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A FAMILY OF SHARP INEQUALITIES FOR SOBOLEV FUNCTIONS

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Abstract. Let $N \geq 5$, Ω be a smooth bounded domain in $\mathbb{R}^N$, $2^* = \frac{2N}{N-2}$, $a > 0$, $S = \inf\big\{\int_{\mathbb{R}^N}|\nabla u|^2 : u \in L^{2^*}(\mathbb{R}^N), \nabla u \in L^2(\mathbb{R}^N), \int_{\mathbb{R}^N}|u|^{2^*} = 1\big\}$ and $||u||^2 = |\nabla u|_2^2 + a|u|_2^2$. We define $2^\flat = \frac{2N}{N-1}$, $2^\# = \frac{2(N-1)}{N-2}$ and consider q such that $2^\flat \leq q \leq 2^\#$. We also define $s = 2 - N + \frac{q}{2^*-q}$ and $t = \frac{2}{N-2}\cdot\frac{1}{2^*-q}$. We prove that there exists an $\alpha_0(q,a,\Omega) > 0$ such that, for all $u \in H^1(\Omega)\setminus\{0\}$,

$$\frac{S}{2^{\frac{2}{N}}}|u|_{2^*}^2 \leq ||u||^2 + \alpha_0\Big(\frac{||u||}{|u|_{2^*}^{2^*/2}}\Big)^s |u|_q^{qt}, \qquad (I)_q$$

where the norms are over Ω. Inequality $(I)_{2^\flat}$ is due to M. Zhu.

1. Introduction

Let $N \geq 5$, Ω be a smooth bounded domain in $\mathbb{R}^N$, $2^\flat = \frac{2N}{N-1}$, $2^\# = \frac{2(N-1)}{N-2}$, $2^* = \frac{2N}{N-2}$, $a > 0$ and $||u||^2 = |\nabla u|_2^2 + a|u|_2^2$. Unless otherwise indicated, norms are over Ω. We recall that the infimum

$$S := \inf_{\substack{u \in L^{2^*}(\mathbb{R}^N)\setminus\{0\} \\ \nabla u \in L^2(\mathbb{R}^N)}} \frac{\int_{\mathbb{R}^N}|\nabla u|^2}{\Big(\int_{\mathbb{R}^N}|u|^{2^*}\Big)^{2/2^*}}$$

is achieved by the Talenti instanton $U(x) := (\frac{N(N-2)}{N(N-2)+|x|^2})^{\frac{N-2}{2}}$.

M. Zhu proved in [23] that there exists $\bar{\alpha}_0 > 0$ such that

$$\frac{S}{2^{\frac{2}{N}}}|u|_{2^*}^2 \leq ||u||^2 + \bar{\alpha}_0|u|_{2^\flat}^2, \qquad (1.1)$$

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for all $u \in H^1(\Omega)$. It was announced by the author in [12] that there exists $\tilde{\alpha}_0 > 0$ such that

$$\frac{S}{2^{\frac{2}{N}}}|u|_{2^*}^2 \leq ||u||^2 + \tilde{\alpha}_0 \frac{||u||}{|u|_{2^*}^{2^*/2}}|u|_{2^\#}^{2^\#}, \tag{1.2}$$

for all $u \in H^1(\Omega) \setminus \{0\}$. In this work we prove a family of inequalities which includes (1.1) and (1.2) as special cases.

The work of M. Zhu was motivated by the works [1] and [19], by Adimurthi and Mancini and by X.J. Wang, respectively. They imply that one cannot expect the existence of a constant $\bar{\alpha}_0$ such that

$$\frac{S}{2^{\frac{2}{N}}}|u|_{2^*}^2 \leq ||u||^2 + \bar{\alpha}_0|u|_2^2,$$

for all $u \in H^1(\Omega)$. In [23], M. Zhu raises the L^2 norm on the right hand side to a higher L^q norm in order to obtain an inequality valid in $H^1(\Omega)$.

The work [12] was motivated by [19], the referred work of X.J. Wang, and by [10], by D.G. Costa and the author. Both [19] and [10] consider the problem

$$\begin{cases} -\Delta u + au + \alpha u^{q-1} = u^{2^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\mathcal{P}$}_{\alpha,q}$$

From [19] we know that if $q < 2^\#$, then problem $(\mathcal{P})_{\alpha,q}$ has a ground state solution for all values of $\alpha \geq 0$. From [10] we know that there exists $\alpha_0 > 0$ such that if $\alpha < \alpha_0$, then problem $(\mathcal{P})_{\alpha,2^\#}$ has a ground state solution and if $\alpha > \alpha_0$, then problem $(\mathcal{P})_{\alpha,2^\#}$ has no ground state solution. The solutions of $(\mathcal{P})_{\alpha,q}$ correspond to critical points of the functional $\Phi_\alpha : H^1(\Omega) \to \mathbb{R}$, defined by

$$\Phi_\alpha(u) := \frac{1}{2}||u||^2 + \frac{\alpha}{q}|u|_q^q - \frac{1}{2^*}|u|_{2^*}^{2^*}. \tag{1.3}$$

We recall that a ground state solution, or least energy solution, of $(\mathcal{P})_{\alpha,q}$ is a function $u \in H^1(\Omega)$ such that

$$\Phi_\alpha(u) = \inf_{\mathcal{N}} \Phi_\alpha.$$

The set $\mathcal{N}$ is the Nehari manifold,

$$\mathcal{N} := \{u \in H^1(\Omega) \setminus \{0\} : \Phi'_\alpha(u)u = 0\}.$$

When $q = 2^\#$, it is possible to determine explicitly the function $\Phi_\alpha|_{\mathcal{N}}$ by solving a quadratic equation. The analysis of [10] takes advantage of this fact. As a by-product it implies a certain inequality (see (15) of [10]). Inequality (1.2) is an improvement of the inequality in [10].

The idea of the proof of inequalities (1.1) and (1.2) is based on an argument by contradiction. Indeed, consider the the functionals $\Psi_\alpha : H^1 \setminus \{0\} \to \mathbb{R}$ defined by

$$\Psi_\alpha(u) = \frac{||u||^2}{|u|_{2^*}^2} + \alpha \frac{|u|_{2^\flat}^2}{|u|_{2^*}^2} \quad \text{or} \quad \Psi_\alpha(u) = \frac{||u||^2}{|u|_{2^*}^2} + \alpha \frac{||u||}{|u|_{2^*}^{2+2^*/2}} |u|_{2^\#}^{2^\#}.$$

