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ON THE BOUNDEDNESS AND DECAY OF MOMENTS OF SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. In this paper we consider the existence and decay of moments of the solutions to the Navier-Stokes equations in the whole space $\mathbf{R}^n$; $n \geq 2$ for existence, $2 \leq n \leq 5$ for decay. The decay obtained is algebraic, of order

$$\int_{\mathbf{R}^n} |x|^k |u|^2\, dx \leq C(t+1)^{-2\mu(1-k/n)}$$

for $0 \leq k \leq n$, for solutions u of appropriate data for which the L^2 norm decays at a rate of order μ. That is for solutions that satisfy $||u(t)||_2 \leq C(t+1)^{-\mu}$. Where $\mu > 1/2$. Such solutions are easy to obtain as for example it suffices for $n = 3$ that the data u_0 is in $L^2 \cap L^1$.

1. Introduction and Notation. We consider the existence and decay of moments of the solutions to the Navier-Stokes equations in the whole space $\mathbf{R}^n$,

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= 0, \\ \operatorname{div} u &= 0, \\ u(\cdot, 0) &= u_0 \in \mathbf{X}. \end{aligned}$$

To study the decay we assume $2 \leq n \leq 5$. The existence is for $n \geq 2$. Here $\mathbf{X}$, to be described below, will be chosen as an appropriate space which

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will insure the existence of the moments. The paper can be divided into two main parts: I. Existence of solutions which have locally finite (and bounded) moments. II. Global boundedness and decay of the moments. To make the paper self-contained we included at the end two appendixes. In Appendix A the more technical parts of existence are described. Appendix B presents some auxiliary estimates necessary to obtain global bounds on the moment

To establish the existence we linearize in explicit form both the convective and the pressure term. To this purpose pressure is expressed in terms of product of Riesz transforms. A sequence $\{u^\ell\}$ of approximating solutions is constructed by letting $v = u^{\ell+1}$ satisfy

$$\begin{aligned} (\mathrm{NS})_\ell \quad v_t - \Delta v + u^\ell \cdot \nabla v + \nabla P(u^\ell, v) &= 0, \\ \operatorname{div} v &= 0, \\ v(0) &= u_0 \end{aligned}$$

with u_0 in an appropriate space. The bilinear operator P is defined by

$$P(u,v) = \sum_{j,k} R_j R_k (u_j v_k)$$

if $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ are functions from $\mathbf{R}^n$ to $\mathbf{R}^n$ and where R_j denotes the Riesz transforms,

$$[R_j f]\hat{}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi)$$

for $j = 1, \dots, n$. When $u^\ell = v$, we recuperate the Navier-Stokes equations since for Navier-Stokes

$$\Delta p = -\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k} (u_j u_k),$$

hence

$$\hat{p}(\xi, t) = -\sum_{j,k} \frac{\xi_j \xi_k}{|\xi|^2} \widehat{u_j u_k}(\xi, t),$$

thus

$$p = \sum_{j,k} R_j R_k (u_j u_k) = P(u,u).$$

The $(NS)_\ell$ linearization is of the type used by Caffarelli, Kohn and Nirenberg in [2], by Kajikiya and Miyakawa in [7], by Leray in [11], and by Sohr, von Wahl and Wiegner in [18]. The advantage of making the linearization explicit is that for the approximations we can use well known properties of the Riesz transforms, such as their boundedness in L^p-spaces (cf. [19]) and in weighted L^p spaces satisfying the Muckenhoupt condition (cf. [6], [20]) to obtain bounds for the solutions of the Navier-Stokes equations and their moments. We expect that our proofs to establish bounds in weighted L^p-spaces, with some modifications, could be used for the approximating solutions constructed by Caffarelli, Kohn and Nirenberg [2], Kajikiya and Miyakawa [7], Sohr, von Wahl and Wiegner [18]. Moreover we point out that the construction we use for the solutions is in some sense similar to the construction used by Kato [8]. The main difference is that we also obtain bounds for the solutions in weighted L^p spaces, provided the data is in such a space. In addition, our proof of existence makes explicit how to obtain bounds for moments of the solutions and moments of the derivatives of higher order (see Lemma 2.2). The study of the decay of the moments of the solutions is not only interesting for its own sake, but it is also a preliminary step to understanding pointwise decay in space and time of solutions to the Navier-Stokes equations. This decay will be described in a forthcoming paper [1].

Different aspects of moments of solutions to the Navier-Stokes equations have been studied by several authors, among them: Caffarelli, Kohn and Nirenberg [2], Konstantin Pileckas [12], H. Sohr and Maria Specovius-Neugebauer [13], and Reinhard Farwig [4]. Questions of decay of solutions to the Navier-Stokes equations in different norms have been studied by, among others, R. Kajikiya and T. Miyakawa [7], T. Kato [8], H. Kozono [9], H. Kozono and T. Ogawa [10], M.E. Schonbek [14], [16], M. Wiegner [22], and Zhang-Linghai [23]. In particular we would like to mention some very interesting results by Takahashi [21]. In order to get his estimates, Takahashi, uses a weighted equation approach. Specifically, in [21] Takahashi gets pointwise decay rates both in time and space for the solutions bounded in some weighted $L^{q,s}$ norms, where $L^{q,s}$ stands for the space time norm

$$\{\int_0^\infty (\int_{\mathbf{R}^n} |u(x,t)|^q dx)^{s/q} dt\}^{1/s}.$$

The organization of this paper is as follows: First we show that "approximating solutions" for data in adequate spaces exist locally in time and belong to L^r and to weighted $L^{r_1}_\nu$ for appropriate r, r_1, and ν. We define here the space $L^{r_1}_\nu$ by

$$f \in L^{r_1}_\nu \quad \text{iff} \quad \Big(\int_{\mathbf{R}^n} |x|^{\nu r_1} |f|^{r_1}\, dx \Big)^{1/r_1} < \infty.$$

For small data it will be shown that $T = \infty$. To obtain the bounds for the weak solutions we pass to the limit. These results in particular hold for $r_1 = 2$ and $\nu \in [0, /2n)$ and as a consequence, the moments

$$M_k(u)(t) = \int_{\mathbf{R}^n} |x|^k |u|^2\, dx$$

are bounded for $0 \le k < n$ provided the data is small. For the derivatives of the solution we show that they remain in in a weighted $L^{r_1}_\nu$ space. Using energy methods we can then prove that the n-th moment is locally bounded. Once more by energy methods, the decay of the moments is deduced. In the case of small initial data of mean zero it is shown that

$$M_k(u)(t) \le C(t+1)^{-(n-k)/2}$$

for all $t \ge 0$, $k = 0, \dots, n$ $(2 \le n \le 5)$.

A few words on notation may be in order. If f is a function from $\mathbf{R}^n \times \mathbf{R}$ to some set Z, we identify it with the function (also denoted by f) from $\mathbf{R}$ to the set of Z-valued functions on $\mathbf{R}^n$ by $f(t)(x) = f(x,t)$; if f is defined on $\mathbf{R}^n \times \mathbf{R}$ operators such as the Riesz transforms, Fourier transform, ∇, div, Δ, etc., operate on the $\mathbf{R}^n$ (space) variables for each fixed value of the $\mathbf{R}$ (time) variable; for example $R_j f(x,t) = [R_j f(t)](x)$, etc. This also holds for the operator P just defined; if u, v depend on x and t, then $P(u,v)(t) = P(u(t), v(t))$. In general, if $\mathcal{X}$ is a Banach space, we write $\|\cdot\|_{\mathcal{X}}$ to denote the norm in $\mathcal{X}$, except if $\mathcal{X} = L^p$, $1 \le p \le \infty$. In this latter case, we write $\|\cdot\|_p$ for the norm. If m is a non-negative integer, we denote by $W^{m,q}$ the usual Sobolev space; $u \in W^{m,q}(\mathbf{R}^n)$ if $D^\alpha u \in L^q$ for $|\alpha| \le m$,

$$\|u\|_{W^{m,q}} = \Big(\sum_{|\alpha| \le m} \|D^\alpha u\|_q^q \Big)^{1/q}.$$

We write $L^p(\mathbf{R}^n)^n$, $W^{m,p}(\mathbf{R}^n)^n$, etc., to denote the space of all vector valued functions $u = (u_1, \dots, u_n)$ such that $u_k \in L^p(\mathbf{R}^n)$ or $u_k \in W^{m,p}(\mathbf{R}^n)$, etc., respectively, for all $k = 1, \dots, n$. We use the same symbol to denote the norm in these vector valued spaces as we do in the component spaces and will occasionally just write L^p, $W^{m,p}$, etc. for for vector valued maps having all their components, respectively, in $L^p(\mathbf{R}^n)$, $W^{m,p}(\mathbf{R}^n)$, etc. As usual, we write $H^m = W^{m,2}$.

2. Admissible solutions. We consider the following method for solving the initial value problem for the Navier-Stokes equations. Define a sequence of functions $\{u_\ell\}$, with $u_\ell = (u_{\ell,1}, \dots, u_{\ell,n}) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ for $\ell = 0, 1, \dots$ inductively by setting $u_0(x,t) \equiv u_0(x)$, the initial value to be satisfied by the solution; assuming u_ℓ defined for some $\ell \geq 0$, define $u_{\ell+1}$ as the solution v of the initial value problem

$$\begin{cases} v_t - \Delta v + u_\ell \cdot \nabla v + \nabla P(u_\ell, v) &= 0\,, \\ v(0) &= u_0\,. \end{cases}$$

Definition. We say u is an *admissible* solution in the interval $[0,T]$ of the Navier-Stokes equations with initial datum u_0, if

$$u \in C([0,T], L^2(\mathbf{R}^n)^n) \cap L^2([0,T], H^1(\mathbf{R}^n)^n)$$

and if the sequence $\{u_\ell\}$ converges to u in $C([0,T], L^2(\mathbf{R}^n))$. The sequence $\{u_\ell\}$ shall be called the *defining sequence* of the admissible solution u.

We prove that for $u_0 \in L^2 \cap L^r$ with $\operatorname{div} u_0 = 0$, where $n < r \leq \infty$, there exists $T > 0$ depending on the L^r norm of u_0 such that the sequence converges in $C([0,T], L^2 \cap L^r)$ and the limit u is a weak solution of the Navier Stokes equations satisfying the energy inequality (Theorem 2.4). We also prove that there is $\delta > 0$ such that if $\|u_0\|_r \leq \delta$, then $T = \infty$.

2.1. The linearized equations. Let $T > 0$. Assume $u = (u_1, \dots, u_n) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ satisfies $u \in L^2([0,T], L^r(\mathbf{R}^n))$, for some r, $n < r \leq \infty$ and

$$\operatorname{div} u = 0.$$

We consider the following initial value problem:

$$v_t - \Delta v + u \cdot \nabla v + \nabla P(u, v) = 0, \tag{1}$$

$$v(0) = v_0, \tag{2}$$

where $v_0 = (v_{01}, \dots, v_{0n}) \in L^2(\mathbf{R}^n)^n$ with $\operatorname{div} v_0 = 0$. We call equation (1) the *Navier Stokes equations linearized with respect to* u, briefly equation (LNS) with respect to u. Let $\mathcal{X}$ be a Banach space. We denote by $C_T(\mathcal{X})$ the Banach space of all continuous functions from $[0,T]$ to $\mathcal{X}$ with the norm

$\|f\|_{C_T(\mathcal{X})} = \sup_{0\le t\le T} \|f(t)\|_{\mathcal{X}}$. If $\mathcal{X} = H^m(\mathbf{R}^n)^n$, we write $\|\cdot\|_{C_T(H^m)}$ for the norm. We use similar conventions in the case of $\mathcal{X} = L^p(\mathbf{R}^n)^n$, $\mathcal{X} = W^{m,p}(\mathbf{R}^n)^n$, etc. Let $\mathcal{X}$ be a Banach space of measurable, locally integrable, $\mathbf{R}^n$-valued functions. We shall say that a mapping $v = (v_1, \dots, v_n) : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is a weak solution of (1), (2) in the interval $[0, T]$ with values in $\mathcal{X}$ if $v \in C_T(\mathcal{X}) = C([0, T], \mathcal{X})$, $v(0) = v_0$, and for every $\phi = (\phi_1, \dots, \phi_n) \in C_0^\infty(\mathbf{R}^n)^n$ the map $t \to (v(t), \phi) : [0, T] \to \mathbf{C}$ is absolutely continuous with

$$\frac{d}{dt}(v(t), \phi) = (v(t), \Delta\phi) + (v(t), u(t) \cdot \nabla\phi) + (P(u, v)(t), \operatorname{div}\phi).$$

We now state an existence theorem (Theorem 2.1) of strong solutions for (1), provided that the initial data and u are sufficiently smooth. For the sake of completeness we include a proof in Appendix A of this paper. We note that by the new uniqueness results of [5] and [3], the solutions we construct as limits of our approximating equations will coincide locally, if the data is large, and globally, if the data is sufficiently small, with the solutions constructed by Kato in [8], which is a consequence of the interpolation of L^3 between L^2 and L^r for $r > n$. The reason for giving our construction instead of using the Kato method directly is that for our approximating solutions we can obtain bounds in weighted spaces L^s_ν for appropriate s and ν. It is here where we need that $r > n$ (see Lemma 2.2). We expect that this probably is a technical difficulty that could be overcome by other methods than ours and it should be sufficient to require $r \ge n$. Our proof follows essentially the steps that Kato is using with the main difference that Kato works in the projection onto divergence fields and thus needs to estimate the Stokes operator. We do not project our solutions hence we work directly with the heat operator and our pressure term is expressed explicitly in terms of Riesz operators. This gives us the advantage, as mentioned above, of using the boundedness of the Riesz operators in the weighted spaces L^s_ν. These bounds are the ones we need when we pass to the limit for our solutions to Navier-Stokes. We believe that, in effect, the process we have done would also lead to the bounds of the Stokes operator. Once this is accomplished one could use these bounds to estimate directly the Kato solutions.

