Advances in Differential Equations Volume 5 (7–9) July–September 2000, pp. 929–975

CODIMENSION-ONE RIEMANN SOLUTIONS: MISSING RAREFACTIONS IN TRANSITIONAL WAVE GROUPS*

Stephen Schecter
Mathematics Department, North Carolina State University, Raleigh, NC 27695

(Submitted by: Reza Aftabizadeh)

Abstract. This paper is the fourth in a series that undertakes a systematic investigation of Riemann solutions of systems of two conservation laws in one spatial dimension. In this paper, three degeneracies that can occur only in transitional wave groups, together with the degeneracies that pair with them, are studied in detail. Conditions for a codimension-one degeneracy are identified in each case, as are conditions for folding of the Riemann solution surface.

1. Introduction. We consider systems of two conservation laws in one space dimension, partial differential equations of the form

$$U_t + F(U)_x = 0 \tag{1.1}$$

with $t > 0$, $x \in \mathbb{R}$, $U(x,t) \in \mathbb{R}^2$, and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a smooth map. The most basic initial-value problem for Equation (1.1) is the *Riemann problem*, in which the initial data are piecewise constant with a single jump at $x = 0$:

$$U(x,0) = \begin{cases} U_L & \text{for } x < 0, \\ U_R & \text{for } x > 0. \end{cases} \tag{1.2}$$

This paper is the fourth in a series of papers in which we study the structure of solutions of Riemann problems; the previous ones are references [6], [7], and [8].

We seek piecewise-continuous, weak solutions of Riemann problems in the scale-invariant form $U(x,t) = \widehat{U}(x/t)$ consisting of a finite number of

*This work was supported in part by the National Science Foundation under Grants DMS-9205535 and DMS-9501255.

Accepted for publication June 1999.

AMS Subject Classifications: 35L65, 35L67, 34C23.

constant parts, continuously changing parts (*rarefaction waves*), and jump discontinuities (*shock waves*). Shock waves occur when

$$\lim_{\xi \to s-} \widehat{U}(\xi) = U_- \neq U_+ = \lim_{\xi \to s+} \widehat{U}(\xi). \tag{1.3}$$

They are required to satisfy the following *viscous profile admissibility criterion*: a shock wave is admissible provided that the ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{1.4}$$

has a heteroclinic solution, or a finite sequence of such solutions, leading from the equilibrium U_- to a second equilibrium U_+.

There are various *types* of rarefaction and shock waves (e.g., 1-family rarefaction waves and classical 1-family shock waves); the *type* of a Riemann solution is the sequence of types of its waves. Our approach to understanding Riemann solutions is to investigate the local structure of the set of Riemann solutions: we consider a particular solution $(\widehat{U}^*, U_L^*, U_R^*, F^*)$ and construct nearby ones. More precisely, we define an open neighborhood $\mathcal{X}$ of $\widehat{U}^*$ in a Banach space of scale-invariant functions $\widehat{U}$, open neighborhoods $\mathcal{U}_L$ and $\mathcal{U}_R$ of U_L^* and U_R^* in $\mathbb{R}^2$, and an open neighborhood $\mathcal{B}$ of F^* in a Banach space of smooth flux functions F. Then our goal is to construct a set $\mathcal{R}$ of Riemann solutions $(\widehat{U}, U_L, U_R, F) \in \mathcal{X} \times \mathcal{U}_L \times \mathcal{U}_R \times \mathcal{B}$ near $(\widehat{U}^*, U_L^*, U_R^*, F^*)$. Near a *structurally stable Riemann solution*, $\widehat{U}$ changes continuously, and its type remains unchanged, when the triple (U_L, U_R, F) varies. Moreover, the left and right states and the speed of each wave in $\widehat{U}$ depend smoothly on (U_L, U_R, F).

In reference [6], we identified a set of sufficient conditions for structural stability of strictly hyperbolic Riemann solutions. Briefly, these conditions have the following character.

(H0) There is a restriction on the sequence of wave types in the solution.

(H1) Each wave satisfies certain nondegeneracy conditions.

(H2) The "wave group interaction condition" is satisfied. In the simplest case, the forward wave curve and the backward wave curve are transverse.

(H3) If a shock wave represented by a connection *to* a saddle is followed by another represented by a connection *from* a saddle, the shock speeds differ.

In reference [7] we began an investigation of the Riemann solutions that occur when one passes to the boundary of the set of structurally stable, strictly hyperbolic Riemann solutions by violating a single condition on this list, but the Riemann solution remains strictly hyperbolic. Under appropriate nondegeneracy conditions, these Riemann solutions constitute a graph over a codimension-one submanifold $\mathcal{S}$ of $\mathcal{U}_L \times \mathcal{U}_R \times \mathcal{B}$. Usually, Riemann data (U_L, U_R, F) on opposite sides of $\mathcal{S}$ give rise to different types of structurally stable Riemann solutions; however, other possibilities can occur. In reference [8] we treated in detail certain codimension-one degeneracies that occur in classical 1-wave and 2-wave groups (and can occur elsewhere).

Structurally stable Riemann solutions can contain transitional wave groups that separate classical slow and fast wave groups. A transitional wave group is either a single shock wave represented by a saddle-to-saddle connection of (1.4), or a transitional analogue of classical composite wave groups. A slow transitional wave group begins with a shock wave represented by a saddle to repeller-saddle connection of (1.4), followed by an adjacent 1-family rarefaction. A fast transitional wave group, which is dual to a slow one, ends with a shock wave represented by a saddle-attractor to saddle connection of (1.4), preceded by an adjacent 2-family rarefaction.

In this paper we study in detail the codimension-one Riemann solutions that occur when the rarefaction that follows the first shock wave in a slow transitional wave group shrinks to zero strength. At the same time we treat the degeneracies that pair with these to continue the Riemann solution. There are three cases, depending on the nature of the wave that follows the rarefaction of zero strength. In the notation of references [6]–[8], these cases are

- two-wave slow transitional wave group: $S \cdot RS\ R_1$;
- three-wave slow transitional wave group: $S \cdot RS\ R_1\ RS \cdot S$;
- long slow transitional wave group: $S \cdot RS\ R_1\ RS \cdot RS\ \dots$.

In the two-wave case, the transitional wave group contains only the saddle to repeller-saddle shock wave and an adjacent slow rarefaction that shrinks to zero strength. In the three-wave case, the rarefaction that shrinks to zero strength is followed by a shock wave represented by a repeller-saddle to saddle connection. In the long case, there are more than three waves in the transititional wave group: the rarefaction that shrinks to zero strength is followed by a shock wave represented by a repeller-saddle to repeller-saddle connection, which is in turn followed by another slow rarefaction, and possibly additional waves.

The structurally stable Riemann solutions into which these metamorphose when the codimension-one boundary is crossed are as follows (reference [7]). In the two-wave and three-wave cases, the transitional wave group is replaced by a single saddle-to-saddle transitional shock wave. In the two-wave case it becomes a shock wave of saddle to repeller-saddle type at the boundary; in the three-wave case, the saddle-to-saddle connection is broken at the boundary by an equilibrium of repeller-saddle type that appears by saddle-node bifurcation. In the long case, the transitional wave group is replaced by one with two fewer waves; at the boundary, the saddle to repeller-saddle connection that begins it is broken by an equilibrium of repeller-saddle type that appears by saddle-node bifurcation.

In the three-wave and long cases it can happen that both types of Riemann solutions are defined for Riemann data on the same side of $\mathcal{S}$, so that we do not have local existence and uniqueness of Riemann solutions.

We do not treat the completely analogous phenomena for fast transitional wave groups.

As far as I know, slow transitional wave groups do not occur in the literature, but fast transitional wave groups have been found in mathematical models for three-phase flow in a porous medium. Two-wave fast transitional wave groups in such models occur in references [3], [9], and [5]. In numerical work, Bell, Trangenstein, and Shubin found what may be a three-wave fast transitional wave group separating a 1-shock and a 2-shock (reference [1], 1014–1015). The authors say that the last shock wave in this group appears to be represented by a connection from an equilibrium with two zero eigenvalues to a saddle, "although numerical dissipation renders a precise assessment difficult." A Riemann solution containing such a shock would not be structurally stable, and hence is unlikely to be observed in numerical work. In reference [4], Keyfitz gives conditions for the existence of a Riemann solution of the type described by Bell, *et al.*

The remainder of the paper is organized as follows. In Sections 2 and 3 we review terminology and results about structurally stable Riemann solutions and codimension-one Riemann solutions from reference [6] and reference [7]. In Section 4 we explain the general approach we will take to analyzing the degeneracies, which mostly comes from reference [8], and we summarize the results. In Sections 5–7 we treat in detail the three missing rarefaction cases that are the subject of this paper.

2. Background on structurally stable Riemann problem solutions.

We consider the system (1.1) with $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, $U(x,t) \in \mathbb{R}^2$, and $F : \mathbb{R}^2 \to \mathbb{R}^2$ a C^2 map. Let

$$\mathcal{U}_F = \left\{ U \in \mathbb{R}^2 \, : \, DF(U) \text{ has distinct real eigenvalues} \right\} \tag{2.1}$$

be the *strictly hyperbolic region* in state-space. We shall call a Riemann solution $\widehat{U}$ *strictly hyperbolic* if $\widehat{U}(\xi) \in \mathcal{U}_F$ for all $\xi \in \mathbb{R}$. In this paper, all Riemann solutions are assumed to be strictly hyperbolic. For $U \in \mathcal{U}_F$, let $\lambda_1(U) < \lambda_2(U)$ denote the eigenvalues of $DF(U)$, and let $\ell_i(U)$ and $r_i(U)$, $i = 1, 2$, denote corresponding left and right eigenvectors with $\ell_i(U) r_j(U) = \delta_{ij}$.

A *rarefaction wave* of *type* R_i is a differentiable map $\widehat{U} : [a, b] \to \mathcal{U}_F$, where $a < b$, such that $\widehat{U}'(\xi)$ is a multiple of $r_i(\widehat{U}(\xi))$ and $\xi = \lambda_i(\widehat{U}(\xi))$ for each $\xi \in [a, b]$. The states $U = \widehat{U}(\xi)$ with $\xi \in [a, b]$ comprise the *rarefaction curve* $\overline{\Gamma}$. The definition of rarefaction wave implies that if $U \in \overline{\Gamma}$, then

$$D\lambda_i(U) r_i(U) = \ell_i(U) D^2 F(U)(r_i(U),\, r_i(U)) \neq 0. \tag{2.2}$$

Condition (2.2) is *genuine nonlinearity* of the ith characteristic line field at U. Assuming (2.2), we can choose $r_i(U)$ such that

$$D\lambda_i(U) r_i(U) = 1. \tag{2.3}$$

In this paper we shall assume that genuine nonlinearity holds along all rarefactions, so we shall assume that along any rarefaction of type R_i, (2.3) holds. The *speed* s of a rarefaction wave of type R_1 is $s = \lambda_1(U_+)$; for a rarefaction wave of type R_2, $s = \lambda_2(U_-)$.

A *shock wave* consists of a *left state* $U_- \in \mathcal{U}_F$, a *right state* $U_+ \in \mathcal{U}_F$, a *speed* s, and a *connecting orbit* Γ, which corresponds to a solution of the ordinary differential equation (1.4) from U_- to U_+. For any equilibrium $U \in \mathcal{U}_F$ of (1.4), the eigenvalues of the linearization at U are $\lambda_i(U) - s$, $i = 1, 2$. We shall use the terminology defined in Table 1 for such an equilibrium. The *type* of a shock wave is determined by the equilibrium types of its left and right states. (For example, w is of type $R \cdot S$ if its connecting orbit joins a repeller to a saddle.)

name	*symbol*	*eigenvalues*
Repeller	R	$+ \quad +$
Repeller-Saddle	RS	$0 \quad +$
Saddle	S	$- \quad +$
Saddle-Attractor	SA	$- \quad 0$
Attractor	A	$- \quad -$

Table 1: Types of equilibria.

An *elementary wave* w is either a rarefaction wave or a shock wave. We write $w : U_- \xrightarrow{s} U_+$ if w has left state U_-, right state U_+, and speed s. Note that an elementary wave also has a *type* T, as defined above. An open subset $\mathcal{N}$ of $\mathbb{R}^2$ is a *neighborhood of* w if $\overline{\Gamma} \subset \mathcal{N}$.

An *allowed sequence of elementary waves* or a *Riemann solution* consists of a sequence of waves $(w_1, w_2, \dots, w_n)$ with increasing wave speeds. We write

$$(w_1, w_2, \dots, w_n) : U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n. \tag{2.4}$$

The *type* of $(w_1, w_2, \dots, w_n)$ is $(T_1, T_2, \dots, T_n)$ if w_i has type T_i. Let

$$(w_1^*, w_2^*, \dots, w_n^*) : U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.5}$$

be a Riemann solution for $U_t + F^*(U)_x = 0$. The Riemann solution (2.5) is *structurally stable* if there are neighborhoods $\mathcal{U}_i$ of U_i^*, $\mathcal{I}_i$ of s_i^*, and $\mathcal{F}$ of F^* in an appropriate Banach space (see [6]), a compact set $\mathcal{K}$ in $\mathbb{R}^2$, and a C^1 map $G : \mathcal{U}_0 \times \mathcal{I}_1 \times \mathcal{U}_1 \times \mathcal{I}_2 \times \ldots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{3n-2}$ with $G(U_0^*, s_1^*, U_1^*, s_2^*, \dots, s_n^*, U_n^*, F^*) = 0$ such that

(P1) $G(U_0, s_1, U_1, s_2, \dots, s_n, U_n, F) = 0$ implies that there exists a Riemann solution $(w_1, w_2, \dots, w_n) : U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n$ for $U_t + F(U)_x = 0$ with successive waves of the same types as those of the wave sequence (2.5) and with each w_i contained in $\operatorname{Int} K$;

(P2) $DG(U_0^*, s_1^*, U_1^*, s_2^*, \dots, s_n^*, U_n^*, F^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dots, \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 = \dot{U}_n, \dot{F} = 0\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

Condition (P2) implies, by the implicit function theorem, that $G^{-1}(0)$ is a graph over $\mathcal{U}_0 \times \mathcal{U}_n \times \mathcal{F}$; $(s_1, U_1, \dots, U_{n-1}, s_n)$ is determined by (U_0, U_n, F). We also require that

(P3) $(w_1, w_2, \dots, w_n)$ can be chosen so each $\overline{\Gamma}_i$ depends continuously on (U_0, U_n, F).

Associated with each type of elementary wave is a *local defining map*, which we use to construct maps G that exhibit structural stability. Let $w^* : U_-^* \xrightarrow{s^*} U_+^*$ be an elementary wave of type T for $U_t + F^*(U)_x = 0$. The local defining map G_T has as its domain a set of the form $\mathcal{U}_- \times \mathcal{I} \times \mathcal{U}_+ \times \mathcal{F}$ (with $\mathcal{U}_\pm$ being neighborhoods of $U_\pm^*$, $\mathcal{I}$ a neighborhood of s^*, and $\mathcal{F}$ a neighborhood of F^*). The range is some $\mathbb{R}^e$; the number e depends only on the wave type T. The local defining map is such that $G_T(U_-^*, s^*, U_+^*, F^*) = 0$. Moreover, if certain *wave nondegeneracy conditions* are satisfied at (U_-^*, s^*, U_+^*, F^*), then there is a neighborhood $\mathcal{N}$ of w^* such that

(D1) $G_T(U_-, s, U_+, F) = 0$ if and only if there exists an elementary wave $w : U_- \xrightarrow{s} U_+$ of type T for $U_t + F(U)_x = 0$ contained in $\mathcal{N}$;

(D2) $DG_T(U_-^*, s^*, U_+^*, F^*)$, restricted to the space $\{(\dot{U}_-, \dot{s}, \dot{U}_+, \dot{F}) : \dot{F} = 0\}$, is surjective.

Condition (D2) implies, by the implicit function theorem, that $G_T^{-1}(0)$ is a manifold of codimension e. In fact,

(D3) w can be chosen so that $\overline{\Gamma}$ varies continuously on $G_T^{-1}(0)$.

We now discuss local defining maps and nondegeneracy conditions for the types of elementary waves that occur in this paper.

First we consider rarefactions. Let $\mathcal{U}_1 = \{U \in \mathcal{U} : D\lambda_1(U) r_1(U) \neq 0\}$. In $\mathcal{U}_1$ we can assume that (2.3) holds with $i = 1$. For each $U_- \in \mathcal{U}_1$, define ψ to be the solution of

$$\frac{\partial \psi}{\partial s}(U_-, s) = r_1(\psi(U_-, s)), \quad \psi(U_-, \lambda_1(U_-)) = U_-.$$

By (2.3), if $\psi(U_-, s) = U$, then $s = \lambda_1(U)$. Thus there is a rarefaction wave of type R_1 for $U_t + F(U)_x = 0$ from U_- to U_+ with speed s if and only if

$$U_+ - \psi(U_-, s) = 0 \tag{2.6}$$

$$s = \lambda_1(U_+) > \lambda_1(U_-). \tag{2.7}$$

Equations (2.6) are defining equations for rarefaction waves of types R_1. The nondegeneracy conditions for rarefaction waves of type R_1, which are implicit in our definition of rarefaction, are the speed inequality (2.7), and the genuine nonlinearity condition (2.2).

