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THE REGULAR ATTRACTOR FOR THE REACTION-DIFFUSION SYSTEM WITH A NONLINEARITY RAPIDLY OSCILLATING IN TIME AND ITS AVERAGING

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Abstract. The longtime behaviour of solutions of a reaction-diffusion system with the nonlinearity rapidly oscillating in time ($f = f(t/\varepsilon, u)$) is studied. It is proved that (under the natural assumptions) this behaviour can be described in terms of global (uniform) attractors $\mathcal{A}^\varepsilon$ of the corresponding dynamical process and that these attractors tend as $\varepsilon \to 0$ to the global attractor $\mathcal{A}^0$ of the averaged autonomous system. Moreover, we give a detailed description of the attractors $\mathcal{A}^\varepsilon$, $\varepsilon \ll 1$, in the case where the averaged system possesses a global Liapunov function.

0. Introduction

We consider the following reaction-diffusion system in a bounded domain $\Omega \subset\subset \mathbb{R}^n$ with a sufficiently smooth boundary:

$$\begin{cases} \partial_t u = a\Delta_x u - f(t/\varepsilon, x, u) + g(x), \\ u\big|_{\partial\Omega} = 0, \quad u\big|_{t=0} = u_0. \end{cases} \tag{0.1}$$

Here $\varepsilon > 0$ is a small parameter, $u = (u^1, \cdots, u^k)$ is an unknown vector-valued function, $f = (f^1, \cdots, f^k)$ and $g = (g^1, \cdots, g^k)$ are given functions and the constant diffusion matrix a is assumed to be diagonal

$$a = \operatorname{diag}\{a_1, \cdots, a_k\}, \quad a_i > 0, \quad i = 1, \cdots, k. \tag{0.2}$$

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The external force g is assumed belonging to the space $L^p(\Omega)$ for a some fixed $p > (n+2)/2$.

A solution of equation (0.1) is a function $u \in W^{(1,2),p}([T, T+1] \times \Omega)$ for every $T \geq 0$ (here and below the symbol $W^{(l_1,l_2),p}(V)$ means an anisotropic Sobolev-Slobodetskii space of functions $w(t,x)$ such that $D_t^{l_1} w$, $D_x^{l_2} w \in L^p(\Omega)$, see e.g. [19]) and satisfies equation (0.1) in the sense of distributions (note that due to the embedding theorem and our choice of the exponent p, $W^{(1,2),p} \subset C$ and, consequently, the nonlinear term in (0.1) is well-defined). The initial value u_0 is supposed to belong to the trace space of $W^{(1,2),p}([0,1] \times \Omega)$ at $t = 0$, i.e.,

$$u_0 \in \Phi_p(\Omega) := W^{2(1-1/p),p}(\Omega) \cap \{u_0\big|_{\partial\Omega} = 0\}. \tag{0.3}$$

(see, e.g. [19]).

The nonlinearity f is assumed to have the polynomial rate of growth with respect to u:

$$|f(z,x,v)| \leq C(1 + |v|^q), \tag{0.4}$$

for some $q > 1$ and C which are independent of $v \in \mathbb{R}^k$, $x \in \Omega$, and $z \in \mathbb{R}$, and to satisfy the anisotropic dissipativity assumption in the following form: there are exponents $p_i \geq \max\{0, pq - 2\}$ such that

$$\sum_{i=1}^{k} f_i(z,x,v) v_i |v_i|^{p_i} \geq -C, \tag{0.5}$$

where the constant C is independent of $z \in \mathbb{R}$, $x \in \Omega$, and $v \in \mathbb{R}^k$ (see [9]).

Moreover, we assume also that the function $f(z,x,v)$ is almost-periodic with respect to z (see Section 2 for precise conditions), continuous with respect to x and sufficiently regular with respect to v, namely,

$$|f'_v(z,x,v)| + |f''_{vv}(z,x,v)| \leq Q(|v|), \tag{0.6}$$

where Q is a monotonic function which is independent of z and x.

The long-time behaviour of solutions of (0.1) in the autonomous case is usually described in terms of global attractors of dynamical systems associated with the problem under consideration, see [2], [14], [24] and references therein.

The case of non-autonomous equations is essentially less understood. In fact up to the moment, there are several natural approaches to extend the attractors theory to the non-autonomous case. One of them is based on reducing the non-autonomous dynamical process to the autonomous one using the skew-product technique. The realization of this approach leads to the so-called uniform attractor $\mathcal{A}^\varepsilon$ of equation (0.1) which is independent of t and is uniform with respect to all nonlinearities $\phi(t/\varepsilon, x, u)$ belonging to the hull $\mathcal{H}(f)$ of the initial nonlinearity f, see [6], [15]. The alternative approach interprets the attractor of the non-autonomous equation (0.1) as a non-autonomous set as well: $\mathcal{A}^\varepsilon_f(t)$, $t \in \mathbb{R}$, see e.g. [8], [18].

The homogenization problems for individual solutions of equations of type (0.1) with rapidly oscillating spatial and temporal terms were investigated in [3], [4], [21], [22], [27] (see also references therein).

The homogenization of attractors were also studied by many authors. See, e.g., [5], [23] for attractors of reaction-diffusion and even hyperbolic equations in non-homogenized spatially-periodic media with asymptotic degeneracy. The case of regular spatially almost-periodic media was considered in [9]. The homogenization aspects for the reaction-diffusion problems with spatial rapid oscillations in subordinated terms (i.e., for $f = f(x/\varepsilon, u)$ or $g = g(x/\varepsilon)$) were considered in [13] and [14]. The temporal averaging of uniform attractors for evolutionary problems was studied in [17] (for the case of 2D Navier-Stokes system and for the nonlinear wave equation with rapidly oscillating in time external forces) and in [26] (for the case of singular perturbed reaction-diffusion system with rapidly oscillating external forces). The homogenization of trajectory attractors associated with ill-posed evolutionary mathematical physics equations (such as 3D Navier-Stokes equations, damped wave equations with supercritical nonlinearities, etc.) is studied in [7].

In the present paper we give a comprehensive study of qualitative and quantitative aspects of global temporal averaging in equations (0.1). Particularly, we show that, under the above assumptions, problems (0.1) possess uniform attractors $\mathcal{A}^\varepsilon$ in Φ_p for every fixed $\varepsilon > 0$. In order to study the behaviour of these attractors as $\varepsilon > 0$, we consider also the averaged problem for (0.1) which obviously has the following form:

$$\begin{cases} \partial_t \overline{u} = a\Delta_x \overline{u} - \overline{f}(x, \overline{u}) + g(x), \\ \overline{u}\big|_{\partial\Omega} = 0, \quad \overline{u}\big|_{t=0} = u_0, \end{cases} \tag{0.7}$$

where

$$\overline{f}(x, v) := \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(h, x, v)\, dv. \tag{0.8}$$

Autonomous equation (0.7) also possesses a (global) attractor $\mathcal{A}^0$, moreover, we have the following upper semi-continuity of the attractor $\mathcal{A}^\varepsilon$.

Theorem 1. *Let the above assumptions hold. Then*

$$\operatorname{dist}_{\Phi_p}\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) \to 0, \quad \textit{as } \varepsilon \to 0,$$

where dist *means a standard non-symmetric Hausdorff distance between subsets of* Φ_p*.*

The main part of our study is devoted to a more detailed investigation of the case where the limit attractor $\mathcal{A}^0$ is regular, it will be so, for example, if equation (0.7) possesses a global Liapunov function and all its equilibria are hyperbolic, see [2]. In this case, the attractor $\mathcal{A}^0$ can be represented as a finite collection of finite dimensional unstable manifolds of these equilibria:

$$\mathcal{A}^0 = \cup_{i=1}^N \mathcal{M}_{z_i}^+, \tag{0.9}$$

where z_i, $i = 1, \cdots, N$ are (hyperbolic) equilibria of (0.7).

The non-autonomous *regular* with respect to ε perturbations of regular attractors associated with reaction-diffusion equations were considered in [13]. We prove that structure (0.9) is in a sense preserves for sufficiently small $\varepsilon > 0$, i.e., that the regular structure of the attractor preserves under rapidly oscillating perturbations as well. To this end, it will be more convenient to use the alternative concept of the nonautonomous attractor of (0.1) under which the attractor is defined to be a set-valued function $t \to \mathcal{A}_f^\varepsilon(t)$.

Theorem 2. *Let the assumptions of Theorem* 1 *hold and let, in addition, the averaged attractor* $\mathcal{A}^0$ *is regular. Then, for every* $\varepsilon < \varepsilon_0$*, there exists a family of compact sets* $\mathcal{A}_f^\varepsilon(t)$*,* $t \in \mathbb{R}$ *in* Φ_p *enjoying the following properties:*

1) *This family is invariant with respect to the dynamical process associated with equation* (0.1)*, i.e.,*

$$U_f^\varepsilon(t,\tau)\mathcal{A}_f^\varepsilon(\tau) = \mathcal{A}_f^\varepsilon(t), \quad \tau \in \mathbb{R}, \quad t \ge \tau, \tag{0.10}$$

where by definition $U_f^\varepsilon(t,\tau)u(\tau) := u(t)$ *and* $u(t)$ *is a solution of* (0.1)*, defined for* $t \ge \tau$*.*

2) *There exist a monotonic function* Q *and a positive constant* γ*, which are independent of* $\varepsilon < \varepsilon_0$*, such that, for every bounded subset* $B \subset \Phi_p$*, the following is true:*

$$\operatorname{dist}_{\Phi_p}\left(U_f^\varepsilon(t+\tau,t)B, \mathcal{A}_f^\varepsilon(t+\tau)\right) \le Q(\|B\|_{\Phi_p})e^{-\gamma\tau}, \tag{0.11}$$

for every $t \in \mathbb{R}$ *and* $\tau \ge 0$*.*

3) *The uniform attractor* $\mathcal{A}^\varepsilon$ *can be expressed in terms of attractors* $\mathcal{A}_f^\varepsilon(t)$ *via*

$$\mathcal{A}^\varepsilon = \left[\cup_{t\in\mathbb{R}} \mathcal{A}_f^\varepsilon(t)\right]_{\Phi_p}, \tag{0.12}$$

where $[\cdot]_{\Phi_p}$ *means the closure in the phase space* Φ_p*.*

As a simple corollary of the proved theorem we derive that

$$\sup_{t\in\mathbb{R}} \operatorname{dist}_{symm,\Phi_p}\left(\mathcal{A}_f^\varepsilon(t),\mathcal{A}^0\right)\to 0 \quad\text{as}\quad \varepsilon\to 0, \tag{0.13}$$

where $\operatorname{dist}_{symm,\Phi_p}$ means a *symmetric* Hausdorff distance for subsets of Φ_p, and, consequently, the family $\mathcal{A}^\varepsilon$ of uniform attractors is upper and lower semicontinuous at $\varepsilon = 0$. The following theorem gives an nonautonomous analogue of decomposition (0.9).

Theorem 3. *Let the assumptions of Theorem* 1 *hold. Then there exist exactly N almost-periodic solutions $z_i^\varepsilon(t)$ of equation* (0.1)*. Moreover, these solutions are hyperbolic, tend to z_i as $\varepsilon \to 0$ and the attractors $\mathcal{A}_f^\varepsilon(t)$ have the following structure:*

$$\mathcal{A}_f^\varepsilon(t) = \cup_{i=1}^N \mathcal{M}^+_{f,z_i^\varepsilon}(t), \tag{0.14}$$

where $\mathcal{M}^+_{f,z_i^\varepsilon}(t)$, $t\in\mathbb{R}$ is a (*non-autonomous*) *unstable manifold of the solution z_i^ε. Note also that $\mathcal{M}^+_{f,z_i^\varepsilon}(t)$ are diffeomorphic to $\mathcal{M}^+_{z_i}$, for every fixed t.*

Thus, under the above assumptions, every solution of equation (0.1) tends to one of the almost-periodic solutions $z_i^\varepsilon(t)$ and the attractor $\mathcal{A}_f^\varepsilon(t)$ consists of these almost-periodic solutions and heteroclinic connections between them in a complete analogy with the structure of the autonomous regular attractors.

The following corollary clarifies the dependence of $\mathcal{A}_f^\varepsilon(t)$ on t.

Corollary 1. *Let $\mathcal{P}$ be a complete metric space of all compact subsets of Φ_p with the symmetric Hausdorff distance as a metric. Then the function $t \to \mathcal{A}_f^\varepsilon(t)$ is almost-periodic as a $\mathcal{P}$-valued function*

$$\mathcal{A}_f^\varepsilon(\cdot)\in AP(\mathbb{R},\mathcal{P}).$$

The rest of the paper is devoted to study the quantitative aspects of convergence (0.13).

Theorem 4. *Let the assumptions of Theorem* 2 *hold and let, in addition, the primitive*

$$F(h,x,u) := \int_0^h [f(h,x,v)-\overline{f}(x,v)]\,dh \tag{0.15}$$

is bounded (*and consequently almost-periodic*) *with respect to h. Then*

$$\sup_{t\in\mathbb{R}} \operatorname{dist}_{symm,\Phi_p}\left(\mathcal{A}_f^\varepsilon(t),\mathcal{A}^0\right)\le C_F\varepsilon^\kappa, \tag{0.16}$$

where $0 < \kappa < 1$ *depends only on the constant* γ *from* (0.11) *and on maximal Liapunov exponent of problem* (0.1) *and the constant* C_F *depends on the equation and, particularly, on the* L^∞*-bounds of function* (0.15).

Note that the assumption on primitive (0.15) is always satisfied for the case of periodic dependence of $f(h,x,v)$ on h. For the case of more general almost-periodic (and even for quasi-periodic) dependence on h, primitive (0.15) may be unbounded indeed (see [21]). Note, however, that even in this case the set of functions f for which (0.15) is bounded is in a sense dense in the space of all functions f. For simplicity, we explain this assertion on an example of the so-called quasiperiodic functions f.

Corollary 2. *Let the assumptions of Theorem* 2 *hold and let, in addition, the function* f *have the following structure:*

$$f(h,x,v) := G(\omega_1 h, \cdots, \omega_m h, x, v), \tag{0.17}$$

where the function G *is* 1*-periodic with respect to the first* m *arguments, and* $\omega = (w_1, \cdots, \omega_m)$ *is a frequency vector associated with* f. *Assume, in addition, that* G *is smooth enough, namely,*

$$G \in C^{2m}(\mathbb{R}^{2m}, C^2(\overline{\Omega} \times \mathbb{R}^k)).$$

Then, estimate (0.13) *remains valid for almost all frequency vectors* $\omega \subset \mathbb{R}^m$ (*with respect to the standard Lebesgue measure in* $\mathbb{R}^m$) :

$$\sup_{t\in\mathbb{R}} \operatorname{dist}_{symm,\Phi_p}\left(\mathcal{A}_f^\varepsilon(t), \mathcal{A}^0\right) \le C(\omega)\varepsilon^\kappa. \tag{0.18}$$

Unfortunately, the constant $C(\omega)$ depends on some Diophantine conditions on the vector ω and, consequently, it is extremely sensitive to small perturbation of ω. That is the reason why, it seems reasonable to obtain the probabilistic analogue of estimate (0.18).

Corollary 3. *Let the assumptions of Corollary* 2 *hold and let the frequencies* ω_i, $i = 1, \cdots, m$ *be independent Gaussian random variables. Then the expectation* $\mathbb{M}$ *of random variable* (0.18) *possesses the following estimate:*

$$\mathbb{M}\Big\{\sup_{t\in\mathbb{R}} \operatorname{dist}_{symm,\Phi_p}\left(\mathcal{A}_f^\varepsilon(t), \mathcal{A}^0\right)\Big\} \le C(\sigma)\varepsilon^\kappa, \tag{0.19}$$

for some constants $0 < \kappa < 1$ *and* $C(\sigma)$ *depending on dispersions of* ω (*but, in contrast of* (0.18)*, this dependence is 'regular'*).

The paper is organized as follows.

The existence and uniqueness of solutions of (0.1) and uniform (with respect to ε) dissipative estimates are derived in Section 1.

The uniform attractors $\mathcal{A}^\varepsilon$ for problem (0.1) are constructed in Section 2. Moreover, their upper semicontinuity at $\varepsilon \to 0$ is verified here.

The convergence of the nonautonomous dynamical systems $\mathcal{U}_f^\varepsilon(t,\tau)$ to the averaged one associated with (0.7) (together with their Frechet derivatives) on a finite interval of time is proved in Section 3. The quantitative bounds for this convergence are derived in Section 4.

Based on these estimates, we investigate in Section 5 the behaviour of nonaveraged system (0.1) near the hyperbolic equilibrium of averaged system (0.7).

The regular structure of the non-averaged nonautonomous attractor $\mathcal{A}_f^\varepsilon(t)$, if $\varepsilon > 0$ is small enough, is established in Section 6.

The quantitive and qualitative results on the convergence $\mathcal{A}_f^\varepsilon(t)$ to $\mathcal{A}^0$ are obtained in Section 7. Moreover, the almost-periodicity of the set-valued function $\mathcal{A}_f^\varepsilon(\cdot)$ is also verified here.

The paper is concluded by Section 8, where we illustrate the obtained results on a number of concrete equations in the form of (0.1).

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1. Uniform (with respect to ε) a priori estimates

In this Section, we derive several auxiliary estimates for the solutions of (0.1) which will be used in the sequel. We start from the dissipative estimate in the phase space Φ_p.

Theorem 1.1. *Let assumptions* (0.2)–(0.6) *hold. Then, the solution* $u(t)$ *of equation* (0.1) *possesses the following estimate:*

$$\|u(t)\|_{\Phi_p} + \|u\|_{W^{(1,2),p}([t,t+1]\times\Omega)} \le Q(\|u_0\|_{\Phi_p})e^{-\alpha t} + Q(\|g\|_{L^p}), \tag{1.1}$$

where the constant $\alpha > 0$ *and the function* Q *are independent of* u_0 *and* $\varepsilon \in (0,\infty)$.

Proof. Let us first derive the uniform estimate of the $L^{pq}(\Omega)$-norm of the solution $u(t)$. To this end, we multiply (following [9]) the lth equation of (0.1) by $u_i(t)|u_i(t)|^{p_i}$ and take a sum over $l \in \{1,\cdots,k\}$ and integrate over $x \in \Omega$. Then, we obtain (after using assumptions (0.2) and (0.5) and after the integration by parts) that

$$\partial_t\Big(\sum_{i=1}^k \frac{1}{p_i+2}\|u_i(t)\|^{p_i+2}_{L^{p_i+2}}\Big) + \sum_{i=1}^k \frac{4a_i(p_i+1)}{(p_i+2)^2}\|\nabla_x(|u_i|^{(p_i+2)/2})\|^2_{L^2}$$

$$\leq C\,\mathrm{vol}(\Omega) + \Big(\sum_{i=1}^{k} \|g_i\|_{L^p}\|u_i(t)\|^{p_i+1}_{L^{p'(p_i+1)}}\Big), \tag{1.2}$$

where p' is a conjugated exponent to p $(1/p + 1/p' = 1)$ and the constant C is the same as in (0.5). Note that, due to the Sobolev's embedding theorem, the second term in the left-hand side of (1.2) can be estimated as follows:

$$\begin{aligned}&\sum_{i=1}^{k} \frac{4a_i(p_i+1)}{(p_i+2)^2}\|\nabla_x(|u_i|^{(p_i+2)/2})\|^2_{L^2}\\ &\geq \alpha\Big(\sum_{i=1}^{k}\frac{1}{p_i+2}\|u_i(t)\|^{p_i+2}_{L^{p_i+2}}\Big) + \alpha\Big(\sum_{i=1}^{k}\|u_i(t)\|^{p_i+2}_{L^{l_n(p_i+2)}}\Big),\end{aligned} \tag{1.3}$$

where $\alpha > 0$ is independent of ε and $l_n := \frac{n}{n-2}$ is the Sobolev's embedding exponent $(W^{1,2} \subset L^{2l_n})$.

Note that, according to our choice of the exponent p $(p > (n+2)/2)$, we have $p' \leq n/(n+2) < l_n$ and, consequently, the last term in the right-hand side of (1.2) can be estimated in the following way:

$$\sum_{i=1}^{k} \|g_i\|_{L^p}\|u_i(t)\|^{p_i+1}_{L^{p'(p_i+1)}} \leq C_1\Big(\sum_{i=1}^{k}\|g_i\|^{p_i+2}_{L^p}\Big) + \alpha\Big(\sum_{i=1}^{k}\|u_i(t)\|^{p_i+2}_{L^{l_n(p_i+2)}}\Big) \tag{1.4}$$

inserting estimates (1.3) and (1.4) in (1.2) we have

$$\begin{aligned}&\partial_t\Big(\sum_{i=1}^{k}\frac{1}{p_i+2}\|u_i(t)\|^{p_i+2}_{L^{p_i+2}}\Big) + \alpha\Big(\sum_{i=1}^{k}\frac{1}{p_i+2}\|u_i(t)\|^{p_i+2}_{L^{p_i+2}}\Big)\\ &\leq C_2\Big(1 + \sum_{i=1}^{k}\|g_i\|^{p_i+2}_{L^p}\Big).\end{aligned} \tag{1.5}$$

The Gronwall's inequality now yields

$$\sum_{i=1}^{k}\|u_i(t)\|^{p_i+2}_{L^{p_i+2}} \leq C_3\Big(\sum_{i=1}^{k}\|u_i(0)\|^{p_i+2}_{L^{p_i+2}}\Big)e^{-\alpha_1 t} + C_4\Big(1 + \sum_{i=1}^{k}\|g_i\|^{p_i+2}_{L^p}\Big), \tag{1.6}$$

for the appropriate constants $\alpha_1 > 0$, C_3 and C_4 which are independent of ε.

Recall now that $\Phi_p \subset C$ and $p_i \geq qp - 2$, consequently, (0.4) and (1.6) imply

$$\|f(t/\varepsilon, x, u(t)\|_{L^p(\Omega)} \leq Q(\|u_0\|_{\Phi_p})e^{-\alpha_1 t} + Q(\|g\|_{L^p}), \tag{1.7}$$

for the appropriate monotonic function Q which is independent of ε.

After obtaining estimate (1.7) for the nonlinearity, we may interpret equation (0.1) as a collection of linear nonhomogeneous heat equations

$$\partial_t u_i - a_i \Delta_x u_i = h_i(t, x), \tag{1.8}$$

where $h(t, x) := -f(t/\varepsilon, x, u(t, x)) + g(x)$. Estimate (1.1) is an immediate corollary of (1.7) and the classical L^p-regularity theory for the heat equation (see e.g. [19]). Theorem 1.1 is proven. □

The following theorem establishes the unique solvability of problem (0.1) and gives a uniform (with respect to ε) estimate for the differences between two solutions of this equation.

Theorem 1.2. *Let the assumptions of Theorem* 1.1 *hold. Then, for every* $u_0 \in \Phi_p$*, problem* (0.1) *has a unique solution* $u(t)$*. Moreover, if* $u_1(t)$ *and* $u_2(t)$ *are two solutions of this problem, then*

$$\|u_1(t) - u_2(t)\|_{\Phi_p} + \|u_1 - u_2\|_{W^{(1,2),p}([t,t+1]\times\Omega)} \le C\|u_1(0) - u_2(0)\|_{\Phi_p} e^{Kt}, \tag{1.9}$$

where the constants C *and* K *depend on* $\|u_i(0)\|_{\Phi_b}$*,* $i = 1, 2$*, but are independent of* ε*.*

Proof. As usual, the existence of a solution for problem (0.1) is a standard corollary of a priori estimate (1.1) and the Leray-Schauder fixed point theorem (see e.g. [19]). So, it only remains to verify (1.9). Indeed, let $v(t) := u_1(t) - u_2(t)$. Then, this function obviously satisfies the equation

$$\partial_t v = a\Delta_x v - l_\varepsilon(t)v, \quad v\big|_{t=0} = u_1(0) - u_2(0), \quad v\big|_{\partial\Omega} = 0, \tag{1.10}$$

where

$$l_\varepsilon(t, x) := \int_0^1 f'_u(t/\varepsilon, x, su_1(t, x) + (1 - s)u_2(t, x))\, dx.$$

Note that, due to (0.6), (1.1) and the fact that $\Phi_p \subset C$, we have the uniform estimate

$$\|l_\varepsilon(t)\|_{L^\infty} \le C := Q(\|u_1(0)\|_{\Phi_p} + \|u_2(0)\|_{\Phi_p}), \tag{1.11}$$

where the function Q is independent of ε. Multiplying now the lth equation of (1.11) by $v_l(t)|v_l(t)|^{p-2}$, taking a sum over l, integrating over $x \in \Omega$ and using estimate (1.11), we derive (analogously to (1.6)) that

$$\|v(t)\|_{L^p} \le C_1\|v(0)\|_{L^p} e^{Kt}, \tag{1.12}$$

where C_1 and K are independent of ε. As in the proof of Theorem 1.1, estimate (1.9) is an immediate corollary of (1.11), (1.12) and the L^p-regularity theory for the heat equation. Theorem 1.2 is proven. □

In conclusion of this Section, we derive the smoothing property for equation (0.1) which is necessary in order to prove the attractor's existence.