Specifically, in appendix A we give a method to construct solutions to the approximating equations described above which converge weakly to solutions of the Navier-Stokes equations. The construction is such that it takes advantage of the form of the pressure and the convective terms; that is, these terms can be written as

$$u\nabla v + \nabla P(u,v) = \nabla[I(u,v) + R_iR_j(u,v)]$$

where R_i are the Riesz operators and I is the identity. We can thus use properties of the operator $I + R_iR_j$, such as it being bounded in L^r_ν. These bounds will yield time dependent bounds for the solutions in L^r_ν for appropriate r and ν, from where uniform bounds for the moments up to order n will follow.

Theorem 2.1. *Let $m \geq 0$ be an integer; let $u \in C_T(W^{m,r}(\mathbf{R}^n)^n)$ and let $v_0 \in L^2 \cap W^{m,q} \cap W^{m,r}(\mathbf{R}^n)^n$, where $n < r < \infty$, $1 < q \leq r$. There exists a unique $v \in C_T(W^{m,q}) \cap C_T(W^{m,r})$ solving (1), (2). If $div\, v_0 = 0$, then $div\, v = 0$, i.e., $div\, v(t) = 0$ for all $t \in [0,T]$, and*

$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(s)\|_2^2\, ds \leq \|v_0\|_2^2 \tag{3}$$

for all $t \in [0,T]$. The solution v is strong and in $C^1([0,T], L^2)$ if $m \geq 3$.

The proof is by a fixed point argument and is accomplished in a series of Lemmas (see appendix A). However, some of the notation required in its proof will be used throughout the paper, thus we record it here and in the appendix.

Let $F(x,t) = F(t)(x) = (4\pi t)^{-n/2}e^{-|x|^2/4t}$ be the fundamental solution of the heat equation in n space variables. If v solves (1), (2), setting $H(u,v) = u\cdot\nabla v + \nabla P(u,v)$, then

$$v(t) = F(t) * v_0 - \int_0^t F(t-s) * H(u,v)(s)\, ds. \tag{4}$$

Let $\varphi \in L^2([0,T], H^1(\mathbf{R}^n, \mathbf{R}^n))$, set

$$\begin{aligned} \mathcal{M}\varphi(t) &= \int_0^t F(t-s) * [u\cdot\nabla\varphi(s) + \nabla P(u,\varphi)(s)]\; ds \qquad (5)\\ &= \int_0^t F(t-s) * H(u,\varphi)(s)\, ds,\\ \mathcal{L}\varphi(t) &= F(t) * v_0 - \mathcal{M}\varphi(t). \qquad (6) \end{aligned}$$

The integral version (4) of the (LNS) with respect to u becomes $v = \mathcal{L}v$; and the solution to the linearized Navier-Stokes is obtained as a fixed point of the operator $\mathcal{L}v$.

For the purpose of the next lemma we introduce real numbers ν, q, r and r_1 which satisfy the following relations:

$$0 \leq \nu < n, \quad 2 \leq r_1 \leq r, \quad 1 \leq q \leq \infty, \tag{7}$$

$$\frac{1}{q} < \frac{\nu}{2} - \frac{n}{2r} + \frac{1}{2}, \quad \frac{1}{r} \leq \frac{1}{r_1} + \frac{\nu}{n} < 1 - \frac{1}{r}; \tag{8}$$

Define σ by

$$\frac{1}{\sigma} = \frac{1}{2}(\frac{1}{r} + \frac{1}{r_1}). \tag{9}$$

The definition implies σ is between r and r_1.

The next Lemma is essential for the results in this paper. It establishes that the solutions and derivatives to the linearized Navier-Stokes equations are bounded in the weighted spaces L^r_ν for appropriate r and ν. It is here that we use heavily the boundedness properties of the Riesz operators in the weighted spaces L^r_ν.

Lemma 2.2. *Assume* (7)-(9) *and assume the function u satisfies*

$$u \in C([0,T], W^{m,r}(\mathbf{R}^n)^n) \cap L^q([0,T], (W^{m,r_1}(\mathbf{R}^n)^n).$$

There exists a constant $K(T,u)$ of the form

$$K(T,u) = C(T)\left(\|u\|_{C_T(W^{m,r})} + \|u\|_{L^q_T(W^{m,r_1})}\right) \tag{10}$$

with $C(T)$ independent of u such that if $D^\alpha v_0 \in L^{r_1}_\nu \cap L^r(\mathbf{R}^n)^n$ for $|\alpha| \leq m$, then the fixed point v of $\mathcal{L}$ satisfies $D^\alpha v \in C([0,T], L^{r_1}_\nu(\mathbf{R}^n))$ for $|\alpha| \leq m$ and

$$\begin{aligned}\|D^\alpha v(t)\|_{L^{r_1}_\nu} \quad \leq \quad & C(T)\Big(\|v_0\|_{W^{m,r_1}} + \sum_{|\beta|\leq m} \|D^\beta v_0\|_{L^{r_1}_\nu}\Big) \\ & +K(T,u)\left(\|v_0\|_{W^{m,r}} + \|v_0\|_{W^{m,r_1}}\right).\end{aligned}$$

Proof. In this proof $C(T)$ denotes a constant depending only on T, n and the exponents r, r_1, ν, q; $K(T,u)$ a constant of the form (10). These constants may differ from formula to formula. Let us write

$$\|g\|_{W^{m.r}_\nu} = \sum_{|\alpha|\leq m} \|D^\alpha g\|_{L^r_\nu}.$$

If $f : \mathbf{R}^n \to \mathbf{R}$, we define $f_{[\nu]}(x) = |x|^\nu |f(x)|$. If $f, g : \mathbf{R}^n \to \mathbf{R}$, then

$$|x|^\nu \, |f * g(x)| \le c_\nu (f_{[\nu]} * |g|(x) + |f| * g_{[\nu]}(x)) \tag{11}$$

where $c_\nu = 2^{\nu-1}$ if $\nu \ge 1$, $c_\nu = 1$ if $0 \le \nu < 1$. Recall that $\mathcal{L} = F(t) * v_0 - \mathcal{M}$ where

$$\mathcal{M}\varphi(t) = \int_0^t F(t-s) * H(u,\varphi)(s)\, ds.$$

As in (55) write

$$[F(t) * H(u,\varphi)(s)]_k = \sum_{j=1}^n F_j(t) * Q_{kj}(u,\varphi)(s),$$

with Q_{kj} given by (56) (as in Appendix A), that is,

$$Q_{kj}(u,\varphi)(s) = u_j(s)\varphi_k(s) + \delta_{jk} \sum_{\mu,\nu} (R_\mu R_\nu (u_\mu(s)\varphi_\nu(s)). \tag{12}$$

Thus,

$$[D^\alpha F(t) * H(u,\varphi)(s)]_k = \sum_{j=1}^n F_j(t) * \tilde{Q}_{kj\alpha}(u,\varphi)(s)$$

where

$$\begin{aligned} &\tilde{Q}_{kj\alpha}(u,\varphi)(s) = D^\alpha Q_{kj}(u,\varphi)(s) \\ &= \sum_{\beta+\gamma=\alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Big\{ D^\beta u_j(s) D^\gamma \varphi_k(s) + \delta_{jk} \sum_{\mu,\nu} (R_\mu R_\nu (D^\beta u_\mu(s) D^\gamma \varphi_\nu(s)) \Big\}. \end{aligned}$$

Multiplying by $|x|^\nu$ we get, by (11)

$$\begin{aligned} &|x|^\nu \, |[D^\alpha F(t) * H(u,\varphi)(s)]_k| \\ &\le c_\nu \Big(\sum_{j=1}^n [F_{j,[\nu]}(t)] * |\tilde{Q}_{kj\alpha}(u,\varphi)(s)| + \sum_{j=1}^n |F_j(t)| * \tilde{Q}_{kj\alpha,[\nu]}(u,\varphi)(s) \Big). \end{aligned} \tag{13}$$

We take L^{r_1} norm. Since $1/r_1 = 2/\sigma - 1/r = 1/r' + 2/\sigma - 1$ and since $\|F_{j,[\nu]}(t)\|_{r'} = Ct^{\nu/2 - n/2r - 1/2}$, the terms in the first sum on the right hand side of (13) can be estimated using Young's inequality by

$$\|[F_{j,[\nu]}(t)] * \tilde{Q}_{kj\alpha}(u,\varphi)(s)\|_{r_1} \le Ct^{\frac{\nu}{2} - \frac{n}{2r} - \frac{1}{2}} \|\tilde{Q}_{kj\alpha}(u,\varphi)(s)\|_{\sigma/2}.$$

It is at this point that we need $r > n \geq 2$, $r_1 > 2$ so that $\sigma > 2$ and thus Riesz transforms are bounded in $L^{\sigma/2}$. Since $\tilde{Q}_{kj\alpha}(u,\varphi)$ consists of the identity operator and products of Riesz transforms applied to products $D^\beta u D^\gamma \varphi$ with $|\beta| + |\gamma| = |\alpha| \leq m$, we get

$$\|[F_{j,[\nu]}(t)] * \tilde{Q}_{kj\alpha}(u,\varphi)(s)\|_{r_1} \leq Ct^{\frac{\nu}{2}-\frac{n}{2r}-\frac{1}{2}}\|u(s)\|_{W^{m,\sigma}}\|\varphi(s)\|_{W^{m,\sigma}}.$$

Since $1/\sigma = (1/2)(1/r + 1/r_1)$ Hölder yields

$$\begin{aligned}&\|[F_{j,[\nu]}(t)] * \tilde{Q}_{kj\alpha}(u,\varphi)(s)\|_{r_1}\\ &\leq Ct^{\frac{\nu}{2}-\frac{n}{2r}-\frac{1}{2}}\|u(s)\|^{1/2}_{W^{m,r}}\|\varphi(s)\|^{1/2}_{W^{m,r}}\|u(s)\|^{1/2}_{W^{m,r_1}}\|\varphi(s)\|^{1/2}_{W^{m,r_1}}.\end{aligned}$$

To estimate the terms in the second sum on the right hand side of (13) we use that $\|F_j(t)\|_{r'} = Ct^{-n/2r-1/2}$ to get by Young's inequality

$$\|F_j(t) * \tilde{Q}_{kj\alpha,[\nu]}(u,\varphi)(s)\|_{r_1} \leq Ct^{-\frac{n}{2r}-\frac{1}{2}}\|\tilde{Q}_{kj\alpha,[\nu]}(u,\varphi)(s)\|_{\sigma/2}.$$

We use again that $\tilde{Q}_{kj\alpha}(u,\varphi)(s)$ is a sum of terms, each of which is of the form $S(D^\beta u D^\gamma \varphi)(s)$, with S either a product of Riesz transforms or the identity operator. Since $\sigma/2 > n/(n-\nu)$, Lemma 5.1 (see Appendix A) implies that all these S operators are bounded in $L^{\sigma/2}_\nu$, hence

$$\begin{aligned}\|\tilde{Q}_{kj\alpha,[\nu]}(u,\varphi)(s)\|_{\sigma/2} &\leq C\sum_{\beta+\gamma=\alpha}\|D^\beta u_k D^\gamma \varphi_\ell\|_{L^{\sigma/2}_\nu}\\ &\leq C\|u(s)\|_{W^{m,r}}\|\varphi(s)\|_{W^{m,r_1}_\nu},\end{aligned}$$

where we used $2/\sigma = 1/r + 1/r_1$ to estimate $\|fg\|_{\sigma/2} \leq \|f\|_r\|g\|_{r_1}$. Thus combining the last inequalities,

$$\begin{aligned}&\|[D^\alpha F(t) * H(u,\varphi)(s)]_k\|_{L^{r_1}_\nu}\\ &\leq Ct^{\frac{\nu}{2}-\frac{n}{2r}-\frac{1}{2}}\|u(s)\|^{1/2}_{W^{m,r}}\|\varphi(s)\|^{1/2}_{W^{m,r}}\|u(s)\|^{1/2}_{W^{m,r_1}}\|\varphi(s)\|^{1/2}_{W^{m,r_1}}\\ &+ Ct^{-\frac{n}{2r}-\frac{1}{2}}\|u(s)\|_{W^{m,r}}\|\varphi(s)\|_{W^{m,r_1}_\nu}.\end{aligned}$$

Replacing t by $t-s$, integrating from 0 to t, we get with $\lambda = \nu/2 - n/(2r) - 1/2$ and recalling that

$$\mathcal{M}\varphi(t) = \int_0^t F(t-s) * H(u,\varphi)(s)\,ds,$$

it follows that

$$\begin{aligned}&\|D^\alpha \mathcal{M}\varphi(t)\|_{L^{r_1}_\nu}\\&\le C\|u\|^{1/2}_{C_T(W^{m,r})}\|\varphi\|^{1/2}_{C_T(W^{m,r})}\int_0^t (t-s)^\lambda \|u(s)\|^{1/2}_{W^{m,r_1}}\|\varphi(s)\|^{1/2}_{W^{m,r_1}}\,ds\\&+C\|u\|_{C_T(W^{m,r})}\int_0^t (t-s)^{-\frac{n}{2r}-\frac12}\|\varphi(s)\|_{W^{r_1,m}_\nu}\,ds\,.\end{aligned}$$