Let (α_k) be any sequence of nonnegative real numbers such that $\alpha_k \to +\infty$. If (1.1) (respectively (1.2)) is false, then, for each k, $\inf_{H^1(\Omega)\setminus\{0\}} \Psi_{\alpha_k} < \frac{S}{2^{\frac{2}{N}}}$. This implies that Ψ_{α_k} has a line of minima (with 0 removed), which are called least energy critical points of Ψ_{α_k}. One of these, u_k, satisfying an appropriate normalization condition, is chosen. Using the blow-up technique, it is possible to prove that there exists a sequence (U_k) of Talenti instantons, concentrating at the boundary of Ω, such that the H^1 norm of the difference between u_k and U_k approaches zero, as $k \to +\infty$. The value of $\Psi_{\alpha_k}(U_k)$ can be used to estimate $\Psi_{\alpha_k}(u_k)$ from below. However, $\Psi_{\alpha_k}(U_k) > \frac{S}{2^{\frac{2}{N}}}$ for large k. This contradicts the hypothesis that $\alpha_0 = +\infty$. We use this argument to prove our family of inequalities. We remark that in the present analysis the functional Φ_α in (1.3) is replaced by $\Phi_\alpha : H^1(\Omega) \setminus \{0\} \to \mathbb{R}$ defined by

$$\Phi_\alpha(u) = \big(\frac{1}{2}||u||^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}\big)\big(1 + \alpha\delta(u)\big)^{\frac{N}{2}},$$

where $\delta : H^1(\Omega) \setminus \{0\} \to \mathbb{R}$, depending on q, is homogeneous of degree zero. This leads to the problem

$$\begin{cases} \Big(1 + \frac{s}{2}\alpha\delta(u)\Big)(-\Delta u + au) & & & \\ \qquad + \frac{qt}{2}\alpha|u|_q^{q(t-1)}|u|^{q-2}u & = & \Big(1 + \Big(1 + s\frac{2^*}{4}\Big)\alpha\delta(u)\Big)|u|^{2^*-2}u & \text{in } \Omega, \\ u & > & 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} & = & 0 & \text{on } \partial\Omega, \end{cases}$$

where $s \in [0,1]$ and $t \in [\frac{2}{2^\flat}, 1]$ are constants which depend on q and N.

Our approach is based on the work [2], due to Adimurthi, Pacella and Yadava. We use [1], [10], [19] and [23], already mentioned. Of course, Talenti [18], Brezis and Nirenberg [7] and P.L. Lions [15] are also of major importance. To our knowledge, Hebey and Vaugon [13] were the first to use a contradiction argument based on blow-up estimates to obtain sharp Sobolev inequalities. We refer to Adimurthi and Yadava [3], Brezis and

Lieb [6], Chabrowski and Willem [8], Li and Zhu [14], Lions, Pacella and Tricarico [16], Z.Q. Wang [20, 21] and M. Zhu [22] for related results.

The organization of this work is as follows. In Section 2, we introduce a family of functionals, derive their associated Euler equations and state our main theorem. In Section 3, arguing by contradiction, we assume that least energy critical points exist for all positive values of α and analyze their asymptotic behavior. In Section 4, we prove our main theorem. Finally, in the Appendix we prove a technical estimate similar to those in Adimurthi and Mancini [1].

2. The functionals and their associated Euler equations

Let $N \geq 5$, $a > 0$, $\alpha \geq 0$ and Ω be a smooth bounded domain in $\mathbb{R}^N$. We regard a as fixed and α as a parameter. Denote the L^p and H^1 norms of u in Ω by

$$|u|_p := \left(\int |u|^p\right)^{\frac{1}{p}} \qquad \text{and} \qquad ||u|| := \left(|\nabla u|_2^2 + a|u|_2^2\right)^{\frac{1}{2}}.$$

Unless otherwise indicated, integrals are over Ω. Let

$$2^* = 2^*(N) := \frac{2N}{N-2}$$

be the critical exponent for the Sobolev embedding $H^1(\Omega) \subset L^q(\Omega)$,

$$2^\flat = 2^\flat(N) := \frac{2N}{N-1} \qquad \text{and} \qquad 2^\# = 2^\#(N) := \frac{2(N-1)}{N-2}.$$

We consider q such that $2^\flat \leq q \leq 2^\#$, and define $s \in [0,1]$ and $t \in [\frac{2}{2^\flat}, 1]$ by

$$s = 2 - N + \tfrac{q}{2^*-q} \tag{2.1}$$

and

$$t = \tfrac{2}{N-2} \cdot \tfrac{1}{2^*-q}. \tag{2.2}$$

We easily check that†

$$qt = \tfrac{2}{N-2} \cdot s + 2. \tag{2.3}$$

Moreover,

$$\begin{aligned} q = 2^\flat \quad &\Longrightarrow \quad s = 0 \quad \text{and} \quad t = \tfrac{2}{2^\flat}, \\ q = 2^\# \quad &\Longrightarrow \quad s = 1 \quad \text{and} \quad t = 1. \end{aligned}$$

†The reader can also verify that $s = (N-1) \times \frac{q-2^\flat}{2^*-q}$ and $t = \frac{s}{N} + \frac{N-1}{N}$.

We recall that the infimum

$$S := \inf_{\substack{u \in L^{2^*}(\mathbb{R}^N) \setminus \{0\} \\ \nabla u \in L^2(\mathbb{R}^N)}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}},$$

which depends on N, is achieved by the Talenti instanton

$$U(x) := \Big(\frac{N(N-2)}{N(N-2)+|x|^2}\Big)^{\frac{N-2}{2}}.$$

This instanton U satisfies

$$-\Delta U = U^{2^*-1}, \tag{2.4}$$

so that

$$\int_{\mathbb{R}^N} |\nabla U|^2 = \int_{\mathbb{R}^N} U^{2^*} = S^{\frac{N}{2}} = [N(N-2)]^{\frac{N}{2}} \omega_N \frac{1}{2^N} \sqrt{\pi} \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N+1}{2}\right)}. \tag{2.5}$$

The value ω_N is the volume of the $N-1$ dimensional unit sphere:

$$\omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}.$$

Substituting this value in the previous equation,

$$S^{\frac{N}{2}} = \frac{\pi^{\frac{N+1}{2}}}{2^{N-1}} \cdot \frac{[N(N-2)]^{\frac{N}{2}}}{\Gamma(\frac{N+1}{2})}.$$

Let $\varepsilon > 0$ and $y \in \mathbb{R}^N$. We define the rescaled instanton

$$U_{\varepsilon,y}(\,\cdot\,) := \varepsilon^{-\frac{N-2}{2}} U\Big(\frac{\cdot\, - y}{\varepsilon}\Big), \tag{2.6}$$

which also satisfies (2.4) and (2.5).