Theorem 1.3. *Let the assumptions of Theorem* 1.1 *hold. Then, the following estimate is valid:*

$$\|u(t)\|_{\Phi_{2p}} \le \frac{t+1}{t}\left(Q(\|u_0\|_{\Phi_p})e^{-\alpha t} + Q(\|g\|_{L^p})\right), \tag{1.13}$$

where the monotonic function Q and the positive constant $\alpha > 0$ are independent of ε.

Proof. Indeed, let $G = G(x) \in W^{2,p}(\Omega) \subset \Phi_{2p}$ be a solution of the following elliptic problem:

$$a\Delta_x G = g, \quad G\big|_{\partial\Omega} = 0 \tag{1.14}$$

and let $w(t) := t(u(t) - G)$. Then, this function obviously satisfies the equation

$$\partial_t w - a\Delta_x w = h(t) := -tf(t/\varepsilon, x, u(t)) + u(t), \quad w\big|_{t=0} = 0. \tag{1.15}$$

Note that, due to estimates (0.4), (1.1) and the embedding $\Phi_p \subset C$, we have

$$\|h(t)\|_{L^{2p}} \le (t+1)\left(Q(\|u_0\|_{\Phi_p})e^{-\alpha t} + Q(\|g\|_{L^p})\right), \tag{1.16}$$

where Q and α are independent of ε. Applying now the L^{2p}-regularity theorem for heat equations to (1.15), we derive that

$$\|w(t)\|_{\Phi_{2p}} \le (t+1)\left(Q_1(\|u_0\|_{\Phi_p})e^{-\alpha t} + Q_1(\|g\|_{L^p})\right). \tag{1.17}$$

Theorem 1.3 is proven. □

2. The attractors

In this Section, we prove that for every ε, equation (0.1) possesses an attractor $\mathcal{A}^\varepsilon$ and prove that these attractors tend (upper semicontinuous) as $\varepsilon \to 0$ to the attractor $\mathcal{A}^0$ of the averaged system (0.1).

We assume that the functions $f(z,x,v)$, $f'_v(z,x,v)$ and $f''_{vv}(z,x,v)$ are almost-periodic with respect to z, for every fixed x and v. The latter means (see, e.g., [20] or [21]) that, for every fixed x and v, the set

$$\{f(z+h,x,v), h \in \mathbb{R}\} \subset\subset C_b(\mathbb{R}, \mathbb{R}^k) \tag{2.1}$$

is a precompact set in $C_b(\mathbb{R})$ (and analogous statements are true for f'_v and f''_{vv}). We also assume that the function f is uniformly continuous with respect to $x \in \Omega$ in the following sense: for every $R \in \mathbb{R}_+$, there exists a function $\alpha_R(\theta)$ such that $\alpha_R \to 0$ as $\theta \to 0$ and

$$|\phi(z,x_1,v_1) - \phi(z,x_2,v_2)| \le \alpha_R(|x_1 - x_2| + |v_1 - v_2|), \tag{2.2}$$

for every $x_1, x_2 \in \Omega$, $z \in \mathbb{R}$ and $v_i \in \mathbb{R}^k$, $|v_i| \le R$ and for $\phi = f$, $\phi = f'_v$ and $\phi = f''_{vv}$.

In order to reformulate our assumptions on f in a more convenient way, we introduce the Frechet space $\mathcal{M}$ which consists of functions $F:\overline{\Omega}\times\mathbb{R}^k\to\mathbb{R}^k$, $F,F'_v,F''_{vv}\in C(\Omega\times\mathbb{R}^k)$ and generated by the following system of seminorms

$$\|F\|_R:=\sup_{x\in\overline{\Omega}}\sup_{v\in B_R}\left(|F(x,v)|+|F'_v(x,v)|+|F''_{vv}(x,v)|\right), \tag{2.3}$$

where $B_R:=\{|v|\le R\}$ an R-ball in $\mathbb{R}^k$. Obviously, the space $\mathcal{M}$ thus defined is a metrizable F-space the convergence in which coincides with the locally compact convergence with respect to $v\in\mathbb{R}^k$.

Lemma 2.1. *Let the assumptions* (2.1) *and* (2.2) *be valid. Then, the function $f(z,x,v)$ interpreted as a map from $\mathbb{R}$ to $\mathcal{M}$ is an almost-periodic* (*in Bochner-Amerio sense*) *function with values in $\mathcal{M}$, i.e., the hull*

$$\mathcal{H}(f):=\{T_hf,h\in\mathbb{R}\}_{C_b(\mathbb{R},\mathcal{M})}\subset\subset C_b(\mathbb{R},\mathcal{M}) \tag{2.4}$$

is compact in $C_b(\mathbb{R},\mathcal{M})$ (*here and below, $(T_hf)(z,x,v):=f(z+h,x,v)$ and $\{\cdot\}_V$ means a closure in the topology of the space V*).

Proof. Indeed, let $\{h_n\}_{n\in\mathbb{N}}$ be an arbitrary sequence. Then, due to assumption (2.1) and the Cantor's diagonal procedure, we may assume without loss of generality that

$$\phi(z,x,v)=\lim_{n\to\infty}f(z+h_n,x,v), \tag{2.5}$$

for every x,v from a dense set. Assumption (2.2) now implies that (2.5) holds, for *every* $x,v\in\overline{\Omega}\times\mathbb{R}^k$. Thus, it remains to check that (2.5) is uniform with respect to $x,v\in\Omega\times B_R$. Assume that it is not so, i.e., there exist sequences $x_n\to x_0$, $v_n\to v_0$ and $z_n\in\mathbb{R}$ such that

$$|f(z_n+h_n,x_n,v_n)-\phi(z_n,x_n,v_n)|\ge\varepsilon_0. \tag{2.6}$$

It also follows from (2.5) that the limit function ϕ satisfies (2.2) as well. Consequently, (2.2) and (2.6) implies that, for a sufficiently large n, we have

$$|f(z_n+h_n,x_0,v_0)-\phi(z_n,x_0,v_0)|\ge\varepsilon_0/2 \tag{2.7}$$

which contradicts to convergence (2.5) with $x=x_0$ and $v=v_0$. Thus, we have proved that convergence (2.5) is uniform with respect to $x,v\in\overline{\Omega}\times B_R$. The convergence of derivatives f'_v and f''_{vv} can be verified analogously. Lemma 2.1 is proven. □

Let us return now to equation (0.1). In order to construct the attractor $\mathcal{A}^\varepsilon$ for this equation we consider (following [6] and [15]) a family of problems of type (0.1) with all of the nonlinearities belonging to the hull of the initial nonlinearity f:

$$\partial_t u = a\Delta_x u - \phi(t/\varepsilon, x, u) + g(x), \ \ \phi \in \mathcal{H}(f), \ \ u\big|_{\partial\Omega} = 0, \ \ u\big|_{t=\tau} = u_\tau. \quad (2.8)$$

Note that assumptions (0.4)-(0.6) on the nonlinearity f obviously remain valid, for every $\phi \in \mathcal{H}(f)$, consequently, due to Theorems 1.1 and 1.2, equations (2.8) are uniquely solvable for every $h \in \mathcal{H}(f)$, $\tau \in \mathbb{R}$ and $u_\tau \in \Phi_p$. Therefore, solving operators

$$U^\varepsilon_\phi(t,\tau) : \Phi_p \to \Phi_p, \ \ U^\varepsilon_\phi(t,\tau)u_\tau := u(t), \quad (2.9)$$

where $u(t)$ is a solution of (2.8) are well defined, for every $h \in \mathcal{H}(f)$ and every $t \geq \tau$ (we indicate in (2.9) the dependence of these operators on ε by the upper index keeping in mind passing to the limit $\varepsilon \to 0$ in the sequel). Moreover, it follows from Theorem 1.1 that

$$\|U^\varepsilon_\phi(t,\tau)u_\tau\|_{\Phi_p} \leq Q(\|u_\tau\|_{\Phi_p})e^{-\alpha(t-\tau)} + Q(\|g\|_{L^p}), \quad (2.10)$$

where the function Q and the constant $\alpha > 0$ are independent of ε, $h \in \mathcal{H}(f)$ and $\tau \in \mathbb{R}$.

Let us now define the extended semigroup $\mathbb{S}_h(\varepsilon) : \Phi_p \times \mathcal{H}(f) \to \Phi_p \times \mathcal{H}(f)$ which corresponds to the family of equations (2.8) using the standard skew product technique (see e.g. [6] or [15]):

$$\mathbb{S}_h(\varepsilon)(v,\phi) := (U^\varepsilon_\phi(h,0)v, T_{h/\varepsilon}\phi), \quad (2.11)$$

where $(T_{h/\varepsilon}f)(t/\varepsilon, x, v) := f((t+h)/\varepsilon, x, v)$. Recall that, by definition, a set $\mathbb{A}^\varepsilon \subset \Phi_p \times \mathcal{H}(f)$ is an attractor of the semigroup $\mathbb{S}_h(\varepsilon)$ if the following assumptions are satisfied:

1. The set $\mathbb{A}^\varepsilon$ is compact in $\Phi_p \times \mathcal{H}(f)$.
2. The set $\mathbb{A}^\varepsilon$ is strictly invariant, i.e., $\mathbb{S}_h(\varepsilon)\mathbb{A}^\varepsilon = \mathbb{A}^\varepsilon$.
3. The set $\mathbb{A}^\varepsilon$ attracts bounded subsets of $\Phi_p \times \mathcal{H}(f)$, i.e., for every bounded in $\Phi_p \times \mathcal{H}(f)$ set $\mathbb{B}$ and for every neighbourhood $\mathcal{O}(\mathbb{A}^\varepsilon)$, there exists a number $T = T(\mathbb{B}, \mathcal{O})$ such that

$$\mathbb{S}_h(\varepsilon)\mathbb{B} \subset \mathcal{O}(\mathbb{A}^\varepsilon), \ \ \forall h \geq T \quad (2.12)$$

(see e.g. [2], [24] for further details).

Theorem 2.1. *Let the assumptions of Theorem* 1.1 *hold and let, in addition,* (2.1) *and* (2.2) *be satisfied. Then, the semigroup* $\mathbb{S}_h(\varepsilon)$*, defined above possesses a* (*global*) *attractor* $\mathbb{A}^\varepsilon \subset \Phi_p \times \mathcal{H}(f)$*. Moreover, the following estimate is true:*

$$\|\mathbb{A}^\varepsilon\|_{\Phi_{2p}\times\mathcal{H}(f)} \leq Q(\|g\|_{L^p}), \quad (2.13)$$

where the function Q *is independent of* ε*.*

Proof. As usual, we should verify the assumptions of the abstract theorem on the attractor's existence (see, e.g., [2]), namely, we should verify that $\mathbb{S}_t(\varepsilon)$ is continuos, for every fixed t, and the fact that it possesses a compact absorbing set in $\Phi_p \times \mathcal{H}(f)$.

The first assumption can be verified in a standard way (see also Theorem 1.2) so, it only remains to verify the existence of a compact absorbing set.

Indeed, it follows from Theorem 1.3 and from the fact that conditions (0.4)–(0.6) remain valid, for every $\phi \in \mathcal{H}(f)$, that

$$\|U^\varepsilon_\phi(t,0)v\|_{\Phi_{2p}} \le \frac{1+t}{t}\left(Q(\|v\|_{\Phi_p})e^{-\alpha t} + Q(\|g\|_{L^p})\right), \tag{2.14}$$

where the function Q and the constant $\alpha > 0$ are independent of ε and $\phi \in \mathcal{H}(f)$. Consequently, the set

$$\mathbb{K} := \{\|u\|_{\Phi_{2p}} \le 2Q(\|g\|_{L^p})\} \times \mathcal{H}(f) \tag{2.15}$$

is a compact (the hull $\mathcal{H}(f)$ is compact due to our assumptions and the embedding $\Phi_{2p} \subset \Phi_p$ is obviously compact) in $\Phi_p \times \mathcal{H}(f)$ absorbing set for the semigroup $\mathbb{S}_t(\varepsilon)$. Thus, according to the abstract theorem for the attractor's existence, the semigroup $\mathbb{S}_t(\varepsilon)$ possesses a global attractor $\mathbb{A}^\varepsilon \subset \mathbb{K}$. Theorem 2.1 is proven. □

Definition 2.1. Define a (uniform) attractor $\mathcal{A}^\varepsilon$ for the initial non-autonomous system (0.1) by projecting $\mathbb{A}^\varepsilon$ to the first component:

$$\mathcal{A}^\varepsilon := \Pi_1 \mathbb{A}^\varepsilon, \tag{2.16}$$

where $\Pi_1(u,\phi) := u$.

Remark 2.1. The attractor $\mathcal{A}^\varepsilon$ admits an internal definition (without using the skew product technique), namely, the set $\mathcal{A}^\varepsilon$ is called a (uniform) attractor for equation (0.1) if the following is true:

1) The set $\mathcal{A}^\varepsilon$ is compact in Φ_p.

2) The set $\mathcal{A}^\varepsilon$ attracts *uniformly* bounded subsets of Φ_p, i.e., for every bounded set $B \subset \Phi_p$ and every neighbourhood $\mathcal{O}(\mathcal{A}^\varepsilon)$ of $\mathcal{A}^\varepsilon$ in Φ_p, there exists a number $T = T(B, \mathcal{O})$ such that

$$U^\varepsilon_f(\tau + t, \tau)B \subset \mathcal{O}(\mathcal{A}^\varepsilon), \quad \forall \tau \in \mathbb{R}, \quad \forall t \ge T. \tag{2.17}$$

3) The set $\mathcal{A}^\varepsilon$ is the minimal one which enjoys 1) and 2).

The equivalence of this definition to the one given above (see Definition 2.1) is verified in [6].

The following corollary gives the description of $\mathcal{A}^\varepsilon$ in terms of bounded solutions of family (2.8).

Corollary 2.1. *Let the above assumptions hold and let* $\mathcal{K}^\varepsilon_\phi$, $\phi\in\mathcal{H}(f)$ *be a collection of all solutions* $u(t)$ *of* (2.8) *which are defined for every* $t\in\mathbb{R}$ *and bounded in* Φ_p, *i.e.,* $\|u(t)\|_{\Phi_p}\le C_u$. *Denote also*

$$\mathcal{K}^\varepsilon_\phi(\tau):=\mathcal{K}^\varepsilon_\phi\big|_{t=\tau}\subset\Phi_p. \tag{2.18}$$

Then, the attractor $\mathcal{A}^\varepsilon$ *possesses the following description:*

$$\mathcal{A}^\varepsilon=\cup_{\phi\in\mathcal{H}(f)}\mathcal{K}^\varepsilon_\phi(0). \tag{2.19}$$

Indeed, description (2.19) is more or less evident corollary of a well known fact that the attractor $\mathbb{A}^\varepsilon$ is generated by all complete bounded trajectories of the semigroup $\mathbb{S}_t(\varepsilon)$, see e.g. [6] for the rigorous proof of (2.19) in an abstract setting.

The rest of this section is devoted to study the behaviour of the obtained attractors $\mathcal{A}^\varepsilon$ as $\varepsilon\to 0$.

First of all, we introduce the averaging of the nonlinear function f by the following formula:

$$\overline{f}(x,v)=\lim_{T\to\infty}\frac{1}{T}\int_0^T f(z,x,v)\,dz. \tag{2.20}$$

This limit exists due to the almost-periodicity assumption (2.1) and Kroneker-Weyl theorem (see e.g. [21]). Moreover, since (due to Lemma 2.1) f is almost periodic with values in $\mathcal{M}$ then, for every $R\in\mathbb{R}_+$,

$$\left\|\frac{1}{T}\int_\tau^{\tau+T} f(z,x,v)\,dz-\overline{f}(x,v)\right\|_R\to 0 \tag{2.21}$$

as $T\to 0$ uniformly with respect to $\tau\in\mathbb{R}$ (the seminorm $\|\cdot\|_R$ is defined by (2.3)). Note also that, due to the fact that the convergence in (2.21) is uniform with respect to $\tau\in\mathbb{R}$, we may replace the function f in (2.20) and (2.21) by any $\phi\in\mathcal{H}(f)$ preserving the convergence to $\overline{f}$. In other words, the averaging $\overline{f}$ is independent of the concrete choice of the representative $\phi\in\mathcal{H}(f)$ ($\overline{\phi}=\overline{f}$, for every $\phi\in\mathcal{H}(f)$).

We now define the averaged system, corresponding to initial system (0.1) as follows:

$$\begin{cases}\partial_t u=a\Delta_x u-\overline{f}(x,u)+g\\ u\big|_{t=0}=u_0,\quad u\big|_{\partial\Omega}=0.\end{cases} \tag{2.22}$$

Note that assumptions (0.4)–(0.6) obviously remain valid for the averaged nonlinearity $\overline{f}(x,v)$, consequently, the results of Theorems 1.1-1.3 and 2.1 also remain valid for limit autonomous problem (2.22).

Theorem 2.2. *Let the above assumptions hold. Then, for every* $u_0 \in \Phi_p$*, problem* (2.22) *has a unique solution* $u(t)$ *which satisfies the following estimate:*

$$\|u(t)\|_{\Phi_p} \le Q(\|u_0\|_{\Phi_p})e^{-\alpha t} + Q(\|g\|_{L^p}), \tag{2.23}$$

where the function Q *and the constant* α *are the same as in Theorem* 1.1. *Moreover, the semigroup* $S_t^0 : \Phi_p \to \Phi_p$ *generated by this equation possesses a global attractor* $\mathcal{A}^0$ *in* Φ_p*, which is bounded in* Φ_{2p}

$$\|\mathcal{A}^0\|_{\Phi_{2p}} \le Q(\|g\|_{L^p}), \tag{2.24}$$

where Q *is the same as in* (2.13)*, and possesses the standard description in terms of bounded solutions defined for all* $t \in \mathbb{R}$*:*

$$\mathcal{A}^0 = \mathcal{K}^0_{\overline{f}}(0) \tag{2.25}$$

(*compare with* (2.18)).

The proof of this theorem repeats word by word the proof of Theorems 1.1 and 2.1 so we omit it here.

The following theorem shows that the attractors $\mathcal{A}^\varepsilon$ of problems (0.1) tend in a sense to the attractor $\mathcal{A}^0$ of averaged problem (2.22) as $\varepsilon \to 0$.

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *hold. Then*

$$\operatorname{dist}_{\Phi_p}\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) \to 0 \quad as \quad \varepsilon \to 0, \tag{2.26}$$

where dist_V *is a non-symmetric Hausdorff distance in* V*, i.e.,*

$$\operatorname{dist}_V(X,Y) = \sup_{x\in X}\inf_{y\in Y}\|x-y\|_V. \tag{2.27}$$

For the proof of the theorem, we need the following simple lemma.

Lemma 2.2. *Let the function* f *satisfy assumptions* (2.1) *and* (2.2)*. Then, for every* $T \in \mathbb{R}$*, every* $u \in C([T,T+1]\times\overline{\Omega},\mathbb{R}^k)$*, every* $\phi \in \mathcal{H}(f)$*, and every* $l \in (1,\infty)$

$$\phi(t/\varepsilon, x, u(t,x)) \rightharpoonup \overline{f}(x,u(t,x)) \ as \ \varepsilon \to 0 \ weakly \ in \ \ L^l([T,T+1]\times\Omega). \tag{2.28}$$

Although the assertion of the lemma is more or less standard, for the reader's convenience, we give below a sketch of the proof. For simplicity, we consider only the case $T=0$ (the general case is analogous).

Let $\phi \in \mathcal{H}(f)$, $u \in C([0,1]\times\Omega)$, $l \in (1,\infty)$ and $\varepsilon_n \to 0$ be arbitrary. Then, due to assumption (0.4), the sequence of functions $\phi(t/\varepsilon_n, x, u(t,x))$ is uniformly bounded in $L^l([0,1]\times\Omega)$ and, consequently, without loss of generality we may assume that it is weakly convergent to some function

$F(t,x)$. Our task is to prove that $F(t,x) = \overline{f}(x,u(t,x))$. To this end, it is sufficient to verify that, for every $\theta \in C([0,1]\times\Omega)$,

$$\int_{x\in\Omega}\int_{t\in[0,1]}[\phi(t/\varepsilon_n,x,u(t,x))\theta(t,x)-\overline{f}(x,u(t,x))\theta(t,x)]\,dx\,dt\to 0 \quad (2.29)$$

as $\varepsilon_n \to 0$. For every $N\in\mathbb{N}$, let us approximate the functions $u(t)$ and $\theta(t)$ by piecewise continuous (with respect to t) ones such that:

$$u^N(t,x) = u(m/N,x) \;\text{ and }\; \theta^N(t,x) := \theta(m/N,x) \;\text{ for } t\in[m/N,(m+1)/N].$$

Then, due to the continuity of u and θ, we have

$$\|u-u^N\|_{L^\infty}\to 0 \text{ and } \|\theta-\theta^N\|_{L^\infty}\to 0 \text{ as } N\to\infty \quad (2.30)$$

and, consequently, (thanks to assumption (0.6) for ϕ and $\overline{f}$) for every $\delta>0$, we may choose $N=N(\delta)$ such that

$$\Big|\int_{x\in\Omega}\int_0^1[\phi(t/\varepsilon_n,x,u(t,x))\theta(t,x)-\overline{f}(x,u(t,x))\theta(t,x)]\,dt\,dx\Big| \le \delta \quad (2.31)$$
$$+\int_{x\in\Omega}\sum_{m=0}^{N-1}\Big|\int_{\frac{m}{N}}^{\frac{m+1}{N}}[\phi(\frac{t}{\varepsilon_n},x,u(\frac{m}{N},x))-\overline{f}(x,u(\frac{m}{N},x))]\,dt\Big|\cdot|\theta(\frac{m}{N},x)|\,dx.$$

After the variable change, $z=t/\varepsilon_n$, we derive that

$$\int_{m/N}^{(m+1)/N}[\phi(t/\varepsilon_n,x,u(m/N,x))-\overline{f}(x,u(m/N,x))]\,dt \quad (2.32)$$
$$=\frac{1}{N}(\varepsilon_nN)\int_{m/(\varepsilon_nN)}^{m/(\varepsilon_nN)+1/(\varepsilon_nN)}[\phi(z,x,u(m/N,x))-\overline{f}(x,u(m/N,x))]\,dt.$$

Applying estimate (2.21) for the function $\phi\in\mathcal{H}(f)$ with $T=1/(N\varepsilon_n)$ and $\tau=m/(\varepsilon_nN)$ in order to estimate the right-hand side of (2.32), we derive that

$$\Big\|\int_{m/N}^{(m+1)/N}[\phi(t/\varepsilon_n,x,u(m/N,x))-\overline{f}(x,u(m/N,x))]\,dt\Big\|_{L^\infty(\Omega)}\to 0 \quad (2.33)$$

as $\varepsilon_n\to 0$. Inserting this estimate to (2.31), we derive (2.29). Lemma 2.2 is proven.