By (8), $\lambda q/(q-1)>-1$ so that we can use Hölder with exponents $q/(q-1)$, $2q$ and $2q$ on the three factors constituting the integrand of the first integral of the right hand side of the last inequality. The integral is then estimated by

$$\begin{aligned}&\int_0^t (t-s)^\lambda \|u(s)\|^{1/2}_{W^{m,r_1}}\|\varphi(s)\|^{1/2}_{W^{m,r_1}}\,ds\\&\le CT^{\lambda+1-1/q}\|u\|^{1/2}_{L^q_T(W^{m,r_1})}\|\varphi\|^{1/2}_{L^q_T(W^{m,r_1})}.\end{aligned}$$

getting

$$\begin{aligned}\|D^\alpha \mathcal{M}\varphi(t)\|_{L^{r_1}_\nu} &\le C(T)(\|u\|_{C_T(W^{m,r})}+\|u\|_{L^q_T(W^{m,r_1})})\\&\quad\times(\|\varphi\|_{C_T(W^{m,r})}+\|\varphi\|_{L^q_T(W^{m,r_1})})\\&\quad+C\|u\|_{C_T(W^{m,r})}\int_0^t (t-s)^{-n/2r-1/2}\|\varphi(s)\|_{W^{m,r_1}_\nu}\,ds.\end{aligned}$$

Assume now $\varphi=v=\mathcal{L}v$ is the fixed point of $\mathcal{L}$. Since $2\le r_1\le r$, we can apply (59) (with $q=r_1$ and $\varphi=v$), used in the proof of Theorem 2.1, to estimate the W^{m,r_1}-norm of $v(s)=F(s)*v_0-\mathcal{M}v(s)$ in terms of $\|u\|_{C_T(W^{m,r})}$ and $\|v_0\|_{W^{m,r_1}}$. It follows that $\|v\|_{L^q_T(W^{m,r_1})}$ has a bound of the type $K(T,u)\|v_0\|_{W^{m,r_1}}$. Similarly, $\|v\|_{C_T(W^{m,r})}$ can be bounded by $K(T,u)\|v_0\|_{W^{m,r}}$. Hence

$$\begin{aligned}&\|D^\alpha \mathcal{M}v(t)\|_{L^{r_1}_\nu} \qquad (14)\\&\le K(T,u)\Big(\|v_0\|_{W^{m,r}}+\|v_0\|_{W^{m,r_1}}+\int_0^t (t-s)^{-\frac{n}{2r}-\frac12}\|v(s)\|_{W^{m,r_1}_\nu}\,ds\Big).\end{aligned}$$

By (11) and Young's inequality,

$$\begin{aligned}\|D^\alpha(F(t)*v_0)\|_{L^{r_1}_\nu} &= \|F(t)*D^\alpha v_0\|_{L^{r_1}_\nu}\\&\le C\left(\|F(t)\|_1\|D^\alpha v_0\|_{L^{r_1}_\nu}+\|F_{[\nu]}\|_1\|D^\alpha v_0\|_{r_1}\right).\end{aligned}$$

Since $\|F(t)\|_1 = 1$ and $\|F_{[\nu]}(t)\|_1 = Ct^{\nu/2}$ for all $t \geq 0$, we see that

$$\|D^\alpha(F(t) * v_0)\|_{L^{r_1}_\nu} \leq C(1+T)^{\frac{\nu}{2}+1}\left(\|v_0\|_{W^{m,r_1}} + \|D^\alpha v_0\|_{L^{r_1}_\nu}\right).$$

Combining with (14) we get for the fixed point of $\mathcal{L}$

$$\begin{aligned}\|v\|_{W^{m,r_1}_\nu} = \|\mathcal{L}\varphi(t)\|_{W^{m,r_1}_\nu} &\leq C(T)\left(\|v_0\|_{W^{m,r_1}} + \|v_0\|_{W^{m,r_1}_\nu}\right)\\ &+ K(T,u)\Big(\|v_0\|_{W^{m,r}} + \|v_0\|_{W^{m,r_1}} + \int_0^t (t-s)^{-\frac{n}{2r}-\frac{1}{2}}\|v(s)\|_{W^{m,r_1}_\nu}\,ds\Big).\end{aligned}$$

The Lemma now follows by Lemma 5.3. (see Appendix A) □

We remark that the main cases of Lemma 2.2 are

1. The case $r_1 = 2$, $q = \infty$. Then (8) is satisfied if $\nu < n/2 - n/r$.
2. The case $r_1 = 2$ and $q = 2$. For (8) we must assume $n/r < \nu < n/2 - n/r$.
3. The case $r_1 = r$, $q = 0$. In this case $\sigma = r$ and (8) requires $r > 2n/(n-\nu)$.

2.2. Existence of admissible solutions. As before, let $n < r \leq \infty$. For simplicity of notation, set $\gamma = \frac{1}{2}(1 - \frac{n}{r})$. Notice that $\gamma > 0$ and we can rewrite the L^r bound for $\mathcal{M}$ (see (62) from Appendix A) in the form

$$\|\mathcal{M}\varphi(t)\|_r \leq C\int_0^t (t-s)^{\gamma-1}\|\varphi(s)\|_r\|u(s)\|_r\,ds. \tag{15}$$

Construct now a sequence $\{u_k\}$ in $C_T(L^2) = C([0,T], L^2(\mathbf{R}^n)^n)$ as follows. Definition of $u_0(x,t)$: With a slight abuse of notation, let $u_0(x,t)$ be the solution of the heat equation of initial datum u_0,

$$u_0(x,t) = F(t) * u_0(x) = \int_{\mathbf{R}^n} F(t, x-y)u_0(y)\,dy$$

for all $x \in \mathbf{R}^n$, $t \in [0,T]$. Assuming u_k defined for some $t \geq 0$ so that $u_k \in C([0,T], L^2 \cap L^r(\mathbf{R}^n))$, let $u_{k+1} = v$ be the solution of the (LNS) system

$$v_t - \Delta v + u_k \cdot \nabla v + \nabla P(u_k, v) = 0, \tag{16}$$

$$\text{div} v = 0 \tag{17}$$

$$v(0) = u_0, \tag{18}$$

The existence and uniqueness of u_{k+1} is guaranteed by Theorem 2.1. Moreover, $u_{k+1} = \mathcal{L}_k u_{k+1} = F(\cdot) * u_0 - \mathcal{M}_k u_{k+1}$, where $\mathcal{M}_k$, $\mathcal{L}_k$ are defined by (5), (6), respectively, with $u = u_k$. By (15) there exists a constant C_r, depending only on the dimension n and on r such that

$$\|\mathcal{M}_k \varphi(t)\|_r \leq C_r T^\gamma \|u_k\|_{C_T(L^r)} \|\varphi\|_{C_T(L^r)},$$

for all t, $0 \leq t \leq T$. Estimating $F(t) * u_0$ by Young's inequality, we get (since $\|F(t)\|_1 = 1$ for all $t > 0$,

$$\|u_{k+1}(t)\|_r \leq \|u_0\|_r + C_r T^\gamma \|u_k\|_{C_T(L^r)} \|u_{k+1}\|_{C_T(L^r)}. \tag{19}$$

Now let $\kappa > 1$ and let Λ be such that

$$\Lambda \geq \kappa \|u_0\|_r. \tag{20}$$

The numbers κ and Λ shall remain fixed in the sequel. We have

Lemma 2.3. *Assume*

$$T \leq (\Lambda C_r)^{-1/\gamma} \big(1 - \frac{1}{\kappa}\big)^{1/\gamma}. \tag{21}$$

Then the sequence $\{u_k\}$ *satisfies* $\sup_{0 \leq t \leq T} \|u_k(t)\|_r \leq \Lambda$ *for* $k = 0, 1, \dots$ *and* $t \leq T$.

Proof. Let $\sigma_k = \sup_{0 \leq t \leq T} \|u_k(t)\|_r$. We proceed by induction on k. The case $k = 0$ is clear from the definition of Λ since $\|u_0(t)\|_r = \|F(t) * u_0\|_r \leq \|u_0\|_r$. Assume thus that the estimate holds for some $k \geq 0$; i.e., assume that $\sigma_k \leq \Lambda$ for some $k > 0$. By (19), taking the sup over $[0, T]$,

$$\sigma_{k+1} \leq \|u_0\|_r + C_r T^\gamma \sigma_k \sigma_{k+1}. \tag{22}$$

Notice that by (21) $C_r T^\gamma \sigma_k \leq C_r T^\gamma \Lambda \leq 1 - \frac{1}{\kappa} < 1$, using this in (22) we get $\sigma_{k+1} \leq \kappa \|u_0\|_r \leq \Lambda$. The lemma is proved. □

We are ready to state and prove our existence theorem. As mentioned before, at least locally, this solution will coincide with the Kato solution.

Theorem 2.4. *Let* $u_0 \in L^2 \cap L^r(\mathbf{R}^n)$*; assume* $\operatorname{div} u_0 = 0$ *and that* T *satisfies* (21)*. There exists* $u \in C([0, T], L^2 \cap L^r(\mathbf{R}^n)) \cap L^2([0, T], H^1(\mathbf{R}^n)^n)$ *such that*

i. *The sequence* $\{u_k\}$ *converges to* u *in* $C([0, T], L^2 \cap L^r(\mathbf{R}^n))$.

ii. u *is a weak solution of the Navier-Stokes equations.*

iii. *u satisfies the energy inequality*

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u(s)\|_2^2\, ds \le \|u_0\|_2^2 \tag{23}$$

Proof. For $k = 1, 2, \ldots$, let us write $\tau_k(t) = \|u_k(t) - u_{k-1}(t)\|_2$, for $0 \le t \le T$, and $\tilde{\tau}_k = \sup_{0\le t\le T} \tau_k(t)$. Then

$$\begin{aligned}\tau_{k+1}(t) &= \|u_{k+1}(t) - u_k(t)\| = \|\mathcal{L}_k u_{k+1}(t) - \mathcal{L}_{k-1} u_k(t)\|_2\\ &= \|\mathcal{M}_k u_{k+1}(t) - \mathcal{M}_{k-1} u_k(t)\|\\ &\le \|\mathcal{M}_k(u_{k+1}(t) - u_k(t))\|_2 + \|(\mathcal{M}_k - \mathcal{M}_{k-1})u_k(t)\|_2.\end{aligned}$$

By (60) (with $m = 0$, $q = 2$),

$$\|\mathcal{M}_k(u_{k+1}(t) - u_k(t))\|_2 \le C\|u_k\|_{C_T(L^r)} \int_0^t (t-s)^{\gamma-1}\tau_{k+1}(s)\, ds;$$

by (61) (also with $m = 0$, $q = 2$), since $\mathcal{M}_k - \mathcal{M}_{k-1}$ is an operator of type $\mathcal{M}$ with $u = u_k - u_{k-1}$,

$$\|(\mathcal{M}_k - \mathcal{M}_{k-1})u_k(t)\|_2 \le C\|u_k\|_{C_T(L^r)} \int_0^t (t-s)^{\gamma-1}\tau_k(s)\, ds.$$

In view of Lemma 2.3 we proved

$$\tau_{k+1}(t) \le C\Lambda \int_0^t (t-s)^{\gamma-1}\tau_k(s)\, ds + C_r\Lambda \int_0^t (t-s)^{\gamma-1}\tau_{k+1}(s)\, ds. \tag{24}$$

By Lemma 5.3,

$$\tau_{k+1}(t) \le M \int_0^t (t-s)^{\gamma-1}\tau_k(s)\, ds,$$

where $M = C\Lambda\Phi_\gamma\left(C\Lambda T^\gamma\Gamma(\gamma)\right)$ and Φ_γ is the function appearing in Lemma 5.3. By induction, we get $\tau_{k+1}(t) \le \frac{\tilde{\tau}_1}{\Gamma(k\gamma+1)}\left(M\Gamma(\gamma)t^\gamma\right)^k$ for $k = 0, 1, \ldots$. Taking suprema over $[0, T]$,

$$\tilde{\tau}_{k+1} \le \frac{\tilde{\tau}_1}{\Gamma(k\gamma+1)}\left(M\Gamma(\gamma)T^\gamma\right)^k \tag{25}$$

for $k = 0, 1, \ldots$. Since the power series $\sum_{k=0}^{\infty} \frac{1}{\Gamma(k\gamma+1)} z^k$ converges to Φ_γ over the whole complex plane, (25) proves the convergence of the series

$$\sum_{k=1}^{\infty} \tilde{\tau}_k = \sum_{k=0}^{\infty} \Big(\sup_{0 \le t \le T} \|u_k(t) - u_{k-1}(t)\|_2 \Big),$$

which is equivalent to the convergence of $\{u_k\}$ in $C([0,T], L^2)$. We denote the limit of the sequence $\{u_k\}$ by u. It is clearly in $C([0,T], L^2(\mathbf{R}^n)^n)$. By Lemma 2.3 we see that $u \in L^\infty([0,T], L^r(\mathbf{R}^n)^n)$; in fact, $\sup_{0 \le t \le T} \|u(t)\|_r \le \Lambda$ by Fatou's Lemma. In order to show that $u \in C([0,T], L^r(\mathbf{R}^n)^n)$, repeat the above argument letting $q = r$ and replacing $\tau_k(t) = \|u_k(t) - u_{k-1}(t)\|_2$, for $0 \le t \le T$, by $\tau_k(t) = \|u_k(t) - u_{k-1}(t)\|_r$, for $0 \le t \le T$, with the corresponding changes for $\tilde{\tau}_k$. The proof we just gave then yields

$$\sum_{k=1}^{\infty} \tilde{\tau}_k = \sum_{k=0}^{\infty} \big(\sup_{0 \le t \le T} \|u_k(t) - u_{k-1}(t)\|_r \big) < \infty,$$

which is equivalent to the convergence of $\{u_k\}$ in $C([0,T], L^r)$. Moreover, multiplying both sides of (16) by u_{k+1} and integrating in space yields,

$$\|u_{k+1}(t)\|_2^2 + 2 \int_0^t \|\nabla u_{k+1}(s)\|_2^2 \, ds \le \|u_0\|_2^2$$

for $k = 0, 1, \ldots$. It follows that $\{\nabla u_k\}$ is bounded in $L^2([0,T], L^2(\mathbf{R}^n))$; there is thus a subsequence converging weakly in $L^2([0,T], L^2)$. Since the limit is necessarily ∇u, we conclude ∇u is in $L^2([0,T], L^2)$, hence $u \in L^2([0,T], H^1)$, and satisfies (23). All that remains to be proved of the theorem is that u is a weak solution of the Navier-Stokes equations. Since u_{k+1} is a weak solution of (1), (2) (with $u = u_k$ and $v_0 = u_0$), we have for $0 \le t \le T$ and $\phi \in C_0^\infty(\mathbf{R}^n)^n$,

$$\begin{aligned} \frac{d}{dt}(u_{k+1}(t), \phi) &= (u_{k+1}(t), \Delta\phi) + (u_{k+1}(t), u_k(t) \cdot \nabla\phi) \\ &\quad + (P(u_k, u_{k+1})(t), \operatorname{div} \phi). \end{aligned}$$

