Advances in Differential Equations Volume 5 (7–9) July–September 2000, pp. 801–832

NONNEGATIVE GLOBAL SOLUTIONS TO A CLASS OF STRONGLY COUPLED REACTION-DIFFUSION SYSTEMS*

Hiroki Hoshino
Fujita Health University College, Toyoake, Aichi 470-1192, Japan

(Submitted by: Hitoshi Ishii)

Abstract. A class of strongly coupled reaction-diffusion systems is studied. First, under some conditions, it is shown that a nonnegative solution exists globally in time. After that, asymptotic behavior of the nonnegative global solution is considered. Especially, when the solution uniformly converges to a steady state with a polynomial rate as time goes to infinity, large-time approximation of the solution is investigated. By the energy method and analytic semigroup theory, it is proved that a global solution for the corresponding system of ordinary differential equations has the role of an asymptotic solution for the reaction-diffusion system and that the spatial average of the global solution to the reaction-diffusion system gives an asymptotic description.

1. Introduction. In this paper, we study a nonnegative solution $(u,v)(t,x)$ to the following strongly coupled reaction-diffusion system under homogeneous Neumann boundary conditions:

$$\begin{cases} u_t = a\Delta u + b\Delta v + f(u)g(v), & \text{in} \quad (0,\infty)\times\Omega, \\ v_t = \qquad\quad d\Delta v - f(u)g(v), & \text{in} \quad (0,\infty)\times\Omega, \end{cases} \tag{1.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{1.2}$$

$$(u,v)(0,x) = (u_0,v_0)(x), \qquad \text{in} \quad \Omega, \tag{1.3}$$

*The research was partially supported by Grant-in-aid for Encouragement of Young Scientists, The Ministry of Education, Science, Sports and Culture, Japan.

Accepted for publication January 1999.
AMS Subject Classifications: 35B35, 35B40, 35K50, 35K57.

where $\Omega \subset \mathbf{R}^N$ $(N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\partial/\partial\nu$ denotes the outward normal derivative to $\partial\Omega$, and f and g are given sufficiently smooth functions.

In the case of $b = 0$ in (1.1), i.e., a weakly coupled reaction-diffusion system, existence and asymptotic behavior of a nonnegative global solution have been considered by Alikakos [1], Masuda [15], Haraux and Youkana [8], Barabanova [4], Hoshino [9], and so on.

Throughout this paper, we assume that the coefficients a, b and d are positive constants and that

$$a \neq d. \tag{1.4}$$

Concerning the initial value $(u_0, v_0)(x)$, the following assumptions are imposed:

$$u_0, v_0 \in L^\infty(\Omega) \quad \text{and} \quad u_0(x), v_0(x) \geq 0 \quad \text{for} \quad x \in \overline{\Omega}. \tag{1.5}$$

The system (1.1)–(1.3) with (1.4) and (1.5) is related to physics, chemistry, biology, etc. We can interpret that (1.1) represents phenomena with non-Fickian diffusion having a nondiagonal diffusion matrix and a balance law:

$$u_t = -\nabla \cdot J_1 + f(u)g(v), \qquad v_t = -\nabla \cdot J_2 - f(u)g(v),$$

where J_1 and J_2 are fluxes; i.e., $J_1 = -\nabla(au + bv)$ and $J_2 = -\nabla(dv)$. In some models, u and v represent temperature or concentration of chemical substances. For the details of the physical or other background, see, e.g., de Groot and Mazur [6], Aris [3] and Kirkaldy [14]. Generally speaking, we cannot always expect that u and v are nonnegative for all $t > 0$, $x \in \overline{\Omega}$, even if the initial values are nonnegative (cf. Otsuka [17]). Hence, it seems important to give sufficient conditions which assure the nonnegativity of $(u, v)(t, x)$ for all $t \geq 0$ and $x \in \overline{\Omega}$.

As regards construction of nonnegative solutions to (1.1)–(1.3), Kirane [13] and Okoya [16] have shown that in the case $a < d$ with $f(u) \geq 0$ for $u \in \mathbf{R}$ and $g(v) = v$, if $u_0(x) \geq (b/(d-a))v_0(x) \geq 0$ for $x \in \overline{\Omega}$, then there is a global solution $(u, v)(t, x)$ such that $u(t, x) \geq (b/(d-a))v(t, x) \geq 0$ for all $t > 0$ and $x \in \overline{\Omega}$, and u is uniformly bounded, and have investigated the uniform convergence properties of $(u, v)(t, x)$. Moreover, we have obtained only an exponentially decaying property of v to 0 in [16]. See also Kanel and Kirane [12].

Some authors, for example Amann [2], Cosner [5] and Redlinger [19], have considered pointwise a priori bounds for solutions not only to such a class of systems having triangular diffusion matrices as (1.1)–(1.3) but also to various strongly coupled reaction-diffusion systems under Neumann or Dirichlet boundary conditions, although the construction of nonnegative solutions has not been investigated therein.

In this paper, we deal with both cases $a \geq b + d$ and $a < d$ under some conditions on f, g and (u_0, v_0) (see Assumptions 2.1–2.3). After we obtain sufficient conditions on the existence of nonnegative global solutions to (1.1)–(1.3) (Theorems 2.1 and 2.2 below), we investigate uniform convergence properties of the nonnegative global solution $(u, v)(t, x)$ to the corresponding constant state $(\overline{u}_0+\overline{v}_0, 0)$ (Theorem 3.1 below), where $\overline{\omega}$ stands for the spatial average of $\omega(x)$:

$$\overline{\omega} = |\Omega|^{-1} \int_\Omega \omega(x)\, dx.$$

When $g(v) = v^n$ with $n \geq 1$, we can obtain rates of the convergence of $(u, v)(t, x)$. In fact, when $n = 1$ (respectively $n > 1$), we will prove that $(u, v)(t, x)$ tends to $(\overline{u}_0 + \overline{v}_0, 0)$ with an exponential rate (respectively a polynomial rate $t^{-1/(n-1)}$). For the details, see Theorem 3.2.

Especially when $g(v) = v^n$ with $n > 1$, we study large-time approximation of the nonnegative global solution $(u, v)(t, x)$ for (1.1)–(1.3) in terms of the nonnegative solution $(U, V)(t)$ to the system

$$\begin{cases} U' = f(U)V^n, & t > 0, \\ V' = -f(U)V^n, & t > 0, \end{cases} \tag{1.6}$$

$$(U, V)(0) = (\overline{u}_0, \overline{v}_0), \tag{1.7}$$

where $' = d/dt$ and $f(u) > 0$ for $u > 0$. The solutions $U(t)$ and $V(t)$ to the problem (1.6) and (1.7) are satisfying

$$U(t) + V(t) = u_\infty, \qquad t \geq 0, \tag{1.8}$$

where

$$u_\infty = \overline{u}_0 + \overline{v}_0, \tag{1.9}$$

and there are positive constants C and C' such that

$$C(1+t)^{-1/(n-1)} \leq u_\infty - U(t) = V(t) \leq C'(1+t)^{-1/(n-1)}, \qquad t \geq 0. \tag{1.10}$$

We can prove that $(U, V)(t)$ is an asymptotic solution to the system (1.1)–(1.3). In fact, the difference between $(u, v)(t, x)$ and $(U, V)(t)$ uniformly decays with the rate $t^{-1/(n-1)-1}$ as $t \to \infty$. See Theorem 4.1. Moreover, we can also show that $(\overline{u}, \overline{v})(t)$ approximates $(u, v)(t, x)$ itself exponentially as $t \to \infty$ (Theorem 4.2 below), where

$$\overline{\omega}(t) = |\Omega|^{-1} \int_{\Omega} \omega(t, x)\, dx.$$

The method of analyzing these approximations is based on the one established by Hoshino and Kawashima [10] and Hoshino [9]. Actually, we see that asymptotic approximations of the nonnegative global solution to (1.1)–(1.3) obtained in this paper are very similar to those of the global solution for the weakly coupled reaction-diffusion system which is the system with $b = 0$ in (1.1) studied by [9].

This paper is organized as follows. In Section 2, a unique nonnegative global solution $(u, v)(t, x)$ to (1.1)–(1.3) with (1.4), (1.5) is obtained. Section 3 is devoted to the uniform convergence properties of $(u, v)(t, x)$ to the corresponding steady state. After that, we restrict ourselves to the case where $g(v) = v^n$ with $n > 1$, that is to say, $(u, v)(t, x)$ uniformly converges to the steady state with a polynomial rate as time goes to infinity, and we consider large-time approximation of $(u, v)(t, x)$ in Section 4. Finally, a few remarks are given in Section 5.

2. Nonnegative global solution. The following assumptions are imposed throughout this paper.

Assumption 2.1.
(i) a, b, d are positive constants and $a \neq d$ (see (1.4)).
(ii) Two given functions f and g are sufficiently smooth.
(iii) $f(u) \geq 0$ for $u \in \mathbf{R}$, $f(u) > 0$ for $u > 0$ and $\lim_{u \to \infty} u^{-1} \log(1 + f(u)) = 0$.
(iv) $g(0) = 0$ and $g(v) > 0$ for $v > 0$.

An example of g is $g(v) = \mathrm{sign}(v)|v|^n$ with $n \geq 1$.

Assumption 2.2. In addition to (1.5), if $a \geq b + d$, then we assume that

$$u_0(x) \geq \frac{b}{a - d}\|v_0\|_\infty - \frac{b}{a - d} v_0(x)$$

for $x \in \overline{\Omega}$. Here, $|\Omega|$ means the volume of Ω, and $\|\omega\|_p = \|\omega\|_{L^p(\Omega)}$.

Assumption 2.3. In addition to (1.5), if $a < d$, then we assume that

$$u_0(x) \geq \frac{b}{d - a} v_0(x) \qquad \text{for} \quad x \in \overline{\Omega}.$$

We frequently use the following two exponents.

Definition 2.1.

$$(r_1, r_2) = \begin{cases} ((a-d)/b, 0), & \text{if} \quad a > d, \\ (0, (d-a)/b), & \text{if} \quad a < d. \end{cases}$$

Applying the analytic semigroup theory to the integral equations

$$\begin{cases} u(t) = \dfrac{1}{r_1+r_2} e^{-td_1A}(r_1u_0+v_0) + \dfrac{1}{r_1+r_2} e^{-td_2A}(r_2u_0-v_0) \\ \qquad + \dfrac{r_1-1}{r_1+r_2} \displaystyle\int_0^t e^{-(t-\tau)d_1A}(f(u)g(v))(\tau)\,d\tau \\ \qquad + \dfrac{r_2+1}{r_1+r_2} \displaystyle\int_0^t e^{-(t-\tau)d_2A}(f(u)g(v))(\tau)\,d\tau, \\ v(t) = \dfrac{r_2}{r_1+r_2} e^{-td_1A}(r_1u_0+v_0) - \dfrac{r_1}{r_1+r_2} e^{-td_2A}(r_2u_0-v_0) \\ \qquad + \dfrac{r_2(r_1-1)}{r_1+r_2} \displaystyle\int_0^t e^{-(t-\tau)d_1A}(f(u)g(v))(\tau)\,d\tau \\ \qquad - \dfrac{r_1(r_2+1)}{r_1+r_2} \displaystyle\int_0^t e^{-(t-\tau)d_2A}(f(u)g(v))(\tau)\,d\tau, \end{cases} \tag{2.1}$$

which are equivalent to the system (1.1)–(1.3), we can get the existence and the regularity of a unique local solution $(u, v)(t, x)$ for (1.1)–(1.3) on $[0, T_{\max}) \times \Omega$. Here, A denotes $-\Delta$ with homogeneous Neumann boundary conditions on $\partial\Omega$, $(d_1, d_2) = (d+br_1, d-br_2)$, and $\{e^{-td_jA}\}_{t\geq 0}$ is the analytic semigroup generated by $-d_jA$ $(j = 1, 2)$. Note that $(d_1, d_2) = (a, d)$ (respectively (d, a)) in the case $a > d$ (respectively $a < d$); i.e., $d_1 = \max\{a, d\}$, $d_2 = \min\{a, d\}$.

