, we use the notations $\widetilde{U}_n := R_n\widehat{U}_{\lambda_n,a_n}$ and $U_n := R_n U_{\lambda_n,a_n}$.

$v_n = (v_n^1, v_n^2, v_n^3)$ satisfies $(1 \le i \le 3)$:

$$\begin{aligned}
\Delta v_n^i &= \Delta \overline{u}_n^i - \Delta U_n^i = -|\nabla \overline{u}_n|^2 \overline{u}_n^i + (H_n, \overline{u}_n)\overline{u}_n^i - H_n^i + |\nabla U_n|^2 U_n^i \\
&= -\sum_{\substack{1\le j\le 3\\ 1\le k\le 2}} \frac{\partial \overline{u}_n^j}{\partial x_k}\Big(\overline{u}_n^i \frac{\partial \overline{u}_n^j}{\partial x_k} - \overline{u}_n^j \frac{\partial \overline{u}_n^i}{\partial x_k}\Big) + (H_n, \overline{u}_n)\overline{u}_n^i - H_n^i \\
&\quad + \sum_{\substack{1\le j\le 3\\ 1\le k\le 2}} \frac{\partial U_n^j}{\partial x_k}\Big(U_n^i \frac{\partial U_n^j}{\partial x_k} - U_n^j \frac{\partial U_n^i}{\partial x_k}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{\substack{1\le j\le 3\\1\le k\le 2}} \frac{\partial v_n^j}{\partial x_k}\Big(\overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_k} - \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_k}\Big) + (H_n,\overline{u}_n)\overline{u}_n^i - H_n^i\\
&\quad + \sum_{\substack{1\le j\le 3\\1\le k\le 2}} \frac{\partial U_n^j}{\partial x_k}\Big(U_n^i\frac{\partial U_n^j}{\partial x_k} - U_n^j\frac{\partial U_n^i}{\partial x_k} - \overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_k} + \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_k}\Big)\\
&\quad + \sum_{\substack{1\le j\le 3\\1\le k\le 2}} \frac{\partial (R_nh_n)^j}{\partial x_k}\Big(\overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_k} - \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_k}\Big),
\end{aligned}
\tag{2.16}
$$

where we have used the equality $|\nabla u|^2u^i = \sum_{\substack{1\le j\le 3\\1\le k\le 2}} \frac{\partial u^j}{\partial x_k}\big(u^i\frac{\partial u^j}{\partial x_k} - u^j\frac{\partial u^i}{\partial x_k}\big)$ for the $\mathbb{S}^2$-valued map u.

Let φ_n^{ij} be the solution of the problem

$$
-\Delta\varphi_n^{ij} = H_n^i\overline{u}_n^j - H_n^j\overline{u}_n^i \ \text{ in } \Omega, \qquad \varphi_n^{ij} = 0 \ \text{ on } \partial\Omega.
$$

Define $A_n^{ij} := \big(\overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_1} - \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_1}, \overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_2} - \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_2}\big)$. Then

$$
\operatorname{div}(A_n^{ij} + \nabla\varphi_n^{ij}) = 0. \tag{2.17}
$$

Define $v_{l,n}$ $(1\le l\le 5)$ by

$$
\begin{cases}
\Delta v_{1,n}^i = \sum_{1\le j\le 3}\nabla v_n^j\cdot(A_n^{ij} + \nabla\varphi_n^{ij}) & \text{in } \Omega\\
v_{1,n}^i = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{2.18}
$$

$$
\begin{cases}
\Delta v_{2,n}^i = -\sum_{1\le j\le 3}\nabla v_n^j\cdot\nabla\varphi_n^{ij} & \text{in } \Omega\\
v_{2,n}^i = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{2.19}
$$

$$
\begin{cases}
\Delta v_{3,n}^i = (H_n,\overline{u}_n)\overline{u}_n^i - H_n^i & \text{in } \Omega\\
v_{3,n}^i = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{2.20}
$$

$$
\begin{cases}
\Delta v_{4,n}^i = \sum_{\substack{1\le j\le 3\\1\le k\le 2}} \frac{\partial U_n^j}{\partial x_k}\Big(U_n^i\frac{\partial U_n^j}{\partial x_k} - U_n^j\frac{\partial U_n^i}{\partial x_k} - \overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_k} + \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_k}\Big) & \text{in } \Omega\\
v_{4,n}^i = 0 & \text{on } \Omega,
\end{cases}
\tag{2.21}
$$

$$
\begin{cases}
\Delta v_{5,n}^i = \sum_{\substack{1\le j\le 3\\1\le k\le 2}} \frac{\partial (R_nh_n)^j}{\partial x_k}\Big(\overline{u}_n^i\frac{\partial \overline{u}_n^j}{\partial x_k} - \overline{u}_n^j\frac{\partial \overline{u}_n^i}{\partial x_k}\Big) & \text{in } \Omega\\
v_{5,n}^i = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{2.22}
$$

(2.17), (2.18) and the estimate in [3] and [7] imply

$$\begin{aligned}\|v_{1,n}\|_{L^\infty(\Omega)} &\le C\|\nabla v_n\|_{L^2(\Omega)}\|A_n^{ij}+\nabla\varphi_n^{ij}\|_{L^2(\Omega)}\\ &\le C\|\nabla v_n\|_{L^2(\Omega)}(\|\nabla\overline{u}_n\|_{L^2(\Omega)}+|H_n|).\end{aligned}\tag{2.23}$$

By the elliptic estimate and Sobolev embedding $W^{2,3/2}(\Omega)\hookrightarrow C^0(\overline{\Omega})$, we have

$$\begin{aligned}\|v_{2,n}\|_{L^\infty(\Omega)} &\le C\|v_{2,n}\|_{W^{2,3/2}(\Omega)}\le C\|\sum_{1\le j\le 3}\nabla v_n^j\cdot\nabla\varphi_n^{ij}\|_{L^{3/2}(\Omega)}\\ &\le C\|\nabla v_n\|_{L^2(\Omega)}\|\nabla\varphi_n^{ij}\|_{L^6(\Omega)}\le C|H_n|\|\nabla v_n\|_{L^2(\Omega)}.\end{aligned}\tag{2.24}$$

$v_{4,n}$ is estimated as $v_{1,n}$. Since

$$\begin{aligned}\operatorname{div}\Big(&U_n^i\frac{\partial U_n^i}{\partial x_1}-U_n^j\frac{\partial U_n^i}{\partial x_1}-\overline{u}_n^i\frac{\partial\overline{u}_n^j}{\partial x_1}+\overline{u}_n^j\frac{\partial\overline{u}_n^i}{\partial x_1},\\ &U_n^i\frac{\partial U_n^i}{\partial x_2}-U_n^j\frac{\partial U_n^i}{\partial x_2}-\overline{u}_n^i\frac{\partial\overline{u}_n^j}{\partial x_2}+\overline{u}_n^j\frac{\partial\overline{u}_n^i}{\partial x_2}\Big)=H_n^j\overline{u}_n^i-H_n^i\overline{u}_n^j,\end{aligned}$$

introducing the solution $v_{4,n}^{ij}\in H_0^1(\Omega)$ of the equation $-\Delta v_{4,n}^{ij}=H_n^j\overline{u}_n^i-H_n^i\overline{u}_n^j$, we have as the estimate of $v_{1,n}$

$$\begin{aligned}&\|v_{4,n}\|_{L^\infty(\Omega)}\le C\|\nabla U_n\|_{L^2(\Omega)}\Big\|U_n\frac{\partial U_n^i}{\partial x_k}-U_n^j\frac{\partial U_n^i}{\partial x_k}-\overline{u}_n^i\frac{\partial\overline{u}_n^j}{\partial x_k}+\overline{u}_n^j\frac{\partial\overline{u}_n^i}{\partial x_k}\\ &+\nabla v_{4,n}^{ij}\Big\|_{L^2(\Omega)}+C\|\nabla U_n\|_{L^2(\Omega)}\|\nabla v_{4,n}^{ij}\|_{L^6(\Omega)}\\ &\le C\|\nabla U_n\|_{L^2(\Omega)}\Big\|U_n\frac{\partial U_n^i}{\partial x_k}-U_n^j\frac{\partial U_n^i}{\partial x_k}-\overline{u}_n^i\frac{\partial\overline{u}_n^j}{\partial x_k}+\overline{u}_n^j\frac{\partial\overline{u}_n^i}{\partial x_k}\\ &+\nabla v_{4,n}^{ij}\Big\|_{L^2(\Omega)}+C|H_n|\\ &\le C\|\nabla U_n\|_{L^2(\Omega)}\Big\|U_n\frac{\partial U_n^i}{\partial x_k}-U_n^j\frac{\partial U_n^i}{\partial x_k}-\overline{u}_n^i\frac{\partial\overline{u}_n^j}{\partial x_k}+\overline{u}_n^j\frac{\partial\overline{u}_n^i}{\partial x_k}\Big\|_{L^2(\Omega)}+C|H_n|\\ &\le C\|\nabla v_n\|_{L^2(\Omega)}+C|H_n|+C\frac{\lambda_n}{d_n},\end{aligned}\tag{2.25}$$

where we have used the defining equation of v_n, (2.15), Lemma 3.7 and the fact that our R_n satisfies $R_n\gamma=\gamma$ (see the proof of (1.3)).

v_5 is estimated as

$$\|v_{5,n}\|_{L^\infty(\Omega)} \le C\frac{\lambda_n}{d_n}. \tag{2.26}$$

Since $v_n = \sum_{l=1}^5 v_{l,n}$, we have

$$\|v_n\|_{L^\infty(\Omega)} \le C\|\nabla v_n\|_{L^2(\Omega)} + C|H_n| + C\frac{\lambda_n}{d_n} \to 0 \quad \text{as } n\to\infty.$$

Remark 2.1. The above proof shows that

$$\begin{aligned} &\|\overline{u}_n - \underline{u}_n - R_n\widehat{U}_{\lambda_n,a_n}\|_{L^\infty(\Omega)} \\ &\le C\|\nabla(\overline{u}_n - \underline{u}_n - R_n\widehat{U}_{\lambda_n,a_n})\|_{L^2(\Omega)} + C|H_n| + C\frac{\lambda_n}{d_n}. \end{aligned} \tag{2.27}$$

A related inequality will be used in the proof of Theorem B. See Section 3, Corollary 3.11.

The following lemma about the small solution $\underline{u}_n$ will be repeatedly used in the proof of Theorem B.

Lemma 2.2. *Assume $\gamma\equiv const.$ Let k be a positive integer. There exists $C_k>0$ (depending only on k) such that*

$$\|\underline{u}_n - \gamma - |H_n|Ee'\|_{C^k(\overline{\Omega})} \le C_k|H_n|^2. \tag{2.28}$$

Here E is defined by (1.5) *and $e' = e-(e,\gamma)\gamma$.*

Proof. Define $\underline{v}_n$ by $\underline{v}_n := \underline{u}_n - \gamma - |H_n|Ee'$. By Theorem A (1),

$$\|\underline{v}_n\|_{C^k(\overline{\Omega})} \to 0 \quad \text{as } n\to\infty. \tag{2.29}$$

Since $\underline{u}_n$ satisfies the equation (1.1), we have

$$\begin{aligned} &\Delta\underline{v}_n + |\nabla\underline{v}_n|^2\underline{u}_n + 2|H_n|(\nabla\underline{v}_n, e')\cdot\nabla E\underline{u}_n + |H_n|^2|\nabla E|^2\underline{u}_n|e'|^2 \\ &-(H_n,\underline{v}_n)\underline{u}_n - (H_n,\gamma)\underline{v}_n - (H_n,\gamma)|H_n|e'E - |H_n|^2E\underline{u}_n|e'|^2 = 0. \end{aligned} \tag{2.30}$$

Since $\underline{v}_n|_{\partial\Omega}=0$, by the elliptic estimate and (2.30),

$$\begin{aligned} \|\underline{v}_n\|_{W^{2,4/3}(\Omega)} &\le C(\||\nabla\underline{v}_n|^2\|_{L^{4/3}(\Omega)} + |H_n|\|\underline{v}_n\|_{W^{1,4/3}(\Omega)} + |H_n|^2) \\ &\le C(\|\nabla\underline{v}_n\|^2_{L^{8/3}(\Omega)} + |H_n|\|\underline{v}_n\|_{W^{1,4/3}(\Omega)} + |H_n|^2). \end{aligned} \tag{2.31}$$

By the Sobolev embedding $W^{2,4/3}(\Omega) \hookrightarrow W^{1,4}(\Omega)$ and (2.29), we have

$$\|\underline{v}_n\|_{W^{2,4/3}(\Omega)} \leq C|H_n|^2 \tag{2.32}$$

for large n.

Iterating a similar argument (using (2.32)) we have by the Sobolev embedding theorem $\|\underline{v}_n\|_{C^k(\overline{\Omega})} \leq C_k|H_n|^2$ for any k and large n. □

3. Proof of Theorem B—the first part. In this section assume $\gamma \equiv$ const. Our purpose in this section is to give the optimal lower bound of the quantity $\mathcal{E}_{H_n}(\overline{u}_n)$. We first prove the following lemma:

Lemma 3.1. *Let $\{\overline{u}_n\}$ and $\{\underline{u}_n\}$ be as in Theorem A. Let α'_n, R'_n, λ'_n, a'_n be as in Corollary 5.2. For simplicity, we delete "′" from $a'_n, \ldots$, etc. Define $w_n := \overline{u}_n - \underline{u}_n - \alpha_n R_n \widehat{U}_{\lambda_n,a_n}$. Here $\underline{u}_n$ and $\overline{u}_n$ are extended to $\mathbb{R}^2 \setminus \overline{\Omega}$ by γ. Then the following expansion holds as $n \to \infty$:*

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n) &= \mathcal{E}_{H_n}(\underline{u}_n) + 8\pi\alpha_n^2 + \int_{\mathbb{R}^2} |\nabla w_n|^2\,dx \\ &\quad + O((|(H_n,\gamma)| + |H_n|^2)\lambda_n^2|\log\lambda_n|) + O\Big((|(H_n,\gamma)| + |H_n|^2)\frac{\lambda_n^2}{d_n^2}\Big) \\ &\quad + O((|(H_n,\gamma)| + |H_n|^2)\|\nabla w_n\|^2_{L^2(\Omega)}).\end{aligned} \tag{3.1}$$

Here $d_n = dist(a_n, \partial\Omega)$.

Proof. Define $W_n \in H_0^1(\Omega;\mathbb{R}^3)$ by $\overline{u}_n = \underline{u}_n + W_n$. Since $\underline{u}_n$ satisfies (1.1), we have

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n) &= \mathcal{E}_{H_n}(\underline{u}_n) + \int_\Omega |\nabla W_n|^2\,dx \\ &\quad + 2\int_\Omega |\nabla\underline{u}_n|^2(\underline{u}_n, W_n)\,dx - 2\int_\Omega (H_n,\underline{u}_n)(\underline{u}_n,W_n)\,dx.\end{aligned} \tag{3.2}$$

Since $1 = |\overline{u}_n|^2 = |\underline{u}_n + W_n|^2 = 1 + 2(\underline{u}_n, W_n) + |W_n|^2$, we have $2(\underline{u}_n, W_n) = -|W_n|^2$. (A similar trick will be used in the later arguments.) Using this, we rewrite (3.2) as

$$\mathcal{E}_{H_n}(\overline{u}_n) = \mathcal{E}_{H_n}(\underline{u}_n) + \int_\Omega |\nabla W_n|^2 - \int_\Omega |\nabla\underline{u}_n|^2|W_n|^2\,dx + \int_\Omega (H_n,\underline{u}_n)|W_n|^2\,dx. \tag{3.3}$$

Since $W_n = \alpha_n R_n \widehat{U}_{\lambda_n,a_n} + w_n$ and $W_n = 0$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, by (5.3) we have

$$\int_\Omega |\nabla W_n|^2\,dx = 8\pi\alpha_n^2 + \int_{\mathbb{R}^2} |\nabla w_n|^2\,dx. \tag{3.4}$$

Next we estimate the integral $\int_\Omega |\nabla \underline{u}_n|^2 |W_n|^2\,dx = \int_\Omega |\nabla \underline{u}_n|^2 (\alpha_n^2 |\widehat{U}_{\lambda_n,a_n}|^2 + 2\alpha_n (R_n \widehat{U}_{\lambda_n,a_n}, w_n) + |w_n|^2)\,dx$. By (2.28),

$$\begin{aligned}\int_\Omega |\nabla \underline{u}_n|^2 |\widehat{U}_{\lambda_n,a_n}|^2\,dx &= O\Big(|H_n|^2 \int_\Omega \frac{\lambda_n^2}{\lambda_n^2 + |x-a_n|^2}\,dx\Big)\\ &= O(|H_n|^2 \lambda_n^2 |\log\lambda_n|).\end{aligned} \tag{3.5}$$

Let h_n' be the harmonic extension of $R_n \widehat{U}_{\lambda_n,a_n}|_{\partial\Omega}$ to Ω. Then

$$\begin{aligned}&\int_\Omega |\widehat{U}_{\lambda_n,a_n}||w_n|\,dx \le \int_\Omega |\widehat{U}_{\lambda_n,a_n}||w_n + \alpha_n h_n'|\,dx + |\alpha_n| \int_\Omega |\widehat{U}_{\lambda_n,a_n}||h_n'|\,dx\\ &\le \|\widehat{U}_{\lambda_n,a_n}\|_{L^2(\Omega)} \|w_n + \alpha_n h_n'\|_{L^2(\Omega)} + |\alpha_n| \|\widehat{U}_{\lambda_n,a_n}\|_{L^2(\Omega)} \|h_n'\|_{L^2(\Omega)}.\end{aligned} \tag{3.6}$$

Since α_n is bounded (see Lemma 3.2), we have

$$\|w_n + \alpha_n h_n'\|_{L^2(\Omega)} \le C\|\nabla(w_n + \alpha_n h_n')\|_{L^2(\Omega)} \le C\|\nabla w_n\|_{L^2(\Omega)} \tag{3.7}$$

and

$$\|\widehat{U}_{\lambda_n,a_n}\|_{L^2(\Omega)} = O\bigg(\Big(\int_\Omega \frac{\lambda_n^2}{\lambda_n^2 + |x-a_n|^2}\,dx\Big)^{1/2}\bigg) = O(\lambda_n |\log\lambda_n|^{1/2}). \tag{3.8}$$

(2.28), (3.6), (3.7), (3.8) and $\|h_n'\|_{L^2(\Omega)} = O\big(\frac{\lambda_n}{d_n}\big)$ imply

$$\begin{aligned}\int_\Omega |\nabla \underline{u}_n|^2 (R_n \widehat{U}_{\lambda_n,a_n}, w_n)\,dx &= O(|H_n|^2 \lambda_n |\log\lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\Omega)})\\ &\quad + O\Big(|H_n|^2 \frac{\lambda_n^2}{d_n} |\log\lambda_n|^{1/2}\Big).\end{aligned} \tag{3.9}$$

Since

$$\|w_n\|_{L^2(\Omega)}^2 \le 2\|w_n + \alpha_n h_n'\|_{L^2(\Omega)}^2 + 2\alpha_n^2 \|h_n'\|_{L^2(\Omega)}^2 \le C\|\nabla w_n\|_{L^2(\Omega)}^2 + C\frac{\lambda_n^2}{d_n^2},$$

we have, using (2.28),

$$\int_\Omega |\nabla \underline{u}_n|^2 |w_n|^2\,dx = O(|H_n|^2 \|\nabla w_n\|^2_{L^2(\Omega)}) + O\Big(|H_n|^2 \frac{\lambda_n^2}{d_n^2}\Big). \tag{3.10}$$

By a similar argument, using (2.28), we obtain

$$\begin{aligned}\int_\Omega (H_n, \underline{u}_n)|W_n|^2\,dx &= O((|(H_n,\gamma)| + |H_n|^2)\lambda_n^2 |\log\lambda_n|)\\ &+ O\Big((|(H_n,\gamma)| + |H_n|^2)\frac{\lambda_n^2}{d_n^2}\Big)\\ &+ O((|(H_n,\gamma)| + |H_n|^2)\lambda_n |\log\lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\Omega)})\\ &+ O((|(H_n,\gamma)| + |H_n|^2)\|\nabla w_n\|^2_{L^2(\Omega)}).\end{aligned} \tag{3.11}$$

