Advances in Differential Equations Volume 8, Number 6, June 2003, Pages 733–768

HAMILTON–JACOBI EQUATIONS WITH MEASURABLE DEPENDENCE ON THE STATE VARIABLE

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Abstract. We study the Hamilton-Jacobi equation

$$H(x, Du) = 0,$$

where $H(x, p)$ is assumed to be measurable in x, quasiconvex and continuous in p. The notion of viscosity solution is adapted to the measurable setting making use of suitable measure–theoretic devices. We obtain integral representation formulae generalizing the ones valid for continuous equations, comparison principles and uniqueness results. We examine stability properties of the new definition and present two approximation procedures: the first one is based on a regularization of the Hamiltonian by mollification and in the second one the approximating sequence is made up by minimizers of certain variational integrals.

1. Introduction

We treat the equation

$$H(x, Du) = 0 \tag{1.1}$$

in the framework of viscosity solution theory. We assume H is measurable in x, continuous and quasiconvex in p. In addition we require a coercitivity condition and the existence of a locally Lipschitz-continuous a.e. subsolution.

The major issue is to adapt the definition of viscosity solution to the measurable setting. The problem has an intrinsic interest from a theoretical

Accepted for publication: December 2002.
AMS Subject Classifications: 35F20, 35D05, 49L25, 53C60.
The research was partially supported by the TMR Network "Viscosity solutions and their applications".

point of view, its solution requires the combination of viscosity solution techniques with measure-theoretic ones, and can be also motivated looking at the applications.

Several models in fact lead to equations with discontinuous Hamiltonians. For example, this happens in geometric optics in the presence of layers, in shape from shading (in [21] it is considered the case of a piecewise Lipschitz continuous Hamiltonian), in combustion phenomena taking place in nonhomogeneous media (see [17] for the case of striated media).

In the second order case, measurable fully nonlinear equations have been already studied using viscosity solution methods (see [6]). However the technique exploited there, which are based on the strong maximum principle, do not apply to first order.

The literature dealing with first order discontinuous equations is not so wide. We basically refer to two papers.

In [18], the authors consider the eikonal equation

$$|Du| = f(x) \tag{1.2}$$

assuming f to be lower semicontinuous. They employ the formula (giving a class of "fundamental solutions" in the continuous case)

$$\begin{aligned} L(x,y) \quad &= \inf \left\{ \int_0^1 f(\xi(t))|\dot{\xi}(t))|dt : \xi(t) \in W^{1,\infty}([0,1],\mathbb{R}^N) \right. \\ &\left. \text{s.t. } \xi(0) = x,\ \xi(1) = y \right\} \end{aligned} \tag{1.3}$$

to build a new definition of solution termed after Monge. However, these solutions do not satisfy the maximality properties holding in the continuous case. Furthermore, this notion is quite indirect since it is not based on test functions.

In [20], the author deals with a class of Hamilton-Jacobi equations related to some control problems, which has nonempty intersection with ours. Equation (1.1) is for instance in such intersection when f is Borel measurable.

As definition of solution, the author considers the one proposed by Ishii (see [3]), based on upper and lower semicontinuous envelopes of the equation. He gives representation formulae for the maximal viscosity subsolution and the minimal viscosity supersolution. Under some special assumptions on the set of discontinuities of the equation, he gets a uniqueness result for the Ishii's solution of the equation verifying certain boundary conditions (see also [21]).

We will not use Ishii's approach, we resort instead to measure theoretic devices. A crucial observation is that what really matters in the analysis of (1.1) is not the whole Hamiltonian, but just the 0-sublevel sets (see [7], [8],

[19])

$$\mathcal{Z}(x) = \left\{ p \in \mathbb{R}^N : H(x,p) \leq 0 \right\}. \tag{1.4}$$

Under our hypotheses the set-valued map $x \mapsto \mathcal{Z}(x)$ is measurable, a.e. convex compact valued.

The starting point of the paper is the modifications of an integral formula, whose particular case is (1.3), which provides in the continuous case a set of "fundamental solutions" to (1.1). The difficulty is that such formula is based on line integrals. The curves, being negligible respect to the Lebesgue measure, are difficult to handle under measurability assumptions.

We overcome it using a key concept introduced in [11], [12] in the study of certain metrics on Lipschitz manifolds: the notion of transversality between curves and sets of 0-Lebesgue measure (see also [1], [5] for other applications of it in problems of calculus of variations).

The integral formula, modified according to this concept, gives a set of functions which are then related to the Hamiltonian, or more precisely to $\mathcal{Z}(x)$, making use of viscosity test functions. The relations satisfied by sub and supertangents are not symmetric. Namely, those verified by supertangents are expressed in terms of density with respect to the variable x, while for subtangents we give relations up to set of 0-Lebesgue measure, i.e., essential with respect to x.

From these properties we derive a definition of viscosity solution for (1.1) in the measurable setting. It reduces to Crandall–Lions one if the Hamiltonian is continuous and, as for convex continuous equations, the notion of viscosity subsolution and a.e. Lipschitz continuous subsolution are equivalent.

However, in contrast with the continuous case, a viscosity solution is not in general an a.e. solution. It can even be a strict subsolution on a set of full measure, see Example 5.3.

This notion of solution makes possible to recover the main properties of viscosity solutions holding in the continuous case, namely existence and uniqueness of solutions of (1.1) satisfying suitable boundary conditions. Here the lack of symmetry between the definitions of supersolution and subsolution plays a crucial role.

As application we present two different approximation procedures which can be useful for a numerical treatment of (1.1).

The first one relies upon a suitable regularization of H through mollification. It yields perturbed continuous Hamilton-Jacobi equations whose viscosity solutions (in the Crandall-Lions sense) converge locally uniformly to the solution of (1.1). This result is obtained by adapting to the new definition of solution standard viscosity stability techniques.

In the second the approximating sequence is made up by the minimizers of certain variational integrals. It generalizes a result given in [4].

The paper is organized as follows. Section 2 includes some terminology, basic definitions and assumptions and the main properties of the set-valued map $\mathcal{Z}(x)$. Section 3 describes the modifications needed to the integral formula. Sections 4 and 5 are devoted to the analysis of the relations between certain functions given by the integral formula and the equation (1.1). In Section 6 the definition of viscosity solution is given and existence and uniqueness results are derived. Finally, Sections 7, 8 concern approximation and stability.

2. Assumptions and preliminaries

Throughout the paper, unless otherwise specified, measure(-able) is meant in the sense of N-dimensional Lebesgue measure. For a measurable set E, $|E|$ will denote its measure. If this measure is zero, the set E will be a called a null set.

For any $x \in \mathbb{R}^N$ and $r > 0$, $B(x, r)$ will stand for the Euclidean ball of center x and radius r. Given a closed subset K and $p_0 \in \mathbb{R}^N$, we set

$$d(p_0, K) = \inf\{|p_0 - p| : p \in K\},$$
$$d^*(p_0, K) = d(p_0, K) - d(p_0, \mathbb{R}^N \setminus K).$$

If K_1, K_2 are compact sets, the Hausdorff distance $d_{\mathcal{H}}$ between them is defined through the formula

$$d_{\mathcal{H}}(K_1, K_2) = \max\Big\{ \max_{p_1 \in K_1} d(p_1, K_2), \max_{p_2 \in K_2} d(p_2, K_1) \Big\}.$$

We will use the term supertangents (subtangents) to signify viscosity test functions from above (below), see [2], [3].

All the curves considered in the sequel will be understood to be Lipschitz continuous. For any x, $y \in \mathbb{R}^N$, $\mathcal{A}_{x,y}$ will denote the set of curves defined in $[0, 1]$ joining x to y. For a curve ξ, $\ell(\xi)$ indicates its Euclidean length.

Our analysis relies upon the map $\mathcal{Z}$ defined in (1.4).

The requirements on H will imply that such set–valued map is a.e. compact convex valued, measurable and locally essentially bounded in the sense specified by the following definitions.

Definition 2.1. A set-valued map $C : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ with $C(x)$ compact for a.e. x, is said *measurable* if $\{x : C(x) \cap K \neq \emptyset\}$ is so for any closed subset K of $\mathbb{R}^N$.

The previous definition can be equivalently stated, see [9], requiring C to be measurable as a map from $\mathbb{R}^N$ to the family of compact subsets of $\mathbb{R}^N$ endowed with the Hausdorff metric.

Definition 2.2. A set-valued map C is said locally essentially bounded provided that for any compact set K there is $R > 0$ verifying

$$C(x) \subset B(0, R) \qquad \text{for a.e. } x \in K. \tag{2.1}$$

The infimum of the R verifying (2.1) will be indicated as $\operatorname{ess\,sup}_K C$.

We proceed to detail the assumptions on H that will hold in the whole paper without any further mention. Additional conditions will be introduced in Sections 6, 7 and 8.

The Hamiltonian $H : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}$ is measurable in x for any p and continuous in p for a.e. x.

Quasiconvexity

$$\begin{gathered} \mathcal{Z}(x) := \{p : H(x,p) \le 0\} \text{ is convex for a.e. } x \\ \partial \mathcal{Z}(x) = \{p : H(x,p) = 0\} \text{ for a.e. } x. \end{gathered} \tag{2.2}$$

Coercitivity

$$\begin{gathered} \text{for any compact subset } K \text{ of } \mathbb{R}^N \text{ there exists } R > 0 \text{ s.t.} \\ \operatorname{ess\,inf} \{H(x,p) : |p| > R,\ x \in K\} > 0. \end{gathered} \tag{2.3}$$

Existence of an a.e. subsolution

$$\begin{gathered} \text{there exists a locally Lipschitz continuous function } \phi \textit{ s.t.} \\ H(x, D\phi(x)) \le 0 \qquad \text{for a.e. } x. \end{gathered} \tag{2.4}$$

If $H(x,p) = |p| - f(x)$, then the above assumptions amount to require f to be measurable, a.e. nonnegative and locally essentially bounded. A subsolution is thus the null function.

Proposition 2.3. *$\mathcal{Z}$ is a measurable locally essentially bounded set-valued map and $\mathcal{Z}(x)$ is a convex, compact set for a.e. x.*

Proof. The convexity and compactness of the values of $\mathcal{Z}$ as well as the essential boundedness come directly from the continuity of $p \mapsto H(x,p)$ for a.e. x and the assumptions (2.2), (2.3).

To show that $\mathcal{Z}$ is measurable, consider first a compact set K and put

$$E = \{x : \mathcal{Z}(x) \cap K \neq \emptyset\}.$$

Let $\{p_n\}_{n\in\mathbb{N}}$ be a dense countable subset of K.