We are interested in studying the C^2 functionals $\Psi_\alpha : H^1(\Omega) \setminus \{0\} \to \mathbb{R}$, defined by

$$\Psi_\alpha(u) := \frac{||u||^2}{|u|_{2^*}^2}\Big(1 + \alpha \frac{|u|_q^{qt}}{||u||^{2-s}|u|_{2^*}^{2^*s/2}}\Big). \tag{2.7}$$

We regard Ψ_α as a restricted functional, in following sense. Consider the functionals β and $\delta : H^1(\Omega) \setminus \{0\} \to \mathbb{R}$, homogeneous of degree zero, defined by

$$\beta(u) := \frac{||u||^2}{|u|_{2^*}^2} \quad \text{and} \quad \delta(u) := \frac{|u|_q^{qt}}{||u||^{2-s}|u|_{2^*}^{2^*s/2}}.$$

We can write Ψ_α in terms of α, β and δ as $\Psi_\alpha = \beta(1+\alpha\delta)$. Consider also the C^2 functionals $\Phi_\alpha : H^1(\Omega)\setminus\{0\} \to \mathbb{R}$, defined by

$$\Phi_\alpha(u) := \Big(\frac{1}{2}||u||^2 - \frac{1}{2^*}|u|_{2^*}^{2^*}\Big)\Big(1+\alpha\frac{|u|_q^{qt}}{||u||^{2-s}|u|_{2^*}^{2^*s/2}}\Big)^{\frac{N}{2}} = \Phi_0(u)(1+\alpha\delta(u))^{\frac{N}{2}}.$$

We recall that the Nehari manifold is

$$\begin{aligned}\mathcal{N} :&= \{u\in H^1(\Omega)\setminus\{0\} : \Phi_\alpha'(u)u = 0\}\\ &= \{u\in H^1(\Omega)\setminus\{0\} : ||u||^2 = |u|_{2^*}^{2^*}\}.\end{aligned}$$

For any $u \in H^1(\Omega)\setminus\{0\}$, there exists a unique $\tau(u) > 0$ such that $\tau(u)u \in \mathcal{N}$. The value of $\tau(u)$ is

$$\tau(u) = \Big(\frac{||u||^2}{|u|_{2^*}^{2^*}}\Big)^{\frac{N-2}{4}} \quad\text{and}\quad \tfrac{1}{N}\Big(\Psi_\alpha(u)\Big)^{\frac{N}{2}} = \Phi_\alpha(\tau(u)u).$$

Next we derive the Euler equation associated to Φ_α. Since

$$\Phi_\alpha' = (1+\alpha\delta)^{\frac{N}{2}-1}\left[\Phi_0'(1+\alpha\delta) + \tfrac{N}{2}\Phi_0\alpha\delta'\right]$$

and

$$\begin{aligned}\delta'(u)(\varphi) = &-(2-s)\frac{\delta(u)}{||u||^2}\int(\nabla u\cdot\nabla\varphi + au\varphi)\\ &+ qt\frac{\delta(u)}{|u|_q^q}\int(|u|^{q-2}u\varphi) - \frac{2^*}{2}s\frac{\delta(u)}{|u|_{2^*}^{2^*}}\int(|u|^{2^*-2}u\varphi),\end{aligned}$$

for all $\varphi\in H^1(\Omega)$, the critical points of Φ_α satisfy

$$\begin{aligned}&\Big[1+\alpha\delta(u)\Big(1-(2-s)\tfrac{N}{4} + (2-s)\tfrac{N-2}{4}\tfrac{|u|_{2^*}^{2^*}}{||u||^2}\Big)\Big]\int(\nabla u\cdot\nabla\varphi + au\varphi)\\ &+\Big[\Big(\tfrac{1}{2}\Big(\tfrac{||u||}{|u|_{2^*}^{2^*/2}}\Big)^s - \tfrac{1}{2^*}\Big(\tfrac{|u|_{2^*}^{2^*/2}}{||u||}\Big)^{2-s}\Big)\Big]\tfrac{qtN}{2}\alpha|u|_q^{q(t-1)}\int(|u|^{q-2}u\varphi) \qquad (2.8)_\alpha\\ &-\Big[1+\alpha\delta(u)\Big(1-s\frac{N}{4}+s\frac{2^*N}{8}\frac{||u||^2}{|u|_{2^*}^{2^*}}\Big)\Big]\int(|u|^{2^*-2}u\varphi) = 0,\end{aligned}$$

for all $\varphi\in H^1(\Omega)$. However, this equation can be simplified. By taking $\varphi = u$, i.e., by differentiating Φ_α along the radial direction, we deduce that $||u||^2 = |u|_{2^*}^{2^*}$. So the critical points of Φ_α satisfy

$$\begin{cases}\big(1+\frac{s}{2}\alpha\delta(u)\big)(-\Delta u + au) + \frac{qt}{2}\alpha|u|_q^{q(t-1)}|u|^{q-2}u & \\ \quad = \big(1+\big(1+s\frac{2^*}{4}\big)\alpha\delta(u)\big)|u|^{2^*-2}u & \text{in } \Omega,\\ \frac{\partial u}{\partial\nu} = 0 & \text{on } \partial\Omega.\end{cases} \qquad (2.9)_\alpha$$

Conversely, we now check that the solutions of $(2.9)_\alpha$ are solutions of $(2.8)_\alpha$, i.e., the solutions of $(2.9)_\alpha$ satisfy

$$||u||^2 = |u|_{2^*}^{2^*}. \tag{2.10}$$

By multiplying $(2.9)_\alpha$ by u and integrating over Ω, we get

$$\Big(1+\tfrac{s}{2}\alpha\delta(u)\Big)||u||^2 + \tfrac{qt}{2}\alpha|u|_q^{qt} = \Big(1+\Big(1+s\tfrac{2^*}{4}\Big)\alpha\delta(u)\Big)|u|_{2^*}^{2^*}$$

or

$$\Big(1+\tfrac{s}{2}\alpha\delta(u)\Big)\Big(\tfrac{||u||}{|u|_{2^*}^{2^*/2}}\Big)^2 + \tfrac{qt}{2}\alpha\delta(u)\Big(\tfrac{||u||}{|u|_{2^*}^{2^*/2}}\Big)^{2-s} = 1+\Big(1+s\tfrac{2^*}{4}\Big)\alpha\delta(u). \tag{2.11}$$

Let $c_1 := 1+\frac{s}{2}\alpha\delta(u)$, $c_2 := \frac{qt}{2}\alpha\delta(u)$ and $\gamma := \frac{||u||}{|u|_{2^*}^{2^*/2}}$. Equation (2.3) implies that $c_1+c_2 = 1+(1+s\frac{2^*}{4})\alpha\delta(u)$. Hence, we can write (2.11) as

$$c_1\gamma^2 + c_2\gamma^{2-s} = c_1+c_2.$$

Therefore, γ has to be one, and the solutions of $(2.9)_\alpha$ are solutions of $(2.8)_\alpha$. The critical points of Ψ_α satisfy

$$\begin{cases} \Big(1+\frac{s}{2}\alpha\delta(u)\Big)\frac{(-\Delta u+au)}{||u||^2} & \\ \qquad +\frac{qt}{2}\alpha\delta(u)\frac{|u|^{q-2}u}{|u|_q^q} = \Big(1+\Big(1+s\frac{2^*}{4}\Big)\alpha\delta(u)\Big)\frac{|u|^{2^*-2}u}{|u|_{2^*}^{2^*}} & \text{in } \Omega,\\ \qquad \frac{\partial u}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

If u is a critical point of Φ_α, then every nonzero multiple of u, in particular u, is a critical point of Ψ_α. Conversely, if u is a critical point of Ψ_α, then $\tau(u)u$ is a critical point of Φ_α. We are interested in proving existence and nonexistence of *least energy* critical points of Φ_α, or equivalently of Ψ_α. We recall that a least energy critical point of Φ_α is a function $u \in H^1(\Omega)\setminus\{0\}$, such that

$$\Phi_\alpha(u) = \inf_{\mathcal{N}}\Phi_\alpha = \inf_{H^1(\Omega)\setminus\{0\}} \tfrac{1}{N}(\Psi_\alpha)^{\frac{N}{2}}.$$

Remark 2.1. System $(2.9)_\alpha$ possesses one and only one constant solution $u \equiv a^{\frac{N-2}{4}}$.