(3.5), (3.9), (3.10) and (3.11) imply (3.1). □

The next lemma gives the expansion of α_n:

Lemma 3.2. *Let α_n, R_n, λ_n and a_n be as in Lemma 3.1. Let* $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ *and* $e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. *We then have (as $n \to \infty$)*

$$\begin{aligned}\alpha_n = 1 &- \frac{1}{2}(E_{x_1}(a_n)(e', R_n e_1) + E_{x_2}(a_n)(e', R_n e_2))\lambda_n |H_n|\\ &+ \frac{1}{4\pi}\int_{\mathbb{R}^2} (R_n \widehat{U}_{\lambda_n,a_n}, (w_n)_{x_1} \wedge (w_n)_{x_2})\,dx + o(|H_n|\lambda_n) + O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big)\\ &+ o(\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)}) + O(|H_n|^2 \|\nabla w_n\|_{L^2(\mathbb{R}^2)})\\ &+ O(|H_n|\lambda_n |\log\lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) + O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}).\end{aligned} \tag{3.12}$$

Here $E_{x_1} = \frac{\partial E}{\partial x_1}$, ... , *etc.*

Proof. By (2.28), $\underline{u}_n$ is relatively homotopic to γ for large n. Thus

$$\int_{\mathbb{R}^2} (\overline{u}_n, (\overline{u}_n)_{x_1} \wedge (\overline{u}_n)_{x_2})\,dx = -4\pi \tag{3.13}$$

for large n. Here $\overline{u}_n$ is extended to $\mathbb{R}^2 \backslash \overline{\Omega}$ by γ. Since $\overline{u}_n = \underline{u}_n + \alpha_n R_n \widehat{U}_{\lambda_n,a_n} + w_n$, we have (for simplicity, we use the notation $\widetilde{U}_n = R_n \widehat{U}_{\lambda_n,a_n}$)

$$
\begin{aligned}
&\int_{\mathbb{R}^2} (\overline{u}_n, (\overline{u}_n)_{x_1} \wedge (\overline{u}_n)_{x_2})\, dx \\
&= \int_{\mathbb{R}^2} (\underline{u}_n + \alpha_n \widetilde{U}_n + w_n, (\underline{u}_n + \alpha_n \widetilde{U}_n + w_n)_{x_1} \wedge (\underline{u}_n + \alpha_n \widetilde{U}_n + w_n)_{x_2})\, dx \\
&= \int_{\mathbb{R}^2} (\underline{u}_n, (\underline{u}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\underline{u}_n, (\underline{u}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\underline{u}_n, (\underline{u}_n)_{x_1} \wedge (w_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\underline{u}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\underline{u}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\underline{u}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\underline{u}_n, (w_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\underline{u}_n, (w_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\underline{u}_n, (w_n)_{x_1} \wedge (w_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\underline{u}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\underline{u}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\underline{u}_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (w_n)_{x_2})\, dx \qquad (3.14) \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (w_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (w_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (w_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (\underline{u}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (w_n, (\underline{u}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (\underline{u}_n)_{x_1} \wedge (w_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (w_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (w_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (w_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (w_n, (w_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (w_n)_{x_1} \wedge (w_n)_{x_2})\, dx.
\end{aligned}
$$

There are 27 terms in (3.14). Write (3.14) as $d_1 + d_2 + \cdots + d_{27}$. Since $\underline{u}_n$ is relatively homotopic to γ for large n, we have

$$d_1 = 0. \tag{3.15}$$

By (5.6) and (2.28),

$$\begin{aligned} d_2 + d_4 &= 2\int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\underline{u}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx = O\big(|H_n|^2 \int_\Omega |\widehat{U}_{\lambda_n, a_n}|\, dx\big) \\ &= O\big(|H_n|^2 \int_\Omega \frac{\lambda_n}{\sqrt{\lambda_n^2 + |x - a_n|^2}}\, dx\big) = O(|H_n|^2 \lambda_n). \end{aligned} \tag{3.16}$$

By (5.4), (5.5) and (2.28) (or (5.6), (5.7) and (2.28)),

$$d_3 + d_7 = O(|H_n|^2 \|\nabla w_n\|_{L^2(\mathbb{R}^2)}). \tag{3.17}$$

By (5.6) and (2.28),

$$\begin{aligned} d_6 + d_8 &= \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\underline{u}_n)_{x_1} \wedge (w_n)_{x_2} + (w_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\ &= O\big(|H_n| \int_\Omega |\widehat{U}_{\lambda_n, a_n}| |\nabla w_n|\, dx\big) = O(|H_n| \|\widehat{U}_{\lambda_n, a_n}\|_{L^2(\Omega)} \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) \\ &= O\big(|H_n| \|\nabla w_n\|_{L^2(\mathbb{R}^2)} \big(\int_\Omega \frac{\lambda_n^2}{\lambda_n^2 + |x - a_n|^2}\, dx\big)^{1/2}\big) \\ &= O(|H_n| \lambda_n |\log \lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\mathbb{R}^2)}). \end{aligned} \tag{3.18}$$

By (5.7) and (2.28),

$$d_9 + d_{21} + d_{25} = O(|H_n| \|\nabla w_n\|^2_{L^2(\mathbb{R}^2)}). \tag{3.19}$$

Obviously, we have

$$d_{10} = (|H_n|^2 \lambda_n) \tag{3.20}$$

$$d_{12} + d_{16} + d_{20} + d_{22} = O(|H_n| \lambda_n |\log \lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) \tag{3.21}$$

and

$$d_{19} = O(|H_n|^2 \|\nabla w_n\|_{L^2(\mathbb{R}^2)}). \tag{3.22}$$

By (5.7),

$$d_{27} = O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}). \tag{3.23}$$

(3.14)–(3.23), (5.6) and $\int_{\mathbb{R}^2}(\widetilde{U}_n, (\widetilde{U}_n)_{x_1} \wedge (\widetilde{U}_n)_{x_2})\, dx = -4\pi$ imply

$$\begin{aligned}
&\int_{\mathbb{R}^2} (\overline{u}_n, (\overline{u}_n)_{x_1} \wedge (\overline{u}_n)_{x_2})\, dx \\
&= \int_{\mathbb{R}^2} (\underline{u}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\underline{u}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\underline{u}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (w_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (\alpha_n \widetilde{U}_n, (w_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (\alpha \widetilde{U}_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + \int_{\mathbb{R}^2} (w_n, (\alpha_n \widetilde{U}_n)_{x_1} \wedge (w_n)_{x_2})\, dx + \int_{\mathbb{R}^2} (w_n, (w_n)_{x_1} \wedge (\alpha_n \widetilde{U}_n)_{x_2})\, dx \\
&\quad + O(|H_n|^2 \lambda_n) + O(|H_n|^2 \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) + O(|H_n| \|\nabla w_n\|^2_{L^2(\mathbb{R}^2)}) \\
&\quad + O(|H_n| \lambda_n |\log \lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) + O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}) \\
&= 3\alpha_n^2 \int_{\mathbb{R}^2} (\underline{u}_n - \gamma), (\widetilde{U}_n)_{x_1} \wedge (\widetilde{U}_n)_{x_2})\, dx - 4\pi \alpha_n^3 \\
&\quad + 3\alpha_n^2 \int_{\mathbb{R}^2} (w_n, (\widetilde{U}_n)_{x_1} \wedge (\widetilde{U}_n)_{x_2})\, dx + 3\alpha_n \int_{\mathbb{R}^2} (\widetilde{U}_n, (w_n)_{x_1} \wedge (w_n)_{x_2})\, dx \\
&\quad + O(|H_n|^2 \lambda_n) + O(|H_n|^2 \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) + O(|H_n| \|\nabla w_n\|^2_{L^2(\mathbb{R}^2)}) \\
&\quad + O(|H_n| \lambda_n |\log \lambda_n|^{1/2} \|\nabla w_n\|_{L^2(\mathbb{R}^2)}) + O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}). \qquad (3.24)
\end{aligned}$$

Since

$$\Delta \widetilde{U}_n = 2\widetilde{U}_n \wedge \widetilde{U}_n = -|\nabla \widetilde{U}_n|^2 U_n \quad (U_n = R_n U_{\lambda_n, a_n}), \qquad (3.25)$$

by (5.3) we have

$$\begin{aligned}
\int_{\mathbb{R}^2} (w_n, (\widetilde{U}_n)_{x_1} \wedge (\widetilde{U}_n)_{x_2})\, dx &= \frac{1}{2} \int_{\mathbb{R}^2} w_n \cdot \Delta \widetilde{U}_n\, dx \\
&= -\frac{1}{2} \int_{\mathbb{R}^2} \nabla w_n \cdot \nabla \widetilde{U}_n\, dx = 0. \qquad (3.26)
\end{aligned}$$

By (3.25),

$$\int_{\mathbb{R}^2}(\underline{u}_n-\gamma,(\widetilde{U}_n)_{x_1}\wedge(\widetilde{U}_n)_{x_2})\,dx=-\frac{1}{2}\int_\Omega|\nabla\widetilde{U}_n|^2(\underline{u}_n-\gamma,U_n)\,dx$$

$$=-4\int_{\mathbb{B}_{d_n}(a_n)}\frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}(\underline{u}_n-\gamma,R_nU_{\lambda_n,a_n})\,dx \tag{3.27}$$

$$-4\int_{\Omega\setminus\mathbb{B}_{d_n}(a_n)}\frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}(\underline{u}_n-\gamma,R_nU_{\lambda_n,a_n})\,dx\ (r=|x-a_n|).$$

Here,

$$\int_{\Omega\setminus\mathbb{B}_{d_n}(a_n)}\frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}(\underline{u}_n-\gamma,R_nU_{\lambda_n,a_n})\,dx \tag{3.28}$$

$$=O\Big(|H_n|\int_{\Omega\setminus\mathbb{B}_{d_n}(a_n)}\frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}\,dx\Big)=O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big).$$

Next we estimate

$$\int_{\mathbb{B}_{d_n}(a_n)}\frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}(\underline{u}_n-\gamma,R_nU_{\lambda_n,a_n})\,dx$$

$$=\int_{\mathbb{B}_{d_n}(a_n)}\frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3}(\underline{u}_n-\gamma,(x_1-a_n^1)R_ne_1$$

$$+(x_2-a_n^2)R_ne_2+\frac{r^2-\lambda_n^2}{2\lambda_n}R_ne_3)\,dx. \tag{3.29}$$

Here, $a_n=(a_n^1,a_n^2)$. By Taylor's formula and (2.28),

$$\underline{u}_n-\gamma=\underline{u}_n(a_n)-\gamma+\nabla\underline{u}_n(a_n)\cdot(x-a_n) \tag{3.30}$$

$$+\frac{1}{2}(\nabla^2\underline{u}_n(a_n)\cdot(x-a_n),(x-a_n))+O(|H_n||x-a_n|^3).$$

By oddness and (2.28)

$$\int_{\mathbb{B}_{d_n}(a_n)}\frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3}(x_1-a_n^1)(\underline{u}_n-\gamma,R_ne_1)\,dx$$

$$=\int_{\mathbb{B}_{d_n}(a_n)}\frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3}(x_1-a_n^1)^2((\underline{u}_n)_{x_1}(a_n),R_ne_1)\,dx$$

$$+O\big(|H_n|\int_{\mathbb{B}_{d_n}(a_n)}\frac{\lambda_n^3r^4}{(\lambda_n^2+r^2)^3}\,dx\big)$$

$$
\begin{aligned}
&= 2\pi((\underline{u}_n)_{x_1}(a_n), R_n e_1)\lambda_n \int_0^{\frac{d_n}{\lambda_n}} \frac{s^3}{(1+s^2)^3}\,ds + O(|H_n|\lambda_n^3|\log\lambda_n|)\\
&= \frac{\pi}{2}\lambda_n((\underline{u}_n)_{x_1}(a_n), R_n e_1) + O\Big(|H_n|\frac{\lambda_n^3}{d_n^2}\Big) + O(|H_n|\lambda_n^3|\log\lambda_n|)\\
&= \frac{\pi}{2}E_{x_1}(a_n)(e', R_n e_1)\lambda_n|H_n| + O(|H_n|^2\lambda_n)\\
&\quad + O\Big(|H_n|\frac{\lambda_n^3}{d_n^2}\Big) + O(|H_n|\lambda_n^3|\log\lambda_n|). \qquad (3.31)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{\mathbb{B}_{d_n}(a_n)} \frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3}(x_2-a_n^2)(\underline{u}_n-\gamma, R_n e_2)\,dx\\
&= \frac{\pi}{2}E_{x_2}(a_n)(e', R_n e_2)\lambda_n|H_n| + O(|H_n|^2\lambda_n)\\
&\quad + O\Big(|H_n|\frac{\lambda_n^3}{d_n^2}\Big) + O(|H_n|\lambda_n^3|\log\lambda_n|) \qquad (3.32)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{B}_{d_n}(a_n)} \frac{2\lambda_n^3}{(\lambda_n^2+r^2)^3}\frac{r^2-\lambda_n^2}{2\lambda_n}(\underline{u}_n-\gamma, R_n e_3)\,dx\\
&= (\underline{u}_n(a_n)-\gamma, R_n e_3)\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^3}{(\lambda_n^2+r^2)^3}(r^2-\lambda_n^2)\,dx\\
&\quad + O\big(|H_n|\int_{\mathbb{B}_{d_n}(a_n)} \frac{\lambda_n^3|r^2-\lambda_n^2|r^2}{(\lambda_n^2+r^2)^3}\,dx\big)\\
&= 2\pi\lambda_n(\underline{u}_n(a_n)-\gamma, R_n e_3)\int_0^{\frac{d_n}{\lambda_n}} \frac{(s^2-1)s}{(1+s^2)^3}\,ds\\
&\quad + O\Big(|H_n|\lambda_n^3\int_0^{\frac{d_n}{\lambda_n}} \frac{s^3|s^2-1|}{(1+s^2)^3}\,ds\Big) \qquad (3.33)\\
&= -2\pi\lambda_n(\underline{u}_n(a_n)-\gamma, R_n e_3)\int_{\frac{d_n}{\lambda_n}}^{\infty} \frac{(s^2-1)s}{(1+s^2)^3}\,ds + O(|H_n|\lambda_n^3|\log\lambda_n|)\\
&\quad \Big(\text{since } \int_0^{\infty} \frac{(s^2-1)s}{(1+s^2)^3}\,ds = 0\Big) = O\big(|H_n|\frac{\lambda_n^3}{d_n^2}\big) + O(|H_n|\lambda_n^3|\log\lambda_n|).
\end{aligned}
$$

Combining (3.27)–(3.33), we have

$$\int_{\mathbb{R}^2}(\underline{u}_n-\gamma,(\widetilde{U}_n)_{x_1}\wedge(\widetilde{U}_n)_{x_2})\,dx=-2\pi(E_{x_1}(a_n)(e',R_ne_1)\tag{3.34}$$
$$+E_{x_2}(a_n)(e',R_ne_2))\lambda_n|H_n|+O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big)+O(|H_n|^2\lambda_n).$$

(3.13), (3.24), (3.26) and (3.34) imply

$$\begin{aligned}4\pi(\alpha_n^3-1)=&-6\pi\alpha_n(E_{x_1}(a_n)(e',R_ne_1)+E_{x_2}(a_n)(e',R_ne_2))\lambda_n|H_n|\\&+3\alpha_n\int_{\mathbb{R}^2}(\widetilde{U}_n,(w_n)_{x_1}\wedge(w_n)_{x_2})\,dx+O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big)\\&+O(|H_n|^2\lambda_n)+O(|H_n|^2\|\nabla w_n\|_{L^2(\mathbb{R}^2)})\\&+O(|H_n|\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)})+O(|H_n|\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\mathbb{R}^2)})\\&+O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}).\end{aligned}\tag{3.35}$$

From (3.35), we deduce that $\alpha_n\to 1$ $(n\to\infty)$ and that

$$\begin{aligned}\alpha_n=1&-\frac{6\pi\alpha_n}{4\pi(\alpha_n^2+\alpha_n+1)}(E_{x_1}(a_n)(e',R_ne_1)+E_{x_2}(a_n)(e',R_ne_2))\lambda_n|H_n|\\&+\frac{3\alpha_n}{4\pi(\alpha_n^2+\alpha_n+1)}\int_{\mathbb{R}^2}(\widetilde{U}_n,(w_n)_{x_1}\wedge(w_n)_{x_2})\,dx+O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big)\\&+O(|H_n|^2\lambda_n)+O(|H_n|^2\|\nabla w_n\|_{L^2(\mathbb{R}^2)})+O(|H_n|\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)})\\&+O(|H_n|\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\mathbb{R}^2)})+O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}).\end{aligned}\tag{3.36}$$

From (3.36), (3.12) follows. □

Combining Lemma 3.1 and Lemma 3.2, we obtain

Corollary 3.3. *As $n\to\infty$, we have the following expansion:*

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n)=&\,8\pi+\mathcal{E}_{H_n}(\underline{u}_n)+\int_{\mathbb{R}^2}|\nabla w_n|^2\,dx\\&+4\int_{\mathbb{R}^2}(R_n\widehat{U}_{\lambda_n,a_n},(w_n)_{x_1}\wedge(w_n)_{x_2})\,dx\\&-8\pi(E_{x_1}(a_n)(e',R_ne_1)+E_{x_2}(a_n)(e',R_ne_2))\lambda_n|H_n|+o(|H_n|^2)\\&+o(|H_n|\lambda_n)+O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big)+o(\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)})\\&+O(|H_n|\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\mathbb{R}^2)})+O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}).\end{aligned}\tag{3.37}$$

To proceed further, we must know more about w_n. To this end, we first show that the blow-up occurs at the interior point of Ω. Next we estimate the order of $\|\nabla w_n\|_{L^2(\Omega)}$ and study the relation between λ_n and $|H_n|$. The following lemma follows from a proof of [12].

Lemma 3.4. *There exist constants $|H_0| > 0$ and $C > 0$ such that for $H \in \mathbb{R}^3 \setminus \{0\}$ with $|H| \leq |H_0|$*

$$E_H(\overline{u}_H) \leq 8\pi + E_H(\underline{u}_H) - C|H|^2. \tag{3.38}$$

Proof. In [12, Lemma 2.1], the inequality $E_H(\overline{u}_H) < 8\pi + E_H(\underline{u}_H)$ is proved. In fact, the same proof and (2.28) prove the inequality (3.38) for small $|H|$. One can also deduce (3.38) from Theorem 4.1. $\square$

The following lemma, due to Sasahara [24], is also useful to estimate $\|\nabla w_n\|_{L^2(\Omega)}$.

Lemma 3.5. *As before, set $\tilde{U} = R\widehat{U}_{\lambda,a}$ ($\lambda > 0$, $a \in \mathbb{R}^2$ and $R \in SO(3)$). Let $W_0(\tilde{U})$ be the subspace of $W := \left\{\phi \in L^1_{\text{loc}}(\mathbb{R}^2;\mathbb{R}^3) : \int_{\mathbb{R}^2} |\nabla\phi|^2\,dx + \frac{|\phi|^2}{(1+|x|^2)^2}\,dx < +\infty\right\}$ spanned by $\tilde{U}$ and the solutions of the equation*

$$\Delta\phi = 2(\tilde{U}_{x_1} \wedge \phi_{x_2} + \phi_{x_1} \wedge \tilde{U}_{x_2}) \quad \text{in } \mathbb{R}^2. \tag{3.39}$$

Then there exists a constant $C > 0$ (independent of λ, a, R) such that

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2\,dx + 4\int_{\mathbb{R}^2} (\tilde{U}, \varphi_{x_1} \wedge \varphi_{x_2})\,dx \geq C\int_{\mathbb{R}^2} |\nabla\varphi|^2\,dx \tag{3.40}$$

for any $\varphi \in W$ with $\int_{\mathbb{R}^2} \nabla\varphi \cdot \nabla\phi\,dx = 0$ ($\phi \in W_0(\tilde{U})$).