Consider x such that $p \mapsto H(x,p)$ is continuous. Then $x \in E$ if and only if

$$\inf_n H(x,p_n) \leq 0.$$

The set E is thus measurable, since the infimum of a countable family of measurable functions is so and the set of x such that $p \mapsto H(x,p)$ fails to be continuous is a null set.

If K is a closed set, not necessarily bounded , write

$$E = \bigcup_n \{x : \mathcal{Z}(x) \cap K \cap B(0,n) \neq \emptyset\}$$

to conclude the proof. □

For any $x \in \mathbb{R}^N$, $q \in \mathbb{R}^N$ we set

$$\sigma(x,q) = \max\{pq : \ p \in \mathcal{Z}(x)\},$$

the support function of the set $\mathcal{Z}(x)$ at q.

Corollary 2.4. *The function $\sigma(x,q)$ is measurable in x for any q and continuous in q for a.e. x.*

Proof. The assertion comes from the definition of support function, the measurability of $\mathcal{Z}$ and the a.e. convexity of $\mathcal{Z}(x)$, see [9]. □

We end the section with the definition of an equivalence relation between Hamiltonians.

Definition 2.5. Two Hamiltonians $H(x,p)$ and $H'(x,p)$, both measurable in x and continuous in p, are said equivalent (in symbols $H \sim H'$) if for any p there exists a null set E_p such that

$$H(x,p) = H'(x,p) \qquad \text{for } x \in \mathbb{R}^N \setminus E_p.$$

Lemma 2.6. *$H \sim H'$ if and only if there exists a null set E verifying*

$$H(x,p) = H'(x,p) \qquad \textit{for } x \in (\mathbb{R}^N \setminus E) \times \mathbb{R}^N.$$

Proof. Assume $H \sim H'$. Let $\{p_n\}_{n\in\mathbb{N}}$ be a countable set dense in $\mathbb{R}^N$ and F the null set of points x for which either $p \mapsto H(x,p)$ or $p \mapsto H'(x,p)$ are not continuous.

The set $E = (\cup_n E_{p_n}) \cup F$ satisfies the statement, where E_{p_n} is as in Definition 2.5.

In fact take $x \notin E$, $p \in \mathbb{R}^N$ and denote by $p_{\overline{n}}$ a subsequence of $\{p_n\}_{n\in\mathbb{N}}$ converging to p. It results

$$H(x,p) = \lim_{\overline{n}\to+\infty} H(x,p_{\overline{n}}) = \lim_{\overline{n}\to+\infty} H'(x,p_{\overline{n}}) = H'(x,p). \qquad \square$$

3. Modification of a formula

The formula we start from is

$$\inf\left\{\int_0^1 \sigma(\xi(t),\dot{\xi}(t))dt : \xi \in \mathcal{A}_{y,x}\right\} \qquad \text{for } x,\, y \in \mathbb{R}^N.$$

It provides in the continuous case a set of fundamental solutions of (1.1). We adapt it to the measurable setting by selecting families of special curves joining y to x.

For this purpose we introduce the key notion of transversality between curves and null sets.

Definition 3.1. A null set E and a curve ξ are said *transversal*, in symbols $\xi \pitchfork E$, if

$$\mathcal{L}^1(\{t:\ \xi(t) \in E\}) = 0,$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure.

For a given curve $\xi : [0,1] \longrightarrow \mathbb{R}^N$ the integral $\int_0^1 \sigma(\xi(t),\dot{\xi}(t))dt$ is not in general well defined. The integrand could even be nonmeasurable, since the composition of a measurable function $(\sigma(x,q))$ and a continuous one $(\xi(t))$ is not necessarily measurable. However we have:

Proposition 3.2. *Given two points y, x and a null set E, there is $\xi \in \mathcal{A}_{y,x}$ verifying*

$$\xi \pitchfork E,$$

$$\ell(\xi) \le 2|x-y|, \tag{3.1}$$

$$\int_0^1 \sigma(\xi(t),\dot{\xi}(t))dt \textit{ is well defined,}$$

$$\int_0^1 \sigma(\xi(t),\dot{\xi}(t))dt \le 2M|x-y| \tag{3.2}$$

with $M = \operatorname{ess\,sup}_{B(y,2|x-y|)} \mathcal{Z}$.

The proof is partly taken from [11], [12].

Proof. It can be assumed that $y = 0$ and $x = e_N$, where e_N is the N-th coordinate vector of $\mathbb{R}^N$. Put

$$g(t) = -t(t-1) \quad \text{for } t \in [0,1]$$

and define a map $\psi : [0,1] \times e_N^\perp \longrightarrow \mathbb{R}^N$ by $\psi(t,z) = te_N + g(t)z$. ψ is a diffeomorphism from $(0,1) \times e_N^\perp$ to its image. For any $z \in e_N^\perp$, consider $\xi_z \in \mathcal{A}_{y,x}$ given by the formula

$$\xi_z(t) = \psi(t,z) \qquad \text{for } t \in [0,1].$$

Fix $z \in e_N^\perp$ and consider the curve ξ obtained through the juxtaposition of the segments joining y to $z/4$, $z/4$ to $e_N + z/4$ and $e_N + z/4$ to x, respectively.

Take into account that g is concave with $\max_{[0,1]} g(x) = 1/4$ and the nonexpansive character of the projection on convex sets to discover that

$$\ell(\xi) = \frac{|z|}{2} + |x-y| \geq \ell(\xi_z). \tag{3.3}$$

Set $B = \psi((0,1) \times (e_N^\perp \cap B(0,1)))$. By (3.3) $B \subset B(y, 2|x-y|)$. Denote by E_1 a null set verifying

$$\mathcal{Z}(x) \subset B(0,M) \qquad \text{for } x \in B \setminus E_1. \tag{3.4}$$

Regularize $\sigma(x,q)$ by partial convolution with respect to x obtaining a sequence $\sigma_\varepsilon(x,q)$ of locally Lipschitz continuous function in (x,q) verifying for $\varepsilon \to 0^+$

$$\sigma_\varepsilon(x,q) \longrightarrow \sigma(x,q) \qquad \text{for } (x,q) \in (\mathbb{R}^N \setminus E_2) \times \mathbb{R}^N, \tag{3.5}$$

where E_2 is a null set.

Let F be the null set given by the union of E, E_1 and E_2. For any curve ζ defined for $t \in [0,1]$, contained in B and transversal to F, it results by (3.5)

$$\sigma_\varepsilon(\zeta(t), \dot{\zeta}(t)) \longrightarrow \sigma(\zeta(t), \dot{\zeta}(t)) \qquad \text{for a.e. } t \in [0,1]$$

and consequently $t \mapsto \sigma(\zeta(t), \dot{\zeta}(t))$ is measurable. Since $\sigma(\zeta(t), \dot{\zeta}(t))$ is also essentially bounded by (3.4), it follows that $\int_0^1 \sigma(\zeta(t), \dot{\zeta}(t))dt$ is well defined. Moreover,

$$\int_0^1 \sigma(\zeta(t), \dot{\zeta}(t))dt \leq M\ell(\zeta) \leq 2M|x-y|. \tag{3.6}$$

Consider $F' = \psi^{-1}(B \cap F)$, which is a null set being ψ is a diffeomorphism. It comes from Fubini's theorem

$$\mathcal{L}^1(\{t : (t,z) \in F'\}) = 0$$

for $\mathcal{L}^{N-1} - a.e.\ z \in e_N^{\perp} \cap B(0,1)$, and

$$\mathcal{L}^1(\{t : \xi_z(t) \in F\}) = \mathcal{L}^1(\{t : \psi(t,z) \in F\}) = \mathcal{L}^1(\{t : (t,z) \in F'\}) = 0. \tag{3.7}$$

By (3.3), (3.6), (3.7), ξ_z verifies the statement, for a.e. $z \in e_N^{\perp} \cap B(0,1)$. □

We set for any x, $y \in \mathbb{R}^N$, $\xi \in \mathcal{A}_{y,x}$, E with $|E| = 0$

$$\mathcal{A}_{y,x}^E = \{\xi \in \mathcal{A}_{y,x} :\ \xi \pitchfork E\}$$

$$I(\xi) = \begin{cases} \int_0^1 \sigma(\xi(t), \dot\xi(t))dt & \text{if the integral is well defined} \\ +\infty & \text{otherwise} \end{cases} \tag{3.8}$$

$$L(y,x) = \sup_{|E|=0} \inf \left\{ I(\xi) : \xi \in \mathcal{A}_{y,x}^E \right\}. \tag{3.9}$$

Proposition 3.3. *L is finite valued and invariant with respect to the equivalence relation given in Definition* 2.5.

Proof. $L(y,x) < +\infty$ for any x, y because of Proposition 3.2. Denote by E the null set where ϕ, see (2.4), is not differentiable and consider $\xi \in \mathcal{A}_{y,x}^E$ with $I(\xi) < +\infty$. It results

$$\sigma(\xi(t), \dot\xi(t)) \ge D\phi(\xi(t))\dot\xi(t) \quad \text{for a.e. } t \in [0,1].$$

Compute

$$\int_0^1 \sigma(\xi(t), \dot\xi(t))dt = \int_0^1 \left[D\phi(\xi(t))\dot\xi(t) + \Big(\sigma(\xi(t), \dot\xi(t)) - D\phi(\xi(t))\dot\xi(t)\Big)\right] dt$$

$$\ge \int_0^1 D\phi(\xi(t))\dot\xi(t)dt = \phi(x) - \phi(y). \tag{3.10}$$

This gives $L(y,x) > -\infty$.

To show the invariance property, observe that $L(y,x)$ can be equivalently defined taking in (3.9) the supremum with respect to the null sets containing a fixed F. In fact for any null set E,

$$\inf \left\{ I(\xi) : \xi \in \mathcal{A}_{y,x}^E \right\} \le \inf \left\{ I(\xi) : \xi \in \mathcal{A}_{y,x}^{E\cup F} \right\}$$

and so

$$\sup_{|E|=0,\, E \supset F} \inf \left\{ I(\xi) : \xi \in \mathcal{A}_{y,x}^E \right\} \le L(y,x)$$

$$\le \sup_{|E|=0} \inf \left\{ I(\xi) : \xi \in \mathcal{A}_{y,x}^{E\cup F} \right\} = \sup_{|E|=0,\, E \supset F} \inf \left\{ I(\xi) : \xi \in \mathcal{A}_{y,x}^E \right\}.$$

The value of L is thus not affected if the Hamiltonian is changed on the set $F \times \mathbb{R}^N$. This concludes the proof in view of Lemma 2.6. $\square$

From the definition of L and arguing as in Proposition 3.2 we can prove:

Proposition 3.4.

i) *For any* x, y, z,

$$L(y,x) \le L(y,z) + L(z,x). \tag{3.11}$$

ii) *For any compact set* K, *there exists* M *such that*

$$L(y,x) \le M|x-y| \quad \text{for any } y,\, x \in K.$$

iii) *For any* x,

$$L(x,x) = 0.$$

iv) *For any fixed* y_0, *the function* $L(y_0,\cdot)$ *is locally Lipschitz continuous.*

Proof. (i) comes from the fact that the juxtaposition of admissible curves yields again an admissible curve.