Our main result is

Theorem 2.2. *Let $N \geq 5$, Ω be a smooth bounded domain in $\mathbb{R}^N$, $a > 0$, $\alpha \geq 0$ and $2^\flat \leq q \leq 2^\#$. There exists a positive real number $\alpha_0 = \alpha_0(q,a,\Omega)$ such that*

(i) *if $\alpha < \alpha_0$, then Ψ_α has a least energy critical point u_α; $\Psi_\alpha(u_\alpha) < \frac{S}{2^{\frac{2}{N}}}$;*

(ii) *if $\alpha > \alpha_0$, then Ψ_α does not have a least energy critical point and*

$$\frac{S}{2^{\frac{2}{N}}} = \inf_{H^1(\Omega)\setminus\{0\}} \Psi_\alpha.$$

This theorem obviously implies that $\frac{S}{2^{\frac{2}{N}}} \leq \Psi_{\alpha_0}$, i.e.,

$$\frac{S}{2^{\frac{2}{N}}}|u|_{2^*}^2 \leq ||u||^2 + \alpha_0\Big(\frac{||u||}{|u|_{2^*}^{2^*/2}}\Big)^s |u|_q^{qt},$$

for all $u \in H^1(\Omega) \setminus \{0\}$.

Remark 2.3. It is easy to check that

$$\Psi_\alpha(1) = a|\Omega|^{\frac{2}{N}}\Big(1 + \frac{\alpha}{a^{\frac{2-s}{2}}|\Omega|^{1-t}}\Big).$$

So, if $a \leq \frac{S}{(2|\Omega|)^{\frac{2}{N}}}$, the least energy critical points of Ψ_α might be constant for α such that $\Psi_\alpha(1) \leq \frac{S}{2^{\frac{2}{N}}}$, i.e.,

$$\alpha \leq |\Omega|^{1-t} \cdot \frac{S/\Big(2|\Omega|\Big)^{\frac{2}{N}} - a}{a^{s/2}}.$$

This simple observation yields the following lower bound for α_0:

$$\alpha_0 \geq |\Omega|^{1-t} \cdot \frac{S/\Big(2|\Omega|\Big)^{\frac{2}{N}} - a}{a^{s/2}}.$$

A second lower bound for α_0 is given in Lemma 4.3.

Remark 2.4. Let $\kappa > 0$. By scaling, we easily check that

$$\alpha_0\big(q, a\,\kappa^2, \frac{\Omega}{\kappa}\big) = \kappa\,\alpha_0(q, a, \Omega).$$

In fact, if $u \in H^1(\Omega)$ and $v : \frac{\Omega}{\kappa} \to \mathbb{R}$ is defined by $v(x) = \kappa^{\frac{N-2}{2}}u(\kappa x)$, then $v \in H^1\Big(\frac{\Omega}{\kappa}\Big)$ satisfies

$$\kappa^2|v|^2_{L^2(\frac{\Omega}{\kappa})} = |u|_2^2, \quad \kappa|v|^{qt}_{L^q(\frac{\Omega}{\kappa})} = |u|_q^{qt}, \quad |v|_{L^{2^*}(\frac{\Omega}{\kappa})} = |u|_{2^*}, \quad |\nabla v|_{L^2(\frac{\Omega}{\kappa})} = |\nabla u|_2.$$

3. Asymptotic behavior of least energy critical points

We consider the minimization problem corresponding to

$$S_\alpha := \inf_{H^1(\Omega)\setminus\{0\}} \Psi_\alpha.$$

From Adimurthi and Mancini [1] and X.J. Wang [19], we know that

$$0 < S_0 < \frac{S}{2^{\frac{2}{N}}}. \tag{3.1}$$

Obviously, S_α is nondecreasing as α increases. Choose any point $P \in \partial\Omega$. By testing Ψ_α with $U_{\varepsilon,P}$ and letting $\varepsilon \to 0$, we conclude that $S_\alpha \leq \frac{S}{2^{\frac{2}{N}}}$ for all $\alpha \geq 0$.

Remark 3.1. If $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$, then S_α is achieved.

We can assume the minimizer is a nonnegative function. In fact, by the maximum principle, a nonnegative minimizer is positive in Ω.

Remark 3.2. The map $\alpha \mapsto S_\alpha$ is continuous on $[0, +\infty)$.

The proof of this remark is similar to the one of Lemma 3.2 of [10].

By the previous remark, the value

$$\alpha_0 := \sup\left\{\alpha \in \mathbb{R} : S_\alpha < \frac{S}{2^{\frac{2}{N}}}\right\} \tag{3.2}$$

is well defined. By (3.1) it is not zero. Remark 3.1 implies:

Remark 3.3. The map $\alpha \mapsto S_\alpha$ is strictly increasing on $[0, \alpha_0]$. If $\alpha \in (\alpha_0, +\infty)$, then Ψ_α does not have a least energy critical point.

Therefore, to prove Theorem 2.2, we just have to establish that α_0 is finite. Arguing by contradiction, we assume that the value α_0 in Theorem 2.2 is infinite and analyze the asymptotic behavior of least energy critical points as $\alpha \to +\infty$.

Lemma 3.4. *The limit of S_α as α tends to $+\infty$ is*

$$\lim_{\alpha\to+\infty} S_\alpha = \frac{S}{2^{\frac{2}{N}}}. \tag{3.3}$$

Suppose $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$ for all $\alpha \geq 0$. Choose a sequence $\alpha_k \to +\infty$ as $k \to +\infty$ and let u_k be a minimizer for Ψ_{α_k} satisfying $(2.9)_{\alpha_k}$. The sequence (u_k) satisfies

$$u_k \rightharpoonup 0 \text{ in } H^1(\Omega),$$

$$\lim_{k\to\infty} |\nabla u_k|_2^2 = \lim_{k\to\infty} |u_k|_{2^*}^{2^*} = \frac{S^{\frac{N}{2}}}{2} \tag{3.4}$$

and

$$\lim_{k\to\infty} \alpha_k \delta(u_k) = 0. \tag{3.5}$$

If we denote by

$$M_k \quad := \quad \max_{\bar{\Omega}} u_k \tag{3.6}$$

and

$$\epsilon_k \quad := \quad M_k^{-\frac{2}{N-2}}, \tag{3.7}$$

then

$$M_k \quad \to \quad +\infty \tag{3.8}$$

and

$$\alpha_k \epsilon_k \quad \to \quad 0, \tag{3.9}$$

as $k \to \infty$*.*

Proof. Suppose $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$ for all $\alpha \geq 0$ and choose a sequence $\alpha_k \to +\infty$ as $k \to +\infty$. Let u_k be a minimizer for Ψ_{α_k} satisfying $(2.9)_{\alpha_k}$, which necessarily exists by Remark 3.1 and rescaling. The functions u_k satisfy

$$\frac{||u_k||^2}{|u_k|_{2^*}^2} = \beta(u_k) < \Psi_{\alpha_k}(u_k) < \frac{S}{2^{\frac{2}{N}}}$$