Assuming now $\operatorname{div} \phi = 0$, integrating from 0 to t,

$$(u_{k+1}(t), \phi) = (u_0, \phi) + \int_0^t \left((u_{k+1}(s), \Delta\phi) + (u_{k+1}(s), u_k(s) \cdot \nabla\phi) \right) ds.$$

Since $\{u_k\}$ converges in $C_T(L^2)$ it is clear that we can take limits and obtain

$$(u(t),\phi) = (u_0,\phi) + \int_0^t ((u(s),\Delta\phi) + (u(s),u(s)\cdot\nabla\phi))\,ds.$$

The theorem is proved. □

2.3. Global existence for small data. In this section we prove that admissible solutions exist globally if the initial datum u_0 has sufficiently small L^r-norm. We assume, as usual, $r > n$, except if $n = 2$ where we need to assume $r \geq 4$. We begin with a simple result on decay for solutions of the heat equation; for convenience we state it in the form of a Lemma.

Lemma 2.5. *There is a constant C_0 such that*

$$\|F(t) * \psi\|_r \leq C_0(1+t)^{-n/2r-1/2}\left(\|\psi\|_{nr/(2n+r)} + \|\psi\|_r\right) \tag{26}$$

for all $\psi \in L^r \cap L^{nr/(2n+r)}(\mathbf{R}^n)$.

Proof. The inequality is trivial for small values of t so that it suffices to prove $\|F(t)*\psi\|_r \leq C_0 t^{-n/2r-1/2}\|\psi\|_{nr/(2n+r)}$ for some constant C_0, $t > 0$, $\psi \in L^q$ where we set $q = nr/(2n+r)$, $s = nr/(nr-n-r)$. Notice that $q \geq 1$ if and only if $r \geq 2n/(n-1)$ which is guaranteed by the assumption $r > n$, except if $n = 2$ which is why we assume $r \geq 4$ if $n = 2$. The assumption $r > n$ also guarantees that $s > 1$. Since $\frac{1}{r} = \frac{1}{q} + \frac{1}{s} - 1$ and $\|F(t)\|_s = Ct^{-n/2+n/2s} = Ct^{-n/2r-1/2}$, the Lemma is an immediate consequence of Young's inequality $\|F(t)*\psi\|_r \leq \|F(t)\|_s\|\psi\|_q$. □

The next theorem proves the existence of global solutions if the data is small.

Theorem 2.6. *Assume $r \geq 4$ if $n = 2$, $r > n$ if $n \geq 3$. Let $u_0 \in L^2(\mathbf{R}^n)^n \cap L^r(\mathbf{R}^n)^n$ and $div\, u_0 = 0$. Let q be as above; i.e., $q = nr/(2n+r)$. There is $\delta > 0$ such that if $\|u_0\|_r < \delta$, then the admissible solution u of initial datum u_0 exists globally. Moreover, there is $\tilde{C}_0 > 0$ such that for β satisfying $1/2 < \beta < n/2r + 1/2$*

$$\|u(t)\|_r \leq \tilde{C}_0(1+t)^{-\beta} \quad \text{for all } t > 0. \tag{27}$$

Proof. See Appendix A Part II. We note again that since $||u_0||_r$ is small, the solution we will be working with is Kato's solution. □

We conclude this section establishing some estimates for the first order derivatives of a global solution of the Navier-Stokes equations. Assume thus

$u \in C([0,\infty), L^2) \cap L^\infty([0,\infty), L^r)$ is a global solution of the Navier-Stokes equations and let $v = \partial u/\partial x_j u$ be a spatial derivative of u. Then v satisfies the equation

$$v_t - \Delta v + u \cdot \nabla v + v \cdot \nabla u + \nabla Q(u,v) = 0, \tag{28}$$

where $Q(u,v) = P(u,v) + P(v,u)$. In addition, $\operatorname{div} v = 0$. We can proceed as in the proof of Theorem 2.1; (28) is essentially Navier Stokes equations linearized with respect to u and v, and v is the fixed point of an operator $\tilde{\mathcal{L}}$, where

$$\tilde{L}\varphi = F(\cdot) * \varphi - \tilde{\mathcal{M}}\varphi,$$
$$\tilde{\mathcal{M}}\varphi(t) = \int_0^t F(t-s) * [u \cdot \nabla v + v \cdot \nabla u + \nabla Q(u,v)](s)\, ds.$$

The difference between the terms $u{\cdot}\nabla v$ and $v{\cdot}\nabla u$ disappears after integrating by parts (as we did in the proof of Theorem 2.1) and and $\tilde{\mathcal{M}}$ satisfies a bound of type (62):

$$\|\tilde{\mathcal{M}}\varphi(t)\|_r \le C \int_0^t (t-s)^{-n/2r-1/2} \|\varphi(s)\|_r \|u(s)\|_r\, ds.$$

It follows that

$$\begin{aligned}\|v(t)\|_r = \|\tilde{\mathcal{L}}v(t)\|_r\| &\le \|F(t) * v_0\|_r + \|\tilde{\mathcal{M}}v(t)\|_r \\ &\le \|F(t) * v_0\|_r + C\|u\|_{L^\infty(L^r)} \int_0^t (t-s)^{-n/2r-1/2} \|\varphi(s)\|_r\, ds.\end{aligned}$$

It follows from Lemma 5.3 that if $v_0 \in L^r$, then $\|v(t)\|_r$ is bounded on finite subintervals of $[0,\infty)$. That is, we proved:

Theorem 2.7. *Let u be an admissible global solution to the Navier-Stokes equations such that $u_0, \nabla u_0 \in L^r$ and*

$$sup_{t \ge 0} \|u(t)\|_r < \infty.$$

Then $\nabla u \in L^\infty([0,T], L^r)$ for all $T > 0$.

A similar argument, combined this time with induction, proves

Theorem 2.8. *Let u be a global admissible solution to the Navier-Stokes equations with initial value $u_0 \in L^r \cap H^1$, $\nabla u_0 \in L^r$ and assume the defining sequence $\{u_k\}$ satisfies*

$$\sup_{k=0,1,\dots;t\geq 0} \|u_k(t)\|_r < \infty.$$

Then for every $T > 0$ there exists $C(T) > 0$ such that

$$\sup_{k=0,1,\dots;0\leq t\leq T} \|\nabla u_k(t)\|_r \leq C(T).$$

3. Moment estimates. We assume u is a global admissible solution of the Navier-Stokes equations with initial value satisfying $u_0 \in L^2 \cap L^r$, where $n < r \leq \infty$ and such that the defining sequence $\{u_k\}$ satisfies

$$\mu(u) = \sup_{k=0,1,2,\dots;t>0} \|u_k(t)\|_r < \infty. \tag{29}$$

By the proof of Theorem 2.6 there is $\delta > 0$ such that taking $\|u_0\|_r \leq \delta$, $n < r$, will result in global existence and the validity of (29).

For $\alpha \geq 0$, $t \geq 0$, we define the moment of order α of u by

$$M_\alpha(u)(t) = \int_{\mathbf{R}^n} |x|^\alpha |u(t)|^2\, dx = \left(\|u(t)\|_{L^2_{\alpha/2}}\right)^2.$$

We want to see that for sufficiently small initial data and for $1 \leq \alpha \leq n$, $2 \leq n \leq 5$, the moments decay in time.

The next theorem gives time dependent bounds for the moments of order $\alpha < (n - 2/r)$. This is a first step to get the decay of the moments. The next step is to get first time dependent bounds up to order n. These bounds are then used to obtain uniform bounds in time.

Theorem 3.1. *Let $0 \leq \alpha < n(1 - 2/r)$. Assume $u_0 \in L^2_{\alpha/2} \cap L^r \cap L^2$ with $\operatorname{div} u_0 = 0$ and such that (29) holds. For every $T > 0$ there is $K(T)$ depending only on T, on $\mu(u)$, and the $L^r \cap L^2_{\alpha/2}$ norm of u_0 such that*

$$M_\alpha(u)(t) \leq K(T) \qquad \textit{for} \quad 0 \leq t \leq T. \tag{30}$$

If in addition to the previous hypotheses, $\alpha > 2n/r$ and $u_0 \in H^1 \cap W^{1,r}$, then

$$M_\alpha(\nabla u)(t) \leq K(T) \qquad \textit{for} \quad 0 \leq t \leq T. \tag{31}$$

Proof. Setting $\nu = \alpha/2$, the theorem is a simple consequence of Lemma 2.2. For (30) we take $r_1 = 2$, $m = 0$, $q = \infty$ to get

$$\|u_{k+1}\|_{L^2_\nu} \le C(T)(\|u_0\|_2 + \|u_0\|_{L^2_\nu}) + K(T, u_k)(\|u_0\|_r + \|u_0\|_2).$$

By (29) and since $\|u_k\|_{C_T(L^2)}$ is bounded independently of k, by the energy inequality (3) in Appendix A, $K(T, u_k)$ is bounded independently of k and the result follows letting $k \to \infty$. For (31) we apply Lemma 2.2 with $r_1 = 2$, $m = 1$, $q = 2$. The assumption $\alpha > 2n/r$ guarantees that (8) holds. We get (with $H^1_\nu = \{f : f, \nabla f \in L^2_\nu\}$)

$$\|u_{k+1}\|_{H^1_\nu} \le C(T)(\|u_0\|_{H^1} + \|u_0\|_{H^1_\nu}) + K(T, u_k)(\|u_0\|_{W^{1,r}} + \|u_0\|_{H^1})$$

where this time $K(T, u_k)$ depends on the $L^2_T(H^1)$ norm of u_k and on the $C_T(W^{1,r})$ norm of u_k. Both of these norms are bounded independently of k. In fact, the $L^2_T(H^1)$ norm of u_k is bounded by the energy inequality (3); the $C_T(W^{1,r})$ norm of u_k can be so bounded by Theorem 2.8. The result follows letting $k \to \infty$. □

We also apply Lemma 2.2 with $r_1 = r$, $q = \infty$, $m = 0$. In this case $\sigma = r$ and (7), (8) hold if $\nu < n(1 - 2/r)$; $K(T, u)$ depends only on T and the L^r norm of u. Thus we get

Theorem 3.2. *Assume $0 \le \nu < n(1 - 2/r)$ and $u_0 \in L^r_\nu \cap L^r \cap L^2$ with $\operatorname{div} u_0 = 0$ and such that (29) holds. For every $T > 0$ there is $K(T)$ depending only on T, on $\mu(u)$, and the $L^r_\nu \cap L^r \cap L^2$ norm of u_0 such that*

$$\int_{\mathbf{R}^n} |x|^{r\nu} |u(t)|^r \, dx \le K(T) \quad \textit{for } 0 \le t \le T.$$

To establish a bound of the moments up to order α we note that a tedious but straightforward computation shows that

$$\begin{aligned} |x|^\alpha u \cdot u_t = -|x|^\alpha |\nabla u|^2 &+ \frac{1}{2}\alpha(n + \alpha - 2)|x|^{\alpha-2}|u|^2 \\ &+ \frac{1}{2}\alpha|x|^{\alpha-2}(x \cdot u)|u|^2 + \alpha|x|^{\alpha-2}(x \cdot u)p + \operatorname{div} E_\alpha \end{aligned} \tag{32}$$

for $\alpha \ge 0$, where $E_\alpha = \frac{1}{2}|x|^\alpha \nabla(|u|^2) - \frac{1}{2}\alpha|x|^{\alpha-2}|u|^2 x - \frac{1}{2}|x|^\alpha |u|^2 u - |x|^\alpha p u$. Integration in space and time would yield a bound for the moments provided we can bound the terms coming from the pressure, convective terms and the

E_α term. For this last term we note that after an integration by parts it follows that we need to be able to estimate the boundary term

$$\liminf_{R\to\infty}\int_{|x|=R}\int_0^T |E_\alpha|\,dt\,dS.$$

This term will be dealt by the following lemma.

Lemma 3.3. *Assume*

$$0<\alpha<(n+1/2)(1-2/r), \tag{33}$$

assume u_0 *satisfies the conditions of Theorems* (3.1), (3.2) *and that* (29) *holds. Then, for every* $T>0$,

$$\liminf_{R\to\infty}\int_{|x|=R}\int_0^T |E_\alpha|\,dt\,dS=0.$$

The proof of this lemma is quite technical and can be found in Appendix B of this paper

Returning to (32) and integrating with respect to space yields formally

$$\begin{aligned}\frac{d}{dt}M_\alpha(u)(t) &= \frac{1}{2}\alpha(n+\alpha-2)M_{\alpha-2}(u)(t)-\int_{\mathbf{R}^n}|x|^\alpha\,|\nabla u|^2\,dx \\ &\quad+\frac{1}{2}\alpha\int_{\mathbf{R}^n}|x|^{\alpha-2}(x\cdot u)|u|^2\,dx+\alpha\int_{\mathbf{R}^n}|x|^{\alpha-2}(x\cdot u)p\,dx.\end{aligned} \tag{34}$$

Assume

(H1) $u_0\in L^r_\beta\cap H^1$, $\quad\operatorname{div}u_0=0$, where $\beta=n(1-\frac{2}{r})$.