Lemma 2.1. $D\psi(U_-, \lambda_1(U_-))(ar_1(U_-) + br_2(U_-), \dot{s})$
$= (\dot{s} - bD\lambda_1(U_-)r_2(U_-))r_1(U_-) + br_2(U_-)$.

Proof. See reference [8]. □

Next we consider shock waves. If there is to be a shock wave solution of $U_t + F(U)_x = 0$ from U_- to U_+ with speed s, we must have that

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0; \tag{E0}$$

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \quad \text{has an orbit from } U_- \text{ to } U_+. \tag{C0}$$

The two-component equation (E0) is a defining equation. In the context of structurally stable Riemann solutions, condition (C0) is an open condition, and therefore is regarded as a nondegeneracy condition, for all but transitional shock waves (those of types $S \cdot S$, $S \cdot RS$, $SA \cdot S$, or $SA \cdot RS$). For these waves, a *separation function* must be defined.

Let us consider $S \cdot RS$ waves. Suppose (1.1) has an $S \cdot RS$ shock wave $w^* : U_-^* \xrightarrow{s^*} U_+^*$. Then for (U_-, s) near (U_-^*, s^*), the differential equation (1.4) has a saddle at U_- with unstable manifold $W_-(U_-, s)$. For $(U_-, s) = (U_-^*, s^*)$, (1.4) has a saddle-node at U_+^*; we denote its center manifold by $W_+(U_-^*, s^*)$. This center manifold perturbs to a family of invariant manifolds $W_+(U_-, s)$. Let $U^*(\tau)$ be the connection of (1.1), with $(U_-, s) = (U_-^*, s^*)$, from U_-^* to U_+^*. Let Σ be a line segment through $U^*(0)$ transverse to $\dot{U}^*(0)$, in the direction V. Then $W_\pm(U_-, s)$ meet Σ in points $\bar{U}_\pm(U_-, s)$, and we define $S(U_-, s)$ by $\bar{U}_-(U_-, s) - \bar{U}_+(U_-, s) = S(U_-, s)V$. See Figure 1.

The family of unstable manifolds $W_-(U_-, s)$ is as smooth as F. The center manifolds $W_+(U_-, s)$ are not uniquely defined. However, if F is C^k, $k < \infty$, then $W_+(U_-, s)$ can be chosen to depend in a C^k manner on (U_-, s) in a neighborhood of (U_-^*, s^*). More precisely, while $\bar{U}_+(U_-, s)$ is uniquely defined for those (U_-, s) for which the differential equation (1.4) has equilibria near U_+^*, $\bar{U}_+(U_-, s)$ is not uniquely defined for other (U_-, s). However, the derivatives of S at (U_-^*, s^*) through order k are independent of the choice.

The construction of a separation function for $S \cdot S$ shock waves is simpler and is discussed in reference [6]. It is as smooth as F.

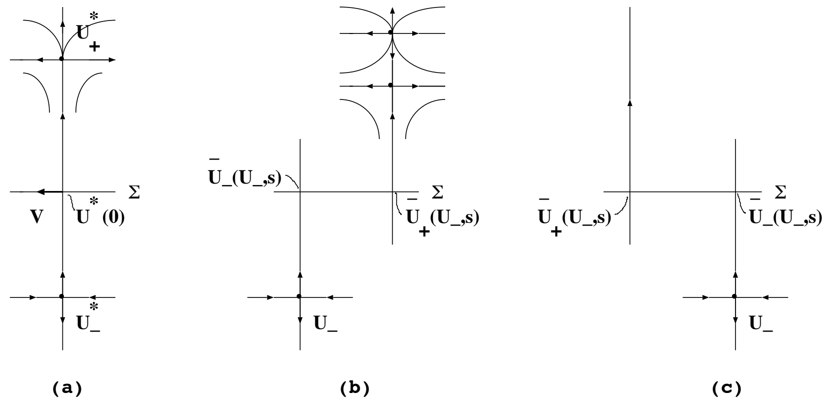

Figure 1: Geometry of the separation function. (a) Phase portrait of $\dot{U} = F(U) - F(U_-^*) - s^*(U - U_-^*)$. (b) Phase portrait of a nearby vector field $\dot{U} = F(U) - F(U_-) - s(U - U_-)$ for which the equilibrium at U_+^* has split into a saddle and a repeller, and for which $S(U_-, s)$ is positive. (c) Phase portrait of a nearby vector field $\dot{U} = F(U) - F(U_-) - s(U - U_-)$ for which the equilibrium at U_+^* has disappeared, and for which $S(U_-, s)$ is negative.

In Table 2 we list additional defining equations and nondegeneracy conditions for the types of shock waves that occur in this paper; the labeling of the conditions is from reference [6]. The wave nondegeneracy conditions are open conditions. Conditions (C1)–(C2) are that the connection Γ is not *distinguished*; for $RS \cdot S$ and $RS \cdot RS$ shock waves, this means that the connection Γ should not lie in the unstable manifold of U_- (i.e., the unique invariant curve tangent to an eigenvector with positive eigenvalue). The transversality condition (T2) is that there is a vector V in $\mathbb{R}^2$ such that the vectors

$$\begin{pmatrix} \ell_1(U_+)(DF(U_-) - sI) \\ D_{U_-}S(U_-, s) \end{pmatrix} V \quad \text{and} \quad \begin{pmatrix} \ell_1(U_+)(U_+ - U_-) \\ \frac{\partial S}{\partial s}(U_-, s) \end{pmatrix} \tag{2.8}$$

are linearly independent.

The geometric meaning of (2.8) is as follows. The system (1.1) has an $S \cdot RS$ shock wave $U_- \xrightarrow{s} U_+$ near a given $S \cdot RS$ shock wave $U_-^* \xrightarrow{s^*} U_+^*$ provided the following system of local defining equations is satisfied:

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0, \quad \lambda_1(U_+) - s = 0, \quad S(U_-, s) = 0.$$

The left-hand side of this system is a map from 5-dimensional $U_- s U_+$-space to $\mathbb{R}^4$. If the nondegeneracy conditions (G13) and (T2) hold at (U_-^*, s^*, U_+^*), then this map has surjective derivative there, so the set of solutions is a curve in U_-sU_+-space; moreover, this curve projects regularly to curves $\mathcal{E}_1$ in U_--space and $\mathcal{E}_2$ in U_+-space. These are the curves of possible left and right states for $S \cdot RS$ shock waves.

type of shock wave	*additional defining equations*	*nondegeneracy conditions*
$RS \cdot S$	$\lambda_1(U_-) - s = 0$ (E2)	$D\lambda_1(U_-)r_1(U_-) \neq 0$ (G2) not distinguished connection (C1)
$RS \cdot RS$	$\lambda_1(U_-) - s = 0$ (E3) $\lambda_1(U_+) - s = 0$ (E4)	$D\lambda_1(U_-)r_1(U_-) \neq 0$ (G3) $D\lambda_1(U_+)r_1(U_+) \neq 0$ (G4) $\ell_1(U_+)(U_+ - U_-) \neq 0$ (B2) not distinguished connection (C2)
$S \cdot S$	$S(U_-, s) = 0$ (S1)	$DS(U_-, s) \neq 0$ (T1)
$S \cdot RS$	$\lambda_1(U_+) - s = 0$ (E13) $S(U_-, s) = 0$ (S2)	$D\lambda_1(U_+)r_1(U_+) \neq 0$ (G13) transversality (T2)

Table 2: Additional defining equations and nondegeneracy conditions for various shock waves.

For the Riemann solution (2.5), let w_i^* have type T_i and local defining map G_{T_i}, with range $\mathbb{R}^{e_i}$. For appropriate neighborhoods $\mathcal{U}_i$ of U_i^*, $\mathcal{I}_i$ of s_i^*, $\mathcal{F}$ of F^*, and $\mathcal{N}_i$ of w_i^*, we can define a map $G : \mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{e_1 + \cdots + e_n}$ by $G = (G_1, \ldots, G_n)$, where

$$G_i(U_0, s_1, \ldots, s_n, U_n, F) = G_{T_i}(U_{i-1}, s_i, U_i, F).$$

The map G is called the *local defining map* of the wave sequence (2.5). Assuming the wave nondegeneracy conditions, if $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$, then for each $i = 1, \ldots, n$, there is an elementary wave $w_i : U_{i-1} \xrightarrow{s_i} U_i$ of type T_i for $U_t + F(U)_x = 0$ contained in $\mathcal{N}_i$, for which $\overline{\Gamma}_i$ is continuous.

We define the *Riemann number* of an elementary wave type T to be $\rho(T) = 3 - e(T)$, where $e(T)$ is the number of defining equations for a wave of type T. For convenience, if w is an elementary wave of type T, we shall write $\rho(w)$ instead of $\rho(T)$.

A *1-wave group* is either a single $R \cdot S$ shock wave or an allowed sequence of elementary waves of the form

$$(R \cdot RS)(R_1 \; RS \cdot RS) \cdots (R_1 \; RS \cdot RS) \, R_1 \, (RS \cdot S), \tag{2.9}$$

where the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

A *transitional wave group* is either a single $S \cdot S$ shock wave or an allowed sequence of elementary waves of one of the forms

$$S \cdot RS(R_1\; RS \cdot RS) \cdots (R_1\; RS \cdot RS)\, R_1\, (RS \cdot S), \tag{2.10}$$

$$(S \cdot SA)\, R_2\, (SA \cdot SA\, R_2) \cdots (SA \cdot SA\, R_2)\, SA \cdot S, \tag{2.11}$$

the terms in parentheses being optional. In cases (2.10) and (2.11), the group is termed *composite*.

A *2-wave group* is either a single $S \cdot A$ shock wave or an allowed sequence of elementary waves of the form

$$(S \cdot SA)\, R_2\, (SA \cdot SA\, R_2) \cdots (SA \cdot SA\, R_2)\, (SA \cdot A), \tag{2.12}$$

where again the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite*.

In reference [6] the following are proved.

Theorem 2.2 (Wave Structure). *Let* (2.5) *be an allowed sequence of elementary waves. Then*

1. $\sum_{i=1}^{n} \rho(w_i^*) \leq 2$;
2. $\sum_{i=1}^{n} \rho(w_i^*) = 2$ *if and only if the following conditions are satisfied.*
   (1) *Suppose that the wave sequence* (2.5) *includes no $SA \cdot RS$ shock waves. Then it consists of one 1-wave group, followed by an arbitrary number of transitional wave groups (in any order), followed by one 2-wave group.*
   (2) *Suppose that the wave sequence* (2.5) *includes $m \geq 1$ shock waves of type $SA \cdot RS$. Then these waves separate $m+1$ wave sequences $g_0, \ldots, g_m$. Each g_i is exactly as in* (1) *with the restrictions that*
   (*a*) *if $i < m$, the last wave in the group has type R_2;*
   (*b*) *if $i > 0$, the first wave in the group has type R_1.*

Theorem 2.3 (Structural Stability). *Suppose that the allowed sequence of elementary waves* (2.5) *has $\sum_{i=1}^{n} \rho(w_i^*) = 2$. Assume that*

(H1) *each wave satisfies the appropriate wave nondegeneracy conditions;*

(H2) *the wave group interaction conditions, as stated precisely in reference [6], are satisfied;*

(H3) *if* w_i^* *is a* $* \cdot S$ *shock wave and* w_{i+1}^* *is an* $S \cdot *$ *shock wave, then* $s_i^* < s_{i+1}^*$.

Then the wave sequence (2.5) *is structurally stable.*

In the remainder of the paper, by a structurally stable Riemann solution we shall mean a sequence of elementary waves that satisfies the hypotheses of Theorem 2.3.

3. Codimension-one Riemann solutions. In order to consider conveniently codimension-one Riemann solutions, the definitions of rarefaction and shock waves in Section 2 must be generalized somewhat.

For the purposes of this paper, a *generalized rarefaction wave* of type R_i has the same definition as a rarefaction of type R_i, except that we allow $a = b$ in the interval of definition $[a, b]$.

A *generalized shock wave* consists of a *left state* U_-, a *right state* U_+ (possibly equal to U_-), a *speed* s, and a sequence of connecting orbits $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2, \ldots, \tilde{\Gamma}_k$ of (1.4) from $U_- = \tilde{U}_0$ to $\tilde{U}_1$, $\tilde{U}_1$ to $\tilde{U}_2, \ldots, \tilde{U}_{k-1}$ to $\tilde{U}_k = U_+$. Note that $\tilde{U}_0, \tilde{U}_1, \ldots, \tilde{U}_k$ must be equilibria of (1.4). We allow for the possibility that $\tilde{U}_{j-1} = \tilde{U}_j$, in which case we assume that $\tilde{\Gamma}_j$ is the trivial orbit $\{\tilde{U}_j\}$.

Associated with each generalized rarefaction or generalized shock wave is a speed s, defined as before, and a curve $\overline{\Gamma}$: the rarefaction curve or the closure of $\tilde{\Gamma}_1 \cup \ldots \cup \tilde{\Gamma}_k$.

A *generalized allowed wave sequence* is a sequence (2.5) of generalized rarefaction and shock waves with increasing wave speeds. If $U_0 = U_L$ and $U_n = U_R$, then associated with a generalized allowed wave sequence $(w_1, w_2, \ldots, w_n)$ is a solution $U(x,t) = \widehat{U}(x/t)$ of the Riemann problem (1.1)–(1.2). Therefore we shall also refer to a generalized allowed wave sequence as a *Riemann solution*.

A generalized allowed wave sequence (2.5) is a *codimension-one Riemann solution* provided that there is a sequence of wave types $(T_1^*, \ldots, T_n^*)$ with $\sum_{i=1}^n \rho(T_i^*) = 2$, neighborhoods $\mathcal{U}_i \subseteq \mathcal{U}$ of U_i^*, $\mathcal{I}_i \subseteq I$ of s_i^*, and $\mathcal{F}$ of F^*, a compact set $\mathcal{K}$ in $\mathbb{R}^2$, and a C^1 map

$$(G, H) : \mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{3n-2} \times \mathbb{R}, \tag{3.1}$$

with $G(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*) = 0$ and $H(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*) = 0$, such that the following conditions, (Q1)–(Q7), are satisfied.

(Q1) If $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$ and $H(U_0, s_1, \ldots, s_n, U_n, F) \geq 0$ then there is a generalized allowed wave sequence

$$(w_1, w_2, \ldots, w_n) : U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n$$

for $U_t + F(U)_x = 0$ with each w_i contained in $\operatorname{Int} K$;

(Q2) if $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$ and $H(U_0, s_1, \ldots, s_n, U_n, F) > 0$, then $(w_1, w_2, \ldots, w_n)$ is a structurally stable Riemann solution of type $(T_1^*, \ldots, T_n^*)$ and G exhibits its structural stability.

(Q3) If $G(U_0, s_1, \ldots, s_n, U_n, F) = 0$ and $H(U_0, s_1, \ldots, s_n, U_n, F) = 0$ then $(w_1, w_2, \ldots, w_n)$ is not a structurally stable Riemann solution.

(Q4) $D(G,H)(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*)$, restricted to some $(3n-1)$-dimensional space of vectors that contains $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \ldots, \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 = \dot{U}_n, \dot{F} = 0\}$, is an isomorphism.

Condition (Q4) implies, by the implicit function theorem, that $(G, H)^{-1}(0)$ is a graph over a codimension-one manifold $\mathcal{S}$ in $\mathcal{U}_0 \times \mathcal{U}_n \times \mathcal{F}$, and $\mathcal{M} := (G, H)^{-1}(\{0\} \times \mathbb{R}_+)$ is a manifold-with-boundary of codimension $3n-2$. We can define maps $\bar{\Gamma}_i : \mathcal{M} \to \mathcal{H}(\operatorname{Int} K)$. We require that

(Q5) $(w_1, w_2, \ldots, w_n)$ can be chosen so that each map $\bar{\Gamma}_i$ is continuous.

(G, H) is again called a *local defining map*.

The surface $\mathcal{S}$ is required to be regularly situated with respect to the foliation of $U_0 U_n F$-space into planes of constant (U_0, F) and planes of constant (U_n, F). More precisely, let

$$\begin{aligned} \Sigma_0 &= \{(\dot{U}_0, \dot{s}_1, \ldots, \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_n = 0 \text{ and } \dot{F} = 0\}, \\ \Sigma_n &= \{(\dot{U}_0, \dot{s}_1, \ldots, \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 \text{ and } \dot{F} = 0\}. \end{aligned}$$

Then we require that one of the following four conditions hold:

($Q6_1$) $D(G, H)(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*)$ restricted to Σ_0 and to Σ_n, respectively, are surjective.