Proof of the theorem. Recall that, due to estimates (2.13) and (2.24) and the compactness of the embedding $\Phi_{2p}\subset\Phi_p$, the family of attractors $\{\mathcal{A}^\varepsilon,\varepsilon\in\mathbb{R}\}$ is precompact in Φ_p. Consequently, it only remains to verify that, for every $\varepsilon_n\to 0$ and $u_n\in\mathcal{A}^{\varepsilon_n}$ such that $u_n\to\overline{u}$ in Φ_p, the limit point $\overline{u}$ belongs to $\mathcal{A}^0$ (see, e.g., [2]). Let us verify this assertion. Indeed, according to description (2.19), there exist the nonlinearities $\phi_n \in \mathcal{H}(f)$ and bounded complete solutions $u_n \in \mathcal{K}^{\varepsilon_n}_{\phi_n}$, i.e., $u_n(t)$, $t \in \mathbb{R}$, satisfying the equations

$$\partial_t u_n = a\Delta_x u_n - \phi_n(t/\varepsilon_n, x, u_n) + g \tag{2.34}$$

such that $u_n(0) = u_n$. Note that, thanks to (1.1) and (2.13), we have the uniform estimate

$$\|u_n\|_{W^{(1,2),p}([T,T+1]\times\Omega)} \le Q(\|g\|_{L^p}), \tag{2.35}$$

where Q is independent of n and $T \in \mathbb{R}$. Recall, that the spaces $W^{(1,2),p}([T, T+1] \times \Omega)$ are reflexive and, consequently (due to the Cantor's diagonal procedure), we may assume without loss of generality that there is a function $\overline{u}(t)$ such that

$$\|\overline{u}\|_{W^{(1,2),p}([T,T+1]\times\Omega)} \le Q(\|g\|_{L^p}) \tag{2.36}$$

and

$$u_n \rightharpoonup \overline{u} \text{ weakly in } W^{(1,2),p}([T, T+1] \times \Omega), \tag{2.37}$$

for every $T \in \mathbb{R}$. Note also that, due to the compactness of the embedding $W^{(1,2),p}([T, T+1] \times \Omega) \subset C([T, T+1] \times \Omega)$, convergence (2.37) implies the strong convergence in C:

$$u_n \to \overline{u} \text{ strongly in } C([T, T+1] \times \Omega), \tag{2.38}$$

for every $T \in \mathbb{R}$. Particularly, $\overline{u}(0) = \overline{u}$ and, consequently, it remains to prove that $\overline{u} \in \mathcal{K}^0_{\overline{f}}$ passing to the limit $n \to \infty$ in equations (2.34) (e.g., in the sense of distributions).

As usual, passing to the limit $n \to \infty$ in the linear terms of (2.34) is an immediate and the only problem is to pass to the limit in the non-linear term. To this end, we recall that the hull $\mathcal{H}(f)$ is compact in $C_b(\mathbb{R}, \mathcal{M})$ and, consequently, without loss of generality, we may assume that $\phi_n \to \phi \in \mathcal{H}(f)$ in this space, and introduce the following identity:

$$\begin{aligned} &\phi_n(t/\varepsilon_n, x, u_n(t,x)) - \overline{f}(x, \overline{u}(t,x)) \\ &= \Big[\phi_n(t/\varepsilon_n, x, u_n(t,x)) - \phi_n(t/\varepsilon_n, x, \overline{u}(t,x))\Big] \\ &\quad + \Big[\phi_n(t/\varepsilon_n, x, \overline{u}(t,x)) - \phi(t/\varepsilon_n, x, \overline{u}(t,x))\Big] \\ &\quad + \Big[\phi(t/\varepsilon_n, x, \overline{u}(t,x)) - \overline{f}(x, \overline{u}(t,x))\Big]. \end{aligned} \tag{2.39}$$

The first term in the right-hand side of (2.39) tends to zero (in $L^\infty([T, T+1] \times \Omega)$) due to convergence (2.38) and assumption (0.6). The second one is also tends to zero due to the convergence $\phi_n \to \phi$ in $C_b(\mathbb{R}, \mathcal{M})$. And finally, the third one tends to zero weakly in $L^p([T,T+1]\times\Omega)$ due to Lemma 2.2. Thus, the limit function $\overline{u}(t)$ satisfies limit equation (2.22) and, consequently, $\overline{u}\in\mathcal{A}^0$. Theorem 2.3 is proven. □

3. The averaging of individual solutions: qualitative aspects

In this Section, we reprove the classical Krylov-Bogolubov's averaging principle for the case of (0.1) and verify that the operators $U^\varepsilon_\phi(t,\tau):\Phi_p\to\Phi_p$, defined in (2.9) converge to $S^0_{t-\tau}$ as $\varepsilon\to0$ *uniformly* with respect to $\phi\in\mathcal{H}(f)$. This result will be essentially used in the next sections in order to clarify the structure of the attractors $\mathcal{A}^\varepsilon$ for small $\varepsilon>0$. We start with the following theorem.

Theorem 3.1. *Let the assumptions of Theorem 2.1 hold. Then, for every $R>0$, there exist a constant K_R and a function $\alpha_R(\varepsilon)$ such that $\alpha_R(\varepsilon)\to0$ as $\varepsilon\to0$ and*

$$\|U^\varepsilon_\phi(\tau+t,\tau)u_0-S^0_tu_0\|_{\Phi_p}\le e^{K_Rt}\alpha_R(\varepsilon) \tag{3.1}$$

uniformly with respect to $\phi\in\mathcal{H}(f)$, $u_0\in B_R$ and $\tau\in\mathbb{R}$.

For the proof of the theorem, we need the improved version of Lemma 2.2.

Lemma 3.1. *Let K_C and K_{L^l}, $1<l<\infty$, be arbitrary compact sets in $C([0,1]\times\Omega)$ and $L^l([0,1]\times\Omega)$ respectively. Then there, exists a function $\alpha(\varepsilon)$ (depending on K_C and K_{L^l}) such that $\alpha(\varepsilon)\to0$ as $\varepsilon\to0$ and*

$$\Big|\int_{[0,1]\times\Omega}[\phi(t/\varepsilon,x,u(t,x))-\overline{f}(x,u(t,x))]\theta(t,x)\,dt\,dx\Big|\le\alpha(\varepsilon), \tag{3.2}$$

for every $\phi\in\mathcal{H}(f)$, $u\in K_C$ and $\theta\in K_{L^l}$.

Proof of the lemma. Indeed, assume that the assertion is wrong. Then, there exist $\phi_n\to\phi\in\mathcal{H}(f)$, $u_n\to u\in K_C$, $\theta_n\to\theta\in K_{L^l}$, and $\varepsilon_n\to0$ such that

$$\Big|\int_{[0,1]\times\Omega}[\phi_n(t/\varepsilon_n,x,u_n(t,x))-\overline{f}(x,u_n(t,x))]\theta_n(t,x)\,dt\,dx\Big|\ge\alpha_0>0. \tag{3.3}$$

Passing to the limit $n\to\infty$ in (3.3) and using conditions (2.2) on $\phi\in\mathcal{H}(f)$, we derive that

$$\Big|\int_{[0,1]\times\Omega}[\phi(t/\varepsilon_n,x,u(t,x))-\overline{f}(x,u(t,x))]\theta(t,x)\,dt\,dx\Big|\ge\alpha_0/2,$$

for $n>>1$ large enough, which contradicts the assertion of Lemma 2.2. Lemma 3.1 is proven. □

Proof of the theorem. We first note that $T_\tau\phi \in \mathcal{H}(f)$, for every $\tau \in \mathbb{R}$ if $\phi \in \mathcal{H}(f)$, consequently, we may verify (3.1) for $\tau = 0$ only.

Let $u_0 \in B_R \subset \Phi_p$ and let $u_\varepsilon(t) := U^\varepsilon_\phi(t,0)u_0$, $u_0(t) := S^0_t u_0$ be the corresponding solutions of non-averaged and averaged equations respectively. We set $v_\varepsilon(t) := u_\varepsilon(t) - u_0(t)$. Then, this function obviously satisfies the equation

$$\begin{aligned} &\partial_t v_\varepsilon - a\Delta_x v_\varepsilon = h_\varepsilon(t,x) := -[\phi(t/\varepsilon, x, u_\varepsilon(t)) - \phi(t/\varepsilon, x, u_0(t))] \\ &- [\phi(t/\varepsilon, x, u_0(t)) - \overline{f}(x, u_0(t))], \quad v_\varepsilon\big|_{t=0} = 0. \end{aligned} \tag{3.4}$$

Multiplying now equation (3.4) by $v_\varepsilon(t)$, integrating over $x \in \Omega$ and using that ϕ'_u is uniformly bounded (thanks to assumption (0.6)), we obtain

$$\partial_t \|v_\varepsilon(t)\|^2_{L^2} - K_R\|v_\varepsilon(t)\|^2_{L^2} \le 2\left([\phi(t/\varepsilon, x, u_0(t)) - \overline{f}(x, u_0(t))], v_\varepsilon(t)\right). \tag{3.5}$$

Moreover, due to estimates (1.1) and (2.23), we have

$$\|u_0\|_{W^{(1,2),p}([T,T+1]\times\Omega)} + \|v_\varepsilon\|_{W^{(1,2),p}([T,T+1]\times\Omega)} \le Q_R \tag{3.6}$$

and, consequently (due to the compactness of embedding $W^{(1,2),p} \subset C$), the sets $\{u_0(t),\, u_0 \in B_R\}$ and $\{v_\varepsilon(t), u_0 \in B_R\}$ are compact in $C([T,T+1]\times\Omega)$ (uniformly with respect to $T \in \mathbb{R}_+$). Therefore, Lemma 3.1 implies the estimate

$$\left|\int_T^{T+1} \left([\phi(t/\varepsilon, x, u_0(t)) - \overline{f}(x, u_0(t))], v_\varepsilon(t)\right)\, dt\right| \le \alpha'_R(\varepsilon) \tag{3.7}$$

where α'_R is independent of $u_0 \in B_R$, $\phi \in \mathcal{H}(f)$, $T \in \mathbb{R}_+$ and tends to zero as $\varepsilon \to 0$. Inserting this estimate into the right-hand side of (3.5) and using the Gronwall's lemma, we derive that

$$\|v_\varepsilon(t)\|^2_{0,2} \le Ce^{K_R t}\alpha'_R(\varepsilon). \tag{3.8}$$

In order to deduce (3.1) from (3.8), we note that the function $h_\varepsilon(t,x)$ in the right-hand side of (3.4) is uniformly bounded in $L^\infty(\mathbb{R}_+\times\Omega)$ (thanks to (0.4) and (3.6)), consequently

$$\int_T^{T+1} \|h_\varepsilon(t)\|^{2p}_{L^{2p}}\, dt \le Q'_R \tag{3.9}$$

uniformly with respect to $\phi \in \mathcal{H}(f)$, $u_0 \in B_R$ and $T \in \mathbb{R}_+$. Applying now the L^{2p}-regularity theorem for heat equations to (3.4) and using (3.9) and that $v_\varepsilon(0) = 0$, we derive

$$\|v_\varepsilon(t)\|_{\Phi_{2p}} \le Q''_R. \tag{3.10}$$

The interpolation inequality

$$\|v_\varepsilon(t)\|_{\Phi_p} \le C\|v_\varepsilon(t)\|_{L^2}^{(1-\gamma)}\|v_\varepsilon(t)\|_{\Phi_{2p}}^{\gamma} \tag{3.11}$$

(with $\gamma = 1 - 1/(2p-1)$) together with (3.8) and (3.10) imply (3.1). Theorem 3.1 is proven. □

In the next sections we will need also the analogue of (3.1) for Frechet derivatives of $U_\phi^\varepsilon(t,\tau)$. To this end, we first formulate the standard result on the differentiability with respect to the initial values.

Theorem 3.2. *Let the assumptions of Theorem* 2.1 *hold. Then, the operator* $U_\phi^\varepsilon(\tau+t,\tau)$ *is Frechet differentiable with respect to the initial values* $u_0 \in \Phi_p$ *and its derivative* $D_{u_0}U_\phi^\varepsilon(\tau+t,\tau)(u_0) \in \mathcal{L}(\Phi_p,\Phi_p)$ *at point* $u_0 \in \Phi_p$ *can be calculated as follows:*

$$D_{u_0}U_\phi^\varepsilon(t+\tau,\tau)(u_0)\xi := v_\xi(t+\tau), \tag{3.12}$$

where $\xi \in \Phi_p$ *is an arbitrary vector and* $v_\xi(t)$ *is a solution of the equation of variations which corresponds to* (0.1)*:*

$$\partial_t v_\xi(t) = a\Delta_x v_\xi(t) - \phi'_u(t/\varepsilon, x, u(t))v_\xi(t), \quad v_\xi(\tau) = \xi, \tag{3.13}$$

where $u(t) := U_\phi^\varepsilon(t,\tau)u_0$*. Moreover, this derivative satisfies the following uniform estimates:*

$$\|D_{u_0}U_\phi^\varepsilon(\tau+t,\tau)(u_0)\|_{\mathcal{L}(\Phi_p,\Phi_p)} \le Q_R e^{K_R t}, \tag{3.14}$$

where Q_R *and* K_R *are independent of* $u_0 \in B_R$*,* $\varepsilon > 0$*,* $\phi \in \mathcal{H}(f)$ *and* $\tau \in \mathbb{R}$*, and*

$$\begin{aligned}&\|U_\phi^\varepsilon(t+\tau,\tau)u_0^1 - U_\phi^\varepsilon(t+\tau,\tau)u_0^2 - D_{u_0}U_\phi^\varepsilon(t+\tau,\tau)(u_0^1)(u_0^1-u_0^2)\|_{\Phi_p}\\ &\le C_R e^{K_R t}\|u_0^1-u_0^2\|_{\Phi_p}\|u_0^1-u_0^2\|_{L^p},\end{aligned} \tag{3.15}$$

where C_R *and* K_R *are independent of* $u_0^1, u_0^2 \in B_R$*,* $\varepsilon > 0$*,* $\phi \in \mathcal{H}(f)$ *and* $\tau \in \mathbb{R}$*.*

The derivation of estimates (3.14) and (3.15) are completely standard so we omit it here (see e.g. [2]). The fact that these estimates are uniform (with respect to ε and ϕ) follows from the fact that estimate (0.6) holds uniformly with respect to $\phi \in \mathcal{H}(f)$.

Corollary 3.1. *Let the assumptions of Theorem* 3.2 *hold. Then the Frechet derivative* $D_{u_0}U_\phi^\varepsilon(t+\tau,\tau)(u_0)$ *is Lipschitz continuous with respect to* u_0*. Moreover,*

$$\|D_{u_0}U_\phi^\varepsilon(t+\tau,\tau)(u_0^1) - D_{u_0}U_\phi^\varepsilon(t+\tau,\tau)(u_0^2)\|_{\mathcal{L}(\Phi_p,\Phi_p)} \le Q_R e^{K_R t}\|u_0^1-u_0^2\|_{\Phi_p} \tag{3.16}$$

where Q_R *and* K_R *are independent of* $u_0^1, u_0^2 \in B_R$, $\phi \in \mathcal{H}(f)$, $\varepsilon > 0$, *and* $\tau \in \mathbb{R}$.

Indeed, (3.16) is an immediate corollary of (3.15).

Remark 3.1. Estimates (3.13)–(3.16) remain true for $\varepsilon = 0$ as well if we define $U_\phi^0(t,\tau) := S_{t-\tau}^0$, where S_t^0 is a semigroup, generated by averaged problem (2.22).

We are now ready to formulate and prove the averaging principle for the Frechet derivatives.

Theorem 3.3. *Let the assumptions of Theorem* 2.1 *hold. Then, for every* $R > 0$, *there exist a constant* $K_R > 0$ *and a function* $\alpha_R(\varepsilon)$, *such that* $\alpha_R(\varepsilon) \to 0$ *as* $\varepsilon \to 0$ *and*

$$\|D_{u_0}U_\phi^\varepsilon(\tau+t,\tau)(u_0) - D_{u_0}S_t^0(u_0)\|_{\mathcal{L}(\Phi_p,\Phi_p)} \le e^{K_R t}\alpha_R(\varepsilon), \tag{3.17}$$

for all $\phi \in \mathcal{H}(f)$, $u_0 \in B_R$, *and* $\tau \in \mathbb{R}$.

Proof. As in Theorem 3.1, we may verify (3.17) for $\tau = 0$ only. Let $u_0 \in B_R \subset \Phi_p$ and $u_0(t) := S_t^0 u_0$, $u_\varepsilon(t) := U_\phi^\varepsilon(t,0)u_0$ be the corresponding solutions of averaged and non-averaged system respectively. Let also $\xi \in \Phi_p$ be an arbitrary vector and $v_\xi^\varepsilon(t) := D_{u_0}U_\phi^\varepsilon(t,0)(u_0)\xi$, $v_\xi^0(t) := D_{u_0}S_t^0(u_0)\xi$ be the corresponding solution of equation of variations, i.e.,

$$\begin{cases}\partial_t v_\xi^\varepsilon(t) - a\Delta_x v_\xi^\varepsilon(t) = -\phi_u'(t/\varepsilon, x, u_\varepsilon(t))v_\xi^\varepsilon(t), \quad v_\xi^\varepsilon(0) = \xi,\\ \partial_t v_\xi^0(t) - a\Delta_x v_\xi^0(t) = -\overline{f}_u'(x, u_0(t))v_\xi^0(t), \quad v_\xi^0(0) = \xi.\end{cases} \tag{3.18}$$

Without loss of generality, we may assume that $\|\xi\|_{\Phi_p} = 1$. Then, analogously to Theorem 1.2, we derive that

$$\|v_\xi^\varepsilon\|_{W^{(1,2),p}([T,T+1]\times\Omega)} + \|v_\xi^0\|_{W^{(1,2),p}([T,T+1]\times\Omega)} \le Q_R e^{K_R T}. \tag{3.19}$$

Let $w_\xi(t) := v_\xi^\varepsilon(t) - v_\xi^0(t)$. Then, this function obviously satisfies the equation

$$\partial_t w_\xi(t) - a\Delta_x w_\xi(t) = \overline{f}_u'(x, u_0(t))v_\xi^0(t) - \phi_u'(t/\varepsilon, x, u_\varepsilon(t))v_\xi^\varepsilon(t), \quad w_\xi(0) = 0. \tag{3.20}$$

In order to estimate the right-hand side of (3.20), we use the following identity:

$$\begin{aligned}&\phi_u'(t/\varepsilon, x, u_\varepsilon(t))v_\xi^\varepsilon(t) - \overline{f}(x, u_0(t))v_\xi^0(t)\\ &= [\phi_u'(t/\varepsilon, x, u_\varepsilon(t)) - \phi_u'(t/\varepsilon, x, u_0(t))]v_\xi^\varepsilon(t)\\ &+ \phi_u'(t/\varepsilon, x, u_0(t))w_\xi(t) + [\phi_u'(t/\varepsilon, x, u_0(t)) - \overline{f}_u'(x, u_0(t))]v_\xi^0(t).\end{aligned} \tag{3.21}$$

Multiplying (3.20) by $w_\xi(t)$, using (3.21) and estimating

$$|([\phi'_u(t/\varepsilon, x, u_\varepsilon(t)) - \phi'_u(t/\varepsilon, x, u_0(t))]v^\varepsilon_\xi(t), w_\xi(t))| \\ \le C_R\|u_\varepsilon(t) - u_0(t)\|_{L^\infty}\|v_\xi(t)\|_{L^\infty}(\|v^\varepsilon_\xi(t)\|_{L^\infty} + \|v^0_\xi(t)\|_{L^\infty}) \le Q_R e^{2K_R t}\alpha_R(\varepsilon) \tag{3.22}$$

(due to (3.1), (3.19) and the embedding $\Phi_p \subset C$), we derive that

$$\begin{aligned} &\partial_t\|w_\xi(t)\|^2_{L^2} - K_R\|w_\xi(t\|^2_{L^2} \le Q_R e^{2K_R t}\alpha_R(\varepsilon) \\ &+ 2\left([\phi'_u(t/\varepsilon, x, u_0(t)) - \overline{f}'_u(x, u_0(t))]v^0_\xi(t), w_\xi(t)\right). \end{aligned} \tag{3.23}$$

In order to estimate the last term in the right-hand side of (3.23) we note that (due to uniform estimate (3.19)) the set $\{v^0_\xi(t)w_\xi(t)e^{-2K_R t}, u_0 \in B_R\}$ is uniformly bounded in $W^{(1,2),p}([T, T+1] \times \Omega)$, $T \in \mathbb{R}$ and, consequently, (due to the compactness of the embedding $W^{(1,2),p} \subset C$) this set of functions is compact in $C([T, T+1] \times \Omega)$ (uniformly with respect to $T \in \mathbb{R}_+$). Thus, according to Lemma 3.1 (applied to the almost periodic function ϕ'_u instead of ϕ), we have

$$\begin{aligned} &\left|\int_T^{T+1}\left([\phi'_u(t/\varepsilon, x, u_0(t)) - \overline{f}'_u(x, u_0(t))]v^0_\xi(t), w_\xi(t)\right)\,dt\right| \\ &\le Q_R e^{2K_R T}\alpha'_R(\varepsilon) \end{aligned} \tag{3.24}$$

where $\alpha'(\varepsilon) \to 0$ as $\varepsilon \to 0$. Inserting this estimate to (3.23) and using the Gronwall's lemma, we finally obtain that

$$\|w_\xi(t)\|_{\Phi_p} \le Q_R e^{2K_R t}\alpha''_R(\varepsilon). \tag{3.25}$$

The derivation of (3.17) from (3.25) is the same as in Theorem 3.1, namely, applying the L^{2p}-regularity theorem for heat equations to relation (3.20) and using (3.19) and that $w_\xi(0) = 0$, we have

$$\|w_\xi(t)\|_{\Phi_{2p}} \le Q_R e^{K_R t}, \tag{3.26}$$

where the constants are independent of $\phi \in \mathcal{H}(f)$, $\varepsilon > 0$ and $u_0 \in B_R$. Interpolation inequality (3.11) together with (3.25) and (3.26) imply (3.17). Theorem 3.3 is proven. □

4. The averaging of individual solutions: quantitative aspects

In this section, we discuss the problem of finding the quantitative estimates for the difference between the corresponding solutions of averaged and non-averaged equations in terms of the parameter ε or (which is the same) of clarifying the rate of convergence of the function $\alpha_R(\varepsilon)$ to zero as $\varepsilon \to 0$. As usual, this problem is closely related with the existence of a bounded z-primitive for the almost periodic function $f(z,x,v) - \overline{f}(x,v)$ with zero mean.

Theorem 4.1. *Let the assumptions of Theorem* 2.1 *hold and let, in addition, the function $f(z,x,v) - \overline{f}(x,v)$ possess a z-primitive $F(z)$ which is almost-periodic with values in $\mathcal{M}$ ($F \in AP(\mathbb{R},\mathcal{M})$, where the space $\mathcal{M}$ is the same as in Section* 2), *i.e.,*

$$F'_z(z,x,v) = f(z,x,v) - \overline{f}(x,v). \tag{4.1}$$

Then

$$\|U^\varepsilon_\phi(t+\tau,\tau)u_0 - S^0_t u_0\|_{L^p} \le C_R \varepsilon^{1/2} e^{K_R t}, \tag{4.2}$$

where C_R and K_R are independent of $\varepsilon > 0$, $\phi \in \mathcal{H}(f)$, $u_0 \in B_R$, and $\tau \in \mathbb{R}$.