(H2) (29) holds.

Then, by Theorem (3.2)

$$\left|\int_{\mathbf{R}^n}|x|^{\alpha-2}(x\cdot u)|u|^2\,dx\right|\le\int_{\mathbf{R}^n}|x|^{\alpha-1}|u|^3\,dx\le\|u\|_{L^r_{\alpha-1}}\|u\|^2_{2r'}<\infty$$

if $0\le\alpha-1<n(1-2/r)$. In fact, recall we are assuming $r\ge3$ so that $2r'\in[2,r]$. Similarly

$$\left|\int_{\mathbf{R}^n}|x|^{\alpha-2}(x\cdot u)p\,dx\right|\le\int_{\mathbf{R}^n}|x|^{\alpha-1}|u||p|\,dx\le\|u\|_{L^r_{\alpha-1}}\|p\|_{r'}<\infty$$

since $\|p\|_{r'} \le C\|u\|_{2r'}^2$. Working with

$$M_{R,\alpha}(u)(t) = \int_{|x|<R} |x|^\alpha |u(t)|^2 \, dx$$

instead of with M_α, we see that

$$\frac{d}{dt} M_{R,\alpha}(u)(t) \le \frac{1}{2}\alpha(n+\alpha-2) M_{R,\alpha-2}(u)(t) + C\|u\|_{L^r_{\alpha-1}} \|u\|_{2r'}^2 + \int_{|x|=R} |E_\alpha| \, dS.$$

We can integrate with respect to t and let $R \to \infty$; by Lemma 3.3 we conclude that $M_\alpha(u)$ is finite if $M_{\alpha-2}(u)$ is finite. We can thus use induction to extend part a of Theorem 3.1 to

Theorem 3.4. *Assume (H1), (H2). Then $M_\alpha(u)(t) < \infty$ and (34) holds for $0 \le \alpha < (n+1/2)(1-2/r)$.*

Remark. The condition $0 \le \alpha < (n+1/2)(1-2/r)$ is necessary to insure that

$$\liminf_{R\to\infty} \int_{|x|=R} \int_0^T |E_\alpha| \, dt \, dS = 0.$$

Remark. The last theorem insures that our bounds are on moments up to order n.

4. Decay of the moments. In this section we assume $2 \le n \le 5$, $u(t)$ is a solution of the Navier-Stokes equations with initial datum u_0 satisfying (H1) of the previous section. We also assume sufficient conditions so that u is a global solution satisfying

$$\|u(t)\|_2 \le C(t+1)^{-\mu} \tag{35}$$

$$\|u(t)\|_\infty \le C(t+1)^{-\mu-n/4} \tag{36}$$

for some $\mu > 1/2$. The constant C in (35) and (36) depends only on the initial datum u_0. It is proved in [15], [22] that (35) holds for global solutions with $\mu = n/4+1/2$ (thus $\mu > 1/2$), if u_0 is in $L^1 \cap H^1$ and has mean zero. It is proved in [17] that (35) implies (36) for sufficiently small data u_0, $2 \le n \le 5$. As remarked after the proof of Theorem 2.6 the admissible solution exists globally if $\|u_0\|_\infty$ is sufficiently small. Obviously (36) implies (29) with $r = \infty$. By Theorems 3.1 all moments of order α with $0 < \alpha < n$ of ∇u are finite; by Theorem 3.4 all moments of u of order α with $0 \le \alpha < n+1/2$ are finite.

Theorem 4.1. *Let u be a global admissible solution of the Navier-Stokes equations with initial datum u_0 satisfying the following hypothesis:*

1. *(H1) of the previous section; i.e., let $\beta = n(1-2/r)$*

$$u_0 \in L^r_\beta \cap L^r \cap H^1, \quad div\, u_0 = 0,$$

2. *Inequalities* (35) *and* (36).

Let $0 \le k \le n$. Then there exists a constant $C \ge 0$ such that

$$M_k(u)(t) \le C(t+1)^{-2\mu(1-\frac{k}{n})} \tag{37}$$

for all $t \ge 0$, $k = 0, 1, \dots, n$.

Proof. Note that the case $k = 0$ is our hypothesis. It suffices to prove that the moment $M_n(u)$ of order n (which, as remarked above, is locally finite) is globally bounded in time. The theorem then follows by Hölder interpolation. In fact, with $1/p = (n-k)/n$, $1/p' = k/n$,

$$\begin{aligned} M_k(u)(t) &= \int_{\mathbf{R}^n} |x|^k |u|^2\, dx \le \left(\int_{\mathbf{R}^n} |u|^2\, dx\right)^{1/p} \left(\int_{\mathbf{R}^n} |x|^n |u|^2\, dx\right)^{1/p'} \\ &= M_0(u)(t)^{1-\frac{k}{n}} M_n(u)(t)^{k/n} \le C(t+1)^{-2\mu(1-\frac{k}{n})} M_n(u)(t)^{k/n}. \end{aligned}$$

We proceed by induction proving there exists $C \ge 0$ such that $M_k(u)(t) \le C$ for all $t \ge 0$, $k = 1, \dots, n$. We begin considering the case $k = 2$, which has to be handled separately. Induction proceeds in steps of 2; i.e., M_k bounded implies M_{k+2} bounded; we also need to consider the case $k = 1$. However, once these bounds are obtained the rest follow by interpolation. We remark in passing that the bound for $k = 1$ was already established by Caffarelli, Kohn and Nirenberg (cf. [2]). By (34),

$$\frac{d}{dt} M_2(u)(t) \le n\|u(t)\|_2^2 - \int_{\mathbf{R}^n} |x|^2 |\nabla u|^2 dx + \int_{\mathbf{R}^n} |x||u|^3 dx + 2\int_{\mathbf{R}^n} |x||u||p| dx \tag{38}$$

and we have

$$\int_{\mathbf{R}^n} |x||u|^3\, dx \le M_2(u)(t)^{1/2} \|u(t)\|_4^2, \tag{39}$$

$$\int_{\mathbf{R}^n} |x||u||p|\, dx \le M_2(u)(t)^{1/2} \|p(t)\|_2^2 \le C M_2(u)(t)^{1/2} \|u(t)\|_4^2. \tag{40}$$

Since $\|u(t)\|_4 \le C(t+1)^{-\mu-n/8}$ by (35) and (36), it follows that

$$\int_{\mathbf{R}^n} |x||u|^3\,dx + 2\int_{\mathbf{R}^n} |x||u||p|\,dx \le CM_2(u)(t)^{1/2}(t+1)^{-\delta}$$

with $\delta = 2\mu + n/4$. Hence, from (38), (39), (40),

$$\frac{d}{dt}M_2(u)(t) \le n(t+1)^{-2\mu} + CM_2(u)(t)^{1/2}(t+1)^{-\delta}. \tag{41}$$

Estimating $M_2^{1/2}(t+1)^{-\delta}$ by $2M_2(t+1)^{-\delta} + 2(1+t)^{-\delta}$, we get

$$\frac{d}{dt}M_2(u)(t) \le n(t+1)^{-2\mu} + C(t+1)^{-\delta} + CM_2(u)(t)(t+1)^{-\delta}. \tag{42}$$

Since the moments are bounded (time dependent) and since $\delta > 1$ it follows by Gronwall's inequality that

$$M_2(u)(t) \le Ce^{\int_0^\infty (t+1)^{-\delta}dt} \le const. \tag{43}$$

Hence case $k = 2$ follows. Assume now $k > 2$. We have by (34),

$$\begin{aligned}\frac{d}{dt}M_k(u)(t) \le{}& \frac{k}{2}(n+k-2)M_{k-2}(u)(t) - \int_{\mathbf{R}^n} |x|^k|\nabla u|^2\,dx \\ &+ \frac{k}{2}\int_{\mathbf{R}^n} |x|^{k-1}|u|^3\,dx + k\int_{\mathbf{R}^n} |x|^{k-1}|u||p|\,dx.\end{aligned} \tag{44}$$

We have

$$\int_{\mathbf{R}^n} |x|^{k-1}|u|^3\,dx \quad\le\quad M_k(u)(t)^{(k-1)/k}\|u(t)\|_{k+2}^{(k+2)/k}, \tag{45}$$

$$\int_{\mathbf{R}^n} |x|^{k-1}|u||p|\,dx \quad\le\quad CM_k(u)(t)^{(k-1)/k}\|u(t)\|_{k+2}^{(k+2)/k}. \tag{46}$$

The first of these inequalities is an immediate consequence of Hölder's inequality (with exponents $k/(k-1)$ and k). For the second inequality notice first that, by Hölder's inequality,

$$\begin{aligned}\int_{\mathbf{R}^n} |x|^{k-1}|u||p|\,dx &\le \Big(\int_{\mathbf{R}^n} |x|^k|u|^2\,dx\Big)^{1/2}\Big(\int_{\mathbf{R}^n} |x|^{k-2}|p|^2\,dx\Big)^{1/2} \\ &= M_k(u)(t)^{1/2}\|p\|_{L^2_\nu}\end{aligned}$$

with $\nu = k/2 - 1$. Then $n/(n-\nu) < 2$ (since $k < n+2$) hence the Riesz transforms are bounded in L^2_ν; since $p = -\sum_{j,k} R_j R_k(u_j u_k)$, we get

$$\begin{aligned}\|p\|_{L^2_\nu} \le C\| \, |u|^2\|_{L^2_\nu} &= C\Big(\int_{\mathbf{R}^n} |x|^{k-2}|u|^4\,dx\Big)^{1/2}\\ &\le M_k(u)(t)^{(k-2)/2k}\|u\|_{k+2}^{(k+2)/k},\end{aligned}$$

where we wrote $|u|^4 = |u|^{2-4/k}|u|^{2+4/k}$ and used Hölder's inequality with exponents $k/(k-2)$, $k/2$. Inequality (46) follows. To continue estimating, we get by Hölder's inequality, (35) and (36)

$$\|u(t)\|_{k+2} \le \|u(t)\|_2^{2/(k+2)}\|u\|_\infty^{k/(k+2)} \le C(t+1)^{-\mu-(nk)/(4k+8)}\,;$$

by Hölder's inequality and (35)

$$M_{k-2}(u)(t) \le M_k(u)(t)^{(k-2)/k}\|u(t)\|_2^{4/k} \le C(t+1)^{-4\mu/k}M_k(u)(t)^{(k-2)/k}$$

so that (44) implies, in view of (45), (46), and the last two inequalities,

$$\begin{aligned}\frac{d}{dt}M_k(u)(t) &\le C_1(t+1)^{-4\mu/k}M_k(u)(t)^{(k-2)/k} \qquad (47)\\ &\quad + C_2(t+1)^{-\mu(k+2)/k-n/4}M_k(u)(t)^{(k-1)/k} - \int_{\mathbf{R}^n}|x|^k|\nabla u|^2\,dx.\end{aligned}$$

Dropping the negative term and estimating the first two terms on the right hand side using $M_k^s \le 1+M_k$ for $s = k/(k-2)$ and $s = k/(k-1)$, respectively, we get

$$\frac{d}{dt}M_k(u)(t) \le C_1(t+1)^{-\rho} + C_2(t+1)^{-\rho}M_k(u)(t)$$

where $\rho = \min\{\frac{4\mu}{k}, \frac{k+2}{k}\mu + \frac{n}{4}\}$. Since $k \le n$ and 4μ, $(k+2)/k\mu + n/4 > \mu + n/4 > n/2 \ge 1$, we see that $\rho > 1$ so that

$$\int_0^t (s+1)^{-\rho}\,ds \le \int_0^\infty (s+1)^{-\rho}\,ds = \frac{1}{\rho-1} < \infty.$$

Integrating from 0 to t we thus get

$$M_k(u)(t) \le C\,(1 + M_k(u)(0)) + C\int_0^t (s+1)^{-\rho}M_k(u)(s)\,ds.$$

By Gronwall's Lemma,

$$M_k(u)(t) \le C\,(1+M_k(u)(0))\,e^{\int_0^\infty (s+1)^{-\rho}\,ds} \le C_o < \infty$$

which completes the proof of the theorem.

5. Appendix A.

5.1. Part I. We start with some observations that will allow us to estimate the pressure term in the integral equation for solutions to Navier-Stokes. We recall that since the Riesz transforms are bounded in $L^p(\mathbf{R}^n)$ for $1 < p < \infty$, there exists for every $p \in (1, \infty)$ a constant A_p such that

$$\|P(u,v)\|_p \leq A_p \| \, |u||v| \, \|_p \tag{48}$$

for all $u, v : \mathbf{R}^n \to \mathbf{R}^n$ such that $|u||v| \in L^p$.