($Q6_2$) $D(G, H)(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*)$ restricted to Σ_n is surjective, and there is a codimension-one manifold $\tilde{\mathcal{S}}$ through (U_0^*, F^*) in (U_0, F)-space such that $\mathcal{S} = \mathcal{U}_n \times \tilde{\mathcal{S}}$;

(Q6$_3$) $D(G,H)(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*)$ restricted to Σ_0 is surjective, and there is a codimension-one manifold $\tilde{\mathcal{S}}$ through (U_n^*, F^*) in (U_n, F)-space such that $\mathcal{S} = \mathcal{U}_0 \times \tilde{\mathcal{S}}$;

(Q6$_4$) there is a codimension-one manifold $\tilde{\mathcal{S}}$ through F^* in F-space such that $\mathcal{S} = \mathcal{U}_0 \times \mathcal{U}_n \times \tilde{\mathcal{S}}$.

When (Q6$_1$), (Q6$_2$) or (Q6$_3$), or (Q6$_4$) holds, then the codimension-one Riemann solution is termed an *intermediate boundary*, a U_L*-boundary* or dual, or an F*-boundary*, respectively.

Finally, we require one of the following conditions to hold:

(Q7$_1$) the linear map

$$DG(U_0^*, s_1^*, \ldots, s_n^*, U_n^*, F^*) \text{ restricted to } \Sigma_0 \cap \Sigma_n \tag{3.2}$$

is an isomorphism. (In this case, $\mathcal{M}$ is a smooth graph over a manifold-with-boundary in $\mathcal{U}_0 \times \mathcal{U}_n \times \mathcal{F}$ with boundary $\mathcal{S}$.)

(Q7$_2$) the linear map (3.2) is not surjective, but the projection of $G^{-1}(0)$ to $\mathcal{U}_0 \times \mathcal{U}_n \times \mathcal{F}$ has a fold along $(G,H)^{-1}(0,0)$.

This case does not arise in the present paper.

A *rarefaction of zero strength* is one whose domain has zero length. A *shock wave of zero strength* is one with $U_L = U_R$ (and hence $\Gamma = \{U_L\}$).

A generalized allowed wave sequence is *minimal* if

1. there are no rarefactions or shock waves of zero strength;
2. no two successive shock waves have the same speed.

Among the minimal generalized allowed wave sequences we include sequences of no waves; these are given by a single $U_0 \in \mathbb{R}^2$, and represent constant solutions of (1.1).

We *shorten* a generalized allowed wave sequence by dropping a rarefaction or shock wave of zero strength, or by amalgamating adjacent shock waves of positive strength with the same speed. Every generalized allowed wave sequence can be shortened to a unique minimal generalized allowed wave sequence. Two generalized allowed wave sequences are *equivalent* if their minimal shortenings are the same. Equivalent generalized allowed wave sequences represent the same solution $U(x,t) = \widehat{U}(x/t)$ of (1.1).

Let $(U_0^*, s_1^*, U_1^*, s_2^*, \dots, s_n^*, U_n^*, F^*)$ be a generalized allowed wave sequence that is a codimension-one Riemann solution of type $(T_1^*, \dots, T_n^*)$. Let $\mathcal{M}$ denote the associated manifold-with-boundary, $\mathcal{M}$ being a graph over the manifold $\mathcal{S}$. Suppose there is an equivalent generalized allowed wave sequence $(U_0^\sharp, s_1^\sharp, U_1^\sharp, s_2^\sharp, \dots, s_m^\sharp, U_m^\sharp, F^*)$ that is a codimension-one Riemann solution in $\partial\mathcal{N}$, $\mathcal{N} = (G^\sharp, H^\sharp)^{-1}(\{0\} \times \mathbb{R}_+)$, where $\operatorname{Int}\mathcal{N}$ consists of structurally stable Riemann solutions of some type $(T_1^\sharp, \dots, T_m^\sharp)$. Suppose in addition that $\partial\mathcal{N}$ is also a graph over $\mathcal{S}$, and the points in $\partial\mathcal{M}$ and $\partial\mathcal{N}$ above the same point in $\mathcal{S}$ are equivalent. Then the codimension-one Riemann solution (2.5) (or its equivalent generalized wave sequence) is said to lie in a *join*.

$\mathcal{M}$ and $\mathcal{N}$ are each graphs over the union of one side of $\mathcal{S}$ and $\mathcal{S}$ itself. If $\mathcal{M}$ and $\mathcal{N}$ are graphs over different sides of $\mathcal{S}$, we have a *regular join*; if $\mathcal{M}$ and $\mathcal{N}$ are graphs over the same side of $\mathcal{S}$, we have a *folded join*. In the case of a folded join, we do not have local existence and uniqueness of Riemann solutions.

4. Missing rarefactions after $S \cdot RS$ waves: General approach and results. Let (2.5) be a Riemann solution of type $(T_1, \dots, T_n)$. We shall say that it satisfies *condition (M)* provided there is an integer k such that $T_{k+1} = S \cdot RS$, $T_{k+2} = R_1$, and all hypotheses of Theorem 2.3 are satisfied, except that the rarefaction $w_{k+2}^* : U_{k+1}^* \xrightarrow{s_{k+2}^*} U_{k+2}^*$ has zero strength.

Thus a one-wave or transitional wave group ends with w_k, and a transitional wave group begins with w_{k+1}. It includes w_{k+1} and w_{k+2}, and may be longer. We have $s_{k+1}^* = s_{k+2}^* = \lambda_1(U_{k+1}^*)$ and $U_{k+1}^* = U_{k+2}^*$. As stated in the introduction, there are three cases to consider:

- two-wave slow transitional wave group: the transitional wave group ends with w_{k+2};
- three-wave slow transitional wave group: w_{k+3} is of type $RS \cdot S$ (and thus ends the transitional wave group);
- long slow transitional wave group: w_{k+3} is of type $RS \cdot RS$ (and thus the transitional wave group has at least one more wave).

Under additional nondegeneracy conditions, we shall show that such a Riemann solution (2.5) is of codimension one and lies in a join. Our arguments will have three steps:

Step 1. We verify that (2.5) is a codimension-one Riemann solution.

Step 2. We construct a Riemann solution equivalent to (2.5) and verify that it too is a codimension-one Riemann solution.

Step 3. We show that the two types of codimension-one Riemann solutions are defined on the same codimension-one surface $\mathcal{S}$ in U_0U_nF-space; the two types of codimension-one Riemann solutions above a given point in $\mathcal{S}$ are equivalent; and the Riemann solution join that we therefore have is of a certain type (intermediate boundary or U_L-boundary, regular or folded join).

All three steps will make use of the local defining map (G, H) of Section 3. However, in the remainder of the paper, we will not show the dependence of (G, H) on F, and we will denote the fixed flux function under consideration by F rather than F^*.

We now discuss the three steps of our arguments in order. We begin with step 1.

In each case the local defining map (G, H) is as follows:

1. G is the map that would be used for structurally stable Riemann solutions of type $(T_1, \dots, T_n)$.
2. We have

$$H(U_0, s_1, \dots, s_n, U_n) = s_{k+2} - \lambda_1(U_{k+1}). \tag{4.1}$$

In order to show that (2.5) is a codimension-one Riemann solution, we must verify (Q1)–(Q7). We will first show (Q7$_1$). Since we are ignoring the dependence of (G, H) on the flux function F, we rewrite (Q7$_1$) as

(A) $DG(U_0^*, s_1^*, \dots, s_n^*, U_n^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot{U}_0, \dot{s}_1, \dots, \dot{s}_n, \dot{U}_n) : \dot{U}_0 = 0 = \dot{U}_n\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

If this holds, then, as in the structurally stable case, the equation $G = 0$ may be solved for $(s_1, U_1, \dots, U_{n-1}, s_n)$ in terms of (U_0, U_n) near $(U_0^*, s_1^*, \dots, s_n^*, U_n^*)$. Let

$$\tilde{H}(U_0, U_n) = H(U_0, s_1(U_0, U_n), \dots, s_n(U_0, U_n), U_n). \tag{4.2}$$

Properties (Q1)–(Q3) and (Q5) follow from a geometric understanding of the situation. We will discuss the geometry in each case in enough detail to make them evident. In addition, we will show that one of the following occurs:

(E1) Both $D_{U_0}\tilde{H}(U_0^*, U_n^*)$ and $D_{U_n}\tilde{H}(U_0^*, U_n^*)$ are nonzero. Thus (Q4) and (Q6_1) are satisfied, so (2.5) is a codimension-one Riemann solution that is an intermediate boundary.

(E2) $\tilde{H}$ is independent of U_n, and $D_{U_0}\tilde{H}(U_0^*, U_n^*) \neq 0$. Thus (Q4) and (Q$6_2$) are satisfied, so (2.5) is a codimension-one Riemann solution that is a U_L-boundary.

Let us discuss the verification of (A) and (E1), which is necessary when $\tilde{H}$ depends on both U_0 and U_n; this occurs in the two-wave and three-wave cases. (The long case, in which $\tilde{H}$ is independent of U_n, is easier.)

In the two-wave case, let $m = k + 2$, and in the three-wave case, let $m = k + 3$, so that the transitional wave group that contains the rarefaction of zero strength ends with w_m. Then a transitional or two-wave group begins with w_{m+1}.

Let $G_1(U_0, s_1, \dots, s_m, U_m)$ and $G_2(U_m, s_{m+1}, \dots, s_n, U_n)$ be the local defining maps for wave sequences of types $(T_1, \dots, T_m)$ and $(T_{m+1}, \dots, T_n)$ respectively, so that $G = (G_1, G_2)$. If the wave sequence (2.5) were structurally stable, we would have

(R1) $DG_1(U_0^*, s_1^*, \dots, s_m^*, U_m^*)$, restricted to the space of vectors $(\dot{U}_0, \dot{s}_1, \dots, \dot{s}_m, \dot{U}_m)$ with $\dot{U}_0 = 0$, is surjective, with one-dimensional kernel spanned by a vector $(\dot{U}_0, \dot{s}_1, \dots, \dot{s}_m, \dot{U}_m)$ with $\dot{U}_m \neq 0$.

(R2) $DG_2(U_m^*, s_{m+1}^*, \dots, s_n^*, U_n^*)$, restricted to the space of vectors $(\dot{U}_m, \dot{s}_{m+1}, \dots, \dot{s}_n, \dot{U}_n)$ with $\dot{U}_n = 0$, is surjective, with one-dimensional kernel spanned by a vector $(\dot{U}_m, \dot{s}_{m+1}, \dots, \dot{s}_n, \dot{U}_n)$ with $\dot{U}_m \neq 0$.

Therefore

(S1) There exist smooth mappings $s_i(U_0, \sigma)$ and $U_i(U_0, \sigma)$, $1 \leq i \leq m$, defined on $\mathcal{U}_0 \times (\sigma^* - \varepsilon, \sigma^* + \varepsilon)$, such that

$$s_i(U_0^*, \sigma^*) = s_i^* \quad \text{and} \quad U_i(U_0^*, \sigma^*) = U_i^*, \tag{4.3}$$

and for each (U_0, σ),

$$U_0 \xrightarrow{s_1(U_0,\sigma)} \cdots \xrightarrow{s_m(U_0,\sigma)} U_m(U_0, \sigma) \tag{4.4}$$

is an admissible wave sequence of type $(T_1,\dots,T_m)$. Moreover,

$$\frac{\partial U_m}{\partial \sigma}(U_0^*,\sigma^*)\neq 0.$$

(S2) There exist smooth mappings $\tilde{s}_i(U_n,\tau)$, $m+1\le i\le n$, and $\tilde{U}_i(U_n,\tau)$, $m\le i\le n-1$, defined on $\mathcal{U}_n\times(\tau^*-\varepsilon,\tau^*-\varepsilon)$, such that

$$\tilde{s}_i(U_n^*,\tau^*)=s_i^* \quad\text{and}\quad \tilde{U}_i(U_n^*,\tau^*)=U_i^*, \tag{4.5}$$

and for each (U_n,τ), $\tilde{U}_m(U_n,\tau)\overset{\tilde{s}_{m+1}(U_n,\tau)}{\longrightarrow}\cdots\overset{\tilde{s}_n(U_n,\tau)}{\longrightarrow}U_n$ is an admissible wave sequence of type $(T_{m+1},\dots,T_n)$. Moreover, $\frac{\partial\tilde{U}_m}{\partial\tau}(U_n^*,\tau^*)\neq 0$.

Of course, $\frac{\partial U_m}{\partial\sigma}(U_0^*,\sigma^*)$ is a multiple of the vector $\dot{U}_m$ given by (R1), and $\frac{\partial\tilde{U}_m}{\partial\tau}(U_n^*,\tau^*)$ is a multiple of the vector $\dot{U}_m$ given by (R2). For a structurally stable Riemann solution, the wave group interaction condition implies that

(S3) $\frac{\partial U_m}{\partial\sigma}(U_0^*,\sigma^*)$ and $\frac{\partial\tilde{U}_m}{\partial\tau}(U_n^*,\tau^*)$ are linearly independent.

Given a wave sequence that satisfies (M), we shall verify (A) as follows. We shall check that (R1) holds. (S1) then holds, except that (4.4) will not be admissible for every (U_0,σ), because for some (U_0,σ) we will have $s_{k+2}-\lambda_1(U_{k+1})<0$. (R2), hence (S2), and (S3) follow from the statement of condition (M). From (R1), (R2), and (S3), it follows that (A) holds. To complete step 1, we must verify (E1), *i.e.*, we must verify that $D_{U_0}\tilde{H}(U_0^*,U_n^*)$ and $D_{U_n}\tilde{H}(U_0^*,U_n^*)$ are nonzero.

Recall that a map is *regular* at a point of its domain if its derivative there is surjective.

In both the two-wave and three-wave cases, $\sigma=s_{k+2}$, and $s_1,\dots,s_{k+1}$ along with $U_1,\dots,U_{k+1}$ are determined by U_0 alone. Then from equations (4.1) and (4.2) we have

$$\tilde{H}(U_0,U_n)=\sigma(U_0,U_n)-\lambda_1(U_{k+1}(U_0)), \tag{4.6}$$

where $\sigma(U_0,U_n)=s_{k+2}(U_0,U_n)$. To verify that $D_{U_n}\tilde{H}(U_0^*,U_n^*)$ is nonzero, we shall use

Proposition 4.1. *Let* (2.5) *satisfy (M) and (A). Assume we are in case* (1) *or* (2). *Let* $\tilde{H}$ *be given by* (4.6). *Then* $D_{U_n}\tilde{H}(U_0^*,U_n^*)$ *is nonzero if and only if* $\tilde{U}_m(U_n,\tau)$ *is regular at* (U_n^*,τ^*).

Proof. See reference [8].

To verify that $D_{U_0}\tilde{H}(U_0^*, U_n^*)$ is nonzero, we shall use

Proposition 4.2. *Let* (2.5) *satisfy (M) and (A). Assume we are in case* (1) *or* (2). *Let* $\tilde{H}$ *be given by* (4.6) *with* $\sigma = s_{k+2}$. *Then* $D_{U_0}\tilde{H}(U_0^*, U_n^*)$ *is nonzero if and only if the equation*

$$DG_1(U_0^*, s_1^*, \dots, s_m^*, U_m^*)(\dot{U}_0, \dot{s}_1, \dots, \dot{s}_m, \dot{U}_m) = 0 \tag{4.7}$$

has a solution $(\dot{U}_0, \dot{s}_1, \dots, \dot{s}_m, \dot{U}_m)$ *such that*

(1) $\dot{U}_m$ *is a multiple of* $\frac{\partial \tilde{U}_m}{\partial \tau}(U_n^*, \tau^*)$. (2) $\dot{s}_{k+2} - D\lambda_1(U_{k+1}^*)\dot{U}_{k+1} \neq 0$.

Proof. See reference [8].

This completes our discussion of step 1 in the verification that a Riemann solution (2.5) satisfying (M) is a codimension-one Riemann solution. We now turn our attention to step 2.

Given a wave sequence (2.5) that satisfies (M), there is a subsequence of two or three waves consisting of the rarefaction of zero strength, the $S \cdot RS$ shock wave that precedes it, and, in the three-wave and long cases, the adjacent wave of the same speed that follows it. Let $\ell = k + 2$ in the two-wave case, $\ell = k + 3$ in the three-wave and long cases. In each case there is a naturally defined generalized shock wave $\tilde{w}_\ell : U_k^* \xrightarrow{s_\ell^*} U_\ell^*$ that can replace the subsequence $(w_{k+1}, \dots, w_\ell)$. The new wave sequence

$$(w_1, \dots, w_k, \tilde{w}_\ell, w_{\ell+1}, \dots, w_n) : U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \xrightarrow{s_\ell^*} U_\ell^* \xrightarrow{s_{\ell+1}^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{4.8}$$

is equivalent to (2.5). We remark:

(1) In the two-wave case, $\tilde{w}_\ell$ is identical to the $S \cdot RS$ shock wave w_k.

(2) In the other cases, $\tilde{w}_\ell$ is a generalized shock wave but not a shock wave. It is constructed by amalgamating the shock waves that preceded and followed the rarefaction of zero strength in the original wave sequence.