Remark 3.6. It is shown in [24] that the solution space of (3.39) is spanned by the maps $\frac{\partial\tilde{U}}{\partial x_i}$ ($i = 1,2$), $\frac{\partial\tilde{U}}{\partial\lambda}$, $\xi_i\tilde{U}$ ($i = 1,2,3$), e_i ($i = 1,2,3$), where ξ_i ($i = 1,2,3$) is a basis of the Lie algebra of $SO(3)$.

Lemma 3.7. *Let R_n, λ_n, a_n be as before. Define h'_n by the solution of*

$$\begin{cases} \Delta h'_n = 0 & \text{in } \Omega \\ h'_n = R_n\widehat{U}_{\lambda_n,a_n} & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}. \end{cases}$$

Then we have (as $n \to \infty$)

$$\int_{\mathbb{R}^2} |\nabla h'_n|^2\,dx = 4\pi\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + O\Big(\frac{\lambda_n^3}{d_n^3}\Big). \tag{3.41}$$

Here the $h^i_{a_n}$ $(i = 1, 2)$ *are defined in* (1.6).

Proof. By the definition of h'_n, we have (set again $\widetilde{U}_n = R_n \widehat{U}_{\lambda_n, a_n}$, $U_n = R_n U_{\lambda_n, a_n}$)

$$
\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla h'_n|^2\, dx = \int_{\mathbb{R}^2} |\nabla \widetilde{U}_n|^2\, dx - \int_{\mathbb{R}^2} |\nabla \widetilde{U}_n - \nabla h'_n|^2\, dx \\
&= 8\pi + \int_\Omega (\Delta \widetilde{U}_n, \widetilde{U}_n - h'_n)\, dx = 8\pi - \int_\Omega |\nabla \widetilde{U}_n|^2 (U_n, \widetilde{U}_n - h'_n)\, dx \\
&= 8\pi - \int_\Omega |\nabla \widetilde{U}_n|^2 (U_{\lambda_n, a_n}, \widehat{U}_{\lambda_n, a_n} - h_n)\, dx \\
&= 8\pi - \int_\Omega |\nabla \widetilde{U}_n|^2 (U_{\lambda_n, a_n}, U_{\lambda_n, a_n} - \gamma - h_n)\, dx \\
&= 8\pi - \int_\Omega |\nabla \widetilde{U}_n|^2\, dx + \int_\Omega |\nabla \widetilde{U}_n|^2 (U_{\lambda_n, a_n}, \gamma)\, dx \\
&\quad + \int_\Omega |\nabla \widetilde{U}_n|^2 (U_{\lambda_n, a_n}, h_n)\, dx. \qquad (3.42)
\end{aligned}
$$

Here h_n is the function defined in (2.14). Here

$$
\begin{aligned}
&\int_\Omega |\nabla \widetilde{U}_n|^2\, dx = \int_\Omega \frac{8\lambda_n^2}{(\lambda_n^2 + r^2)^2}\, dx \quad (r = |x - a_n|) \qquad (3.43) \\
&= 8\pi - \int_{\mathbb{R}^2 \setminus \Omega} \frac{8\lambda_n^2}{(\lambda_n^2 + r^2)^2}\, dx = 8\pi - 8\lambda_n^2 \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{r^4}\, dx + O\Big(\frac{\lambda_n^4}{d_n^4}\Big), \\
&\int_\Omega |\nabla \widetilde{U}_n|^2 (U_{\lambda_n, a_n}, \gamma)\, dx = \int_\Omega \frac{8\lambda_n^2 (r^2 - \lambda_n^2)}{(\lambda_n^2 + r^2)^3}\, dx \\
&= -\int_{\mathbb{R}^2 \setminus \Omega} \frac{8\lambda_n^2 (r^2 - \lambda_n^2)}{(\lambda_n^2 + r^2)^3}\, dx \qquad \Big(\text{since } \int_{\mathbb{R}^2} \frac{8\lambda_n^2 (r^2 - \lambda_n^2)}{(\lambda_n^2 + r^2)^3}\, dx = 0\Big) \\
&= -8\lambda_n^2 \int_{\mathbb{R}^2 \setminus \Omega} \frac{1}{r^4}\, dx + O\Big(\frac{\lambda_n^4}{d_n^4}\Big), \qquad (3.44)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega |\nabla \widetilde{U}_n|^2 (U_{\lambda_n, a_n}, h_n)\, dx = \int_\Omega \frac{16\lambda_n^3}{(\lambda_n^2 + r^2)^3} (x_1 - a_n^1) h_n^1\, dx \\
&+ \int_\Omega \frac{16\lambda_n^3}{(\lambda_n^2 + r^2)^3} (x_2 - a_n^2) h_n^2\, dx + \int_\Omega \frac{8\lambda_n^2 (r^2 - \lambda_n^2)}{(\lambda_n^2 + r^2)^3} h_n^3\, dx
\end{aligned}
$$

$$= 4\pi\Big(\frac{\partial h_n^1}{\partial x_1}(a_n) + \frac{\partial h_n^2}{\partial x_2}(a_n)\Big)\lambda_n + O\Big(\frac{\lambda_n^3}{d_n^3}\Big)$$
$$= 4\pi\Big(\frac{\partial h_{a_n}^1}{\partial x_1}(a_n) + \frac{\partial h_{a_n}^2}{\partial x_2}(a_n)\Big)\lambda_n^2 + O\Big(\frac{\lambda_n^3}{d_n^3}\Big), \tag{3.45}$$

where $a_n = (a_n^1, a_n^2)$, $h_n = (h_n^1, h_n^2, h_n^3)$.

In calculating (3.45), we have used a similar argument used in (3.31), (3.32) and (3.33). From (3.42)–(3.45), we obtain (3.41). □

To prove that the blow-up point is in the interior of Ω, the following lemma is useful:

Lemma 3.8. *Let h_a^1, h_a^2 be as in* (1.6). *Set $d = \mathrm{dist}(a, \partial\Omega)$. Then we have*

$$\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a) = \frac{1}{d^2} + o\Big(\frac{1}{d^2}\Big) \quad \text{as } d \to 0. \tag{3.46}$$

Proof. Assume d is small enough. Define $\Omega_d := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > d\}$. By definition, $a \in \partial\Omega_d$. Let ν be the outer normal of $\partial\Omega_d$ at a. Set $\overline{a} = a + 2d\nu$. Define

$$g^i(x) := \frac{2(x_i - \overline{a}_i - 2\nu_i(x - \overline{a}, \nu))}{|x - \overline{a}|^2} \quad (i = 1, 2).$$

g^i $(i = 1, 2)$ is harmonic in Ω. Moreover, $h_a^i(x) - g^i(x) = o\big(\frac{1}{d}\big)$ on $\partial\Omega$ (as $d \to 0$). Thus by the maximum principle, we have

$$\|h_a^i - g^i\|_{L^\infty(\Omega)} = o\Big(\frac{1}{d}\Big) \quad (\text{as } d \to 0). \tag{3.47}$$

Then by the elliptic estimate, see [8, Theorem 4.8], we have (using (3.47))

$$d|\nabla h_a^i(a) - \nabla g^i(a)| \le C\|h_a^i - g^i\|_{L^\infty(\Omega)} = o\Big(\frac{1}{d}\Big). \tag{3.48}$$

(3.48) implies

$$\frac{\partial h^1}{\partial x_1}(a) + \frac{\partial h^2}{\partial x_2}(a) = \frac{\partial g^1}{\partial x_1}(a) + \frac{\partial g^2}{\partial x_2}(a) + o\Big(\frac{1}{d^2}\Big) = \frac{1}{d^2} + o\Big(\frac{1}{d^2}\Big) \quad \text{as } d \to 0.$$

As a corollary of Corollary 3.3, Lemma 3.4, Lemma 3.5, Lemma 3.7, Lemma 3.8 and Corollary 5.2, we have

Corollary 3.9. *There exists a constant $C > 0$ such that $d_n \geq C$ for all n. In particular, the blow-up point is in the interior of Ω.*

Proof. Suppose for some subsequence (still denoted by $\{d_n\}$) $d_n \to 0$. By Corollary 3.3, Lemma 3.5, Remark 3.6 and Corollary 5.2, there exists a constant $C_i > 0$ $(i = 1, 2)$ such that

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi &+ \mathcal{E}_{H_n}(\underline{u}_n) + C_1\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)} - C_2\lambda_n|H_n| \\ &+ o(|H_n|^2) + O\Big(|H_n|\frac{\lambda_n^2}{d_n^2}\Big) + o(\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)}) \\ &+ O(|H_n|\lambda_n|\log\lambda_n|^{1/2}\|\nabla w_n\|_{L^2(\mathbb{R}^2)}) + O(\|\nabla w_n\|^3_{L^2(\mathbb{R}^2)}).\end{aligned} \tag{3.49}$$

Since $\|\nabla w_n\|_{L^2(\mathbb{R}^2)} \to 0$ (as $n \to \infty$), (3.49) implies

$$\mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + C_3\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)} - C_4\lambda_n|H_n| + o(|H_n|^2) + o\Big(\frac{\lambda_n^2}{d_n^2}\Big). \tag{3.50}$$

Since $\|\nabla w_n\|_{L^2(\mathbb{R}^2)} = \|\nabla w_n\|_{L^2(\Omega)} + |\alpha_n|\|\nabla\widetilde{U}_n\|_{L^2(\mathbb{R}^2\setminus\Omega)} \geq |\alpha_n|\|\nabla h'_n\|_{L^2(\mathbb{R}^2)}$, by (3.41), (3.46) and (3.50),

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n) &\geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + C_5\frac{\lambda_n^2}{d_n^2} - C_4\lambda_n|H_n| + o(|H_n|^2) + o\Big(\frac{\lambda_n^2}{d_n^2}\Big) \\ &\geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + C_6\frac{\lambda_n^2}{d_n^2} - C_4\lambda_n|H_n| + o(|H_n|^2) \\ &\geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - C_7 d_n^2|H_n|^2 + o(|H_n|^2) \\ &\geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + o(|H_n|^2) \quad \text{as } n \to \infty.\end{aligned} \tag{3.51}$$

Here, we have used the inequality $C_6\frac{\lambda_n^2}{d_n^2} - C_4\lambda_n|H_n| \geq -C_7 d_n^2|H_n|^2$ for some constant $C_7 > 0$. But (3.51) contradicts (3.38). This proves Corollary 3.9. $\square$

We also obtain

Corollary 3.10. *There exists a constant $C > 0$ such that $C^{-1} \leq \frac{\lambda_n}{|H_n|} \leq C$ for all n. We also have $\|\nabla w_n\|_{L^2(\mathbb{R}^2)} = O(|H_n|)$ and $\alpha_n = 1 + O(|H_n|^2)$ as $n \to \infty$.*

Proof. By Corollary 3.9, $d_n \geq C > 0$ for some $C > 0$; thus we may drop d_n in the expansion (3.37). Then by (3.37),

$$\mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + C\lambda_n^2 - C\lambda_n|H_n| + o(|H_n|^2). \tag{3.52}$$

By (3.38) and (3.52),

$$C\lambda_n^2 - C\lambda_n|H_n| + o(|H_n|^2) \leq -C|H_n|^2. \tag{3.53}$$

By (3.53), $C\lambda_n^2 - C\lambda_n|H_n| \leq 0$ for large n, so we have

$$\lambda_n \leq C|H_n| \tag{3.54}$$

for large n. Also by (3.53), $-C\lambda_n|H_n| \leq -C|H_n|^2$ for large n and we have

$$\lambda_n \geq C|H_n| \tag{3.55}$$

for large n. (3.54) and (3.55) imply the first part of Corollary 3.10.

Since $C^{-1} \leq \frac{\lambda_n}{|H_n|} \leq C$, (3.37) implies

$$\mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + C\|\nabla w_n\|^2_{L^2(\mathbb{R}^2)} - C|H_n|^2 \tag{3.56}$$

for large n. (3.56) and (3.38) imply $\|\nabla w_n\|_{L^2(\mathbb{R}^2)} = O(|H_n|)$. $\alpha_n = 1 + O(|H_n|^2)$ follows from (3.12). $\square$

As a corollary of Remark 2.1 and Corollary 3.10, we have

Corollary 3.11. *We have* $\|w_n\|_{L^\infty(\Omega)} = O(|H_n|)$.

Proof. This is noted in Remark 2.1. We need some modification of the argument of the proof of (1.4), since in the present case α_n is involved. However, we know by Corollary 3.10, $\alpha_n = 1 + O(|H_n|^2)$. Using this, the proof proceeds essentially the same as in the proof of (1.4). We leave the details to the reader. $\square$

We also need to know the asymptotic behavior of R_n.

Lemma 3.12. *We have* $R_n\gamma = \gamma + O(|H_n|)$ *as* $n \to \infty$. *In particular, if* $R_n \to R$, *then* $R\gamma = \gamma$.

Remark 3.13. Since $R_n \in SO(3)$ and $SO(3)$ is compact, we can always find a subsequence of $\{R_n\}$ (still denoted by $\{R_n\}$) and $R \in SO(3)$ such that $R_n \to R$ as $n \to \infty$.

Proof of Lemma 3.12. Let α_n, R_n, λ_n and a_n be as before. Define v_n by $\overline{u}_n = \underline{u}_n + R_n\widehat{U}_{\lambda_n,a_n} + v_n$. By Corollary 3.10 and Corollary 3.11, $\alpha_n = 1 + O(|H_n|^2)$, $\|\nabla w_n\|_{L^2(\Omega)} = O(|H_n|)$ and $\|w_n\|_{L^\infty(\Omega)} = O(|H_n|)$. Thus

$$\|v_n\|_{L^\infty(\Omega)} = O(|H_n|), \quad \|\nabla v_n\|_{L^2(\Omega)} = O(|H_n|). \tag{3.57}$$

Since $\overline{u}_n$ satisfies (1.1), we have

$$\|\Delta\overline{u}_n + |\nabla\overline{u}_n|^2\overline{u}_n\|_{L^\infty(\Omega)} = O(|H_n|). \tag{3.58}$$

On the other hand,

$$\begin{aligned}\Delta\overline{u}_n + |\nabla\overline{u}_n|^2\overline{u}_n &= \Delta\underline{u}_n + R_n\Delta U_{\lambda_n,a_n} + \Delta v_n \\ &\quad + |\nabla\underline{u}_n + R_n\nabla U_{\lambda_n,a_n} + \nabla v_n|^2(\underline{u}_n + R_n\widehat{U}_{\lambda_n,a_n} + v_n) \\ &= \Delta v_n + |\nabla U_{\lambda_n,a_n}|^2(\gamma - R_n\gamma) + f_n,\end{aligned} \tag{3.59}$$

where

$$\begin{aligned}f_n = \Delta\underline{u}_n &+ |\nabla U_{\lambda_n,a_n}|^2(\underline{u}_n - \gamma) + |\nabla U_{\lambda_n,a_n}|^2 v_n + (|\nabla\underline{u}_n|^2 + |\nabla v_n|^2 \\ &+ 2(\nabla\underline{u}_n, R_n\nabla U_{\lambda_n,a_n}) + 2(\nabla\underline{u}_n, \nabla v_n) \\ &+ 2(R_n\nabla U_{\lambda_n,a_n}, \nabla v_n))(\underline{u}_n + R_n\widehat{U}_{\lambda_n,a_n} + v_n).\end{aligned}$$

In (3.59), we have used $\underline{u}_n + R_n\widehat{U}_{\lambda_n,a_n} + v_n = (\underline{u}_n - \gamma) + R_nU_{\lambda_n,a_n} + (\gamma - R_n\gamma) + v_n$. (Recall that by our convention $\gamma = e_3$.) By (2.28) and (3.57),

$$\|f_n\|_{L^1(\Omega)} = O(|H_n|). \tag{3.60}$$

(3.58), (3.59) and (3.60) imply that

$$g_n := \Delta v_n + |\nabla U_{\lambda_n,a_n}|^2(\gamma - R_n\gamma) \tag{3.61}$$

satisfies

$$\|g_n\|_{L^1(\Omega)} = O(|H_n|). \tag{3.62}$$

We know that the blow-up point a_∞ is in the interior of Ω (see Corollary 3.9). Thus $d = \mathrm{dist}(a_\infty, \partial\Omega) > 0$ and $\mathbb{B}_{d/2}(a_n) \Subset \Omega$ for large n. Let $\xi_n \in \mathbb{R}^3$ be such that $(\gamma - R_n\gamma, \xi_n) = |\gamma - R_n\gamma|$ and $|\xi_n| = 1$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that $\varphi(r) \equiv 1$ in $|r| \le 1$, $\varphi \equiv 0$ in $|r| \ge 2$ and $|\varphi(r)| \le 1$ in $\mathbb{R}$. Define $\varphi_n \in C_0^\infty(\Omega; \mathbb{R}^3)$ by $\varphi_n := \xi_n\varphi\big(\frac{4|x-a_n|}{d}\big)$. φ_n satisfies $\varphi_n \equiv \xi_n$ in $\mathbb{B}_{d/4}(a_n)$, $\varphi_n \equiv 0$ in $\Omega \setminus \mathbb{B}_{d/2}(a_n)$, $\|\varphi_n\|_{L^\infty(\Omega)} \le 1$ and $\|\nabla\varphi_n\|_{L^\infty(\Omega)} \le C$ (= independent of n). Using φ_n as a test function, we have, by (3.57), (3.61) and (3.62),

$$\begin{aligned}O(|H_n|) &= \int_\Omega |\nabla U_{\lambda_n,a_n}|^2(\gamma - R_n\gamma, \varphi_n)\,dx \\ &= \int_{\mathbb{B}_{d/4}(a_n)} |\nabla U_{\lambda_n,a_n}|^2(\gamma - R_n\gamma, \varphi_n)\,dx \\ &\quad + \int_{\Omega\setminus\mathbb{B}_{d/4}(a_n)} |\nabla U_{\lambda_n,a_n}|^2(\gamma - R_n\gamma, \varphi_n)\,dx \\ &= |\gamma - R_n\gamma| \int_{\mathbb{B}_{d/4}(a_n)} |\nabla U_{\lambda_n,a_n}|^2\,dx + O\bigg(\int_{\Omega\setminus\mathbb{B}_{d/4}(a_n)} |\nabla U_{\lambda_n,a_n}|^2\,dx\bigg).\end{aligned} \tag{3.63}$$

Here

$$\int_{\Omega\setminus\mathbb{B}_{d/4}(a_n)} |\nabla U_{\lambda_n,a_n}|^2\,dx \le \int_{\mathbb{R}^2\setminus\mathbb{B}_{d/4}(a_n)} \frac{8\lambda_n^2}{(\lambda_n^2+r^2)^2}\,dx = O(\lambda_n^2) = O(|H_n|^2). \tag{3.64}$$

(3.63) and (3.64) imply $\gamma - R_n\gamma = O(|H_n|)$ as $n\to\infty$. This completes the proof. □

By Corollary 3.10 and Lemma 3.12, there exist $\beta > 0$ and $\wp \in \mathbb{R}^3$ such that, as $n\to\infty$,

$$\lambda_n = \beta|H_n| + o(|H_n|), \tag{3.65}$$

$$R_n\gamma = \gamma + |H_n|\wp + o(|H_n|). \tag{3.66}$$

The next lemma plays an important role.