Proof. As before, we may assume without loss of generality that $\tau = 0$. Note also that, since F is almost periodic with values in $\mathcal{M}$, we have the uniform estimate:

$$|\mathcal{F}(z,x,v)| + |\mathcal{F}'_v(z,x,v)| + |\mathcal{F}''_{vv}(z,x,v)| \le Q(|v|), \tag{4.3}$$

where the monotonic function Q is independent of $\mathcal{F} \in \mathcal{H}(F)$, $z \in \mathbb{R}$, and $x \in \Omega$. Moreover, it is not difficult to verify passing to the limit in (4.1) that, for every $\phi \in \mathcal{H}(f)$, there exists $\mathcal{F} := \mathcal{F}_\phi \in \mathcal{H}(F)$ such that

$$\mathcal{F}'_z(z,x,v) = \phi(z,x,v) - \overline{f}(x,v). \tag{4.4}$$

Let now $u_0 \in B_R$ and $u_\varepsilon(t) := U^\varepsilon_\phi(t,0)u_0$, $u_0(t) := S^0_t u_0$ be the corresponding solutions of the non-averaged and averaged problem respectively. Then, the function $v_\varepsilon(t) := u_\varepsilon(t) - u_0(t)$ obviously satisfies equation (3.4). Multiplying the i-th equation of (3.4) by $v^i_\varepsilon(t)|v^i_\varepsilon(t)|^{p-2}$ integrating over $x \in \Omega$, $t \in [0,T]$ and taking a sum over $i = 1,\cdots,k$, we derive (analogously to (1.6), (1.12) and (3.5)) that

$$\begin{aligned} \|v_\varepsilon(T)\|^p_{L^p} \le K_R \int_0^T \|v_\varepsilon(t)\|^p_{L^p}\,dt \\ + \sum_{i=1}^k \int_0^T \left([\phi_i(t/\varepsilon,x,u_0(t)) - \overline{f}_i(x,u_0(t))], v^i_\varepsilon(t)|v^i_\varepsilon(t)|^{p-2}\right)\,dt. \end{aligned} \tag{4.5}$$

Integrating by parts in the integrals in the right-hand side of (4.5) and using assumption (4.4), we derive

$$\int_0^T \left([\phi_i(t/\varepsilon,x,u_0(t)) - \overline{f}_i(x,u_0(t))], v^i_\varepsilon(t)|v^i_\varepsilon(t)|^{p-2}\right)\,dt$$

$$= \varepsilon \left(\mathcal{F}_i(T/\varepsilon, x, u_0(T)), v_\varepsilon^i(T)|v_\varepsilon^i(T)|^{p-2}\right)$$
$$- (p-1)\varepsilon \int_0^T \left(\mathcal{F}_i(t/\varepsilon, x, u_0(t)), \partial_t v_\varepsilon^i(t)|v_\varepsilon^i(t)|^{p-2}\right)\, dt \tag{4.6}$$
$$- \varepsilon \int_0^T \left((\mathcal{F}_i)'_v(t/\varepsilon, x, u_0(t))\partial_t u_0(t), v_\varepsilon^i(t)|v_\varepsilon^i(t)|^{p-2}\right)\, dt.$$

Let us estimate every term in the right-hand side of (4.6). Indeed, it follows from (4.3), (2.23) and from the Hölder inequality that

$$\varepsilon| \left(\mathcal{F}_i(t/\varepsilon, x, u_0(T)), v_\varepsilon^i(T)|v_\varepsilon^i(T)|^{p-2}\right)| \tag{4.7}$$
$$\le C\varepsilon^p \|\mathcal{F}(T/\varepsilon, x, u_0(T))\|_{L^p}^p + 1/(2k)\|v_\varepsilon(T)\|_{L^p}^p \le Q_R\varepsilon^p + 1/(2k)\|v_\varepsilon(T)\|_{L^p}^p.$$

Analogously,

$$\varepsilon\Big| \int_0^T \left((F_i)'_v(t/\varepsilon, x, u_0(t))\partial_t u_0(t), v_\varepsilon^i(t)|v_\varepsilon^i(t)|^{p-2}\right)\, dt\Big| \tag{4.8}$$
$$\le C\varepsilon^p \int_0^T \|\mathcal{F}'_v(t/\varepsilon, x, u_0(t))\|_{L^\infty}^p \|\partial_t u_0(t)\|_{L^p}^p\, dt + \int_0^T \|v_\varepsilon(t)\|_{L^p}^p\, dt$$
$$\le Q_R T\varepsilon^p + \int_0^T \|v_\varepsilon(t)\|_{L^p}^p\, dt.$$

And finally,

$$\varepsilon\Big| \int_0^T \left(\mathcal{F}_i(t/\varepsilon, x, u_0(t)), \partial_t v_\varepsilon^i(t)|v_\varepsilon^i(t)|^{p-2}\right)\, dt\Big| \tag{4.9}$$
$$\le C\varepsilon^{p/2} \int_0^T \|\mathcal{F}(t/\varepsilon, x, u_0(t))\|_{L^p}^p \|\partial_t v_\varepsilon(t)\|_{L^p}^p\, dt + \int_0^T \|v_\varepsilon(t)\|_{L^p}^p\, dt$$
$$\le Q_R T\varepsilon^{p/2} + \int_0^T \|v_\varepsilon(t)\|_{L^p}^p\, dt.$$

Here we have used the fact that v_ε is bounded in $W^{(1,2),p}([T, T+1]\times\Omega)$. Inserting these estimates into the right-hand side of (4.6), we derive the inequality

$$\|v_\varepsilon(T)\|_{L^p}^p \le Q'_R T\varepsilon^{p/2} + K'_R \int_0^T \|v_\varepsilon(t)\|_{L^p}^p\, dt. \tag{4.10}$$

The Gronwall's lemma implies now estimate (4.2). Theorem 4.1 is proven. □

Corollary 4.1. *Let the assumptions of Theorem* 4.1 *hold. Then*

$$\|U_\phi^\varepsilon(t+\tau, \tau)u_0 - S_t^0 u_0\|_{\Phi_p} \le C_R \varepsilon^{1/2(2p-1)} e^{K_R t}, \tag{4.11}$$

where C_R *and* K_R *are independent of* $\varepsilon > 0$, $\phi \in \mathcal{H}(f)$, $u_0 \in B_R$, *and* $\tau \in \mathbb{R}$.

Indeed, (4.11) is an immediate corollary of (4.2) and interpolation inequality (3.11).

Let us now discuss the assumptions of the proved theorem. We first note that in the case where $f(z,x,v)$ is *periodic* with respect to z, assumption (4.1) is automatically satisfied. Indeed, since the *periodic* function $f(z,x,v) - \overline{f}(x,v)$ has a zero mean, a function

$$F(z,x,v) := \int_0^z [f(z,x,v) - \overline{f}(x,v)]\,dz \tag{4.12}$$

will be bounded, continuous and periodic with respect to z. However, for general cases where $f(z,x,v)$ is almost-periodic (or quasi-periodic) with respect to z function (4.12) may be unbounded with respect to z (see e.g. [21]). Such strange behaviour of (4.12) is possible due to the fact that in contrast to the periodic case the almost-periodic function may have arbitrarily small Fourier modes and, consequently, small denominators may appear in (4.12) after the integration.

Thus, verifying whether or not integral (4.12) is bounded (or (which is the same) is almost-periodic) with respect to z is a rather complicated problem (see [21] for further examples and explanations). That is why, we give below only several sufficient conditions, which are formulated in terms of Fourier amplitudes and Fourier modes of the function $f(z,x,v)$. Recall (see [21]), that an almost-periodic function $f(z)$ can be expanded into Fourier series:

$$f(z,x,v) \sim \sum_{k\in\mathbb{Z}} A_{\omega_k}(x,v)e^{iw_k z}, \tag{4.13}$$

where $\{\omega_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ are Fourier modes and $A_{\omega_k} \in \mathcal{M}$ are the corresponding Fourier amplitudes which can be found by the following formula:

$$A_k(x,v) := \lim_{T\to\infty} \frac{1}{T}\int_0^T f(z,x,v)e^{-i\omega_k z}\,dz. \tag{4.14}$$

Particularly, $A_0(x,v) \equiv \overline{f}(x,v)$.

Proposition 4.1. *Assume that, for every* $R > 0$

$$\sum_{k\in\mathbb{Z},\,\omega_k\neq 0} (1+|\omega_k|^{-1})\|A_{\omega_k}\|_R < Q_R < \infty \tag{4.15}$$

(*the norm* $\|\cdot\|_R$ *is defined by* (2.3)). *Then, function* (4.12) *is bounded and almost-periodic with respect to* z (*with values in* $\mathcal{M}$) *and, consequently, satisfies the assumptions of Theorem* 4.1.

Proof. Indeed, in this case, the function f is obviously represented by uniformly convergent series (4.13) and, consequently,

$$F(z,x,v) = \sum_{k\in\mathbb{Z},\,\omega_k\neq 0} A_{\omega_k}(x,v)\frac{1}{\omega\,k}\left(e^{i\omega_k z}-1\right). \tag{4.15'}$$

Condition (4.15) guarantees the uniform convergence of (4.15′) and the estimate

$$\|F(z,x,v)\|_R \le 2Q_R.$$

Proposition 4.1 is proven. □

Let us now consider the particular case where f has only finite number m of rationally independent Fourier modes (i.e., f is quasiperiodic with respect to z). In this case, there exists a vector $\omega = (\omega^1,\cdots,\omega^m)\in\mathbb{R}^m$ of rationally independent modes such that every mode ω_k can be represented in the following form:

$$\omega_k = (\omega, l_k) := \sum_{i=1}^m \omega^i l_k^i,\quad l_k\in\mathbb{Z}^m. \tag{4.16}$$

It is well known (see, e.g., [21]) that (4.16) implies existence of a function $G\in C(\mathbb{T}^m,\mathcal{M})$ (where $\mathbb{T}^m := \mathbb{R}^m/\mathbb{Z}^m$ is an m-dimensional torus), such that

$$f(z,x,v) = G(\omega^1 z,\cdots,\omega^m z;x,v),\quad G(\phi;x,v)\sim\sum_{l\in\mathbb{Z}^m} A_l(x,v)e^{i(\phi,l)} \tag{4.17}$$

with $A_l := A_{(\omega,l)}$ in notations of (4.13).

Proposition 4.2. *Assume that the vector of frequencies $\omega\in\mathbb{R}^m$ satisfies the following Diophantine condition:*

$$|(\omega,l)| \ge C_\omega |l|^{1-m-\delta}, \tag{4.18}$$

for some positive $\delta<1$. Assume also that the function G in (4.17) *is sufficiently smooth with respect to ϕ, namely,*

$$G\in C^{2m}(\mathbb{T}^m,\mathcal{M}). \tag{4.19}$$

Then the function f, defined by (4.17) *has a quasi-periodic primitive F which satisfies condition* (4.1) *of Theorem* 4.1.

Proof. Indeed, assumption (4.19) implies that

$$\|A_l\|_R \le Q_R/|l|^{2m} \tag{4.20}$$

where Q_R is independent of l. Consequently, according to (4.18) and (4.20),

$$\sum_{k\in\mathbb{Z},\omega_k\neq 0}\left(1+|w_k|^{-1}\right)\|A_{\omega_k}\|_R \le \sum_{l\in\mathbb{Z}^m,l\neq 0}\left(1+|(\omega,l)|^{-1}\right)\|A_l\|_R$$
$$\le Q_R\sum_{l\in\mathbb{Z}^m,l\neq 0}(1+C_\omega|l|^{m-1+\delta})|l|^{-2m} \le Q'_R\sum_{l\in\mathbb{Z}^m,l\neq 0}|l|^{-m-1+\delta}<\infty.$$

Proposition 4.2 is proven. □

Remark 4.1. Note that the set of frequencies $\omega\in\mathbb{R}^m$ for which Diophantine condition (4.18) is violated has a zero Lebesgue measure (see, e.g., [1]). Thus, in a sense a bounded primitive exists for almost all sufficiently regular ((4.19) is assumed to be valid) quasiperiodic functions with zero mean.

Recall that Theorem 4.1 implies that the upper bound for difference between the corresponding solutions of non-averaged and averaged equations in L^p has at least an order $\varepsilon^{1/2}$ with respect to ε. In order to obtain the lower bounds we consider the following simplest example:

$$y'(t)+y(t)=\sin\frac{t}{\varepsilon},\quad y\in\mathbb{R},\quad y(0)=0 \tag{4.21}$$

which has a solution

$$y_\varepsilon(t)=\frac{\varepsilon}{1+\varepsilon^2}\big(-\sin\frac{t}{\varepsilon}+\varepsilon\cos\frac{t}{\varepsilon}-\varepsilon e^{-t}\big). \tag{4.22}$$

The corresponding solution of the averaged equation is obviously $y_0(t)\equiv 0$ and, consequently, the lower bound for the rate of convergence is ε^1 with respect to ε. The next theorem shows that it is possible to obtain the upper bound with the sharp rate of convergence ε^1 with respect to ε if the nonlinearity f satisfies the assumptions of Proposition 4.1.

Theorem 4.2. *Let the assumptions of Theorem* 2.1 *hold and let, in addition, the non-linear function f satisfy condition* (4.15)*. Then*

$$\|U^\varepsilon_\phi(t+\tau,\tau)u_0-S^0_t u_0\|_{L^p(\Omega)}\le C_R e^{K_R t}\,\varepsilon, \tag{4.23}$$

where C_R and K_R are independent of $\varepsilon>0$, $\phi\in\mathcal{H}(f)$, $u_0\in B_R$, and $\tau\in\mathbb{R}$.

Proof. As before, without loss of generality, we may assume that $\tau=0$. Note also that the Fourier modes ω_k and the modulus $|A_{\omega_k}(x,v)|$ of Fourier amplitudes are the same for every $\phi\in\mathcal{H}(f)$ and, consequently, assumption (4.15) is satisfied uniformly with respect to $\phi\in\mathcal{H}(f)$. Thus, without loss of generality, we may also assume $\phi=f$.

Let $u_0 \in B_R$ and $u_\varepsilon(t) := U_f^\varepsilon(t,0)u_0$, $u_0(t) := S_t^0 u_0$ be the corresponding solutions of the non-averaged and averaged equations. Then, a function $v_\varepsilon(t) := u_\varepsilon(t) - u_0(t)$ obviously satisfies the equation

$$\begin{aligned}\partial_t v_\varepsilon(t) - a\Delta_x v_\varepsilon(t) = -[f(t/\varepsilon, x, u_\varepsilon(t)) - f(t/\varepsilon, x, u_0(t))]\\ -[f(t/\varepsilon, x, u_0(t)) - \overline{f}(x, u_0(t))]\,, \quad v_\varepsilon(0) = 0.\end{aligned} \tag{4.24}$$

We now introduce a function $w_\varepsilon(t)$ as a solution of an auxiliary equation

$$\partial_t w_\varepsilon(t) - a\Delta_x w_\varepsilon(t) = h_\varepsilon(t) := [f(t/\varepsilon, x, u_0(t))) - \overline{f}(x, u_0(t))], \quad w_\varepsilon(0) = 0. \tag{4.25}$$

In order to solve (4.25), we expand $h_\varepsilon(t)$ in Fourier series (4.13)

$$h_\varepsilon(t,x) = \sum_{\omega_k \neq 0} A_{\omega_k}(x, u_0(t,x)) e^{i\omega_k t/\varepsilon}. \tag{4.25$'$}$$

The uniform convergence of (4.25′) is guaranteed by assumption (4.15). Let us introduce a function

$$\theta_\varepsilon(t,x) := \sum_{\omega_k \neq 0} e^{i\omega_k t/\varepsilon} \left(-i\frac{\omega_k}{\varepsilon} - a\Delta_x\right)^{-1} A_k(x, u_0(t,x)). \tag{4.26}$$

We note that (4.26) satisfies (4.25) if $u_0 \equiv u_0(x)$ is independent of t. In order to estimate (4.26), we recall that the operator $-a\Delta_x$ generates an analytic semigroup in $L^p(\Omega)$ (see, e.g., [16]). Therefore, the following estimate holds for its resolvent

$$|\lambda| \cdot \|(\lambda - a\Delta_x)^{-1}\|_{L^p \to L^p} \le C_p, \quad \forall \lambda \in i\mathbb{R}. \tag{4.27}$$

Particularly, (4.27) implies that

$$\left\| \left(-i\frac{\omega_k}{\varepsilon} - a\Delta_x\right)^{-1} \right\|_{L^p \to L^p} \le C_p \frac{\varepsilon}{|\omega_k|} \tag{4.28}$$

and, consequently, due to (4.15) and (2.23),

$$\|\theta_\varepsilon(t)\|_{L^p} \le C_p \varepsilon \sum_{\omega_k \neq 0} |\omega_k|^{-1} \|A_{\omega_k}(x, u_0(t,x))\|_{L^p} \le Q'_R\, \varepsilon, \tag{4.29}$$

for the appropriate Q'_R. Let now $W_\varepsilon(t) := w_\varepsilon(t) - \theta_\varepsilon(t)$. Then, this function satisfies the equation

$$\partial_t W_\varepsilon(t) - a\Delta_x W_\varepsilon(t) = H_\varepsilon(t), \quad W_\varepsilon(0) = -\theta_\varepsilon(0) \tag{4.30}$$

where

$$H_\varepsilon(t) := -\sum_{\omega_k\neq 0} e^{i\omega_k t/\varepsilon}\left(-i\frac{\omega_k}{\varepsilon} - a\Delta_x\right)^{-1}\big[(A_k)'_v(x,u_0(t,x))\partial_t u_0(t,x)\big]. \tag{4.31}$$

Using now (4.28), (4.15) and (2.23) and arguing as before, we derive the estimate

$$\|H_\varepsilon\|_{L^p([T,T+1]\times\Omega)} \le Q'_R\,\varepsilon\|\partial_t u_0\|_{L^p([T,T+1]\omega)} \le Q''_R\,\varepsilon, \tag{4.32}$$

where Q''_R is independent of T. Having estimates (4.29) and (4.32), we may deduce from (4.30) (multiplying the ith equation by $W^i_\varepsilon|W^i_\varepsilon|^{p-2}$ and arguing analogously to (1.6)) that

$$\|W_\varepsilon(t)\|_{L^p} \le C_R\,\varepsilon, \tag{4.33}$$

for some $C_R > 0$. Combining (4.29) and (4.33), we finally deduce that

$$\|w_\varepsilon(t)\|_{L^p} \le C'_R\,\varepsilon. \tag{4.34}$$

We are now ready to finish the proof of the theorem. To this end, we introduce one more function $V_\varepsilon(t) := v_\varepsilon(t) + w_\varepsilon(t)$ which obviously satisfies the equation

$$\partial_t V_\varepsilon(t) - a\Delta_x V_\varepsilon(t) + l_\varepsilon(t)V_\varepsilon(t) = l_\varepsilon(t)w_\varepsilon(t), \quad V_\varepsilon(0) = 0, \tag{4.35}$$

where

$$l_\varepsilon(t,x) := \int_0^1 f'_v(t/\varepsilon, x, su_\varepsilon(t) + (1-s)u_0(t))\,ds.$$

Recall that, due to (0.6), (1.1) and (2.23), we have $\|l_\varepsilon(t,x)\|_{L^\infty} \le Q_R$, consequently, (4.34) implies the estimate

$$\|l_\varepsilon w_\varepsilon\|_{L^p([T,T+1]\times\Omega)} \le C''_R\,\varepsilon. \tag{4.36}$$

Using (4.36) and arguing as in (1.6), we derive from (4.35) the following estimate:

$$\|V_\varepsilon(t)\|_{L^p} \le C_R e^{K_R t}\,\varepsilon, \tag{4.37}$$

for the appropriate constants C_R and Q_R. This estimate together with (4.34) imply (4.23). Theorem 4.2 is proven. □

Remark 4.2. Note that, arguing as in the proof of Theorem 4.2, we may verify that (4.23) remains true with the exponent p replaced by any *finite* $r \ge p$.

Remark 4.3. Note that the constant K_R in (4.23) depends only on constants and functions introduced in (0.4)–(0.6) and is independent of frequencies ω_l. But the constant C_R in (4.23) depends on sum (4.15) (consequently, depends on small denominators and is extremely sensitive to small perturbations of ω_l) in the following way:

$$C_R \le C'_R \sum_{k\in\mathbb{Z},\,\omega_k\neq 0} (1+|\omega_k|^{-1})\|A_{\omega_k}\|_R, \tag{4.38}$$

where C'_R is already independent of ω_l. Moreover, if (instead of convergence (4.15)) we only have that

$$\sum_{k\in\mathbb{Z},\,\omega_k\neq 0} (1+|\omega_k|^{-1+\delta})\|A_{\omega_k}\|_R < Q'_R < \infty, \tag{4.39}$$

for some $0 \le \delta < 1$, then arguing as in the proof of Theorem 4.3, but using instead of (4.28) a weaker estimate

$$\left\| \left(-i\frac{\omega_k}{\varepsilon} - a\Delta_x\right)^{-1} \right\|_{L^p\to L^p} \le C_p \left(\frac{\varepsilon}{|\omega_k|}\right)^{1-\delta}, \tag{4.40}$$

we derive the following analogue of (4.23):

$$\|U^\varepsilon_\phi(t+\tau,\tau)u_0 - S^0_t u_0\|_{L^p(\Omega)} \le L_R e^{K_R t}\,\varepsilon^{1-\delta}, \tag{4.41}$$

where the constant K_R is also independent of ω_l and the constant L_R satisfies the following analogue of (4.38):

$$L_R \le C''_R \sum_{k\in\mathbb{Z},\,\omega_k\neq 0} (1+|\omega_k|^{-1+\delta})\|A_{\omega_k}\|_R, \tag{4.42}$$

where C''_R is already independent of ω_l. We will essentially use this simple observation in Section 7 in order to obtain the probabilistic interpretation for the quantitative averaging of regular attractors.

5. The averaging near a hyperbolic equilibrium

This Section is devoted to the detailed study of the behaviour of nonaveraged system (0.1) in a neighbourhood of a hyperbolic equilibrium of averaged system (2.22). To be more precise, we assume that there is $z_0 = z_0(x) \in \Phi_p$ which satisfies the equation

$$a\Delta_x z_0 - \overline{f}(x,z_0) = g, \quad z_0\big|_{\partial\Omega} = 0. \tag{5.1}$$

We consider the equation of variation which corresponds to the equilibrium z_0 (see Theorem 3.2):

$$\partial_t w = \mathcal{L}_{z_0} w := a\Delta_x w - \overline{f}'_v(x,z_0)w, \quad w(t)\in\Phi_p,\ w(0)=\xi. \tag{5.2}$$

It is well known that the resolvent of the operator $\mathcal{L}_{z_0}$ is compact in $L^p(\Omega)$ (particularly, the spectrum of $\mathcal{L}_{z_0}$ is discrete) and generates an analytic semigroup in $L^p(\Omega)$

$$D_{v_0} S^0_t(z_0)\xi = e^{\mathcal{L}_{z_0} t}\xi. \tag{5.3}$$

Our main assumption for this section is that the equilibrium z_0 is hyperbolic, i.e., the spectrum of $\mathcal{L}_{z_0}$ does not intersect the imaginary axis:

$$\sigma(\mathcal{L}_{z_0}) \cap i\mathbb{R} = \varnothing. \tag{5.4}$$

Let $\Pi_- : L^p \to L^p$ and $\Pi_+ : L^p \to L^p$ be the spectral projectors which correspond to the stable ($\{\operatorname{Re}\lambda < 0\}$) and to the unstable ($\{\operatorname{Re}\lambda > 0\}$) parts of the spectrum of $\mathcal{L}_{z_0}$ respectively and let $V^- := \Pi_- L^p$ and $V^+ := \Pi_+ L^p$ be the corresponding spectral subspaces (see [16]). Then, the subspaces $V^\pm$ are invariant with respect to $e^{\mathcal{L}_{z_0} t}$:

$$e^{\mathcal{L}_{z_0} t} V^\pm = V^\pm, \quad L^p = V^+ + V^-, \quad V^+ \cap V^- = \{0\} \tag{5.5}$$

and, moreover, the space V^+ is finite dimensional $\kappa(z_0) := \dim V^+ < \infty$ (recall that $\kappa(z_0)$ is called the unstable index of the equilibrium z_0). The following classical result describes the behaviour of a linearized system near the hyperbolic equilibrium.

Proposition 5.1. *Let $z_0 \in \Phi_p$ be a hyperbolic equilibrium. Then, there exist $C > 0$ and $\alpha > 0$ such that*

$$\begin{cases} \|e^{\mathcal{L}_{z_0} t}\xi_-\|_{L^p} \le Ce^{-\alpha t}\|\xi_-\|_{L^p}, & \forall \xi_- \in V^-, \\ \|e^{\mathcal{L}_{z_0} t}\xi_+\|_{L^p} \ge C^{-1}e^{\alpha t}\|\xi_+\|_{L^p}, & \forall \xi_+ \in V^+. \end{cases} \tag{5.6}$$

Moreover, if $\xi_+ \in \Phi_p \cap V^+$ (resp. $\xi_- \in \Phi_p \cap V^-$), then estimates (5.6) *remain valid with L^p replaced by Φ_p.*

For the proof of the proposition, see, e.g. [16].

The main task of this section is to obtain the description, analogous to (5.6), for non-averaged system (0.1) for sufficiently small ε in a small (but independent of ε) neighbourhood of z_0 (surely, the linear subspaces $V^\pm$ will be replaced by the appropriate nonlinear manifolds). We start by constructing an almost-periodic solution of (0.1), which corresponds to the equilibrium z_0 as $\varepsilon = 0$.