We shall need the following property of the Riesz transforms. If $\nu \in \mathbf{R}$, $1 \leq p \leq \infty$, we'll say that a mapping $f : \mathbf{R}^n \to \mathbf{R}$ is in $L^p_\nu(\mathbf{R}^n)$ if and only if $x \to |x|^\nu f(x)$ is in L^p. We say $f \in L^p_\nu(\mathbf{R}^n)^n$ if $f = (f_1, \dots, f_n)$ and $f_j \in L^p_\nu(\mathbf{R}^n)$ for $j = 1, \dots, n$. In either case, we define the norm $\| \cdot \|_{L^p_\nu}$ as the L^p norm of $x \to |x|^\nu f(x)$ so that L^p_ν is a weighted L^p-space with the weight $w(x) = |x|^{p\nu}$. We have:

Lemma 5.1. *Let ν, q be real numbers such that $0 \leq \nu < n$, $n/(n-\nu) < q < \infty$. The Riesz transforms $R_1, \dots, R_n$ are bounded operators in $L^q_\nu(\mathbf{R}^n)$.*

Proof. If $w : \mathbf{R}^n \to [0, \infty)$ then the Riesz transforms are bounded in the weighted L^q space $L^q(wdx)$ if (and only if) the weight w is in the Muckenhoupt class A_q; that is, if and only if there exists a constant C such that

$$|Q|^{-q} \int_Q w(x)\, dx \left(\int_Q w(x)^{-\frac{1}{q-1}}\, dx \right)^{q-1} \leq C$$

for all cubes Q in $\mathbf{R}^n$ of sides parallel to the axes, where $|Q|$ denotes the Lebesgue measure of Q. We refer to [6] for details, specifically the theorem in Chapter 4, Section III. Since $L^q_\nu = L^q(wdx)$ with $w(x) = |x|^{\nu q}$ it is easy to see (cf. [6], p.21) that $w \in A_q$ if and only if $-n < \nu q < n(q-1)$. Restricting this condition to $\nu \geq 0$, we get exactly the inequality of the statement of the lemma. □

The next theorem was stated in Section 2 as Theorem 2.1. It establishes the existence of solutions to the approximating linearizing equations. The theorem here is renumbered according to the order of the section.

Theorem 5.2. *Let $m \geq 0$ be an integer; let let $u \in C_T(W^{m,r}(\mathbf{R}^n))$ and let $v_0 \in W^{m,q} \cap W^{m,r}(\mathbf{R}^n)$, where $n < r < \infty$, $1 < q \leq r$. There exists a unique $v \in C_T(W^{m,q}) \cap C_T(W^{m,r})$ solving (1), (2). If $\operatorname{div} v_0 = 0$, then $\operatorname{div} v = 0$, i.e., $\operatorname{div} v(t) = 0$ for all $t \in [0,T]$ and*

$$\|v(t)\|_2^2 + 2\int_0^t \|\nabla v(s)\|_2^2\, ds \leq \|v_0\|_2^2 \tag{49}$$

for all $t \in [0,T]$. The solution v is strong and in $C^1([0,T], L^2)$ if $m \geq 3$.

5.2. Proof of the Existence Theorem . Let

$$F(x,t) = F(t)(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$$

be the fundamental solution of the heat equation in n space variables. If v solves (1), (2), setting $H(u,v) = u \cdot \nabla v + \nabla P(u,v)$, then we recall

$$v(t) = F(t) * v_0 - \int_0^t F(t-s) * H(u,v)(s)\, ds. \tag{50}$$

If $\varphi \in L^2([0,T], H^1(\mathbf{R}^n, \mathbf{R}^n))$, we set

$$\begin{aligned} \mathcal{M}\varphi(t) &= \int_0^t F(t-s) * [u \cdot \nabla \varphi(s) + \nabla P(u,\varphi)(s)]\; ds \qquad (51)\\ &= \int_0^t F(t-s) * H(u,\varphi)(s)\, ds, \\ \mathcal{L}\varphi(t) &= F(t) * v_0 - \mathcal{M}\varphi(t). \qquad (52) \end{aligned}$$

Then the integral version (50) of the (LNS) with respect to u becomes $v = \mathcal{L}v$. In showing that $\mathcal{L}$ has a fixed point, the following function plays a role. Let $b > 0$; for $z \in \mathbf{C}$ define

$$\Phi_b(z) = \sum_{m=0}^{\infty} \frac{1}{\Gamma(mb+1)} z^m.$$

The series has infinite radius of convergence, so Φ_b is an entire function. Moreover, Φ_b is positive valued and increasing on $[0,\infty)$. Notice that $\Phi_1(z) = e^z$. The function Φ_b appears in the following elementary Gronwall style lemma; we supply a proof for the sake of completeness.

Lemma 5.3. *Let $\psi \in C([0,T])$, assume $\psi \geq 0$ and satisfies*

$$\psi(t) \leq A + C \int_0^t (t-s)^{b-1} \psi(s)\, ds$$

for all $t \in [0,T]$, where $b > 0$, $A, C \geq 0$. Then $\psi(t) \leq A\Phi_b(Ct^b\Gamma(b))$ for all $t \in [0,T]$. In particular, if $A = 0$, then $\psi \equiv 0$.

Proof. We show by a straightforward induction that

$$\psi(t) \leq A \sum_{k=0}^{m-1} \frac{(Ct^b\Gamma(b))^k}{\Gamma(kb+1)} + M \frac{(Ct^b\Gamma(b))^m}{\Gamma(mb+1)} \tag{53}$$

for $m = 0, 1, \ldots$, $t \in [0,T]$. Let $M = \sup_{0 \leq t \leq T} \psi(t)$. Then

$$\psi(t) \leq A + CM \int_0^t (t-s)^{b-1}\, ds = A + CMt^b \frac{\Gamma(b)}{\Gamma(b+1)}$$

which is (53) for $m = 0$. Using

$$\int_0^t (t-s)^{b-1} s^{bk}\, ds = t^{b(k+1)} B(b, bk+1) = t^{b(k+1)} \frac{\Gamma(b)\Gamma(bk+1)}{\Gamma(b(k+1)+1)},$$

where B is the Beta function, it is easy to see that if (53) holds for some m it also holds for $m+1$. This establishes (53) from which the lemma follows at once. $\square$

We will be using the following variant of the Banach-Kakutani fixed point theorem.

Lemma 5.4. *Let $\mathcal{X}$ be a Banach space, let $T > 0$ and for every $(\tau, \sigma) \in [0,T] \times [0,T]$ let $K(\tau, \sigma)$ be a bounded linear map from $\mathcal{X}$ to $\mathcal{X}$ such that $(\tau, \sigma) \to K(\tau, \sigma)v$ is measurable from $[0,T] \times [0,T]$ to $\mathcal{X}$ for every $v \in \mathcal{X}$. Assume also there exists $C \geq 0$, $a > -1$, such that*

$$\|K(\tau, \sigma)v\|_{\mathcal{X}} \leq C\tau^a \|v\|_{\mathcal{X}}$$

and such that

$$\|[K(\tau, \sigma) - K(\tau, \sigma')]v\|_{\mathcal{X}} \leq C\tau^a \delta(|\sigma - \sigma'|)\|v\|_{\mathcal{X}}$$

for all $v \in \mathcal{X}$, $\tau, \sigma, \sigma' \in [0,T]$ where $\lim_{\epsilon \to 0} \delta(\epsilon) = 0$. Let $\psi \in C_T(\mathcal{X}) = C([0,T], \mathcal{X})$ and define $\mathcal{T}\varphi : [0,T] \to \mathcal{X}$ by

$$\mathcal{T}\varphi(t) = \psi(t) + \int_0^t K(t-s, s)\varphi(s)\, ds\,.$$

Then $\mathcal{T}$ is a map from $C_T(\mathcal{X})$ into itself which has a unique fixed point. Moreover, if v is that fixed point, then

$$\|v(t)\| \leq \|\psi\|_{C_T(\mathcal{X})} \Phi_b \left(Ct^{a+1}\Gamma(a+1) \right) \quad \text{for } 0 \leq t \leq T. \tag{54}$$

Proof. We write $\mathcal{T} = \psi + S$, where

$$S\varphi(t) = \int_0^t K(t-s, s)\varphi(s)\, ds = \int_0^t K(s, t-s)\varphi(t-s)\, ds\,.$$

To see $\mathcal{T}\varphi \in C_T(\mathcal{X})$ it suffices to see that $S\varphi \in C_T(\mathcal{X})$. We have

$$S\varphi(t) - S\varphi(t') = \int_0^t K(s, t-s)[\varphi(t-s) - \varphi(t'-s)]\, ds$$
$$+ \int_0^t [K(s, t-s) - K(s, t'-s)]\varphi(t'-s)\, ds + \int_t^{t'} K(s, t'-s)\varphi(t'-s)\, ds.$$

Taking norms in $\mathcal{X}$,

$$\|S\varphi(t) - S\varphi(t')\|_{\mathcal{X}} \leq C \int_0^t s^a \|\varphi(t-s) - \varphi(t'-s)\|_{\mathcal{X}}\, ds$$
$$+ C\delta(|t-t'|) \int_0^t s^a \|\varphi(t'-s)\|_{\mathcal{X}}\, ds + C\left|\int_t^{t'} s^a \|\varphi(t'-s)\|_{\mathcal{X}}\, ds\right|.$$

The continuity of $S\varphi$ is immediate from these estimates, since s^a is integrable over $[0,T]$. Assume now φ_1, φ_2 are fixed points of $\mathcal{T}$ in $C_T(\mathcal{X})$. We have for $t \in [0,T]$

$$\varphi_1(t) - \varphi_2(t) = \mathcal{T}\varphi_1(t) - \mathcal{T}\varphi_2(t) = \int_0^t K(t-s,s)[\varphi_1(s) - \varphi_2(s)]\, ds,$$

Taking norms in $\mathcal{X}$ and setting $\psi(t) = \|\varphi_1(t) - \varphi_2(t)\|_{\mathcal{X}}$,

$$\psi(t) \leq C \int_0^t (t-s)^a \psi(s)\, ds$$

for all $t \in [0,T]$. Since $a > -1$, we can apply Lemma 5.3 (with $b = a+1$) to conclude $\psi \equiv 0$; i.e., $\varphi_1 \equiv \varphi_2$, proving the uniqueness of the fixed point. To see that a fixed point exists, define a sequence $\{\varphi_k\}$ in $C_T(\mathcal{X})$ by $\varphi_0 = 0$ and, assuming φ_k defined for some $k \geq 0$, set $\varphi_{k+1} = \mathcal{T}\varphi_k$. We get for $k \geq 1$

$$\|\varphi_{k+1}(t) - \varphi_k(t)\|_{\mathcal{X}} \leq C \int_0^t (t-s)^a \|\varphi_k(s) - \varphi_{k-1}(s)\|_{\mathcal{X}}\, ds$$

so that, by induction, with $b = a+1$,

$$\|\varphi_{k+1}(t) - \varphi_k(t)\|_{\mathcal{X}} \leq \frac{(Ct^b\Gamma(b))^k}{\Gamma(kb+1)} \sup_{0\leq t\leq T} \|\psi(t)\|_{\mathcal{X}}$$

for $0 \leq t \leq T$, $k = 1, 2, \ldots$. Since the series of general term $(Ct^b\Gamma(b))^k/\Gamma(kb+1)$ converges to $\Phi_b(Ct^b\Gamma(b))$, the existence of a fixed point satisfying (54) follows. □

We consider the integrand of $\mathcal{M}$ defined in (5) in more detail. Set

$$F_j(t)(x) = F_j(x,t) = \frac{\partial F}{\partial x_j}(x,t) = -\frac{1}{2}(4\pi)^{-n/2} t^{-n/2-1} x_j e^{-|x|^2/4t}$$

for $j = 1, \ldots, n$. Since $\operatorname{div} u = 0$, we see integrating by parts that the k-th component of $F(t) * (u \cdot \nabla\varphi)$ is

$$[F(t) * (u \cdot \nabla\varphi)]_k = F(t) * (u \cdot \nabla\varphi_k) = F(t) * \operatorname{div}(\varphi_k u) = \sum_{j=1}^n F_j(t) * (u_j \varphi_k).$$

Similarly, integrating by parts, the k-th component of $F(t) * \nabla P(u,\varphi)$ works out to

$$[F(t) * \nabla P(u,\varphi)]_k = \sum_{\mu,\nu} F_k(t) * (R_\mu R_\nu(u_\mu \varphi_\nu)).$$

We thus have that the k-th component of $F(t) * H(u, \varphi)(s)$ is

$$[F(t) * H(u, \varphi)(s)]_k = \sum_{j=1}^{n} F_j(t) * Q_{kj}(u, \varphi)(s) \tag{55}$$

where

$$Q_{kj}(u, \varphi)(s) = u_j(s)\varphi_k(s) + \delta_{jk} \sum_{\mu,\nu} (R_\mu R_\nu (u_\mu(s)\varphi_\nu(s)). \tag{56}$$

Since the Riesz transforms commute with derivations and are bounded in L^p for $1 < p < \infty$, for every such p there is $C = C_p$ such that for all α with $\alpha \le m$ we get

$$\|D^\alpha Q_{kj}(u, \varphi)(s)\|_p = C \sum_{\mu,\nu} \|D^\alpha (u_\mu(s)\varphi_\nu(s))\|_p.$$

We take p defined by $1/p = 1/r + 1/q$ (q, r as in the statement of Theorem 2.1). By Leibnitz' formula and by Hölder's inequality,

$$\|D^\alpha Q_{kj}(u, \varphi)(s)\|_p \le C\|u(s)\|_{W^{m,r}} \|\varphi(s)\|_{W^{m,q}}, \tag{57}$$

for all $s \in [0, T]$, $|\alpha| \le m$. Since $1/q = 1/p - 1/r = 1/r' + 1/p - 1$ and

$$\|F_j(t)\|_{r'} = C_r t^{-\frac{n}{2r} - \frac{1}{2}}$$

(where C_r is a constant depending on r) we get by Young's inequality,

$$\begin{aligned} \|F(t) * H(u, \varphi)(s)\|_{W^{m,q}} &\le \sum_{|\alpha| \le m} \|D^\alpha (F(t) * H(u, \varphi)(s))\|_q \\ &\le C \sum_{|\alpha| \le m} \sum_{k,j} \|F_j(t) * D^\alpha (Q_{kj}(u, \varphi)(s))\|_q \\ &\le Ct^{-n/2r - 1/2} \sum_{|\alpha| \le m} \sum_{k,j} \|D^\alpha (Q_{kj}(u, \varphi)(s))\|_p. \end{aligned}$$