Lemma 3.13. *Let h'_n be as in Lemma 3.7. Define $v_n \in H^1_0(\Omega;\mathbb{R}^3)$ by $v_n := \overline{u}_n - \underline{u}_n - \alpha_n R_n \widehat{U}_{\lambda_n,a_n} + \alpha_n h'_n$. Then*

$$\|\nabla v_n\|_{L^2(\Omega)} = o(|H_n|) \quad \text{as } n\to\infty. \tag{3.67}$$

Proof. Since $SO(3)$ is compact, there exists a subsequence of $\{R_n\}$ (still denoted by $\{R_n\}$) such that $R_n \to R$. By Lemma 3.12,

$$R\gamma = \gamma. \tag{3.68}$$

Let $h' = Rh$, $h = \begin{pmatrix} h^1_{a_\infty} \\ h^2_{a_\infty} \\ 0 \end{pmatrix}$. Then by (3.65) and the maximum principle, we have

$$\|h'_n - \beta|H_n|h'\|_{L^\infty(\Omega)} = o(|H_n|). \tag{3.69}$$

Since $\overline{u}_n$ satisfies (1.1), we have (after some calculation)

$$\begin{aligned} &\Delta v_n + |\nabla U_{\lambda_n,a_n}|^2(\underline{u}_n - R_n\gamma) - |\nabla U_{\lambda_n,a_n}|^2 h'_n + |\nabla U_{\lambda_n,a_n}|^2 v_n \\ &\quad + 2(R_n\nabla U_{\lambda_n,a_n}, \nabla v_n)R_n U_{\lambda_n,a_n} + |\nabla v_n|^2\overline{u}_n + F_n = 0, \end{aligned} \tag{3.70}$$

where (as before, set $\widetilde{U}_n = R_n\widehat{U}_{\lambda_n,a_n}$, $U_n = R_n U_{\lambda_n,a_n}$)

$$\begin{aligned} F_n &= |\nabla\underline{u}_n|^2(\alpha_n\widetilde{U}_n - \alpha_n h'_n + v_n) + 2(\nabla\underline{u}_n, \alpha_n\nabla U_n)\overline{u}_n \\ &\quad - 2(\nabla\underline{u}_n, \alpha_n\nabla h'_n)\overline{u}_n + 2(\nabla\underline{u}_n, \nabla v_n)\overline{u}_n \\ &\quad - 2(\alpha_n\nabla U_n, \alpha_n\nabla h'_n)\overline{u}_n + \alpha_n^2|\nabla h'_n|^2\overline{u}_n \\ &\quad - 2(\alpha_n\nabla h'_n, \nabla v_n)\overline{u}_n - (H_n, \alpha_n\widetilde{U}_n)\overline{u}_n + (H_n, \alpha_n h'_n)\overline{u}_n - (H_n, v_n)\overline{u}_n \\ &\quad - (\alpha_n - 1)|\nabla U_n|^2 U_n + \{\alpha_n^2|\nabla U_n|^2\overline{u}_n - |\nabla U_n|^2(\underline{u}_n + \widetilde{U}_n - h'_n + v_n)\} \\ &\quad + 2(\nabla U_n, \nabla v_n)(\underline{u}_n - R_n\gamma - h'_n + v_n) \\ &\quad + 2\{\alpha_n(\nabla U_n, \nabla v_n)\overline{u}_n - (\nabla U_n, \nabla v_n)(\underline{u}_n + \widetilde{U}_n - h'_n + v_n)\}. \end{aligned} \tag{3.71}$$

Define $V_n := |H_n|^{-1}v_n$. Then by (3.57), there exists $C > 0$ (independent of n) such that

$$\|\nabla V_n\|_{L^2(\Omega)} \le C, \quad \|V_n\|_{L^\infty(\Omega)} \le C. \tag{3.72}$$

We shall show that $\|\nabla V_n\|_{L^2(\Omega)} = o(1)$ as $n \to \infty$. By (3.70), V_n satisfies the following equation:

$$\begin{aligned} &\Delta V_n + 2(\nabla U_n, \nabla V_n)U_n + |\nabla U_n|^2 V_n + |H_n||\nabla V_n|^2 \overline{u}_n \\ &+ |\nabla U_n|^2 |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |\nabla U_n|^2 |H_n|^{-1} h_n' + |H_n|^{-1}F_n = 0 \quad \text{in } \Omega. \end{aligned} \tag{3.73}$$

Define $\mu_n^{-1} := \|\nabla V_n\|_{L^\infty(\Omega)}$. Let $b_n \in \overline{\Omega}$ be such that $\mu_n^{-1} = |\nabla V_n(b_n)|$. Define $\check{V}_n(x) := V_n(\mu_n x + b_n)$ and $\Omega_n := \{\mu_n^{-1}(x - b_n) : x \in \Omega\}$. Then

$$\|\nabla \check{V}_n\|_{L^2(\Omega_n)} \le C, \quad \|\nabla \check{V}_n\|_{L^\infty(\Omega_n)} \le 1 \tag{3.74}$$

for some $C > 0$ independent of n. $\check{V}_n$ satisfies

$$\begin{aligned} &\Delta \check{V}_n + 2(\nabla \check{U}_n, \nabla \check{V}_n)\check{U}_n + |\nabla \check{U}_n|^2 \check{V}_n + |H_n||\nabla \check{V}_n|^2 \check{\overline{u}}_n \\ &+ |\nabla \check{U}_n|^2 |H_n|^{-1}(\check{\underline{u}}_n - R_n\gamma) - |\nabla \check{U}_n|^2 |H_n|^{-1} \check{h}_n' + \mu_n^2 |H_n|^{-1}\check{F}_n = 0 \quad \text{in } \Omega_n. \end{aligned} \tag{3.75}$$

Here $\check{U}_n(x) = U_n(\mu_n x + b_n)$, $\check{\underline{u}}_n(x) = \underline{u}_n(\mu_n x + b_n)$, ... , etc. Note that $\check{U}_n(x) = R_n U_0\big(\frac{\mu_n}{\lambda_n}\big(x - \frac{a_n - b_n}{\mu_n}\big)\big)$.

Define $\gamma_n := \mu_n \lambda_n^{-1}$, $c_n = \mu_n^{-1}(a_n - b_n)$. We decompose the proof into several cases.

Case (1): μ_n^{-1} is bounded. In this case, $\{V_n\}$ is equibounded and equicontinuous. By Arzelà-Ascoli, there exists a subsequence of $\{V_n\}$ (still denoted by $\{V_n\}$) such that $V_n \to V_\infty$ in $C^0(\overline{\Omega})$. Pass to the limit $n \to \infty$ in (3.73). One can verify that $|H_n|^{-1}F_n \to 0$ in $L^1(\Omega)$, $2(\nabla U_n, \nabla V_n)U_n \to 0$ in $L^1(\Omega)$, $|H_n||\nabla V_n|^2\overline{u}_n \to 0$ in $L^1(\Omega)$ and $|\nabla U_n|^2 V_n + |\nabla U_n|^2|H_n|^{-1}(\underline{u}_n - R_n\gamma) - |\nabla U_n|^2|H_n|^{-1}h_n' \to 8\pi(V_\infty(a_\infty) + E(a_\infty)e' - \wp - \beta h'(a_\infty))\delta_{a_\infty}$ in $\mathcal{D}'(\Omega)$. (Use (3.69) and $|\nabla U_n|^2 \to 8\pi\delta_{a_\infty}$ in $\mathcal{D}'(\Omega)$.) Thus we have

$$\Delta V_\infty + 8\pi(V_\infty(a_\infty) + E(a_\infty)e' - \wp - \beta h'(a_\infty))\delta_{a_\infty} = 0 \quad \text{in } \Omega. \tag{3.76}$$

If $V_\infty(a_\infty) + E(a_\infty)e' - \wp - \beta h'(a_\infty) \ne 0$, then $V_\infty = -4(V_\infty(a_\infty) + E(a_\infty)e' - \wp - \beta h'(a_\infty)) \log|x - a_\infty| +$ smooth function in Ω. This is a contradiction, since V_∞ is continuous in Ω. Thus

$$V_\infty(a_\infty) + E(a_\infty)e' - \wp - \beta h'(a_\infty) = 0. \tag{3.77}$$

Then, multiplying (3.73) by V_n and integrating over Ω, we have

$$\begin{aligned}\int_\Omega |\nabla V_n|^2\,dx \le \left|2\int_\Omega (\nabla U_n,\nabla V_n)(U_n,V_n)\,dx\right| + |H_n|\left|\int_\Omega |\nabla V_n|^2(\overline{u}_n,V_n)\,dx\right| \\ + \left|\int_\Omega |\nabla U_n|^2(V_n + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n', V_n)\,dx\right| \\ + |H_n|^{-1}\left|\int_\Omega (F_n,V_n)\,dx\right|. \qquad (3.78)\end{aligned}$$

We now estimate four terms in (3.78).

$$\left|\int_\Omega (\nabla U_n,\nabla V_n)(U_n,V_n)\,dx\right| \le C\int_\Omega |\nabla U_n|\,dx = O(\lambda_n|\log\lambda_n|), \qquad (3.79)$$

$$|H_n|\left|\int_\Omega |\nabla V_n|^2(\overline{u}_n,V_n)\,dx\right| = O(|H_n|), \qquad (3.80)$$

$$|H_n|^{-1}\left|\int_\Omega (F_n,V_n)\,dx\right| \le C|H_n|^{-1}\int_\Omega |F_n|\,dx = o(1). \qquad (3.81)$$

Since μ_n^{-1} is bounded, we have, by Taylor's formula,

$$\begin{aligned} &V_n(x) + |H_n|^{-1}(\underline{u}_n(x) - R_n\gamma) - |H_n|^{-1}h_n'(x) \qquad (3.82)\\ &= V_n(a_n) + |H_n|^{-1}(\underline{u}_n(a_n) - R_n\gamma) - |H_n|^{-1}h_n'(a_n) + O(|x-a_n|).\end{aligned}$$

(3.82) and (3.77) imply

$$\begin{aligned}&\left|\int_\Omega |\nabla U_n|^2(V_n + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n', V_n)\,dx\right| \\ &\le |V_n(a_n) + |H_n|^{-1}(\underline{u}_n(a_n) - R_n\gamma) - |H_n|^{-1}h_n'(a_n)|\int_\Omega |\nabla U_n|^2\,dx \\ &+ O\left(\int_\Omega |\nabla U_n|^2|x-a_n|\,dx\right) = o(1). \qquad (3.83)\end{aligned}$$

(3.78), (3.79), (3.80), (3.81) and (3.83) imply

$$\int_\Omega |\nabla V_n|^2\,dx = o(1) \quad \text{as } n\to\infty.$$

Therefore in this case, we have $\|\nabla V_n\|_{L^2(\Omega)} = o(1)$.

In the following, we assume $\mu_n \to 0$ as $n \to \infty$. Since $\mu_n \to 0$, for some subsequence of $\{n\}$ (still denoted by $\{n\}$) $\Omega_n \to \Omega_\infty$, where $\Omega_\infty = \mathbb{R}^2$ if $\mu_n^{-1} d(b_n, \partial\Omega) \to \infty$, otherwise $\Omega_\infty =$ half-plane in $\mathbb{R}^2$.

Case (2): $\gamma_n \to 0$. Since $\{\check{V}_n\}$ is equibounded and equicontinuous and $\{\nabla \check{V}_n\}$ is bounded in $L^2(\Omega_n)$, for some subsequence (still denoted by $\{\check{V}_n\}$)

$$\check{V}_n \to \check{V}_\infty \quad \text{in } C^0_{\mathrm{loc}}(\Omega_\infty), \quad \nabla \check{V}_n \rightharpoonup \nabla \check{V}_\infty \quad \text{weakly in } L^2(\Omega_\infty). \tag{3.84}$$

By (3.75), one can also verify that

$$\Delta \check{V}_n \to 0 \quad \text{in } L^\infty_{\mathrm{loc}}(\Omega_\infty). \tag{3.85}$$

(3.84), (3.85) and the elliptic estimate imply

$$\check{V}_n \to \check{V}_\infty \quad \text{in } C^1_{\mathrm{loc}}(\overline{\Omega_\infty}) \tag{3.86}$$

and

$$\begin{cases} \Delta \check{V}_\infty = 0 & \text{in } \Omega_\infty \\ \check{V}_\infty = 0 & \text{on } \partial\Omega_\infty \text{ if } \partial\Omega_\infty \neq \emptyset. \end{cases} \tag{3.87}$$

In any case, (3.87) implies $\check{V}_\infty \equiv$ const. This contradicts $|\nabla \check{V}_\infty(0)| = \lim_{n\to\infty} |\nabla \check{V}_n(0)| = 1$. Thus the case $\gamma_n \to 0$ does not occur.

Case (3): There exists $C > 0$ such that $C^{-1} \le \gamma_n \le C$. In this case, there exists a subsequence of $\{\gamma_n\}$ (still denoted by $\{\gamma_n\}$) such that $\gamma_n \to \Gamma$ for some $0 < \Gamma < \infty$. There are two possibilities:

(3-a): $\{c_n\}$ is not bounded: In this case, the same argument as in the proof of the case (2) derives the contradiction. Thus this case does not occur.

(3-b): $\{c_n\}$ is bounded: In this case, there exists a subsequence of $\{c_n\}$ (still denoted by $\{c_n\}$) such that $c_n \to p$ for some $p \in \mathbb{R}^3$. As in the case (2), (3.84) holds. Since $|H_n||\nabla \check{V}_n|^2 \overline{u}_n \to 0$ in $L^\infty_{\mathrm{loc}}(\Omega_\infty)$, $|\mu_n^2 |H_n|^{-1} \check{F}_n \to 0$ in $L^\infty_{\mathrm{loc}}(\Omega_\infty)$, and $\Delta \check{V}_n$ is bounded in $L^\infty(\Omega_n)$ by (3.75), by elliptic regularity theory, (3.86) also holds and $\check{V}_\infty$ satisfies

$$\begin{aligned} &\Delta \check{V}_\infty + |\nabla(RU_{\Gamma^{-1},p})|^2 (\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty)) \\ &+ 2(\nabla(RU_{\Gamma^{-1},p}), \nabla \check{V}_\infty) RU_{\Gamma^{-1},p} = 0 \quad \text{in } \Omega_\infty. \end{aligned} \tag{3.88}$$

We claim $\Omega_\infty = \mathbb{R}^2$. In fact, since c_n is bounded, $b_n \to a_\infty$. Since $a_\infty \in \Omega$, $\mu_n^{-1} d(b_n, \partial\Omega) \to \infty$ and $\Omega_\infty = \mathbb{R}^2$. Rewriting (3.88), we have

$$\begin{aligned} &\Delta(\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty)) \\ &+ |\nabla(RU_{\Gamma^{-1},p})|^2 (\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty)) \\ &+ 2(\nabla(RU_{\Gamma^{-1},p}), \nabla(\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty))) RU_{\Gamma^{-1},p} = 0 \quad \text{in } \mathbb{R}^2. \end{aligned} \tag{3.89}$$

We claim

$$(RU_{\Gamma^{-1},p}(x), \check{V}_\infty(x) + E(a_\infty)e' - \wp - \beta h'(a_\infty)) = 0 \quad \text{in } \mathbb{R}^2. \tag{3.90}$$

In fact, by (3.57)

$$\begin{aligned}\overline{u}_n &= \underline{u}_n + \alpha_n \widetilde{U}_n - \alpha_n h'_n + v_n \\ &= U_n + |H_n|(V_n + Ee' - \wp - \beta h') + o(|H_n|)\end{aligned} \tag{3.91}$$

in the $L^\infty(\Omega)$-sense. Since $|\underline{u}_n|^2 = |U_n|^2 = 1$, we have

$$1 = 1 + 2|H_n|(U_n, V_n + Ee' - \wp - \beta h') + o(|H_n|)$$

and $(U_n, V_n + Ee' - \wp - \beta h') = o(1)$ in the $L^\infty(\Omega)$-sense. Thus

$$(\check{U}_n, \check{V}_n + \check{E}e' - \wp - \beta \check{h}') = o(1) \tag{3.92}$$

in the $L^\infty(\Omega_n)$-sense. Since $b_n \to a_\infty$, passing to the limit in (3.92), we have (3.90).

(3.89) and (3.90) show that $\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty)$ is a Jacobi field along $RU_{\Gamma^{-1},p}$. Then, by Lemma 5.4,

$$\begin{aligned}&\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty) \\ &\in \operatorname{span}\Big\langle \frac{\partial(RU_{\Gamma^{-1},p})}{\partial x_1}, \frac{\partial(RU_{\Gamma^{-1},p})}{\partial x_2}, \frac{\partial(RU_{\Gamma^{-1},p})}{\partial \lambda}, \\ &\qquad \xi_1 RU_{\Gamma^{-1},p}, \xi_2 RU_{\Gamma^{-1},p}, \xi_3 RU_{\Gamma^{-1},p}\Big\rangle.\end{aligned} \tag{3.93}$$

On the other hand, by (5.3), we have

$$\begin{aligned}0 &= \int_{\mathbb{R}^2} \nabla V_n \cdot \nabla\Big(\frac{\partial U_n}{\partial x_1}\Big)\,dx - \alpha_n |H_n|^{-1} \int_{\mathbb{R}^2} \nabla h'_n \cdot \nabla\Big(\frac{\partial U_n}{\partial x_1}\Big)\,dx \\ &= \int_{\mathbb{R}^2} \nabla \check{V}_n \cdot \nabla\Big(\frac{\partial \check{U}_n}{\partial x_1}\Big)\,dx - \alpha_n |H_n|^{-1} \int_{\mathbb{R}^2} \nabla \check{h}'_n \cdot \nabla\Big(\frac{\partial \check{U}_n}{\partial x_1}\Big)\,dx.\end{aligned} \tag{3.94}$$

Here V_n is extended to $\mathbb{R}^2 \setminus \overline{\Omega}$ by 0 and h'_n is extended to $\mathbb{R}^2 \setminus \overline{\Omega}$ by $\widetilde{U}_n$.

Since $|H_n|^{-1} \nabla \check{h}'_n \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^2)$, taking the limit $n \to \infty$ in (3.94), we have

$$\int_{\mathbb{R}^2} \nabla \check{V}_\infty \cdot \nabla\Big(\frac{\partial(RU_{\Gamma^{-1},p})}{\partial x_1}\Big)\,dx = 0. \tag{3.95}$$

By a similar argument, we obtain

$$0 = \int_{\mathbb{R}^2} \nabla \check{V}_\infty \cdot \nabla\Big(\frac{\partial(RU_{\Gamma^{-1},p})}{\partial x_2}\Big)\, dx = \int_{\mathbb{R}^2} \nabla \check{V}_\infty \cdot \nabla\Big(\frac{\partial(RU_{\Gamma^{-1},p})}{\partial \lambda}\Big)\, dx$$
$$= \int_{\mathbb{R}^2} \nabla \check{V}_\infty \cdot \nabla(\xi_i RU_{\Gamma^{-1},p})\, dx \ (i = 1, 2, 3). \tag{3.96}$$

(3.93), (3.95) and (3.96) imply $\check{V}_\infty + E(a_\infty)e' - \wp - \beta h'(a_\infty) \equiv 0$. This is a contradiction, since $\check{V}_n \to \check{V}_\infty$ in $C^1_{\text{loc}}(\mathbb{R}^2)$ and $|\nabla \check{V}_n(0)| = 1$. Thus the case (3-b) does not occur.

Finally, we consider the case

Case (4): γ_n is not bounded. We may assume (after passing to a subsequence if necessary) $\gamma_n \to \infty$. There are two possibilities:

(4-a): $\{c_n\}$ is not bounded: In this case, the same argument as in case (2) and case (3-a) derives the contradiction.