We will check that there is a shock type $\widehat{T}_\ell$ such that (4.8) is a codimension-one Riemann solution in the boundary of the structurally stable Riemann solutions of type $(T_1, \dots, T_k, \widehat{T}_\ell, T_{\ell+1}, \dots, T_m)$. In order to do this, we will construct another local defining map (G, H) for Riemann solutions of type $(T_1, \dots, T_k, \widehat{T}_\ell, T_{\ell+1}, \dots, T_m)$ exhibiting a certain degeneracy in the wave of type $\widehat{T}_\ell$. For this new map, we will verify (A), *i.e.*, $(Q7_1)$, at the point $(U_0^*, s_1^*, \dots, s_k^*, U_k^*, s_\ell^*, U_\ell^*, s_{\ell+1}^*, \dots, s_n^*, U_n^*)$, and, if we verified (E$i$) for the map (G, H) associated with (2.5), we will verify that the corresponding condition (Ei) for the new map (G, H). Thus (Q4) and (Q6$_i$) hold for the new map; as in step 1, (Q1)–(Q3) and (Q5) follow from a geometric understanding of the situation.

This completes our discussion of step 2. As to step 3, in each case it follows from our construction in step 2 that the two types of codimension-one Riemann solutions are defined on the same codimension-one surface $\mathcal{S}$ in U_0U_nF-space, and that the two types of codimension-one Riemann solutions above a given point in $\mathcal{S}$ are equivalent. In the different cases we shall not discuss these facts; we shall, however, discuss the type of join that occurs; see reference [8], pp. 17–18.

Let us return to the map G of step 1. Suppose that instead of splitting the map G into G_1 and G_2 at U_m, we split it at U_k. The wave group interaction condition will then imply a condition analogous to (S3), but for vectors based at U_k^* instead of at U_m^*. More precisely, there are a parameter σ and smooth mappings $s_i(U_0, \sigma)$ and $U_i(U_0, \sigma)$, $1 \le i \le k$, such that (4.3) holds, and for each (U_0, σ), $U_0 \overset{s_1(U_0,\sigma)}{\longrightarrow} \cdots \overset{s_k(U_0,\sigma)}{\longrightarrow} U_k(U_0, \sigma)$ is an admissible wave sequence of type $(T_1, \dots, T_k)$. The vector $\frac{\partial U_k}{\partial \sigma}(U_0^*, \sigma^*)$ is nonzero. Similarly, there are a parameter τ and smooth mappings $\tilde{s}_i(U_n, \tau)$, $k+1 \le i \le n$, and $\tilde{U}_i(U_n, \tau)$, $k \le i \le n-1$, such that (4.5) holds, and for each (U_n, τ) such that $s_{k+2} - \lambda_1(U_{k+1}) \ge 0$, $\tilde{U}_k(U_n, \tau) \overset{\tilde{s}_{k+1}(U_n,\tau)}{\longrightarrow} \cdots \overset{\tilde{s}_n(U_n,\tau)}{\longrightarrow} U_n$ is an admissible wave sequence of type $(T_{k+1}, \dots, T_n)$. The vector $\frac{\partial \tilde{U}_k}{\partial \tau}(U_n^*, \tau^*)$ turns out to be the projection to $\dot{U}_k$-space of a nonzero vector in the kernel of the set of linearized defining equations for shock waves of type $S \cdot RS$ near $(U_k^*, s_{k+1}^*, U_{k+1}^*)$. The wave group interaction condition says that the vectors $\frac{\partial U_k}{\partial \sigma}(U_0^*, \sigma^*)$ and $\frac{\partial \tilde{U}_k}{\partial \tau}(U_n^*, \tau^*)$ are transverse. Geometrically, this means that the curve $U_k(U_0^*, \sigma)$ is transverse to the curve of left states of $S \cdot RS$ shock waves through U_k^*. Algebraically, it is equivalent to requiring that the vectors

$$\begin{pmatrix} \ell_1(U_{k+1}^*)(DF(U_k^*) - s_{k+1}^* I) \\ D_{U_k} S(U_k^*, s_{k+1}^*) \end{pmatrix} \frac{\partial U_k}{\partial \sigma}(U_0^*, \sigma^*) \text{ and } \begin{pmatrix} \ell_1(U_{k+1}^*)(U_{k+1}^* - U_k^*) \\ \frac{\partial S}{\partial s_{k+1}}(U_k^*, s_{k+1}^*) \end{pmatrix}$$

$$\text{are linearly independent.} \tag{4.9}$$

This transversality condition is thus implied by condition (M). Compare (2.8).

In Table 3 we show the three missing rarefaction cases that are the subject of this paper; "STRG" stands for "slow transitional wave group," and the words "predecessor" and "successor" denote the waves preceding and following the rarefaction of zero strength with the same wave speed. Whether the missing rarefaction gives rise to an F-, U_L-, U_R-, or intermediate boundary depends on the location of the missing rarefaction in the wave sequence (see the end of Section 3 in reference [8]); the table gives the possibilities. In each case, only the first boundary type listed is studied. Each case gives rise to a join, which in two cases may be folded; this information is also given in the table. In the remainder of the paper we state the conditions under which these results hold and give proofs.

case	*predecessor*	*successor*	*boundary type*	*join type for 1st poss.*
Two-wave STRG	$S \cdot RS$	none	intermediate, U_L, U_R, or F	regular
Three-wave STRG	$S \cdot RS$	$RS \cdot S$	intermediate, U_L, U_R, or F	regular or folded
Long STRG	$S \cdot RS$	$RS \cdot RS$	U_L or F	regular or folded

Table 3: Three missing rarefaction cases.

5. Missing rarefaction in a two-wave slow transitional wave group .

Theorem 5.1. *Let* (2.5) *be a Riemann solution of type* $(T_1, \ldots, T_n)$ *that satisfies condition (M), and in addition assume* $T_{k+3} \neq RS \cdot *$. *Assume*

(1) *The backward wave curve mapping* $\tilde{U}_{k+2}(U_n, \tau)$ *is regular at* (U_n^*, τ^*).

(2) *The forward wave curve mapping* $U_k(U_0, \sigma)$ *is regular at* (U_0^*, σ^*).

(3) *The only solution of the system*

$$(DF(U_k^*) - s_{k+1}^* I)\dot{U}_k + \dot{s}_{k+1}(U_{k+1}^* - U_k^*) = 0,$$
$$DS(U_k^*, s_{k+1}^*)(\dot{U}_k, \dot{s}_{k+1}) = 0$$

is $(\dot{U}_k, \dot{s}_{k+1}) = (0, 0)$.

(4) *In the case* $k = 1$, *the numerator of expression* (5.20) *below is nonzero. (If* $k > 1$, *the numerator of an analogous expression must be nonzero.)*

Then (2.5) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(T_1, \dots, T_k, S \cdot S, T_{k+3}, \dots, T_n)$ *because the* $S \cdot S$ *shock wave becomes an* $S \cdot RS$ *shock wave. Riemann solution* (2.5) (*and its equivalent*) *lies in a regular join that is an intermediate boundary.*

Proof. From the general theory of reference [6], the system of equations for wave sequences of type $(T_1, \dots, T_k)$ can be solved near $(U_0^*, s_1^*, \dots, s_k^*, U_k^*)$ for $(s_1, U_1, \dots, s_k, U_k)$ in terms of U_0 and a variable σ.

However, we shall assume for simplicity that $k = 1$. Thus a slow transitional wave group of (2.5) is $U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^*$, with $T_2 = S \cdot RS$, $T_3 = R_1$, $T_4 \neq RS \cdot *$. We have $s_2^* = s_3^* = \lambda_1(U_2^*)$ and $U_2^* = U_3^*$.

Step 1. We note that $(U_0, s_1 \dots, s_3, U_3)$ near $(U_0^*, s_1^*, \dots, s_3^*, U_3^*)$ represents an admissible wave sequence of type $(T_1, S \cdot RS, R_1)$ if and only if

$$U_1 - \eta(U_0, s_1) = 0, \tag{5.1}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{5.2}$$

$$\lambda_1(U_2) - s_2 = 0, \tag{5.3}$$

$$S(U_1, s_2) = 0, \tag{5.4}$$

$$U_3 - \psi(U_2, s_3) = 0, \tag{5.5}$$

$$s_3 - \lambda_1(U_2) \geq 0. \tag{5.6}$$

Here $U_1 = \eta(U_0, s_1)$ is the one-wave curve; in general $U_k = \eta(U_0, \sigma)$ would be the transformed one-wave curve. The function $S(U_1, s_2)$ is the separation function defined in Section 2.

Let $G(U_0, s_1, \dots, s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, S \cdot RS, R_1, T_4, \dots, T_n)$ near $(U_0^*, s_1^*, \dots, s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, \dots, s_3, U_3)$ is given by the left-hand sides of (5.1)–(5.5), and $G_2(U_3, s_4, \dots, s_n, U_n)$ is the local defining map for wave sequences of type $(T_4, \dots, T_n)$. The linearization of (5.1)–(5.5) at $(U_0^*, s_1^*, \dots, s_3^*, U_3^*)$ is

$$\dot U_1 - D\eta(U_0^*, s_1^*)(\dot U_0, \dot s_1) = 0, \tag{5.7}$$

$$(DF(U_2^*) - s_2^* I)\dot U_2 - (DF(U_1^*) - s_2^* I)\dot U_1 - \dot s_2(U_2^* - U_1^*) = 0, \tag{5.8}$$

$$D\lambda_1(U_2^*)\dot U_2 - \dot s_2 = 0, \tag{5.9}$$

$$DS(U_1^*, s_2^*)(\dot U_1, \dot s_2) = 0, \tag{5.10}$$

$$\dot U_3 - D\psi(U_2^*, s_3^*)(\dot U_2, \dot s_3) = 0. \tag{5.11}$$

Solutions of (5.7)–(5.11) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2, \dot{s}_3, \dot{U}_3) = (0, 0, 0, 0, 0, 1, r_1(U_2^*)). \tag{5.12}$$

Thus (R1) holds. Since (R2) and (S3) follow from condition (M), (A) holds.

Solutions of (5.1)–(5.6) near $(U_0^*, s_1^*, \dots, s_3^*, U_3^*)$ are parametrized by U_0 and s_3 as follows:

$$s_1 = s_1(U_0), \tag{5.13}$$
$$U_1 = U_1(U_0), \tag{5.14}$$
$$s_2 = s_2(U_0), \tag{5.15}$$
$$U_2 = U_2(U_0), \tag{5.16}$$
$$U_3 = \psi(U_2, s_3),\ s_3 \geq \lambda_1(U_2).$$

Here (5.13)–(5.16) is the solution of (5.1)–(5.4), and $s_2(U_0) = \lambda_1(U_2(U_0))$.

The left-hand side of (5.6) is the map H for this situation, so $\tilde{H}(U_0, U_n) = s_3(U_0, U_n) - \lambda_1(U_2(U_0))$. The second part of (E1) holds by Proposition 4.1.

To verify the first part of (E1) using Proposition 4.2, recall that $\widetilde{U}_3(U_n, \tau)$ is the backward wave curve mapping, and let

$$\frac{\partial \widetilde{U}_3}{\partial \tau}(U_n^*, \tau^*) = \alpha r_1(U_2^*) + \beta r_2(U_2^*). \tag{5.17}$$

We set $\dot{U}_1 = a r_1(U_1^*) + b r_2(U_1^*)$, $\dot{U}_2 = c r_1(U_2^*) + d r_2(U_2^*)$, and, motivated by Proposition 4.2, we set $\dot{U}_3 = \alpha r_1(U_2^*) + \beta r_2(U_2^*)$. We multiply (5.8) and (5.11) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then (5.8)–(5.11) become the system

$$\begin{aligned}
&- a(\lambda_1(U_1^*) - s_2^*)\ell_1(U_2^*)r_1(U_1^*) - b(\lambda_2(U_1^*) - s_2^*)\ell_1(U_2^*)r_2(U_1^*) \\
&\qquad - \dot{s}_2 \ell_1(U_2^*)(U_2^* - U_1^*) = 0, \\
&(\lambda_2(U_2^*) - s_2^*)d - a(\lambda_1(U_1^*) - s_2^*)\ell_2(U_2^*)r_1(U_1^*) \\
&\qquad - b(\lambda_2(U_1^*) - s_2^*)\ell_2(U_2^*)r_2(U_1^*) - \dot{s}_2 \ell_2(U_2^*)(U_2^* - U_1^*) = 0, \\
&D\lambda_1(U_2^*)(c r_1(U_2^*) + d r_2(U_2^*)) - \dot{s}_2 = 0, \\
&DS(U_1^*, s_2^*)(a r_1(U_1^*) + b r_2(U_1^*), \dot{s}_2) = 0, \\
&\dot{s}_3 - d D\lambda_1(U_2^*) r_2(U_2^*) = \alpha, && (5.18) \\
&d = \beta. && (5.19)
\end{aligned}$$

We have used Lemma 2.1 in (5.18)–(5.19).

Simplifying the notation, this system becomes

$$\begin{aligned} -Aa - Bb - C\dot{s}_2 &= 0, \\ Dd - Ea - Fb - G\dot{s}_2 &= 0, \\ c + Hd - \dot{s}_2 &= 0, \\ Ia + Jb + K\dot{s}_2 &= 0, \\ \dot{s}_3 - Hd &= \alpha, \\ d &= \beta. \end{aligned}$$

Here A through K have the obvious meanings. By assumption (3) of the theorem, we have $\mathcal{D} = \begin{vmatrix} A & B & C \\ E & F & G \\ I & J & K \end{vmatrix} \neq 0$. Therefore this system can be solved uniquely for $(a, b, \dot{s}_2, c, d, \dot{s}_3)$. Then we calculate that

$$\dot{s}_3 - D\lambda_1(U_2^*)\dot{U}_2 = \frac{\mathcal{D}\alpha + (H\mathcal{D} - IDB + DJA)\beta}{\mathcal{D}}, \tag{5.20}$$

which is nonzero by assumption (4) of the theorem. Thus the hypotheses of Proposition 4.2 are satisfied provided we can choose $(\dot{U}_0, \dot{s}_1)$ to satisfy (5.7) with $\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*)$; we can do this by assumption (2) of the theorem.

Step 2. We note that if (U_0, s_1, U_1, s, U) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ represents an admissible wave sequence of type $(T_1, S \cdot S)$ or $(T_1, S \cdot RS)$, then we must have

$$U_1 - \eta(U_0, s_1) = 0, \tag{5.21}$$

$$F(U) - F(U_1) - s(U - U_1) = 0, \tag{5.22}$$

$$S(U_1, s) = 0, \tag{5.23}$$

$$\lambda_1(U) - s \leq 0. \tag{5.24}$$

The inequality (5.24) simply says that an eigenvalue of $\dot{U} = F(U) - F(U_1) - s(U - U_1)$ at U is nonpositive. Solutions of (5.22)–(5.23) with $\lambda_1(U) - s < 0$ represent $S \cdot S$ shock waves; those with $\lambda_1(U) - s = 0$ represent $S \cdot RS$ shock waves. See Figure 2.

Let $G(U_0, s_1, U_1, s, U, s_4, U_4, s_5, \dots, s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, S \cdot S, T_4, \dots, T_n)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*, s_4^*, U_4^*,$

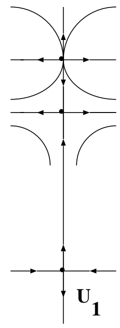

Figure 2: Phase portrait of $\dot U = F(U) - F(U_1) - s(U - U_1)$ for a value of (U_1, s) for which the equilibrium at U_2^* has split into a saddle and a repeller, and for which $S(U_1, s) = 0$. If U is the saddle, then $\lambda_1(U) - s$ is negative and we have an $S \cdot S$ shock wave; if U is the repeller, then $\lambda_1(U) - s$ is positive and there is no connection from U_1 to U.

$s_5^*, \ldots, s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, U_1, s, U)$ is given by the left-hand side of (5.21)–(5.23), and $G_2(U, s_4, U_4, s_5, \ldots, s_n, U_n)$ is as in step 1. The linearization of (5.21)–(5.23) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ is

$$\dot U_1 - D\eta(U_0^*, s_1^*)(\dot U_0, \dot s_1) = 0, \tag{5.25}$$

$$(DF(U_2^*) - s_2^* I)\dot U - (DF(U_1^*) - s_2^* I)\dot U_1 - \dot s(U_2^* - U_1^*) = 0, \tag{5.26}$$

$$DS(U_1^*, s_2^*)(\dot U_1, \dot s) = 0. \tag{5.27}$$

Solutions of (5.25)–(5.27) with $\dot U_0 = 0$ form a one-dimensional space spanned by the vector

$$(\dot U_0, \dot s_1, \dot U_1, \dot s, \dot U) = (0, 0, 0, 0, r_1(U_2^*)). \tag{5.28}$$

To see this, we set $\dot U = c r_1(U_2^*) + d r_2(U_2^*)$, and multiply (5.26) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. We obtain

$$-\ell_1(U_2^*)(DF(U_1^*) - s_2^* I)\dot U_1 - \dot s \ell_1(U_2^*)(U_2^* - U_1^*) = 0, \tag{5.29}$$

$$(\lambda_2(U_2^*) - s_2^*)d - \ell_2(U_2^*)(DF(U_1^*) - s_2^* I)\dot U_1 - \dot s \ell_2(U_2^*)(U_2^* - U_1^*) = 0. \tag{5.30}$$

We look for solutions of the system (5.25), (5.29), (5.30), (5.27) with $\dot U_0 = 0$. Solutions of (5.25) with $\dot U_0 = 0$ have the form $\dot U_1 = \dot s_1 W$, where W is a fixed vector. However, when we set $\dot U_1 = \dot s_1 W$ in (5.29) and (5.27), we obtain a system of two equations in $\dot{s}_1$ and $\dot{s}$ whose only solution is $\dot{s}_1 = \dot{s} = 0$. (This follows from (4.9).) Then (5.30) implies that $d = 0$. The result follows.