By (57)

$$\|F(t) * H(u, \varphi)(s)\|_{W^{m,q}} \le Ct^{-n/2r - 1/2} \|u(s)\|_{W^{m,r}} \|\varphi(s)\|_{W^{m,q}}. \tag{58}$$

We have:

Lemma 5.5. *Let $u \in C([0, T], W^{m,r}(\mathbf{R}^n)^n$ and let $v_0 \in W^{m,q} \cap W^{m,r}(\mathbf{R}^n)^n$. Then $\mathcal{L}$ defined by (6) is a map from $C([0, T], W^{m,q} \cap W^{m,r}(\mathbf{R}^n)^n)$ to itself with a unique fixed point in $C([0, T], W^{m,q} \cap W^{m,r}(\mathbf{R}^n)^n)$.*

Proof. Immediate from Lemma 5.4 with $\mathcal{X} = W^{m,q}(\mathbf{R}^n)^n \cap W^{m,r}(\mathbf{R}^n)^n$, $K(\tau,\sigma)\varphi = F(\tau) * H(u,\varphi)(\sigma)$, and $\psi = F(\cdot) * v_0$. The assumed estimates of Lemma 5.4, with $a = -n/(2r) - 1/2 > -1$ are a consequence of (58). Notice also that if $v_0 \in W^{m,q}$, then $F(\cdot) * v_0 \in C_T(W^{m,q})$ and $\|F(\cdot) * v_0\|_{C_T(W^{m,q})} \le \|v_0\|_{W^{m,q}}$ for $1 < q \le r$. □

Notice that (58) implies

$$\|\mathcal{M}\varphi(t)\|_{W^{m,q}} \le C \int_0^t (t-s)^{-n/2r-1/2} \|u(s)\|_{W^{m,r}} \|\varphi(s)\|_{W^{m,q}}\, ds \tag{59}$$

for $1 < q \le r$, $t \ge 0$, $\varphi \in C([0,T], W^{m,q}(\mathbf{R}^n)^n)$. An immediate consequence is

$$\|\mathcal{M}\varphi(t)\|_{W^{m,q}} \le C \|u\|_{C_T(W^{m,r})} \int_0^t (t-s)^{-n/2r-1/2} \|\varphi(s)\|_{W^{m,q}}\, ds\,. \tag{60}$$

The roles of u and φ are symmetric in all these computations (which is, in last instance, due to the fact that because $\operatorname{div} u = 0$ we have

$$(u\cdot\nabla\varphi)_j = u\cdot\nabla(\varphi_j) = \operatorname{div}(\varphi_j u)\,).$$

We can thus switch the roles of u and φ to get in case $u(t) \in W^{m,q}$, $1 < q \le r$

$$\|\mathcal{M}\varphi(t)\|_{W^{m,q}} \le C \|\varphi\|_{C_T(W^{m,r})} \int_0^t (t-s)^{-n/2r-1/2} \|u(s)\|_{W^{m,q}}\, ds \tag{61}$$

for all $t \in [0,T]$, all $\varphi \in C([0,T], W^{m,q})$. We will use (59) several times with $m = 0$, $q = r$, we thus restate it in this case as

$$\|\mathcal{M}\varphi(t)\|_r \le C \int_0^t (t-s)^{-n/2r-1/2} \|\varphi(s)\|_r \|u(s)\|_r\, ds. \tag{62}$$

For the rest of this section, v denotes the fixed point of $\mathcal{L}$. Until further notice we assume $q = 2$ so that $W^{m,q} = H^m$. We have

Lemma 5.6. *The fixed point v is a weak solution of* (1), (2) *if $m \ge 1$. If $m \ge 3$, then $v \in C^1([0,T], L^2)$ and*

$$\frac{d}{dt}v(t) = \Delta v(t) - u(t)\cdot\nabla v(t) - \nabla P(u,v)(t) \tag{63}$$

for all $t \in [0,T]$, so that v is a strong solution of (1), (2) *in this case.*

Proof. Defining $R\psi$ for $\psi \in L^2([0,T], L^p(\mathbf{R}^n))$ or for $\psi = (\psi_1, \dots, \psi_n) \in L^2([0,T], L^p(\mathbf{R}^n)^n)$ (where $1 \le p \le \infty$) by

$$R\psi(t) = \int_0^t F(t-s) * \psi(s) ds = \int_0^t F(s) * \psi(t-s) ds\,,$$

we can write $v(t) = F(t) * v_0 + RH(u,v)(t)$. Taking p so that $1/p = 1/2 + 1/r$, hence $1/2 = 1/r' + 1/p - 1$ and Young's inequality implies $\|F(t-s) * \psi(s)\|_2 \leq \|F(t-s)\|_{r'}\|\psi(s)\|_p$. Since $\|F(t)\|_{r'} = Ct^{-n/2r}$ for all $t > 0$, some constant C depending on r, and since $n/r < 1$, we get by Hölder's inequality

$$\|R\psi(t)\|_2 \leq C\int_0^t (t-2)^{-n/2r}\|\psi(s)\|_p\,ds \leq CT^{\frac{1}{2}(1-\frac{n}{r})}\|\psi\|_{L^2([0,T],L^p)}.$$

If $\psi \in C^\infty(\mathbf{R}^n \times [0,T])$, with $\psi(t) \in C_0^\infty(\mathbf{R}^n)$ for all $t \in [0,T]$ then, clearly, $R\psi \in C^\infty$ and the last estimate can be written in the form

$$\|R\psi\|_{C([0,T],L^2)} \leq CT^{\frac{1}{2}(1-\frac{n}{r})}\|\psi\|_{L^2([0,T],L^p)}. \tag{64}$$

Since the set of these ψ's is dense in $L^2([0,T],L^p)$, we proved that $\psi \in L^2([0,T], L^p(\mathbf{R}^n))$ implies $R\psi \in C([0,T],L^2(\mathbf{R}^n))$ and (64) holds. Taking once more a smooth ψ, we have

$$\frac{d}{dt}R\psi(t) = \Delta R\psi(t) + \psi(t) = R(\Delta\psi)(t) + \psi(t) \tag{65}$$

for all $t \in [0,T]$. Using (64) in (65), we get

$$\|\frac{d}{dt}R\psi\|_{C([0,T],L^2)} \leq C(T)\|\psi\|_{L^2([0,T],W^{2,p})} + \|\psi\|_{C([0,T],L^2)}. \tag{66}$$

By density we see that if $\psi \in C([0,T],L^2) \cap L^2([0,T],W^{2,p})$, then $R\psi \in C^1([0,T],L^2)$ and (65), (66) hold. We now specialize to the case $\psi = H(u,v) = u\cdot\nabla v + \nabla P(u,v)$. Since $u \in C([0,T],W^{m,r})$, $v \in C([0,T],H^m)$, it is easy to see that $H(u,v) \in C([0,T],W^{m-1,p})$ where, as above, $1/p = 1/r + 1/2$. We conclude that if $m \geq 3$, then $v = F(\cdot) * v_0 - RH(u,v)$ is in $C^1([0,T],L^2)$ and satisfies (63). If we only have $m \geq 1$, we can approximate $\psi = H(u,v)$ in $C([0,T],L^p)$ by a sequence $\{\psi_k\}$ in $C([0,T],W^{3,p})$. Setting $v_k = F(\cdot) * v_0 - R\psi_k$, we get $v - v_k = R(\psi_k - \psi)$ which, by (64), converges to 0 in $C([0,T],L^2)$. If $\varphi \in C_0^\infty(\mathbf{R}^n)^n$, then

$$(v_k(t),\varphi) = (v_0,\varphi) + \int_0^t [(v_k(t),\Delta\varphi) - (\psi_k(t),\varphi)]\,dt.$$

Letting $k \to \infty$, we see v is a weak solution of the (LNS) equations. □

To continue, we need a somewhat technical lemma.

Lemma 5.7. *Let $\{u_k\}$ be a sequence in $C([0,T],L^r(\mathbf{R}^n)^n)$ converging to u and let $\{v_0^k\}$ be a sequence in $L^2(\mathbf{R}^n)^n$ converging to v_0. For $k = 1,2,\ldots$ define $\mathcal{M}_k$ by*

$$\mathcal{M}_k\varphi(t) = \int_0^t F(t-s) * H(u_k,\varphi)(s)\,ds,$$

set $\mathcal{L}_k = F(\cdot)*v_0^k - \mathcal{M}_k$. Then all conditions of Lemma 5.5 are satisfied with $m = 0$; let $v_k \in C([0,T],L^2)$ be the fixed point of $\mathcal{L}_k$, $k = 1,2,\ldots$. Then $\{v_k\}$ converges to v, the fixed point of $\mathcal{L}$, in $C([0,T],L^2)$.

Proof. We have for $t \in [0,T]$

$$v(t) - v_k(t) = \mathcal{L}v(t) - \mathcal{L}_k v_k(t) = F(t) * (v_0 - v_0^k) - \mathcal{M}(v - v_k)(t) - \tilde{\mathcal{M}}_k v_k(t),$$

where $\tilde{\mathcal{M}}_k = \mathcal{M} - \mathcal{M}_k$ is an operator of the same type as $\mathcal{M}$ (defined by (5) except that u is replaced by $u - u_k$. Taking L^2 norms, estimating $\|F(t) * (v_0 - v_0^k)\|_2$ by Young's inequality, $\|\mathcal{M}(v - v_k)(t)\|_2$ by (60) (with $m = 0$, so $C_T(W^{m,r}) = C([0,T], L^r)$, $H^m = L^2$), and $\|\tilde{\mathcal{M}}_k v_k(t)\|_2$ by (62), we get

$$\begin{aligned}\|v(t) - v_k(t)\|_2 \le \|v_0 - v_0^k\|_2 + C\|u\|_{C([0,T],L^r)} \int_0^t (t-s)^{-n/2r-1/2} \|v(s) - v_k(s)\|_2 ds \\ + C(T)\|u - u_k\|_{C([0,T],L^r)} \|v_k\|_2.\end{aligned}$$

Since

$$\int_0^t (t-s)^{-n/2r-1/2}\, ds \le CT^{1/2-n/2r} = C(T)$$

for all $t \in [0,T]$, Gronwall's lemma implies

$$\|v(t) - v_k(t)\|_2 \le \left(\|v_0 - v_0^k\|_2 + C(T)\|u - u_k\|_{C([0,T],L^r)}\|v_k\|_2\right) e^{C(T)}$$

for all $t \in [0,T]$. The Lemma follows. $\square$

Assume now $\operatorname{div} v_0 = 0$. We can approximate v_0 in L^2 by a sequence $\{v_0^k\}$ in $C_0^\infty(\mathbf{R}^n)^n$ with $\operatorname{div} v_0^k = 0$ for all k,we approximate u in $C([0,T], L^r)$ by a sequence $\{u_k\}$ in $C([0,T], C_0^\infty(\mathbf{R}^n)^n)$. By the previous lemma, if v_k is the fixed point corresponding to v_0^k and u_k, then $\{v_k\}$ converges to v in $C([0,T], L^2)$. By Lemma 5.6, $v_k \in C^1([0,T], L^2)$ and

$$v_k'(t) - \Delta v_k = -u_k \cdot \nabla v_k - \nabla P(u_k, v_k). \tag{67}$$

We have if $\operatorname{div} u = 0$

$$\Delta P(u,v) = -\sum_{j,k} \frac{\partial^2}{\partial x_j \partial x_k}(u_k v_j) = -\operatorname{div}(u \cdot \nabla v). \tag{68}$$

In fact, by the definition of P and of the Riesz transforms,

$$\widehat{\Delta P(u,v)}(\xi,t) = \sum_{j,k} \xi_j \xi_k \widehat{u_j v_k}(\xi,t)$$

so that the first equality in (68) follows by taking the inverse Fourier transform. The second equality is a consequence of $\operatorname{div} u = 0$. Taking divergence of both sides in (67) we get

$$(\operatorname{div} v_k)'(t) - \Delta(\operatorname{div} v_k) = -\operatorname{div}(u_k \cdot \nabla v_k) - \Delta P(u_k, v_k) = 0,$$

by (68), since $\operatorname{div} u_k = 0$. Thus $\operatorname{div} v_k$ satisfies the (homogeneous) heat equation with initial datum $\operatorname{div} v_k(0) = \operatorname{div} v_0^k = 0$, hence is identically 0. Letting $k \to \infty$, we proved that $v \in C([0,T],\mathcal{H})$ if $\operatorname{div} v_0 = 0$.

Finally, if we multiply both sides of (67) by v_k and integrate over x then the terms $(v_k, u_k \cdot v_k)$ and $(v_k, \nabla P(u_k, v_k))$ vanish because $\operatorname{div} v_k = 0$, and we get

$$\frac{d}{dt}\|v_k(t)\|_2^2 + 2\|\nabla v_k(t)\|_2^2 = 0.$$

Integrating from 0 to t,

$$\|v_k(t)\|_2^2 + 2\int_0^t \|\nabla v_k(s)\|_2^2\, ds = \|v_0^k\|_2^2$$

for all $t \in [0,T]$. Letting $k \to \infty$ we get (49) by Fatou's Lemma. The proof of Theorem 2.1 is complete.