(4-b): $\{c_n\}$ is bounded: There exists a subsequence of $\{c_n\}$ (still denoted by $\{c_n\}$) such that $c_n \to p$ for some $p \in \mathbb{R}^2$. Note that, as in the case (3-b), $\Omega_\infty = \mathbb{R}^2$. (3.84) also holds in this case. Passing to the limit $n \to \infty$ in (3.75), we obtain

$$\Delta \check{V}_\infty + 8\pi(\check{V}_\infty(p) + E(a_\infty)e' - \wp - \beta h'(a_\infty))\delta_p = 0 \quad \text{in } \mathbb{R}^2. \tag{3.97}$$

By (3.84), $\check{V}_\infty$ is continuous in $\mathbb{R}^2$. Thus the same argument as in case (1) shows that

$$\check{V}_\infty(p) + E(a_\infty)e' - \wp - \beta h'(a_\infty) = 0. \tag{3.98}$$

We return to equation (3.73). Multiplying (3.73) by $V_n - \wp$, we have

$$\begin{aligned}
&\Big|\int_\Omega (\Delta V_n, V_n - \wp)\, dx\Big| \\
&\le 2\Big|\int_\Omega (\nabla U_n, \nabla V_n)(U_n, V_n - \wp)\, dx\Big| + |H_n|\Big|\int_\Omega |\nabla V_n|^2(\overline{u}_n, V_n - \wp)\, dx\Big| \\
&\quad + \Big|\int_\Omega |\nabla U_n|^2(V_n + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n', V_n - \wp)\, dx\Big| \\
&\quad + |H_n|^{-1}\Big|\int_\Omega (F_n, V_n - \wp)\, dx\Big|.
\end{aligned} \tag{3.99}$$

Now, we estimate the four terms in the right side of (3.99). One can easily verify

$$|H_n|\Big|\int_\Omega |\nabla V_n|^2(\overline{u}_n, V_n - \wp)\, dx\Big| = O(|H_n|) = o(1) \tag{3.100}$$

and

$$|H_n|^{-1}\left|\int_\Omega (F_n, V_n - \wp)\,dx\right| = o(1). \tag{3.101}$$

To estimate the third term, note that by Taylor's formula

$$\begin{aligned} &V_n(x) + |H_n|^{-1}(\underline{u}_n(x) - R_n\gamma) - |H_n|^{-1}h_n'(x) \\ &= V_n(a_n) + |H_n|^{-1}(\underline{u}_n(a_n) - R_n\gamma) - |H_n|^{-1}h_n'(a_n) + O(\mu_n^{-1}|x - a_n|). \end{aligned} \tag{3.102}$$

Thus we have

$$\begin{aligned} &\left|\int_\Omega |\nabla U_n|^2 (V_n + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n', V_n - \wp)\,dx\right| \\ &\le C|V_n(a_n) + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n'(a_n)| \int_\Omega |\nabla U_n|^2\,dx \\ &\quad + O\left(\mu_n^{-1}\int_\Omega |\nabla U_n|^2 |x - a_n|\,dx\right) \\ &\le C|\check{V}_n(c_n) + |H_n|^{-1}(\underline{u}_n(a_n) - R_n\gamma) - |H_n|^{-1}h_n'(a_n)| \int_\Omega |\nabla U_n|^2\,dx \\ &\quad + O(\mu_n^{-1}\lambda_n) \quad (\text{since } V_n(a_n) = \check{V}_n(c_n)) \\ &= o(1) \quad (\text{by } \gamma_n = \mu_n\lambda_n^{-1} \to \infty \text{ and } (3.98)). \end{aligned} \tag{3.103}$$

Next, we estimate the first term in (3.99). As in (3.92), we have $(U_n, V_n + Ee' - \wp - \beta h') = o(1)$ in the $L^\infty(\Omega)$-sense. Thus

$$\begin{aligned} (U_n, V_n - \wp) &= -(U_n, Ee' - \beta h') + o(1) \\ &= -(\widetilde{U}_n, Ee' - \beta h') - (R_n\gamma, Ee' - \beta h') + o(1) \\ &= -(\widetilde{U}_n, Ee' - \beta h') - (\gamma, Ee' - \beta h') + o(1) \\ &= -(\widetilde{U}_n, Ee' - \beta h') + o(1) \end{aligned} \tag{3.104}$$

in the $L^\infty(\Omega)$-sense, where we have used Lemma 3.12 and $(\gamma, Ee' - \beta h') = 0$.

By (3.104), we have

$$\begin{aligned} &\left|\int_\Omega (\nabla U_n, \nabla V_n)(U_n, V_n - \wp)\,dx\right| \le C\mu_n^{-1}\int_\Omega |\nabla U_n||\widetilde{U}_n|\,dx + o(1) \\ &\le C\mu_n^{-1}\int_\Omega \frac{\lambda_n}{\lambda_n^2 + r^2} \cdot \frac{\lambda_n}{\sqrt{\lambda_n^2 + r^2}}\,dx + o(1) \\ &\le C\mu_n^{-1}\int_\Omega \frac{\lambda_n^2}{(\lambda_n^2 + r^2)^{3/2}}\,dx + o(1) \\ &\le O(\mu_n^{-1}\lambda_n) + o(1) = o(1) \quad (\text{since } \gamma_n = \mu_n\lambda_n^{-1} \to \infty). \end{aligned} \tag{3.105}$$

(3.99), (3.100), (3.101), (3.103) and (3.105) imply

$$\left|\int_\Omega (\Delta V_n, V_n - \wp)\,dx\right| = o(1) \tag{3.106}$$

as $n \to \infty$. By integration by parts, we have

$$-\int_\Omega (\Delta V_n, V_n - \wp)\,dx = \int_\Omega |\nabla V_n|^2\,dx + \int_{\partial\Omega}\Big(\frac{\partial V_n}{\partial n}, \wp\Big)\,d\sigma. \tag{3.107}$$

(3.106), (3.107) and Lemma 5.5 imply

$$\int_\Omega |\nabla V_n|^2\,dx = o(1).$$

Summing up the four cases (1)–(4), we have (3.67). □

By Corollary 3.3 and Lemma 3.13, we obtain

Corollary 3.14. *We have the following expansion as* $n \to \infty$ *:*

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n) = 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + 4\pi\Big(\frac{\partial h^1_{a_n}}{\partial x_2}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 \\ - 8\pi(E_{x_1}(a_n)(e', R_n e_1) + E_{x_2}(a_n)(e', R_n e_2))\lambda_n |H_n| + o(|H_n|^2).\end{aligned} \tag{3.108}$$

Proof. We must calculate

$$\int_{\mathbb{R}^2} |\nabla w_n|^2\,dx + 4\int_{\mathbb{R}^2} (\widetilde{U}_n, (w_n)_{x_1} \wedge (w_n)_{x_2})\,dx. \tag{3.109}$$

Here, as before, $\widetilde{U}_n = R_n \widehat{U}_{\lambda_n, a_n}$ and $w_n = -\alpha_n h'_n + v_n$ (h'_n and v_n are as in Lemma 3.13). By Corollary 3.10 and Lemma 3.13,

$$\begin{aligned}(3.109) = &\int_\Omega |\nabla h'_n|^2\,dx + 4\int_\Omega (\widetilde{U}_n, (h'_n)_{x_1} \wedge (h'_n)_{x_2})\,dx \\ &+ \int_{\mathbb{R}^2\setminus\Omega} |\nabla \widetilde{U}_n|^2 + 4\int_{\mathbb{R}^2\setminus\Omega} (\widetilde{U}_n, (\widetilde{U}_n)_{x_1} \wedge (\widetilde{U}_n)_{x_2})\,dx + o(|H_n|^2).\end{aligned} \tag{3.110}$$

We have already shown that (see Lemma 3.7)

$$\int_\Omega |\nabla h'_n|^2\,dx + \int_{\mathbb{R}^2\setminus\Omega} |\nabla \widetilde{U}_n|^2\,dx = 4\pi\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + o(|H_n|^2). \tag{3.111}$$

On the other hand,

$$4\int_\Omega (\widetilde{U}_n, (h'_n)_{x_1} \wedge (h'_n)_{x_2})\,dx = O\Big(|H_n|^2 \int_\Omega |\widetilde{U}_n|\,dx\Big) = O(|H_n|^2\lambda_n) = o(|H_n|^2) \tag{3.112}$$

and

$$\begin{aligned} &4\int_{\mathbb{R}^2\setminus\Omega} (\widetilde{U}_n, (\widetilde{U}_n)_{x_1} \wedge (\widetilde{U}_n)_{x_2})\,dx = -2\int_{\mathbb{R}^2\setminus\Omega} |\nabla \widetilde{U}_n|^2 (U_n, \widetilde{U}_n)\,dx \\ &= -2\int_{\mathbb{R}^2\setminus\Omega} |\nabla\widetilde{U}_n|^2\,dx + 2\int_{\mathbb{R}^2\setminus\Omega} |\nabla\widetilde{U}_n|^2 (U_{\lambda_n,a_n}, \gamma)\,dx \\ &= -16\int_{\mathbb{R}^2\setminus\Omega} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}\,dx + 16\int_{\mathbb{R}^2\setminus\Omega} \frac{\lambda_n^2}{(\lambda_n^2+r^2)^2}\cdot\frac{r^2-\lambda_n^2}{\lambda_n^2+r^2}\,dx \\ &= -16\lambda_n^2\int_{\mathbb{R}^2\setminus\Omega}\frac{1}{r^4}\,dx + 16\lambda_n^2\int_{\mathbb{R}^2\setminus\Omega}\frac{1}{r^4}\,dx + O(\lambda_n^4) \\ &= O(\lambda_n^4) = o(|H_n|^2). \end{aligned} \tag{3.113}$$

Combining (3.110), (3.111), (3.112) and (3.113), we have

$$(3.109) = 4\pi\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 + o(|H_n|^2). \tag{3.114}$$

(3.37) and (3.114) imply (3.108). Here we note that from (3.111), $\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n) > 0$. This is used in the following. □

The next theorem is the main result of this section:

Theorem 3.15. *We have*

$$\mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - 4\pi\frac{|\nabla E(a_n)|^2|e'|^2}{\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)}|H_n|^2 + o(|H_n|^2)$$

as $n \to \infty$.

Proof. By Corollary 3.14 and the fact that

$$\max\{E_{x_1}(a_n)(e', Re_1) + E_{x_2}(a_n)(e', Re_2) : R \in SO(3)\} = |\nabla E(a_n)||e'|,$$

we have

$$\begin{aligned} \mathcal{E}_{H_n}(\overline{u}_n) \geq 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + 4\pi\Big(\frac{\partial h^1_{a_n}}{\partial x_1}(a_n) + \frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\Big)\lambda_n^2 \\ - 8\pi|\nabla E(a_n)||e'|\lambda_n|H_n| + o(|H_n|^2). \end{aligned} \tag{3.115}$$

The assertion of the Theorem 3.15 follows from $4\pi\big(\frac{\partial h^1_{a_n}}{\partial x_2}(a_n)+\frac{\partial h^2_{a_n}}{\partial x_2}(a_n)\big)\lambda_n^2-8\pi|\nabla E(a_n)||e'|\lambda_n|H_n| \geq -4\pi\frac{|\nabla E(a_n)|^2|e'|^2}{\frac{\partial h^1_{a_n}}{\partial x_1}(a_n)+\frac{\partial h^2_{a_n}}{\partial x_2}(a_n)}|H_n|^2$. □

4. Proof of Theorem B—the second part. In this section, we assume $\gamma \equiv$ const. Our purpose in this section is to prove the following theorem, and we complete the proof of Theorem B.

Theorem 4.1. *Let $\{H_n\} \subset \mathbb{R}^3$ be as in Theorem B. Let $a \in \Omega$ be given. Then there exists $\psi_n \in \Phi_{-1}$ such that*

$$\mathcal{E}_{H_n}(\psi_n) = 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - 4\pi\frac{|\nabla E(a)|^2|e'|^2}{\frac{\partial h^1_a}{\partial x_1}(a)+\frac{\partial h^2_a}{\partial x_2}(a)}|H_n|^2 + o(|H_n|^2)$$

as $n \to \infty$.

In this section, for simplicity, we will use the same notations which have been used in Section 3. However, the author believes that this causes no confusion.

To prove Theorem 4.1, we first construct $\psi_n \in H^1_\gamma(\Omega : \mathbb{S}^2)$. For this, define

$$\lambda_n = \frac{|\nabla E(a)||e'|}{\frac{\partial h^1_a}{\partial x_1}(a)+\frac{\partial h^2_a}{\partial x_2}(a)}|H_n|. \tag{4.1}$$

Next remark that

$$\begin{aligned}&\max_{R\in SO(3)}(E_{x_1}(a)(e', Re_1) + E_{x_2}(a)(e', Re_2))\\ =&\max_{\substack{R\in SO(3)\\ R\gamma=\gamma}}(E_{x_1}(a)(e', Re_1) + E_{x_2}(a)(e', Re_2)) = |\nabla E(a)||e'|.\end{aligned}$$

Thus there exists $R \in SO(3)$ with $R\gamma = \gamma$ such that

$$(E_{x_1}(a)(e', Re_1) + E_{x_2}(a)(e', Re_2)) = |\nabla E(a)||e'|. \tag{4.2}$$

ψ_n is defined by a solution of the following problem:

$$\|\nabla(\psi_n - \underline{u}_n - R\widehat{U}_{\lambda_n,a})\|_{L^2(\Omega)} = \inf_{\psi\in H^1_\gamma(\Omega:\mathbb{S}^2)}\|\nabla(\psi - \underline{u}_n - R\widehat{U}_{\lambda_n,a})\|_{L^2(\Omega)}. \tag{4.3}$$

By direct minimization, one can easily verify that the solution of problem (4.3) exists and is smooth (see [21]). Note that the solution of problem (4.3) is not in general unique. ψ_n is defined to be one of its solutions.

In the following, we study the properties of ψ_n and seek the asymptotic expansion of $\mathcal{E}_{H_n}(\psi_n)$. The strategy is very much like the one of the proof of the expansion of $\mathcal{E}_{H_n}(\overline{u}_n)$; however, we will see that the analysis of ψ_n is somewhat easier than that of $\overline{u}_n$.

We begin with the following:

Lemma 4.2. *We have*

$$\|\nabla(\psi_n - \underline{u}_n - R\widehat{U}_{\lambda_n,a})\|_{L^2(\Omega)} = O(|H_n|) \tag{4.4}$$

as $n \to \infty$.

Proof. Let $h_{\lambda_n,a}$ be the solution of the problem

$$\Delta h_{\lambda_n,a} = 0 \text{ in } \Omega, \qquad h_{\lambda_n,a} = \widehat{U}_{\lambda_n,a} \text{ on } \partial\Omega.$$

Define $\psi'_n := \frac{\underline{u}_n + R\widehat{U}_{\lambda_n,a} - Rh_{\lambda_n,a}}{|\underline{u}_n + R\widehat{U}_{\lambda_n,a} - Rh_{\lambda_n,a}|}$ for large n. One can easily verify that $\psi'_n \in H^1_\gamma(\Omega : \mathbb{S}^2)$ and $\|\nabla(\psi'_n - \underline{u}_n - R\widehat{U}_{\lambda_n,a})\|_{L^2(\Omega)} = O(|H_n|)$ as $n \to \infty$. (4.4) follows from this and the definition of ψ_n. $\square$

Lemma 4.2 implies, in particular, that $\psi_n \in \Phi_{-1}$ for large n. Next, we consider the following problem:

$$\inf_{\alpha \in \mathbb{R}\setminus\{0\}} \|\nabla(\psi_n - \underline{u}_n - \alpha R\widehat{U}_{\lambda_n,a})\|_{L^2(\mathbb{R}^2)}. \tag{4.5}$$

Here ψ_n and $\underline{u}_n$ are extended to $\mathbb{R}^2 \setminus \overline{\Omega}$ by γ.

An argument similar to the proof of Lemma 5.1 shows that (4.5) is attained by some $\alpha_n \in \mathbb{R} \setminus \{0\}$. Set $\psi_n = \underline{u}_n + \alpha_n R\widehat{U}_{\lambda_n,a} + w_n$. Then, as in Corollary 5.2, we obtain

$$\int_{\mathbb{R}^2} \nabla(RU_{\lambda_n,a}) \cdot \nabla w_n \, dx = 0. \tag{4.6}$$

By Lemma 4.2, we also have

$$\|\nabla w_n\|_{L^2(\mathbb{R}^2)} = O(|H_n|). \tag{4.7}$$

Then the proof of Lemma 3.2 shows that

$$\begin{aligned}\alpha_n = 1 &- \frac{1}{2}(E_{x_1}(a)(e', Re_1) + E_{x_2}(a)(e', Re_2))\lambda_n|H_n| \\ &+ \frac{1}{4\pi}\int_{\mathbb{R}^2} (R\widehat{U}_{\lambda_n,a}, (w_n)_{x_1} \wedge (w_n)_{x_2})\, dx + o(|H_n|^2)\end{aligned} \tag{4.8}$$

as $n \to \infty$. In particular

$$\alpha_n = 1 + O(|H_n|^2). \tag{4.9}$$

The analogue of Lemma 3.13 is the following:

Lemma 4.3. *Define* v_n *by* $v_n = \psi_n - \underline{u}_n - \alpha_n R\widehat{U}_{\lambda_n,a} + \alpha_n R h_{\lambda_n,a}$, *where* $h_{\lambda_n,a}$ *is defined in the proof of Lemma 4.2. Then we have*

$$\|\nabla v_n\|_{L^2(\Omega)} = o(|H_n|)$$

as $n \to \infty$.

Proof. The proof of this lemma follows the same steps as in the proof of Lemma 3.13. In fact, in the present case the proof is somewhat easier. For simplicity, we use the notations $\widetilde{U}_n = R\widehat{U}_{\lambda_n,a}$, $U_n = RU_{\lambda_n,a}$ and $h'_n = Rh_{\lambda_n,a}$. We first derive the equation satisfied by v_n. By (4.3), the following holds:

$$\Delta(\psi_n - \underline{u}_n - \widetilde{U}_n) = -(|\nabla\psi_n|^2 + (\Delta\underline{u}_n, \psi_n) + (\Delta U_n, \psi_n))\psi_n. \tag{4.10}$$

Then after some calculation, we have

$$\begin{aligned} &\Delta v_n + 2(\nabla U_n, \nabla v_n)U_n + 2(\nabla U_n, \nabla v_n)v_n \\ &+ |\nabla v_n|^2\psi_n - |\nabla U_n|^2(U_n, (\underline{u}_n - \gamma) - \alpha_n h'_n + v_n)U_n + G_n = 0, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} G_n = {}& (\alpha_n - 1)|\nabla U_n|^2 U_n + |\nabla\underline{u}_n|^2\psi_n + \alpha_n^2|\nabla h'_n|^2\psi_n + 2(\nabla\underline{u}_n, \alpha_n\nabla U_n)\psi_n \\ &- 2(\nabla\underline{u}_n, \alpha_n\nabla h'_n)\psi_n + 2(\nabla\underline{u}_n, \nabla v_n)\psi_n - 2(\alpha_n\nabla U_n, \alpha_n\nabla h'_n)\psi_n \\ &- 2(\alpha_n\nabla h'_n, \nabla v_n)\psi_n + 2(\alpha_n\nabla U_n, \nabla v_n)(-\alpha_n h'_n + (1-\alpha_n)\gamma + (\underline{u}_n - \gamma)) \\ &+ 2\alpha_n(\alpha_n - 1)(\nabla U_n, \nabla v_n)U_n + 2(\alpha_n - 1)(\nabla U_n, \nabla v_n)v_n \\ &+ (\Delta\underline{u}_n, \underline{u}_n + \alpha_n\widetilde{U}_n - \alpha_n h'_n + v_n)\psi_n \\ &- |\nabla U_n|^2(U_n, (\underline{u}_n - \gamma) - \alpha_n h'_n + v_n)((\underline{u}_n - \gamma) + (1-\alpha_n)\gamma - \alpha_n h'_n + v_n) \\ &- (\alpha_n - 1)|\nabla U_n|^2(U_n, (\underline{u}_n - \gamma) - \alpha_n h'_n + v_n)U_n \\ &+ (\alpha_n^2 - \alpha_n)|\nabla U_n|^2\psi_n + (\alpha_n - 1)|\nabla U_n|^2(U_n, \gamma)\psi_n. \end{aligned} \tag{4.12}$$

We claim that for any $0 < q < 1$,

$$\|v_n\|_{L^\infty(\Omega)} = O(|H_n|^q) \tag{4.13}$$

as $n \to \infty$. To prove this claim, first note that for any $1 < t$, we have

$$\int_\Omega |\nabla U_n|^{2t}\,dx = O\left(\int_\Omega \frac{\lambda_n^{2t}}{(\lambda_n^2 + r^2)^{2t}}\,dx\right) = O(\lambda_n^{2-2t}) = O(|H_n|^{2-2t}). \tag{4.14}$$

By (4.7), the fact that $\|\nabla h_n'\|_{L^2(\Omega)} = O(|H_n|)$ and Sobolev embedding, we have

$$\|\nabla v_n\|_{L^2(\Omega)} = O(|H_n|) \quad \text{and} \quad \|v_n\|_{L^r(\Omega)} = O(|H_n|)\ (1 < \forall r < \infty). \tag{4.15}$$