Under (1.4) and (1.5) and Assumptions 2.1 and 2.2, global existence of a nonnegative solution $(u, v)(t, x)$ can be proved in the case $a > d$.

Theorem 2.1. *Suppose $a \geq b + d$. Under Assumptions 2.1 and 2.2, there exists a unique nonnegative solution $(u, v)(t, x)$ to the system (1.1)–(1.3) for all $t \geq 0$. This solution is smooth for $t > 0$. Furthermore, $v(t, x)$ satisfies*

$$0 \leq v(t, x) \leq \|v_0\|_\infty, \qquad t > 0, \quad x \in \overline{\Omega}, \tag{2.2}$$

there exists a constant $M > 0$ *such that*

$$\frac{b}{a-d}\|v(t)\|_\infty - \frac{b}{a-d}v(t,x) \le u(t,x) \le M$$

for all $t > 0$ *and* $x \in \overline{\Omega}$, *and* $(u,v)(t,x)$ *satisfies*

$$\overline{u}(t) + \overline{v}(t) = u_\infty, \qquad t \ge 0, \tag{2.3}$$

where u_∞ *is defined by* (1.9).

When $a < d$, we can get the following results.

Theorem 2.2. *If* $a < d$ *is assumed, then under Assumptions* 2.1 *and* 2.3 *there exists a unique nonnegative global solution* $(u,v)(t,x)$ *to the system* (1.1)–(1.3). *This solution is smooth for* $t > 0$, $v(t,x)$ *fulfills the boundedness* (2.2), *there is a constant* $\tilde{M}$ *such that* $\frac{b}{d-a}v(t,x) \le u(t,x) \le \tilde{M}$ *for all* $t > 0$ *and* $x \in \overline{\Omega}$, *and* $(u,v)(t,x)$ *satisfies* (2.3) *for all* $t \ge 0$.

Diagonalization of the triangular diffusion matrix in (1.1) and the well-known comparison theorem to parabolic equations (see, e.g., Pao [18] or Smoller [21]) are essential to the proofs of Theorems 2.1 and 2.2.

Define (w,z) by

$$(w,z) = (r_1 u + v, r_2 u - v). \tag{2.4}$$

Obviously, we see that (2.4) gives

$$(u,v) = \Big(\frac{1}{r_1+r_2}w + \frac{1}{r_1+r_2}z, \frac{r_2}{r_1+r_2}w - \frac{r_1}{r_1+r_2}z\Big), \tag{2.5}$$

so that substituting (2.4) and (2.5) into (1.1)–(1.3) yields

$$\begin{cases} w_t = d_1\Delta w + (r_1-1)f\big(\frac{1}{r_1+r_2}w + \frac{1}{r_1+r_2}z\big)g\big(\frac{r_2}{r_1+r_2}w - \frac{r_1}{r_1+r_2}z\big), \\ z_t = d_2\Delta z + (r_2+1)f\big(\frac{1}{r_1+r_2}w + \frac{1}{r_1+r_2}z\big)g\big(\frac{r_2}{r_1+r_2}w - \frac{r_1}{r_1+r_2}z\big), \end{cases} \tag{2.6}$$

$$\text{in} \quad (0,\infty)\times\Omega,$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{2.7}$$

$$(w,z)(0,x) = (w_0,z_0)(x) = (r_1u_0 + v_0, r_2u_0 - v_0)(x), \qquad \text{in} \quad \Omega. \tag{2.8}$$

Proof of Theorem 2.1. When $a > d$, it follows from Definition 2.1 that (2.6) becomes

$$\begin{cases} w_t = a\Delta w + (r_1 - 1)f(\frac{1}{r_1}w + \frac{1}{r_1}z)g(-z), \\ z_t = d\Delta z + f(\frac{1}{r_1}w + \frac{1}{r_1}z)g(-z), \end{cases} \quad \text{in} \quad (0,\infty)\times\Omega. \tag{2.9}$$

Now we consider the system (2.9) with (2.7), (2.8). Let M_1 be a positive constant. Then the comparison theorem for initial boundary value problems of parabolic equations and Assumptions 2.1 and 2.2 assure that if $-M_1 \le z_0(x) \le 0$, then $-M_1 \le z(t,x) \le 0$ for $t \in (0, T_{\max})$ and $x \in \overline{\Omega}$; that is to say, if $0 \le v_0(x) \le M_1$, then $0 \le v(t,x) \le M_1$ for $t \in (0, T_{\max})$ and $x \in \overline{\Omega}$.

Next, the assumption $a \ge b + d$ implies $r_1 \ge 1$. Hence, with use of the comparison theorem again, we see that if $w_0(x) \ge M_2$, where $M_2 \ge M_1$ is a constant, then $w(t,x) \ge M_2$ for $t \in (0, T_{\max})$ and $x \in \overline{\Omega}$. Therefore, if $r_1 u_0(x) + v_0(x) \ge M_2 \ge M_1$, then $r_1 u(t,x) + v(t,x) \ge M_2 \ge M_1$ for $t \in (0, T_{\max})$ and $x \in \overline{\Omega}$, which implies

$$r_1 u(t,x) \ge M_1 - v(t,x) \ge 0, \qquad t \in (0, T_{\max}), \quad x \in \overline{\Omega}.$$

In fact, it is sufficient to choose $M_1 = M_2 = \|v_0\|_\infty$.

In order to obtain the uniform boundedness of $u(t,x)$, it is enough to show the existence of constants $C_1, C_2 > 0$ depending only on $\|v_0\|_\infty$ such that

$$\frac{d}{dt}\int_\Omega [1 + C_1\{v(t,x) + v(t,x)^2\}]e^{C_2 u(t,x)}\,dx \le 0, \qquad t \in (0, T_{\max}),$$

due to the argument in Haraux and Kirane [7] and Haraux and Youkana [8].

Proof of Theorem 2.2. In the case $a < d$, we find that (2.6) becomes

$$\begin{cases} w_t = d\Delta w - f(\frac{1}{r_2}w + \frac{1}{r_2}z)g(w), \\ z_t = a\Delta z + (r_2 + 1)f(\frac{1}{r_2}w + \frac{1}{r_2}z)g(w), \end{cases} \quad \text{in} \quad (0,\infty)\times\Omega \tag{2.10}$$

on account of Definition 2.1. Thus, by applying the comparison theorem to (2.10), (2.7), (2.8) again, we obtain that if $0 \le w_0(x) \le M_3$, then $0 \le w(t,x) \le M_3$ on $(0, T_{\max}) \times \overline{\Omega}$, and if $z_0(x) \ge M_4 \ge 0$, then $z(t,x) \ge M_4$ on $(0, T_{\max}) \times \overline{\Omega}$. In other words, $0 \le v(t,x) \le M_3$, $r_2 u(t,x) - v(t,x) \ge M_4$, $t > 0$, $x \in \overline{\Omega}$. Note that it is possible to take $M_3 = \|v_0\|_\infty$, $M_4 = 0$. We can also get the uniform boundedness of $u(t,x)$ by virtue of the discussion in [8]. See also [13].

3. Uniform convergence properties. This section is devoted to the discussion of asymptotic behavior of the nonnegative global solution $(u,v)(t,x)$ to the problem (1.1)–(1.3) obtained in the previous section.

Theorem 3.1. *Suppose that* $a \geq b+d$ *or* $a < d$. *Let* $(u,v)(t,x)$ *be the nonnegative global solution to* (1.1)–(1.3) *obtained by Theorem* 2.1 *or* 2.2. *Then,* $(u,v)(t,x) \to (u_\infty, 0)$ *as* $t \to \infty$ *uniformly in* $x \in \Omega$, *where* $u_\infty = \overline{u}_0 + \overline{v}_0$ *(see* (1.9)*).*

Proof. Green's formula and (1.1)–(1.3) give

$$\|v(t)\|_2^2 + 2d\int_0^t \|\nabla v(\tau)\|_2^2\,d\tau + 2\int_0^t\int_\Omega vf(u)g(v)\,dx\,d\tau = \|v_0\|_2^2, \qquad t>0,$$

from which it follows that $\int_0^t \|\nabla v(\tau)\|_2^2\,d\tau$ is uniformly bounded. Similarly, Green's formula and Schwarz's inequality yield

$$\begin{aligned}\|u(t)\|_2^2 &+ a\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau \\ &\leq \|u_0\|_2^2 + \frac{b^2}{a}\int_0^t \|\nabla v(\tau)\|_2^2\,d\tau + 2\int_0^t\int_\Omega uf(u)g(v)\,dx\,d\tau.\end{aligned}$$

Hence, one can see that $\int_0^t \|\nabla u(\tau)\|_2^2\,d\tau$ is uniformly bounded on account of an equality,

$$\int_0^t\int_\Omega f(u)g(v)\,dx\,d\tau = \int_\Omega u(t,x)\,dx - \int_\Omega u_0(x)\,dx,$$

obtained by integrating (1.1) over $(0,t)\times\Omega$ and the uniform boundedness of $u(t,x)$. From (2.1), we see that $t \mapsto \|\nabla u(t)\|_2$ and $t \mapsto \|\nabla v(t)\|_2$ are uniformly continuous on $[\delta,\infty)$ for every $\delta > 0$, so that we get $\lim_{t\to\infty} \|\nabla u(t)\|_2 = \lim_{t\to\infty} \|\nabla v(t)\|_2 = 0$.

With the help of (2.3) and the Poincaré inequality (see, e.g., Smoller [21])

$$\lambda\|\omega - \overline{\omega}\|_2^2 \leq \|\nabla\omega\|_2^2 \text{ for } \omega \in W^{1,2}(\Omega) \text{ satisfying } \partial\omega/\partial\nu = 0 \text{ on } \partial\Omega, \tag{3.1}$$

where λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$, we see that the same manner as in [9] leads us to the conclusion of the theorem. □

Here and hereafter, the discussion is concentrated upon the case $g(v) = v^n$ with $n \geq 1$. The next theorem is the result on rates of the convergence obtained in Theorem 3.1.

Theorem 3.2. *Let $g(v) = v^n$ with $n \geq 1$ and let $(u, v)(t, x)$ be the nonnegative global solution to* (1.1)–(1.3) *obtained in Theorem* 2.1 *or* 2.2.

(i) *If $n = 1$, then there exist positive constants T and K such that*

$$\|u(t) - u_\infty\|_\infty \leq \begin{cases} Ke^{-r(t-T)}, & \text{if} \quad d_2\lambda \neq f(u_\infty), \\ K(1 + t - T)e^{-r(t-T)}, & \text{if} \quad d_2\lambda = f(u_\infty), \end{cases}$$

$$\|v(t)\|_\infty \leq Ke^{-f(u_\infty)(t-T)}$$

for $t \geq T$, where $d_2 = d - br_2 = \min\{a, d\}$, λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$, $u_\infty = \overline{u}_0 + \overline{v}_0$ (see (1.9)*), and $r = \min\{d_2\lambda, f(u_\infty)\} = \min\{a\lambda, d\lambda, f(u_\infty)\}$.*

(ii) *If $n > 1$, then there exist positive constants T and K such that*

$$\|(u - u_\infty, v)(t)\|_\infty \leq K(1 + t - T)^{-1/(n-1)}, \qquad t \geq T,$$

where

$$\|(\omega_1, \omega_2)\|_p = (\|\omega_1\|_p^2 + \|\omega_2\|_p^2)^{1/2}, \tag{3.2}$$

for $\omega_1, \omega_2 \in L^p(\Omega)$, $1 \leq p \leq \infty$.