(ii) is a consequence of (3.2), (iii) is directly derived from (ii) and (3.10). Finally (iv) follows from (i) and (ii). $\square$

The role of the supremum in the formula defining L is essential for the invariance proved in Proposition 3.3. However, if we do without such a property and fix a representative H, we are able to show:

Proposition 3.5. *There exists a null set* F (*depending on the representative* H) *such that for any* x, y

$$I(\xi) < +\infty \qquad \text{for any } \xi \in \mathcal{A}^F_{y,x} \tag{3.12}$$

$$L(y,x) = \inf\left\{I(\xi) : \xi \in \mathcal{A}^F_{y,x}\right\}. \tag{3.13}$$

Proof. Let F_n, M_n be a sequence of null sets and positive constants, respectively, with $\mathcal{Z}(x) \subset B(0,M_n)$ for $x \in B(0,n) \setminus F_n$. Let F_0 be a null set verifying for $\varepsilon \to 0^+$ $\sigma_\varepsilon \longrightarrow \sigma$ pointwise in $(\mathbb{R}^N \setminus F_0) \times \mathbb{R}^N$, where σ_ε is defined as in the proof of Proposition 3.2. Set $\tilde F = F_0 \cup (\cup_n F_n)$.

If a curve ξ is transversal to $\tilde F$, then $t \mapsto \sigma(\xi(t),\dot\xi(t))$ is measurable and essentially bounded, hence $I(\xi)$ is finite.

Fix x, y. For any $k \in \mathbb{N}$, a null set $E_k \supset \tilde F$ (see the proof of Proposition 3.3) can be chosen which depends on the representative H verifying

$$L(y,x) \le \inf\left\{I(\xi) : \xi \in \mathcal{A}^{E_k}_{y,x}\right\} + \frac{1}{k}.$$

Set $E = \cup_k E_k$ and fix k_0. It results

$$L(y,x) \geq \inf \left\{ I(\xi) : \xi \in \mathcal{A}^E_{y,x} \right\} \geq \inf \left\{ I(\xi) : \xi \in \mathcal{A}^{E_{k_0}}_{y,x} \right\} \geq L(y,x) - \frac{1}{k_0}.$$

This implies, being k_0 arbitrary,

$$L(y,x) = \inf \left\{ I(\xi) : \xi \in \mathcal{A}^E_{y,x} \right\}. \tag{3.14}$$

Consequently, for a given countable set B dense in $\mathbb{R}^N$, a null set $F \supset \tilde{F}$ can be selected in such a way that (3.14) holds for any x, $y \in B$.

Fix x_0, $y_0 \in \mathbb{R}^N$, $\xi \in \mathcal{A}^F_{y_0,x_0}$, $\varepsilon > 0$ and select x, $y \in B$ such that

$$\max \left\{ L(x_0,x), L(y,y_0) \right\} < \varepsilon. \tag{3.15}$$

Then $\xi_1 \in \mathcal{A}^F_{y,y_0}$, $\xi_2 \in \mathcal{A}^F_{x_0,x}$ can be found with

$$I(\xi_1) < \varepsilon \quad , \quad I(\xi_2) < \varepsilon. \tag{3.16}$$

A curve $\eta \in \mathcal{A}^F_{y,x}$ is defined by juxtaposition of ξ_1, ξ, ξ_2.

It results by (3.15), (3.16) and (3.11)

$$\begin{aligned} L(y_0,x_0) &\leq L(y_0,y) + L(y,x) + L(x,x_0) \leq I(\eta) + 2\varepsilon \\ &\leq I(\xi_1) + I(\xi) + I(\xi_2) + 2\varepsilon \leq I(\xi) + 4\varepsilon. \end{aligned}$$

This inequality implies the thesis since ε is arbitrary. □

Taking into account (3.9), we see that (3.13) still holds if F is replaced by any null set containing F. This fact will be exploited in the sequel.

Given an open set Ω, we define for any $y, x \in \Omega$ the quantity $L^\Omega(y,x)$ as in (3.9), but considering only curves contained in Ω. An analogous of Proposition 3.5 holds for L^Ω.

We have:

Proposition 3.6. *Let v be a locally a.e. Lipschitz continuous subsolution of* (1.1) *in Ω. Then*

$$v(x) - v(y) \leq L^\Omega(y,x) \tag{3.17}$$

for any y, x.

Proof. Denote by E the null set of points where v is not differentiable and select a null set $F \supset E$ verifying

$$L^\Omega(y,x) = \inf\{ I(\xi) : \ \xi \in \mathcal{A}^F_{y,x} \ \text{ s.t. } \xi(t) \in \Omega \text{ for any } t \} \tag{3.18}$$

for any y, x. The equality

$$v(x) - v(y) = \int_0^1 \frac{d}{dt} v(\xi(t)) dt$$

holds for any curve ξ contained in Ω. If in addition $\xi \pitchfork F$, then

$$\frac{d}{dt}v(\xi(t)) = Dv(\xi(t))\dot{\xi}(t) \le \sigma(\xi(t), \dot{\xi}(t))$$

for a.e. t. Therefore, $v(x) - v(y) \le I(\xi)$ and the thesis comes from (3.18). □

4. Subsolution properties of L

The main result we get here, is that $L(y_0, \cdot)$ is an a.e. subsolution of (1.1). Indeed we show something more. Namely, we prove that any element of the superdifferential of $L(y_0, \cdot)$ at a given point belongs to a certain convex compact set whose definition relies on the local behaviour of $\mathcal{Z}$, up to a set of density 0.

We start recalling the definition of density as well as other notions based on it, see [13], [22] for a general treatment of these topics.

For any subset E and $x \in \mathbb{R}^N$ the density $\mathcal{D}_x(E)$ of E at x is given by the formula

$$\mathcal{D}_x(E) = \lim_{r \to 0} \frac{|E \cap B(x,r)|}{|B(x,r)|}.$$

For any x_0, $\mathcal{Z}_0$ compact set, we write

$$\operatorname{ap}\lim_{x \to x_0} \mathcal{Z}(x) = \mathcal{Z}_0,$$

where ap lim stands for approximate limit, to signify

$$\mathcal{D}_{x_0}(\{x : \ d_{\mathcal{H}}(\mathcal{Z}(x), \mathcal{Z}_0) < \varepsilon\}) = 1 \qquad \text{for any } \varepsilon > 0.$$

$\mathcal{Z}$ is said approximately continuous at x_0 provided that $\mathcal{Z}(x_0)$ is compact and

$$\operatorname{ap}\lim_{x \to x_0} \mathcal{Z}(x) = \mathcal{Z}(x_0).$$

The usual local characterization of measurability holds for set-valued maps, see [13]. $\mathcal{Z}(x)$ is thus approximately continuous for a.e. x.

For a measurable function, the approximate limsup is defined by

$$\operatorname{ap}\limsup_{x \to x_0} f(x) = \inf\{a \in \mathbb{R} : \ \mathcal{D}_{x_0}(\{x : \ f(x) \le a\}) = 1\}. \tag{4.1}$$

We extend it to the set-valued map $\mathcal{Z}$ putting

$$\begin{aligned} \operatorname{ap}\limsup\nolimits_{x \to x_0} \mathcal{Z}(x) \quad &= \bigcap \big\{ K \text{ convex, compact} : \\ &\mathcal{D}_{x_0}(\{x : \ \mathcal{Z}(x) \subset K\}) = 1 \big\}. \end{aligned} \tag{4.2}$$

In the sequel we will denote by $\overline{f}(x_0)$, $\overline{\mathcal{Z}}(x_0)$ the quantities defined in (4.1), (4.2), respectively.

Proposition 4.1.

i) *If* $\mathcal{Z}$ *is approximately continuous at* x_0*, then* $\overline{\mathcal{Z}}(x_0) = \mathcal{Z}(x_0)$

ii) *If* $H(x,p) = |p| - f(x)$*, then* $\overline{\mathcal{Z}}(x) = B(0, \overline{f}(x))$ *for any* x*.*

Proof. Since $\mathcal{Z}$ is a.e. convex valued and

$$\text{ap}\lim_{x \to x_0} \mathcal{Z}(x) = \mathcal{Z}(x_0)$$

we have, also,

$$\text{ap}\lim_{x \longrightarrow x_0} \mathcal{Z}(x) = \text{co}(\mathcal{Z}(x_0)),$$

where co is the convex hull. Therefore, $\mathcal{Z}(x_0)$ is convex by the uniqueness of the approximate limit. One has $\mathcal{D}_{x_0}(\{x : \ \mathcal{Z}(x) \subset \mathcal{Z}(x_0) + B(0,\delta)\}) = 1$ for any $\delta > 0$, and $\mathcal{Z}(x_0) + B(0,\delta)$ is convex and compact. Consequently, $\overline{\mathcal{Z}}(x_0) \subset \mathcal{Z}(x_0)$. In the case of strict inclusion, there should exists a convex compact subset K with

$$\mathcal{D}_{x_0}(\{x : \ \mathcal{Z}(x) \subset K\}) = 1, \quad \mathcal{Z}(x_0) \setminus K \neq \emptyset. \tag{4.3}$$

This should imply the existence of a positive constant ε verifying

$$d_{\mathcal{H}}(K', \mathcal{Z}(x_0)) \geq \varepsilon \tag{4.4}$$

for any closed subset K' of K. (4.3), (4.4) contradict $\mathcal{Z}(x_0)$ being the approximate limit of $\mathcal{Z}$ for $x \longrightarrow x_0$.

Let us now prove (ii) of the statement. Since

$$\mathcal{Z}(x) = B(0, f(x)) \qquad \text{for a.e. } x \tag{4.5}$$

we have from (4.1) $\overline{\mathcal{Z}}(x_0) \subset B(0, \overline{f}(x_0))$. In the case of strict inclusion, one can argue as in the first part and exploit (4.5) to find a constant a with $a < \overline{f}(x_0)$ such that $\mathcal{D}_{x_0}\{x : \ f(x) \leq a\}) = 1$. This contradicts (4.1). □

The next proposition is crucial for proving Theorem 4.3 about C^1– supertangents of $L(y_0, \cdot)$.