Thus we have, for any $1 < p < 2$,

$$\|(\nabla \underline{u}_n, \alpha_n \nabla U_n)\psi_n\|_{L^p(\Omega)} = O(|H_n|\|\nabla U_n\|_{L^p(\Omega)}) = o(|H_n|), \tag{4.16}$$

$$\|(\nabla \underline{u}_n, \nabla v_n)\psi_n\|_{L^p(\Omega)} = O(|H_n|^2), \tag{4.17}$$

$$\begin{aligned}
&\|(\alpha_n \nabla U_n, \nabla v_n)(-\alpha_n h_n' + (1-\alpha_n)\gamma + (\underline{u}_n - \gamma))\|_{L^p(\Omega)} \\
&= O(|H_n|\||\nabla U_n||\nabla v_n|\|_{L^p(\Omega)}) = O(|H_n|\|\nabla U_n\|_{L^{\frac{2p}{2-p}}(\Omega)}\|\nabla v_n\|_{L^2(\Omega)}) \\
&= O(|H_n|^{2/p}),
\end{aligned} \tag{4.18}$$

$$\begin{aligned}
&\||\nabla U_n|^2(U_n, (\underline{u}_n - \gamma) - \alpha_n h_n' + v_n)((\underline{u}_n - \gamma) \\
&+ (1-\alpha_n)\gamma - \alpha_n h_n' + v_n)\|_{L^p(\Omega)} \\
&\le C(|H_n|^2\||\nabla U_n|^2\|_{L^p(\Omega)} + |H_n|\||\nabla U_n|^2 v_n\|_{L^p(\Omega)} + \||\nabla U_n|^2|v_n|^2\|_{L^p(\Omega)}) \\
&= O(|H_n|^{2/p}) + O(|H_n|\||\nabla U_n|^2\|_{L^{\frac{p}{2-p}}(\Omega)}\|v_n\|_{L^{\frac{p}{p-1}}(\Omega)}) \\
&+ O(\||\nabla U_n|^2\|_{L^{\frac{p}{2-p}}(\Omega)}\||v_n|^2\|_{L^{\frac{p}{p-1}}(\Omega)}) = O(|H_n|^{2/p}) + O(|H_n|^{\frac{4}{p}-2}).
\end{aligned} \tag{4.19}$$

Other terms in G_n are easily estimated, and from (4.14)–(4.19), we have for p close to 1:

$$\|G_n\|_{L^p(\Omega)} = O(|H_n|). \tag{4.20}$$

We also have

$$\|(\nabla U_n, \nabla v_n)U_n\|_{L^p(\Omega)} \le \|\nabla v_n\|_{L^2(\Omega)}\|\nabla U_n\|_{L^{\frac{2p}{2-p}}(\Omega)} = O(|H_n|^{\frac{2}{p}-1}), \tag{4.21}$$

$$\|(\nabla U_n, \nabla v_n)v_n\|_{L^p(\Omega)} = O(|H_n|^{\frac{2}{p}-1}), \tag{4.22}$$

$$\||\nabla v_n|^2\psi_n\|_{L^p(\Omega)} \le \|\nabla v_n\|_{L^2(\Omega)}\|\nabla v_n\|_{L^{\frac{2p}{2-p}}(\Omega)} \le C|H_n|\|v_n\|_{W^{2,p}(\Omega)}, \tag{4.23}$$

$$\begin{aligned}
&\||\nabla U_n|^2(U_n, (\underline{u}_n - \gamma) - \alpha_n h_n' + v_n)U_n\|_{L^p(\Omega)} \\
&\le C|H_n|\||\nabla U_n|^2\|_{L^p(\Omega)} + C\||\nabla U_n|^2|v_n|\|_{L^p(\Omega)} \\
&\le C|H_n|^{\frac{2}{p}-1} + C\||\nabla U_n|^2\|_{L^{\frac{p}{2-p}}(\Omega)}\|v_n\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&= O(|H_n|^{\frac{2}{p}-1}) + O(|H_n|^{\frac{4}{p}-3}).
\end{aligned} \tag{4.24}$$

By the L^p-elliptic estimate, (4.11) and (4.20)–(4.24), we have

$$\|v_n\|_{W^{2,p}(\Omega)} \le C|H_n|\|v_n\|_{W^{2,p}(\Omega)} + O(|H_n|^{\frac{4}{p}-3}). \tag{4.25}$$

Thus by the Sobolev embedding $W^{2,p}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, we have (for large n)

$$\|v_n\|_{L^\infty(\Omega)} = O(|H_n|^{\frac{4}{p}-3}). \tag{4.26}$$

Since $1 < p < 2$ is arbitrary, (4.13) follows from (4.26).

We continue the proof of the lemma. Define

$$I_n := G_n - |\nabla U_n|^2 (U_n, (\underline{u}_n - \gamma) - \alpha_n h_n' + v_n) U_n.$$

Then (4.11) is written as

$$\Delta v_n + 2(\nabla U_n, \nabla v_n) U_n + 2(\nabla U_n, \nabla v_n) v_n + |\nabla v_n|^2 \psi_n + I_n = 0. \tag{4.27}$$

By the fact that $R\gamma = \gamma$ and (4.9), we have

$$\psi_n = \underline{u}_n + \alpha_n \widetilde{U}_n - \alpha_n h_n' + v_n = U_n + (\underline{u}_n - \gamma) - \alpha_n h_n' + v_n + O(|H_n|^2) \tag{4.28}$$

in the $L^\infty(\Omega)$-sense. From (4.28) and (4.13), we have

$$1 = |\psi_n|^2 = 1 + 2(U_n, (\underline{u}_n - \gamma) - \alpha_n h_n' + v_n) + O(|H_n|^p)$$

for $0 < \forall p < 2$. Thus

$$\|(U_n, (\underline{u}_n - \gamma) - \alpha_n h_n' + v_n)\|_{L^\infty(\Omega)} = O(|H_n|^p) \tag{4.29}$$

for $0 < \forall p < 2$.

From now on, we proceed as in the proof of Lemma 3.13. Define $V_n := |H_n|^{-1} v_n$. V_n satisfies

$$\begin{aligned} &\Delta V_n + 2(\nabla U_n, \nabla V_n) U_n + 2|H_n|(\nabla U_n, \nabla V_n) V_n \\ &+ |H_n||\nabla V_n|^2 \psi_n + |H_n|^{-1} I_n = 0 \quad \text{in } \Omega. \end{aligned} \tag{4.30}$$

Set $\mu_n^{-1} := \|\nabla V_n\|_{L^\infty(\Omega)}$. As in the proof of Lemma 3.13, we decompose the proof into four cases.

Case (1): μ_n^{-1} is bounded. In this case, $\|\nabla V_n\|_{L^\infty(\Omega)}$ is bounded. We claim

$$\|I_n\|_{L^p(\Omega)} = o(|H_n|) \tag{4.31}$$

for $p > 1$ close to 1. In fact, by (2.28), $\|\Delta \underline{u}_n + |H_n| e'\|_{L^\infty(\Omega)} = O(|H_n|^2)$. Since $(e', \gamma) = 0$, we have

$$\begin{aligned}
&\|(\Delta \underline{u}_n, \underline{u}_n + \alpha_n \widetilde{U}_n - \alpha_n h_n' + v_n)\psi_n\|_{L^p(\Omega)} \\
&= \|(\Delta \underline{u}_n, (\underline{u}_n - \gamma) + \alpha_n \widetilde{U}_n - \alpha_n h_n' + v_n)\psi_n\|_{L^p(\Omega)} + O(|H_n|^2) \\
&\le C|H_n| \| |\underline{u}_n - \gamma| + \alpha_n |\widetilde{U}_n| + \alpha_n |h_n'| + |v_n| \|_{L^p(\Omega)} + O(|H_n|^2) \\
&= O(|H_n|^2 + |H_n| \|\widetilde{U}_n\|_{L^p(\Omega)}) = O(|H_n|^2).
\end{aligned}$$

Other terms in I_n are estimated, using (4.29), as in the estimate of $\|G_n\|_{L^p(\Omega)}$ in (4.20), and we have (4.31).

Multiplying (4.30) by V_n and integrating over Ω, we have, using the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ for any $1 < r < \infty$ and $\|\nabla V_n\|_{L^2(\Omega)} \le C$ for some $C > 0$,

$$\begin{aligned}
\int_\Omega |\nabla V_n|^2 \, dx &\le C \int_\Omega |\nabla U_n| |\nabla V_n| |V_n| \, dx + C|H_n| \int_\Omega |\nabla U_n| |\nabla V_n| |V_n|^2 \, dx \\
&\quad + C|H_n| \int_\Omega |\nabla V_n|^2 |V_n| \, dx + |H_n|^{-1} \int_\Omega |I_n| |V_n| \, dx \\
&\le C \|V_n\|_{L^\infty(\Omega)} \int_\Omega |\nabla U_n| \, dx + C|H_n| \|V_n\|_{L^\infty(\Omega)}^2 \int_\Omega |\nabla U_n| \, dx \\
&\quad + C|H_n| \|V_n\|_{L^\infty(\Omega)} + |H_n|^{-1} \|I_n\|_{L^p(\Omega)} \|V_n\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\le C|H_n| |\log |H_n|| \|V_n\|_{L^\infty(\Omega)} + C|H_n|^2 |\log |H_n|| \|V_n\|_{L^\infty(\Omega)}^2 \\
&\quad + C|H_n| \|V_n\|_{L^\infty(\Omega)} + o(1) = o(1) \text{ (by (4.13))}.
\end{aligned}$$

Thus in this case, we have, as $n \to \infty$,

$$\int_\Omega |\nabla V_n|^2 \, dx = o(1).$$

In the following, we assume $\mu_n \to 0$ as $n \to \infty$.

Set $\check{V}_n(x) = V_n(\mu_n x + b_n)$, where $\mu_n^{-1} = |\nabla V_n(b_n)|$. Define $\Omega_n = \{\mu_n^{-1}(x - b_n) : x \in \Omega\}$. $\check{V}_n$ satisfies (3.74) and

$$\begin{aligned}
&\Delta \check{V}_n + 2(\nabla \check{U}_n, \nabla \check{V}_n)\check{U}_n + 2|H_n|(\nabla \check{U}_n, \nabla \check{V}_n)\check{V}_n \\
&+ |H_n| |\nabla \check{V}_n|^2 \check{\psi}_n + \mu_n^2 |H_n|^{-1} \check{I}_n = 0 \quad \text{in } \Omega_n.
\end{aligned} \tag{4.32}$$

Here, again, $\check{U}_n(x) = U_n(\mu_n x + b_n), \dots$, etc.

Also define $\gamma_n = \mu_n \lambda_n^{-1}$, $c_n = \mu_n^{-1}(a_n - b_n)$. As in the proof of Lemma 3.13, for some subsequence (still denoted by the same symbol) $\Omega_n \to \Omega_\infty$, where $\Omega_\infty = \mathbb{R}^2$ if $\mu_n^{-1} d(b_n, \partial\Omega) \to \infty$; otherwise $\Omega_\infty =$ half plane in $\mathbb{R}^2$.

Case (2): $\gamma_n \to 0$**.** Using (4.13), one can verify, by the same argument as in the proof of the case (2) of Lemma 3.13,

$$\Delta(\check{V}_n - \check{V}_n(0)) = \Delta \check{V}_n \to 0 \quad \text{in } L^\infty(\Omega_\infty). \tag{4.33}$$

By the mean value theorem and the fact that $\|\nabla \check{V}_n\|_{L^\infty(\Omega_n)} \le 1$, $|\check{V}_n - \check{V}_n(0)|$ is $L^\infty_{\text{loc}}(\Omega_\infty)$-bounded. Thus by the elliptic estimate, there exists $V'_\infty \in C^1(\Omega_\infty)$ with $\int_{\Omega_\infty} |\nabla V'_\infty|^2 \, dx < \infty$ such that $\check{V}_n - \check{V}_n(0) \to V'_\infty$ in $C^1_{\text{loc}}(\overline{\Omega_\infty})$ and

$$\begin{cases} \Delta V'_\infty = 0 & \text{in } \Omega_\infty \\ V'_\infty = 0 & \text{on } \partial\Omega_\infty, \text{ if } \partial\Omega_\infty \ne \emptyset. \end{cases}$$

From this, $V'_\infty \equiv$ const. However, this contradicts $|\nabla(\check{V}_n(0) - \check{V}_n(b_n))| = |\nabla \check{V}_n(0)| = 1$. Thus the case $\gamma_n \to 0$ does not occur.

Case (3): There exists $C > 0$ **such that** $C^{-1} \le \gamma_n \le C$**.** We may assume that $\gamma_n \to \Gamma$ for some $\Gamma > 0$. As in the proof of Lemma 3.13, there are two possibilities:

(3-a): $\{c_n\}$ *is not bounded:* By an argument similar to that of case (2), one can verify that this case does not occur.

(3-b): $\{c_n\}$ *is bounded:* We may assume that $c_n \to p$ for some $p \in \mathbb{R}^3$. In this case, $b_n \to a$ and $\Omega_n \to \mathbb{R}^2$, since $\mu_n^{-1} d(b_n, \partial\Omega) \to \infty$. As in case (2), by elliptic regularity, there exists $V'_\infty \in C^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} |\nabla V'_\infty|^2 \, dx < \infty$ such that $\check{V}_n - \check{V}_n(0) \to V'_\infty$ in $C^1_{\text{loc}}(\mathbb{R}^2)$. For simplicity, set $\beta = \frac{|\nabla E(a)||e'|}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}$. Then by (4.1), $\lambda_n = \beta |H_n|$.

We first prove that $\check{V}_n$ itself converges in $C^1_{\text{loc}}(\mathbb{R}^2)$. In fact, by (4.29) (take $p > 1$ there), we have

$$(\check{U}_n, \check{E}e' - \beta \check{h}' + \check{V}_n) = o(1) \quad \text{in } \Omega_n. \tag{4.34}$$

Here h' is defined in the proof of Lemma 3.13. (4.34) implies

$$(\check{U}_n, \check{V}_n(0)) = -(\check{U}_n, \check{E}e' - \beta \check{h}' + (\check{V}_n - \check{V}_n(0))) + o(1). \tag{4.35}$$

In particular, $\lim_{n\to\infty} (\check{U}_n(x), \check{V}_n(0))$ exists for all $x \in \mathbb{R}^2$. From this, $\lim_{n\to\infty} \check{V}_n(0)$ exists and

$$\lim_{n\to\infty} (\check{U}_n(x), \check{V}_n(0)) = -(RU_{\Gamma^{-1},p}(x), E(a)e' - \beta h'(a) + V'_\infty(x)) \quad \text{in } \mathbb{R}^2. \tag{4.36}$$

Define $V''_\infty := V'_\infty + \lim_{n\to\infty} \check{V}_n(0)$. Then by (4.36),

$$(RU_{\Gamma^{-1},p}(x), E(a)e' - \beta h'(a) + V''_\infty(x)) = 0 \quad \text{in } \mathbb{R}^2. \tag{4.37}$$

That is, $E(a)e' - \beta h'(a) + V''_\infty$ is a vector field along $RU_{\Gamma^{-1},p}$.

Passing to the limit $n \to \infty$ in (4.32), we have

$$\Delta V''_\infty + 2(\nabla(RU_{\Gamma^{-1},p}), \nabla V''_\infty) RU_{\Gamma^{-1},p} = 0 \quad \text{in } \mathbb{R}^2$$

or equivalently,

$$\begin{aligned}&\Delta(V''_\infty + E(a)e' - \beta h'(a)) \\ &+ 2(\nabla(RU_{\Gamma^{-1},p}), \nabla(V''_\infty + E(a)e' - \beta h'(a)))RU_{\Gamma^{-1},p} = 0 \quad \text{in } \mathbb{R}^2.\end{aligned} \tag{4.38}$$

Define $\tilde{V}_\infty := V''_\infty \circ \Pi$, where $\Pi : \mathbb{S}^2 \to \mathbb{R}^2$ is the stereographic projection from the north pole n of $\mathbb{S}^2$. By the conformal invariance of the equation (4.38), $\tilde{V}_\infty$ satisfies

$$\begin{aligned}&\Delta(\tilde{V}_\infty + E(a)e' - \beta h'(a)) \\ &+ 2(\nabla(RU_{\Gamma^{-1},p} \circ \Pi), \nabla(\tilde{V}_\infty + E(a)e' - \beta h'(a)))RU_{\Gamma^{-1},p} \circ \Pi = 0\end{aligned} \tag{4.39}$$

in $\mathbb{S}^2 \setminus \{n\}$. Since $\int_{\mathbb{R}^2} |\nabla V''_\infty|^2\,dx = \int_{\mathbb{S}^2\setminus\{n\}} |\nabla \tilde{V}_\infty|^2\,dx < +\infty$ and the 2-capacity of $\{n\}$ is 0, (4.39) holds on $\mathbb{S}^2$.

We claim $\tilde{V}_\infty \in W^{2,2}(\mathbb{S}^2)$ (and then $\tilde{V}_\infty \in C^\infty(\mathbb{S}^2)$ by elliptic regularity). To prove this we need only to show that $\tilde{V}_\infty \in L^2(\mathbb{S}^2)$, since if $\tilde{V}_\infty \in L^2(\mathbb{S}^2)$, then (4.39), $2(\nabla(RU_{\Gamma^{-1},p}\circ\Pi), \nabla(\tilde{V}_\infty + E(a)e' - \beta h'(a)))RU_{\Gamma^{-1},p}\circ\Pi \in L^2(\mathbb{S}^2)$ and the elliptic regularity theory show that $\tilde{V}_\infty \in W^{2,2}(\mathbb{S}^2)$.

Now we prove $\tilde{V}_\infty \in L^2(\mathbb{S}^2)$. In the following argument, $\check{V}_n$ is extended to $\mathbb{R}^2 \setminus \overline{\Omega_n}$ by 0. First, by the Poincaré inequality, there exists a constant $m_n \in \mathbb{R}^3$ such that

$$\int_{\mathbb{S}^2} |\check{V}_n \circ \Pi - m_n|^2\,dx \le C \int_{\mathbb{S}^2} |\nabla(\check{V}_n \circ \Pi)|^2\,dx = C \int_{\mathbb{R}^2} |\nabla \check{V}_n|^2\,dx \le C.$$

Thus there exists $g \in L^2(\mathbb{S}^2; \mathbb{R}^3)$ such that $\check{V}_n \circ \Pi - m_n \rightharpoonup g$ weakly in $L^2(\mathbb{S}^2)$. We claim m_n convergence to some $m_\infty \in \mathbb{R}^3$. In fact, let $\zeta_n \in \mathbb{R}^3$ be such that $|\zeta_n| = 1$ and $(m_n, \zeta_n) = |m_n|$. We may assume that $\zeta_n \to$

ζ_∞ for some unit vector ζ_∞. Let $\varphi \in C^\infty(\mathbb{S}^2)$ be such that $\operatorname{supp}(\varphi) \subset$ southern hemisphere of $\mathbb{S}^2$, $\varphi \geq 0$ and $\int_{\mathbb{S}^2} \varphi = 1$. Set $\varphi_n = \zeta_n \varphi$. Then

$$\int_{\mathbb{S}^2} (\check{V}_n \circ \Pi, \varphi_n) - \int_{\mathbb{S}^2} (m_n, \varphi_n) \to \int_{\mathbb{S}^2} (g, \zeta_\infty \varphi) \tag{4.40}$$

as $n \to \infty$. Obviously, the right-hand side of (4.40) is finite. On the other hand, since $\check{V}_n \to V''_\infty$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$, by the condition of the support of φ, $\int_{\mathbb{S}^2} (\check{V}_n \circ \Pi, \varphi_n) \to \int_{\mathbb{S}^2} (\tilde{V}_\infty, \zeta_\infty \varphi)$, and this limit is finite. Since $\int_{\mathbb{S}^2} (m_n, \varphi_n) = |m_n|$, $|m_n|$ is bounded, and for some subsequence (still denoted by $\{m_n\}$) $m_n \to m_\infty$ for some $m_\infty \in \mathbb{R}^3$. Then $\check{V}_n \circ \Pi \rightharpoonup g + m_\infty$ weakly in $L^2(\mathbb{S}^2)$. Since $\check{V}_n \circ \Pi \to \tilde{V}_\infty$ in $C^1_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{n\})$, $\tilde{V}_\infty = g + m_\infty \in L^2(\mathbb{S}^2)$, and the claim is proved.