Since the last component of (5.28) is nonzero, (R1) holds. Since the last component of (5.28) agrees with the last component of (5.12), it follows easily that (A) holds.

Solutions of (5.21)–(5.23) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ can be parametrized by U_0 and a parameter t near 0 as follows: $s_1 = s_1(U_0, t)$, $U_1 = U_1(U_0, t)$, $s = s(U_0, t)$, $U = U(U_0, t)$, with $(s_1(U_0^*, 0), U_1(U_0^*, 0), s(U_0^*, 0), U(U_0^*, 0)) = (s_1^*, U_1^*, s_2^*, U_2^*)$ and

$$(\frac{\partial s_1}{\partial t}(U_0^*, 0), \frac{\partial U_1}{\partial t}(U_0^*, 0), \frac{\partial s}{\partial t}(U_0^*, 0), \frac{\partial U}{\partial t}(U_0, 0)) = (0, 0, 0, r_1(U_2^*)).$$

From (5.24), the map H for this situation is $s - \lambda_1(U)$. The second half of (E1) is now verified as in Section 8 of reference [8].

To verify the first half of (E1), we note that by the proof of Proposition 4.2, the first half of (E1) holds if and only if there is a solution $(\dot{U}_0, \dot{s}_1, \dot{U}_2, \dot{s}, \dot{U})$ of (5.25)–(5.27) with $\dot{U} = \alpha r_1(U_2^*) + \beta r_2(U_2^*)$, α and β given by (5.17), for which $\dot{s} - D\lambda_1(U_2^*)\dot{U}$ is nonzero.

Because of assumption (2) of the theorem, we need only consider (5.26)–(5.27). Thus in these equations we set $\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*)$ and set $\dot{U}$ equal to the above expression. We multiply (5.26) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then (5.26)–(5.27) become the system

$$\begin{aligned} -Aa - Bb - C\dot{s} &= 0, \\ D\beta - Ea - Fb - G\dot{s} &= 0, \\ Ia + Jb + K\dot{s} &= 0. \end{aligned}$$

Here the capital letters have the same meaning as before.

By assumption (3), $\mathcal{D} \neq 0$, so this system can be solved uniquely for $(a, b, \dot{s})$. We find that $\dot{s} - D\lambda_1(U_2^*)\dot{U}$ equals expression (5.20), which is nonzero by assumption (4).

Step 3. In step 1, $U_3 = \psi(U_2(U_0), s_3)$ is defined for $s_3 \geq \lambda_1(U_2(U_0))$, while in step 2, $U(U_0, t)$ is defined for $t \leq 0$ as in Section 7 of reference [8]. Since the last components of (5.28) and (5.12) agree, the join is regular; see reference [8].

Remark 5.2. The triples (U_1, s_2, U_2) near (U_1^*, s_2^*, U_2^*) such that there is an $S \cdot RS$ shock wave from U_1 to U_2 with speed s_2 satisfy (5.2)–(5.4), with linearization (5.8)–(5.10). As noted in Section 3, the nondegeneracy conditions for $S \cdot RS$ shock waves guarantee that (5.8)–(5.10) have a one-dimensional kernel, so that the solutions of (5.2)–(5.4) form a curve through (U_1^*, s_2^*, U_2^*). These conditions also guarantee that this curve projects to smooth curves $\mathcal{E}_1$ in U_1-space and $\mathcal{E}_2$ in U_2-space. Assumption (3) of the theorem is equivalent to requiring that $\mathcal{E}_2$ is transverse at U_2^* to the 1-rarefaction there: its tangent vector $cr_1(U_2^*) + dr_2(U_2^*)$ has $d \neq 0$. Assumption (4) says that, in addition, $\mathcal{E}_2$ is transverse to the backward wave curve $\tilde{U}_3(U_n^*, \tau)$ at $U_2^* = U_3^*$. Assumptions (3) and (4) are thus natural geometric requirements for the existence of a codimension-one Riemann solution of the desired type. Since the backward wave curve $\tilde{U}_3(U_n^*, \tau)$ is also transverse at $U_2^* = U_3^*$ to the 1-rarefaction there by condition (M), the geometry of wave curves in the vicinity of $U_2^* = U_3^*$ is as pictured in Figure 3. We note that assumptions (3) and (4) are used only to verify the first part of (E1) in both steps of the proof.

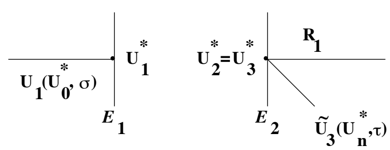

Figure 3: The curves $\mathcal{E}_1$ and $\mathcal{E}_2$ and nearby wave curves.

6. Missing rarefaction in a three-wave slow transitional wave group.

Theorem 6.1. *Let (2.5) be a Riemann solution of type $(T_1, \ldots, T_n)$ that satisfies condition (M), and in addition assume $T_{k+3} = RS \cdot S$. Assume*

(1) *The backward wave curve mapping $\tilde{U}_{k+3}(U_n, \tau)$ is regular at (U_n^*, τ^*).*

(2) *The forward wave curve mapping $U_k(U_0, \sigma)$ is regular at (U_0^*, σ^*).*

(3) *The only solution of the system*

$$(DF(U_k^*) - s_{k+1}^* I)\dot{U}_k + \dot{s}_{k+1}(U_{k+1}^* - U_k^*) = 0,$$
$$DS(U_k^*, s_{k+1}^*)(\dot{U}_k, \dot{s}_{k+1}) = 0$$

is $(\dot{U}_k, \dot{s}_{k+1}) = (0, 0)$.

(4) *In the case $k = 1$, the numerator of expression* (6.21) *below is nonzero.* (*In the case $k > 1$, an analogous assumption holds.*)

(5) *In the case $k = 1$, if $(0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ is a nonzero vector in the kernel of* (6.37)–(6.39) *below, then $\dot{U}$ is a nonzero vector that is transverse to $\frac{\partial \tilde{U}_4}{\partial \tau}(U_n^*, \tau^*)$.* (*In the case $k > 1$, an analogous assumption holds.*)

(6) $\ell_1(U_{k+2}^*)(U_{k+3}^* - U_{k+2}^*)$ *is nonzero.*

(7) *The system*

$$\begin{aligned}(DF(U_k^*) - s_{k+3}^* I)\dot{U}_k + \dot{s}(U_{k+3}^* - U_k^*) \\ = (DF(U_{k+3}^*) - s_{k+3}^* I)\frac{\partial \widetilde{U}_{k+3}}{\partial \tau}(U_n^*, \tau^*), \\ DS(U_k^*, s_{k+3}^*)(\dot{U}_k, \dot{s}) = 0\end{aligned}$$

has either a unique solution or no solution.

Then (2.5) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type $(T_1, \dots, T_k, S \cdot S, T_{k+4}, \dots, T_n)$ because the connection of the $S \cdot S$ shock wave develops an equilibrium of type RS that breaks it. Riemann solution* (2.5) *(and its equivalent) lie in a join that is an intermediate boundary. The join may be regular or folded.*

Proof. As in Section 5, we shall assume for simplicity that $k = 1$. Then (2.5) has a slow transitional wave group $U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^* \xrightarrow{s_4^*} U_4^*$ with $T_2 = S \cdot RS$, $T_3 = R_1$, $T_4 = RS \cdot S$. We have $s_2^* = s_3^* = s_4^* = \lambda_1(U_2^*)$ and $U_2^* = U_3^*$.

Step 1. We note that $(U_0, s_1, \dots, s_4, U_4)$ near $(U_0^*, s_1^*, \dots, s_4^*, U_4^*)$ represents an admissible wave sequence of type $(T_1, S \cdot RS, R_1, RS \cdot S)$ if and only if

$$U_1 - \eta(U_0, s_1) = 0, \tag{6.1}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{6.2}$$

$$\lambda_1(U_2) - s_2 = 0, \tag{6.3}$$

$$S(U_1, s_2) = 0, \tag{6.4}$$

$$U_3 - \psi(U_2, s_3) = 0, \tag{6.5}$$

$$F(U_4) - F(U_3) - s_4(U_4 - U_3) = 0, \tag{6.6}$$

$$\lambda_1(U_3) - s_4 = 0, \tag{6.7}$$

$$s_3 - \lambda_1(U_2) \geq 0. \tag{6.8}$$

Here $\eta(U_0, s_1)$ and $S(U_1, s_2)$ are defined as in Section 5.

Let $G(U_0, s_1, \dots, s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, S \cdot RS, R_1, RS \cdot S, T_5, \dots, T_n)$ near $(U_0^*, s_1^*, \dots, s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, \dots, s_4, U_4)$ is given by the left-hand sides of (6.1)–(6.7), and $G_2(U_4, s_5, \dots, s_n, U_n)$ is the local defining map for wave sequences of type $(T_5, \dots, T_n)$. The linearization of (6.1)–(6.7) at $(U_0^*, s_1^*, \dots, s_4^*, U_4^*)$ is

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{6.9}$$

$$(DF(U_2^*) - s_2^* I)\dot{U}_2 - (DF(U_1^*) - s_2^* I)\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{6.10}$$

$$D\lambda_1(U_2^*)\dot{U}_2 - \dot{s}_2 = 0, \tag{6.11}$$

$$DS(U_1^*, s_2^*)(\dot{U}_1, \dot{s}_2) = 0, \tag{6.12}$$

$$\dot{U}_3 - D\psi(U_2^*, s_3^*)(\dot{U}_2, \dot{s}_3) = 0, \tag{6.13}$$

$$(DF(U_4^*) - s_4^* I)\dot{U}_4 - (DF(U_3^*) - s_4^* I)\dot{U}_3 - \dot{s}_4(U_4^* - U_3^*) = 0, \tag{6.14}$$

$$D\lambda_1(U_3^*)\dot{U}_3 - \dot{s}_4 = 0. \tag{6.15}$$

Solutions of (6.9)–(6.15) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$\begin{aligned}&(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2, \dot{s}_3, \dot{U}_3, \dot{s}_4, \dot{U}_4)\\ &= (0, 0, 0, 0, 0, 1, r_1(U_2^*), 1, (DF(U_4^*) - s_4^* I)^{-1}(U_4^* - U_3^*)).\end{aligned} \tag{6.16}$$

Thus (R1) holds. Since (R2) and (S3) follow from condition (M), (A) holds.

Solutions of (6.1)–(6.8) near $(U_0^*, s_1^*, \dots, s_4^*, U_4^*)$ are parametrized by U_0 and s_3 as follows: $s_1 = s_1(U_0)$, $U_1 = U_1(U_0)$, $s_2 = s_2(U_0)$, $U_2 = U_2(U_0)$, $U_3 = \psi(U_2(U_0), s_3)$, $s_3 \geq \lambda_1(U_2(U_0))$, $s_4 = s_3$, $U_4 = \varphi(U_3, s_4)$. Here $(s_1(U_0), U_1(U_0), s_2(U_0), U_2(U_0))$ is the solution of (6.1)–(6.4) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$, and $U_4 = \varphi(U_3, s_4)$ is the solution of (6.6) near (U_3^*, s_4^*, U_4^*).

The left-hand side of (6.8) is the map H for this situation, so $\tilde{H}(U_0, U_n) = s_3(U_0, U_n) - \lambda_1(U_2(U_0))$. The second part of (E1) holds by Proposition 4.1.

To verify the first part of (E1) using Proposition 4.2, recall that $\widetilde{U}_4(U_n, \tau)$ is the backward wave curve mapping, and let

$$\frac{\partial \widetilde{U}_4}{\partial \tau}(U_n^*, \tau^*) = \alpha r_1(U_4^*) + \beta r_2(U_4^*). \tag{6.17}$$

We set $\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*)$, $\dot{U}_2 = cr_1(U_2^*) + dr_2(U_2^*)$, $\dot{U}_3 = er_1(U_3^*) + fr_2(U_3^*)$, and, motivated by Proposition 4.2, we set

$$\dot{U}_4 = \alpha r_1(U_4^*) + \beta r_2(U_4^*). \tag{6.18}$$

We multiply (6.10) and (6.13)–(6.14) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then (6.10)–(6.15) become the system

$$\begin{aligned}
&- a(\lambda_1(U_1^*) - s_2^*)\ell_1(U_2^*)r_1(U_1^*) - b(\lambda_2(U_1^*) - s_2^*)\ell_1(U_2^*)r_2(U_1^*)\\
&\qquad - \dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*) = 0,\\
&(\lambda_2(U_2^*) - s_2^*)d - a(\lambda_1(U_1^*) - s_2^*)\ell_2(U_2^*)ar_1(U_1^*)\\
&- b(\lambda_2(U_1^*) - s_2^*)\ell_2(U_2^*)r_2(U_1^*) - \dot{s}_2\ell_2(U_2^*)(U_2^* - U_1^*) = 0,\\
&D\lambda_1(U_2^*)(cr_1(U_2^*) + dr_2(U_2^*)) - \dot{s}_2 = 0,\\
&DS(U_1^*, s_2^*)(ar_1(U_1^*) + br_2(U_1^*), \dot{s}_2) = 0,\\
&e - (\dot{s}_3 - dD\lambda_1(U_2^*)r_2(U_2^*)) = 0, && (6.19)\\
&f - d = 0, && (6.20)\\
&\alpha(\lambda_1(U_4^*) - s_4^*)\ell_1(U_3^*)r_1(U_4^*) + \beta(\lambda_2(U_4^*) - s_4^*)\ell_1(U_3^*)r_2(U_4^*)\\
&- \dot{s}_4\ell_1(U_3^*)(U_4^* - U_3^*) = 0,\\
&\alpha(\lambda_1(U_4^*) - s_4^*)\ell_2(U_3^*)r_1(U_4^*) + \beta(\lambda_2(U_4^*) - s_4^*)\ell_2(U_3^*)r_2(U_4^*)\\
&- (\lambda_2(U_3^*) - s_4^*)f - \dot{s}_4\ell_2(U_3^*)(U_4^* - U_3^*) = 0,\\
&D\lambda_1(U_3^*)(er_1(U_3^*) + fr_2(U_3^*)) - \dot{s}_4 = 0.
\end{aligned}$$

We have used Lemma 2.1 in (6.19)–(6.20).

Simplifying the notation, this system becomes

$$\begin{aligned}
-Aa - Bb - C\dot{s}_2 &= 0\\
Dd - Ea - Fb - G\dot{s}_2 &= 0\\
c + Hd - \dot{s}_2 &= 0\\
Ia + Jb + K\dot{s}_2 &= 0\\
e + Hd - \dot{s}_3 &= 0\\
f - d &= 0\\
-N\dot{s}_4 &= -L\alpha - M\beta\\
-Df - R\dot{s}_4 &= -P\alpha - Q\beta\\
e + Hf - \dot{s}_4 &= 0.
\end{aligned}$$

Here the capital letters have the obvious meanings. As in Section 5, assumption (3) of the theorem implies that $\mathcal{D} = \begin{vmatrix} A & B & C \\ E & F & G \\ I & J & K \end{vmatrix} \neq 0$; moreover, $D \neq 0$, and by assumption (6) of the theorem, $N \neq 0$. Therefore this system can be solved uniquely for $(a, b, \dot{s}_2, c, d, \dot{s}_3, e, f, \dot{s}_4)$. Then we calculate that

$$\begin{aligned}&\dot{s}_3 - D\lambda_1(U_2^*)\dot{U}_2 = \dot{s}_3 - \dot{s}_2 \\ &= \frac{(\mathcal{D}L + (RL - NP)(BI - AJ))\alpha + (\mathcal{D}M + (RM - QN)(BI - AJ))\beta}{\mathcal{D}N}.\end{aligned} \tag{6.21}$$

This is nonzero by assumption (4). Thus, as in Section 5, the hypotheses of Proposition 4.2 are satisfied.

Step 2. We look for points (U_0, s_1, U_1, s, U) near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ that represent admissible wave sequences of type $(T_1, S \cdot S)$.

We begin by studying the four-parameter family of differential equations

$$\dot{U}_2 = F(U_2) - F(\eta(U_0, s_1)) - s_2(U_2 - \eta(U_0, s_1)), \tag{6.22}$$

near $(U_0, s_1, s_2, U_2) = (U_0^*, s_1^*, s_2^*, U_2^*)$.