Theorem 5.1. *Let the assumptions of Theorem* 2.1 *hold and let $z_0 \in \Phi_p$ be a hyperbolic equilibrium for averaged system* (2.22)*. Then, there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that, for every $\phi \in \mathcal{H}(f)$ and every $\varepsilon < \varepsilon_0$, equation* (2.8) *has a unique solution $u^\varepsilon_\phi(t)$, which is defined for all $t \in \mathbb{R}$ and belongs to the*

δ-neighbourhood of z_0 for all t, i.e., this solution is uniquely defined by the assumption

$$\sup_{t\in\mathbb{R}} \|u^\varepsilon_\phi(t) - z_0\|_{\Phi_p} \le \delta. \tag{5.7}$$

Moreover, this solution is almost-periodic with respect to t with the same frequency basis as $f(t/\varepsilon, x, v)$ and tends to z_0 as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0}\sup_{t\in\mathbb{R}} \|u^\varepsilon_\phi(t) - z_0\|_{\Phi_p} = 0. \tag{5.8}$$

Proof. In order to simplify the notations, we only consider below the case $\phi = f$, although all the estimates, obtained below, are uniform with respect to $\phi \in \mathcal{H}(f)$.

Assume that the solution $u(t) := u^\varepsilon_f(t)$ is already constructed. Then, it obviously satisfies the following relation:

$$u(m+1) = U^\varepsilon_f(m+1, m)u(m), \quad \forall m \in \mathbb{R}. \tag{5.9}$$

Conversely, if the function $u(m)$, $m \in \mathbb{Z}$ satisfies (5.9), then the function $u(\tau) := U^\varepsilon_f(\tau, m)u(m)$, $\tau \in [m, m+1]$ satisfies equation (0.1). Thus, instead of solving (0.1) it is sufficient to solve (5.9). Denote,

$$S^\varepsilon(\tau) := U^\varepsilon_f(\tau+1, \tau) := S^0_1 + K^\varepsilon(\tau). \tag{5.10}$$

Then, it follows from Theorems 3.1–3.3 that

$$\|K^\varepsilon(\tau)v_0\|_{\Phi_p} + \|D_{v_0}K^\varepsilon(\tau)(v_0)\|_{L(\Phi_p,\Phi_p)} \le \alpha_R(\varepsilon), \quad \|v_0\|_{\Phi_p} \le R, \tag{5.11}$$

where the function α_R is independent of τ and tends to 0 as $\varepsilon \to 0$. Moreover, due to Theorem 3.2, we have also the estimate

$$\|D_{v_0}K^\varepsilon(\tau)(v^1_0) - D_{v_0}K^\varepsilon(\tau)(v^2_0)\|_{L(\Phi_p,\Phi_p)} \le C_R\|v^1_0 - v^2_0\|_{\Phi_p}, \tag{5.12}$$

for $v^1_0, v^2_0 \in B_R$. Recall also that, due to Theorem 3.2, we have

$$\|S^0_1 v_0 - z_0\|_{\Phi_p} + \|D_{v_0}S^0_1(v_0) - \mathbb{L}\|_{L(\Phi_p,\Phi_p)} \le C_R\|v_0 - z_0\|_{\Phi_p}, \tag{5.13}$$

for $\|v_0\| \le R$ and $\mathbb{L} := D_{v_0}S^0_1 = e^{\mathcal{L}_{z_0}}$.

Estimates (5.11)–(5.13) allow to apply the implicit function theorem to relation (5.9). Indeed, let

$$\mathcal{R}(\Phi_p) := C_b(\mathbb{Z}, \Phi_p) \tag{5.14}$$

be the space of all bounded sequences with values in Φ_p. Consider a function

$$\mathcal{F} : \mathcal{R}(\Phi_b) \times \mathbb{R}_+ \to \mathcal{R}(\Phi_b), \quad \mathcal{F}(w,\varepsilon)_m := w_m - S^1_0 w_{m-1} - K^\varepsilon(m-1)v_{m-1}. \tag{5.15}$$

It follows from (5.10)–(5.14) that $\mathcal{F}(z_0, 0) = 0$, that $\mathcal{F}$ is Frechet differentiable with respect to the first argument and its derivative $D_w\mathcal{F}(w, \varepsilon)$ is continuous at $(w, \varepsilon) = (z_0, 0)$. Thus, it only remains to verify that the linear operator

$$D_w\mathcal{F}(0,0) : \mathcal{R}(\Phi_p) \to \mathcal{R}(\Phi_p), \quad [D_w\mathcal{F}(0,0)w]_m = w_m - \mathbb{L}w_{m-1} \tag{5.16}$$

is invertible. The following standard lemma shows that this is an immediate corollary of the fact that z_0 is hyperbolic.

Lemma 5.1. *Let z_0 be a hyperbolic equilibrium. Then, operator* (5.16) *is invertible in $\mathcal{R}(\Phi_p)$ and in $\mathcal{R}(L^p)$, i.e., for every $h \in \mathcal{R}(\Phi_p)$ (resp. $h \in \mathcal{R}(L^p)$), the equation*

$$v_m = \mathbb{L}v_{m-1} + h_{m-1} \tag{5.17}$$

has a unique bounded solution $v := \mathbb{T}h \in \mathcal{R}(\Phi_p)$ (resp. $v \in \mathcal{R}(L^p)$). Moreover, the solving operator $\mathbb{T}$ satisfies the following estimate:

$$\|\mathbb{T}\|_{\mathcal{R}(\Phi_p)\to\mathcal{R}(\Phi_p)} + \|\mathbb{T}\|_{\mathcal{R}(L^p)\to\mathcal{R}(L^p)} \le C. \tag{5.18}$$

Proof. Indeed, let $v_m^\pm := \Pi_\pm v_m$, $h_m^\pm := \Pi_\pm h_m$, and $\mathbb{L}^\pm := \Pi_\pm\mathbb{L}$. Then, obviously,

$$v_m^\pm = \mathbb{L}v_{m-1}^\pm + h_{m-1}^\pm. \tag{5.19}$$

It now follows from Proposition 5.1 that a unique bounded solution of (5.19) is defined by the following expression:

$$v_m^+ := -\sum_{l=m}^{+\infty} \left(\mathbb{L}^+\right)^{m-1-l} h_l^+, \quad v_m^- := \sum_{l=-\infty}^{m} \left(\mathbb{L}^-\right)^{m-1-l} h_l^-. \tag{5.20}$$

The convergence of series (5.20) and estimate (5.18) are immediate corollaries of estimates (5.6). Lemma 5.1 is proven. □

We are now ready to finish the proof of the theorem. Indeed, all conditions of the implicit function theorem is verified for function (5.15) and, consequently, there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ in the δ-neighbourhood of z_0, there exists a unique function $u^\varepsilon \in \mathcal{R}(\Phi_p)$ such that

$$\mathcal{F}(u^\varepsilon, \varepsilon) \equiv 0, \quad , \|u^\varepsilon - z_0\|_{\mathcal{R}(\Phi_p)} \le \delta \tag{5.21}$$

or (which is the same) $u^\varepsilon(m)$, $m \in \mathbb{Z}$ satisfies (5.9). Moreover, $u^\varepsilon \to z_0$ in $\mathcal{R}(\Phi_p)$ as $\varepsilon \to 0$. Thus, the existence of a bounded solution $u^\varepsilon(m) = u_f^\varepsilon(m)$ of equation (5.9) is proven. Analogously, we can establish the existence of the solution $u_\phi^\varepsilon(m)$ for every nonlinearity ϕ belonging to the hull $\mathcal{H}(f)$ of the initial nonlinearity f. Moreover, it can be easily proven using the explicit construction of u^ε given above that $u^\varepsilon_\phi(m)$ depends continuously on $\phi \in \mathcal{H}(f)$. The desired solution $u^\varepsilon_\phi(t)$, $t \in \mathbb{R}$, can be now defined as follows:

$$u^\varepsilon_\phi(t) := u^\varepsilon_{T_{t/\varepsilon}\phi}(0).$$

The almost-periodicity of this solution follows from the fact that the function $\phi \to u^\varepsilon_\phi(0)$ is continuous and that the flow $T_{t/\varepsilon} : \mathcal{H}(f) \to \mathcal{H}(f)$ is almost-periodic (since f is assumed to be almost-periodic, see [21] for the details). Theorem 5.1 is proven. □

Remark 5.1. The results similar to the proved above are often referred as 'global averaging principle' or 'the second Bogolubov's theorem'.

The next corollary gives the quantitative estimate for the rate of convergence in (5.8) under the assumptions of Section 4.

Corollary 5.1. *Let the assumptions of Theorems* 5.1 *and* 4.1 *hold. Then*

$$\sup_{t\in\mathbb{R}} \|u^\varepsilon_\phi(t) - z_0\|_{L^p} \le C\varepsilon^{1/2}, \tag{5.22}$$

where C is independent of $\phi \in \mathcal{H}(f)$. Moreover, let, in addition, the assumptions of Theorem 4.3 *hold. Then the improved version of* (5.22) *is valid*:

$$\sup_{t\in\mathbb{R}} \|u^\varepsilon_\phi(t) - z_0\|_{L^p} \le C'\varepsilon, \tag{5.23}$$

where C' is independent of $\varepsilon \le \varepsilon_0$ and $\phi \in \mathcal{H}(f)$.

Proof. As in the proof of Theorem 5.1 we only consider the case $\phi = f$, moreover, we only derive below estimate (5.23). Estimate (5.22) is completely analogous only instead of (4.23) one should use (4.2).

Note that the function $u^\varepsilon(m) := u^\varepsilon_f(m)$, $m \in \mathbb{Z}$ satisfies the relation

$$\begin{aligned} u^\varepsilon(m) - z_0 = \mathbb{L}(u_\varepsilon(m-1) - z_0) + [S^0_1(u^\varepsilon(m-1) - S^0_1(z_0) \\ - D_{v_0}S^0_1(z_0)(u^\varepsilon(m-1) - z_0)] + K^\varepsilon(m-1)(u^\varepsilon(m-1)) \end{aligned} \tag{5.24}$$

(see (5.9)–(5.13)). According to (4.23), we have

$$\|K^\varepsilon(m-1)(u^\varepsilon(m-1))\|_{L^p} \le C\varepsilon. \tag{5.25}$$

Moreover, due to Theorem 3.2, the second term in the right-hand side of (5.24) can be estimated as follows:

$$\begin{aligned} \|S^0_1(u^\varepsilon(m-1) - S^0_1(z_0) - D_{v_0}S^0_1(z_0)(u^\varepsilon(m-1) - z_0)\|_{L^p} \\ \le C\|u^\varepsilon - z_0\|_{\mathcal{R}(\Phi_p)}\|u^\varepsilon - z_0\|_{\mathcal{R}(L^p)}. \end{aligned} \tag{5.26}$$

Applying the operator $\mathbb{T}$, defined in Lemma 5.1 to both sides of (5.24) and using estimates (5.18), (5.25) and (5.26), we derive that

$$\|u^\varepsilon - z_0\|_{\mathcal{R}(L^p)} \le C_1\|u^\varepsilon - z_0\|_{\mathcal{R}(\Phi_p)}\|u^\varepsilon - z_0\|_{\mathcal{R}(L^p)} + C_2\varepsilon. \tag{5.27}$$

Recall that, according to Theorem 5.1, $\|u^\varepsilon - z_0\|_{\mathcal{R}(\Phi_p)} \to 0$ as $\varepsilon \to 0$. Consequently, for a sufficiently small $\varepsilon > 0$, (5.27) implies the estimate

$$\|u^\varepsilon - z_0\|_{\mathcal{R}(L^p)} \le 2C_2\varepsilon.$$

Corollary 5.1 is proven. □

Introduce now the perturbed unstable sets $M^+_{z_0,\delta}(\phi,\varepsilon)$, $\phi \in \mathcal{H}(f)$, of the equilibrium z_0 by the following standard expression:

$$\begin{aligned} M^+_{z_0,\delta}(\phi,\varepsilon) := \{u(0)\,:\, &\exists u(t),\, t\le 0 \text{ is a backward solution}\\ &\text{of problem (2.8), such that } \|u(t)-z_0\|_{\Phi_p} \le \delta,\ t \le 0\}. \end{aligned} \tag{5.28}$$

The following theorem shows that sets (5.28) are smooth finite dimensional submanifolds of Φ_p if δ and ε are small enough.

Theorem 5.2. *Let the assumptions of Theorem* 5.1 *hold. Then, there exist* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon < 0$ *and every* $\phi \in \mathcal{H}(f)$, *sets* (5.28) *are* $\kappa(z_0)$*-dimensional* C^1*-submanifolds, which are diffeomorphed to the unstable subspace* V^+*. As usual, these manifolds can be represented as graphs of* C^1 *functions* $\mathcal{N}_{\phi,\varepsilon} : V^+ \to V^-$ *:*

$$M^+_{z_0,\delta}(\phi,\varepsilon) = V_\delta(z_0) \cap \{u_0 := u^\varepsilon_\phi(0) + v^+_0 + v^-_0 \,:\, v^\pm_0 \in V^\pm,\ v^-_0 = \mathcal{N}_{\phi,\varepsilon}(v^+_0)\} \tag{5.29}$$

where $V_\delta(z_0)$ *is a* δ*-neighbourhood of* z_0 *in* Φ_p*, and functions* $\mathcal{N}_{\phi,\varepsilon}$ *are continuous with respect to* $\phi \in \mathcal{H}(f)$*, for every fixed* $\varepsilon > 0$*. Moreover, for every* $u_0 \in M^+_{z_0,\delta}(\phi,\varepsilon)$ *there exists a unique backward solution* $u(t) \in V_\delta(z_0)$*,* $u(0) = u_0$*, of problem* (2.8) *and this solution satisfies the estimate*

$$\|u(t) - u^\varepsilon_\phi(t)\|_{\Phi_p} \le Ce^{(\alpha/2)t}\|u(0) - u^\varepsilon_\phi(0)\|_{\Phi_p}, \quad t \le 0, \tag{5.30}$$

where $\alpha > 0$ *is defined in Proposition* 5.1, C *is independent of* ε*,* u_0 *and* ϕ*, and the solution* $u^\varepsilon_\phi(t)$ *is constructed in Theorem* 5.1.

Proof. As in the proof of Theorem 5.1, we only consider below the case $\phi = f$ (the continuity of the obtained manifolds with respect to $\phi \in \mathcal{H}(f)$ will be clear from the construction given below). Moreover, in order to construct manifold (5.28) we first prove that the set of all backward solutions of (0.1) which belong to $V_\delta(z_0)$ is in a fact a smooth manifold in the corresponding functional space. We seek for this backward solution in the form $u^\varepsilon_f(m) +$

$w(m)$, $m \in \mathbb{Z}_-$, where u_f^ε is defined in Theorem 5.1 and $w \in \mathcal{R}^-(\Phi_p) := C_b(\mathbb{Z}_-, \Phi_p)$ satisfies the following relation (compare with (5.24)):

$$\begin{aligned} w(m) &= \mathbb{L}w(m-1) \\ &+ [S_1^0(w(m-1) + u_f^\varepsilon(m-1)) - S_1^0(u_f^\varepsilon(m-1)) - D_{v_0}S_1^0(z_0)w(m-1)] \\ &+ [K^\varepsilon(m-1)(w(m-1) + u_f^\varepsilon(m-1)) - K^\varepsilon(m-1)(u_f^\varepsilon(m-1))], \end{aligned} \tag{5.31}$$

where $m = 0, -1, \dots$. To transform relation (5.31), we need the following space

$$\mathcal{R}_\beta^-(\Phi_p) := \{v \in \mathcal{R}^-(\Phi_p) \,:\, \sup_{m \in \mathbb{Z}^-} e^{-\beta m}\|w(m)\|_{\Phi_p} < \infty\}, \quad \beta \in \mathbb{R}$$

and simple analogue of Lemma 5.1 for the spaces $\mathcal{R}_\beta^-(\Phi_p)$.

Lemma 5.2. *Let the assumptions of Lemma* 5.1 *hold. Then, for every* $h \in \mathcal{R}_\beta^-(\Phi_p)$, $|\beta| \le \alpha/2$, *and for every* $v_0^+ \in V^+$ *the problem*

$$v(m) = \mathbb{L}v(m-1) + h(m-1), \quad \Pi_+ v(0) = v_0^+ \tag{5.32}$$

has a unique solution $v := \mathbb{I}v_0^+ + \mathbb{T}^- h$, *where* $\mathbb{I} : V^+ \to \mathcal{R}_\beta^-(\Phi_p)$ *is a bounded linear operator, which corresponds to the solution of* (5.32) *with* $h \equiv 0$, *and* $\mathbb{T}^- : \mathcal{R}_\beta^-(\Phi_p) \to \mathcal{R}_\beta^-(\Phi_p)$ *is a bounded linear operator, which corresponds to the solution of* (5.32) *with* $v_0^+ = 0$.

We left the rigorous proof of the lemma (which is based on estimates (5.6) and is completely analogous to the proof of Lemma 5.1) to the reader.

We are now ready to finish the proof of the theorem. Applying Lemma 5.2 to equation (5.31) and denoting the second and the third term in the right-hand side of (5.31) by $\mathbb{S}(w(m-1), \varepsilon)$ and $\mathbb{K}(w(m-1), \varepsilon)$ respectively, we derive the following relation for w:

$$w = \mathbb{I}v_0^+ + \mathbb{T}^-\mathbb{S}(w, \varepsilon) + \mathbb{T}^-\mathbb{K}(w, \varepsilon), \tag{5.33}$$

where $v_0^+ = \Pi_+ w(0)$. As in the proof of Theorem 5.1, we are going to apply the implicit function theorem to relation (5.33). Indeed, introduce a function

$$\mathcal{F} : \mathcal{R}^-(\Phi_p) \times V^+ \times \mathbb{R}_+ \to \mathcal{R}^-(\Phi_p)$$

by the relation

$$\mathcal{F}(w, v_0^+, \varepsilon) := w - \mathbb{I}v_0^+ - \mathbb{T}^-\mathbb{S}(w, \varepsilon) - \mathbb{T}^-\mathbb{K}(w, \varepsilon). \tag{5.34}$$

Using estimates (5.13), we derive that

$$\begin{aligned} \|\mathbb{S}(w(m), \varepsilon)\|_{\Phi_p} &\le C_1\|w(m)\|_{\Phi_p}^2 + C_2\|w(m)\|_{\Phi_p}\|u_f^\varepsilon(m) - z_0\|_{\Phi_p} \\ \|D_{v_0}\mathbb{S}(w(m), \varepsilon)\|_{\Phi_p \to \Phi_p} &\le C_3(\|w(m)\|_{\Phi_p} + \|u_f^\varepsilon(m) - z_0\|_{\Phi_p}), \end{aligned} \tag{5.35}$$

for the appropriate constants C_i which depend only on $\|w(m)\|_{\Phi_p}$. Analogously, it follows from (5.11) that

$$\|\mathbb{K}(w(m),\varepsilon)\|_{\Phi_p} + \|D_{v_0}\mathbb{K}(w(m),\varepsilon)\|_{\Phi_p\to\Phi_p}\|w(m)\|_{\Phi_p} \le \|w(m)\|_{\Phi_p}\alpha_R(\varepsilon). \tag{5.36}$$

Estimates (5.35)–(5.36) imply that $\mathcal{F}, D_w\mathcal{F}, D_{v_0^+}\mathcal{F}$ are continuous at $(0,0,0)$ and that $\mathcal{F}_w(0,0,0) = \mathrm{Id}$. Therefore, due to the implicit function theorem, there exists $\delta > 0$, $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ and every v_0 belonging to a sufficiently small neighbourhood $V(0)^+$ of zero in V^+, there exists a *unique* solution w of

$$\mathcal{F}(w, v_0^+, \varepsilon) = 0 \ \text{ such that } \ \|w\|_{\mathcal{R}^-(\Phi_p)} \le \delta. \tag{5.37}$$

Moreover, this solution has the form: $w = W(v_0^+,\varepsilon) : V(0)^+ \to \mathcal{R}^-(\Phi_p)$, where W is a C^1-function with respect to v_0^+. Note that (5.37) is equivalent to (5.31) and, consequently,

$$M^+_{\delta,z_0}(f,\varepsilon) = V_\delta(z_0) \cap \{u_0 = u_f^\varepsilon(0) + W(v_0^+,\varepsilon)(0) : \quad v_0 \in V(0)^+\}. \tag{5.38}$$

Recall also that, due to definitions of the operators $\mathbb{I}$ and $\mathbb{T}^-$ and relation (5.37), we have

$$\Pi_+ W(v_0^+,\varepsilon)(0) \equiv v_0^+. \tag{5.39}$$

Thus, (5.29) is satisfied with

$$\mathcal{N}_{f,\varepsilon}(v_0^+) := \Pi_- W(v_0^+,\varepsilon)(0).$$

It only remains to verify the exponential stabilization (5.30). To this end, we note that, due to (5.35)-(5.36) and Lemma 5.2, function (5.34) is well-defined as

$$\mathcal{F} : \mathcal{R}^-_{\alpha/2}(\Phi_b) \times V^+ \times \mathbb{R}_+ \to \mathcal{R}^-_{\alpha/2}(\Phi_b), \tag{5.40}$$

where $\alpha > 0$ is the same as in Proposition 5.1. Moreover, applying the implicit function theorem to relation (5.37) with $w \in \mathcal{R}^-_{\alpha/2}$ (and taking into account the uniqueness part), we derive that

$$\|W(v_0^+,\varepsilon)\|_{\mathcal{R}^-_{\alpha/2}(\Phi_p)} \le C\|v_0^+\|_{\Phi_p}. \tag{5.41}$$

Estimate (5.41) implies (5.30). Theorem 5.2 is proven. □

Note now, that taking $\varepsilon = 0$ in Theorem 5.2, we obtain the unstable manifold $M^+_{z_0,\delta}(\overline{f},0)$ for averaged system (2.22). Moreover, it follows from the construction given in the proof of the theorem that

$$\|\mathcal{N}_{\phi,\varepsilon}(v_0^+) - \mathcal{N}_{\overline{f},0}(v_0^+)\|_{\Phi_p} + \|D_{v_0^+}\mathcal{N}_\phi(v_0^+) - D_{v_0}\mathcal{N}_{\overline{f},0}(v_0^+)\|_{V_0^+\to V_0^-} \le \alpha(\varepsilon), \tag{5.42}$$

for $v_0^+ \in V^+(0)$ where $\alpha(\varepsilon) \to 0$ as $\varepsilon \to 0$. Note also, that

$$\Pi_- D_{v_0^+} \mathcal{F}(0,0,0) = 0$$

and, consequently, $D_{v_0^+} \mathcal{N}_{\overline{f},0}(0) = 0$. Therefore, we have the estimate

$$\|\mathcal{N}_{\overline{f},0}(v_0^+)\|_{\Phi_p} \le C\|v_0^+\|_{\Phi_p}^2. \tag{5.43}$$

Let us now formulate several corollaries of the proved theorem.

Corollary 5.2. *Let the assumptions of Theorem* 5.2 *hold and let* $u(t)$, $t \le 0$ *be a backward solution of problem* (0.1) *such that* $\|u(t) - z_0\|_{\Phi_p} \le \delta$, *for every* $t \le 0$. *Then* $u(0) \in M_{z_0,\delta}^+(f,\varepsilon)$ *and, consequently,* $u(t)$ *exponentially stabilizes to* $u_f^\varepsilon(t)$ *as* $t \to -\infty$ ((5.30) *is satisfied*).

Corollary 5.3. *Let the assumptions of Theorem* 5.2 *hold. Then, for every* $t \ge 0$, *we have the embeddings*

$$M_{z_0,\delta}^+(T_{h/\varepsilon}\phi, \varepsilon) \subset U_f^\varepsilon(h,0) M_{z_0,\delta}^+(\phi,\varepsilon). \tag{5.44}$$

Moreover, if $u(0) \in M_{z_0,\delta}^+(\phi,\varepsilon)$ *and* $\|u(t) - z_0\|_{\Phi_p} \le \delta$ *for* $t \le N$, *then*

$$u(t) := U_\phi^\varepsilon(t,0)u(0) \in M_{z_0,\delta}^+(T_{t/\varepsilon}\phi,\varepsilon) \text{ for } t \le N. \tag{5.45}$$

Indeed, (5.44) and (5.45) follow immediately from definition (5.28) of the unstable sets.

Remark 5.2. Note that (it follows from (5.41)) estimate (5.30) can be improved in the following way:

$$\|u(t) - u_\phi^\varepsilon(t)\|_{\Phi_p} \le e^{(\alpha/2)t}\|\Pi_+(u(0) - u_\phi^\varepsilon(0))\|_{\Phi_p}. \tag{5.46}$$

Remark 5.3. Replacing $t \to -\infty$ by $t \to +\infty$ in definition (5.28) and arguing analogously to Theorem 5.2, we may define and study the *stable* manifolds $M_{z_0,\delta}^-(\phi,\varepsilon)$, which consist of initial data for solutions tending exponentially to $u^\varepsilon(t)$ as $t \to +\infty$. We do not give a proper consideration of these manifolds because only the unstable manifolds are significant for constructing the regular attractor (see the next Section).