Proposition 4.2. *For any* x_0*,* q*,* K *convex compact subset of* $\mathbb{R}^N$ *with*

$$\mathcal{D}_{x_0}(\{x : \ \mathcal{Z}(x) \subset K\}) = 1$$

it results

$$\limsup_{t \longrightarrow 0^+} \frac{L(x_0 + tq, x_0)}{t} \leq \sigma_K(-q),$$

where σ_K *is the support function of* K*.*

Proof. Assume $|q| = 1$ and fix $\varepsilon > 0$ small. For t small enough it results

$$|\{x: \mathcal{Z}(x) \nsubseteq K\} \cap B(x_0, 2t)| < \omega_{N-1} \varepsilon t^N \tag{4.6}$$

with ω_{N-1} indicating the $(N-1)$-Lebesgue measure of $(N-1)$-dimensional unit ball. Fix such a t and put

$$E = \{x: \mathcal{Z}(x) \nsubseteq K\} \cap B(x_0, 2t), \quad M = \operatorname{ess\,sup}_{B(x_0,1)} \mathcal{Z}(x).$$

Select a null set F for which (3.13) is verified and

$$\mathcal{Z}(x) \subset B(0, M) \qquad \text{for } x \in B(x_0, 1) \setminus F.$$

Set

$$B = B(0, \varepsilon^{1/2(N-1)} t) \cap q^\perp, \quad \xi_y(s) = x_0 + (1-s)tq + y$$

for $y \in B$, $s \in [0,1]$. By Fubini's theorem

$$|E| \geq \int_B \mathcal{L}^1(\xi_y([0,1]) \cap E) dy \tag{4.7}$$

and $\xi_y \pitchfork F$ for $\mathcal{L}^{N-1}$-a.e. $y \in B$. If $\mathcal{L}^1(\xi_y([0,1]) \cap E) \geq \varepsilon^{1/2} t$ for $\mathcal{L}^{N-1}$-a.e. $y \in B$, then by (4.7), it should result

$$|E| \geq \varepsilon^{1/2} t \omega_{N-1} \varepsilon^{1/2} t^{N-1} = \omega_{N-1} \varepsilon t^N,$$

which should contradict (4.6).

Hence, there exists $y_0 \in B$ with $\xi_{y_0}([0,1]) \cap E$, and so $\{s: \xi_{y_0}(s) \in E\}$, $\mathcal{L}^1$-measurable, verifying

$$\mathcal{L}^1(\xi_{y_0}([0,1]) \cap E) < \varepsilon^{1/2} t. \tag{4.8}$$

Consequently,

$$\mathcal{L}^1(\xi_{y_0}([0,1]) \setminus E) \geq (1 - \varepsilon^{1/2}) t. \tag{4.9}$$

By Proposition 3.2, there are $\xi_1 \in \mathcal{A}^F_{x_0+tq, x_0+tq+y_0}$, $\xi_2 \in \mathcal{A}^F_{x_0+y_0, x_0}$ verifying

$$\ell(\xi_i) \leq 2\varepsilon^{1/2(N-1)} t \tag{4.10}$$

for $i = 1, 2$. Define $\xi \in \mathcal{A}^F_{x_0+tq, x_0}$ through the juxtaposition of ξ_1, ξ_{y_0}, ξ_2. Recall that $\xi(s) \in B(x_0, 2t)$ for any s and (4.6), (4.8), (4.10), to compute

$$\begin{aligned} &L(x_0 + tq, x_0) \leq I(\xi) \\ &\leq 4\varepsilon^{1/2(N-1)} tM + \int_{\{\xi_{y_0} \in E\}} \sigma(\xi_{y_0}, \dot{\xi}_{y_0}) ds + \int_{\{\xi_{y_0} \notin E\}} \sigma_K(\dot{\xi}_{y_0}) ds \\ &\leq 4\varepsilon^{1/2(N-1)} tM + M\varepsilon^{1/2} t + \mathcal{L}^1(\xi_{y_0}([0,1]) \setminus E) \sigma_K(-q) t. \end{aligned}$$

The thesis is then obtained taking into account that ε is arbitrary and (4.9). □

In the remainder of this section we will set $u = L(y_0, \cdot)$.

Theorem 4.3. *For any x_0 and any C^1-supertangent ψ to u at x_0, it results*

$$D\psi(x_0) \in \overline{\mathcal{Z}}(x_0). \tag{4.11}$$

Proof. Let K be a convex compact subset of $\mathbb{R}^N$ verifying

$$\mathcal{D}_{x_0}(\{x : \ \mathcal{Z}(x) \subset K\}) = 1. \tag{4.12}$$

Take $q \in \mathbb{R}^N$ and compute, recalling Proposition 4.2 and (3.11)

$$D\psi(x_0)q = \lim_{h\to 0^+} \frac{\psi(x_0) - \psi(x_0 - hq)}{h} \le \liminf_{h\to 0^+} \frac{L(x_0 - hq, x_0)}{h} \le \sigma_K(q).$$

Since K is convex, the previous inequality implies that $D\psi(x_0) \in K$. The assertion is thus proved in view of the definition of $\overline{\mathcal{Z}}(x_0)$ given in (4.2). □

Remark 4.4. Consider an open set Ω. Then it is apparent that Proposition 4.2 and Theorem 4.3 hold with L^Ω and $u = L^\Omega(y_0, \cdot)$ in place of L and $u = L(y_0, \cdot)$, respectively.

We make use of Theorem 4.3 and Proposition 3.6 to show:

Corollary 4.5. *u is the maximal a.e. locally Lipschitz continuous subsolution of* (1.1) *vanishing at y_0.*

Proof. By Proposition 3.4, u is locally Lipschitz continuous and $u(y_0) = 0$. By Proposition 4.1 and Theorem 4.3, u is an a.e. subsolution of (1.1). The maximality property comes from Proposition 3.6. □

Corollary 4.6. *Let v a a.e. locally Lipschitz continuous subsolution of* (1.1)*. The relation* (4.11) *holds true for any x_0 and any ψ C^1-supertangent to v at x_0.*

Proof. Take a convex compact subset K verifying (4.12), $q \in \mathbb{R}^N$, and compute taking into account Proposition 3.6

$$\begin{aligned} D\psi(x_0)q &= \lim_{h\to 0^+} \frac{\psi(x_0) - \psi(x_0 - hq)}{h} \\ &\le \liminf_{h\to 0^+} \frac{v(x_0) - v(x_0 - hq)}{h} \le \liminf_{h\to 0^+} \frac{L(x_0 - hq, x_0)}{h} \le \sigma_K(q). \end{aligned}$$

The thesis comes as in Theorem 4.3. □

The argument of the previous result, Proposition 3.6 and Remark 4.4 show:

Proposition 4.7. *Given an open set Ω, a locally Lipschitz continuous function v is subsolution of (1.1) in Ω if and only if*

$$v(x) - v(y) \leq L^{\Omega}(y, x).$$

Corollary 4.8. *Assume H to be l.s.c. in x. Then*

$$L(y, x) = \inf\{I(\xi) : \ \xi \in \mathcal{A}_{y,x}\} \tag{4.13}$$

for any y, x.

Proof. Denote by $T(y, x)$ the right hand side of (4.13). For any y_0, $T(y_0, \cdot)$ is locally Lipschitz continuous because of the coercitivity assumption (2.3) and (3.11) holds with T in place of L. Since

$$T(y, x) \leq L(y, x)$$

for any y, x, then by Proposition 4.7 $T(y_0, \cdot)$ is an a.e. subsolution of (1.1).

Let v be a locally Lipschitz continuous a.e. subsolution of (1.1) vanishing at y_0, then

$$v(x) = \int_0^1 \frac{d}{dt} v(\xi(t)) dt$$

for any x, $\xi \in \mathcal{A}_{y_0,x}$. It results (see [10])

$$\frac{d}{dt} v(\xi(t)) = p(t)\dot{\xi}(t) \tag{4.14}$$

for a.e. t, and a suitable element $p(t)$ of the (Clarke) generalized gradient

$$\partial v(\xi(t)) = \mathrm{co}\big\{p \in \mathbb{R}^N : \text{ there exists a sequence } x_n \text{ where } v \text{ is differentiable s.t. } \xi(t) = \lim_n x_n \text{ and } p = \lim_n Dv(x_n)\big\}.$$

By the assumptions on H, $\mathcal{Z}$ is u.s.c. and since it is convex valued, one has

$$\partial v(x) \subset \mathcal{Z}(x) \quad \text{ for any } x. \tag{4.15}$$

(4.14), (4.15) imply

$$\frac{d}{dt} v(\xi(t)) \leq \sigma(\xi(t), \dot{\xi}(t)) \text{ for a.e. } t.$$

Consequently

$$v \leq T(y_0, \cdot).$$

This maximality property gives the thesis in view of the Corollary 4.5. □

5. Supersolution properties of L

In this section our aim is to investigate the properties of Lipschitz continuous subtangents to $u := L(y_0, \cdot)$, for any fixed y_0.

To do that, we consider the signed distance $(x, p) \mapsto d^*(p, \mathcal{Z}(x))$, which plays the role of a canonical Hamiltonian for the equation (1.1). It enjoys all the properties of H listed in Section 2, see [9] for results ensuring its measurable dependence from x.

We start by deriving an equivalent expression for $d^*(p, \mathcal{Z}(x))$ that we will use for a regularization procedure in Section 7.

Proposition 5.1. *For a.e. x and any p, it results*

$$d^*(p, \mathcal{Z}(x)) = \sup_{|q|=1} \{pq - \sigma(x, q)\}. \tag{5.1}$$

Proof. Fix p_0, and x_0 such that $\mathcal{Z}(x_0)$ is convex and compact. Denote by p' the projection of p_0 on $\partial \mathcal{Z}(x)$ and for any unit vector q set

$$\Sigma_q = \{p : pq = \sigma(x, q)\}.$$

Assume first that $p_0 \notin \mathcal{Z}(x)$ and take q such that the hyperplane Σ_q separates p_0 and $\mathcal{Z}(x)$. Then

$$p_0 q - \sigma(x, q) = d(p_0, \Sigma_q) \le d^*(p_0, \mathcal{Z}(x)). \tag{5.2}$$

On the other hand set $q' = \frac{p_0 - p'}{|p_0 - p'|}$ and compute

$$p_0 q' - \sigma(x, q') = p_0 q' - p' q' = d^*(p_0, \mathcal{Z}(x)). \tag{5.3}$$

(5.2) and (5.3) give (5.1).