Proof. (i) For any $\varepsilon \in (0, f(u_\infty))$, there is a constant $T > 0$ such that

$$0 < f(u_\infty) - \varepsilon \leq f(u(t)) \tag{3.3}$$

for $t \geq T$, and the Green's formula gives

$$\int_\Omega v^{p-1} \Delta v \, dx \leq 0 \tag{3.4}$$

for $p \geq 1$. Then, multiplying the second equation of (1.1) by v^{p-1} and integrating the resulting expression, we get

$$\frac{d}{dt}\|v(t)\|_p^p \leq -p\{f(u_\infty) - \varepsilon\}\|v(t)\|_p^p, \qquad t > T,$$

which implies

$$\|v(t)\|_p \leq \|v(T)\|_p e^{-\{f(u_\infty)-\varepsilon\}(t-T)}, \qquad t \geq T \tag{3.5}$$

for all $p \in [1, \infty)$. Accordingly,

$$\|v(t)\|_\infty \le \|v(T)\|_\infty e^{-\{f(u_\infty)-\varepsilon\}(t-T)}, \qquad t \ge T. \tag{3.6}$$

Define two bounded linear operators P_0 and P_+ by

$$P_0\omega = \overline{\omega} = |\Omega|^{-1} \int_\Omega \omega(x)\,dx, \quad P_+\omega = \omega - P_0\omega,$$

respectively. Then (2.3) and (3.5) yield

$$|(P_0u)(t) - u_\infty| = |(P_0v)(t)| \le \overline{v}(T)e^{-\{f(u_\infty)-\varepsilon\}(t-T)}, \tag{3.7}$$

for $t \ge T$. The P_+-part of u satisfies

$$\begin{aligned}(P_+u)(t) = {} & \frac{1}{r_1+r_2} e^{-(t-T)d_1A}(r_1(P_+u)(T) + (P_+v)(T)) \\ & + \frac{1}{r_1+r_2} e^{-(t-T)d_2A}(r_2(P_+u)(T) - (P_+v)(T)) \\ & + \frac{r_1-1}{r_1+r_2} \int_T^t e^{-(t-\tau)d_1A}(P_+(f(u)g(v)))(\tau)\,d\tau \\ & + \frac{r_2+1}{r_1+r_2} \int_T^t e^{-(t-\tau)d_2A}(P_+(f(u)g(v)))(\tau)\,d\tau,\end{aligned} \tag{3.8}$$

with $g(v) = v$, where $d_1 = d + br_1 = \max\{a, d\}$, $d_2 = d - br_2 = \min\{a, d\}$, and A stands for $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$. The analytic semigroup $\{e^{-tA}\}_{t\ge0}$ has the following $L^p - L^q$ estimate:

$$\|e^{-tA}(P_+\omega)\|_p \le Cm(t)^{-(N/2)(1/q-1/p)}e^{-\lambda t}\|P_+\omega\|_q, \tag{3.9}$$

where $1 \le q \le p \le \infty$, $m(t) = \min\{t, 1\}$ and λ is the smallest positive eigenvalue of $-A$. Refer to, e.g., Rothe [20]. Combining (3.8) and (3.9), we obtain

$$\begin{aligned}\|(P_+u)(t)\|_p \le {} & C\|(u,v)(T)\|_p e^{-d_2\lambda(t-T)} \\ & + C\int_T^t m(t-\tau)^{-(N/2)(1/q-1/p)}e^{-d_2\lambda(t-\tau)}\|v(\tau)\|_q\,d\tau\end{aligned} \tag{3.10}$$

for $1 \le q \le p \le \infty$ and $t \ge T$. Let p and q satisfy

$$0 \le \frac{N}{2}(\frac{1}{q} - \frac{1}{p}) < 1. \tag{3.11}$$

We find that the second term of the right-hand side of (3.10) are bounded above by

$$C\|v(T)\|_q \int_T^t m(t-\tau)^{-(N/2)(1/q-1/p)} e^{-d_2\lambda(t-\tau)} e^{-\{f(u_\infty)-\varepsilon\}(\tau-T)}\, d\tau \\ \le C\|v(T)\|_q e^{-r_0(t-T)} \tag{3.12}$$

with use of (3.5), where

$$r_0 = \begin{cases} d_2\lambda, & \text{if} \quad d_2\lambda < f(u_\infty), \\ f(u_\infty) - \varepsilon, & \text{if} \quad d_2\lambda \ge f(u_\infty). \end{cases}$$

Take $q \in (N/2, \infty]$ and $p = \infty$ that satisfy (3.11). Thus, it follows from (3.7), (3.10) and (3.12) that

$$\|u(t) - u_\infty\|_\infty \le C\|(u,v)(T)\|_\infty e^{-r_0(t-T)}, \qquad t \ge T. \tag{3.13}$$

Now, we have a sharper estimate

$$f(u_\infty) - Ke^{-r_0(t-T)} \le f(u(t)) \le f(u_\infty) + Ke^{-r_0(t-T)} \tag{3.14}$$

for $t \ge T$ than (3.3) by virtue of (3.13). Then, we find that for every $p \in [1, \infty]$ a similar calculation gives

$$\|v(t)\|_p \le e^{K/r_0}\|v(T)\|_p e^{-f(u_\infty)(t-T)}, \qquad t \ge T \tag{3.15}$$

being sharper than (3.6) by means of (3.14) instead of (3.3). Hence, by (3.15), $(P_0u)(t) - u_\infty$ can be estimated as

$$|(P_0u)(t) - u_\infty| \le C|(P_0v)(T)|e^{-f(u_\infty)(t-T)}, \qquad t \ge T,$$

which is sharper than (3.7). Moreover, the estimate of the integral in the second term of the right-hand side of (3.10) is improved as

$$\int_T^t m(t-\tau)^{-(N/2)(1/q-1/p)} e^{-d_2\lambda(t-\tau)} e^{-f(u_\infty)(\tau-T)}\, d\tau \\ \le \begin{cases} Ce^{-r(t-T)}, & \text{if} \quad d_2\lambda \ne f(u_\infty), \\ C(1+t-T)e^{-r(t-T)}, & \text{if} \quad d_2\lambda = f(u_\infty) \end{cases}$$

for $t \geq T$, where $r = \min\{d_2\lambda, f(u_\infty)\} = \min\{a\lambda, d\lambda, f(u_\infty)\}$. Consequently,

$$\|u(t) - u_\infty\|_\infty \leq \begin{cases} Ke^{-r(t-T)}, & \text{if} \quad d_2\lambda \neq f(u_\infty), \\ K(1+t-T)e^{-r(t-T)}, & \text{if} \quad d_2\lambda = f(u_\infty). \end{cases}$$

This completes the proof of (i).

(ii) When $n > 1$, we see that multiplying the equation for v in (1.1) by v^{p-1} and integrating the resulting expression over Ω implies

$$\frac{d}{dt}\|v(t)\|_p^p \leq -p\{f(u_\infty) - \varepsilon\}\|v(t)\|_{p+n-1}^{p+n-1}, \qquad t > T,$$

with use of (3.3) and (3.4). Therefore, it can be shown that

$$\|v(t)\|_p \leq |\Omega|^{1/p}\|v_0\|_\infty[1+(n-1)\|v_0\|_\infty^{n-1}(f(u_\infty)-\varepsilon)(t-T)]^{-1/(n-1)}, \quad t \geq T \tag{3.16}$$

for $p \in [1, \infty)$, from which it holds that

$$\|v(t)\|_\infty \leq K(1+t-T)^{-1/(n-1)}, \qquad t \geq T.$$

Next, it is derived from (2.3) and (3.16) with $p = 1$ that

$$|(P_0u)(t) - u_\infty| = |(P_0v)(t)| \leq K(1+t-T)^{-1/(n-1)}, \qquad t \geq T.$$

In the same way as obtaining (3.10),

$$\begin{aligned} \|(P_+u)(t)\|_p \leq\ & C\|(u,v)(T)\|_p e^{-d_2\lambda(t-T)} \\ & +C\int_T^t m(t-\tau)^{-(N/2)(1/q-1/p)}e^{-d_2\lambda(t-\tau)}\|v(\tau)\|_{qn}^n \, d\tau \end{aligned}$$

for $1 \leq q \leq p \leq \infty$, where (3.8) with $g(v) = v^n$ $(n > 1)$ and (3.9) were used. Take $p = \infty$ and $q \in (N/2, \infty)$. Then there results

$$\|(P_+u)(t)\|_\infty \leq K(1+t-T)^{-n/(n-1)}, \qquad t \geq T$$

from (3.16). This proves (ii) of Theorem 3.2. □

The following Theorem 3.3 gives a rough estimate of exponential decay of $\|(P_+u, P_+v)(t)\|_\infty$ in the case of $n > 1$.

Theorem 3.3. *If $g(v) = v^n$ with $n > 1$, then there exist constants $T, K > 0$ such that*

$$\|(u - \overline{u}, v - \overline{v})(t)\|_\infty \leq K(1 + t - T)^K e^{-\min\{a,d\}\lambda(t-T)}$$

for $t \geq T$.

Proof. For $1 \leq p \leq \infty$, one can get

$$\|(P_+(f(u)v^n))(\tau)\|_p \leq K\|v(t)\|_\infty^{n-1}\|(P_+u, P_+v)(t)\|_p$$

by virtue of Taylor's expansion of $f(u)$ and $v^n = (P_0v + P_+v)^n$, and the boundedness of P_0 and P_+. Observe that the P_+-part of v satisfies

$$\begin{aligned}(P_+v)(t) = {} & \frac{r_2}{r_1 + r_2} e^{-(t-T)d_1A}(r_1(P_+u)(T) + (P_+v)(T)) \\ & - \frac{r_1}{r_1 + r_2} e^{-(t-T)d_2A}(r_2(P_+u)(T) - (P_+v)(T)) \\ & + \frac{r_2(r_1 - 1)}{r_1 + r_2} \int_T^t e^{-(t-\tau)d_1A}(P_+(f(u)g(v)))(\tau)\, d\tau \\ & - \frac{r_1(r_2 + 1)}{r_1 + r_2} \int_T^t e^{-(t-\tau)d_2A}(P_+(f(u)g(v)))(\tau)\, d\tau.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\|(P_+u, P_+v)(t)\|_p \leq {} & Ce^{-d_2\lambda(t-T)}\|(P_+u, P_+v)(T)\|_p \\ & + K\int_T^t e^{-d_2\lambda(t-\tau)}\|v(\tau)\|_\infty^{n-1}\|(P_+u, P_+v)(\tau)\|_p\, d\tau \\ \leq {} & Ce^{-d_2\lambda(t-T)}\|(P_+u, P_+v)(T)\|_p \\ & + K\int_T^t e^{-d_2\lambda(t-\tau)}(1 + \tau - T)^{-1}\|(P_+u, P_+v)(\tau)\|_p\, d\tau\end{aligned}$$

owing to (3.8) and Theorem 3.2 (ii). Thus, it is easily seen from Gronwall's inequality that Theorem 3.3 holds true. For the details of the proof, we refer the reader to [9]. □

4. Asymptotic approximation. In this section, we study large-time approximation of $(u, v)(t, x)$ in terms of the solution $(U, V)(t)$ to the problem (1.6) and (1.7) when the nonnegative global solution $(u,v)(t,x)$ for the strongly coupled system (1.1)–(1.3) uniformly tends to $(u_\infty,0)$ as $t\to\infty$ with the polynomial rate $t^{-1/(n-1)}$ by restricting ourselves to the case $g(v)=v^n$ with $n>1$ (Theorem 3.2 (ii)).

Theorem 4.1. *Suppose that $g(v)=v^n$ with $n>1$. Let $(u,v)(t,x)$ be the nonnegative global solution to* (1.1)–(1.3) *obtained in Theorem* 2.1 *or* 2.2. *Then $(u,v)(t,x)=(U,V)(t)+O(t^{-1/(n-1)-1})$ uniformly in $x\in\Omega$ as $t\to\infty$.*

At the same time, we can prove the following theorem, which gives information about the exponential decay of the difference between $(u,v)(t,x)$ and $(\overline{u},\overline{v})(t)$ that is sharper than Theorem 3.3.