Note also, that arguing as in Corollary 5.1, we may estimate the rate of convergence of $\alpha(\varepsilon)$ to zero in (5.42) if the assumptions of Section 4 are satisfied. For example, let the assumptions of Theorems 5.1 and 4.3 be satisfied. Then the following estimate is valid:

$$\|\mathcal{N}_{\phi,\varepsilon}(v_0^+) - \mathcal{N}_{\overline{f},0}(v_0^+)\|_{L^p} + \|D_{v_0^+}\mathcal{N}_{\phi,\varepsilon}(v_0^+) - D_{v_0^+}\mathcal{N}_{\overline{f},0}(v_0^+)\|_{V^+\to L^p} \le C\varepsilon, \tag{5.47}$$

where C is independent of $\phi \in \mathcal{H}(f)$ and of $v_0^+ \in V(0)^+$. Since estimate (5.47) is also not necessary for the quantitative averaging of regular attractors (see Section 7), we left its rigorous proof to the reader.

We conclude this section by proving that every solution of (0.1) tends exponentially to the appropriate unstable manifold while it remains in a δ-neighbourhood of z_0.

Theorem 5.3. *Assume that the assumptions of Theorem* 5.2 *hold. Then there exists* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon < \varepsilon_0$, $\phi \in \mathcal{H}(f)$ *and every trajectory* $u(t) := U_\phi^\varepsilon(t,0)u_0$ *which belongs to the* δ*-neighbourhood of* z_0 *for* $t \le N$, *there exists a trajectory* $v(t) := U_\phi^\varepsilon(t,0)v_0$ *which belongs to the unstable manifold* ($v_0 \in M^+_{z_0,\delta}(\phi,\varepsilon)$) *such that*

$$\|u(t) - v(t)\|_{\Phi_p} \le Ce^{-(\alpha/2)t}\|u_0 - v_0\|_{\Phi_p}, \quad t \le N, \tag{5.48}$$

where the constant C *is independent of* $\phi \in \mathcal{H}(f)$, $\varepsilon \le \varepsilon_0$, u_0, *and* $N \in [0,\infty]$.

Proof. As before, we only consider the case $\phi = f$. We claim that, for a sufficiently small δ and ε, there exists a unique $v_0 \in M^+_{z_0,\delta}(f,\varepsilon)$ such that the corresponding solution $v(t)$ satisfies the condition

$$\Pi_+ v(N) = \Pi_+ u(N). \tag{5.49}$$

Then due to assumptions of the theorem

$$\|\Pi_+(v(N) - u_f^\varepsilon(N))\|_{\Phi_p} \le C_+\|u(N) - u_f^\varepsilon(N)\|_{\Phi_p} \le C_+\delta. \tag{5.50}$$

Moreover, we claim that the solution $v(t)$ thus obtained satisfies all assumptions of the theorem. In order to verify these assertions, we need the following analogue of Lemmata 5.1–5.2 for the case of a finite interval.

Lemma 5.3. *Let the assumptions of Lemma* 5.1 *hold and let* $N \in \mathbb{N}$. *Consider the problem*

$$\begin{cases} w(m) = \mathbb{L}w(m-1) + h(m-1), \quad m = 1, \cdots, N, \\ \Pi_- w(0) = w_-, \quad \Pi_+ w(N) = w_+. \end{cases} \tag{5.51}$$

Define, analogously to (5.14), *the spaces of finite sequences*

$$\mathcal{R}_\beta^{[0,N]} := \big\{w \in C_b(\{0,\cdots,N\},\Phi_p) : \max_{0\le m\le N} e^{\beta m}\|w(m)\|_{\Phi_p} < \infty\big\}. \tag{5.52}$$

Then, problem (5.51) *defines three bounded linear operators:*

$$\mathbb{T}_{[0,N]} : \mathcal{R}_{\alpha/2}^{[0,N]} \to \mathcal{R}_{\alpha/2}^{[0,N]}, \quad \mathbb{I}^+_{[0,N]} : V^+ \to \mathcal{R}_{\alpha/2}^{[0,N]}, \quad \mathbb{I}^-_{[0,N]} : V^- \to \mathcal{R}_0^{[0,N]}, \tag{5.53}$$

(where $\alpha > 0$ is the same as in Proposition 5.1). The first one ($h \to w$) solves (5.51) with zero boundary conditions and arbitrary right-hand side h, the second one ($w_+ \to w$) solves (5.51) with $h = 0$, $w_- = 0$ and arbitrary w_+, and the third one ($w_- \to w$) corresponds to the case $h = 0$, $w_+ = 0$ and arbitrary w_-. Thus, a unique solution w of (5.51) can be represented in the following form

$$w = \mathbb{T}_{[0,N]}h + \mathbb{I}^+_{[0,N]}w_+ + \mathbb{I}^-_{[0,N]}w_-. \tag{5.54}$$

Moreover, the norms of these operators are bounded uniformly with respect to $N \in \mathbb{N}$

$$\|\mathbb{T}_{[0,N]}\|_{\mathcal{R}^{[0,N]}_{\alpha/2}\to\mathcal{R}^{[0,N]}_{\alpha/2}} + \|\mathbb{I}^+_{[0,N]}\|_{V^+\to\mathcal{R}^{[0,N]}_{\alpha/2}} + \|\mathbb{I}^-_{[0,N]}\|_{V^-\to\mathcal{R}^{[0,N]}_0} \le C, \tag{5.55}$$

where C is independent of $N \in \mathbb{N}$.

The assertion of the lemma can be easily verified analogously to the proof of Lemma 5.1 so we omit the rigorous proof here.

Let us first check the existence of a solution $v(t)$. As before, we seek for it in the form $v(m) = w(m) + u^\varepsilon_f(m)$, $m = 0, \cdots, N$. Then, the sequence $w(m)$ satisfies (5.31) with the following boundary conditions:

$$\Pi_- w(0) = \mathcal{N}_{f,\varepsilon}(\Pi_+ w(0)), \quad \Pi_+ w(N) = \Pi_+(u(N) - u^\varepsilon_f(N)) := w_N, \tag{5.56}$$

where the function $\mathcal{N}_{f,\varepsilon}$ is defined in Theorem 5.2. According to Lemma 5.3, we may rewrite (5.31) in the following form (compare with (5.33)):

$$\begin{aligned}\mathcal{F}(w, w_N, \varepsilon) := w - \mathbb{I}^-_{[0,N]}\mathcal{N}_{f,\varepsilon}(\Pi_+ w(0)) - \mathbb{I}^+_{[0,N]}w_N \\ - \mathbb{T}_{[0,N]}\mathbb{S}(w,\varepsilon) - \mathbb{T}_{[0,N]}\mathbb{K}(w,\varepsilon) = 0,\end{aligned} \tag{5.57}$$

where $\mathcal{F} : \mathcal{R}^{[0,N]}_0 \times V_+ \times \mathbb{R}_+ \to \mathcal{R}^{[0,N]}_0$. It is not difficult to verify, using estimates (5.35)-(5.36), (5.42)–(5.43) and (5.55), that function (5.57) satisfies all assumptions of the implicit function theorem and, consequently, there exist $\delta > 0$ and $\varepsilon_0 > 0$ such that, for every $w_N \in V_+$, $\|w_N\|_{\Phi_p} \le \delta$ and every $\varepsilon < \varepsilon_0$, there exists a unique solution $w(m)$ of (5.57) such that

$$\|w\|_{\mathcal{R}^{[0,N]}_0} \le C\|w_N\|_{\Phi_p} \le C\delta. \tag{5.58}$$

Moreover, since norms (5.55) are independent of $N \in \mathbb{N}$, the constants $\delta > 0$, $\varepsilon_0 > 0$, and C in (5.58) are also independent of N.

Thus, the solution $v(t)$, which satisfies (5.49) and belongs to the unstable manifold is constructed. So, it only remains to verify inequality (5.48). To this end, we introduce functions $\widehat{w}(m) := u(m) - u^\varepsilon_f(m)$ and $\theta(m) := w(m) -$

$\widehat{w}(m)$. Since the function $w(m)$ also satisfies (5.31), we derive (analogously to (5.57), but taking into the account $\Pi_+\theta(N)=0$) that

$$\theta = \mathbb{I}^{-}_{[0,N]}\Pi_-\theta(0) + \mathbb{T}_{[0,N]}\left(\mathbb{S}(w,\varepsilon) - \mathbb{S}(\widehat{w},\varepsilon)\right) + \mathbb{T}_{[0,N]}\left(\mathbb{K}(w,\varepsilon) - \mathbb{K}(\widehat{w},\varepsilon)\right). \tag{5.59}$$

It is not difficult to verify (analogously to (5.35) and (5.36)) that

$$\begin{aligned} \|\mathbb{S}(w(m),\varepsilon) - \mathbb{S}(\widehat{w}(m),\varepsilon)\|_{\Phi_p} + \|\mathbb{K}(w(m),\varepsilon) - \mathbb{K}(\widehat{w}(m)\|_{\Phi_p} \\ \le C\|\theta(m)\|_{\Phi_p}\left(\alpha'(\varepsilon) + \|w(m)\|_{\Phi_p} + \|\widehat{w}(m)\|_{\Phi_p}\right), \end{aligned} \tag{5.60}$$

where $C>0$ and $\alpha'(\varepsilon)\to 0$ as $\varepsilon\to 0$. Particularly, (5.60), (5.58) and the fact that $u(t)$ belongs to δ-neighborhood of z_0 for $t\in[0,N]$ imply that

$$\|\mathbb{S}(w,\varepsilon) - \mathbb{S}(\widehat{w},\varepsilon)\|_{\mathcal{R}^{[0,N]}_{\alpha/2}} + \|\mathbb{K}(w,\varepsilon) - \mathbb{K}(\widehat{w},\varepsilon)\|_{\mathcal{R}^{[0,N]}_{\alpha/2}} \le C_1\|\theta\|_{\mathcal{R}^{[0,N]}_{\alpha/2}}(\alpha(\varepsilon)+\delta) \tag{5.61}$$

where C_1 is independent of N. Applying (5.61) and (5.55) to (5.59), we derive the estimate

$$\|\theta\|_{\mathcal{R}^{[0,N]}_{\alpha/2}} \le C'\|\theta(0)\|_{\Phi_p} + C''(\alpha'(\varepsilon)+\delta)\|\theta\|_{\mathcal{R}^{[0,N]}_{\alpha/2}}, \tag{5.62}$$

where C' and C'' are independent of N. Taking now $\delta>0$ and $\varepsilon_0\ge\varepsilon$ small enough, we derive from (5.62) that

$$\|\theta\|_{\mathcal{R}^{[0,N]}_{\alpha/2}} \le C_3\|\theta(0)\|_{\Phi_p}$$

or (which is the same)

$$\|u(m)-v(m)\|_{\Phi_p} \le C_3 e^{-(\alpha/2)m}\|u(0)-v(0)\|_{\Phi_p}, \quad m=0,\cdots,N. \tag{5.63}$$

Estimate (5.48) is an immediate corollary of (5.63). Theorem 5.3 is proven.

Corollary 5.4. *Let the assumptions of Theorem* 5.3 *hold and let, in addition, a solution* $u(t)=U^{\varepsilon}_{\phi}(t,0)u_0$, $\varepsilon<\varepsilon_0$ *and* $\phi\in\mathcal{H}(f)$, *remains in the* δ*-neighbourhood of* z_0 *for every* $t\ge 0$. *Then, this solution stabilizes exponentially to* u^{ε}_{ϕ} *as* $t\to+\infty$, *i.e.,*

$$\|u(t)-u^{\varepsilon}_{\phi}(t)\|_{\Phi_p} \le Ce^{-(\alpha/2)t}\|u(0)-u^{\varepsilon}_{\phi}(0)\|_{\Phi_p}, \quad t\ge 0. \tag{5.64}$$

Indeed, the solution $u(t)$ satisfies all assumptions of Theorem 5.3 with $N=\infty$, consequently, there exists a solution $v(t)$, which belongs to the appropriate unstable manifold and satisfies (5.48) for every $t\ge 0$. Particularly, $v(t)$ remains in the δ-neighbourhood of z_0 for every $t\ge 0$. But, according to (5.30), every solutions belonging to the unstable manifold except of $u^{\varepsilon}_{\phi}(t)$ goes out from this neighbourhood for a sufficiently large t.

Therefore, $v(t) \equiv u^\varepsilon_\phi(t)$ and then (5.64) coincides with (5.38). Corollary 5.4 is proven.

6. Regular attractors

In this Section, we consider the case where averaged system (2.22) possesses a global Liapunov function, it will be so for instance if the averaged nonlinearity $\overline{f}(x,v)$ has a gradient structure

$$\overline{f}(x,v) = \nabla_v \overline{F}(x,v), \quad \overline{F} \in C(\Omega \times \mathbb{R}^k). \tag{6.1}$$

In this case, under the natural (generic) assumption that all equilibria of (2.22) are hyperbolic, we have the following detailed description for the regular structure of $\mathcal{A}^0$.

Proposition 6.1. *Let the assumptions of Theorem* 2.2 *hold and let, in addition, the semigroup $S^0_t : \Phi_p \to \Phi_p$ possess a global Liapunov function. Assume also that the set of its equilibria*

$$\mathcal{R} := \{z_0 \in \Phi_p, \quad a\Delta_x z_0 - \overline{f}(x,z_0) + g = 0\,\} \tag{6.2}$$

is finite and all the equilibria are hyperbolic. Then, the attractor $\mathcal{A}^0$ is a finite collection of finite dimensional unstable manifolds, which corresponds to every equilibrium $z_0 \in \mathcal{R}$:

$$\mathcal{A}^0 = \cup_{z_0 \in \mathcal{R}} \mathcal{M}^+_{z_0}, \tag{6.3}$$

where $\dim M^+_{z_0} = \kappa(z_0)$ *and* $M^+_{z_0} \sim \mathbb{R}^{\kappa(z_0)}$ *and every solution* $u(t) \in \mathcal{K}_{\overline{f}}$ *belonging to the attractor is a heteroclinic orbit, connecting two different equilibria* $z^-_0, z^+_0 \in \mathcal{R}$*:*

$$\lim_{t\to-\infty} \|u(t) - z^-_0\|_{\Phi_p} = 0, \quad \lim_{t\to+\infty} \|u(t) - z^+_0\|_{\Phi_p} = 0, \quad z^+_0 \neq z^-_0. \tag{6.4}$$

Moreover, the attractor $\mathcal{A}^0$ is exponential in the following sense: there exist a positive constant $\alpha > 0$ and a monotonic function Q such that, for every bounded subset $B \subset \Phi_p$, we have

$$\operatorname{dist}_{\Phi_p}\left(S^0_t B, \mathcal{A}^0\right) \le Q(\|B\|_{\Phi_p})e^{-\alpha t}. \tag{6.5}$$

The proof of Proposition 6.1 is given, e.g. in [2]. (In a fact, we reprove below this proposition for more general non-autonomous case, see also [13]).

The main task of this section is to verify that description (6.3)–(6.5) remains valid (after minor changes) for non-averaged equation (0.1) if $\varepsilon < \varepsilon_0$ is small enough. To this end, we note that, due to Theorem 5.1 and the fact that $\mathcal{R}$ is finite, there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$, every $z_0 \in \mathcal{R}$ and every $\phi \in \mathcal{H}(f)$, equation (2.8) possesses a unique solution

$u^\varepsilon_{\phi,z_0}(t)$, $t\in\mathbb{R}$, which remains in δ_0-neighbourhood of the equilibrium z_0, this solution is almost-periodic with respect to t and tends to z_0 as $\varepsilon\to 0$. Thus, it is natural to define

$$\mathcal{R}^\varepsilon_\phi := \{u^\varepsilon_{\phi,z_0},\quad z_0\in\mathcal{R}\} \tag{6.6}$$

and interpret it as the set of all 'equilibria' of non-averaged problem (2.8) which correspond to the equilibria set $\mathcal{R}$ as $\varepsilon\to 0$. We are now ready to formulate the analogue of (6.4) for the non-averaged equation.

Theorem 6.1. *Let the assumptions of Theorem* 2.1 *and Proposition* 6.1 *hold. Then, there exists $\varepsilon_0>0$ such that, for $\varepsilon<\varepsilon_0$, every solution $u(t) := U^\varepsilon_\phi(t,0)u_0$, $t\ge 0$, $\phi\in\mathcal{H}(f)$ of equation* (2.8) *stabilizes to one of 'equilibria'* (6.6):

$$\lim_{t\to+\infty}\|u(t)-u^\varepsilon_{\phi,z_0}(t)\|_{\Phi_p}=0,\quad \textit{for some } z_0=z_0(u)\in\mathcal{R}. \tag{6.7}$$

Moreover, every complete bounded solution $u(t)\in\mathcal{K}^\varepsilon_\phi$ of (2.8) *is a heteroclinic orbit between to different 'equilibria' from* (6.6):

$$\lim_{t\to-\infty}\|u(t)-u^\varepsilon_{\phi,z_0^-}(t)\|_{\Phi_p}=0,\quad \lim_{t\to+\infty}\|u(t)-u^\varepsilon_{\phi,z_0^+}(t)\|_{\Phi_p}=0,\quad z_0^+\ne z_0^-. \tag{6.8}$$

Particularly, there are not any almost-periodic solutions of (2.8) *except of* (6.6).

For the proof of the theorem, we need the non-autonomous analogue of the following two lemmata which play a fundamental role in the proof of Proposition 6.1 (in the autonomous case).

Lemma 6.1. *Let the assumptions of Theorem* 6.1 *hold. Then, for every $\delta<\delta_0$ there exists $\varepsilon_0:=\varepsilon_0(\delta)>0$ such that, for every $\varepsilon<\varepsilon_0$ and every $R\in\mathbb{R}_+$, there is a positive number $T=T(\delta,R)$ such that, for every solution $u(t):=U^\varepsilon_\phi(t,0)u_0$, $\|u_0\|_{\Phi_p}\le R$, $\phi\in\mathcal{H}(f)$, we have*

$$\left[\cup_{s\in[0,T]}\{u(s)\}\right]\cap\left[\cup_{z_0\in\mathcal{R}}V_\delta(z_0)\right]\ne\varnothing, \tag{6.9}$$

where $V_\delta(z_0):=\{u_0\in\Phi_p,\ \|u_0-z_0\|_{\Phi_p}\le\delta\}$.

Assertion (6.9) means that every solution $u(t)$ visits a δ-neighbourhood of some equilibrium $z_0\in\mathcal{R}$ while $t\in[0,T]$.

Proof. Indeed, assume that the assertion of the lemma is wrong. Then, there exist $\varepsilon_n\to 0$, $\phi_n\in\mathcal{H}(f)$, $u_n^0\in V_R(0)$ and $T_n\to+\infty$ such that

$$\mathrm{dist}_{\Phi_p}(u_n(t),\mathcal{R})\ge\delta_0>0,\ \ t\in[0,T_n],\quad u_n(t):=U^\varepsilon_{\phi_n}(t,0)u_n^0. \tag{6.10}$$

We now set $\tilde{\phi}_n := T_{-T_n/(2\varepsilon)}\phi_n$ and $\tilde{u}_n(t) := U^\varepsilon_{\tilde{\phi}_n}(t, -T_n/2)u^0_n$. Then, it follows from (6.10) that

$$\operatorname{dist}_{\Phi_p}(\tilde{u}_n(t), \mathcal{R}) \geq \delta_0 > 0,\ t \in [-T_n/2, T_n/2]. \tag{6.11}$$

Recall, that $u^0_n \in V_R(0)$, consequently, due to Theorem 1.3

$$\|\tilde{u}_n\|_{W^{(1,2),2p}([T,T+1]\times\Omega)} \leq C, \quad T \geq 1 - T_n/2. \tag{6.12}$$

Therefore (due to compactness of the embedding $W^{(1,2),2p} \subset C(\Phi_p)$ and that $T_n \to \infty$), without loss of generality we may assume that $u_n \to u_0$ in the spaces $C([T, T+1], \Phi_p)$ for every $T \in \mathbb{R}$. Arguing as in the proof of Theorem 2.3, we establish that $u_0(t)$ is a complete bounded solution of averaged equation (2.22). From the other side, passing to the limit $n \to \infty$ in (6.11), we derive that

$$\operatorname{dist}_{\Phi_p}(u_0(t), \mathcal{R}) \geq \delta_0 > 0, \quad t \in \mathbb{R} \tag{6.13}$$

which contradicts the existence of a global Liapunov function for averaged problem (2.22) (see also (6.4)). Lemma 6.1 is proven. □

Lemma 6.2. *Assume that the assumptions of Theorem* 6.1 *hold. Let* $\delta_0 > 0$ *be a sufficiently small number and let* $L : \Phi_p \to \mathbb{R}$ *be a Liapunov function of the averaged system* (2.22)*. Then, there exist* $\delta' < \delta_0$ *and* $\varepsilon_0 = \varepsilon_0(\delta')$ *such that the following statements are valid if* $\varepsilon < \varepsilon_0$*: let* $z_0 \in \mathcal{R}$*,* $u_0 \in V_{\delta'}(z_0)$*, and* $u(t) := U^\varepsilon_\phi(t, 0)u_0$*,* $\phi \in \mathcal{H}(f)$*, be a solution of* (2.8)*, then*

1) *if* $u(T) \in V_{\delta'}(z'_0)$*, for some* $z'_0 \in \mathbb{R}$*,* $z'_0 \neq z_0$ *and* $T > 0$*, then necessarily* $L(z'_0) < L(z_0)$*;*

2) *if* $u(T) \notin V_{\delta_0}(z_0)$*, for some* $T > 0$*, then* $u(t) \notin V_{\delta'}(z_0)$*, for every* $t \geq T$*.*

Proof. Let us only verify the first statement of the lemma (the second one can be verified analogously). Indeed, let this statement to be wrong, i.e., there exist an equilibrium $z'_0 \in \mathbb{R}$ with

$$L(z'_0) \geq L(z_0), \tag{6.14}$$

sequences $\delta_n \to 0$, $\varepsilon_n \to 0$, $u^n_0 \in V_{\delta_n}(z_0)$, $T_n \in \mathbb{R}_+$, and $\phi_n \in \mathcal{H}(f)$, such that the corresponding solutions $u_n(t) := U^{\varepsilon_n}_{\phi_n}(t, 0)u^n_0$ satisfy

$$u_n(T_n) \in V_{\delta_n}(z'_0). \tag{6.15}$$

Let $\delta_0 > 0$ be such that $V_{\delta_0}(z^i_0) \cap V_{\delta_0}(z^j_0) = \varnothing$, for $z^i_0, z^j_0 \in \mathcal{R}$, $z^i_0 \neq z^j_0$. It follows from (6.15) that there exist $0 < t_n < T_n$ such that

$$\|u_n(t_n) - z_0\|_{\Phi_p} = \delta_0, \quad \|u_n(t) - z_0\|_{\Phi_p} < \delta_0, \quad t < t_n. \tag{6.16}$$

Moreover, thanks to $u_n(0) \to z_0$, one can easily verify using Theorem 3.1 that $t_n \to \infty$ as $n \to \infty$. Shifting the time variable $t \to t + t_n$, we obtain a new family of solutions

$$\tilde{u}_n(t) := U^{\varepsilon_n}_{\tilde{\phi}_n}(t, -t_n)u^n_0, \quad \tilde{\phi}_n := T_{-t_n/\varepsilon_n}\phi_n \tag{6.17}$$

such that

$$\tilde{u}_n(-t_n) \to z_0, \quad \|\tilde{u}_n(0) - z_0\|_{\Phi_p} = \delta_0 > 0, \quad \tilde{u}_n(\tilde{T}_n) \to z'_0, \quad \tilde{T}_n := T_n - t_n > 0. \tag{6.18}$$

Passing now to the limit $n \to \infty$ in (6.18) and arguing as in Lemma 6.1, we obtain a complete bounded solution $\tilde{u}(t)$ of averaged equation (2.22) such that

$$\tilde{u}(t) \to z_0, \quad \text{as} \quad t \to -\infty \quad \text{and} \quad \|\tilde{u}(0) - z_0\|_{\Phi_p} = \delta_0.$$