Assume that $p_0 \in \mathcal{Z}(x)$, then since Σ_q is a supporting hyperplane to $\mathcal{Z}(x)$, it results

$$p_0 q - \sigma(x, q) = -d(p_0, \Sigma_q) \le d^*(p_0, \mathcal{Z}(x)) \tag{5.4}$$

take q' with $p' q' = \sigma(x, q')$ to get

$$p_0 q' - \sigma(x, q') = d^*(p_0, \mathcal{Z}(x)). \tag{5.5}$$

This ends the proof. □

For any measurable function g, the essential limsup at a point x_0 is given by the formula

$$\operatorname{ess}\limsup_{x \to x_0} g(x) = \inf_{\varepsilon > 0} \{\operatorname{ess\,sup}_{B(x_0, \varepsilon)} g\}.$$

We proceed to state the main result of the section.

Theorem 5.2. *For $x_0 \neq y_0$ and ψ Lipschitz continuous subtangent to u at x_0, it results*

$$\operatorname{ess}\limsup_{x \longrightarrow x_0} d^*(D\psi(x), \mathcal{Z}(x)) \geq 0.$$

Proof. Assume that $\psi(x_0) = u(x_0)$. If the assertion is not true, there should be positive constants $s < |y_0 - x_0|$ and ε such that

$$d^*(D\psi(x), \mathcal{Z}(x)) \leq -\varepsilon \tag{5.6}$$

for $x \in B(x_0, s) \setminus E$, with E suitable null set.

It Thus we can select a positive η for which the inequality (5.6) still holds in $B(x_0, s) \setminus E$, with $\psi - \eta| \cdot - x_0|^2$ and $\frac{\varepsilon}{2}$ in place of ψ and ε, respectively. Therefore, ψ can be assumed without loss of generality to be a strict subtangent. Consequently,

$$\psi(x) - u(x) < -\delta \tag{5.7}$$

for any $x \in \partial B(x_0, s)$ and a suitable $\delta > 0$.

Consider a null set F, containing E and the set of the points where ψ is not differentiable, such that

$$L(y, x) = \inf\{I(\xi) : \ \xi \in \mathcal{A}^F_{y,x}\}$$

for any y, x. Therefore,

$$I(\xi) \leq L(y_0, x_0) + \frac{\delta}{2}$$

for a suitable $\xi \in \mathcal{A}^F_{y_0,x_0}$. Set $t_0 = \max\{t : \xi(t) \in \partial B(x_0, s)\}$, $z_0 = \xi(t_0)$. It results

$$\begin{aligned} \int_{t_0}^1 \sigma(\xi, \dot{\xi})dt &\leq L(y_0, x_0) - \int_0^{t_0} \sigma(\xi, \dot{\xi})dt + \frac{\delta}{2} \\ &\leq L(y_0, x_0) - L(y_0, z_0) + \frac{\delta}{2}. \end{aligned} \tag{5.8}$$

Use (5.7), (5.8) to obtain

$$\psi(x_0) - \psi(z_0) \geq u(x_0) - u(z_0) + \delta \geq \int_{t_0}^1 \sigma(\xi, \dot{\xi})dt + \frac{\delta}{2}.$$

Take into account that ψ is differentiable at $\xi(t)$ for a.e. t to derive

$$\int_{t_0}^1 \left[D\psi(\xi)\dot{\xi} - \sigma(\xi, \dot{\xi}) \right] dt \geq \frac{\delta}{2}.$$

From this we have that

$$D\psi(\xi)\dot{\xi} > \sigma(\xi, \dot{\xi})$$

and so

$$d^*(D\psi(\xi(t)), \mathcal{Z}(\xi(t))) > 0 \tag{5.9}$$

in a subset of $[t_0, 1]$ of positive measure. Since $\xi(t) \in B(x_0, s)$ for any $t \in (t_0, 1]$ and $\xi \pitchfork E$, (5.9) contradicts (5.6). ☐

In contrast with the continuous case, u is not necessarily a.e. solution of (1.1). We slightly modify the interesting example given in [12], to show that it can be actually a strict subsolution even on a set of full measure.

The same phenomenon can happen for more general viscosity solutions of (1.1), in the sense of the definition of the next section.

Example 5.3. We construct an a.e. positive function f defined in $\mathbb{R}^2$ such that

$$L(y_0, x_0) = 0 \qquad \text{for any } y_0, x_0 \tag{5.10}$$

with L corresponding to the Hamiltonian $H(x, p) = |p| - f(x)$.

For this purpose, we consider the set of points of $\mathbb{R}^2$ with at least one rational coordinate, and denote it by F.

We select a sequence of open sets F_n verifying $F_{n+1} \subset F_n$, $F \subset F_n$, $|F_n| \le \frac{1}{n}$ for any n. Then we set

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^2 \setminus F_1 \\ 1/n & \text{if } x \in F_n \setminus F_{n+1} \\ 0 & \text{if } x \in \cap_n F_n. \end{cases}$$

Then f is measurable and $\{x : f(x) = 0\} = \cap_n F_n$ is a null set.

Fix a null set $E \subset \mathbb{R}^2$, $y_0 = (y_1, y_2)$, $x_0 = (x_1, x_2)$, $\varepsilon > 0$, n with $\frac{1}{n} < \varepsilon$. Take two rational numbers a, b verifying $|y_2 - b| < \varepsilon$, $|x_1 - a| < \varepsilon$. Set

$$y' = (y_1, b)\ ,\ x' = (a, x_2)$$
$$B = \{x' + te_2 : t \in [\min(0, b - x_2), \max(0, b - x_2)]\} \subset F$$
$$A = \{y' + te_1 : t \in [\min(0, a - y_1), \max(0, a - y_1)]\} \subset F.$$

Since $A \cup B$ is compact, one can find $\delta \in (0, \varepsilon)$ such that

$$B_\delta = \{x : d(x, B) < \delta\} \subset F_n, \quad A_\delta = \{x : d(x, A) < \delta\} \subset F_n.$$

By Fubini's theorem there is a curve ξ, defined in $[0, 1]$, made up by the union of a vertical and an horizontal line, such that

$$\xi \pitchfork E$$
$$\xi(t) \in A_\delta \cup B_\delta \quad \text{ for any } t$$
$$\ell(\xi) \le |x_1 - y_1| + |x_2 - y_2| + 2\varepsilon$$
$$|\xi(0) - y_0| < 2\varepsilon\ ,\ |\xi(1) - x_0| < 2\varepsilon.$$

Therefore, since E is arbitrary, it results

$$L(y_0, x_0) \le 4\varepsilon + \varepsilon(|x_1 - y_1| + |x_2 - y_2|) + 2\varepsilon^2$$

and so (5.10) is proved, since also ε is arbitrary.

6. Viscosity solutions

The properties shown for $L(y_0, \cdot)$ in Sections 4 and 5, will be taken here as basis for the the definition of viscosity solution of the measurable equation (1.1).

Definition 6.1. A u.s.c. function u is said to be a viscosity subsolution of (1.1) provided that for any x_0 and any C^1-supertangent ψ to u at x_0, it results

$$D\psi(x_0) \in \overline{\mathcal{Z}}(x_0).$$

Definition 6.2. A l.s.c. function v is said to be a viscosity supersolution of (1.1) if

$$\operatorname{ess}\limsup_{x \to x_0} d^*(D\varphi(x), \mathcal{Z}(x)) \ge 0$$

for any x_0, any φ Lipschitz continuous subtangent to v at x_0.

A continuous function is said to be a viscosity solution of (1.1) if it is at the same time sub and supersolution.

From now on we will omit the term viscosity before sub- (super) solution.

Remark 6.3. The $\operatorname{ess}\limsup$ has been already used in [6] in the definition of viscosity solution for a class of nonlinear second order equations.

The definition of solution takes a simple form for the eikonal equation (1.2). In this case in fact, thanks to Proposition 4.1.(ii), a solution u verifies

$$|D\psi(x_0)| \le \operatorname{ap}\limsup_{x \longrightarrow x_0} f(x), \quad \operatorname{ess}\limsup_{x \longrightarrow x_0} |D\varphi(x)| - f(x) \ge 0$$

for any x_0, any ψ,φ C^1-supertangent and Lipschitz continuous subtangent to u at x_0, respectively.

The definition of subsolution allows us to recover in the measurable setting a property holding when H is continuous and quasiconvex in p.

Proposition 6.4. *A function u is a subsolution of* (1.1) *if and only if it is a a.e. locally Lipschitz continuous subsolution of* (1.1)*.*

Proof. Assume u to be a subsolution. Since the map $\mathcal{Z}$ is locally essentially bounded, then, given an open bounded set Ω, there is an $M > 0$ verifying

$$\overline{\mathcal{Z}}(x) \subset B(0, M) \quad \text{ for any } x \in \Omega.$$

This implies

$$|Du| \leq M \qquad \text{in } \Omega$$

in the viscosity sense. Consequently, u is locally Lipschitz continuous, see [3]. From the equality

$$\mathcal{Z}(x) = \overline{\mathcal{Z}}(x) \qquad \text{for a.e. } x$$

we get that u is an a.e. subsolution.

The converse assertion is the statement of Corollary 4.6. □

The Definition 6.2 reduces to the Crandall-Lions one in the case where the Hamiltonian is continuous.

Proposition 6.5. *Assume H to be continuous in both arguments. A l.s.c. function v is a supersolution of* (1.1) *in the sense of Crandall–Lions if and only if it verifies Definition* 6.2.

Proof. Let v be a Crandall–Lions supersolution and take a Lipschitz continuous subtangent ψ to v at a point x_0. Then according to [7], Proposition 4.3

$$d^*(p_0, \mathcal{Z}(x_0)) \geq 0 \tag{6.1}$$

for a suitable $p_0 \in \partial\psi(x_0)$.

Assume for purposes of contradiction that

$$d^*(D\psi(x), \mathcal{Z}(x)) \leq -\varepsilon \tag{6.2}$$

for $x \in B(x_0, r) \setminus E$, with E null set and r, ε positive constants.

According to [10], Theorem 2.5.1 p_0 can be expressed as a finite convex combination of vectors p_i verifying for any i

$$p_i = \lim_n D\psi(x_n^i) \quad \text{ with } x_n^i \notin E \quad \text{ for any } n.$$

Consequently by the continuity of $(x, p) \mapsto d^*(p, \mathcal{Z}(x))$ and (6.2)

$$d^*(p_i, \mathcal{Z}(x_0)) \leq -\varepsilon \quad \text{for any } i.$$

This implies thanks to the convexity of $\mathcal{Z}(x_0)$

$$d^*(p_0, \mathcal{Z}(x_0)) \leq -\varepsilon$$

which contradicts (6.1).