and

$$||u_k||^2 = |u_k|_{2^*}^{2^*}, \tag{3.10}$$

because of (2.10). Together, $||u_k||^{\frac{4}{N}} < \frac{S}{2^{\frac{2}{N}}}$, the sequence (u_k) is bounded in $H^1(\Omega)$. The definition of Ψ_α (equality (2.7)) and (3.10) imply that

$$\alpha_k \frac{||u_k||^s |u_k|_q^{qt}}{|u_k|_{2^*}^{2+2^*s/2}} = \alpha_k \frac{|u_k|_q^{qt}}{|u_k|_{2^*}^2} < \frac{S}{2^{\frac{2}{N}}},$$

for all positive integers k. If we combine this inequality with the fact that the norms $|u_k|_{2^*}$ are uniformly bounded we deduce that $u_k \rightharpoonup 0$ in $H^1(\Omega)$. We can assume that $u_k \to 0$ a.e. on Ω, and $|\nabla u_k|^2 \rightharpoonup \mu$ and $|u_k|^{2^*} \rightharpoonup \nu$ in the sense of measures on $\bar{\Omega}$. So,

$$\lim_{k\to\infty} |\nabla u_k|_2^2 = ||\mu|| \quad \text{and} \quad \lim_{k\to\infty} |u_k|_{2^*}^{2^*} = ||\nu||,$$

where $\frac{S}{2^{\frac{2}{N}}}||\nu||^{\frac{2}{2^*}} \leq ||\mu||$. Now equality (3.3) follows from

$$\frac{S}{2^{\frac{2}{N}}} \leq \frac{||\mu||}{||\nu||^{\frac{2}{2^*}}} = \lim_{k\to\infty} \beta(u_k) \leq \lim_{k\to\infty} \Psi_{\alpha_k}(u_k) = \lim_{k\to\infty} S_{\alpha_k} \leq \frac{S}{2^{\frac{2}{N}}}. \qquad (3.11)$$

Taking the limit of both sides of (3.10) as $k \to +\infty$,

$$||\nu|| = ||\mu||. \qquad (3.12)$$

Combining (3.11) and (3.12), $||\mu|| = ||\nu|| = \frac{S^{\frac{N}{2}}}{2}$, or (3.4).

Equalities (3.4) imply there exists a constant c such that

$$|u_k|_{2^*} \geq c > 0, \qquad (3.13)$$

for all positive integers k. Another consequence of (3.11) is $\lim_{k\to\infty} \beta(u_k) = \frac{S}{2^{\frac{2}{N}}}$ and so $\lim_{k\to\infty} \alpha_k \beta(u_k)\delta(u_k) = 0$. However,

$$\lim_{k\to\infty} \alpha_k \beta(u_k)\delta(u_k) = \frac{S}{2^{\frac{2}{N}}} \lim_{k\to\infty} \alpha_k \delta(u_k).$$

Equality (3.5) follows.

Combining (3.5),

$$\alpha_k \delta(u_k) = \alpha_k \frac{|u_k|_q^{qt}}{|u_k|_{2^*}^2}$$

and the fact that the norms $|u_k|_{2^*}$ are uniformly bounded, we also get

$$\lim_{k\to\infty} \alpha_k |u_k|_q^{qt} = 0. \qquad (3.14)$$

But, from (2.2),

$$\begin{aligned} |u_k|_q^{qt} &= \Big(\int u_k^q\Big)^t = M_k^{qt}\Big[\int \Big(\frac{u_k}{M_k}\Big)^q\Big]^t \geq M_k^{qt}\Big[\int \Big(\frac{u_k}{M_k}\Big)^{2^*}\Big]^t \\ &= M_k^{(q-2^*)t}\Big(\int u_k^{2^*}\Big)^t = M_k^{-\frac{2}{N-2}}|u_k|_{2^*}^{2^*t} = \epsilon_k |u_k|_{2^*}^{2^*t}. \end{aligned}$$

This, (3.13) and (3.14) imply (3.8) and (3.9). □

Remark 3.5. Suppose that (α_k) converges to a positive real number and $S_{\alpha_k} \nearrow \frac{S}{2^{\frac{2}{N}}}$. Let $u_k \in H^1(\Omega)$ be a minimizer for Ψ_{α_k} satisfying $(2.9)_{\alpha_k}$and suppose $u_k \rightharpoonup 0$ in $H^1(\Omega)$. The previous argument shows that (3.4), (3.5), (3.8) and (3.9) hold.

Lemma 3.6. *Suppose* $S_{\alpha_k} < \frac{S}{2^{\frac{2}{N}}}$ *and either* $\alpha_k \to +\infty$*, or the hypotheses of Remark* 3.5 *hold. Let* $u_k \in H^1(\Omega)$ *be a positive minimizer for* Ψ_{α_k} *satisfying* $(2.9)_{\alpha_k}$*. Then*

$$\lim_{k\to\infty} |\nabla u_k - \nabla U_{\epsilon_k,P_k}|_2 = 0 \tag{3.15}$$

and $P_k \in \partial\Omega$*, for large* k*, where* P_k *is such that* $u_k(P_k) = M_k$*, and* M_k *and* ϵ_k *are as in* (3.6) *and* (3.7)*, respectively.*

Proof. We use the Gidas and Spruck blow up technique [11]. Let $\Omega_k := (\Omega - P_k)/\epsilon_k$ and $v_k : \Omega_k \to \mathbb{R}$ be defined by $v_k(x) := \epsilon_k^{\frac{N-2}{2}} u_k(\epsilon_k x + P_k)$. We can assume that $P_k \to P_0$ and $\Omega_k \to \Omega_\infty$. We let

$$L = \lim_{k\to+\infty} \operatorname{dist}(P_k, \partial\Omega)/\epsilon_k \in [0, +\infty].$$

From

$$|v_k|^{qt}_{L^q(\Omega_k)} = \epsilon_k^{\frac{N-2}{2}qt} \epsilon_k^{-Nt} |u_k|^{qt}_q = \epsilon_k^{-1} |u_k|^{qt}_q,$$

we deduce that

$$\delta(u_k) = \epsilon_k \delta(v_k), \tag{3.16}$$

where the norms in $\delta(v_k)$ are computed in Ω_k. Also,

$$\begin{aligned} |v_k|^{q(t-1)}_{L^q(\Omega_k)} v_k^{q-1}(x) &= \epsilon_k^{\frac{N-2}{2}q(t-1)} \epsilon_k^{-N(t-1)} \epsilon_k^{\frac{N-2}{2}(q-1)} |u_k|^{q(t-1)}_q u_k^{q-1}(\epsilon_k x + P_k) \\ &= \epsilon_k^{\frac{N}{2}} |u_k|^{q(t-1)}_q u_k^{q-1}(\epsilon_k x + P_k). \end{aligned}$$