Lemma 6.2. *There is a function $\gamma(U_0, s_1, s_2) = 0$, defined near $(U_0, s_1, s_2, U_2) = (U_0^*, s_1^*, s_2^*, U_2^*)$, such that* (6.22) *undergoes a saddle-node bifurcation near U_2^* when the surface $\gamma = 0$ is crossed. For $\gamma > 0$ (respectively $= 0$, < 0) there are no (respectively $1, 2$) equilibria of* (6.22) *near U_2^*. We may take*

$$\begin{aligned}&D\gamma(U_0^*, s_1^*, s_2^*)(\dot{U}_0, \dot{s}_1, \dot{s}_2) \\ &= -\ell_1(U_2^*)(DF(U_1^*) - s_2^* I)D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) - \dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*).\end{aligned} \tag{6.23}$$

Proof. We first consider the degenerate equilibria of (6.22) near $(U_0^*, s_1^*, s_2^*, U_2^*)$. They satisfy the system

$$F(U_2) - F(\eta(U_0, s_1)) - s_2(U_2 - \eta(U_0, s_1)) = 0, \tag{6.24}$$

$$\lambda_1(U_2) - s_2 = 0. \tag{6.25}$$

If (1) the linearization of this system at $(U_0, s_1, s_2, U_2) = (U_0^*, s_1^*, s_2^*, U_2^*)$ is surjective, and (2) the projection of the three-dimensional kernel to $\dot{U}_0\dot{s}_1\dot{s}_2$-space is regular, then there is a three-dimensional surface in $U_0 s_1 s_2$-space, tangent at $(U_0^*, s_1^*, s_2^*, U_2^*)$ to the projected kernel, such that (6.22) undergoes a saddle-node bifurcation near U_2^* when the surface is crossed.

Linearizing (6.24)–(6.25), we obtain

$$(DF(U_2^*) - s_2^* I)\dot{U}_2 - (DF(U_1^*) - s_2^* I)D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{6.26}$$
$$D\lambda_1(U_2^*)\dot{U}_2 - \dot{s}_2 = 0.$$

Let $\dot{U}_2 = cr_1(U_2^*) + dr_2(U_2^*)$, and multiply (6.26) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. We obtain

$$-\ell_1(U_2^*)(DF(U_1^*) - s_2^* I)D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) - \dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*) = 0, \tag{6.27}$$

$$(\lambda_2(U_2^*) - s_2^*)d - \ell_2(U_2^*)(DF(U_1^*) - s_2^* I)D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) - \dot{s}_2\ell_2(U_2^*)(U_2^* - U_1^*) = 0, \tag{6.28}$$

$$c + dD\lambda_1(U_2^*)r_2(U_2^*) - \dot{s}_2 = 0. \tag{6.29}$$

If we set $\dot{U}_0 = 0$ in (6.27)–(6.29), we obtain a system of three linear equations in the four unknowns $(\dot{s}_1, \dot{s}_2, c, d)$. It is enough to show that this system can be solved in terms of $\dot{s}_1$ or $\dot{s}_2$ for the remaining variables. The reduced system is

$$-\ell_1(U_2^*)(DF(U_1^*) - s_2^* I)\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)\dot{s}_1 - \dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*) = 0, \tag{6.30}$$

$$(\lambda_2(U_2^*) - s_2^*)d - \ell_2(U_2^*)(DF(U_1^*) - s_2^* I)\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)\dot{s}_1 - \dot{s}_2\ell_2(U_2^*)(U_2^* - U_1^*) = 0, \tag{6.31}$$

together with (6.29). By (4.9), the coefficients of $\dot{s}_1$ and $\dot{s}_2$ in (6.30) are not both 0. The desired conclusion follows.

Let $\gamma(U_0, s_1, s_2) = 0$ be the surface of parameter values for which degenerate equilibria near U_2^* occur. We may take $D\gamma(U_0^*, s_1^*, s_2^*)(\dot{U}_0, \dot{s}_1, \dot{s}_2)$ to be the left side of (6.27) or its opposite. To see which, we must determine on which side of the surface there are equilibria of (6.22) near U_2^*.

Let $(U_0, s_1, s_2, U_2) = (U_0^*, s_1(\mu), s_2(\mu), U_2(\mu))$ be a curve of solutions of (6.24) with $s_1(\mu) = s_1^* + \ell_1(U_2^*)(DF(U_1^*) - s_2^* I)\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)\alpha(\mu)$, $s_2(\mu) = s_2^* + \ell_1(U_2^*)(U_2^* - U_1^*)\alpha(\mu)$, $U_2(0) = U_2^*$, and $\alpha(0) = 0$. We must take $\dot{U}_2(0)$ to be a multiple of $r_1(U_2^*)$; we choose $\dot{U}_2(0) = r_1(U_2^*)$. After some computation we find that $\dot{\alpha} = 0$ and

$$\ddot{\alpha}(0) = \frac{1}{(\ell_1(U_2^*)(DF(U_1^*) - s_2^* I)\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*))^2 + (\ell_1(U_2^*)(U_2^* - U_1^*))^2}.$$

Thus to obtain that $\gamma > 0$ when there are no equilibria near U_2^* we take $D\gamma(U_0^*, s_1^*, s_2^*)$ to be given by (6.25).

Next we construct a separation function $\tilde{S}(U_1, s)$, (U_1, s) near (U_1^*, s_4^*), which can be used to study connections of

$$\dot{U} = F(U) - F(U_1) - s(U - U_1) \tag{6.32}$$

from U_1 to the saddle $U(U_1, s)$ of (6.32) near U_4^*. We first recall that as in Section 2, to study connections from U_1 to equilibria near U_2^*, we can define a vector V, a transversal Σ, invariant manifolds $W_\pm(U_1, s_2)$, points $\bar{U}_\pm(U_1, s_2)$ in Σ, and a separation function $S(U_1, s_2)$. If (6.32) has equilibria near U_2^* we define $\tilde{U}_+(U_1, s) = \bar{U}_+(U_1, s)$. If it has no equilibria near U_2^*, we define $\tilde{U}_+(U_1, s)$ to be the intersection of the stable manifold of $U(U_0, s)$ with Σ. We then define $\bar{U}_-(U_1, s) - \tilde{U}_+(U_1, s) = \tilde{S}(U_1, s)V$. See Figure 4.

Lemma 6.3. *The family of invariant manifolds $W_+(U_1, s_2)$ can be chosen so that the functions $\tilde{S}(U_1, s)$ and $S(U_1, s)$ agree on a neighborhood of (U_1^*, s_4^*).*

This amounts to choosing $W_+(U_1, s_2)$ to be the stable manifold of the equilibrium near U_4^* whenever there are no equilibria of (6.32) near U_2^*. This choice is permissible; see reference [2], Proposition 2.8, p. 1217.

We note that if (U_0, s_1, U_1, s, U) near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ represents an admissible wave sequence of type $(T_1, S \cdot S)$, then we must have

$$U_1 - \eta(U_0, s_1) = 0, \tag{6.33}$$

$$F(U) - F(U_1) - s(U - U_1) = 0, \tag{6.34}$$

$$\tilde{S}(U_1, s) = 0, \tag{6.35}$$

$$\gamma(U_0, s_1, s) \geq 0. \tag{6.36}$$

Let $G(U_0, s_1, U_1, s, U, s_5, U_5, s_6, \ldots, s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, S \cdot S, T_5, \ldots, T_n)$ near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*, s_5^*, U_5^*,$

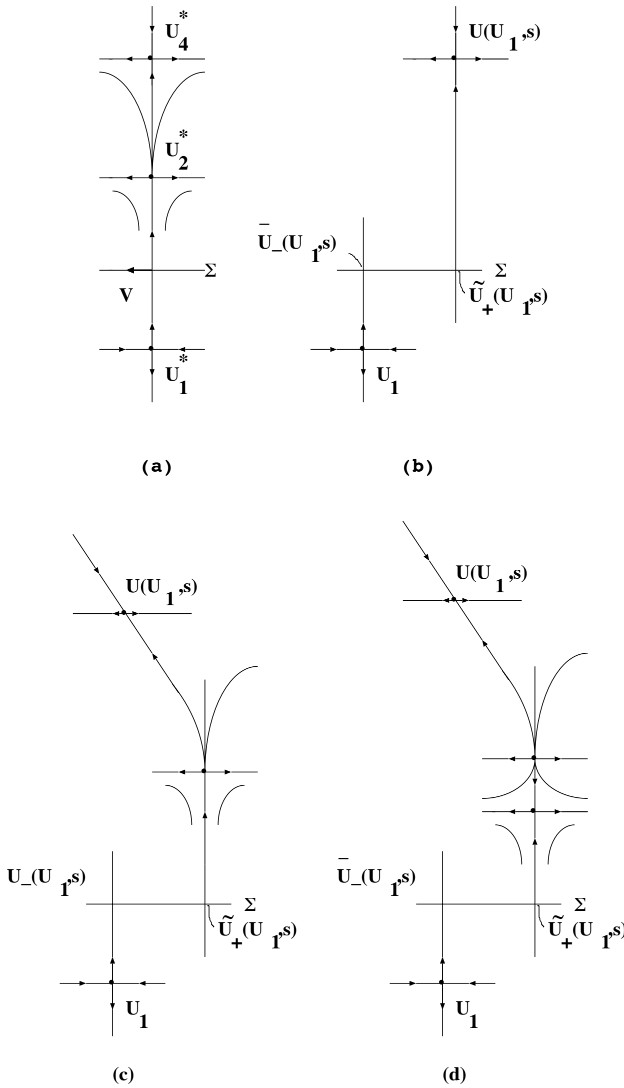

Figure 4: Geometry of the separation function in Section 6. (a) Phase portrait of $\dot{U} = F(U) - F(U_1^*) - s_4^*(U - U_1^*)$. (b) Phase portrait of $\dot{U} = F(U) - F(U_1) - s(U - U_1)$ for a value of (U_1, s) for which the equilibrium at U_2^* has disappeared, and for which S is positive. (c) Phase portrait for a value of (U_1, s) for which there is a repeller-saddle near U_2^* and for which S is positive. (d) Phase portrait for a value of (U_1, s) for which the equilibrium at U_2^* has split into a saddle and a repeller, and for which S is positive.

$s_6^*, \dots, s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, U_1, s, U)$ is given by the left-hand side of (6.33)–(6.35), and $G_2(U, s_5, U_5, s_6, \dots, s_n, U_n)$ is as in step 1. The linearization of (6.33)–(6.35) at $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ is

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{6.37}$$

$$(DF(U_4^*) - s_4^* I)\dot{U} - (DF(U_1^*) - s_4^* I)\dot{U}_1 - \dot{s}(U_4^* - U_1^*) = 0, \tag{6.38}$$

$$D\tilde{S}(U_1^*, s_4^*)(\dot{U}_1, \dot{s}) = 0. \tag{6.39}$$

Case 1. $\frac{\partial \tilde{S}}{\partial s}(U_1^*, s_4^*) \neq 0$. In this case, solutions of (6.37)–(6.39) with $\dot{U}_0 = 0$ form a one-dimensional space that is spanned by the vector $(0, 1, \dot{U}_1, \dot{s}, \dot{U})$, where $\dot{U}_1$ is obtained by solving (6.37) with $\dot{U}_0 = 0$ and $\dot{s}_1 = 1$; $\dot{s}$ is then obtained by solving (6.39), and $\dot{U}$ is obtained by solving (6.38). Thus

$$\dot{U} = (DF(U_4^*) - s_4^* I)^{-1} \cdot \tag{6.40}$$

$$\{(DF(U_1^*) - s_4^* I) - (U_4^* - U_1^*)(\frac{\partial \tilde{S}}{\partial s}(U_1^*, s_4^*))^{-1} D_{U_0}\tilde{S}(U_1^*, s_4^*)\}\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*).$$

This is a nonzero vector by assumption (5). Solutions of (6.33)–(6.36) near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ are parametrized by U_0 and s_1 as follows: $U_1 = \eta(U_0, s_1)$, $s = s(U_0, s_1)$, $U = U(U_0, s_1)$, $\gamma(U_0, s_1, s) \geq 0$.

Case 2. $\frac{\partial \tilde{S}}{\partial s}(U_1^*, s_4^*) = 0$. In this case, solutions of (6.37)–(6.39) with $\dot{U}_0 = 0$ form a one-dimensional space that is spanned by the vector $(0, 0, 0, 1, \dot{U})$ with

$$\dot{U} = (DF(U_4^*) - s_4^* I)^{-1}(U_4^* - U_1^*). \tag{6.41}$$

Solutions of (6.33)–(6.36) near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ are parametrized by U_0 and s as follows: $s_1 = s_1(U_0, s)$, $U_1 = U_1(U_0, s)$, $U = U(U_0, s)$, $\gamma(U_0, s_1, s) \geq 0$. In either case, (R1) holds. (R2) follows from condition (M), and (A) then follows from assumption (5).

From (6.36), the map H for this situation is $\gamma(U_0, s_1, s)$. To verify the second half of (E1), consider separately the two cases.

In Case 1, we have $\tilde{H}(U_0, U_n) = \gamma(U_0, s_1(U_0, U_n), s(U_0, s_1(U_0, U_n)))$. Therefore,

$$\begin{aligned} D_{U_n}\tilde{H}(U_0^*, U_n^*) &= \frac{\partial \gamma}{\partial s_1}(U_0^*, s_1^*, s_4^*) D_{U_n} s_1(U_0^*, U_n^*) \\ &\quad + \frac{\partial \gamma}{\partial s}(U_0^*, s_1^*, s_4^*)\frac{\partial s}{\partial s_1}(U_0^*, s_1^*) D_{U_n} s_1(U_0^*, U_n^*) \end{aligned}$$

$$= \{-\ell_1(U_2^*)(DF(U_1^*) - s_2^* I))\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)$$
$$+ \ell_1(U_2^*)(U_2^* - U_1^*)\frac{D_{U_1}\tilde{S}(U_0^*, s_4^*)\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*)}{\frac{\partial \tilde{S}}{\partial s}(U_0^*, s_4^*)}\} D_{U_n} s_1(U_0^*, U_n^*).$$

By (4.9), this is a nonzero multiple of $D_{U_n} s_1(U_0^*, U_n^*)$. The latter can be proved to be nonzero as in the proof of Theorem 4.2 of reference [8].

Thus the second half of (E1) is verified in Case 1. The argument for Case 2 is left to the reader.

To verify the first half of (E1), we note that by the argument used to prove Proposition 4.2, the first half of (E1) holds if and only if there is a solution $(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ of (6.37)–(6.39) such that

(a) $\dot{U}$ is a multiple of $\frac{\partial \tilde{U}_4}{\partial \tau}(U_n^*, \tau^*)$,

(b) $D\gamma(U_0^*, s_1^*, s_2^*)(\dot{U}_0, \dot{s}_1, \dot{s}) \neq 0$.

In the system (6.37)–(6.39), we set $\dot{U}_1 = a r_1(U_1^*) + b r_2(U_1^*)$ and, motivated by Proposition 4.2, we set $\dot{U} = \epsilon\alpha r_1(U_4^*) + \epsilon\beta r_2(U_4^*)$. Note that from (6.23), (6.37), the formula for $\dot{U}_1$ just given, and the definitions of A, B, and C, we have

$$D\gamma(U_0^*, s_1^*, s_2^*)(\dot{U}_0, \dot{s}_1, \dot{s}) = -Aa - Bb - C\dot{s}. \tag{6.42}$$

To find the solutions $(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ of (6.37)–(6.39) that satisfy (a), we multiply (6.38) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then (6.38)–(6.39) become the system

$$\begin{aligned}
&\epsilon\alpha(\lambda_1(U_4^*) - s_4^*)\ell_1(U_2^*)r_1(U_4^*) + \epsilon\beta(\lambda_2(U_4^*) - s_4^*)\ell_1(U_2^*)r_2(U_4^*)\\
&- a(\lambda_1(U_1^*) - s_4^*)\ell_1(U_2^*)r_1(U_1^*) - b(\lambda_2(U_1^*) - s_4^*)\ell_1(U_2^*)r_2(U_1^*)\\
&- \dot{s}\ell_1(U_2^*)(U_4^* - U_1^*) = 0, \qquad (6.43)\\
&\epsilon\alpha(\lambda_1(U_4^*) - s_4^*)\ell_2(U_2^*)r_1(U_4^*) + \epsilon\beta(\lambda_2(U_4^*) - s_4^*)\ell_2(U_2^*)r_2(U_4^*)\\
&- a(\lambda_1(U_1^*) - s_4^*)\ell_2(U_2^*)r_1(U_1^*) - b(\lambda_2(U_1^*) - s_4^*)\ell_2(U_2^*)r_2(U_1^*)\\
&- \dot{s}\ell_2(U_2^*)(U_4^* - U_1^*) = 0, \qquad (6.44)\\
&D\tilde{S}(U_1^*, s_4^*)(a r_1(U_1^*) + b r_2(U_1^*), \dot{s}) = 0. \qquad (6.45)
\end{aligned}$$

Since $\ell_1(U_2^*)(U_4^* - U_1^*) = \ell_1(U_2^*)(U_4^* - U_2^*) + \ell_1(U_2^*)(U_2^* - U_1^*) = N + C$ and $\ell_2(U_2^*)(U_4^* - U_1^*) = \ell_2(U_2^*)(U_4^* - U_2^*) + \ell_2(U_2^*)(U_2^* - U_1^*) = R + G$, (6.43)–(6.45) become the system

$$\begin{aligned} L\epsilon\alpha + M\epsilon\beta - Aa - Bb - (N + C)\dot{s} &= 0, \\ P\epsilon\alpha + Q\epsilon\beta - Ea - Fb - (R + G)\dot{s} &= 0, \\ Ia + Jb + K\dot{s} &= 0. \end{aligned}$$

There are now two cases to consider. Let

$$\tilde{\mathcal{D}} = \begin{vmatrix} A & B & N + C \\ E & F & R + G \\ I & J & K \end{vmatrix}.$$

Case 1. $\tilde{\mathcal{D}} \neq 0$. In this case, we set $\epsilon = 1$, solve for $(a, b, \dot{s})$ in terms of (α, β), and substitute into (6.42). We find that $D\gamma(U_0^*, s_1^*, s_2^*)(\dot{U}_0, \dot{s}_1, \dot{s})$ is given by the negative of (6.21), which is nonzero by assumption (4) of the theorem.