Theorem 4.2. *Assume the same conditions as in Theorem* 4.1. *In the case where $a>d$* (*respectively $a<d$*), *Assumptions* 2.1 *and* 2.2 (*respectively* 2.1 *and* 2.3) *are imposed. Then $(\overline{u},\overline{v})(t)$ gives an asymptotic description to the nonnegative solution $(u,v)(t,x)$ as follows.*

(i) *When $a>b+d$ or $a<d$, fix $\varepsilon>0$. Then,*

$$(u,v)(t,x)=(\overline{u},\overline{v})(t)+O(t^{\varepsilon}e^{-\min\{a,d\}\lambda t})$$

uniformly in $x\in\Omega$ as $t\to\infty$, where λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$.

(ii) *If $a=b+d$, then*

$$(u,v)(t,x)=(\overline{u},\overline{v})(t)+O(e^{-d\lambda t})$$

uniformly in $x\in\Omega$ as $t\to\infty$.

Let $(\phi,\psi)(t,x)$ satisfy

$$\begin{aligned} u(t,x)-u_\infty &= (U(t)-u_\infty)(1+\phi(t,x))=-V(t)(1+\phi(t,x)),\\ v(t,x) &= V(t)(1+\psi(t,x)), \end{aligned} \tag{4.1}$$

where $(U,V)(t)$ solves (1.6) and (1.7), and (1.8) was used. Hence, it is derived from (1.1)–(1.3), (1.6), (1.7) and (4.1) that in $(0,\infty)\times\Omega$,

$$\begin{aligned} &\phi_t=a\Delta\phi-b\Delta\psi-V^{n-1}\big\{-f(u_\infty-V)\phi-Vf_u(u_\infty-V)\phi\\ &\quad+nf(u_\infty-V)\psi+h(\phi,\psi)\big\},\\ &\psi_t=d\Delta\psi-V^{n-1}\big\{-Vf_u(u_\infty-V)\phi+(n-1)f(u_\infty-V)\psi+h(\phi,\psi)\big\}, \end{aligned} \tag{4.2}$$

$$\frac{\partial \phi}{\partial \nu} = \frac{\partial \psi}{\partial \nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{4.3}$$

$$\begin{cases} \phi(0,x) = \phi_0(x) = -\dfrac{u_0(x)-\overline{u}_0}{\overline{v}_0}, \\ \psi(0,x) = \psi_0(x) = \dfrac{v_0(x)-\overline{v}_0}{\overline{v}_0}, \end{cases} \qquad \text{in} \quad \Omega, \tag{4.4}$$

where $f_u = df/du$, and $h(\phi,\psi)$ is defined by

$$\begin{aligned} &-f(u_\infty - V)(1+\phi) + f(u_\infty - V(1+\phi))(1+\psi)^n \\ &\quad = -f(u_\infty - V)\phi - Vf_u(u_\infty - V)\phi + nf(u_\infty - V)\psi + h(\phi,\psi). \end{aligned}$$

Note that we have

$$\begin{aligned} &-f(u_\infty - V)(1+\psi) + f(u_\infty - V(1+\phi))(1+\psi)^n \\ &\qquad = -Vf_u(u_\infty - V)\phi + (n-1)f(u_\infty - V)\psi + h(\phi,\psi), \end{aligned}$$

and $h(\phi,\psi) = O(|\phi|^2 + |\psi|^2)$ as $(\phi,\psi) \to (0,0)$. Observe that we have $(P_0\phi)(t) = (P_0\psi)(t)$ for all $t \geq 0$ by (4.1) and (2.3), and $P_0\phi_0 = P_0\psi_0 = 0$ on account of (4.4). The following Theorem 4.3 implies Theorems 4.1 and 4.2 in the case where the initial perturbation from the mean value is small enough.

Theorem 4.3. (i) *In both cases where $a > b+d$ and $a < d$, fix $\varepsilon > 0$. Then there exists a constant $\delta_0 > 0$ such that if $\|(\phi_0,\psi_0)\|_\infty \leq \delta_0$, then $(\phi,\psi)(t,x)$ satisfying* (4.2)–(4.4) *is estimated as follows*:

$$\begin{aligned} \|(\phi,\psi)(t)\|_\infty &\leq C(\varepsilon)\|(\phi_0,\psi_0)\|_\infty(1+t)^{-1}, \\ \|(P_+\phi, P_+\psi)(t)\|_\infty &\leq C(\varepsilon)\|(\phi_0,\psi_0)\|_\infty(1+t)^{1/(n-1)+\varepsilon}e^{-\min\{a,d\}\lambda t} \end{aligned}$$

for $t \geq 0$, where $C(\varepsilon)$ is a positive constant depending on ε and $C(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

(ii) *When $a = b+d$, there exists a constant $\delta_0 > 0$ such that if $\|(\phi_0,\psi_0)\|_\infty \leq \delta_0$, then*

$$\begin{aligned} \|(\phi,\psi)(t)\|_\infty &\leq C\|(\phi_0,\psi_0)\|_\infty(1+t)^{-1}, \\ \|(P_+\phi, P_+\psi)(t)\|_\infty &\leq C\|(\phi_0,\psi_0)\|_\infty(1+t)^{1/(n-1)}e^{-d\lambda t} \end{aligned}$$

for $t \geq 0$, where C is a positive constant.

Remark 4.1. From (4.1), the following relations hold:

$$u(t,x)-\overline{u}(t) = -V(t)\cdot(P_+\phi)(t,x),$$
$$v(t,x)-\overline{v}(t) = V(t)\cdot(P_+\psi)(t,x).$$

A series of lemmas is necessary for us in order to prove Theorem 4.3.

Lemma 4.4. There exists a nondecreasing function $L(r)$ on $[0,\infty)$ such that if we set $K(t)=L(\sup_{0\le\tau\le t}\|(\phi,\psi)(\tau)\|_\infty)$, then for every $p\in[1,\infty]$,

$$\|(h(\phi,\psi))(t)\|_p \le K(t)\|(\phi,\psi)(t)\|_{2p}^2,$$
$$\|(P_+h(\phi,\psi))(t)\|_p \le K(t)\|(\phi,\psi)(t)\|_\infty\|(P_+\phi,P_+\psi)(t)\|_p.$$

Proof. Thanks to the smoothness of f and Taylor's expansion, we have

$$\begin{aligned}
h(\phi,\psi) &= (1/2)V^2f_{uu}(u_\infty - V(1+\theta\phi))(1+\theta\psi)^n\phi^2\\
&\quad - nVf_u(u_\infty - V(1+\theta\phi))(1+\theta\psi)^{n-1}\phi\psi\\
&\quad + (1/2)n(n-1)f(u_\infty - V(1+\theta\phi))(1+\theta\psi)^{n-2}\psi^2\\
&= (1/2)V^2f_{uu}(u_\infty - V - \theta V(P_0\phi) - \theta V(P_+\phi))\\
&\quad \times\{1+\theta(P_0\psi)+\theta(P_+\psi)\}^n(P_0\phi+P_+\phi)^2\\
&\quad - nVf_u(u_\infty - V - \theta V(P_0\phi) - \theta V(P_+\phi))\\
&\quad \times\{1+\theta(P_0\psi)+\theta(P_+\psi)\}^{n-1}(P_0\phi+P_+\phi)(P_0\psi+P_+\psi)\\
&\quad + (1/2)n(n-1)f(u_\infty - V - \theta V(P_0\phi) - \theta V(P_+\phi))\\
&\quad \times\{1+\theta(P_0\psi)+\theta(P_+\psi)\}^{n-2}(P_0\psi+P_+\psi)^2,
\end{aligned}$$

where $\theta\in(0,1)$, $f_u=df/du$ and $f_{uu}=d^2f/du^2$, so that we get the conclusions. □

Definition 4.1. Let $\sigma>0$. For $1\le p\le\infty$,

$$\begin{aligned}
I_p &= \|(\phi_0,\psi_0)\|_p,\\
M_p(t) &= \sup_{0\le\tau\le t} V(\tau)^{-(n-1)}\|(\phi,\psi)(\tau)\|_p,\\
M_\infty^0(t) &= \sup_{0\le\tau\le t} V(\tau)^{-(n-1)}|(P_0\phi,P_0\psi)(\tau)|,\\
M_{p,\sigma}^+(t) &= \sup_{0\le\tau\le t} V(\tau)^{1+\sigma}e^{d_2\lambda\tau}\|(P_+\phi,P_+\psi)(\tau)\|_p,\\
M_p^+(t) &= \sup_{0\le\tau\le t} V(\tau)e^{d_2\lambda\tau}\|(P_+\phi,P_+\psi)(\tau)\|_p,
\end{aligned}$$

where $V(t)$ is the solution for (1.6) and (1.7), $d_2 = d - br_2 = \min\{a, d\}$, and λ is the smallest positive eigenvalue of $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$.

Evidently, Hölder's inequality yields

$$I_q \le CI_p,\ M_q(t) \le CM_p(t),\ M^+_{q,\sigma}(t) \le CM^+_{p,\sigma}(t),\ M^+_q(t) \le CM^+_p(t) \quad (4.5)$$

for $1 \le q \le p \le \infty$, where $C = |\Omega|^{1/q-1/p}$.

Hereafter, $CK(t)$ is identified with $K(t)$ for simplicity, provided that C is independent of σ.

Lemma 4.5. *Suppose that* $1 \le p \le 2$.

(i) *Let* $\sigma > 0$. *If* $a > b + d$ *or* $a < d$, *then*

$$M^+_{p,\sigma}(t) \le C(\sigma)I_2 + C(\sigma)K(t)M_\infty(t)M^+_{2,\sigma}(t)$$

for $t \ge 0$. *Here,* $C(\sigma)$ *is a constant which depends on* σ *and* $C(\sigma) \to \infty$ *as* $\sigma \to \infty$.

(ii) *If* $a = b + d$, *then* $M^+_p(t) \le CI_2 + K(t)M_\infty(t)M^+_2(t)$ *for* $t \ge 0$.

Definition 4.2. $(\xi, \eta) = (r_1\phi - \psi, r_2\phi + \psi)$, where r_1 and r_2 are defined by Definition 2.1.