5.3. Part II. In this Subsection we present the proof of Theorem 2.6 which will be relabeled according to the ordering of the section

Theorem 5.8. *Assume $r \geq 4$ if $n = 2$, $r > n$ if $n \geq 3$. Let $u_0 \in L^2) \cap L^r$ and $div u_0 = 0$. Let q be as above; i.e., $q = nr/(2n+r)$. There is $\delta > 0$ such that if $\|u_0\|_r < \delta$, then the admissible solution u of initial datum u_0 exists globally. Moreover, there is $\tilde{C}_0 > 0$ such that for β satisfying $1/2 < \beta < n/2r + 1/2$*

$$\|u(t)\|_r \leq \tilde{C}_0(1+t)^{-\beta} \quad \text{for all } t > 0. \tag{69}$$

Proof. Notice first that since $q = nr/(2n+r) < r$ we have that $u_0 \in L^q$ since by hypothesis $u_0 \in L^2) \cap L^r$. Thus let $\|u_0\|_q \leq C_0$ for some (fixed) given C_0. Let $\{u_k\}$ be the defining sequence of the solution constructed on the interval $[0,\infty)$. Note that $u_0 \in L^2) \cap L^r$ Since $u_0(t)$, the first term in our sequence, was defined as the solution to the heat equations with data u_0 it will be in $C((0,\infty) : W^{m,r})$. Hence by theorem 2.1 it follows that u_1 will have the necessary smoothness, by induction all the terms of the sequence $\{u_k\}$ will be bounded in L^r. Unfortunately the bounds obtained this way are not uniform. Thus we will have to use the smallness of the data to get uniform bounds.

By (26) we have

$$\|F(t) * u_0\|_r \leq C_0(1+t)^{-n/2r-1/2} \leq C_0(1+t)^{-\beta}.$$

Applying (62) (with $u = u_k$, $\varphi = u_{k+1}$) we get

$$\begin{aligned} \|u_{k+1}(t)\|_r = \|\mathcal{L}_k u_{k+1}(t)\|_r &\leq \|F(t) * u_0\|_r + \|\mathcal{M}_k u_{k+1}\|_r \\ &\leq C_0(1+t)^{-\beta} + C\int_0^t (t-s)^{-\frac{n}{2r}-\frac{1}{2}} \|u_k(s)\|_r \|u_{k+1}(s)\|_r\, ds. \end{aligned} \tag{70}$$

We emphasize that C depends only on n, r. *Claim* If $1-\sigma<\beta<\sigma<1$, then

$$\lim_{T\to\infty}(1+T)^{\beta}\int_0^T(T-s)^{-\sigma}(1+s)^{-2\beta}\,ds=0.$$

In fact, changing variables we get

$$\begin{aligned}(1+T)^{\beta}\int_0^T(T-s)^{-\sigma}(1+s)^{-2\beta}\,ds&=(1+T)^{\beta}T^{1-\sigma}\int_0^1(1-s)^{-\sigma}(1+Ts)^{-2\beta}\,ds\\&\le(1+\frac{1}{T})^{\beta}\int_0^1(1-s)^{-\sigma}(\frac{T}{1+Ts})^{\beta-\sigma+1}(1+Ts)^{-\beta-\sigma+1}\,ds\\&\le(1+\frac{1}{T})^{\beta}\int_0^1(1-s)^{-\sigma}s^{-\beta+\sigma-1}(1+Ts)^{-\beta-\sigma+1}\,ds.\end{aligned}$$

Since $-\beta+\sigma-1>-1$ and $-\beta-\sigma+1<0$, the last integrand is uniformly bounded by the function $s\to(1-s)^{-\sigma}s^{-\beta+\sigma-1}\in L^1(0,1)$ and converges pointwise to 0 as $T\to\infty$. The *claim follows* from Lebesgue's dominated convergence theorem.

We apply the claim with β as in the statement of the theorem, $\sigma=n/2r+1/2$ to find T_0 such that

$$4CC_0(T+1)^{\beta}\int_0^T(T-s)^{-\sigma}(1+s)^{-2\beta}\,ds<1 \tag{71}$$

whenever $T\ge T_0$. Next we select $\Lambda>0$ small enough so as to satisfy

$$\Lambda<2C_0(1+T_0)^{-\beta}, \tag{72}$$

$$T_0\le(\Lambda C_r)^{-1/\gamma}(\frac{1}{2})^{1/\gamma}, \tag{73}$$

where C_r is the same constant appearing in (21) and $\gamma=(1/2)(1-n/r)$ as before. We *claim*: If u_0 satisfies

$$\|u_0\|_r\le\frac{1}{2}\Lambda, \tag{74}$$

then

$$\|u_k(t)\|_r\le 2C_0(1+t)^{-\beta}\qquad\text{for all}\quad t>0 \tag{75}$$

for $k=0,1,2,\ldots$ In fact, (75) is certainly true for $k=0$, since $u_0(t)$ is the solution to the heat equation. We proceed by induction assuming it proved for some $k\ge0$. Assume that (75) is false if we replace k by $k+1$ and let

$$T_1=\sup\{T\,|\,\|u_{k+1}(t)\|_r\le 2C_0(1+t)^{-\beta}\,\text{for}0\le t\le T\}.$$

Since $u_{k+1}:[0,\infty)\to L^r$ is continuous, we see that $T_1>0$, while our assumption that (75) fails to hold for $k+1$ implies $T_1<\infty$. We notice first that $T_1\ge T_0$. In fact, by (73, T_0 satisfies (21) with $\kappa = 2$ while (74) implies u_0 satisfies (20)(also with $\kappa = 2$). Lemma 2.3 then implies that $\|u_{k+1}(t)\|_r \leq \Lambda$ for $0 \leq t \leq T_0$ hence, by (72),

$$\|u_{k+1}(t)\|_r(1+t)^\beta < 2C_0(\frac{1+t}{1+T_0})^\beta \leq 2C_0$$

for $0 \leq t \leq T_0$. We now apply (70) with $t = T_1$; since both $\|u_k(s)\|_r$ and $\|u_{k+1}(s)\|_r$ are bounded by $2C_0(1+t)^{-\beta}$ in $[0, T_1]$, the definition of T_1 (and the continuity of u_k from $[0,\infty)$ to L^r) implies

$$\begin{aligned} 2C_0(1+T_1)^{-\beta} &= \|u_{k+1}(T_1)\|_r \\ &\leq C_0(1+T_1)^{-\beta} + 4CC_0^2 \int_0^{T_1} (T_1 - s)^{-n/2r-1/2} s^{-2\beta}\, ds. \end{aligned}$$

Multiplying by $(1+T_1)^\beta$ gives since $T_1 \geq T_0$

$$2C_0 \leq C_0 + C_0[4CC_0(1+T_1)^\beta \int_0^{T_1} (T_1 - s)^{-n/2r-1/2} s^{-2\beta}\, ds] < 2C_0$$

which is a contradiction. *The claim is established.* We take $\delta = (1/2)\Lambda$. The sequence $\{u_k\}$ defining the admissible solution of initial value u_0 with $\|u_0\| \leq \delta$ will then satisfy (75) for all k. If we look at the proof of our theorem of existence of admissible solutions (Theorem 2.4) we see that the assumption that T satisfies (21) is used only to invoke lemma 2.3 . Since $\|u_k\|_r \leq \Lambda$ in $[0, T_0]$ for all $k \geq 0$, T_0 as above, we have

$$\|u_k\|_r \leq \Lambda_1 = \max\{\Lambda, 2C_0(1+T_0)^{-\beta}\}.$$

It follows that (24) of Theorem 2.4 holds with Λ replaced by Λ_1 for arbitrary $t \in [0,\infty)$. The rest of the proof of the theorem is valid for arbitrary $T > 0$, completing the proof of the existence part of our theorem. Since (27) is immediate from (75), we are done. □

6. Appendix B.

In this section we establish the proof of Lemma 3.3 . The lemma here is renumbered according to the ordering of the section.

Lemma 6.1. *Assume*

$$0 < \alpha < (n+1/2)(1-2/r), \tag{76}$$

assume u_0 *satisfies the conditions of Theorems* 3.1, 3.2 *and that* (29) *holds. Then, for every* $T > 0$,

$$\liminf_{R\to\infty} \int_{|x|=R} \int_0^T |E_\alpha|\, dt\, dS = 0.$$

Proof. We have

$$\int_0^T |E_\alpha|\,dt \le \sum_{i=1}^4 \int_0^T g_i\,dt$$

where $g_1 = \frac{1}{2}|x|^\alpha \nabla(|u|^2)$, $g_2 = \frac{1}{2}\alpha|x|^{\alpha-1}|u|^2$, $g_3 = \frac{1}{2}|x|^\alpha|u|^3$, $g_4 = |x|^\alpha|p||u|$.
Claim: For $i = 1, 2, 3, 4$, we have $g_i \in L^{q_i}_{\sigma_i}$ where

$$\sigma_i \ge \frac{n-1}{q_i'}, \quad \frac{1}{q_i} + \frac{1}{q_i'} = 1. \tag{77}$$

In fact, the first term g_1 is bounded by a sum of terms of the form $|x|^\alpha u_j \nabla u_k$; by Theorem 3.2 $u_j \in L^r_\nu$ for $0 \le \nu < n(1-2/r)$ while $\nabla u_k \in L^2_\beta$ for $n/r < \beta < (n/2)(1-2/r)$ by Theorem 3.1 (applied with $\alpha = \beta/2$). It follows that $g_1 \in L^{2r/(r+2)}_{\nu+\beta-\alpha}$. Take $q_1 = 2r/(r+2)$, $\sigma_1 = \nu+\beta-\alpha$, where we take $\nu+\beta$ sufficiently close to $(3n/2)(1-2/r)$ (as we may) so that

$$\sigma_1 = \nu+\beta-\alpha \ge (\frac{3n}{2} - n - \frac{1}{2})(1-\frac{2}{r}) = (\frac{n-1}{2})(1-\frac{2}{r}) = \frac{n-1}{q_1'}$$

and (77) holds for $i = 1$. Using now $u \in L^r_\nu$ we see $g_2 \in L^{r/2}_{2\nu-\alpha+1}$, $g_3 \in L^{r/3}_{3\nu-\alpha}$. To deal with g_3 we assume (as we may) from now on that $r \ge 3$. Taking ν sufficiently close to $n(1-2/r)$ it is now immediate that (77) holds for $i = 2, 3$. Concerning g_4 we use $u \in L^r_\nu \cap L^r$ to get $u_j u_k \in L^{r/2}_\nu$. Since $\nu < n(1-2/r)$ is equivalent to $r/2 > n/(n-\nu)$, the Riesz transforms are bounded operators in $L^{r/2}_\nu$ by Lemma 5.1. Recalling that $p = P(u,u) = \sum_{j,k} R_j R_k(u_j u_k)$ we have $p \in L^{r/2}_\nu$, hence $|x|^\alpha pu \in L^{r/3}_{2\nu-\alpha}$. Taking $\sigma_4 = 2\nu-\alpha$, $q_4 = r/3$ it is again easy to see that they satisfy (77) if α satisfies (76) and ν is sufficiently close to $n(1-2/r)$. The *claim is established*. Because all our bounds are uniform in bounded intervals of t, we also proved

$$G_i(T) = \int_0^T g_i\,dt \in L^{q_i}_{\sigma_i}$$

for $i = 1, 2, 3, 4$. Assume now $\epsilon > 0$, $R_1 \ge 1$ are given. There is then $R_0 > R_1$ such that

$$\int_{|x|>R_0} |x|^{\sigma_i q_i}|G_i|^{q_i}\,dx < \epsilon/5$$

for $i = 1, 2, 3, 4$. For $i = 1, 2, 3, 4$ let

$$S_i = \{R : R_0 \le R \le R_0 + 1 \int_{|x|=R} |x|^{\sigma_i q_i}|G_i|^{q_i}\,dS \ge \epsilon.\}$$

With $|S|$ denoting one dimensional Lebesgue measure of the set S, we have

$$\epsilon|S_i| \leq \int_{R_0}^{R_0+1} \int_{|x|=R} |x|^{\sigma_i q_i} |G_i|^{q_i} dS\, dR \leq \int_{|x|>R_0} |x|^{\sigma_i q_i} |G_i|^{q_i} dx < \epsilon/5;$$

thus, $|S_i| \leq 1/5$ for $i = 1, 2, 3, 4$, hence $[R_0, R_0 + 1] \setminus \cup_{i=1}^4 S_i \neq \emptyset$. We proved: For every $\epsilon > 0$. $R_1 \geq 1$, there exists $R > R_1$ such that

$$\int_{|x|=R} |x|^{\sigma_i q_i} |G_i|^{q_i}\, dS < \epsilon$$

for $i = 1, 2, 3, 4$. Now by Hölder,

$$\begin{aligned} \int_{|x|=R} |G_i(x)|\, dS &= R^{-\sigma_i} \int_{|x|=R} |x|^{\sigma_i} |G_i(x)|\, dS \\ &\leq R^{-\sigma_i} \Big(\int_{|x|=R} dS \Big)^{1/q_i'} \Big(\int_{|x|=R} |x|^{q_i \sigma_i} |G_i(x)|^{q_i}\, dS \Big)^{1/q_i} \\ &= C R^{-\sigma_i + (n-1)/q_i'} \Big(\int_{|x|=R} |x|^{q_i \sigma_i} |G_i(x)|^{q_i}\, dS \Big)^{1/q_i} < C\epsilon^{1/q_i} \end{aligned}$$

for $i = 1, 2, 3, 4$, where C does not depend on R; the last inequality is due to the fact that $R \geq 1$ and $-\sigma_i + (n-1)/q_i' \leq 0$. It follows that

$$\int_{|x|=R} \int_0^T |E_\alpha|\, dS < C \sum_{i=1}^4 \epsilon^{q_i};$$

since ϵ is arbitrary, the lemma is proved.

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