Case 2. $\tilde{\mathcal{D}} = 0$. By assumption (7), there are no solutions unless $\epsilon = 0$. Let $(a, b, \dot{s})$ be a nontrivial solution with $\epsilon = 0$. We must have $\dot{s} \neq 0$, since otherwise $(a, b, \dot{s})$ would be a nontrivial solution of the system

$$\begin{aligned} Aa + Bb + C\dot{s} &= 0, \\ Ea + Fb + G\dot{s} &= 0, \\ Ia + Jb + K\dot{s} &= 0; \end{aligned}$$

this is impossible because $\mathcal{D} \neq 0$. Using (6.42), and the first equation of our system with $\epsilon = 0$, we find that

$$D\gamma(U_0^*, s_1^*, s_2^*)(\dot{U}_0, \dot{s}_1, \dot{s}) = -Aa - Bb - C\dot{s} = N\dot{s} \neq 0.$$

Step 3. Since the last component of (6.16) and the last component of (6.40) or (6.41) need not be parallel, by reference [8] the join need not be regular.

Remark 6.4. The observation that the vectors discussed in step 3 need not be parallel implies that the transformed one-wave curve need not continue smoothly through the degeneracy, which is related to the need for assumption (4) of the theorem in step 2 of the proof. This assumption is transversality of the curve $U = U(U_0^*, s)$, defined in step 2, to the backward wave curve $\tilde{U}_4(U_n^*, \tau)$. Assumption (5) can be reformulated as follows.

Let $(w_1, \dots, w_k, \tilde{w}_{k+3}, w_{k+4}, \dots, w_n) : U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \xrightarrow{s_{k+3}^*} U_{k+3}^* \xrightarrow{s_{k+4}^*} \cdots \xrightarrow{s_n^*} U_n^*$ denote the wave sequence equivalent to (2.5), in which the subsequence $(w_{k+1}, w_{k+2}, w_{k+3})$ has been replaced by a single generalized shock wave $\tilde{w}_{k+3}$ of type $S \cdot S$ from U_k^* to U_{k+3}^*. (The wave is a generalized shock wave because it is represented by a sequence of two connecting orbits.) Then assumption (5) says that this new wave sequence satisfies the wave group transversality condition. See reference [6], pp. 340–341, for a discussion of the wave group transversality condition for shock waves of type $S \cdot S$.

Remark 6.5. Assumptions (3) and (4) have the following geometric interpretation. The quadruples (U_1, s_2, U_2, U_4) such that there is a shock wave of type $S \cdot RS$ from U_1 to U_2 with speed s_2, and a shock wave of type $RS \cdot S$ from U_2 to U_4 with the same speed, form a curve $\mathcal{E}$ through $(U_1^*, s_2^*, U_2^*, U_4^*)$. This curve may be found by solving (6.2)–(6.4) and (6.6) with $U_3 = U_2$ and $s_4 = s_2$. This curve projects to curves $\mathcal{E}_1$, $\mathcal{E}_2$, and $\mathcal{E}_4$ through U_1^*, U_2^*, and U_4^* respectively: for each $U_1 \in \mathcal{E}_1$ there is a speed s_2 and points $U_2 \in \mathcal{E}_2$ and $U_4 \in \mathcal{E}_4$ such that there is an $S \cdot RS$ shock wave from U_1 to U_2 with speed s_2 and an $RS \cdot S$ shock wave from U_2 to U_4 with the same speed. As in the previous section, assumption (3) guarantees that $\mathcal{E}_2$ is transverse at U_2^* to the 1-rarefaction there. This 1-rarefaction, parametrized by its speed s_3, is transformed by $RS \cdot S$ shock waves to a curve $U_4(U_0^*, s_3)$ through U_4^*, as discussed in step 1 of the proof. Assumption (3) then guarantees that $\mathcal{E}_4$ is then transverse at U_4^* to the curve $U_4(s_3)$. Assumption (4) says that $\mathcal{E}_4$ is transverse to the backward wave curve $\tilde{U}_4(U_n^*, \tau)$ at U_4^*. As in the previous section, we conclude that assumptions (3) and (4) are natural geometric requirements for the existence of a codimension-one Riemann solution of the desired type. Since the backward wave curve $\tilde{U}_4(U_n^*, \tau)$ is also transverse at U_4^* to the curve $U_4(U_0^*, s_3)$ by condition (M), the geometry of wave curves in the vicinity of U_4^* is as pictured in Figure 5. As in the previous section, we note that assumptions (3) and (4) are used only to verify the first part of (E1) in both steps of the proof.

Assumptions (6) and (7) are used in the verification of (E1) in steps 1 and 2 respectively. However, we do not have natural geometric interpretations for these assumptions.

Remark 6.6. If the Lax admissibility criterion is used, then, in the variations of Figure 4(c) and (d) for which $S = 0$, the $S \cdot S$ shock waves from U_1 to the distant saddle become admissible.

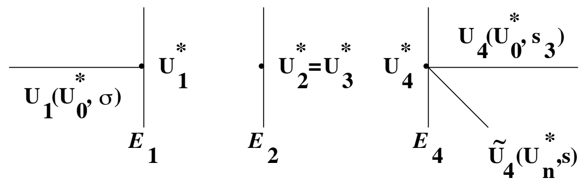

Figure 5: The curves $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_4$, and nearby wave curves.

7. Missing rarefaction in a long slow transitional wave group.

Theorem 7.1. *Let* (2.5) *be a Riemann problem solution of type* $(T_1, \dots, T_n)$ *that satisfies condition (M), and in addition assume* $T_{k+3} = RS \cdot RS$, *so* $T_{k+4} = R_1$. *Assume the following:*

(1) *The forward wave curve mapping* $U_k(U_0, \rho_k)$ *is regular at* (U_0^*, ρ_k^*).

(2) *In the case* $k = 1$, *the expression* $\mathcal{G}$ *defined below is nonzero.* (*In the case* $k > 1$, *an analogous expression must be nonzero.*)

(3) *In the case* $k = 1$, *the expression* $\mathcal{K}$ *defined below is nonzero.* (*In the case* $k > 1$, *an analogous expression must be nonzero.*)

Then (2.5) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(T_1, \dots, T_k, S \cdot RS, R_1, T_{k+5}, \dots, T_n)$ *because the connection of the* $S \cdot RS$ *shock develops an equilibrium of type* RS *that breaks it. Riemann solution* (2.5) (*and its equivalent*) *lie in a join that is a* U_L*-boundary. The join is regular* (*respectively folded*) *if* $\mathcal{H}\mathcal{K}\ell_1(U_{k+3}^*)(U_{k+3}^* - U_{k+2}^*)$ *is positive* (*respectively negative*), *where* $\mathcal{H}$ *is a nonzero expression defined below.*

Proof. As in Sections 5 and 6, we shall assume for simplicity that $k = 1$. Then (2.5) has a slow transitional wave group $U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^* \xrightarrow{s_4^*} U_4^* \xrightarrow{s_5^*} U_5^*$ with $T_2 = S \cdot RS$, $T_3 = R_1$, $T_4 = RS \cdot RS$, $T_5 = R_1$; it may be longer. We have $s_2^* = s_3^* = s_4^*$ and $U_2^* = U_3^*$.

Step 1. We note that $(U_0, s_1, \dots, s_4, U_4)$ near $(U_0^*, s_1^*, \dots, s_4^*, U_4^*)$ represents an admissible wave sequence of type $(T_1, S \cdot RS, R_1, RS \cdot RS)$ if and only if

$$U_1 - \eta(U_0, s_1) = 0, \tag{7.1}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{7.2}$$

$$\lambda_1(U_2) - s_2 = 0, \tag{7.3}$$

$$S(U_1, s_2) = 0, \tag{7.4}$$
$$U_3 - \psi(U_2, s_3) = 0, \tag{7.5}$$
$$s_3 - \lambda_1(U_2) \geq 0, \tag{7.6}$$
$$F(U_4) - F(U_3) - s_4(U_4 - U_3) = 0, \tag{7.7}$$
$$\lambda_1(U_3) - s_4 = 0, \tag{7.8}$$
$$\lambda_1(U_4) - s_4 = 0. \tag{7.9}$$

Here $\eta(U_0, s_1)$ and $S(U_1, s_2)$ are defined as in Section 5.

Let $G(U_0, s_1, \ldots, s_n, U_n)$ be the local defining map for wave sequences of type $(T_1, S \cdot RS, R_1, RS \cdot RS, R_1, T_6, \ldots, T_n)$ near $(U_0^*, s_1^*, \ldots, s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, \ldots, s_4, U_4)$ is given by the left-hand sides of equations (7.1)–(7.5) and (7.7)–(7.9), and $G_2(U_4, s_5, \ldots, s_n, U_n)$ is the local defining map for wave sequences of type $(R_1, T_6, \ldots, T_n)$. From the theory of reference [6],

$$DG_1(U_0^*, s_1^*, \ldots, s_4^*, U_4^*), \text{ restricted to } \left\{ (\dot{U}_0, \dot{s}_1, \ldots, \dot{s}_4, \dot{U}_4) \,:\, \dot{U}_0 = 0 \right\}, \text{ is an isomorphism,} \tag{7.10}$$

and

$$DG_2(U_4^*, s_5^*, \ldots, s_n^*, U_n^*), \text{ restricted to } \{ (\dot{U}_4, \dot{s}_5, \ldots, \dot{s}_n, \dot{U}_n) \,:\, \dot{U}_4 = \dot{U}_n = 0 \}, \text{ is an isomorphism.} \tag{7.11}$$

Therefore (A) holds. From (7.10), we can solve equations (7.1)–(7.5) and (7.7)–(7.9) for $(s_1, U_1, \ldots, s_4, U_4)$ in terms of U_0 near $(U_0^*, s_1^*, \ldots, s_4^*, U_4^*)$. Since a solution of $G = 0$ represents a Riemann solution of the desired type if and only if $s_3 - \lambda_1(U_2) = s_3 - s_2 \geq 0$, we now study $\tilde{H}(U_0) := s_3(U_0) - s_2(U_0)$. To verify (E2), we calculate $D\tilde{H}(U_0^*)\dot{U}_0$ by linearizing equations (7.1)–(7.5) and (7.7)–(7.9) at $(U_0^*, s_1^*, \ldots, s_4^*, U_4^*)$ and solving for $\dot{s}_3 - \dot{s}_2$ in terms of $\dot{U}_0$.

Linearizing equations (7.1)–(7.5) and (7.7)–(7.9) yields

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{7.12}$$
$$(DF(U_2^*) - s_2^* I)\dot{U}_2 - (DF(U_1^*) - s_2^* I)\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{7.13}$$
$$D\lambda_1(U_2^*)\dot{U}_2 - \dot{s}_2 = 0, \tag{7.14}$$
$$DS(U_1^*, s_2^*)(\dot{U}_1, \dot{s}_2) = 0, \tag{7.15}$$

$$\dot{U}_3 - D\psi(U_2^*, s_3^*)(\dot{U}_2, \dot{s}_3) = 0, \tag{7.16}$$

$$(DF(U_4^*) - s_4^* I)\dot{U}_4 - (DF(U_3^*) - s_4^* I)\dot{U}_3 - \dot{s}_4(U_4^* - U_3^*) = 0, \tag{7.17}$$

$$D\lambda_1(U_3^*)\dot{U}_3 - \dot{s}_4 = 0, \qquad D\lambda_1(U_4^*)\dot{U}_4 - \dot{s}_4 = 0.$$

In this system let

$$\dot{U}_0 = ir_1(U_0^*) + jr_2(U_0^*), \tag{7.18}$$

$$\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*), \tag{7.19}$$

$$\dot{U}_2 = cr_1(U_2^*) + dr_2(U_2^*), \tag{7.20}$$

$$\dot{U}_3 = er_1(U_3^*) + fr_2(U_3^*), \tag{7.21}$$

$$\dot{U}_4 = gr_1(U_4^*) + hr_2(U_4^*). \tag{7.22}$$

We multiply (7.12) by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$, (7.13) and (7.16) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$, and (7.17) by $\ell_1(U_4^*)$. We get

$$a - \ell_1(U_1^*)D\eta(U_0^*, s_1^*)(ir_1(U_0^*) + jr_2(U_0^*), \dot{s}_1) = 0, \tag{7.23}$$

$$b - \ell_2(U_1^*)D\eta(U_0^*, s_1^*)(ir_1(U_0^*) + jr_2(U_0^*), \dot{s}_1) = 0, \tag{7.24}$$

$$\begin{aligned} -a(\lambda_1(U_1^*) - s_2^*)\ell_1(U_2^*)r_1(U_1^*) - b(\lambda_2(U_1^*) - s_2^*)\ell_1(U_2^*)r_2(U_1^*) \\ -\dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*) = 0, \end{aligned} \tag{7.25}$$

$$\begin{aligned} (\lambda_2(U_2^*) - s_2^*)d - a(\lambda_1(U_1^*) - s_2^*)\ell_2(U_2^*)r_1(U_1^*) \\ -b(\lambda_2(U_1^*) - s_2^*)\ell_2(U_2^*)r_2(U_1^*) - \dot{s}_2\ell_2(U_2^*)(U_2^* - U_1^*) = 0, \end{aligned} \tag{7.26}$$

$$D\lambda_1(U_2^*)(cr_1(U_2^*) + dr_2(U_2^*)) - \dot{s}_2 = 0, \tag{7.27}$$

$$DS(U_1^*, s_2^*)(ar_1(U_1^*) + br_2(U_1^*), \dot{s}_2) = 0, \tag{7.28}$$

$$e - (\dot{s}_3 - dD\lambda_1(U_2^*)r_2(U_2^*)) = 0, \tag{7.29}$$

$$f - d = 0, \tag{7.30}$$

$$-f(\lambda_2(U_3^*) - s_4^*)\ell_1(U_4^*)r_2(U_3^*) - \dot{s}_4\ell_1(U_4^*)(U_4^* - U_3^*) = 0, \tag{7.31}$$

$$(\lambda_2(U_4^*) - s_4^*)h - f(\lambda_2(U_3^*) - s_4^*)\ell_2(U_4^*)r_2(U_3^*) - \dot{s}_4\ell_2(U_4^*)(U_4^* - U_3^*) = 0, \tag{7.32}$$

$$D\lambda_1(U_3^*)(er_1(U_3^*) + fr_2(U_3^*)) - \dot{s}_4 = 0, \tag{7.33}$$

$$D\lambda_1(U_4^*)(gr_1(U_4^*) + hr_2(U_4^*)) - \dot{s}_4 = 0. \tag{7.34}$$

Let

$$\eta_{ij} = \ell_i(U_1^*)D\eta(U_0^*, s_1^*)r_j(U_0^*), \quad i,j = 1,2,$$
$$\eta_{i3} = \ell_i(U_1^*)\frac{\partial \eta}{\partial s_1}(U_0^*, s_1^*), \quad i = 1,2.$$

Let $A, \dots, K$ have the same meaning as in the previous two sections. Let $S = \ell_1(U_4^*)r_1(U_3^*)$, $T = \ell_1(U_4^*)r_2(U_3^*)$, $U = \ell_1(U_4^*)(U_4^* - U_3^*)$, $V = \lambda_2(U_4^*) - s_4^*$, $W = \ell_2(U_4^*)r_1(U_3^*)$, $X = \ell_2(U_4^*)r_2(U_3^*)$, $Y = \ell_2(U_4^*)(U_4^* - U_3^*)$, $Z = D\lambda_1(U_4^*)r_2(U_4^*)$. Then (7.23)–(7.34) become

$$a - (\eta_{11}i + \eta_{12}j + \eta_{13}\dot{s}_1) = 0, \tag{7.35}$$
$$b - (\eta_{21}i + \eta_{22}j + \eta_{23}\dot{s}_1) = 0, \tag{7.36}$$
$$-Aa - Bb - C\dot{s}_2 = 0, \tag{7.37}$$
$$Dd - Ea - Fb - G\dot{s}_2 = 0, \tag{7.38}$$
$$c + Hd - \dot{s}_2 = 0, \tag{7.39}$$
$$Ia + Jb + K\dot{s}_2 = 0, \tag{7.40}$$
$$e + Hd - \dot{s}_3 = 0,$$
$$f - d = 0,$$
$$-DTf - U\dot{s}_4 = 0,$$
$$Vh - DXf - Y\dot{s}_4 = 0,$$
$$e + Hf - \dot{s}_4 = 0,$$
$$g + Zh - \dot{s}_4 = 0.$$

This system can be solved for $(\dot{s}_1, a, b, \dot{s}_2, c, d, \dot{s}_3, e, f, \dot{s}_4, g, h)$ in terms of (i,j). Let $\mathcal{G} = ITBG - ITCF + TKAF - TAGJ - TKBE + TJCE + IUB - UAJ$, $\mathcal{H} = (A\eta_{13} + B\eta_{23})K - (I\eta_{13} + J\eta_{23})C$. Then we find that

$$\begin{aligned} D\tilde{H}(U_0^*)(ir_1(U_0^*) + jr_2(U_0^*)) &= \dot{s}_3 - \dot{s}_2 \\ &= \frac{\mathcal{G}((\eta_{11}\eta_{23} - \eta_{13}\eta_{21})i + (\eta_{12}\eta_{23} - \eta_{13}\eta_{22})j)}{U\mathcal{H}}. \end{aligned} \tag{7.41}$$

Note that:

- $\mathcal{G} \neq 0$ by assumption (2).
- Since $D\eta(U_0^*, s_1^*)$ is surjective by assumption (1), either $\eta_{11}\eta_{23} - \eta_{13}\eta_{21}$ or $\eta_{12}\eta_{23} - \eta_{13}\eta_{22}$ is nonzero. Thus there exist (i,j) such that the determinant in the numerator of expression (7.41) is nonzero.