Obviously, we have

$$(\phi, \psi) = \big(\frac{1}{r_1 + r_2}\xi + \frac{1}{r_1 + r_2}\eta, -\frac{r_2}{r_1 + r_2}\xi + \frac{r_1}{r_1 + r_2}\eta\big). \quad (4.6)$$

Proof of Lemma 4.5. It is derived from Definition 4.1 and (4.6) that

$$\begin{aligned}\xi_t = d_1\Delta\xi + V^{n-1}\Big[&\big\{1 + \frac{r_2(r_1 - 1)n}{r_1 + r_2}\big\}f(u_\infty - V)\xi \\ &- \frac{r_1(r_1 - 1)n}{r_1 + r_2}f(u_\infty - V)\eta + \frac{r_1 - 1}{r_1 + r_2}Vf_u(u_\infty - V)\xi \\ &+ \frac{r_1 - 1}{r_1 + r_2}Vf_u(u_\infty - V)\eta - (r_1 - 1)h(\phi, \psi)\Big], \end{aligned} \quad (4.7)$$

where $d_1 = d + br_1 = \max\{a, d\}$. In the same way, it is derived that

$$\begin{aligned}\eta_t = d_2\Delta\eta + V^{n-1}\Big[&\frac{r_2(r_2 + 1)n}{r_1 + r_2}f(u_\infty - V)\xi + \big\{1 - \frac{r_1(r_2 + 1)n}{r_1 + r_2}\big\}f(u_\infty - V)\eta \\ + \frac{r_2 + 1}{r_1 + r_2}&Vf_u(u_\infty - V)\xi + \frac{r_2 + 1}{r_1 + r_2}Vf_u(u_\infty - V)\eta - (r_2 + 1)h(\phi, \psi)\Big], \end{aligned} \quad (4.8)$$

where $d_2 = d - br_2 = \min\{a, d\}$. Therefore, we find that $(P_+\xi, P_+\eta)(t, x)$ satisfies the following system:

$$\begin{cases} (P_+\xi)_t = d_1\Delta(P_+\xi) + V^{n-1}\Big[\big\{1 + \frac{r_2(r_1-1)n}{r_1+r_2}\big\}f(u_\infty - V)(P_+\xi) \\ \quad -\frac{r_1(r_1-1)n}{r_1+r_2}f(u_\infty - V)(P_+\eta) + \frac{r_1-1}{r_1+r_2}Vf_u(u_\infty - V)(P_+\xi) \\ \quad +\frac{r_1-1}{r_1+r_2}Vf_u(u_\infty - V)(P_+\eta) - (r_1-1)(P_+h(\phi,\psi))\Big], \\ (P_+\eta)_t = d_2\Delta(P_+\eta) + V^{n-1}\Big[\frac{r_2(r_2+1)n}{r_1+r_2}f(u_\infty - V)(P_+\xi) \\ \quad +\big\{1 - \frac{r_1(r_2+1)n}{r_1+r_2}\big\}f(u_\infty - V)(P_+\eta) + \frac{r_2+1}{r_1+r_2}Vf_u(u_\infty - V)(P_+\xi) \\ \quad +\frac{r_2+1}{r_1+r_2}Vf_u(u_\infty - V)(P_+\eta) - (r_2+1)(P_+h(\phi,\psi))\Big], \end{cases}$$

$$\text{in} \quad (0,\infty)\times\Omega, \tag{4.9}$$

$$\frac{\partial(P_+\xi)}{\partial\nu} = \frac{\partial(P_+\eta)}{\partial\nu} = 0, \qquad \text{on} \quad (0,\infty)\times\partial\Omega, \tag{4.10}$$

$$\begin{cases} (P_+\xi)(0,x) = (P_+\xi_0)(x) = \xi_0(x), \\ (P_+\eta)(0,x) = (P_+\eta_0)(x) = \eta_0(x), \end{cases} \qquad \text{in} \quad \Omega \tag{4.11}$$

from (4.7), (4.8), (4.3) and (4.4).

Let $\gamma > 0$ be a parameter. After multiplying both sides of the first (respectively second) equation in (4.9) by $P_+\xi$ (respectively $P_+\eta$), carrying integration by parts over Ω gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\gamma\|(P_+\xi)(t)\|_2^2 + \|(P_+\eta)(t)\|_2^2\right\} = -d_1\gamma\|\nabla(P_+\xi)(t)\|_2^2 - d_2\|\nabla(P_+\eta)(t)\|_2^2 \\ &+ V(t)^{n-1}f(u_\infty - V)\Big[\big\{1 + \frac{r_2(r_1-1)n}{r_1+r_2}\big\}\gamma\|(P_+\xi)(t)\|_2^2 \\ &+ \big\{\frac{r_2(r_2+1)}{r_1+r_2} - \frac{r_1(r_1-1)}{r_1+r_2}\gamma\big\}n\int_\Omega (P_+\xi)(P_+\eta)\,dx \\ &+ \big\{1 - \frac{r_1(r_2+1)n}{r_1+r_2}\big\}\|(P_+\eta)(t)\|_2^2\Big] + V(t)^n f_u(u_\infty - V)\Big[\frac{r_1-1}{r_1+r_2}\gamma\|(P_+\xi)(t)\|_2^2 \\ &+ \big(\frac{r_1-1}{r_1+r_2}\gamma + \frac{r_2+1}{r_1+r_2}\big)\int_\Omega (P_+\xi)(P_+\eta)\,dx + \frac{r_2+1}{r_1+r_2}\|(P_+\eta)(t)\|_2^2\Big] \\ &- V(t)^{n-1}\int_\Omega (P_+h)\{(r_1-1)\gamma(P_+\xi) + (r_2+1)(P_+\eta)\}\,dx, \end{aligned} \tag{4.12}$$

where (4.10) was used.

The first and second terms of the right-hand side of (4.12) in which the gradient operators are appearing are estimated as follows:

$$-d_1\gamma\|\nabla(P_+\xi)(t)\|_2^2 - d_2\|\nabla(P_+\eta)(t)\|_2^2 \le -d_2\lambda\{\gamma\|(P_+\xi)(t)\|_2^2 + \|(P_+\eta)(t)\|_2^2\},$$

where the Poincaré inequality (3.1) was applied.

Secondly, Schwarz's inequality yields

$$\begin{aligned}&\{1+\frac{r_2(r_1-1)n}{r_1+r_2}\}\gamma\|(P_+\xi)(t)\|_2^2 + \Big\{\frac{r_2(r_2+1)}{r_1+r_2}\\&-\frac{r_1(r_1-1)}{r_1+r_2}\gamma\Big\}n\int_\Omega (P_+\xi)(P_+\eta)\,dx + \{1-\frac{r_1(r_2+1)n}{r_1+r_2}\}\|(P_+\eta)(t)\|_2^2\\&\le \Big[\Big\{1+\frac{r_2(r_1-1)n}{r_1+r_2}\Big\}\gamma + \Big|\frac{r_2(r_2+1)}{r_1+r_2} - \frac{r_1(r_1-1)}{r_1+r_2}\gamma\Big|n\cdot\frac{\theta}{2}\Big]\|(P_+\xi)(t)\|_2^2\\&+\Big[\Big|\frac{r_2(r_2+1)}{r_1+r_2} - \frac{r_1(r_1-1)}{r_1+r_2}\gamma\Big|n\cdot\frac{1}{2\theta} + \Big\{1-\frac{r_1(r_2+1)n}{r_1+r_2}\Big\}\Big]\|(P_+\eta)(t)\|_2^2.\end{aligned}$$

Here, $\theta > 0$ is taken to satisfy

$$\begin{aligned}&\Big\{1+\frac{r_2(r_1-1)n}{r_1+r_2}\Big\}\gamma + \Big|\frac{r_2(r_2+1)}{r_1+r_2} - \frac{r_1(r_1-1)}{r_1+r_2}\gamma\Big|n\cdot\frac{\theta}{2}\\&=\gamma\Big[\Big|\frac{r_2(r_2+1)}{r_1+r_2} - \frac{r_1(r_1-1)}{r_1+r_2}\gamma\Big|n\cdot\frac{1}{2\theta} + \Big\{1-\frac{r_1(r_2+1)n}{r_1+r_2}\Big\}\Big],\end{aligned}$$

that is to say, $|r_2(r_2+1) - r_1(r_1-1)\gamma|\theta^2 + 2\{r_2(r_1-1) + r_1(r_2+1)\}\gamma\theta - |r_2(r_2+1) - r_1(r_1-1)\gamma|\gamma = 0$.

When $a > b+d$ or $a < d$, $r_2(r_2+1) - r_1(r_1-1)\gamma \neq 0$ holds true. Then $\theta = \frac{\theta_1}{\theta_2}$, where $\theta_1 = -\{r_2(r_1-1) + r_1(r_2+1)\}\gamma + [\{r_2(r_1-1) + r_1(r_2+1)\}^2\gamma^2 + \{r_2(r_2+1) - r_1(r_1-1)\gamma\}^2\gamma]^{1/2}$ and $\theta_2 = |r_2(r_2+1) - r_1(r_1-1)\gamma|$. This choice of θ gives

$$\begin{aligned}k(\gamma) &\equiv \Big[\Big\{1+\frac{r_2(r_1-1)n}{r_1+r_2}\Big\}\gamma + \Big|\frac{r_2(r_2+1) - r_1(r_1-1)\gamma}{r_1+r_2}\Big|n\cdot\frac{\theta}{2}\Big]\frac{1}{\gamma}\\&= 1 + \frac{n}{2}\cdot\frac{r_2(r_1-1) - r_1(r_2+1)}{r_1+r_2}\\&+\frac{n}{2}\cdot\frac{[\{r_2(r_1-1)+r_1(r_2+1)\}^2 + \{r_2(r_2+1) - r_1(r_1-1)\gamma\}^2\gamma^{-1}]^{1/2}}{r_1+r_2}.\end{aligned} \tag{4.13}$$

If $a > b + d$, then $r_2 = 0$ from Definition 2.1, so that (4.13) becomes

$$k(\gamma) = 1 + \frac{n}{2}\left\{-1 + \sqrt{1 + (r_1 - 1)^2\gamma}\right\} > 1. \tag{4.14}$$

In the case where $a < d$, we have $r_1 = 0$ from Definition 2.1. Then (4.13) is

$$k(\gamma) = 1 + \frac{n}{2}\left\{-1 + \sqrt{1 + (r_2 + 1)^2\gamma^{-1}}\right\} > 1. \tag{4.15}$$

When $a = b+d$, $(r_1, r_2) = (1, 0)$ holds true, so that $r_2(r_2+1) - r_1(r_1-1)\gamma = 0$. Hence,

$$\begin{aligned}
&\Big\{1 + \frac{r_2(r_1-1)n}{r_1+r_2}\Big\}\gamma\|(P_+\xi)(t)\|_2^2 + \Big\{\frac{r_2(r_2+1)}{r_1+r_2} \\
&\quad - \frac{r_1(r_1-1)}{r_1+r_2}\gamma\Big\}n\int_\Omega (P_+\xi)(P_+\eta)\,dx + \Big\{1 - \frac{r_1(r_2+1)n}{r_1+r_2}\Big\}\|(P_+\eta)(t)\|_2^2 \\
&\quad \le k(\gamma)\left\{\gamma\|(P_+\xi)(t)\|_2^2 + \|(P_+\eta)(t)\|_2^2\right\}
\end{aligned}$$

for $t \ge 0$, where

$$k(\gamma) = \begin{cases} (4.14), & \text{if} \quad a > b + d, \\ (4.15), & \text{if} \quad a < d, \\ 1, & \text{if} \quad a = b + d. \end{cases} \tag{4.16}$$

Thirdly, the absolute value of the terms including $V^n f_u(u_\infty - V)$ in (4.12) are estimated as follows. It holds from the Schwarz inequality that

$$\begin{aligned}
&\Big|\frac{r_1-1}{r_1+r_2}\gamma\|(P_+\xi)(t)\|_2^2 + \big(\frac{r_1-1}{r_1+r_2}\gamma + \frac{r_2+1}{r_1+r_2}\big) \\
&\qquad \times \int_\Omega (P_+\xi)(P_+\eta)\,dx + \frac{r_2+1}{r_1+r_2}\|(P_+\eta)(t)\|_2^2\Big| \\
&\le \Big(\frac{|r_1-1|}{r_1+r_2}\gamma + \big|\frac{r_1-1}{r_1+r_2}\gamma + \frac{r_2+1}{r_1+r_2}\big| \cdot \frac{\kappa}{2}\Big)\|(P_+\xi)(t)\|_2^2 \\
&+ \Big(\big|\frac{r_1-1}{r_1+r_2}\gamma + \frac{r_2+1}{r_1+r_2}\big| \cdot \frac{1}{2\kappa} + \frac{r_2+1}{r_1+r_2}\Big)\|(P_+\eta)(t)\|_2^2,
\end{aligned}$$

where $\kappa > 0$ is chosen to satisfy

$$|(r_1-1)\gamma + (r_2+1)|\kappa^2 - 2\{(r_2+1) - |r_1-1|\}\gamma\kappa - |(r_1-1)\gamma + (r_2+1)|\gamma = 0.$$