Thus,

$$\begin{aligned} \Delta v_k(x) &= \epsilon_k^{\frac{N+2}{2}} \Delta u_k(\epsilon_k x + P_k) \\ v_k^{2^*-1}(x) &= \epsilon_k^{\frac{N+2}{2}} u_k^{2^*-1}(\epsilon_k x + P_k) \\ \epsilon_k^2 v_k(x) &= \epsilon_k^{\frac{N+2}{2}} u_k(\epsilon_k x + P_k) \\ \epsilon_k |v_k|^{q(t-1)}_{L^q(\Omega_k)} v_k^{q-1}(x) &= \epsilon_k^{\frac{N+2}{2}} |u_k|^{q(t-1)}_q u_k^{q-1}(\epsilon_k x + P_k). \end{aligned}$$

The functions v_k satisfy

$$\left\{ \begin{array}{rcll} \Big(1 + \frac{s}{2}\epsilon_k \alpha_k \delta(v_k)\Big)(-\Delta v_k + a\epsilon_k^2 v_k) & & & \\ + \frac{qt}{2}\epsilon_k \alpha_k |v_k|^{q(t-1)}_{L^q(\Omega_k)} v_k^{q-1} & & & \\ -\Big(1 + \Big(1 + s\frac{2^*}{4}\Big)\epsilon_k \alpha_k \delta(v_k)\Big) v_k^{2^*-1} & = & 0 & \text{in } \Omega_k, \\ 0 < v_k \le v_k(0) & = & 1 & \text{in } \Omega_k, \\ \frac{\partial v_k}{\partial \nu} & = & 0 & \text{on } \partial\Omega_k. \end{array} \right. \tag{3.17}$$

Suppose that $L = +\infty$. Then $\Omega_\infty = \mathbb{R}^N$. We use (3.5), (3.9) (which obviously implies $\epsilon_k \to 0$), (3.16) and

$$|v_k|^q_{L^q(\Omega_k)} = \int_{\Omega_k} v_k^q \geq \int_{\Omega_k} v_k^{2^*} = \int u_k^{2^*} = |u_k|^{2^*}_{2^*} \geq c^{2^*}, \tag{3.18}$$

(from (3.13)). By the elliptic estimates in [4], $v_k \to v$ in $C^2_{\mathrm{loc}}(\Omega_\infty)$ where v satisfies

$$\begin{cases} -\Delta v = v^{2^*-1} & \text{in } \Omega_\infty, \\ 0 < v \leq v(0) = 1 & \text{in } \Omega_\infty. \end{cases}$$

By lower semicontinuity of the norm, $v \in L^{2^*}(\Omega_\infty)$ and $\nabla v \in L^2(\Omega_\infty)$. Therefore, $v = U$. From (3.4),

$$S^{\frac{N}{2}} = \int_{\mathbb{R}^N} |\nabla U|^2 \leq \lim_{k\to\infty} |\nabla u_k|_2^2 = \frac{S^{\frac{N}{2}}}{2},$$

which is impossible. So L is finite. This implies that $P_0 \in \partial\Omega$. Without loss of generality, we assume that $P_0 = 0$ and that in a neighborhood $B_R(0) = \{x \in \mathbb{R}^N : |x| < R\}$ of 0 the sets Ω and $\partial\Omega$ are described by

$$\Omega \cap B_R(0) = \{(x', x_N) \in B_R(0) : x_N > g(x')\},$$
$$\partial\Omega \cap B_R(0) = \{(x', x_N) \in B_R(0) : x_N = g(x')\},$$

where $g : B_R(0) \cap \{(0, x_N) : x_N \in \mathbb{R}\} \to \mathbb{R}$ is such that $g(0) = 0$ and $\nabla g(0) = 0$. We make the change of coordinates associated to the map $\psi = (\psi_1, \ldots, \psi_N) : B_R(0) \to \mathbb{R}^N$, with

$$\begin{aligned} \psi_i(x) &= x_i - \frac{g(x') - x_N}{1 + |\nabla g(x')|^2} \cdot \frac{\partial g}{\partial x_i}(x'), \quad \text{for } 1 \leq i \leq N-1, \\ \psi_N(x) &= x_N - g(x'). \end{aligned}$$

The determinant of the Jacobian of ψ at 0 is 1. We can choose $R_0 > 0$ and an open neighborhood $V \subset B_R(0)$ of zero, such that

$\psi : V \to B_{R_0}(0)$ is a diffeomorphism,
$\psi : \Omega \cap V \to B_{R_0}(0)_+ := \{(y', y_N) \in B_{R_0} : y_N > 0\}$,
$\psi : \partial\Omega \cap V \to \{(y', y_N) \in B_{R_0} : y_N = 0\}$.

If $u : V \to \mathbb{R}$ is smooth and $v : B_{R_0}(0)_+ \to \mathbb{R}$ is such that $v(y) = u(\psi^{-1}(y))$, then

$$(\Delta u)(\psi^{-1}(y)) = \sum_{i,j=1}^N a_{i,j}(y) \frac{\partial^2 v}{\partial y_i \partial y_j}(y) + \sum_{i=1}^N b_i(y) \frac{\partial v}{\partial y_i}(y),$$

$$\frac{\partial u}{\partial \nu}(\psi^{-1}(y)) = d(y)\frac{\partial v}{\partial y_N}(y) \quad \text{on } y_N = 0,$$

with $a_{i,j}$, b_i and d smooth functions, $a_{i,j}(y) = \delta_{i,j} + O(|y|)$ and

$$d(y) = 1 + |(\nabla g)[(\psi^{-1}(y))']|^2 \geq 1.$$

As above, $(\psi^{-1}(y))'$ denotes the first $N-1$ coordinates of $\psi^{-1}(y)$. We let $Q_k = \psi(P_k)$ and denote by $(Q_k)_N$ the N-th coordinate of Q_k. We also let $B_k = (B_{R_0}(0)_+ - Q_k)/\epsilon_k$. We define $w_k : B_{R_0}(0)_+ \to \mathbb{R}$ by $w_k(y) = u_k(\psi^{-1}(y))$ and $\tilde{w}_k : B_k \to \mathbb{R}$ by $\tilde{w}_k(x) = \epsilon_k^{\frac{N-2}{2}} w_k(\epsilon_k x + Q_k)$. The functions $\tilde{w}_k$ satisfy

$$\begin{cases} \Big(1 + \frac{s}{2}\epsilon_k\alpha_k\delta(v_k)\Big)\times \\ \Big(-\sum_{i,j=1}^N \tilde{a}_{i,j,k}\frac{\partial^2 \tilde{w}_k}{\partial x_i \partial x_j} - \sum_{i=1}^N \epsilon_k \tilde{b}_{i,k}\frac{\partial \tilde{w}_k}{\partial x_i} + a\epsilon_k^2 \tilde{w}_k\Big) \\ \qquad + \frac{qt}{2}\epsilon_k\alpha_k |v_k|_{L^q(\Omega_k)}^{q(t-1)} \tilde{w}_k^{q-1} \\ \qquad - \Big(1 + \Big(1 + s\frac{2^*}{4}\Big)\epsilon_k\alpha_k\delta(v_k)\Big)\tilde{w}_k^{2^*-1} = 0 & \text{in } B_k, \\ 0 < \tilde{w}_k \leq \tilde{w}_k(0) = 1 & \text{in } B_k, \\ \frac{\partial \tilde{w}_k}{\partial x_N} = 0 & \text{on } \underline{\partial} B_k, \end{cases}$$

with $\underline{\partial} B_k = \partial B_k \cap (\mathbb{R}^{N-1} \times \{-(Q_k)_N/\epsilon_k\})$,

$$\tilde{a}_{i,j,k}(x) = a_{i,j}(\epsilon_k x + Q_k) = \delta_{i,j} + O(|\epsilon_k x + Q_k|) \tag{3.19}$$

and $\tilde{b}_{i,k}(x) = b_i(\epsilon_k x + Q_k)$.