Recall now that (2.22) possesses a global Liapunov function L, consequently (see also (6.4)), $\tilde{u}(t)$ is a heteroclinic orbit, i.e., there exists $z^1_0 \in \mathcal{R}$ such that

$$\tilde{u}(t) \to z^1_0 \text{ as } t \to +\infty \text{ and, consequently, } L(z^1_0) < L(z_0). \tag{6.19}$$

A priori, we have two possibilities: 1) $z^1_0 = z'_0$ and 2) $z^1_0 \neq z'_0$. The first one contradicts assumption (6.14). So, it only remains to investigate the second one. In this case, we obviously have a sequence $\tilde{T}'_n < \tilde{T}_n$, $\tilde{T}_n \to +\infty$, such that

$$\tilde{u}_n(\tilde{T}'_n) \to z^1_0, \quad \tilde{u}_n(\tilde{T}_n) \to z'_0, \quad \tilde{T}_n > \tilde{T}'_n. \tag{6.20}$$

Arguing now as before, we derive from (6.20) the existence of the other heteroclinic orbit $\tilde{u}^1(t)$ of equation (2.22), such that

$$\tilde{u}^1(t) \to z^1_0, \quad \text{as} \quad t \to -\infty \quad \text{and} \quad u^1(t) \to z^2_0 \text{ as } t \to +\infty.$$

As before the equality $z^2_0 = z'_0$ is prohibited by (6.14). Thus, repeating the above procedure, we obtain a sequence of heteroclinic connections for (2.22):

$$z_0 \to z^1_0 \to z^2_0 \to \cdots, \quad L(z_0) > L(z^1_0) > L(z^2_0) > \cdots . \tag{6.21}$$

Recall now that, due to our assumptions, the set $\mathcal{R}$ is finite, therefore, sequence (6.21) cannot be infinite. Consequently, we must have $z'_0 = z^N_0$ for some N (because we have from the very beginning $u_n(T_n) \to z'_0$ for some $T_n \to \infty$). Assertion (6.21) implies now that $L(z_0) > L(z'_0)$ which contradicts (6.14). Lemma 6.2 is proven. □

We are now ready to complete the proof of the theorem. Indeed, it follows from Theorem 1.1 that it is sufficient to verify (6.7) only for $u_0 \in V_{R_0}(0)$ where $R_0 > 0$ is large enough. Fix now a constant $\delta' > 0$ in such way that, for every $z_0 \in \mathcal{R}$, the local description of the dynamics (given in Theorems 5.1–5.3) works in $V_{\delta_0}(z_0)$ if $\varepsilon < \varepsilon_0$ (it is possible to do because we have only a finite number of equilibria). Moreover, fix also $\delta' \le \delta_0/2$ in Lemma 6.2 and $T := T(\delta', R_0)$ from Lemma 6.1 (in other words, we assume that ε_0 is small enough that the assertions of Lemmata 6.1 and 6.2 hold with constants fixed above). We claim that, for every $\varepsilon < \varepsilon_0$, we have stabilization (6.7). Indeed, $u(t) := U^\varepsilon_\phi(t,0)u_0$ be an arbitrary solution of (2.8) such that $u_0 \in V_{R_0}(0)$. Then, it follows from Lemma 6.1 that there exists a time $T_1 \le T$ and an equilibrium z_0^1 such that

$$u(T_1) \in V_{\delta'}(z_0^1). \tag{6.22}$$

After the first visiting the δ'-neighbourhood of the equilibrium z_0^1 a solution u a priori has two possibilities: 1) $u(t)$ remains inside of $V_{\delta_0}(z_0)$, for every $t \ge T_1$; 2) there is a moment $T_1' > T_1$ such that $u(T_1') \in \partial V_{\delta_0}(z_0^1)$. In the first case, due to Corollary 5.4, $u(t)$ stabilizes exponentially to $u^\varepsilon_{\phi,z_0^1}(t)$ as $t \to +\infty$ and, consequently, (6.7) holds with $z_0^- = z_0^1$.

Consider now the second possibility. In this case, due to Lemma 6.2, our solution $u(t)$ never returns in $V_{\delta'}(z_0^1)$, for $t > T_1'$. Applying Lemma 6.1 again, we derive that there exists $T_2 > T_1'$, $T_2 - T_1' \le T$ and an equilibrium $z_0^2 \in \mathcal{R}$, such that

$$u(T_2) \in V_{\delta'}(z_0^2), \quad \text{and} \quad L(z_0^1) > L(z_0^2). \tag{6.23}$$

Repeating the above arguments, we construct finally the monotone increasing sequences $T_i > 0$, $i = 1, \cdots, N$ and $T_i' > T_i$, $i = 1, \cdots N-1$, $T_{i+1} - T_i' \le T$, and a sequence of equilibria z_0^i, $i = 1, \cdots, N$ such that

$$\begin{aligned} u(T_i) \in V_\delta(z_0^i),\ \ i = 1, \cdots, N, \quad u(T_i') \in \partial V_{\delta_0}(z_0^i), \quad i = 1, \cdots N-1, \\ L(z_0^1) > L(z_0^2) > \cdots > L(z_0^N). \end{aligned} \tag{6.24}$$

Note that the last inequality of (6.24) together with the finiteness of $\mathcal{R}$ implies that this sequence cannot be infinite, therefore there exists $N = N(u) \le N_{max} := \#\mathcal{R}$ such that

$$u(t) \in V_{\delta_0}(z_0^N), \quad \text{for} \quad t \ge T_N. \tag{6.25}$$

Including (6.25) together with Corollary 5.4 implies (6.7). Assertion (6.8) can be verified analogously only we should use also Corollary 5.2 in order to verify the convergence as $t \to -\infty$. Theorem 6.1 is proven.

Let us now verify the analogue of (6.3) for the non-averaged system. Note, however, that, in contrast to limit equation (2.22), equations (2.8) are non-autonomous for $\varepsilon > 0$, consequently, the global unstable manifolds $M^+_{z_0,f}(t,\varepsilon)$ and the non-autonomous analogue $\mathcal{A}^\varepsilon_\phi(t)$ of the regular attractor $\mathcal{A}^0$ also naturally depend on t and $\phi \in \mathcal{H}(f)$.

Definition 6.1. Define the non-autonomous regular attractor $\mathcal{A}^\varepsilon_\phi(t)$ by the following expression:

$$\mathcal{A}^\varepsilon_\phi(t) := \mathcal{K}^\varepsilon_\phi(t), \quad t \in \mathbb{R}, \quad \phi \in \mathcal{H}(f), \tag{6.26}$$

where the kernels $\mathcal{K}^\varepsilon_\phi(t)$ have been defined in (2.18). Moreover, for every $z_0 \in \mathcal{R}$ define the global unstable sets $M^+_{z_0,\phi}(\tau,\varepsilon)$ in the following natural way:

$$\begin{aligned} M^+_{z_0,\phi}(\tau,\varepsilon) := \{u_\tau \in \Phi_p \,:\, \exists u \in \mathcal{K}^\varepsilon_\phi, \quad u(\tau) = u_\tau, \\ \lim_{t\to-\infty} \|u(t) - u^\varepsilon_{\phi,z_0}(t)\|_{\Phi_p} = 0\} \end{aligned} \tag{6.27}$$

Obviously,

$$U^\varepsilon_\phi(t,\tau)M^+_{z_0,\phi}(\tau,\varepsilon) = M^+_{z_0,\phi}(t,\varepsilon), \quad t \geq \tau. \tag{6.28}$$

The next two theorems show that the object, defined by (6.26) is indeed a natural non-autonomous analogue of the (autonomous) regular attractor $\mathcal{A}^0$ of limit system (2.22) (see Proposition 6.1).

Theorem 6.2. *Let the assumptions of Theorem* 6.1 *hold. Then, for a sufficiently small $\varepsilon < \varepsilon_0$, the sets $M^+_{z_0,\phi}(\tau,\varepsilon)$ defined by* (6.27) *are finite dimensional C^1-submanifolds of Φ_p which are diffeomorphed to $\mathbb{R}^{\kappa(z_0)}$, for every fixed $\tau \in \mathbb{R}$. Moreover, attractor* (6.26) *is a finite collection of these unstable manifolds corresponding to equilibria $z_0 \in \mathcal{R}$:*

$$\mathcal{A}^\varepsilon_\phi(t) = \cup_{z_0\in\mathcal{R}} M^+_{z_0,\phi}(t,\varepsilon), \quad t \in \mathbb{R} \tag{6.29}$$

(*compare with* (6.3)).

Proof. Indeed, description (6.29) is an immediate corollary of (6.27) and Theorem 6.1. So, it remains to verify that unstable sets (6.27) are indeed C^1-submanifolds of Φ_p. For simplicity, we verify this fact only for $\phi = f$ and $\tau = 0$ (the other cases can be considered analogously). To this end, we use the standard reasonings which are based on the following evident representation of the set (6.27):

$$M^+_{z_0,f}(0,\varepsilon) = \cup_{n=0}^\infty M_n, \quad M_n := U^\varepsilon_{T_{-n/\varepsilon}f}(0,-n)M^+_{z_0,\delta}(T_{-n/\varepsilon}f,\varepsilon), \tag{6.30}$$

where the local unstable manifolds $M^+_{z_0,\delta}(\phi,\varepsilon)$, $\delta > 0$ are defined in (5.28). Note that (5.44) implies the embedding

$$M_n \subset M_{n+1}. \tag{6.31}$$

Note also that M_0 is a submanifold of Φ_p if $\delta > 0$ is small enough (due to Theorem 5.2). Let us verify that every M_n is also a submanifold. To this end, we recall the following standard lemma.

Lemma 6.3. *Let the assumptions of Theorem* 2.1 *hold. Then,*

1) *the operator* $U^\varepsilon_\phi(t,\tau):\Phi_p\to\Phi_p$ *is injective for every* $t\ge\tau$, $\varepsilon\ge 0$, *and* $\phi\in\mathcal{H}(f)$;

2) *The kernel of its Frechet derivative*

$$\ker\{D_{u_0}U^\varepsilon_\phi(t,\tau)(u_0)\}=\{0\}, \tag{6.31}$$

for every $u_0\in\Phi_p$.

The assertions of this lemma are immediate corollaries of a standard theory of backward uniqueness for parabolic boundary problems (see e.g. [2]).

Recall now that the manifold M_0 is finite and $\kappa(z_0)$-dimensional, consequently the assertions 1) and 2) are enough in order to verify that

$$U^\varepsilon_{T_{-n/\varepsilon}f}(0,-n):M_0\to M_n$$

is a diffeomorphism. Thus, M_n, $n\in\mathbb{N}$ are $\kappa(z_0)$-dimensional manifolds (diffeomorphed to $\mathbb{R}^{\kappa(z_0)}$) as well. Formulae (6.30) and (6.31) now imply that $M^+_{z_0,f}(0,\varepsilon)$ also possesses a structure of a C^1-manifold, diffeomorphed to $\mathbb{R}^{\kappa(z_0)}$. So, it remains to verify that the topology induced on $M^+_{z_0,f}(0,\varepsilon)$ coincides with the topology, induced by its embedding to Φ_p. But this fact is an immediate corollary of Lemma 6.2, which prohibits the recurrent (homoclinic) orbits near z_0. Theorem 6.2 is proven. □

Let us now verify that the obtained regular attractors $\mathcal{A}^\varepsilon_\phi(t)$ are uniformly with respect to $\varepsilon<\varepsilon_0$ exponential ones.

Theorem 6.3. *Let the assumptions of Theorem* 6.1 *hold. Then, there exist positive constants* $\gamma>0$, $\varepsilon_0>0$ *and a monotone function* Q *such that, for every* $\varepsilon<\varepsilon_0$, $\phi\in\mathcal{H}(f)$, $\tau\in\mathbb{R}$, *and bounded subset* $B\subset\Phi_p$ *the following estimate is valid:*

$$\operatorname{dist}_{\Phi_p}\left(U^\varepsilon_\phi(t+\tau,\tau)B,\mathcal{A}^\varepsilon_\phi(t+\tau)\right)\le Q\left(\|B\|_{\Phi_p}\right)e^{-\gamma t}. \tag{6.32}$$

Proof. As in the autonomous case (see [2]), the proof of estimate (6.32) is based on the following lemma.

Lemma 6.4. *Let the assumptions of Theorem* 6.1 *hold and let* $u(t):=U^\varepsilon_\phi(t,0)u_0$ *be a solution of* (2.8) *such that*

$$\|u(t)-z_0\|_{\Phi_p}\le\delta_0\ \ \text{for}\ \ t\in[0,N], \tag{6.33}$$

for some equilibrium $z_0\in\mathcal{R}$. *Then,*

$$\operatorname{dist}_{\Phi_p}\left(u(t),\mathcal{A}^\varepsilon_\phi(t)\right)\le Ce^{-\beta t}\left(\operatorname{dist}_{\Phi_p}\left(u_0,\mathcal{A}^\varepsilon_\phi(0)\right)\right)^\theta,\ \ t\in[0,N], \tag{6.34}$$

where positive constants C, $0<\theta<1$ *and* β *are independent of* u, $\varepsilon\le\varepsilon_0$, $\phi\in\mathcal{H}(f)$, $N\in\mathbb{R}_+$, *and* $z_0\in\mathcal{R}$.

Proof of the lemma. Indeed, according to Theorem 5.3 there exists $v(t)\in M^+_{z_0,\phi}(t,\varepsilon)$ such that estimate (5.48) is satisfied, consequently,

$$\operatorname{dist}_{\Phi_p}\left(u(t),\mathcal{A}^\varepsilon_\phi(t)\right)\le C_1e^{-(\alpha/2)t},\quad t\le N \tag{6.35}$$

(where $\alpha>0$ is a minimum of the constants $\alpha=\alpha(z_0)$, introduced in Proposition 5.1, over $z_0\in\mathcal{R}$). From the other side, it follows from estimate (1.9) that

$$\operatorname{dist}_{\Phi_p}\left(u(t),\mathcal{A}^\varepsilon_\phi(t)\right)\le C_2e^{Kt}\operatorname{dist}_{\Phi_p}\left(u_0,\mathcal{A}^\varepsilon_\phi(0)\right) \tag{6.36}$$

(where the constant $K>0$ is also uniform with respect to u_0, ϕ, and $\varepsilon\le\varepsilon_0$). Combining (6.35) and (6.36), we derive for $t\le N$ that

$$\operatorname{dist}_{\Phi_p}\left(u(t),\mathcal{A}^\varepsilon_\phi(t)\right)\le C_3e^{-\alpha/4t}\min\{e^{(K+\alpha/4)t}\operatorname{dist}_{\Phi_p}\left(u_0,\mathcal{A}^\varepsilon_\phi(0)\right),e^{-\alpha/4t}\}. \tag{6.36$'$}$$

Computing the minimum in the right-hand side of (6.36′), we obtain the estimate

$$\min\{e^{(K+\alpha/4)t}\operatorname{dist}_{\Phi_p}\left(u_0,\mathcal{A}^\varepsilon_\phi(0)\right),e^{-\alpha/4t}\}\le\left(\operatorname{dist}_{\Phi_p}\left(u_0,\mathcal{A}^\varepsilon_\phi(0)\right)\right)^\theta \tag{6.37}$$

where $\theta:=\alpha/(2(\alpha+2K))$. Lemma 6.4 is proven. □

We are now ready to prove (6.32). Indeed, due to uniform estimate (2.10) it is sufficient to verify it only for $B=V_R(0)$, for a sufficiently large fixed R, and, due to the translation invariants, we may assume also that $\tau=0$. Let $u_0\in V_R(0)$, $\varepsilon<\varepsilon_0$, $\phi\in\mathcal{H}(f)$, and $u(t):=U^\varepsilon_\phi(t,0)u_0$ be an arbitrary solution of (2.8). Let δ_0 and $\delta'>0$ and $T=T(\delta')$ be the same as in the proof of Lemma 6.2. Fix now the sequence z_0^i, $i=1,\cdots,N(u)\le N_{max}=\#\mathcal{R}$, of equilibria and the sequences of times T_i, $i=1,\cdots,N$, T_i', $i=1,\cdots,N$ such that $T_N'=\infty$ and the following conditions are satisfied:

$$\begin{cases}1.\ \ u(t)\in V_{\delta_0}(z_0^i),\quad \text{for } t\in[T_i,T_i'],\ i=1,\cdots,N,\\ 2.\ \ T_1\le T,\quad 0<T_{i+1}-T_i'\le T,\quad i=1,\cdots,N-1.\end{cases} \tag{6.38}$$

The existence of such sequences is verified in the proof of Lemma 6.4. Thus, on the intervals $t\in[T_i,T_i']$, $i=1,\cdots,N$, thanks to Lemma 6.4, we have the estimate

$$\operatorname{dist}_{\Phi_p}\left(u(t),\mathcal{A}^\varepsilon_\phi(t)\right)\le Ce^{-\beta(t-T_i)}\left(\operatorname{dist}_{\Phi_p}\left(u(T_i),\mathcal{A}^\varepsilon_\phi(T_i)\right)\right)^\theta. \tag{6.39}$$

From the other side, for $t \in [T_i', T_{i+1}]$, $i = 0, \cdots, N-1$ (where $T_0' = 0$) we have the estimate, analogous to (6.36):

$$\operatorname{dist}_{\Phi_p}\left(u(t), \mathcal{A}^\varepsilon_\phi(t)\right) \le Ce^{KT} \operatorname{dist}_{\Phi_p}\left(u(T_i'), \mathcal{A}^\varepsilon_\phi(T_i')\right). \tag{6.40}$$

Iterating formulae (6.39) and (6.40) and using the obvious estimates

$$\operatorname{dist}_{\Phi_p}\left(u_0, \mathcal{A}^\varepsilon_\phi(0)\right) \le R_0; \quad e^{-\theta^i \beta T_i - \beta(t-T_i)} \le e^{-\beta\theta^i t},\ t \ge T_i,$$

we derive that

$$\operatorname{dist}_{\Phi_p}\left(u(t), \mathcal{A}^\varepsilon_\phi(t)\right) \le R_0 \left(C^2 e^{KT}\right)^{N_{max}} e^{-\theta^{N_{max}}\beta t}, t \ge 0. \tag{6.41}$$

Since all the constants in (6.41) are independent of the choice of the trajectory $u(t)$, estimate (6.41) implies (6.32). Theorem 6.3 is proven. □

Remark 6.1. Recall that (in Section 2), we studied the uniform attractor $\mathcal{A}^\varepsilon$ for system (0.1) (see Theorem 2.1 and Definition 2.1), the construction of which is based on a reducing the initial non-autonomous dynamics to the autonomous one by a skew-product technique. Definition (6.26) of the non-autonomous attractor $\mathcal{A}^\varepsilon_\phi(t)$ corresponds to an alternative generalization of the attractor's concept to the non-autonomous case, which is widely used for the study the long-time behaviour of stochastic equations (see e.g. [18]). Under this approach the attractor of a non-autonomous dynamical system $\{U^\varepsilon_f(t,\tau) : \Phi_p \to \Phi_p, t \ge \tau\}$ (which corresponds to initial problem (0.1)) is defined to be a family of sets $\{\mathcal{A}^\varepsilon_f(t),\ t \in \mathbb{R}\}$ which enjoys the following properties:

1. $\mathcal{A}^\varepsilon_f(t)$ is compact in Φ_p, for every $t \in \mathbb{R}$;
2. The family $\{\mathcal{A}^\varepsilon_f(t), t \in \mathbb{R}\}$ is strictly invariant, i.e., $U^\varepsilon_f(t,\tau)\mathcal{A}^\varepsilon(\tau) = \mathcal{A}^\varepsilon_f(t)$;
3. This family possesses a pull-back attraction property, i.e., for every $t \in \mathbb{R}$ and every bounded subset $B \subset \Phi_p$, we have

$$\lim_{\tau \to +\infty} \operatorname{dist}_{\Phi_p}\left(U^\varepsilon_f(t, t-\tau)B, \mathcal{A}^\varepsilon_f(t)\right) = 0. \tag{6.42}$$

It is known (see [6]) that the non-autonomous attractor, thus defined coincides with the one, defined in (6.26) using the kernel sections $\mathcal{K}^\varepsilon_f(t)$.

Note however, that (in general) the convergence in (6.42) is not uniform with respect to $t \in \mathbb{R}$ and (therefore) the forward convergence $U^\varepsilon_f(t+\tau, t)B \to \mathcal{A}^\varepsilon_f(t+\tau)$ as $\tau \to +\infty$ may be violated at all. Theorem 6.3 shows that we have this uniformness and the forward exponential attraction property, for the case of rapidly oscillated perturbations of autonomous regular attractors.

Note, in addition, that, in this case, we have a clear relation between the uniform attractor $\mathcal{A}^\varepsilon$ and the non-autonomous one $\mathcal{A}^\varepsilon_f(t)$:

$$\mathcal{A}^\varepsilon = \left[\cup_{t\in\mathbb{R}} \mathcal{A}^\varepsilon_f(t)\right]_{\Phi_p} \tag{6.43}$$

which also may be violated for general non-autonomous systems.

7. The averaging of regular attractors

In this section, based on the results on previous section, we give a more comprehensive study of the convergence $\mathcal{A}^\varepsilon \to \mathcal{A}^0$ for the case where the limit attractor $\mathcal{A}^0$ is regular. We start with the following theorem:

Theorem 7.1. *Let the assumptions of Theorems* 2.1 *and* 6.1 *hold. Then, for every $\phi \in \mathcal{H}(f)$ and every $T \in \mathbb{R}$, the symmetric distance between the attractors $\mathcal{A}^\varepsilon_\phi(T)$ and $\mathcal{A}^0$ possesses the following estimate:*

$$\operatorname{dist}_{symm,\Phi_p}\left(\mathcal{A}^\varepsilon_\phi(T), \mathcal{A}^0\right) \le C\left(\alpha_{R_{max}}(\varepsilon)\right)^\kappa, \tag{7.1}$$

where $\alpha_R(\varepsilon)$ is the same as in Theorem 3.1, *$R_{max} = sup_{\varepsilon\ge 0}\|\mathcal{A}^\varepsilon\|_{\Phi_b} < \infty$, and $C > 0$, $0 < \kappa < 1$ are certain positive constants which depend only on equation* (0.1)*, but are independent of ϕ and T.*

Proof. As before, we only check the assertion of the theorem, for the case $\phi = f$ (the other cases are analogous).

Let $v_0 \in \mathcal{A}^\varepsilon_f(T)$, for a fixed $T \in \mathbb{R}$. Then, by definition, there exists a bounded complete solution $u^\varepsilon \in \mathcal{K}^\varepsilon_f$ such that $u^\varepsilon(T) = v_0$. Let $u^0_M(t)$ be a solution of limit problem (2.22), defined for $t \ge T - M$ and satisfying the initial condition $u^0_M(T-M) := u^\varepsilon(T-M)$, where $M > 0$ is a positive number, which will be fixed below. Then, from the one side, due to Theorem 3.1, we have

$$\|v_0 - u^0_M(T)\|_{\Phi_b} \le e^{K_{R_{max}}M}\alpha_{R_{max}}(\varepsilon). \tag{7.2}$$

From the other side, since the limit attractor $\mathcal{A}^0$ is exponential (due to our assumptions and Theorem 6.3), we have

$$\operatorname{dist}_{\Phi_p}\left(u^0_M(T), \mathcal{A}^0\right) \le C' e^{-\gamma t}, \tag{7.3}$$

where $\gamma > 0$ is defined in Theorem 6.3. Combining estimates (7.2) and (7.3) and taking into account that $M > 0$ is arbitrary, we derive

$$\operatorname{dist}_{\Phi_p}\left(v_0, \mathcal{A}^0\right) \le \min_{M>0}\left(e^{K_{R_{max}}M}\alpha_{R_{max}}(\varepsilon) + C' e^{-\gamma t}\right). \tag{7.4}$$

Computing the minimum in the right-hand side of (7.4) and reminding that $v_0 \in \mathcal{A}_f^\varepsilon(T)$ is arbitrary, we obtain

$$\operatorname{dist}_{\Phi_b}\left(\mathcal{A}_f^\varepsilon(T), \mathcal{A}^0\right) \le C\alpha_{R_{max}}(\varepsilon)^\kappa, \tag{7.5}$$

where $\kappa := \frac{\gamma}{K_{R_{max}}+\gamma}$.