Then multiplying (4.39) by $\tilde{V}_\infty + E(a)e' - \beta h'(a)$ and integrating over $\mathbb{S}^2$, we have, by (4.37),

$$\int_{\mathbb{R}^2} |\nabla(V''_\infty + E(a)e' - \beta h'(a))|^2 \, dx = 0.$$

Thus $V''_\infty \equiv$ const. But this contradicts the fact that $|\nabla \check{V}_n(0)| = 1$ and that $\check{V}_n \to V''_\infty$ in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$. Therefore, the case (3-b) does not occur.

Case (4): γ_n is not bounded. We may assume that $\gamma_n \to \infty$. Multiplying (4.30) by V_n and integrating over Ω, we have

$$\begin{aligned} \int_\Omega |\nabla V_n|^2 \, dx \leq {} & 2 \left| \int_\Omega (\nabla U_n, \nabla V_n)(U_n, V_n) \, dx \right| \\ & + 2|H_n| \left| \int_\Omega (\nabla U_n, \nabla V_n) |V_n|^2 \, dx \right| \\ & + |H_n| \int_\Omega |\nabla V_n|^2 |V_n| \, dx + |H_n|^{-1} \int_\Omega |I_n||V_n| \, dx \\ \leq {} & 2 \left| \int_\Omega (\nabla U_n, \nabla V_n)(U_n, V_n) \, dx \right| + o(1). \end{aligned} \tag{4.41}$$

By (4.29) and (4.9), we have

$$(U_n, V_n) = -(U_n, Ee' - \beta h') + o(1) \tag{4.42}$$

in the $L^\infty(\Omega)$-sense. Thus by (4.41)

$$\begin{aligned} \int_\Omega |\nabla V_n|^2 \, dx &\leq 2 \int_\Omega |\nabla U_n||\nabla V_n||(U_n, Ee' - \beta h')| \, dx + o(1) \\ &\leq O\left(\mu_n^{-1} \int_\Omega |\nabla U_n||\widetilde{U}_n| \, dx \right) = O\left(\mu_n^{-1} \int_\Omega \frac{\lambda_n^2}{(\lambda_n^2 + r^2)^{3/2}} \right) \\ &= O(\mu_n^{-1} \lambda_n) = O(\gamma_n^{-1}) = o(1). \end{aligned} \tag{4.43}$$

In the calculation of (4.43), we have used the fact that $R\gamma = \gamma$ and $(U_n, Ee' - \beta h') = (\widetilde{U}_n, Ee' - \beta h')$. Summing up the four cases (1)–(4), we have $\|\nabla v_n\|_{L^2(\Omega)} = o(|H_n|)$ as $n \to \infty$. $\square$

Now the proof of Theorem 4.1 is easy. Using (4.6), (4.8), (4.9) and Lemma 4.3, the same argument as in the proof of Corollary 3.14 gives the expansion

$$\begin{aligned}\mathcal{E}_{H_n}(\psi_n) = 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) + 4\pi\Big(\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)\Big)\lambda_n^2 \\ - 8\pi(E_{x_1}(a)(e', Re_1) + E_{x_2}(a)(e', Re_2))\lambda_n |H_n| + o(|H_n|^2).\end{aligned} \tag{4.44}$$

By (4.44) and our choice of λ_n (see (4.1)) and R (see (4.2)), we have the conclusion of Theorem 4.1.

Remark 4.4. The proof of Lemma 4.3 is somewhat easier than the proof of Lemma 3.13 because of the simplicity of the equation satisfied by ψ_n and R with $R\gamma = \gamma$ independent of n in the case of ψ_n.

We are now ready to prove Theorem B.

Proof of Theorem B (1). Let $a \in \Omega$. Let ψ_n be as in Theorem 4.1. Since $\psi_n \in \Phi_{-1}$ for large n, by the minimality of $\overline{u}_n$, we have $\mathcal{E}_{H_n}(\overline{u}_n) \le \mathcal{E}_{H_n}(\psi_n)$ for large n. Then by Theorem 3.15 and Theorem 4.1, we obtain

$$\begin{aligned}8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - 4\pi \frac{|\nabla E(a_n)|^2 |e'|^2}{\frac{\partial h_{a_n}^1}{\partial x_1}(a_n) + \frac{\partial h_{a_n}^2}{\partial x_2}(a_n)} |H_n|^2 + o(|H_n|^2) \\ \le 8\pi + \mathcal{E}_{H_n}(\underline{u}_n) - 4\pi \frac{|\nabla E(a)|^2 |e'|^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)} |H_n|^2 + o(|H_n|^2)\end{aligned} \tag{4.45}$$

as $n \to \infty$. Since $a_n \to a_\infty$ $(n \to \infty)$, by (4.45) we have

$$\frac{|\nabla E(a_\infty)|^2}{\frac{\partial h_{a_\infty}^1}{\partial x_1}(a_\infty) + \frac{\partial h_{a_\infty}^2}{\partial x_2}(a_\infty)} \ge \frac{|\nabla E(a)|^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}. \tag{4.46}$$

In (4.46), $a \in \Omega$ is arbitrary. Therefore we have

$$\frac{|\nabla E(a_\infty)|^2}{\frac{\partial h_{a_\infty}^1}{\partial x_1}(a_\infty) + \frac{\partial h_{a_\infty}^2}{\partial x_2}(a_\infty)} = \sup_{a \in \Omega} \frac{|\nabla E(a)|^2}{\frac{\partial h_a^1}{\partial x_1}(a) + \frac{\partial h_a^2}{\partial x_2}(a)}.$$

This completes the proof of Theorem B (1). $\square$

To prove Theorem B (2), we need the following:

Lemma 4.5. *Let λ_n be as in Section 3. Then we have*

$$\lim_{n\to\infty}\frac{\lambda_n}{|H_n|}=\frac{|\nabla E(a_\infty)||e'|}{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)}.$$

Proof. By Corollary 3.10, there exists $C>0$ such that $C^{-1}\le\frac{\lambda_n}{|H_n|}\le C$ for all n. Thus there exist a subsequence of $\{n\}$ (still denoted by $\{n\}$) and $\eta>0$ such that $\frac{\lambda_n}{|H_n|}\to\eta$ as $n\to\infty$. We have to show that $\eta=\frac{|\nabla E(a_\infty)||e'|}{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)}$. By Theorem 3.15 and Theorem 4.1, we have (see also the proof of Theorem B (1))

$$\mathcal{E}_{H_n}(\overline{u}_n)=8\pi+\mathcal{E}_{H_n}(\underline{u}_n)-4\pi\frac{|\nabla E(a_\infty)|^2|e'|^2}{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)}|H_n|^2+o(|H_n|^2). \tag{4.47}$$

We also have, by Corollary 3.14, that fact that $a_n\to a_\infty$ and that $\frac{\lambda_n}{|H_n|}\to\eta$,

$$\begin{aligned}\mathcal{E}_{H_n}(\overline{u}_n)&=8\pi+\mathcal{E}_{H_n}(\underline{u}_n)+4\pi\Big(\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)\Big)\eta^2|H_n|^2\\&\quad-8\pi(E_{x_1}(a_\infty)(e',Re_1)+E_{x_2}(a_\infty)(e',Re_2))\eta|H_n|^2+o(|H_n|^2)\\&\ge 8\pi+\mathcal{E}_{H_n}(\underline{u}_n)+4\pi\Big(\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)\Big)\eta^2|H_n|^2\\&\quad-8\pi|\nabla E(a_\infty)||e'|\eta|H_n|^2+o(|H_n|^2).\end{aligned} \tag{4.48}$$

(4.47) and (4.48) imply

$$\begin{aligned}&4\pi\Big(\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty)+\frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)\Big)\eta^2-8\pi|\nabla E(a_\infty)||e'|\eta\\&\le-4\pi\frac{|\nabla E(a_\infty)|^2|e'|^2}{\frac{\partial h^1}{\partial x_1}(a_\infty)+\frac{\partial h^2}{\partial x_2}(a_\infty)}.\end{aligned}$$

From this we have $\eta=\frac{|\nabla E(a_\infty)||e'|}{\frac{\partial h^1}{\partial x_1}(a_\infty)+\frac{\partial h^2}{\partial x_2}(a_\infty)}$. So far, we have only proved the convergence of a subsequence of $\big\{\frac{\lambda_n}{|H_n|}\big\}$. The convergence of the full sequence follows from the uniqueness of the possible limit via standard argument. $\square$

Proof of Theorem B (2). Let $\overline{\lambda}_n = \|\nabla \overline{u}_n\|^{-1}_{L^\infty(\Omega)}$. Then $\overline{\lambda}_n \to 0$. Define $\overline{v}_n(x) := \overline{u}_n(\overline{\lambda}_n x + \overline{a}_n)$, where $\overline{a}_n \in \overline{\Omega}$ and $|\overline{u}_n(\overline{a}_n)| = \overline{\lambda}_n^{-1}$. Obviously,

$$\overline{a}_n \to a_\infty \quad \text{as } n \to \infty. \tag{4.49}$$

$\overline{v}_n$ satisfies

$$\Delta \overline{v}_n + |\nabla \overline{v}_n|^2 \overline{v}_n - \overline{\lambda}_n^2 (H_n, \overline{v}_n)\overline{v}_n + \overline{\lambda}_n^2 H_n = 0 \quad \text{in } \overline{\lambda}_n^{-1}(\Omega - \overline{a}_n). \tag{4.50}$$

As in the proof of Theorem A (2-b-iii), $\overline{\lambda}_n^{-1} d(\overline{a}_n, \partial\Omega) \to \infty$ and $\overline{\lambda}_n^{-1}(\Omega - \overline{a}_n) \to \mathbb{R}^2$ as $n \to \infty$. Moreover, by (4.50), the fact that $\|\nabla \overline{v}_n\|_{L^\infty(\overline{\lambda}_n^{-1}(\Omega - \overline{a}_n))} \leq 1$ and elliptic regularity, there exists a subsequence of $\{\overline{v}_n\}$ (still denoted by $\{\overline{v}_n\}$) and $\overline{v}_\infty \in C^\infty(\mathbb{R}^2; \mathbb{S}^2)$ such that

$$\overline{v}_n \to \overline{v}_\infty \quad \text{in } C^\infty_{\text{loc}}(\mathbb{R}^2). \tag{4.51}$$

Since $\int_\Omega |\nabla \overline{u}_n|^2 \, dx$ is bounded, $\int_{\mathbb{R}^2} |\nabla \overline{v}_\infty|^2 \, dx < \infty$, and by (4.50) and (4.51) we have

$$\Delta \overline{v}_\infty + |\nabla \overline{v}_\infty|^2 \overline{v}_\infty = 0 \quad \text{in } \mathbb{R}^2. \tag{4.52}$$

(4.52) shows that $\overline{v}_\infty$ is a harmonic map from $\mathbb{R}^2$ to $\mathbb{S}^2$, and by the removable singularity theorem of Sacks-Uhlenbeck [23], $\overline{v}_\infty$ is extended as a harmonic map from $\mathbb{S}^2$ to $\mathbb{S}^2$. We also denote the extended map as $\overline{v}_\infty$. The similar argument in the proof of Theorem A (2-b-iii) shows that the degree of $\overline{v}_\infty$ is -1 and there exist $R \in SO(3)$, $\lambda > 0$ and $a \in \mathbb{R}^2$ such that $\overline{v}_\infty = RU_{\lambda,a}$ and

$$\lim_{n\to\infty} \Big\| \nabla \Big(\overline{u}_n - RU_{\lambda,a}\Big(\frac{\cdot - \overline{a}_n}{\overline{\lambda}_n} \Big) \Big) \Big\|_{L^2(\Omega)} = 0. \tag{4.53}$$

Since $\overline{v}_\infty = RU_{\lambda,a}$ and $|\nabla \overline{v}_\infty(0)| = \|\nabla \overline{v}_\infty\|_{L^\infty(\mathbb{R}^2)} = 1$, $a = 0$ and $\lambda = 2\sqrt{2}$.

Let α_n, R_n, λ_n and a_n be as in Section 3. By (1.3) and (4.53),

$$\lim_{n\to\infty} \Big\| \nabla \Big(RU_{2\sqrt{2},0}\Big(\frac{\cdot - \overline{a}_n}{\overline{\lambda}_n} \Big) - \alpha_n R_n U_{\lambda_n, a_n} \Big) \Big\|_{L^2(\Omega)} = 0. \tag{4.54}$$

(4.54) implies that $\frac{\lambda_n}{2\sqrt{2}\overline{\lambda}_n} = 1 + o(1)$ as $n \to \infty$. Then by Lemma 4.5,

$$\lim_{n\to\infty} \frac{|H_n|}{\overline{\lambda}_n} = \frac{2\sqrt{2}}{|e'|} \cdot \frac{\frac{\partial h^1_{a_\infty}}{\partial x_1}(a_\infty) + \frac{\partial h^2_{a_\infty}}{\partial x_2}(a_\infty)}{|\nabla E(a_\infty)|}. \tag{4.55}$$

Note that (4.55) holds only for a subsequence of $\{\frac{|H_n|}{\overline{\lambda}_n}\}$. However, (4.55) also holds for the full sequence because of the uniqueness of the possible limit. (4.55) is equivalent to the assertion of Theorem B (2). Thus we complete the proof. □

5. Technical lemmas. In this section, we prove some lemmas used in Section 3 and Section 4.

Lemma 5.1. *Let* $\gamma \equiv$ *const. and* $\delta > 0$. *Set* $\mathcal{M} := \{\alpha R U_{\lambda,a} : \alpha \in \mathbb{R} \setminus \{0\},\ \lambda > 0,\ a \in \overline{\Omega},\ R \in SO(3)\}$. *Let* $u \in H^1_\gamma(\Omega;\mathbb{S}^2)$. *Assume that* u *is extended to* $\mathbb{R}^2 \setminus \overline{\Omega}$ *by* γ. *Define* $d(u,\mathcal{M})$ *by*

$$d(u,\mathcal{M}) := \inf\{\|\nabla(u-\varphi)\|_{L^2(\mathbb{R}^2)} : \varphi \in \mathcal{M}\}.$$

Assume that $d(u,\mathcal{M})^2 < \|\nabla u\|^2_{L^2(\mathbb{R}^2)} - \delta$. *Then there exist* $\lambda > 0$, $a \in \overline{\Omega}$, $R \in SO(3)$, *and* $\alpha \in \mathbb{R} \setminus \{0\}$ *such that* $d(u,\mathcal{M}) = \|\nabla(u - \alpha R U_{\lambda,a})\|_{L^2(\mathbb{R}^2)}$.

Proof. Let $(\alpha_k, R_k, \lambda_k, a_k) \in (\mathbb{R}\setminus\{0\}) \times SO(3) \times (0,\infty) \times \overline{\Omega}$ be a minimizing sequence. For large k, we have

$$\|\nabla(u - \alpha_k R_k U_{\lambda_k,a_k})\|^2_{L^2(\mathbb{R}^2)} < \|\nabla u\|^2_{L^2(\mathbb{R}^2)} - \frac{\delta}{2}.$$

From this, we have

$$-2\alpha_k(\nabla u, R_k \nabla U_{\lambda_k,a_k})_{L^2(\mathbb{R}^2)} + 8\pi\alpha_k^2 < -\frac{\delta}{2}. \tag{5.1}$$

Thus α_k is bounded, and for some subsequence (still denoted by the same symbol), $\alpha_k \to \alpha$ and $\alpha \neq 0$. Assume $\lambda_k \to +\infty$ or $\lambda_k \to 0$. Then $\nabla U_{\lambda_k,a_k} \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^2)$. This contradicts (5.1). Thus for some subsequence (still denoted by the same symbol), $\lambda_k \to \lambda > 0$. Since $SO(3)$ and $\overline{\Omega}$ are compact, for some subsequence (still denoted by the same symbol), $R_k \to R \in SO(3)$ and $a_k \to a \in \overline{\Omega}$. Then $d(u,\mathcal{M}) = \|\nabla(u - \alpha R U_{\lambda,a})\|_{L^2(\mathbb{R}^2)}$. □

As a corollary of Lemma 5.1 and (1.3), we have

Corollary 5.2. *For large* n, *there exists* $\alpha'_n \in \mathbb{R} \setminus \{0\}$, $\lambda'_n > 0$, $a'_n \in \Omega$, $R'_n \in SO(3)$ *such that*

$$\|\nabla(\overline{u}_n - \underline{u}_n - \alpha'_n R'_n U_{\lambda'_n,a'_n})\|_{L^2(\mathbb{R}^2)} \to 0 \quad (n \to \infty). \tag{5.2}$$

Moreover, for $w_n := \overline{u}_n - \underline{u}_n - \alpha'_n R'_n U_{\lambda'_n,a'_n}$, $\varphi_n = R'_n U_{\lambda'_n,a'_n}$ *and* $\xi \in \mathfrak{so}(3)$, *we have*

$$\begin{aligned}\int_{\mathbb{R}^2} \nabla w_n \cdot \nabla \varphi_n \, dx &= \int_{\mathbb{R}^2} \nabla w_n \cdot \nabla\Big(\frac{\partial \varphi_n}{\partial x_i}\Big)\, dx \ (i=1,2)\\ = \int_{\mathbb{R}^2} \nabla w_n \cdot \nabla\Big(\frac{\partial \varphi_n}{\partial \lambda}\Big)\, dx &= \int_{\mathbb{R}^2} \nabla w_n \cdot \nabla(\xi\varphi_n)\, dx = 0.\end{aligned} \tag{5.3}$$

Here $\mathfrak{so}(3)$ *is the Lie algebra of* $SO(3)$ *and* $\underline{u}_n$ *and* $\overline{u}_n$ *are extended to* $\mathbb{R}^2 \setminus \overline{\Omega}$ *by* γ.

Proof. By (1.3), we can apply Lemma 5.1 to $\overline{u}_n - \underline{u}_n$. Then there exist $\alpha_n' \in \mathbb{R} \setminus \{0\}$, $R_n' \in SO(3)$, $a_n \in \overline{\Omega}$ and $\lambda_n' > 0$ such that

$$d(\overline{u}_n - \underline{u}_n, \mathcal{M}) = \|\nabla(\overline{u}_n - \underline{u}_n - \alpha_n' R_n' U_{\lambda_n', a_n'})\|_{L^2(\mathbb{R}^2)}.$$

We need to show that $a_n \in \Omega$. An argument similar to the proof of Theorem A (2-b-iii) shows that $(\lambda_n')^{-1}\mathrm{dist}(a_n', \partial\Omega) \to \infty$. In particular, $a_n' \in \Omega$ for large n. (5.3) follows from the minimality of $(\alpha_n', R_n', a_n', \lambda_n')$. $\square$

The proof of the next lemma can be found, for example, in [4].

Lemma 5.3.