If $(r_1-1)\gamma+(r_2+1)\neq 0$, then

$$\kappa=\frac{\{(r_2+1)-|r_1-1|\}\gamma+\sqrt{\{(r_2+1)-|r_1-1|\}^2\gamma^2+|(r_1-1)\gamma+(r_2+1)|^2\gamma}}{|(r_1-1)\gamma+(r_2+1)|},$$

and hence

$$\begin{aligned}\ell(\gamma)&\equiv\Big(\frac{|r_1-1|}{r_1+r_2}\gamma+\frac{1}{r_1+r_2}|(r_1-1)\gamma+(r_2+1)|\cdot\frac{\kappa}{2}\Big)\cdot\frac{1}{\gamma}\\&=\frac{|r_1-1|+(r_2+1)}{2(r_1+r_2)}+\frac{\sqrt{\{(r_2+1)-|r_1-1|\}^2+|(r_1-1)\gamma+(r_2+1)|^2\gamma^{-1}}}{2(r_1+r_2)}.\end{aligned}\tag{4.17}$$

Accordingly, together with the case $(r_1-1)\gamma+(r_2+1)=0$, we get

$$\begin{aligned}&\Big|\frac{r_1-1}{r_1+r_2}\gamma\|(P_+\xi)(t)\|_2^2+\Big(\frac{r_1-1}{r_1+r_2}\gamma+\frac{r_2+1}{r_1+r_2}\Big)\int_\Omega(P_+\xi)(P_+\eta)\,dx\\&+\frac{r_2+1}{r_1+r_2}\|(P_+\eta)(t)\|_2^2\Big|\le\ell(\gamma)\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\},\end{aligned}$$

where

$$\ell(\gamma)=\begin{cases}(4.17), & \text{if}\quad (r_1-1)\gamma+(r_2+1)\neq 0,\\ \dfrac{r_2+1}{r_1+r_2}, & \text{if}\quad (r_1-1)\gamma+(r_2+1)=0.\end{cases}\tag{4.18}$$

One can see that the absolute value of the final term in the right-hand side of (4.12) is bounded above by $\sqrt{2\max\{(r_1-1)^2\gamma,(r_2+1)^2\}}V(t)^{n-1}\|(P_+h)(t)\|_2$ $\times\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\}^{1/2}$ with use of Schwarz's inequality.

Consequently, combining the above calculations yield

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\}\le -d_2\lambda\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\}\\&+k(\gamma)V(t)^{n-1}f(u_\infty-V(t))\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\}\\&+\ell(\gamma)V(t)^n|f_u(u_\infty-V(t))|\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\}\\&+\sqrt{2\max\{(r_1-1)^2\gamma,(r_2+1)^2\}}V(t)^{n-1}\|(P_+h)(t)\|_2\\&\times\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\}^{1/2}\end{aligned}\tag{4.19}$$

for $t>0$, where $k(\gamma)$ (respectively $\ell(\gamma)$) is the constant defined by (4.16) (respectively (4.18)). Define a function $E(t)$ by

$$E(t)\equiv\left\{\gamma\|(P_+\xi)(t)\|_2^2+\|(P_+\eta)(t)\|_2^2\right\}^{1/2}.$$

Then, (4.19) yields

$$E'(t) \le -[d_2\lambda - k(\gamma)V(t)^{n-1}f(u_\infty - V(t)) - \ell(\gamma)V(t)^n|f_u(u_\infty - V(t))|]E(t) + \sqrt{2\max\{(r_1-1)^2\gamma, (r_2+1)^2\}}V(t)^{n-1}\|(P_+h)(t)\|_2,$$

from which it follows that

$$E'(t) + \left[d_2\lambda - k(\gamma)V(t)^{n-1}f(u_\infty - V(t)) - C_1\ell(\gamma)(1+t)^{-n/(n-1)}\right]E(t) \le C_2(\gamma)V(t)^{n-1}\|(P_+h)(t)\|_2, \qquad t > 0,$$

where C_1 is a positive constant such that $|V(t)^n f_u(u_\infty - V(t))| \le C_1(1+t)^{-n/(n-1)}$ for $t \ge 0$ and $C_2(\gamma) = \sqrt{2\max\{(r_1-1)^2\gamma, (r_2+1)^2\}}$. Thus $E(t)$ is estimated as

$$E(t) \le \Big(\frac{V(t)}{\overline{v}_0}\Big)^{-k(\gamma)} e^{-d_2\lambda t}e^{(n-1)C_1\ell(\gamma)}E(0) + C_2(\gamma)e^{(n-1)C_1\ell(\gamma)}\int_0^t \Big(\frac{V(t)}{V(\tau)}\Big)^{-k(\gamma)} e^{-d_2\lambda(t-\tau)}V(\tau)^{n-1}\|(P_+h)(t)\|_2\,d\tau,$$

which implies

$$V(t)^{k(\gamma)}e^{d_2\lambda t}E(t) \le \overline{v}_0{}^{k(\gamma)}e^{(n-1)C_1\ell(\gamma)}E(0) + C_2(\gamma)e^{(n-1)C_1\ell(\gamma)}\int_0^t V(\tau)^{n-1+k(\gamma)}e^{d_2\lambda\tau}\|(P_+h)(\tau)\|_2\,d\tau. \tag{4.20}$$

If $a > b + d$ or $a < d$, then we take $\sigma = k(\gamma) - 1 > 0$, so that in both cases we find that the integral appearing in (4.20) can be estimated as follows by means of Lemma 4.4:

$$\begin{aligned}&\int_0^t V(\tau)^{n-1+k(\gamma)}e^{d_2\lambda\tau}\|(P_+h)(\tau)\|_2\,d\tau \\ \le&\int_0^t V(\tau)^{n-1+k(\gamma)}e^{d_2\lambda\tau}K(\tau)\|(\phi,\psi)(\tau)\|_\infty\|(P_+\phi, P_+\psi)(\tau)\|_2\,d\tau \qquad (4.21)\\ \le& K(t)M_\infty(t)M_{2,\sigma}^+(t)\int_0^t V(\tau)^{2(n-1)}\,d\tau \le K(t)M_\infty(t)M_{2,\sigma}^+(t).\end{aligned}$$

If $a = b + d$, then $\sigma = k(\gamma) - 1 = 0$ for every $\gamma > 0$, and the estimate of the integral in (4.20) can be obtained as

$$\int_0^t V(\tau)^n e^{d_2\lambda\tau}\|(P_+h)(\tau)\|_2\,d\tau \le K(t)M_\infty(t)M_2^+(t), \qquad t \ge 0. \tag{4.22}$$

By means of (4.6) and the notation (3.2), we obtain

$$\|(P_+\phi, P_+\psi)(t)\|_2 \le C(r_1, r_2)\|(P_+\xi, P_+\eta)(t)\|_2 \le C(r_1, r_2)\max\{1, \gamma^{-1/2}\}E(t)$$

and $E_0 \le \max\{1, \gamma^{1/2}\}C(r_1, r_2)\|(\phi_0, \psi_0)\|_2$. Therefore, by combining (4.20) with (4.21) or (4.22), and by taking, e.g., $\gamma = 1$ in the case $a = b + d$, we see that Lemma 4.5 holds true in the case $p = 2$. For $1 \le p < 2$, the inequality $\|\omega\|_p \le |\Omega|^{1/p-1/2}\|\omega\|_2$ for $\omega \in L^2(\Omega)$ assures the validity of Lemma 4.5. This completes the proof. $\square$

Lemma 4.6. *$M^+_{\infty,\sigma}(t) \le C(\sigma)I_\infty + C(\sigma)K(t)M_\infty(t)M^+_{\infty,\sigma}(t)$ for $t \ge 0$, provided that $a > b + d$ or $a < d$. If $a = b + d$, then $M^+_\infty(t) \le CI_\infty + K(t)M_\infty(t)M^+_\infty(t)$.*

Proof. This lemma can be shown via analyzing $(\xi, \eta)(t, x)$ as well as the previous lemma. It follows from (4.9)–(4.11) that

$$\begin{aligned}
&(P_+\xi)(t) = e^{-td_1A}(P_+\xi_0) \qquad (4.23)\\
&+ \int_0^t e^{-(t-\tau)d_1A}V(\tau)^{n-1}\Big[\Big\{1 + \frac{r_2(r_1-1)n}{r_1+r_2}\Big\}f(u_\infty - V(\tau))(P_+\xi)(\tau)\\
&- \frac{r_1(r_1-1)n}{r_1+r_2}f(u_\infty - V(\tau))(P_+\eta)(\tau) + \frac{r_1-1}{r_1+r_2}V(\tau)f_u(u_\infty - V(\tau))(P_+\xi)(\tau)\\
&+ \frac{r_1-1}{r_1+r_2}V(\tau)f_u(u_\infty - V(\tau))(P_+\eta)(\tau) - (r_1-1)(P_+h(\phi,\psi))(\tau)\Big]\,d\tau,\\
&(P_+\eta)(t) = e^{-td_2A}(P_+\eta_0) \qquad (4.24)\\
&+ \int_0^t e^{-(t-\tau)d_2A}V(\tau)^{n-1}\Big[\frac{r_2(r_2+1)n}{r_1+r_2}f(u_\infty - V(\tau))(P_+\xi)(\tau)\\
&+ \Big\{1 - \frac{r_1(r_2+1)n}{r_1+r_2}\Big\}f(u_\infty - V(\tau))(P_+\eta)(\tau)\\
&- \frac{r_2+1}{r_1+r_2}V(\tau)f_u(u_\infty - V(\tau))(P_+\xi)(\tau)\\
&+ \frac{r_2+1}{r_1+r_2}V(\tau)f_u(u_\infty - V(\tau))(P_+\eta)(\tau) - (r_2+1)(P_+h(\phi,\psi))(\tau)\Big]\,d\tau,
\end{aligned}$$

for $t > 0$, where $d_1 = d + br_1 = \max\{a, d\}$, $d_2 = d - br_2 = \min\{a, d\}$, and A is $-\Delta$ with homogeneous Neumann boundary condition on $\partial\Omega$. Applying the $L^p - L^q$ estimate (3.9) to (4.23) and (4.24) yields

$$\begin{aligned}&\|(P_+\xi, P_+\eta)(t)\|_p Ce^{-d_2\lambda t}\|(P_+\xi_0, P_+\eta_0)\|_p\\&+ C\int_0^t m(t-\tau)^{-(N/2)(1/q-1/p)}e^{-d_2\lambda(t-\tau)}V(\tau)^{n-1}\|(P_+\xi, P_+\eta)(\tau)\|_q d\tau\\&+ C\int_0^t e^{-d_2\lambda(t-\tau)}V(\tau)^{n-1}\|(P_+h)(\tau)\|_p d\tau, \qquad t>0 \qquad (4.25)\end{aligned}$$

for $1 \le q \le p \le \infty$. Here, p and q are required to satisfy (3.11); i.e., $(N/2)(1/q-1/p) < 1$. If we take σ to be positive, then we see that multiplying both sides of (4.25) by $V(t)^{1+\sigma}e^{d_2\lambda t}$ gives

$$\begin{aligned}&V(t)^{1+\sigma}e^{d_2\lambda t}\|(P_+\xi, P_+\eta)(t)\|_p \qquad (4.26)\\&\le CI_p + CV(t)^{1+\sigma}\int_0^t m(t-\tau)^{-(N/2)(1/q-1/p)}e^{d_2\lambda\tau}V(\tau)^{n-1}\\&\times \|(P_+\xi, P_+\eta)(\tau)\|_q\, d\tau + CV(t)^{1+\sigma}\int_0^t e^{d_2\lambda\tau}V(\tau)^{n-1}\|(P_+h)(\tau)\|_p\, d\tau\end{aligned}$$

for $t > 0$. We can easily show that the second term in the right-hand side of (4.26) is estimated by

$$CM_{q,\sigma}^+(t)V(t)^{1+\sigma}\int_0^t m(t-\tau)^{-(N/2)(1/q-1/p)}V(\tau)^{n-2-\sigma}\, d\tau \le CM_{q,\sigma}^+(t)$$

according to (3.11) and $m(t-\tau) = \min\{t-\tau, 1\}$ for $t \ge \tau$. It is derived from Lemma 4.1 that

$$\begin{aligned}&V(t)^{1+\sigma}\int_0^t e^{d_2\lambda\tau}V(\tau)^{n-1}\|(P_+h)(\tau)\|_p\, d\tau\\&\le CV(t)^{1+\sigma}\int_0^t e^{d_2\lambda\tau}V(\tau)^{n-1}K(\tau)\|(\phi,\psi)(\tau)\|_\infty\|(P_+\phi, P_+\psi)(\tau)\|_p\, d\tau\\&\le K(t)M_\infty(t)M_{p,\sigma}^+(t)V(t)^{1+\sigma}\int_0^t V(\tau)^{2n-3-\sigma}d\tau \le K(t)M_\infty(t)M_{p,\sigma}^+(t).\end{aligned}$$

Combine (4.26) with these two estimates. If $a > b+d$ or $a < d$, then

$$M_{p,\sigma}^+(t) \le CI_p + CM_{q,\sigma}^+(t) + K(t)M_\infty(t)M_{p,\sigma}^+(t), \qquad t \ge 0 \qquad (4.27)$$

for $1 \le q \le p \le \infty$ with (3.11).