Let now $v_0 \in \mathcal{A}^0$ and $u_0 \in \mathcal{K}^0$ be the corresponding complete bounded solution. Define again $u^\varepsilon(t)$, $t \ge T - M$ as a solution of nonhomogenized problem (0.1) with the initial condition $u^\varepsilon(T-M) := u_0(T-M)$. Then, analogously, from Theorem 3.1, we deduce that

$$\|v_0 - u^\varepsilon(T)\|_{\Phi_b} \le e^{K_{R_{max}}M}\alpha_{R_{max}}(\varepsilon). \tag{7.6}$$

From the other side, according to Theorem 6.3, we have

$$\operatorname{dist}_{\Phi_b}\left(v_0, \mathcal{A}_f^\varepsilon(T)\right) \le C' e^{-\gamma M}, \tag{7.7}$$

where $\gamma > 0$ is independent of ε. Combining (7.6) and (7.7) and fixing M in an optimal way, we derive that

$$\operatorname{dist}_{\Phi_b}\left(\mathcal{A}^0, \mathcal{A}_f^\varepsilon(T)\right) \le C\alpha_{R_{max}}(\varepsilon)^\kappa. \tag{7.8}$$

It remains to note that (7.5) and (7.8) imply (7.1). Theorem 7.1 is proven. □

Corollary 7.1. *Let the assumptions of Theorem* 7.1 *hold. Then, for every $\phi \in \mathcal{H}(f)$ and every $T \in \mathbb{R}$, the family of sets $\{\mathcal{A}_\phi^\varepsilon(T), \varepsilon \in [0, \varepsilon_0]\}$ is upper and lower semicontinuous at $\varepsilon = 0$. Moreover,*

$$\sup_{\phi\in\mathcal{H}(f)}\sup_{T\in\mathbb{R}}\operatorname{dist}_{symm,\Phi_b}\left(\mathcal{A}_\phi^\varepsilon(T), \mathcal{A}^0\right) \to 0 \quad \text{as } \varepsilon \to 0 \tag{7.9}$$

and, consequently,

$$\operatorname{dist}_{symm,\Phi_b}\left(\mathcal{A}^\varepsilon, \mathcal{A}^0\right) \to 0 \quad \text{as} \quad \varepsilon \to 0, \tag{7.10}$$

where $\mathcal{A}^\varepsilon$ is a uniform attractor, associated with equation (0.1).

Assume now that the assumptions of Section 4 are satisfied, then Theorem 7.1 implies the following estimate for quantity (7.1).

Corollary 7.2. *Let the assumptions of Theorems* 7.1 *and* 4.1 *hold. Then, for every $\phi \in \mathcal{H}(f)$ and every $T \in \mathbb{R}$, we have*

$$\operatorname{dist}_{symm,\Phi_b}\left(\mathcal{A}_\phi^\varepsilon(T), \mathcal{A}^0\right) \le C_f\varepsilon^{\kappa'}, \tag{7.11}$$

where $0 < \kappa' < 1$ and $C_f > 0$ are independent of $\phi \in \mathcal{H}(f)$ and $T \in \mathbb{R}$.

Particularly, if the nonlinearity f is quasiperiodic with m independent frequencies $\omega \in \mathbb{R}^m$ with respect to t and smooth enough (i.e., (4.17) and

(4.19) are satisfied), then, due to Proposition 4.2, estimate (7.11) holds for almost every frequency vector $\omega \subset \mathbb{R}^m$. Note however, that the constant C_f in this case depends on C_ω in Diophantine condition (4.18) ($C_f \sim C_\omega^{\kappa'}$) and, consequently, it is extremely sensitive to small perturbations of the frequency vector $\omega \in \mathbb{R}^m$. Thus, since in applications only an approximate value of the frequency vector $\omega \in \mathbb{R}^m$ is usually known, the quantitative estimate of distance between homogenized and nonhomogenized attractors in the form (7.11) seems senseless. One is the most natural way to overcome this problem is to give a probabilistic interpretation of estimate (7.1). Under this approach, the frequency vector $\omega \in \mathbb{R}^m$ is treated as a random variable and one investigates the expectation of random variable (7.1). For simplicity, we only consider the case of Gaussian random variable ω although the concrete type of distribution function seems to be non essential for the estimates formulated below.

Theorem 7.2. *Let the assumptions of Theorem* 7.1 *be valid. Assume also that ω_i, $i = 1, \cdots, m$ be independent Gaussian random variables with expectations m_i and dispersions σ_i. Let, in addition, the nonlinearity f has the form*

$$f(t, x, u) := G(\omega_1 t, \cdots, \omega_m t, x, u), \tag{7.12}$$

where the fixed deterministic function $G : \mathbb{T}^m \to \mathcal{M}$ satisfies the condition

$$G \in C^m (\mathbb{T}^m, \mathcal{M}). \tag{7.13}$$

Then, the attractors $\mathcal{A}^\varepsilon_\phi(t)$ can be also interpreted as random variables. Moreover, the expectation $\mathbb{M}$ of symmetric distance (7.1) *possesses the following estimate:*

$$\mathbb{M}\left\{ \sup_{\phi \in \mathcal{H}(f)} \sup_{T \in \mathbb{R}} \operatorname{dist}_{symm,\Phi_p} \left(\mathcal{A}^\varepsilon_\phi(T), \mathcal{A}^0\right) \right\} \le C(\sigma)\varepsilon^\kappa, \tag{7.14}$$

where $C(\sigma)$ and $0 < \kappa < 1$ are some constants, and in contrast to (7.11) *the dependence of C on σ and G is regular.*

Proof. Indeed, for fixed ω arguing as in the proof of Theorem 7.1, but using estimate (4.41) instead of (4.2), we derive the estimate

$$\operatorname{dist}_{symm,\Phi_p} \left(\mathcal{A}^\varepsilon_\phi(T), \mathcal{A}^0\right) \le C L^{\kappa_1}_{R_{max}} \varepsilon^\kappa, \tag{7.15}$$

where $0 < \kappa < 1$, $0 < \kappa_1 < 1$,

$$L_{R_{max}} := \sum_{l \in \mathbb{Z}^m,\, l \neq 0} (1 + |(w, l)|^{-1+\delta}) \|A_l\|_{R_{max}}, \tag{7.16}$$

$\delta > 0$ is a small fixed parameter, and the Fourier coefficients A_l are defined in (4.17) (see Remark 4.3). Let us estimate random variable (7.15). First of all, we note that, due to (7.13) (see also (4.20)),

$$L_{R_{max}} \leq C_1 \sum_{l\in\mathbb{Z}^m,\, l\neq 0} (1 + |(\omega, l)|^{-1+\delta})|l|^{-m} \tag{7.17}$$

and, consequently, we have

$$\begin{aligned} &\mathbb{M}\Big\{ \sup_{\phi\in\mathcal{H}(f)} \sup_{T\in\mathbb{R}} \operatorname{dist}_{symm,\Phi_p} \left(\mathcal{A}^\varepsilon_\phi(T), \mathcal{A}^0\right) \Big\} \leq \\ &\leq C_2\varepsilon^\kappa \Big(\sum_{l\in\mathbb{Z}^m,\, l\neq 0} \mathbb{M}\Big\{|(\omega, l)|^{-1+\delta}\Big\}|l|^{-m}\Big)^{\kappa_1} \end{aligned} \tag{7.18}$$

(here, we have implicitly used the fact that $\kappa_1 < 1$). So, it remains to prove that the series in the right-hand side of (7.18) is finite. To this end, we recall that (w, l) is also a Gaussian random variable with expectation $m(l) := \sum_{i=1}^m l_l m_i$ and with dispersion $\sigma^2(l) := \sum_{i=1}^m |l_i|^2\sigma_i^2$. Consequently,

$$\begin{aligned} &\mathbb{M}\big\{|(\omega, l)|^{-1+\delta}\big\} = (2\pi\sigma(l))^{-1/2} \int_R |z|^{-1+\delta} e^{-(z-m(l))^2/(2\sigma(l))}\, dz \\ &\leq \sigma(l)^{(-1+\delta)/2} \int_R |z|^{-1+\delta} e^{-z^2/2}\, dz \leq C_\delta |\sigma|^{(-1+\delta)/2}|l|^{(-1+\delta)/2}. \end{aligned} \tag{7.19}$$

Here we have essentially used the fact that $\delta > 0$. Inserting (7.19) in the right-hand (7.18), we derive estimate (7.14). Theorem 7.2 is proven. □

Remark 7.1. Note that the exponent κ_1 in (7.15) is less than $1/(2p-1) < 1/2$ (see (4.11)), consequently, arguing as in the proof of Theorem 7.2, we may deduce that

$$\mathbb{M}\Big\{ \sup_{\phi\in\mathcal{H}(f)} \sup_{T\in\mathbb{R}} |\operatorname{dist}_{symm,\Phi_p} \left(\mathcal{A}^\varepsilon_\phi(T), \mathcal{A}^0\right)|^2 \Big\} \leq C'\varepsilon^{2\kappa} \tag{7.20}$$

and, consequently, the dispersion $\mathbb{D}$ of the random variable (7.1) possesses the estimate:

$$\mathbb{D}\Big\{ \sup_{\phi\in\mathcal{H}(f)} \sup_{T\in\mathbb{R}} \operatorname{dist}_{symm,\Phi_p} \left(\mathcal{A}^\varepsilon_\phi(T), \mathcal{A}^0\right) \Big\} \leq C'\varepsilon^\kappa \tag{7.21}$$

which is analogous to (7.14).

In conclusion of this section, we investigate the dependence of the attractors $\mathcal{A}^\varepsilon_f(t)$ on t under the assumptions of Theorem 6.3. We first recall that, according to (6.29), these attractors consist of a finite collection of the corresponding unstable manifolds $\mathcal{M}^+_{z_0,f}(t,\varepsilon)$ and the dependence of these manifolds on t is almost periodic. Note however that, in contrast of $\mathcal{A}^\varepsilon_f(t)$, these manifolds are not compact (and not closed) in the phase space Φ_p and, consequently, these almost-periodicities are not very informative. In order to overcome this disadvantage, we prove below the almost-periodicity of the function $t \to \mathcal{A}^\varepsilon_f(t)$ as a set-valued function. To this end, we need to introduce the following space.

Definition 7.1. Let B_R be an R-ball in the space Φ_{2p}, which contains all attractors $\mathcal{A}^\varepsilon_\phi(t)$, $t \in \mathbb{R}$, $\phi \in \mathcal{H}(f)$ (such R exists due to Theorem 2.1). Consider, a space $\mathcal{P}$ of all closed subsets of B_R endowed by the symmetric Hausdorff distance as a metric:

$$\mathcal{P} := \{B \subset B_R : \ B = \overline{B}, \ d(B_1, B_2) := \operatorname{dist}_{symm,\Phi_p}(B_1, B_2)\}. \tag{7.22}$$

Then, as known, $(\mathcal{P}, d)$ thus defined is a complete metric space.

Theorem 7.3. *Let the assumptions of Theorem* 6.3 *hold. Then, for every* $\varepsilon < \varepsilon_0$ *and every* $\phi \in \mathcal{H}(f)$, *the function* $t \to \mathcal{A}^\varepsilon_\phi(t)$ *is continuous and almost-periodic as a function with values in* $\mathcal{P}$:

$$\mathcal{A}^\varepsilon_\phi(\cdot) \in AP(\mathbb{R}, \mathcal{P}). \tag{7.23}$$

Proof. As before, we restrict ourselves to prove the theorem only for $\phi = f$ (the case of arbitrary $\phi \in \mathcal{H}(f)$ is analogous).

Let $\tilde f := T_s f$, s and τ be arbitrary positive numbers and let $u^\varepsilon_f(t)$, $v^\varepsilon_{\tilde f}$ be solutions of (0.1) with the nonlinearities f and $\tilde f$ and with initial values $u^\varepsilon_f(t-\tau) = u_\tau$ and $v^\varepsilon_{\tilde f}(t-\tau) = v_\tau$ respectively. Then

$$\|u^\varepsilon_f(t) - v^\varepsilon_{\tilde f}(t)\|_{\Phi_b} \le Ce^{K\tau}\left(\|f - \tilde f\|_R + \|u_\tau - v_\tau\|_{\Phi_p}\right) \tag{7.24}$$

if $\|u_\tau\|_{\Phi_p}, \|v_\tau\|_{\Phi_p} \le R$ (see (2.3) for the definition of $\|\cdot\|_R$). Indeed, the function $w(t) := u^\varepsilon_f(t) - v^\varepsilon_{\tilde f}(t)$ satisfies the equation

$$\begin{aligned}\partial_t w(t) - a\Delta_x w(t) = &-[f(t/\varepsilon, x, u^\varepsilon_f(t)) - f(t/\varepsilon, x, v^\varepsilon_{\tilde f}(t))] \\ &+[\tilde f(t/\varepsilon, x, v^\varepsilon_{\tilde f}(t)) - f(t/\varepsilon, x, v^\varepsilon_{\tilde f}(t))].\end{aligned} \tag{7.25}$$

Then, according to (2.23) and (0.6), the first term in the right-hand of (7.25) can be estimated via

$$|f(t/\varepsilon, x, u^\varepsilon_f(t)) - f(t/\varepsilon, x, v^\varepsilon_{\tilde f}(t))| \le Q_R|u^\varepsilon_f(t) - v^\varepsilon_{\tilde f}(t)| \tag{7.26}$$

and the second term can be estimated by

$$|\tilde f(t/\varepsilon, x, v^\varepsilon_{\tilde f}(t)) - f(t/\varepsilon, x, v^\varepsilon_{\tilde f}(t))| \le \|f - \tilde f\|_R. \tag{7.27}$$

Using estimates (7.26) and (7.27) and arguing as in the proof of Theorem 1.2, we derive estimate (7.24).

Let us now verify now the assertion of the theorem. To this end, we verify that the function $\mathcal{A}_f^\varepsilon(t)$ also has a relatively dense set of μ-almost periods, for every $\mu > 0$. We derive this fact from the estimate

$$\operatorname{dist}_{symm,\Phi_p}\left(\mathcal{A}_f^\varepsilon(t+s),\mathcal{A}_f^\varepsilon(t)\right) \le C_1\|f - T_{s/\varepsilon}f\|_R^\kappa, \tag{7.28}$$

for some $0 < \kappa < 1$ and $C_1 > 0$ which are independent of $t, s \in \mathbb{R}$. Indeed, (7.28) implies, from the one side, that $\mathcal{A}_f^\varepsilon(t)$ is uniformly continuous with values in $\mathcal{P}$ since f is uniformly continuous in $\mathcal{M}$. From the other side, (7.28) implies that every μ-almost period T_μ/ε of $f(t/\varepsilon, x, u)$ is simultaneously an $C_1\mu^\kappa$-almost period of $\mathcal{A}_f^\varepsilon(t)$ and, consequently, $\mathcal{A}_f^\varepsilon(t)$ is indeed almost-periodic with values in $\mathcal{P}$ thanks to Bochner-Amerio criteria.

Thus, it remains to prove estimate (7.28). Indeed, let $u_0 \in \mathcal{A}_f^\varepsilon(T)$ and $u_f^\varepsilon(t) \in \mathcal{A}_f^\varepsilon(t)$ be a complete bounded solution of (0.1) such that $u_f^\varepsilon(T) = u_0$. Let $v_{\tilde f}^\varepsilon(t)$, $t \ge T - \tau$ be a solution of (0.1) with the right-hand side $\tilde f := T_{s/\varepsilon}f(t/\varepsilon, x, u)$ and with the initial condition $v_{\tilde f}^\varepsilon(T-\tau) := u_f^\varepsilon(t-\tau)$. Then, from the one side thanks to (7.24), we have

$$\|u_0 - v_{\tilde f}^\varepsilon(T)\|_{\Phi_p} \le Ce^{K\tau}\|f - T_{s/\varepsilon}f\|_R. \tag{7.29}$$

From the other side, thanks to (6.32), we have

$$\operatorname{dist}_{\Phi_p}\left(v_{\tilde f}^\varepsilon(T),\mathcal{A}_f^\varepsilon(T+s)\right) \le Ce^{-\gamma\tau}. \tag{7.30}$$

Combining (7.29) and (7.30) and fixing τ in an optimal way (as in the proof of Theorem 7.1), we obtain that

$$\operatorname{dist}_{\Phi_p}\left(\mathcal{A}_f^\varepsilon(T),\mathcal{A}_f^\varepsilon(T+s)\right) \le C_1\|f - T_{s/\varepsilon}f\|_R^\kappa \tag{7.31}$$

uniformly with respect to $T \in \mathbb{R}$. The opposite estimate

$$\operatorname{dist}_{\Phi_p}\left(\mathcal{A}_f^\varepsilon(T+s),\mathcal{A}_f^\varepsilon(T)\right) \le C_1\|f - T_{s/\varepsilon}f\|_R^\kappa$$

can be proven analogously. Estimate (7.28) is proven. Theorem 7.3 is proven. □

Remark 7.2. It can be easily verified that, under the assumptions of the previous theorem, the hull $\mathcal{H}(\mathcal{A}_f^\varepsilon(\cdot))$ of the almost-periodic function $\mathcal{A}_f^\varepsilon(t)$ in $C_b(\mathbb{R},\mathcal{P})$ possesses the following description:

$$\mathcal{H}(\mathcal{A}_f^\varepsilon(\cdot)) = \{\mathcal{A}_\phi^\varepsilon(\cdot),\ \phi \in \mathcal{H}(f)\} \tag{7.32}$$

and, consequently (analogously to (6.43)), we have

$$\mathcal{A}^\varepsilon = \mathcal{H}(\mathcal{A}_f^\varepsilon(\cdot))\big|_{t=0}. \tag{7.33}$$

8. Examples

In this concluding section, we give several concrete examples of equations of the form (0.1) for which the conditions of Section 7 are satisfied. We start with the most simple case of a scalar equation

Example 8.1. Let $k = 1$. Consider a scalar equation

$$\partial_t u = a\Delta_x u - f(t/\varepsilon, x, u) + g(x),\ \ x \in \Omega \subset\subset \mathbb{R}^n,\ \ u\big|_{\partial\Omega} = 0. \tag{8.1}$$

Assume that the function $f(h, x, u)$ is almost-periodic with values in $\mathcal{M}$ (see Section 2), satisfies the standard condition

$$f(h, x, v) \ge 0, \quad \text{for}\ \ |v| > L, \tag{8.2}$$

for a sufficiently large fixed L and $h \in \mathbb{R}$, $x \in \Omega$, and growth restriction (0.4). Then, our dissipativity assumption (0.5) is satisfied for every p and, consequently, the assumptions of Sections 2 and 3 are fulfilled for equation (8.1) (in a fact growth restriction (0.4) is also can be omitted for the scalar case due to the maximum principle). Moreover, the limit homogenized equation

$$\partial_t \overline{u} = a\Delta_x \overline{u} - \overline{f}(x, \overline{u}) + g(x),\ \ x \in \Omega \subset\subset \mathbb{R}^n,\ \ \overline{u}\big|_{\partial\Omega} = 0 \tag{8.3}$$

is autonomous and has a gradient form. Thus, for generic $g \in L^p(\Omega)$ this limit equation possesses a regular attractor (see e.g. [2]). Therefore, the assumptions of Section 6 are also satisfied for generic external forces $g \in L^p(\Omega)$.

Example 8.2. In the following example, we consider a system of two coupled parabolic equations, which is a nonautonomous analogue of the so-called FitzHugh-Nagumo system:

$$\begin{cases} \partial_t u_1 = d_1\Delta_x u_1 - f(t/\varepsilon, x, u_1) - u_2 + g_1(x), \\ \partial_t u_2 = d_2\Delta_x u_2 - \gamma u_2 + \delta u_1 + g_2(x), \\ u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega} = 0, \end{cases} \tag{8.4}$$

where $u = (u_1, u_2)$, $x \in \Omega \subset\subset \mathbb{R}^n$, $d_1, d_2, \gamma, \delta > 0$ are positive parameters. Assume also that the nonlinear interaction function $f(h, x, v)$ is almost periodic with values in $\mathcal{M}$ (see Section 2) and satisfies the following assumptions:

$$\begin{cases} 1.\ \ C(1 + |v|^{q+1}) \ge f(h, x, v) \cdot v \ge -C_1 + C_2|v|^{q+1}, \quad q > 1 \\ 2.\ \ f'_v(h, x, v) \ge -K, \end{cases} \tag{8.5}$$

for some constants C, C_i, and K. Then, a simple calculation reveals that anisotropic dissipativity assumption (0.5) is satisfied for every $p_1 = p_2 > 0$ and, consequently, the results of Sections 1, 2, and 3 hold for system (8.4). Assume now, in addition, that

$$\gamma > K. \tag{8.6}$$

Then, the limit averaged problem

$$\begin{cases} \partial_t \overline{u}_1 = d_1 \Delta_x \overline{u}_1 - \overline{f}(x, \overline{u}_1) - \overline{u}_2 + g_1(x), \\ \partial_t \overline{u}_2 = d_2 \Delta_x \overline{u}_2 - \gamma \overline{u}_2 + \delta \overline{u}_1 + g_2(x), \\ \overline{u}_1\big|_{\partial\Omega} = \overline{u}_2\big|_{\partial\Omega} = 0 \end{cases} \tag{8.7}$$

possesses a global Liapunov function in the form

$$\begin{aligned} L(\overline{u}_1, \overline{u}_2) := \int_\Omega \frac{|\partial_t \overline{u}_1|^2}{2} + \frac{|\partial_t \overline{u}_2|^2}{2\delta} + \gamma d_1 \frac{|\nabla_x \overline{u}_1|^2}{2} - \gamma d_2 \frac{|\nabla_x \overline{u}_2|^2}{2\delta} \\ + \gamma \overline{F}(x, \overline{u}_1) + \gamma \overline{u}_1 \cdot \overline{u}_2 - \gamma^2 \frac{|\overline{v}|^2}{2\delta} - \gamma g_1 \overline{u}_1 + \frac{\gamma}{\delta} g_2 \overline{u}_2 \, dx \end{aligned} \tag{8.8}$$

where $\overline{F}(x,v) := \int_0^v \overline{f}(x,v)\,dv$, see [12] for details. Thus, in this case, for generic $g = (g_1, g_2) \in L^p(\Omega)^2$ the attractor of averaged problem (8.7) is regular, consequently, the assumptions of Section 6 are satisfied for (8.4) and, therefore, for sufficiently small ε, the nonautonomous attractor of it is also regular, almost-periodic and tends to the attractor of limit equation (8.7) as $\varepsilon \to 0$ (see Theorems 6.3, 7.1 and 7.3).

Example 8.3. In the last example, we consider the following generalization of the Lotka-Volterra system:

$$\begin{cases} \partial_t u_i(t) = a_i \Delta_x u_i(t) - f_i(u_i(t)) - u_i(t) \left(\sum_{j=1}^k b_{i,j}(t/\varepsilon,) u_j^2(t) \right) + g_i(x), \\ u_i\big|_{\partial\Omega} = 0, \quad i = 1, \cdots, k \end{cases} \tag{8.9}$$

where $a_i > 0$, $b_{ij}(h)$ are scalar almost periodic functions, such that $b_{ij}(h) \geq 0$ and the functions f_i satisfy the assumptions

$$1. \ \ f \in C^2(\mathbb{R}, \mathbb{R}), \quad 2. \ \ C(1 + |u_i|^{q_i}) \geq f_i(u_i) \cdot u_i \geq -C_1 + C_2 |u_i|^{q_i}, \tag{8.10}$$

for some fixed vector $(q_1, \cdots, q_k)$ satisfying $q_i > 1$. Then, dissipativity assumption (0.5) is obviously satisfied, for every $p_1 = \cdots = p_k > 0$. Consequently, the results of Sections 1-3 hold for system (8.9). Moreover, the limit averaged equation

$$\begin{cases} \partial_t \overline{u}_i(t) = a_i \Delta_x \overline{u}_i(t) - f_i(\overline{u}_i(t)) - \overline{u}_i(t) \left(\sum_{j=1}^k \overline{b}_{i,j} \overline{u}_j^2(t) \right) + g_i(x), \\ \overline{u}_i \big|_{\partial\Omega} = 0, \quad i = 1, \cdots, k \end{cases} \tag{8.11}$$

has a gradient form if $\overline{b}_{ij} = \overline{b}_{ji}$ and, consequently, its attractor is regular for generic $g \in [L^p(\Omega)]^k$. Thus, the results of Sections 5-7 are also valid for equation (8.9) for generic external forces g.

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