The other part of the assertion is immediate. □

To obtain comparison and uniqueness principles for (1.1), we need, as in the continuous case, to strengthen the condition (2.4). We require:

Existence of a strict a.e. subsolution

$$\begin{gathered}\text{there exists a locally Lipschitz continuous function } \phi \\ \text{s.t. for any compact set } K\text{, it can be determined } \varepsilon > 0 \text{ with} \\ d^*(D\phi(x), \mathcal{Z}(x)) \le -\varepsilon \qquad \text{for a.e. } x \in K.\end{gathered} \tag{6.3}$$

Such a condition will be assumed in the remainder of the section.

For the eikonal equation (1.2), this is equivalent to require that for any compact set K,

$$m \le f \le M \quad \text{a.e. in } K$$

for suitable positive constants m, M.

A consequence of (6.3) is that the definition of supersolution can be expressed using the gauge function

$$\gamma(x,p) = \inf\left\{\lambda > 0 : \frac{p}{\lambda} + \left(1 - \frac{1}{\lambda}\right) D\phi(x)\right\} \tag{6.4}$$

instead of $d^*(\cdot, \mathcal{Z}(\cdot))$, as it will be clear in the forthcoming Proposition 6.6.

γ is defined for a.e. x and any p and it is measurable with respect to x for any p and continuous in p for a.e. x, see [9]. Moreover, it enjoys the following homogeneity property

$$\gamma(\lambda p + (1-\lambda) D\phi(x)) = \lambda \gamma(x,p) \tag{6.5}$$

for a.e x, for any p and $\lambda > 0$.

Such function has been already used in [7], [8] to define some special supersolutions of a class of degenerate Hamilton–Jacobi equations.

Proposition 6.6. *A l.s.c. function v is supersolution of* (1.1) *if and only if*

$$\operatorname{ess}\limsup_{x \to x_0} \gamma(x, D\varphi(x)) \ge 1$$

for any x_0, any φ Lipschitz continuous subtangent to v at x_0.

This property will be used in a forthcoming comparison principle, as well as in Section 7. The proof comes directly from the following lemma:

Lemma 6.7. *Let Ω be a bounded open subset, φ a Lipschitz continuous function defined on it. Then*

$$\gamma(x, D\varphi(x)) \le 1 - \delta \tag{6.6}$$

for a.e. $x \in \Omega$ and a certain $\delta \in (0,1)$ if and only if

$$d^*(D\varphi(x), \mathcal{Z}(x)) \le -\delta' \tag{6.7}$$

for a.e. $x \in \Omega$ *and a suitable* $\delta' > 0$.

Proof. Take ε such that

$$d^*(D\phi(x), \mathcal{Z}(x)) \le -\varepsilon \tag{6.8}$$

for $x \in \Omega \setminus F$, with $|F| = 0$. Define E as the union of F and the set of points x where $\mathcal{Z}(x)$ is not a convex compact set.

Assume that (6.6) holds for a certain $x \in \Omega \setminus E$ and set $p_0 = D\varphi(x)$. Then

$$p' := \frac{1}{1-\delta}(p_0 - \delta D\phi(x)) \in \mathcal{Z}(x).$$

Record the relation

$$p_0 = (1-\delta)p' + \delta D\phi(x) \tag{6.9}$$

and consider p with $|p_0 - p| \le \varepsilon\delta$. Derive from (6.9)

$$|\frac{1-\delta}{\delta}p' + D\phi(x) - \frac{1}{\delta}p| \le \varepsilon$$

and so by (6.8)

$$p'' := \frac{1}{\delta}p - \frac{1-\delta}{\delta}p' \in \mathcal{Z}(x).$$

Therefore,

$$p = \delta p'' + (1-\delta)p' \in \mathcal{Z}(x).$$

Then (6.7) is obtained with $\delta' = \varepsilon\delta$.

To show the converse assertion, first set $M = \operatorname{ess\,sup}_\Omega \mathcal{Z}$, then consider $x \in \Omega$, where (6.7) holds, ϕ is differentiable and $\mathcal{Z}(x)$ is convex and compact, with $\mathcal{Z}(x) \subset B(0, M)$. Set $p_0 = D\varphi(x)$, $\lambda = \gamma(x, p_0)$. It results

$$\delta' \le |p_0 - \frac{1}{\lambda}p_0 - (1 - \frac{1}{\lambda})D\phi(x)| \le 2M(\frac{1}{\lambda} - 1).$$

Consequently,

$$\frac{1}{\lambda} \ge \frac{\delta' + 2M}{2M} \quad \text{and} \quad \lambda \le 1 - \frac{\delta'}{2M + \delta'}.$$

Therefore, (6.6) is proved with $\delta = \delta'/(2M + \delta')$. □

We detail some results when the equation is posed in an open bounded set Ω. The argument of the next Theorem goes back to [14].

Theorem 6.8. *Let* $v \in \mathrm{LSC}(\overline{\Omega})$, $u \in \mathrm{USC}(\overline{\Omega})$ *be a supersolution and an a.e. subsolution of* (1.1) *in* Ω, *respectively. Then*

$$\max_{\overline{\Omega}}\{u - v\} = \max_{\partial\Omega}\{u - v\}.$$

Proof. If the statement is not true, then the maximum of $u - v$ in $\overline{\Omega}$ should be in Ω. Recall that u is locally Lipschitz continuous in Ω and define

$$u_\delta = (1 - \delta)u + \delta\phi$$

for $\delta \in (0, 1)$.

Since u_δ converges uniformly to u for δ going to 0, then for δ small enough u_δ is subtangent to v at a point $x_\delta \in \Omega$. Then by (6.5)

$$\gamma(x, Du_\delta(x)) = (1 - \delta)\gamma(x, Du(x)) \leq 1 - \delta$$

for a.e. $x \in \Omega$. This is in contrast to v being supersolution in Ω and Proposition 6.4. □

We now consider a continuous datum g on $\partial\Omega$ and define

$$w(x) = \min \{g(y) + L(x, y) : y \in \partial\Omega\}. \tag{6.10}$$

We will require g to satisfy the following compatibility condition, see [15]

$$g(x) - g(y) \leq L(y, x) \tag{6.11}$$

for any y, $x \in \partial\Omega$.

Theorem 6.9. *If* (6.11) *holds, then* w *is the unique solution of* (1.1) *in* Ω *equaling* g *on* $\partial\Omega$.

Proof. Let ψ a C^1–supertangent to w at a point $x_0 \in \Omega$, then for any q

$$\begin{aligned} D\psi(x_0)q_0 \quad & \lim_{h \longrightarrow 0^+} \frac{\psi(x_0) - \psi(x_0 - hq)}{h} \\ & \leq \liminf_{h \longrightarrow 0^+} \frac{L(y_h, x_0) - L(y_h, x_0 - hq)}{h}, \end{aligned} \tag{6.12}$$

where y_h realizes the minimum in (6.10) for $x = x_0 - hq$. From (6.12) it can be derived arguing as in Theorem 4.3 that $D\psi(x_0) \in \overline{\mathcal{Z}}(x_0)$.

Finally, w is a supersolution in Ω because it is minimum of supersolutions. Exploiting condition (6.11) it can be easily seen that it equals g on $\partial\Omega$. The asserted uniqueness comes from Theorem 6.8. □

7. Approximation by regularization

Here we apply our definition of solution for (1.1) to show that a certain regularization procedure for the Hamiltonian yields a stability property for the solutions of the corresponding equations.

Since the approximated Hamiltonians are continuous, the solutions of the measurable equation are locally uniform limit of viscosity solutions in the Crandall–Lions sense.

We start by giving a stability result for subsolutions of (1.1), we consider a sequence of approximate equations

$$H_\varepsilon(x, Du) = 0 \tag{7.1}$$

and denote by $\mathcal{Z}_\varepsilon$, σ_ε, I_ε, L_ε the quantities corresponding to $\mathcal{Z}$, σ, I, L, respectively. We require (2.2), (2.3), (2.4) to hold for (7.1) with obvious adaptations.

We assume that for any compact subset K of $\mathbb{R}^N$ and for any ε, there exists $R > 0$ such that

$$\operatorname{ess\,inf}\{H_\varepsilon(x,p) : \ |p| > R, \ x \in K\} > 0 \tag{7.2}$$

and

$$H(x,p) \le \inf\{\liminf H_\varepsilon(x,p_\varepsilon) : \ p_\varepsilon \longrightarrow p\} \tag{7.3}$$

for a.e. x and for any p. Using (7.2) and (7.3) we can prove (see [11] for related results):

Proposition 7.1. *Let u_ε be a subsolution of (7.1) in an open set Ω and assume the sequence u_ε to be locally equibounded. Then u_ε converges locally uniformly, up to a subsequence, to a subsolution u of (1.1) in Ω.*

Proof. Denote by E a null set such that $\mathcal{Z}_\varepsilon(x) \subset B(0,R)$ for any ε, a suitable $R > 0$ and $x \in \Omega \setminus E$.

Take $x \in \Omega \setminus E$ such that (7.3) holds, and consider a sequence p_ε with $p_\varepsilon \in \mathcal{Z}_\varepsilon(x)$ for any ε. Then p_ε converges to an element p, up to a subsequence, and by (7.3), $p \in \mathcal{Z}(x)$. This implies the relation

$$\lim_\varepsilon \max_{p \in \mathcal{Z}_\varepsilon(x)} d(p, \mathcal{Z}(x)) = 0$$

for a.e. x, and consequently

$$\limsup_{\varepsilon \longrightarrow 0} \sigma_\varepsilon(x,q) \le \sigma(x,q) \tag{7.4}$$

for a.e. x and any q.

Denote by $F \supset E$ a null set such that (7.4) holds in its complement and Proposition 3.5 is verified for L^Ω and any L^Ω_ε. Consider a curve ξ contained in Ω, defined in $[0,1]$ and transversal to F. Since

$$\sigma_\varepsilon(\xi(t), \dot{\xi}(t)) \le R|\dot{\xi}|$$

for any ε, for a.e. $t \in [0,1]$, Fatou's Lemma and (7.4) can be used to get

$$\int_0^1 \sigma(\xi(t), \dot{\xi}(t))dt \ge \limsup_{\varepsilon \longrightarrow 0} \int_0^1 \sigma_\varepsilon(\xi(t), \dot{\xi}(t))dt$$

and so

$$L^{\Omega}(y,x) \geq \limsup_{\varepsilon \longrightarrow 0} L^{\Omega}_{\varepsilon}(y,x) \tag{7.5}$$

for any y, x. The sequence u_ε is locally equi-Lipschitz-continuous by (7.2) and being locally equibounded, it converges up to a subsequence to a locally Lipschitz-continuous function u.