When $N = 1, 2, 3$, take $(p, q) = (\infty, 2)$, which satisfies (3.11). Then we find that (4.27), together with Lemma 4.5, asserts

$$\begin{aligned} M^{+}_{\infty,\sigma}(t) &\le CI_\infty + CM^{+}_{2,\sigma}(t) + K(t)M_\infty(t)M^{+}_{\infty,\sigma}(t) \\ &\le C(\sigma)I_\infty + C(\sigma)K(t)M_\infty(t)M^{+}_{\infty,\sigma}(t) \end{aligned}$$

owing to (4.5).

Now consider the case $N \ge 4$. Define the finite, increasing sequence $\{p_j\}_{j=0,1,\dots,N}$ by $p_0 = 1$, $p_N = \infty$ and $1/p_{j-1} - 1/p_j = 1/N$ for $j = 1, 2, \dots, N$. If one chooses $p = p_j$ and $q = p_{j-1}$ in (4.27), then we get

$$M^{+}_{p_j,\sigma}(t) \le CI_\infty + CM^{+}_{p_{j-1},\sigma}(t) + K(t)M_\infty(t)M^{+}_{\infty,\sigma}(t)$$

for $j = 1, 2, \cdots, N$ with the help of (4.5). Therefore, we obtain

$$M^{+}_{p_j,\sigma}(t) \le CI_\infty + CM^{+}_{1,\sigma}(t) + K(t)M_\infty(t)M^{+}_{\infty,\sigma}(t),$$

which gives

$$M^{+}_{\infty,\sigma}(t) \le C(\sigma)I_\infty + C(\sigma)K(t)M_\infty(t)M^{+}_{\infty,\sigma}(t)$$

by virtue of Lemma 4.5 with $p = 1$. We can also treat the case $a = b + d$ by setting $\sigma = 0$ formally in the above calculations. This completes the proof of Lemma 4.6. □

Concerning the estimate for the P_0-part of $(\phi, \psi)(t, x)$, the following lemma holds true.

Lemma 4.7. *For $t \ge 0$, $M^{0}_{\infty}(t) \le K(t)M_2(t)^2$.*

Proof. It follows from (4.1) and (2.3) that $(P_0\phi)(t) = (P_0\psi)(t)$ for all $t \ge 0$. Then, applying the operator P_0 to both sides of (4.2) gives

$$\begin{aligned} (P_0\psi)'(t) = &-(n-1)V(t)^{n-1}f(u_\infty - V(t))(P_0\psi)(t) \\ &+ V(t)^n f_u(u_\infty - V(t))(P_0\psi)(t) - V(t)^{n-1}(P_0h)(t). \end{aligned}$$

Accordingly, this equation implies

$$(P_0\psi)(t) = -f(u_\infty - V(t))V(t)^{n-1} \int_0^t f(u_\infty - V(\tau))^{-1}(P_0h)(\tau)\, d\tau,$$

where the fact $P_0\psi_0 = 0$ was used. Therefore,

$$V(t)^{-(n-1)}|(P_0\psi)(t)| \le C\int_0^t \|h(\tau)\|_1\,d\tau \le C\int_0^t K(\tau)\|(\phi,\psi)(\tau)\|_2^2\,d\tau$$
$$\le K(t)M_2(t)^2\int_0^t V(\tau)^{2(n-1)}\,d\tau,$$

from which the conclusion follows. This completes the proof of Lemma 4.7.

Proof of Theorem 4.3. In the cases where $a > b+d$ and $a < d$, combining Lemmas 4.5–4.7 and (4.5) and writing $X(t) = M_\infty^0(t) + M_{\infty,\sigma}^+(t)$, we see that for all $t \ge 0$,

$$X(t) \le C(\sigma)X(0) + C(\sigma)L(C(\sigma))X(t)^2.$$

Therefore there exists a constant $\delta_0 > 0$ such that if $X(0) \le \delta_0$, then $X(t) \le C(\sigma)X(0)$ for all $t \ge 0$. This shows that

$$|(P_0\phi, P_0\psi)(t)| \le C(\sigma)I_\infty V(t)^{n-1}$$
$$\|(P_+\phi, P_+\psi)(t)\|_\infty \le C(\sigma)I_\infty V(t)^{-1-\sigma}e^{-d_2\lambda t}$$

for $t \ge 0$. Similarly, in the case $a = b + d$,

$$|(P_0\phi, P_0\psi)(t)| \le CI_\infty V(t)^{n-1}$$
$$\|(P_+\phi, P_+\psi)(t)\|_\infty \le CI_\infty V(t)^{-1-\sigma}e^{-d_2\lambda t}$$

for $t \ge 0$. Thus, there result the conclusions of Theorem 4.3. □

Remark 4.2. It should be noted that when $a > b + d$ (respectively $a < d$), $\lim_{\gamma\to 0} k(\gamma) = 1$ (respectively $\lim_{\gamma\to\infty} k(\gamma) = 1$) by (4.16), namely, $\sigma \to 0$ as $\gamma \to 0$ (respectively $\gamma \to \infty$). Moreover, if $a > b + d$ (respectively $a < d$), then $r_1 > 1$ (respectively $r_1 = 0$), and hence we obtain $\lim_{\gamma\to 0} \ell(\gamma) = \infty$ (respectively $\lim_{\gamma\to\infty} \ell(\gamma) = \infty$) according to (4.18). If $a = b + d$, then we have $k(\gamma) = 1$ and we can take $\ell(\gamma) = 1 + \sqrt{2}$ by taking $\gamma = 1$.

Proofs of Theorem 4.1 and Theorem 4.2. In the case where I_∞ is sufficiently small, we can immediately obtain Theorems 4.1 and 4.2 by Theorem 4.3 with $\varepsilon = \sigma/(n-1)$ and Remark 4.1 after the statement of Theorem 4.3. When I_∞ is large, we apply the approach in Section 5 of Hoshino [9] to our system (1.1)–(1.3); namely, the following lemmas are helpful in proving the theorems.

Lemma 4.8. *For the solution* (u, v) *to* (1.1)–(1.3),

$$\lim_{t\to\infty}\Big\|\Big(\frac{u-\overline{u}}{\overline{v}},\frac{v-\overline{v}}{\overline{v}}\Big)(t)\Big\|_\infty = 0.$$

This lemma is proved by the comparison theorem to parabolic equations and Theorem 3.3. Choose T_0 such that $\big\|\big(\frac{u-\overline{u}}{\overline{v}},\frac{v-\overline{v}}{\overline{v}}\big)(T_0)\big\|_\infty \le \delta_0$, where δ_0 is the number appearing in Theorem 4.3. Lemma 4.8 assures the choice of such T_0.

Lemma 4.9. *Let* V *be the solution for* (1.6), (1.7) *with* $V(0) = \overline{v}_0$, *and let* $\tilde{V}$ *be the solution for*

$$\frac{d\tilde{V}}{dt} = -f(u_\infty - \tilde{V})\tilde{V}^n, \qquad t > T_0$$

with $0 < \tilde{V}(T_0) < u_\infty$. *Then* $|V(t) - \tilde{V}(t)| = O(t^{-1/(n-1)-1})$ *as* $t \to \infty$.

This lemma can be shown by considering the behavior of a function $W(t)$ defined by $\tilde{V}(t) = (1 + W(t))V(t)$.

Regard T_0 as the initial time, and let $\tilde{V}$ be the solution for

$$\frac{d\tilde{V}}{dt} = -f(u_\infty - \tilde{V})\tilde{V}^n, \quad t > T_0, \qquad \tilde{V}(T_0) = \overline{v}(T_0).$$

Lemma 4.8 assures the existence of such a T_0. For $t \ge T_0$, define $(\tilde{\phi}, \tilde{\psi})$ by the following expressions, which are similar to (4.1):

$$u(t,x) = u_\infty - \tilde{V}(t)(1+\tilde{\phi}(t,x)), \qquad v(t,x) = \tilde{V}(t)(1+\tilde{\psi}(t,x)).$$

One can obtain estimates of $(\tilde{\phi}, \tilde{\psi})$ similar to those obtained in Theorem 4.3:

$$\begin{aligned}\|(\tilde{\phi},\tilde{\psi})(t)\|_\infty &\le C(\sigma)\|(\tilde{\phi},\tilde{\psi})(T_0)\|_\infty(1+t-T_0)^{-1},\\ \|(P_+\tilde{\phi},P_+\tilde{\psi})(t)\|_\infty &\le C(\sigma)\|(\tilde{\phi},\tilde{\psi})(T_0)\|_\infty(1+t-T_0)^{1/(n-1)+\sigma}e^{-d_2\lambda(t-T_0)}\end{aligned}$$

for $t \ge T_0$ on account of $\|(\tilde{\phi},\tilde{\psi})(T_0)\|_\infty \le \delta_0$. Furthermore, we can show Theorems 4.1 and 4.2 with use of

$$\begin{aligned}u(t,x) - u_\infty &= -V(t) + (V(t) - \tilde{V}(t)) - \tilde{V}(t)\tilde{\phi}(t,x),\\ v(t,x) &= V(t) + (\tilde{V}(t) - V(t)) + \tilde{V}(t)\tilde{\psi}(t,x),\end{aligned}$$

and Lemma 4.9 in the general case $u_0, v_0 \in L^\infty(\Omega)$ with $u_0, v_0 \geq 0$. □

For the details of the argument, we refer the reader to [9].

5. Concluding remarks. First, when $a \geq b + d$, as is easily seen from the proof of Theorem 2.1, there is a possibility that $v(t, x)$ is nonnegative but $u(t, x)$ may have negative parts, although $(u, v)(t, x)$ is bounded. Actually, with a slight modification to the conditions on f, if $0 \leq v_0(x) \leq M_1$ and $r_1 u_0(x) + v_0(x) \geq M_2$, then there exists a constant M such that

$$r_1^{-1}(M_2 - v(t,x)) \leq u(t,x) \leq M, \qquad t \geq 0, \quad x \in \overline{\Omega},$$

where M_2 and M are not necessarily positive. In the case $a < d$, we can show the boundedness of u:

$$r_2^{-1}(v(t,x) + M_4) \leq u(t,x) \leq \tilde{M}, \qquad t \geq 0, \quad x \in \overline{\Omega}.$$

Secondly, the following system can be also studied:

$$\begin{cases} u_t = a\Delta u - b\Delta v + f(u)g(v), \\ v_t = \qquad\quad d\Delta v - f(u)g(v), \end{cases} \qquad \text{in} \quad (0,\infty) \times \Omega \tag{5.1}$$

with (1.2) and (1.3), where a, b, d are positive constants and $a \neq d$. Let r_1 and r_2 be the exponents defined by Definition 2.1, that is, if $a > d$ (respectively $a < d$), then $(r_1, r_2) = ((a-d)/b, 0)$ (respectively $(0, (d-a)/b)$). Then, it follows from (5.1) that

$$\begin{aligned} (r_1 u - v)_t &= (d + br_1)\Delta(r_1 u - v) + (r_1 + 1)f(u)g(v), \\ (r_2 u + v)_t &= (d - br_2)\Delta(r_2 u + v) + (r_2 - 1)f(u)g(v), \end{aligned}$$

where $d + br_1 = \max\{a, d\}$, $d - br_2 = \min\{a, d\}$. The counterpart of Theorem 2.1 (respectively 2.2) can be stated as the following Theorem 5.1 (i) (respectively (ii)).