(1) *Assume* $u, v \in H^1(\Omega) \cap L^\infty(\Omega)$, $w \in H^1(\Omega)$. *In addition assume* $u \wedge v = 0$ *on* $\partial\Omega$ *or* $w = 0$ *on* $\partial\Omega$. *Then*

$$\int_\Omega (u, v_{x_1} \wedge w_{x_2} + w_{x_1} \wedge v_{x_2})\, dx = \int_\Omega (v, u_{x_1} \wedge w_{x_2} + w_{x_1} \wedge u_{x_2})\, dx. \quad (5.4)$$

(2) *Assume* $u \in H^1(\Omega)$ *and* $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$. *Then*

$$\left| \int_\Omega (w, u_{x_1} \wedge u_{x_2})\, dx \right| \le C \|\nabla w\|_{L^2(\Omega)} \|\nabla u\|^2_{L^2(\Omega)}. \quad (5.5)$$

(3) *Assume* $u, v, w \in L^\infty(\mathbb{R}^2)$ *with* $\|\nabla u\|_{L^2(\mathbb{R}^2)}, \|\nabla v\|_{L^2(\mathbb{R}^2)}, \|\nabla w\|_{L^2(\mathbb{R}^2)} < \infty$. *Then*

$$\int_{\mathbb{R}^2} (u, v_{x_1} \wedge w_{x_2} + w_{x_1} \wedge v_{x_2})\, dx = \int_{\mathbb{R}^2} (v, u_{x_1} \wedge w_{x_2} + w_{x_1} \wedge u_{x_2})\, dx. \quad (5.6)$$

(4) *Assume* $u, w \in L^\infty(\mathbb{R}^2)$ *with* $\|\nabla u\|_{L^2(\mathbb{R}^2)}, \|\nabla w\|_{L^2(\mathbb{R}^2)} < \infty$. *Then*

$$\left| \int_{\mathbb{R}^2} (w, u_{x_1} \wedge u_{x_2})\, dx \right| \le C \|\nabla w\|_{L^2(\mathbb{R}^2)} \|\nabla u\|^2_{L^2(\mathbb{R}^2)}. \quad (5.7)$$

Proof. (1) and (2) are proved in [4]. (3) and (4) follow from (1), (2) and a density argument. $\square$

Lemma 5.4. *Let $R \in SO(3)$, $\lambda > 0$ and $a \in \mathbb{R}^2$. Let* $W(RU_{\lambda,a}) := \{V \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \int_{\mathbb{R}^2} |\nabla V|^2\,dx < \infty,\quad \int_{\mathbb{R}^2} \frac{|V|^2}{(1+|x|^2)^2}\,dx < \infty,\quad (RU_{\lambda,a}, V) = 0$ *almost everywhere in* $\mathbb{R}^2\}$. *Define the Jacobi operator* $J_{RU_{\lambda,a}}$ *at* $RU_{\lambda,a}$ *by* $J_{RU_{\lambda,a}}V := \Delta V + |\nabla(RU_{\lambda,a})|^2 V + 2(\nabla(RU_{\lambda,a}), \nabla V)RU_{\lambda,a}$. *Then* $\ker J_{RU_{\lambda,a}} \cap W(RU_{\lambda,a})$ *is spanned by* $\frac{\partial(RU_{\lambda,a})}{\partial x_1}$, $\frac{\partial(RU_{\lambda,a})}{\partial x_2}$, $\frac{\partial(RU_{\lambda,a})}{\partial \lambda}$, $\xi_1 RU_{\lambda,a}$, $\xi_2 RU_{\lambda,a}$ *and* $\xi_3 RU_{\lambda,a}$, *where* ξ_i $(1 \le i \le 3)$ *is a basis of the Lie algebra of* $SO(3)$.

Remark. The Jacobi operator $J_{RU_{\lambda,a}}$ at $RU_{\lambda,a}$ is the linearization of the operator $u \mapsto \Delta u + |\nabla u|^2 u$ at $RU_{\lambda,a}$.

Proof of Lemma 5.4. The proof is reduced to the case $R = \mathrm{id}$, $\lambda = 1$ and $a = 0$. In fact, $V \in \ker J_{RU_{\lambda,a}} \cap W(RU_{\lambda,a})$ if and only if $V'(x) := R^{-1}V(\lambda x + a) \in \ker J_{U_0} \cap W(U_0)$. Thus we prove the lemma in the case $R = \mathrm{id}$, $\lambda = 1$ and $a = 0$.

Since U_0 satisfies $\Delta U_0 + |\nabla U_0|^2 U_0 = 0$ in $\mathbb{R}^2$, one can easily verify that six vector fields $\frac{\partial U_0}{\partial x_1}$, $\frac{\partial U_0}{\partial x_2}$, $\frac{\partial U_0}{\partial \lambda}$, $\xi_1 U_0$, $\xi_2 U_0$, $\xi_3 U_0$ are in $\ker J_{U_0} \cap W(U_0)$ and linearly independent. We complete the proof if we show that $\dim(\ker J_{U_0} \cap W(U_0)) = 6$. Note that $V \in W(U_0)$ if and only if $V \circ \Pi \in H^1(\mathbb{S}^2)$ and $(V \circ \Pi(x), U_0 \circ \Pi(x)) = 0$ almost everywhere in $\mathbb{S}^2$, where $\Pi : \mathbb{S}^2 \to \mathbb{R}^2$ is the stereographic projection from the north pole. Note also that $U_0 = \Pi^{-1}$ and $U_0 \circ \Pi = \mathrm{id} : \mathbb{S}^2 \to \mathbb{S}^2$. Thus by the conformal invariance of the problem, we complete the proof if we show that $\dim(\ker J_{\mathrm{id}_{\mathbb{S}^2}} \cap \{V \in H^1(\mathbb{S}^2) : (\mathrm{id}_{\mathbb{S}^2}(x), V(x)) = 0$ almost everywhere in $\mathbb{S}^2\}) = 6$. However, this is proved in [28]. Thus we complete the proof. $\square$

Lemma 5.5. *Let V_n be as in the case* (4-b) *of Lemma* 3.13. *Then we have*

$$\int_{\partial\Omega} \left|\frac{\partial V_n}{\partial n}\right|^2 d\sigma = o(1) \quad \text{as } n \to \infty.$$

Proof. Let ζ be a smooth function defined in $\mathbb{R}^2$ such that $\zeta \equiv 0$ in $\{x \in \mathbb{R}^2 : |x| \le 1\}$, $\zeta \equiv 1$ in $\{x \in \mathbb{R}^2 : |x| \ge 2\}$, $0 \le \zeta \le 1$ in $\mathbb{R}^2$. By Corollary 3.9, the blow-up point a_∞ is in the interior of Ω. Thus $d = d(a_\infty, \partial\Omega) > 0$ and $\mathbb{B}_{d/4}(a_n) \subset \mathbb{B}_{d/2}(a_\infty) \subset \Omega$ for large n. For large n, put $\zeta_n(x) := \zeta\big(\frac{4(x-a_n)}{d}\big)$. Define $\overline{V}_n = \zeta_n V_n$. By (3.73), $\overline{V}_n$ satisfies

$$\begin{aligned}&\Delta\overline{V}_n - 2\nabla\zeta_n \cdot \nabla V_n - \Delta\zeta_n V_n + 2\zeta_n(\nabla U_n, \nabla V_n)U_n + \zeta_n|H_n||\nabla V_n|^2\overline{u}_n\\ &+ \zeta_n|\nabla U_n|^2(V_n + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n')\\ &+ \zeta_n|H_n|^{-1}F_n = 0 \quad \text{in } \Omega.\end{aligned} \tag{5.8}$$

We claim

$$\|\nabla V_n\|_{L^p(\Omega)} = o(1) \ (1 \le \forall p < 2). \tag{5.9}$$

In fact, as in the proof of (3.100), (3.101) and (3.103), we have

$$\begin{aligned} &\||\nabla U_n|^2(V_n + |H_n|^{-1}(\underline{u}_n - R_n\gamma) - |H_n|^{-1}h_n') \\ &+ |H_n||\nabla V_n|^2\underline{u}_n + |H_n|^{-1}F_n\|_{L^1(\Omega)} = o(1). \end{aligned} \tag{5.10}$$

Since $2(\nabla U_n, \nabla V_n)U_n$ is bounded in $L^1(\Omega)$, there exist a subsequence of $\{n\}$ (still denoted by $\{n\}$) and a Radon measure μ in Ω such that

$$2(\nabla U_n, \nabla V_n)U_n \rightharpoonup \mu \tag{5.11}$$

in the sense of measures. On the other hand, for any $\epsilon > 0$, we have

$$\begin{aligned} &\left|\int_{\Omega\setminus\mathbb{B}_\epsilon(a_n)} 2(\nabla U_n, \nabla V_n)U_n\, dx\right| = O\left(\int_{\Omega\setminus\mathbb{B}_\epsilon(a_n)} |\nabla U_n||\nabla V_n|\, dx\right) \\ &= O\left(\mu_n^{-1}\int_{\Omega\setminus\mathbb{B}_\epsilon(a_n)} \frac{\lambda_n}{\lambda_n^2 + r^2}\, dx\right) \\ &= O\left(\mu_n^{-1}\int_\epsilon^{\mathrm{diam}\Omega} \frac{\lambda_n r}{\lambda_n^2 + r^2}\, dr\right) = O(\mu_n^{-1}\lambda_n) = o(1) \end{aligned} \tag{5.12}$$

since $\gamma_n = \mu_n\lambda_n^{-1} \to \infty$. From (5.12), we obtain $\mathrm{supp}(\mu) \subset \{a_\infty\}$, and there exists $D \in \mathbb{R}^3$ such that $\mu = D\delta_{a_\infty}$. Since $\{V_n\}$ is $H_0^1(\Omega)$-bounded, we may assume that $V_n \rightharpoonup V_\infty$ weakly in $H^1(\Omega)$ for some $V_\infty \in H_0^1(\Omega)$. Then (3.73), (5.10) and (5.11) show that

$$\Delta V_\infty + D\delta_{a_\infty} = 0 \quad \text{in } \Omega. \tag{5.13}$$

Thus $V_\infty = DG(\cdot, a_\infty)$, where G is a Green's function of $-\Delta$ in Ω. In this case, however, $V_\infty \in H_0^1(\Omega)$ if and only if $D = 0$. Thus $\mu = 0$. Then (5.9) follows from Lemma 5.6 below.

We return to (5.8). By the elliptic estimate, we have

$$\begin{aligned} &\|\overline{V}_n\|_{W^{2,4/3}(\Omega)} \le C(\|\overline{V}_n\|_{L^{4/3}(\Omega)} + \|\nabla\zeta_n \cdot \nabla V_n\|_{L^{4/3}(\Omega)} + \|\Delta\zeta_n V_n\|_{L^{4/3}(\Omega)} \\ &+ \|\zeta_n|\nabla U_n||\nabla V_n|\|_{L^{4/3}(\Omega)} + |H_n|\|\zeta_n|\nabla V_n|^2\|_{L^{4/3}(\Omega)} \\ &+ \|\zeta_n|\nabla U_n|^2\|_{L^{4/3}(\Omega)} + |H_n|^{-1}\|\zeta_n F_n\|_{L^{4/3}(\Omega)}). \end{aligned} \tag{5.14}$$

By Hölder's inequality,

$$\|\zeta_n|\nabla U_n||\nabla V_n|\|_{L^{4/3}(\Omega)} \leq \|\zeta_n|\nabla U_n|\|_{L^4(\Omega)}\|\nabla V_n\|_{L^2(\Omega)} = O(\|\zeta_n|\nabla U_n|\|_{L^4(\Omega)}). \tag{5.15}$$

Here

$$\begin{aligned}\|\zeta_n|\nabla U_n|\|^4_{L^4(\Omega)} &\leq \int_{\Omega\setminus\mathbb{B}_{d/4}(a_n)} |\nabla U_n|^4\,dx \\ &= O\left(\int_{\Omega\setminus\mathbb{B}_{d/4}(a_n)} \frac{\lambda_n^4}{(\lambda_n^2+r^2)^4}\,dx\right) = O(\lambda_n^4).\end{aligned} \tag{5.16}$$

(5.15) and (5.16) imply

$$\|\zeta_n|\nabla U_n||\nabla V_n|\|_{L^{4/3}(\Omega)} = O(\lambda_n) = o(1). \tag{5.17}$$

Similarly, we have

$$\|\zeta_n|\nabla U_n|^2\|_{L^{4/3}(\Omega)} = O(\lambda_n^2) = o(1) \tag{5.18}$$

and

$$|H_n|^{-1}\|\zeta_n F_n\|_{L^{4/3}(\Omega)} = o(1). \tag{5.19}$$

By Hölder's inequality and the Sobolev embedding $W^{1,4/3}(\Omega) \hookrightarrow L^4(\Omega)$,

$$\begin{aligned}&|H_n|\|\zeta_n|\nabla V_n|^2\|_{L^{4/3}(\Omega)} \leq |H_n|\|\zeta_n|\nabla V_n|\|_{L^4(\Omega)}\|\nabla V_n\|_{L^2(\Omega)} \\ &\leq C|H_n|\|\zeta_n|\nabla V_n|\|_{L^4(\Omega)} \leq C|H_n|(\|\nabla\overline{V}_n\|_{L^4(\Omega)} + \||\nabla\zeta_n||V_n|\|_{L^4(\Omega)}) \\ &\leq C|H_n|\|\overline{V}_n\|_{W^{2,4/3}(\Omega)} + C|H_n|\|V_n\|_{L^4(\Omega)}.\end{aligned} \tag{5.20}$$

(5.9), (5.14), (5.17)–(5.20) and Sobolev embedding imply

$$\|\overline{V}_n\|_{W^{2.4/3}(\Omega)} = o(1). \tag{5.21}$$

Then by the trace theorem $W^{1,4/3}(\Omega) \hookrightarrow L^2(\partial\Omega)$, we have

$$\int_{\partial\Omega}\left|\frac{\partial V_n}{\partial n}\right|^2 d\sigma = \int_{\partial\Omega}\left|\frac{\partial\overline{V}_n}{\partial n}\right|^2 d\sigma \leq C\|\overline{V}_n\|^2_{W^{2,4/3}(\Omega)} = o(1).$$

Thus we complete the proof. □

Lemma 5.6. *Let $\{g_n\}$ be a bounded sequence in $L^1(\Omega)$. Assume that $g_n \rightharpoonup 0$ in the sense of measures in Ω. Let φ_n be a solution of the problem*

$$\begin{cases} \Delta\varphi_n = g_n & \text{in } \Omega \\ \varphi_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Then for any $1 \le p < 2$, we have $\|\nabla\varphi_n\|_{L^p(\Omega)} = o(1)$ as $n \to \infty$.

Proof. Let $1 < p < 2$. It is well-known that $\{\varphi_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Let p' be such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then $p' > 2$. Let $\varPhi_n \in W^{-1,p'}(\Omega)$ be such that $\|\varPhi_n\|_{W^{-1,p'}(\Omega)} = 1$ and $\langle \varPhi_n, \varphi_n\rangle \ge \frac{1}{2}\|\varphi_n\|_{W^{1,p}(\Omega)}$, where $\langle\cdot,\cdot\rangle$ is the dual pairing defined on $W^{-1,p'}(\Omega) \times W^{1,p}(\Omega)$. Let $\phi_n \in W_0^{1,p'}(\Omega)$ be the solution of the problem $\Delta\phi_n = \varPhi_n$ in Ω. By elliptic regularity, since $\{\varPhi_n\}$ is bounded in $W^{-1,p'}(\Omega)$, $\{\phi_n\}$ is bounded in $W_0^{1,p'}(\Omega)$. Since $p' > 2$, by the Sobolev embedding $W_0^{1,p'}(\Omega) \hookrightarrow C^0(\overline{\Omega})$, there exist a subsequence of $\{\phi_n\}$ (still denoted by $\{\phi_n\}$) and $\phi_\infty \in C^0(\overline{\Omega})$ such that $\|\phi_n - \phi_\infty\|_{C^0(\overline{\Omega})} = o(1)$. Then we have

$$\begin{aligned} \frac{1}{2}\|\varphi_n\|_{W^{1,p}(\Omega)} &\le \langle \varPhi_n, \varphi_n\rangle \le \langle \Delta\phi_n, \varphi_n\rangle = \langle \phi_n, \Delta\varphi_n\rangle = \langle \phi_n, g_n\rangle \\ &\le \langle \phi_n - \phi_\infty, g_n\rangle + \langle \phi_\infty, g_n\rangle \\ &\le \|\phi_n - \phi_\infty\|_{L^\infty(\Omega)}\|g_n\|_{L^1(\Omega)} + \langle \phi_\infty, g_n\rangle \to 0. \end{aligned} \tag{5.22}$$

So far, (5.22) holds only for a subsequence of $\{\varphi_n\}$. However, it also holds for the full sequence because of the uniqueness of the possible limit. Thus the proof is completed. □

Remark. After completion of this work, the author found a simplified proof of Theorem 3.15 relying only on energy comparison arguments. This is presented in [13]. The method of this paper and [13] is applied to the asymptotic analysis of H-systems; see [14] and [15].

Acknowledgments. I would like to thank the referee for pointing out several mistakes in the first version of this paper.

REFERENCES

[1] F. Alouges and A. Soyeur, *On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness*, Nonlinear Anal. T. M. A. **18** (1992), 1071–1084.

[2] F. Bethuel, H. Brezis and F. Hélein, *Ginzburg-Landau vortices*, Birkhäuser, Boston–Basel–Berlin, 1994.

[3] F. Bethuel and J. M. Ghidaglia, *Improved regularity of solutions to elliptic equations involving Jacobians and applications*, J. Math. Pures Appl. **72** (1993), 441–474.

[4] H. Brezis and J. M. Coron, *Multiple solutions of H-systems and Rellich's conjecture*, Commun. Pure and Appl. Math. **37** (1984), 149–187.

[5] H. Brezis and J. M. Coron, *Large solutions for harmonic maps in two dimensions*, Commun. Math. Phys. **92** (1983), 203–215.

[6] H. Brezis, J. M. Coron and E. Lieb, *Harmonic maps with defects*, Commun. Math. Phys. **107** (1986), 649–705.

[7] S. Chanillo and Y. Y. Li, *Continuity of solutions of uniformly elliptic equations in* $\mathbb{R}^2$, Manuscripta Math. **77** (1992), 415–433.

[8] D. Gilberg and Trudinger, *Elliptic Partial Differential Equations of Second Order, Grundlehren Math. Wiss.*, vol. 224, Springer, Berlin and New York, 1983.

[9] B. Guo and M. C. Hong, *The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps*, Calc. Var. **1** (1993), 311–334.

[10] F. Hélein, *Régularité des applications faiblement harmoniques entre une surface et une ariété riemannienne*, C. R. Acad. Sci. Paris **70** (1991), 591–596.

[11] M. C. Hong, *The Landau-Lifshitz equation with the external field—a new extension for harmonic maps with values in* $\mathbb{S}^2$, Math. Z. **220** (1995), 171–188.

[12] M. C. Hong and L. Lemaire, *Multiple solutions of the static Landau-Lifshitz equation from* $\mathbb{B}^2$ *to* $\mathbb{S}^2$, Math. Z. **220** (1995), 295–306.

[13] T. Isobe, *On the construction of solutions of the Landau-Lifshitz equation*, to appear in Differ. and Integral Eq.

[14] ———, *On the asymptotic analysis of solutions of the H-systems I&II*, to appear in Adv. Differ. Eq.

[15] ———, *Classification of blow-up points and multiplicity of solutions of the H-systems*, preprint.

[16] J. Jost, *The Dirichlet problem for harmonic maps from a surface with nonconstant boundary values*, J. Differ. Geom. **19** (1984), 393–401.

[17] L. Lemaire, *Applications harmoniques de surfaces riemanniennes*, J. Differ. Geom. **13** (1978), 51–78.

[18] F. H. Lin, *Solutions of Ginzburg-Landau equations and critical points of the renormalized energy*, Ann. Inst. Henri Poincaré, Analyse non linéaire **12** (1995), 599–622.

[19] P. L. Lions, *The concentration compactness principle in the calculus of variations. The limit case. Part I*, Rev. Mat. Iberoamericano **1.1** (1985), 145–201.

[20] ———, *The concentration compactness principle in the calculus of variations. The limit case. Part II*, Rev. Mat. Iberoamericano **1.2** (1985), 45–121.

[21] W. M. Ni and I. Takagi, *On the shape of least energy solutions to a semilinear Neumann problem*, Commun. Pure and Appl. Math. **45** (1991), 819–851.

[22] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent*, J. Funct. Anal. **89** (1990), 1–52.

[23] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math. **113** (1981), 1–24.

[24] Y. Sasahara, *An asymptotic analysis for large solutions of H-systems*, Adv. Math. Sci. and Appl. **5** (1995), 219–237.

[25] R. Schoen and K. Uhlenbeck, *A regularity theory for harmonic maps*, J. Differ. Geom. **18** (1983), 253–268.

[26] Y. Shen and S. Yan, *Constant boundary value problem for a quasilinear elliptic system*, Manuscripta Math. **79** (1993), 99–112.

[27] Y. Shen and S. Yan, *Existence and nonexistence of nontrivial axially symmetric solution for quasilinear elliptic systems with inhomogeneous term*, Commun. in P. D. E. **18** (1993), 1477–1491.

[28] R. T. Smith, *The second variation formula for harmonic mappings*, Proc. A. M. S. **47** (1975), 229–236.

[29] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z. **1984**, 511–517.

[30] ———, *Large H-surfaces via the mountain pass lemma*, Math. Ann. **270** (1985), 441–459.

[31] ———, *On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2-dimensions*, Differential and Integral Eq. **7** (1994), 1613–1624.