Since u_ε is a subsolution of (7.1) in Ω, it results from Proposition 4.7 that

$$u_\varepsilon(y) - u_\varepsilon(x) \leq L^{\Omega}_{\varepsilon}(y,x)$$

for any y, x. Let ε go to 0 and use (7.5) to discover

$$u(y) - u(x) \leq L^{\Omega}(y,x)$$

which gives the thesis in view of Proposition 4.7. □

It is worth noting that to obtain general stability results for supersolutions as the previous one for subsolutions, it is necessary to require some L^∞-convergence of H_ε to H with respect to x. Such assumptions imply that H is continuous in x once the approximated Hamiltonians are so. This is quite unsatisfactory in our setting.

To illustrate this issue let us consider the eikonal equation (1.2) and the approximate equations

$$|Du(x)| = f_\varepsilon(x). \tag{7.6}$$

Given an open set Ω, it is not difficult to show using the representation formulae for solution or viscosity solution techniques that if f_ε converge to f locally in $L^\infty(\Omega)$, then any locally equibounded sequence of solutions of (7.6) in Ω converges locally uniformly, up to a subsequence, to a solution of (1.2) in Ω.

However, this is not true anymore if we weaken a bit the convergence of f_ε to f, as the following example shows.

Example 7.2. Consider a sequence f_ε of continuous functions verifying

$$f_\varepsilon(x) = \begin{cases} 1 & \text{if } |x_N| \geq \varepsilon \\ 1/2 & \text{if } x_N = 0 \end{cases} \tag{7.7}$$

and

$$1/2 \leq f_\varepsilon \leq 1 \quad \text{ in } \mathbb{R}^N.$$

Since $f_\varepsilon \longrightarrow 1$ a.e., then $f_\varepsilon \longrightarrow 1$ in $L^k_{loc}(\mathbb{R}^N)$ for any k. One has

$$L(y,x) = |y-x| \qquad \text{for any } y,\, x \tag{7.8}$$

while

$$L_\varepsilon(y,x) = \frac{1}{2}|y-x| \tag{7.9}$$

for y, x with $y_N = x_N = 0$.

From (7.8), (7.9) it is easy to realize, taking into account the representation formulae of Section 6, that a sequence of solutions of the perturbed equations does not converge in general to a solution of (1.2).

Remark 7.3. Stability of supersolutions can be recovered assuming the following monotonicity property $\theta_\varepsilon \mathcal{Z}(x) \subset \mathcal{Z}_\varepsilon(x)$ for a.e. x, any ε, where θ_ε is a sequence converging to 1, see [12]. In [7] the previous condition is used for stability properties in a class of degenerate Hamilton–Jacobi equations.

We proceed to outline the announced regularization procedure. We set

$$H_\varepsilon(x,p) = \sup_{|q|=1} \{pq - \sigma_\varepsilon(x,q)\} \tag{7.10}$$

for any $\varepsilon > 0$, x, p, where σ_ε is the partial mollification of σ with respect to x through a standard mollifier ρ_ε, namely

$$\sigma_\varepsilon(x,q) = \int_{\mathbb{R}^N} \sigma(y,q)\rho_\varepsilon(x-y)dy.$$

σ_ε is locally Lipschitz-continuous in (x,p) and so the regularized Hamiltonian H_ε is continuous in both arguments. Since σ_ε is positively homogeneous and convex in p for any x, it is the support function of the set-valued map $\mathcal{Z}_\varepsilon$, therefore by Proposition 5.1

$$H_\varepsilon(x,p) = d^*(p, \mathcal{Z}_\varepsilon(x)) \quad \text{ for a.e. } x, \text{ any } p$$

and L_ε is defined by formula (4.13), where $I(\xi)$ is given by (3.8) with σ_ε in place of σ.

We notice that if $H(x,p) = |p| - f(x)$, then $H_\varepsilon(x,p) = |p| - f_\varepsilon(x)$ with $f_\varepsilon = f * \rho_\varepsilon$.

Theorem 7.4. *Assume u_ε to be a solution of (7.1) in an open set Ω with $\{u_\varepsilon\}$ locally equibounded. Then u_ε converges locally uniformly, up to a subsequence, to a solution u of (1.1) in Ω.*

Proof. We set

$$H(x,p) = \sup_{|q|=1} \{pq - \sigma(x,q)\} = d^*(x, \mathcal{Z}(x)) \tag{7.11}$$

for a.e. x, any p.

The condition (7.2) holds true since H verifies (2.3) and by the very definition of σ_ε. Recall that

$$\sigma_\varepsilon(x,q) \longrightarrow \sigma(x,q) \tag{7.12}$$

for a.e. x and any q, and take x_0 for which (7.12) holds true.

Fix p_0 and $\delta > 0$. Then consider a sequence p_ε converging to p_0 and a unit vector q_0 such that

$$H(x_0,p_0) \leq p_0 q_0 - \sigma(x_0,q_0) + \delta. \tag{7.13}$$

It results

$$\begin{aligned}\liminf_{\varepsilon \longrightarrow 0} H_\varepsilon(x_0,p_\varepsilon) &\geq \lim_{\varepsilon \longrightarrow 0}(p_\varepsilon q_0 - \sigma_\varepsilon(x_0,q_0)) \\ = p_0 q_0 - \sigma(x_0,q_0) &\geq H(x_0,p_0) - \delta.\end{aligned}$$

This shows that (7.3) holds since δ is arbitrary. Therefore, u_ε converges locally uniformly, up to a subsequence, to a subsolution u of (1.1) in Ω, in light of Proposition 7.1.

To show that u is also a supersolution, consider a Lipschitz continuous function ψ subtangent to u at $x_0 \in \Omega$. Assume for purpose of contradiction that

$$H(x, D\psi) < -\delta \tag{7.14}$$

for a.e. $x \in B(x_0, r) \subset \Omega$, with δ, r positive constants.

By (7.14), it can be assumed that ψ is a strict subtangent. Taking this into account and that $\psi_\varepsilon = \psi * \rho_\varepsilon$ and u_ε converge uniformly to ψ and u, respectively, to discover the existence of a sequence x_ε satisfying

$$x_\varepsilon \longrightarrow x_0 \tag{7.15}$$

$$\psi_\varepsilon \text{ is subtangent to } u_\varepsilon \text{ at } x_\varepsilon. \tag{7.16}$$

It results by (7.11), (7.14)

$$\begin{aligned}H_\varepsilon(x, D\psi_\varepsilon(x)) = \sup_{|q|=1}\{&D\psi_\varepsilon(x)q - \sigma_\varepsilon(x,q)\} = \\ \sup_{|q|=1}\{&(D\psi(\cdot)q - \sigma(\cdot,q)) * \rho_\varepsilon\} \leq -\delta\end{aligned}$$

for $x \in B(x_0, r/2)$, $\varepsilon < r/2$. By (7.15), (7.16) this is a contradiction with u_ε being a supersolution of (7.1). □

We derive a result on the convergence of L_ε to L. We emphasize that the argument of the next theorem, as well as of that of the previous one, is based on viscosity solutions stability techniques, and does not make a direct use of the formulae of L_ε and L.

As far as we know, it does not seem easy to get a proof working on the integral representations.

We need further to strengthen the condition (6.3) requiring

Existence of a global strict a.e. subsolution

$$\begin{array}{l}\text{there exists a locally Lipschitz continuous function } \phi \text{ with} \\ d^*(D\phi(x), \mathcal{Z}(x)) \le -\delta \quad \text{for a.e. } x \text{ and a suitable } \delta > 0.\end{array} \tag{7.17}$$

Assumption (7.17) will be assumed for the next two results.

Theorem 7.5. *For any fixed y_0, $L_\varepsilon(y_0, \cdot)$ converges to $L(y_0, \cdot)$ locally uniformly.*

The proof of the theorem relies upon the following uniqueness result:

Proposition 7.6. *For any y_0, $u := L(y_0, \cdot)$ is the unique solution of* (1.1) *in $\mathbb{R}^N \setminus \{y_0\}$ verifying*

$$u(y_0) = 0 \tag{7.18}$$

$$\lim_{|x| \longrightarrow +\infty} u(x) - \phi(x) = +\infty. \tag{7.19}$$

Proof. Let ϕ be a subsolution verifying (7.17), it can be supposed $\phi(y_0) = 0$. Denote by E the set where ϕ is not differentiable and take $\xi \in \mathcal{A}^E_{y_0,x}$ with $I(\xi) < +\infty$. It results by (7.17)

$$\int_0^1 \sigma(\xi, \dot{\xi}) dt \ge \int_0^1 \left(D\phi(\xi)\dot{\xi} + \delta|\dot{\xi}| \right) dt \ge \phi(x) + \delta|y_0 - x|$$

for any x. Consequently,

$$u(x) - \phi(x) \ge \delta|y_0 - x|$$

which proves (7.19). (7.18) is immediate.

Assume for purposes of contradiction the existence of w violating the asserted uniqueness.

Use the Kruzkov transform to define

$$\tilde{u} = 1 - \exp(\phi - u) \ , \ \tilde{w} = 1 - \exp(\phi - w).$$

Both these functions vanish at y_0 and have limit 1 at infinity. They are locally Lipschitz continuous a.e. subsolutions of

$$v + \tilde{\gamma}(\cdot, Dv) = 1 \tag{7.20}$$

with

$$\tilde{\gamma}(x,p) = \gamma(x, p + \exp(\phi(x) - u(x))D\phi(x))$$

for a.e. x and any p, see (6.4) for the definition of γ. Moreover, they verify

$$\begin{array}{l} \text{ess} \limsup_{x\to x_0} \tilde\gamma(x, D\psi(x)) \geq 1 - \tilde u(x_0) \\ \text{ess} \limsup_{x\to x_0} \tilde\gamma(x, D\psi(x)) \geq 1 - \tilde w(x_0) \end{array} \tag{7.21}$$

for any $x_0 \neq y_0$ and for any Lipschitz continuous function ψ subtangent to u, respectively to w, at x_0.