We use again (3.5), (3.9), (3.16), (3.18), and we also use (3.19). By elliptic regularity theory, $\tilde{w}_k \to w$ in $C^2_{\mathrm{loc}}(\bar{B}_\infty)$ where $B_\infty = \{(x', x_N) \in \mathbb{R}^N : x_N > -L\}$ and

$$\begin{cases} -\Delta w = w^{2^*-1} & \text{in } B_\infty, \\ 0 < w \leq w(0) = 1 & \text{in } B_\infty, \\ \frac{\partial w}{\partial x_N} = 0 & \text{on } \partial B_\infty. \end{cases}$$

We deduce that $w = U$. Moreover, L has to be zero.

Suppose $P_k \notin \partial\Omega$ for large k. Since $\nabla \tilde{w}_k(0) = 0$ and $\frac{\partial \tilde{w}_k}{\partial x_N} = 0$ on $\partial B_k \cap (\mathbb{R}^{N-1} \times \{-(Q_k)_N/\epsilon_k\})$, by the mean value theorem there exists $r_k \in \mathbb{R}$, with $-(Q_k)_N/\epsilon_k < r_k < 0$ such that $\frac{\partial^2 \tilde{w}_k}{\partial x_N^2}(0, r_k) = 0$. Recalling that $\tilde{w}_k \to w$ in $C^2_{\mathrm{loc}}(\bar{B}_\infty)$, it follows that $\frac{\partial^2 w}{\partial x_N^2}(0) = 0$. This is impossible because $w = U$ and $\frac{\partial^2 U}{\partial x_N^2}(0) < 0$. We conclude that $P_k \in \partial\Omega$ for large k.

Returning to (3.17),

$$v_k \to v \text{ in } C^2_{\mathrm{loc}}(\Omega_\infty) \tag{3.20}$$

where $\Omega_\infty = \mathbb{R}^N_+$ and

$$\begin{cases} -\Delta v = v^{2^*-1} & \text{in } \Omega_\infty, \\ 0 < v \le v(0) = 1 & \text{in } \Omega_\infty, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega_\infty. \end{cases}$$

So $v = U$. Finally, from (3.4), (3.20) and

$$\int_{\mathbb{R}^N_+} |\nabla U|^2 = \frac{S^{\frac{N}{2}}}{2},$$

we deduce (3.15). □

As in [2] and [5], let $\mathcal{M} := \{CU_{\varepsilon,y}, C \in \mathbb{R}, \varepsilon > 0, y \in \partial\Omega\}$ and $d(u, \mathcal{M}) := \inf\{|\nabla(u-V)|_2, V \in \mathcal{M}\}$. The set $\mathcal{M} \setminus \{0\}$ is a manifold of dimension $N+1$. The tangent space $T_{C_l,\varepsilon_l,y_l}(\mathcal{M})$ at $C_l U_{\varepsilon_l,y_l}$ is given by

$$T_{C_l,\varepsilon_l,y_l}(\mathcal{M}) = \text{span}\,\left\{U_{\varepsilon,y}, C\frac{\partial}{\partial\varepsilon}U_{\varepsilon,y}, C\frac{\partial}{\partial\tau_i}U_{\varepsilon,y}, 1 \le i \le N-1\right\}_{(C_l,\varepsilon_l,y_l)}$$

where $T_x(\partial\Omega) = \text{span}\{\tau_1, \dots, \tau_{N-1}\}$.

As in Lemma 3.6, let $u_k \in H^1(\Omega)$ be a positive minimizer for Ψ_{α_k} satisfying $(2.9)_{\alpha_k}$. For large k, the infimum $d(u_k, \mathcal{M})$ is achieved:

$$d(u_k, \mathcal{M}) = |\nabla(u_k - C_k U_{\varepsilon_k,y_k})|_2 \text{ for } C_k U_{\varepsilon_k,y_k} \in \mathcal{M}. \tag{3.21}$$

Furthermore,

$$C_k = 1 + o(1) \tag{3.22}$$

$y_k \to P_0$ and $\varepsilon_k/\epsilon_k \to 1$ (see Lemma 1 of [5] and Lemma 2.3 of [2]). From (3.9),

$$\alpha_k \varepsilon_k \to 0. \tag{3.23}$$

We define $w_k := u_k - C_k U_{\varepsilon_k,y_k}$, so that

$$\int \nabla U_{\varepsilon_k,y_k} \cdot \nabla w_k = 0. \tag{3.24}$$

On the one hand, from (3.15),

$$\lim_{k\to\infty} |\nabla(u_k - C_k U_{\varepsilon_k,y_k})|_2 = 0.$$

On the other hand, from Poincaré's inequality, and the fact that both the average of u_k and the average of $C_k U_{\varepsilon_k,y_k}$, in Ω, converge to zero,

$$\lim_{k\to\infty} |u_k - C_k U_{\varepsilon_k,y_k}|_{2^*} = 0.$$

Together,

$$\lim_{k\to\infty} ||w_k|| = 0. \tag{3.25}$$

Our next aim is the lower bound for $|\nabla w_k|_2^2 + c\alpha_k \int U_{\varepsilon_k,y_k}^{qt-2} w_k^2$ in Lemma 3.11 where c is a constant. To obtain that lower bound we consider two eigenvalue problems. The first one can be regarded as the limit of the second, in a sense made precise below.

Lemma 3.7. (Bianchi and Egnell [5], Rey [17]) *The eigenvalue problem*

$$\begin{cases} -\Delta\varphi = \mu U^{2^*-2}\varphi & \text{in } \mathbb{R}^N_+, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\mathbb{R}^N_+, \\ \int_{\mathbb{R}^N_+} U^{2^*-2}\varphi^2 < \infty \end{cases} \tag{3.26}$$

admits a discrete spectrum $\mu_1 < \mu_2 \le \mu_3 \le \cdots$ such that $\mu_1 = 1$, $\mu_2 = \mu_3 = \cdots = \mu_N = 2^* - 1$ and $\mu_{N+1} > 2^* - 1$. The eigenspaces V_1 and $V_{(2^*-1)}$, corresponding to 1 *and* $(2^* - 1)$, *are given by*

$$V_1 = \textit{span}\ U, \quad V_{(2^*-1)} = \textit{span}\left\{ \left.\frac{\partial U_{1,y}}{\partial y_i}\right|_{y=0},\ \textit{for } 1 \le i \le N-1 \right\}.$$