• In the denominator of expression (7.41), $U \neq 0$ by nondegeneracy condition (B2) for $RS \cdot RS$ shock waves.

• By (4.9), $\begin{pmatrix} A & B \\ I & J \end{pmatrix}\begin{pmatrix} \eta_1 3 \\ \eta_2 3 \end{pmatrix}$ and $\begin{pmatrix} C \\ K \end{pmatrix}$ are linearly independent. Thus $\mathcal{H} \neq 0$. Thus $D\tilde{H}(U_0^*)$ is a nonzero vector, so that (E2) holds. Therefore $\mathcal{C} = \{U_0 : \tilde{H}(U_0) = 0\}$ is a smooth curve near U_0^*, and for (U_0, U_n) near (U_0^*, U_n^*), a solution of type $(T_1, S \cdot RS, R_1, RS \cdot RS, R_1, T_6, \dots, T_n)$ exists provided U_0 is on the side of $\mathcal{C}$ to which this vector points.

Step 2. Next we consider the point $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*, s_5^*, U_5^*, s_6^* \dots, s_n^*, U_n^*)$ in $\mathbb{R}^{3n-4}$. We shall investigate the existence of nearby points $(U_0, s_1, U_1, s, U, s_5, U_5, s_6, \dots, s_n, U_n)$ that represent Riemann solutions of type $(T_1, S \cdot RS, R_1, T_6, \dots, T_n)$.

We first define $\gamma(U_0, s_1, s_2)$ as in Section 6. From (6.23) we obtain

$$D\gamma(U_0^*, s_1^*, s_2^*)(ir_1(U_0^*) + jr_2(U_0^*), \dot{s}_1, \dot{s}_2) \\ = -A(\eta_{11}i + \eta_{12}j + \eta_{13}\dot{s}_1) - B(\eta_{21}i + \eta_{22}j + \eta_{23}\dot{s}_1) - C\dot{s}_2. \quad (7.42)$$

Next we construct a separation function $\tilde{S}(U_1, s)$, (U_1, s) near (U_1^*, s_4^*), which can be used to study connections of

$$\dot{U} = F(U) - F(U_1) - s(U - U_1) \quad (7.43)$$

from U_1 to equilibria near U_4^*. As in the previous section, to study connections from U_1 to equilibria near U_2^*, we can define a vector V, a transversal Σ, invariant manifolds $W_\pm(U_1, s_2)$, points $\bar{U}_\pm(U_1, s_2)$ in Σ, and a separation function $S(U_1, s_2)$. If (7.43) has equilibria near U_2^* we define $\tilde{U}_+(U_1, s) = \bar{U}_+(U_1, s)$. If (7.43) has no equilibria near U_2^*, we note that the center manifold through U_4^* that is present when $(U_1, s) = (U_1^*, s_4^*)$ extends to a family of invariant manifolds $\tilde{W}_+(U_1, s)$, and we define $\tilde{U}_+(U_1, s)$ to be the intersection of this manifold with Σ. We then define $\bar{U}_-(U_1, s) - \tilde{U}_+(U_1, s) = \tilde{S}(U_1, s)V$. See Figure 6. $\tilde{S}(U_1, s)$ is smooth and agrees with $S(U_1, s)$ near (U_1^*, s_4^*) if $W_+(U_1, s)$ is chosen correctly.

We note that if (U_0, s_1, U_1, s, U) near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ represents an

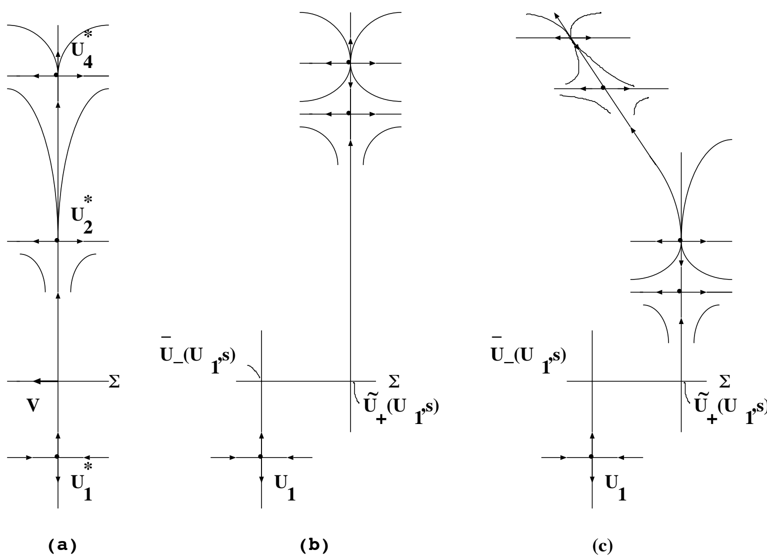

Figure 6: Geometry of the separation function in Section 7. (a) Phase portrait of $\dot{U} = F(U) - F(U_1^*) - s_4^*(U - U_1^*)$. (b) Phase portrait of $\dot{U} = F(U) - F(U_1) - s(U - U_1)$ for a value of (U_1, s) for which the equilibrium at U_2^* has disappeared, the equilibrium at U_4^* has split into a saddle and a repeller, and S is positive. (c) Phase portrait for a value of (U_1, s) for which the equilibria at U_2^* and U_4^* have both split into a saddle and a repeller, and for which S is positive.

admissible wave sequence of type $(T_1, S \cdot RS)$, then we must have

$$U_1 - \eta(U_0, s_1) = 0 \tag{7.44}$$

$$F(U) - F(U_1) - s(U - U_1) = 0 \tag{7.45}$$

$$\lambda_1(U) - s = 0 \tag{7.46}$$

$$\tilde{S}(U_1, s) = 0 \tag{7.47}$$

$$\gamma(U_0, s_1, s) \geq 0. \tag{7.48}$$

We shall see that (7.44)–(7.47) can be solved for (s_1, U_1, s, U) in terms of U_0 near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$; we denote the solution $(\hat{s}_1(U_0), \hat{U}_1(U_0), \hat{s}(U_0), \hat{U}(U_0))$. Once $U = \hat{U}(U_0)$ is found, the remainder of the Riemann solution is obtained by solving for $(s_5, U_5, \ldots, U_{n-1}, s_n)$ in terms of (U, U_n).

We have a Riemann solution of the desired type if and only if the function $\tilde{H}(U_0) := \gamma(U_0, \hat{s}_1(U_0), \hat{s}(U_0)) \geq 0$. To calculate $D\tilde{H}(U_0^*)\dot{U}_0$ with $\dot{U}_0 = ir_1(U_0^*) + jr_2(U_0^*)$, we linearize (7.44)–(7.47) at $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$, solve the linearized equations for $(\dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ in terms of $\dot{U}_0$, and substitute the formulas for $\dot{s}_1$ and $\dot{s}$ into (7.42).

The linearization of (7.44)–(7.47) at $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$ is

$$\dot{U}_1 - D\eta(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{7.49}$$

$$(DF(U_4^*) - s_4^* I)\dot{U}_2 - (DF(U_1^*) - s_4^* I)\dot{U}_1 - \dot{s}(U_4^* - U_1^*) = 0, \tag{7.50}$$

$$D\lambda_1(U_4^*)\dot{U} - \dot{s} = 0, \tag{7.51}$$

$$D\tilde{S}(U_1^*, s_4^*)(\dot{U}_1, \dot{s}) = 0. \tag{7.52}$$

We make the substitutions (7.18)–(7.20) and

$$\dot{U} = g r_1(U_4^*) + h r_2(U_4^*).$$

We multiply (7.49) by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$, and (7.50) by $\ell_1(U_4^*)$ and $\ell_2(U_4^*)$. Then (7.49)–(7.52) become (7.35)–(7.36) and

$$\begin{aligned}
&- a(\lambda_1(U_1^*) - s_4^*)\ell_1(U_4^*) r_1(U_1^*) - b(\lambda_2(U_1^*) - s_4^*)\ell_1(U_4^*) r_2(U_1^*) \\
&- \dot{s}\ell_1(U_4^*)(U_4^* - U_1^*) = 0,
\end{aligned} \tag{7.53}$$

$$\begin{aligned}
&(\lambda_2(U_4^*) - s_4^*)h - a(\lambda_1(U_1^*) - s_4^*)\ell_2(U_4^*) r_1(U_1^*) \\
&- b(\lambda_2(U_1^*) - s_4^*)\ell_2(U_4^*) r_2(U_1^*) - \dot{s}\ell_2(U_4^*)(U_4^* - U_1^*) = 0,
\end{aligned} \tag{7.54}$$

$$D\lambda_1(U_4^*)(g r_1(U_4^*) + h r_2(U_4^*)) - \dot{s} = 0, \tag{7.55}$$

$$D\tilde{S}(U_1^*, s_4^*)(a r_1(U_1^*) + b r_2(U_1^*), \dot{s}) = 0. \tag{7.56}$$

We calculate

$$\begin{aligned}
&(\lambda_1(U_1^*) - s_4^*)\ell_1(U_4^*) r_1(U_1^*) = (\lambda_1(U_1^*) - s_4^*)\ell_1(U_4^*)(\ell_1(U_2^*) r_1(U_1^*) \\
&\quad \cdot r_1(U_2^*) + \ell_2(U_2^*) r_1(U_1^*) \cdot r_2(U_2^*)) = AS + ET, \\
&(\lambda_2(U_1^*) - s_4^*)\ell_1(U_4^*) r_2(U_1^*) = (\lambda_2(U_1^*) - s_4^*)\ell_1(U_4^*)(\ell_1(U_2^*) r_2(U_1^*) \\
&\quad \cdot r_1(U_2^*) + \ell_2(U_2^*) r_2(U_1^*) \cdot r_2(U_2^*)) = BS + FT, \\
&\ell_1(U_4^*)(U_4^* - U_1^*) = \ell_1(U_4^*)(U_4^* - U_3^*) + \ell_1(U_4^*)(U_2^* - U_1^*) \\
&= U + \ell_1(U_4^*)\{\ell_1(U_2^*)(U_2^* - U_1^*) r_1(U_2^*) + \ell_2(U_2^*)(U_2^* - U_1^*) r_2(U_2^*)\} \\
&= U + CS + GT.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(\lambda_1(U_1^*) - s_4^*)\ell_2(U_4^*) r_1(U_1^*) &= AW + EX, \\
(\lambda_2(U_1^*) - s_4^*)\ell_2(U_4^*) r_2(U_1^*) &= BW + FX, \\
\ell_2(U_4^*)(U_4^* - U_1^*) &= Y + CW + GX.
\end{aligned}$$

Then (7.49)–(7.52) become

$$a-(\eta_{11}i+\eta_{12}j+\eta_{13}\dot{s}_1)=0, \tag{7.57}$$
$$b-(\eta_{21}i+\eta_{22}j+\eta_{23}\dot{s}_1)=0, \tag{7.58}$$
$$-(AS+ET)a-(BS+FT)b-(U+CS+GT)\dot{s}=0, \tag{7.59}$$
$$Vh-(AW+EX)a-(BW+FX)b-(Y+CW+GX)\dot{s}=0, \tag{7.60}$$
$$g+Zh-\dot{s}=0, \tag{7.61}$$
$$Ia+Jb+K\dot{s}=0. \tag{7.62}$$

Let

$$\mathcal{K}=((AS+ET)\eta_{13}+(BS+FT)\eta_{23})K+(I\eta_{13}+J\eta_{23})(U+CS+GT).$$

By assumption (3), $\mathcal{K}\neq 0$, so we can solve (7.57)–(7.62) for $(\dot{s}_1,a,b,\dot{s},g,h)$ in terms of (i,j). We substitute the formulas for $\dot{s}_1$ and $\dot{s}$ into (7.42). We find that

$$D\tilde{H}(U_0^*)(ir_1(U_0^*)+jr_2(U_0^*))=-\frac{\mathcal{G}((\eta_{11}\eta_{23}-\eta_{13}\eta_{21})i+(\eta_{12}\eta_{23}-\eta_{13}\eta_{22})j)}{\mathcal{K}}. \tag{7.63}$$

Since $\mathcal{G}\neq 0$, $\mathcal{K}\neq 0$, and $D\eta(U_0^*,s_1^*)$ is surjective, there exist (i,j) such that this expression is nonzero.

Thus $D\tilde{H}(U_0^*)$ is a nonzero vector, so that (E2) holds. Therefore $\mathcal{C}=\{U_0:\tilde{H}(U_0)=0\}$ is a smooth curve near U_0^*, and for (U_0,U_n) near (U_0^*,U_n^*), a solution of type $(T_1,S\cdot RS,R_1,T_6,\dots,T_n)$ exists provided U_0 is on the side of $\mathcal{C}$ to which this vector points.

Step 3. It is easy to see that the curves $\mathcal{C}$ defined in step 1 and step 2 coincide. We obtain the final conclusion of the theorem by comparing (7.41) and (7.63).

Remark 7.2. Assumption (2) has the following geometric interpretation. The triplets (U_1,s_2,U_2) such that there is a shock wave of type $S\cdot RS$ from U_1 to U_2 with speed s_2 form a curve $\mathcal{E}$ through (U_1^*,s_2^*,U_2^*): the solutions of (7.2)–(7.4). Similarly, the triplets (U_3,s_4,U_4) such that there is a shock wave of type $RS\cdot RS$ from U_3 to U_4 with speed s_4 form a curve $\tilde{\mathcal{E}}$ through (U_3^*,s_4^*,U_4^*): the solutions of (7.7)–(7.9). The curve $\mathcal{E}$ projects to a curve $\mathcal{E}_2$ through U_2^*; the curve $\tilde{\mathcal{E}}$ projects to a curve $\mathcal{E}_3$ through U_3^*. Assumption

(2) says that $\mathcal{E}_2$ and $\mathcal{E}_3$ meet transversally at $U_2^* = U_3^*$. This is a natural geometric requirement for codimension-one Riemann solutions of the desired type. It is used in both steps 1 and 2 of the proof.

Remark 7.3. Assumption (3) can be reformulated as follows. Let $(w_1, \dots, w_k, \tilde{w}_{k+3}, w_{k+4}, \dots, w_n) : U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \xrightarrow{s_{k+3}^*} U_{k+3}^* \xrightarrow{s_{k+4}^*} \cdots \xrightarrow{s_n^*} U_n^*$ denote the wave sequence equivalent to (2.5), in which the subsequence $(w_{k+1}, w_{k+2}, w_{k+3})$ has been replaced by a single generalized shock wave $\tilde{w}_{k+3}$ of type $S \cdot RS$ from U_k^* to U_{k+3}^*. (The wave is a generalized shock wave because it is represented by a sequence of two connecting orbits.) Then assumption (3) says that $\tilde{w}_{k+3}$ satisfies (4.9).

Remark 7.4. If the Lax admissibility criterion is used, then, in the variation of Figure 6(c) for which $S = 0$, the $S \cdot RS$ shock wave from U_1 to the distant saddle becomes admissible.

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