In view of the maximality property of u proved in Corollary 4.5, it comes that $m := \inf_{\mathbb{R}^N}\{\tilde w - \tilde u\}$ is negative and it is attained at a certain point $x_0 \neq y_0$. Then $\tilde u$ is subtangent at x_0 and

$$\tilde\gamma(x, D\tilde u(x)) \leq 1 - \tilde u(x) \leq 1 - \tilde w(x_0) + \frac{m}{2}$$

for a.e. x in a suitable neighborhood of x_0. This contradicts (7.21). □

Proof of the Theorem 7.5. Set

$$\phi_\varepsilon = \phi * \rho_\varepsilon \ , \ u_\varepsilon = L_\varepsilon(y_0, \cdot)$$

Arguing as in Theorem 7.4, it can be deduced

$$H_\varepsilon(x, D\phi_\varepsilon(x)) \leq -\delta \quad \text{ for a.e } x,$$

for any $\varepsilon > 0$ and consequently, as in Proposition 7.6

$$u_\varepsilon(x) - \phi_\varepsilon(x) \geq \delta|y_0 - x|. \tag{7.22}$$

By Theorem 7.4 a subsequence, still denoted by u_ε, converges locally uniformly to a solution v of (1.1) in $\mathbb{R}^N \setminus \{y_0\}$ with $v(y_0) = 0$.

Derive from (7.22) and the local uniform convergence of ϕ_ε to ϕ

$$\lim_{|x|\longrightarrow+\infty} v(x) - \phi(x) = +\infty.$$

The thesis thus comes from Proposition 7.6. □

Other convergence results can be obtained using the previous theorem and the representation formulae of Section 6. We will not detail them here.

8. Variational approximation

In this section we consider an open bounded set $\Omega \subset \mathbb{R}^N$ and the sequence of variational integrals

$$J_n(u) = \int_\Omega (\frac{1}{n} G(x, Du)^n - g)dx \tag{8.1}$$

defined in the Sobolev space $W_0^{1,n}(\Omega)$.

We exploit the results of Section 6 to study the asymptotic behaviour of the sequence u_n of the minimizers of J_n and to identify its limit. It generalizes a result given in [4] with $G(x,p) = |p|$.

We require g to be a positive function belonging to $L^\infty(\Omega)$. We assume the variational kernel G measurable in x for any p and the map $p \mapsto G(x,p)$ positively homogeneous, strictly convex and differentiable, for a.e. x. Moreover, the inequalities

$$r|p| \leq G(x,p) \leq R|p| \tag{8.2}$$

must be satisfied for a.e. $x \in \Omega$, $p \in \mathbb{R}^N$ and certain positive constants R, r.

We relate (8.1) to the equation (1.1) with

$$H(x,p) = G(x,p) - 1. \tag{8.3}$$

The Hamiltonian given by the previous formula satisfies the assumptions of Section 2 and (6.3) with $\phi = 0$.

Our assumptions guarantee that the functionals in (8.1) are well defined and finite. Moreover, see [16], there exists for any n a unique minimizer u_n of J_n in $W_0^{1,n}(\Omega)$ and it is characterized by the property of being weak solution of the associated Euler equation

$$-\mathrm{div}(G(x,Du)^{n-1} D_p G(x,Du)) = g \qquad \text{in } \Omega \tag{8.4}$$

satisfying homogeneous Dirichlet boundary conditions.

Theorem 8.1. *The sequence u_n converges uniformly in $\overline{\Omega}$ to the solution of* (1.1) *vanishing on $\partial\Omega$.*

For the proof of the Theorem we need some preliminary Lemmas. We will denote by $|\cdot|_n$ the norm in $L^n(\Omega)$ or in $L^n(\Omega,\mathbb{R}^N)$.

Lemma 8.2. *The sequence $|Du_n|_n$ is bounded.*

Proof. Exploit that u_n is a weak solution of (8.4) and take u_n itself as a test function to obtain

$$\int_\Omega \left(G(x,Du_n)^{n-1} D_p G(x,Du_n) Du_n\right) dx = \int_\Omega g u_n dx$$

and by the the Euler formula for homogeneous functions

$$\int_\Omega G(x,Du_n)^n dx = \int_\Omega g u_n dx.$$

From this, using (8.2) and the Poincaré inequality, derive

$$\int_\Omega r^n |Du_n|^n dx \leq C_n |g|_{n^*} |Du_n|_n, \tag{8.5}$$

where C_n is a suitable constant and n^* verifies $1/n + 1/n^* = 1$. Therefore,

$$\Big(\int_\Omega |Du_n|^n dx\Big)^{\frac{n-1}{n}} \le \frac{C_n}{r^n}|g|_{n^*}$$

and

$$|Du_n|_n \le \Big(\frac{C_n}{r^n}|g|_{n^*}\Big)^{\frac{1}{n-1}}. \tag{8.6}$$

This gives the thesis taking into account that the constant C_n in (8.5) can be taken less or equal the diameter of Ω. □

Lemma 8.3. *The functions u_n are equibounded and converge uniformly in $\overline{\Omega}$, up to a subsequence, to $u \in W_0^{1,\infty}(\Omega)$.*

Proof. Fix $m > N$, by Hoelder inequality

$$|Du_n|_m \le |Du_n|_n \, |\Omega|^{\frac{n}{m}\frac{1}{n-m}} \tag{8.7}$$

for $n > m$. By Morrey's inequality, the functions u_n are Hoelder-continuous in Ω with exponent $1 - N/m$ and Hoelder constant of the form $C|Du_n|_m$ where C depends only on Ω, m and N.

Consequently, by (8.7) and Lemma 8.2 the sequence u_n is equicontinuous and, since u_n vanishes on $\partial\Omega$, equibounded. Hence, it converges, up to a subsequence, uniformly in $\overline{\Omega}$ to a function u. Observe that, up to a subsequence,

$$u_n \rightharpoonup u \qquad \text{in } W_0^{1,m}(\Omega)$$

and so by the weak lower semicontinuity of norms

$$|Du|_m \le \liminf_{n\to\infty} |Du_n|_m.$$

From this derive employing (8.6) and (8.7)

$$|Du|_m \le \liminf_{n\to\infty} \Big(\frac{C_n}{r^n}|g|_{n^*}\Big)^{\frac{1}{n-1}} |\Omega|^{\frac{n}{m}\frac{1}{n-m}}$$

and

$$|Du|_m \le \frac{1}{r}|\Omega|^{\frac{1}{m}}.$$

The previous argument can be repeated for any $m > N$ with the same limit function u. Therefore, the sequence $|Du|_m$ is bounded and so $Du \in L^\infty(\Omega)$. This concludes the proof. □

Lemma 8.4. *The function u verifies*

$$H(x, Du) \le 0 \qquad \textit{for a.e. } x \in \Omega.$$

Proof. Assume for purposes of contradiction that

$$|\{x \in \Omega : \ H(x, Du) > 0\}| > 0,$$

then there is $\delta > 0$ for which

$$|\{x \in \Omega : \ H(x, Du) > \delta\}| := \varepsilon > 0. \tag{8.8}$$

Set

$$E_n = |\{x \in \Omega : \ H(x, Du_n) > \delta\}|. \tag{8.9}$$

We claim that

$$\lim_{n \longrightarrow \infty} n^2 |E_n| = 0. \tag{8.10}$$

If (8.10) is not true, there should be a positive constant a and a subsequence of (8.9), still denoted by E_n, verifying

$$|E_n| > \frac{a}{n^2}.$$

Therefore,

$$\int_\Omega \frac{1}{n} G(x, Du)^n dx \geq \frac{a}{n^2} \frac{(1+\delta)^n}{n} \longrightarrow_{n \to \infty} +\infty \tag{8.11}$$

while

$$\int_\Omega g u_n dx \leq M \tag{8.12}$$

for any n and a suitable $M > 0$, being the sequence u_n equibounded.

From (8.11), (8.12) it comes

$$\lim_{n \to \infty} J_n(u_n) = +\infty.$$

This cannot be for u_n are minimizers of J_n in $W_0^{1,n}(\Omega)$ and so verify

$$J_n(u_n) \leq J_n(0) = 0$$

for any n. By (8.10), n_0 can be selected such that

$$\Big| \bigcup_{n > n_0} E_n \Big| \leq \sum_{n > n_0} \frac{1}{n^2} < \varepsilon. \tag{8.13}$$

For any fixed $m > N$

$$Du_n \rightharpoonup Du \qquad \text{in } L^m(\Omega, \mathbb{R}^N)$$

up to a subsequence. Hence, a sequence of convex combinations of Du_n

$$v_k = \sum_i \lambda_i^k Du_{n_i^k}$$

converges strongly to Du in $L^m(\Omega,\mathbb{R}^N)$ and so, up to a subsequence, a.e. pointwise. It can be assumed

$$n_i^k > n_0 \qquad \text{for any } i,\, k.$$

Consequently, by the a.e. convexity of G it results

$$H(x,Du(x)) = \lim_k H(x,v_k(x)) \le \liminf_k \sum_i \lambda_i^k H(x,Du_{n_i^k}(x)) \le \delta$$

for a.e. $x \in \Omega \setminus \bigcup_{n>n_0} E_n$. Hence,

$$|\{x\in\Omega:\ H(x,Du)>\delta\}| \le |\bigcup_{n>n_0} E_n|$$

which is impossible in view of (8.8) and (8.13). □

Proof of the Theorem 8.1. Denote by w the viscosity solution of (1.1) vanishing on $\partial\Omega$. By Proposition 6.9, w is maximal in the class of function v Lipschitz-continuous in Ω verifying

$$\begin{cases} H(x,Dv)\le 0 & \text{a.e. in } \Omega \\ v=0 & \text{on } \partial\Omega. \end{cases}$$

Argue by contradiction assuming that $u \ne w$. Being g positive, it follows that

$$-\int_\Omega gwdx < -\int_\Omega gudx.$$

Then there is $\delta>0$ satisfying

$$\delta - \int_\Omega gwdx < -\int_\Omega gu_n dx \tag{8.14}$$

$$\int_\Omega \frac{1}{n} G(x,Dw)^n dx \le \frac{1}{n}|\Omega| < \delta \tag{8.15}$$

if n is large enough. For such an n it results by (8.14) and (8.15)

$$J_n(w) < \delta - \int_\Omega gwdx < -\int_\Omega gu_n dx < J_n(u_n)$$

which is impossible. Therefore, $w=u$ and the whole sequence u_n converges to it uniformly in $\overline{\Omega}$. □

Remark 8.5. In [4], Theorem 8.1 is proved with $G(x,p)=|p|$. Moreover, the strong convergence of u_n to u in $W_0^{1,m}(\Omega)$ for any m is obtained, exploiting that u verifies

$$|Du| = 1 \qquad \text{a.e. in } \Omega$$

and using Clarkson and Hoelder inequalities.

In our setting however the situation is different since a strict inequality

$$G(x, Du) < 1$$

can hold on a subset of Ω of positive measure, see Example 5.3, and so the previous argument cannot be used to obtain the same convergence result. So far it is not clear if a different argument may work or instead the property is false.

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