Theorem 5.1. (i) *Suppose that $a + b \leq d$, and suppose also that f and g satisfy Assumption* 2.1. *Then, if $v_0(x) \geq 0$ and $\frac{b}{d-a}\|v_0\|_\infty - \frac{b}{d-a}v_0(x) \leq u_0(x)$ for $x \in \overline{\Omega}$, then there is a nonnegative global solution $(u, v)(t, x)$ to the problem* (5.1) *with* (1.2), (1.3) *satisfying $0 \leq v(t, x) \leq \|v_0\|_\infty$ for $t > 0$ and $x \in \overline{\Omega}$. Moreover, there is a positive constant M such that*

$$0 \leq \frac{b}{d-a}\|v(t)\|_\infty - \frac{b}{d-a}v(t,x) \leq u(t,x) \leq M, \qquad (t,x) \in (0,\infty) \times \overline{\Omega}.$$

(ii) *In the case* $a > d$, *if* $0 \le \frac{b}{a-d} v_0(x) \le u_0(x)$ *for* $x \in \overline{\Omega}$, *and* f *and* g *satisfy Assumption* 2.1, *then the problem* (5.1) *with* (1.2), (1.3) *has a nonnegative global solution* $(u, v)(t, x)$ *satisfying* $0 \le v(t, x) \le \|v_0\|_\infty$ *for* $t > 0$, $x \in \overline{\Omega}$, *and there exists a positive constant* $\tilde{M}$ *such that*

$$0 \le \frac{b}{a-d} v(t, x) \le u(t, x) \le \tilde{M}, \qquad (t, x) \in (0, \infty) \times \overline{\Omega}.$$

As regards uniform convergence properties of the nonnegative solution $(u, v)(t, x)$ to (5.1), with (1.2) and (1.3), we can obtain results like the theorems in Section 3. When $g(v) = v^n$ with $n > 1$, large-time approximation of $(u, v)(t, x)$ can be also investigated. The functions $\phi(t, x)$ and $\psi(t, x)$ defined by (4.1) satisfy

$$\begin{aligned} \phi_t &= a\Delta\phi + b\Delta\psi \\ &\quad - V^{n-1}\left\{-f(u_\infty - V)\phi - V f_u(u_\infty - V)\phi + n f(u_\infty - V)\psi + h(\phi, \psi)\right\}, \\ \psi_t &= d\Delta\psi \\ &\quad - V^{n-1}\left\{-V f_u(u_\infty - V)\phi + (n-1) f(u_\infty - V)\psi + h(\phi, \psi)\right\}, \end{aligned} \tag{5.2}$$

in $(0, \infty) \times \Omega$, (4.3) and (4.4), where $h(\phi, \psi)$ is the same function as the one appearing in (4.2). It is easily seen that the approach established in Section 4 is applicable to analyzing the large-time approximation of the nonnegative solution $(u, v)(t, x)$ to (5.1) with (1.2) and (1.3); that is to say, we can obtain this approximation in consideration of decay rates of the solution $(\phi, \psi)(t, x)$ to the problem (5.2) with (4.3), (4.4). The counterparts of Theorems 4.1 and 4.2 can be stated as follows.

Theorem 5.2. (i) *Suppose that* $g(v) = v^n$ *with* $n > 1$ *and* $a \ne d$. *Let* $(u, v)(t, x)$ *be the nonnegative global solution to the problem* (5.1) *with* (1.2), (1.3). *Then* $(u, v)(t, x) = (U, V)(t) + O(t^{-1/(n-1)-1})$ *uniformly in* $x \in \Omega$ *as* $t \to \infty$, *where* $(U, V)(t)$ *is the solution for the problem* (1.6) *and* (1.7).

(ii) *The difference between* $(u, v)(t, x)$ *and the spatial average* $(\overline{u}, \overline{v})(t)$ *decays with an exponential rate. In fact, when* $a + b < d$ *or* $a < d$, *fix* $\varepsilon > 0$. *Then* $(u, v)(t, x) = (\overline{u}, \overline{v})(t) + O(t^\varepsilon e^{-\min\{a,d\}\lambda t})$ *uniformly in* $x \in \Omega$ *as* $t \to \infty$, *where* λ *is the smallest positive eigenvalue of* $-\Delta$ *with homogeneous Neumann boundary condition on* $\partial\Omega$. *Moreover, if* $a + b = d$, *then*

$$(u, v)(t, x) = (\overline{u}, \overline{v})(t) + O(e^{-d\lambda t})$$

uniformly in $x \in \Omega$ *as* $t \to \infty$.

Finally, we give a remark on the following more general system than (1.1)–(1.3):

$$\begin{cases} u_t = a\Delta u + b\Delta v + f(u)g(v), \\ v_t = c\Delta u + d\Delta v - f(u)g(v), \end{cases} \quad \text{in} \quad (0,\infty)\times\Omega \tag{5.3}$$

with (1.2) and (1.3). Here a, b, c, d are supposed to be positive constants satisfying $ad > bc$, (1.5) is assumed, and f, g are continuous functions and

$$f(u), g(v) \geq 0 \qquad \text{for} \quad u, v \in \mathbf{R}. \tag{5.4}$$

Definition 5.1. Two positive exponents r_1 and r_2 are defined by

$$r_1 = \frac{(a-d) + \sqrt{(d-a)^2 + 4bc}}{2b}, \qquad r_2 = \frac{-(a-d) + \sqrt{(d-a)^2 + 4bc}}{2b}.$$

Recently, Kanel and Kirane [11] have studied the problem (5.3) with (1.2) and (1.3) under the conditions that $d < a < b + d$, c is small enough so that $r_1 < 1$ and $|a - b + c - d|$ is also small and $f(u) = u^n$ with an odd natural number n, $g(v) = v$ which are different from f, g in (5.4). Actually, it has been shown in [11] that if $r_1 u_0(x) + v_0(x) \geq 0$, $r_2 u_0(x) - v_0(x) \geq 0$ and $r_1 < 1$, then $u(t,x)$ is nonnegative for $t \geq 0$, $x \in \Omega$, and $(u,v)(t,x)$ is uniformly bounded.

Let (w,z) be $(w,z) = (r_1 u + v, r_2 u - v)$. Then, the system (5.3) with (1.2), (1.3) becomes the same system as (2.6)–(2.8), where $d_1 = d + br_1$ and $d_2 = d - br_2$. Note that the assumption $a \neq d$ implies $d_2 > 0$.

Under (5.4), we can show that if $w_0(x) \geq M_1$ (respectively $w_0(x) \leq M_1$), then $w(t,x) \geq M_1$ (respectively $w(t,x) \leq M_1$) for $t > 0$ and $x \in \overline{\Omega}$, provided that $r_1 \geq 1$ (respectively $r_1 \leq 1$), and that if $z_0(x) \geq M_2$, then $z(t,x) \geq M_2$ for $t > 0$ and $x \in \overline{\Omega}$. Therefore, in the case $r_1 \geq 1$, choosing $M_1 = M_2 = 0$, if $r_1 u_0(x) + v_0(x) \geq 0$ and $r_2 u_0(x) - v_0(x) \geq 0$, then $r_1 u(t,x) + v(t,x) \geq 0$ and $r_2 u(t,x) - v(t,x) \geq 0$ for $t > 0$, $x \in \overline{\Omega}$, and hence $u(t,x) \geq 0$. In the case $r_1 \leq 1$, if $r_1 u_0(x) + v_0(x) \leq 0$ and $r_2 u_0(x) - v_0(x) \geq 0$, then we get $r_1 u(t,x) + v(t,x) \leq 0$ and $r_2 u(t,x) - v(t,x) \geq 0$ for $t > 0$, $x \in \overline{\Omega}$, and hence $v(t,x) \leq 0$.

REFERENCES

[1] N. D. Alikakos, *L^p-bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations, 4 (1979), 827–868.

[2] H. Amann, *Global existence for semilinear parabolic systems*, J. Reine Angew. Math., 360 (1985), 47–83.

[3] R. Aris, "The Mathematical Theory of Diffusion and Reaction in Permeable Catalysis, Vol. 1," Oxford University Press, Oxford, 1975.

[4] A. Barabanova, *On the global existence of solutions of a reaction-diffusion equation with exponential nonlinearity*, Proc. Amer. Math. Soc., 122 (1994), 827–831.

[5] C. Cosner, *Pointwise bounds for strongly coupled semi-linear systems of parabolic partial differential equations*, Indiana Univ. Math. J., 30 (1981), 607–619.

[6] S. R. de Groot and P. Mazur, "Non-equilibrium Thermodynamics," North-Holland, Amsterdam, 1962.

[7] A. Haraux and M. Kirane, *Estimations C^1 pour des problèmes paraboliques semi-linéaires*, Ann. Fac. Sci. Toulouse, 5 (1983), 265–280.

[8] A. Haraux and A. Youkana, *On a result of K. Masuda concerning reaction-diffusion equations*, Tôhoku Math. J., 40 (1988), 159–163.

[9] H. Hoshino, *Rate of convergence of global solutions for a class of reaction-diffusion systems and the corresponding asymptotic solutions*, Adv. Math. Sci. Appl., 6 (1996), 177–195.

[10] H. Hoshino and S. Kawashima, *Asymptotic equivalence of a reaction-diffusion system to the corresponding system of ordinary differential equations*, Math. Models Meth. Appl. Sci., 5 (1995), 813–834.

[11] J. I. Kanel, and M. Kirane, *Pointwise a priori estimates for a strongly coupled system of reaction-diffusion equations with a balance law*, Math. Mech. Appl. Sci., 21 (1998), 1227–1232.

[12] J. I. Kanel and M. Kirane, *Global solutions to a reaction diffusion system*, preprint.

[13] M. Kirane, *Global bounds and asymptotics for a system of reaction-diffusion equations*, J. Math. Anal. Appl., 138 (1989), 1172–1189.

[14] J. S. Kirkaldy, *Diffusion in multicomponent metallic systems*, Canadian J. Physics, 35 (1957), 435–440.

[15] K. Masuda, *On the global existence and asymptotic behavior of solutions of reaction diffusion equations*, Hokkaido Math. J., 12 (1983), 360–370.

[16] S. S. Okoya, *Boundedness for a system of reaction-diffusion equations 1*, Mathematika, 41 (1994), 293–300.

[17] K. Otsuka, *On the positivity of the fundamental solutions for parabolic systems*, J. Math. Kyoto Univ., 28 (1988), 119–132.

[18] C. V. Pao, "Nonlinear Parabolic and Elliptic Equations," Plenum Press, New York, 1992.

[19] R. Redlinger, *Pointwise a priori bounds for strongly coupled semilinear problems*, Indiana Univ. Math. J., 36 (1987), 441–454.

[20] F. Rothe, "Global Solutions of Reaction-Diffusion Systems," Lecture Notes in Math., 1072, Springer-Verlag, Berlin, 1984.

[21] J. Smoller, "Shock Waves and Reaction-Diffusion Equations" Springer-Verlag, Berlin, 1983.