Now we let $\varepsilon > 0$, $\nu_\varepsilon > 0$, and $y_\varepsilon \in \partial\Omega$ with $\lim_{\varepsilon\to 0} y_\varepsilon = y_0$. Let $\{\varphi_{i,\varepsilon}\}_{i=1}^\infty$ be a complete set of orthogonal eigenfunctions with eigenvalues $\mu_{1,\varepsilon} < \mu_{2,\varepsilon} \le \mu_{3,\varepsilon} \le \cdots$ for the weighted eigenvalue problem

$$\begin{cases} -\Delta\varphi + \nu_\varepsilon U_{\varepsilon,y_\varepsilon}^{qt-2}\varphi = \mu U_{\varepsilon,y_\varepsilon}^{2^*-2}\varphi & \text{in } \Omega, \\ \frac{\partial\varphi}{\partial\nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\varphi_{1,\varepsilon} > 0$ and

$$\int_\Omega U^{2^*-2}\varphi_{i,\varepsilon}\varphi_{j,\varepsilon} = \delta_{i,j}.$$

Let $\Omega_\varepsilon := (\Omega - y_\varepsilon)/\varepsilon$. The sets Ω_ε converge to a half space as $\varepsilon \to 0$. For a function v on Ω, we define $\tilde{v}$ on Ω_ε by $\tilde{v}(x) := \varepsilon^{\frac{N-2}{2}} v(\varepsilon x + y_\varepsilon)$.

The relation between these eigenvalue problems and the one considered in Lemma 3.7 is given in

Lemma 3.8. *Suppose $y_\varepsilon \in \partial\Omega$, $\lim_{\varepsilon\to 0} y_\varepsilon = y_0$, $\lim_{\varepsilon\to 0} \varepsilon^{2-s}\nu_\varepsilon = 0$ and the sets Ω_ε converge to $\mathbb{R}^N_+$. Then, up to a subsequence,*

$$\lim_{\varepsilon\to 0} \mu_{i,\varepsilon} = \mu_i \tag{3.27}$$

and

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} U^{2^*-2}(\tilde{\varphi}_{i,\varepsilon} - \tilde{\varphi}_i)^2 = 0, \tag{3.28}$$

for all positive integers i. The functions μ_i and $\tilde{\varphi}_i$ satisfy

$$\begin{cases} -\Delta\tilde{\varphi}_i = \mu_i U^{2^*-2}\tilde{\varphi}_i & \text{in } \mathbb{R}^N_+, \\ \frac{\partial\tilde{\varphi}_i}{\partial\nu} = 0 & \text{on } \partial\mathbb{R}^N_+, \\ \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_i^2 = 1, & \end{cases}$$

and the functions $\tilde{\varphi}_i$ are supposed extended to $\mathbb{R}^N$ by reflection. In particular, from the previous lemma, $\mu_1 = 1$, $\tilde{\varphi}_1 = CU$ for some constant $C > 0$, $\mu_i = 2^* - 1$ for $2 \le i \le N$ and $\mu_{N+1} > 2^* - 1$. Also, $\{\tilde{\varphi}_i\}_{i=2}^N$ is in the span of $\{\partial U_{1,y}/\partial y_i|_{y=0}$, for $1 \le i \le N-1\}$.

We postpone the proof, since it requires the following Lemma and Remark.

Lemma 3.9. *Suppose $y_\varepsilon \in \bar{\Omega}$, $\varphi_\varepsilon \in H^1(\Omega)$,*

$$\int U^{qt-2}_{\varepsilon,y_\varepsilon}\varphi_\varepsilon^2 \to 0 \qquad \text{and} \qquad \int |\nabla\varphi_\varepsilon|^2 \to 0,$$

as $\varepsilon \to 0$. Then

$$\int U^{2^*-2}_{\varepsilon,y_\varepsilon}\varphi_\varepsilon^2 \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. We denote the average of φ_ε in Ω by $\bar{\varphi}_\varepsilon$. By Poincaré's inequality, $|\varphi_\varepsilon - \bar{\varphi}_\varepsilon|_{2^*} \to 0$. The limits in this proof are taken as ε approaches zero. So we can write $\varphi_\varepsilon = \bar{\varphi}_\varepsilon + \eta_\varepsilon$, with $\eta_\varepsilon \to 0$ in L^{2^*}. We know that

$$\int U^{qt-2}_{\varepsilon,y_\varepsilon}(\bar{\varphi}_\varepsilon^2 + 2\bar{\varphi}_\varepsilon\eta_\varepsilon + \eta_\varepsilon^2) = o(1)$$

and we estimate the three terms on the left hand side. There exists a $b > 0$ such that

$$\int U^{qt-2}_{\varepsilon,y_\varepsilon}\bar{\varphi}_\varepsilon^2 \ge b\bar{\varphi}_\varepsilon^2\varepsilon^s.$$

Also,

$$\Big|\int U^{qt-2}_{\varepsilon,y_\varepsilon}\eta_\varepsilon\bar{\varphi}_\varepsilon\Big| \le |\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\Big(\int U^{(qt-2)\frac{2N}{N+2}}_{\varepsilon,y_\varepsilon}\Big)^{\frac{N+2}{2N}} \le C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^s.$$

If $2^\flat \le q < 2^\#$, then

$$\int U^{qt-2}_{\varepsilon,y_\varepsilon}\eta_\varepsilon^2 \le |\eta_\varepsilon|^2_{2^*}\Big(\int U^{(qt-2)\frac{N}{2}}_{\varepsilon,y_\varepsilon}\Big)^{\frac{2}{N}} \le C|\eta_\varepsilon|^2_{2^*}\varepsilon^s. \tag{3.29}$$

If $q = 2^\#$, then

$$\int U^{qt-2}_{\varepsilon,y_\varepsilon}\eta_\varepsilon^2 \le |\eta_\varepsilon|^2_{2^*}\Big(\int U^{\frac{N}{N-2}}_{\varepsilon,y_\varepsilon}\Big)^{\frac{2}{N}} \le C|\eta_\varepsilon|^2_{2^*}\varepsilon|\log\varepsilon|^{\frac{2}{N}}. \tag{3.30}$$

Thus,

$$b\bar{\varphi}_\varepsilon^2\varepsilon^s \leq C|\bar{\varphi}_\varepsilon|\varepsilon^s + o(1).$$

This shows that $\bar{\varphi}_\varepsilon\varepsilon^{\frac{s}{2}}$ is bounded. But if $\bar{\varphi}_\varepsilon\varepsilon^{\frac{s}{2}}$ is bounded this shows that

$$\bar{\varphi}_\varepsilon\varepsilon^{\frac{s}{2}} \to 0. \tag{3.31}$$

We want to prove that

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\bar{\varphi}_\varepsilon^2 + 2\bar{\varphi}_\varepsilon\eta_\varepsilon + \eta_\varepsilon^2) = o(1).$$

For the first term on the left hand side we have, by (3.31),

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\bar{\varphi}_\varepsilon^2 \leq C\bar{\varphi}_\varepsilon^2\varepsilon^2 \to 0.$$

For the third term we have

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\eta_\varepsilon^2 \leq C|\eta_\varepsilon|_{2^*}^2 \to 0.$$

We claim that the remaining term also converges to zero. This will prove the lemma. For the second term we have the estimate

$$\zeta_\varepsilon := \left|\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\bar{\varphi}_\varepsilon\eta_\varepsilon\right| \leq |\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\left(\int U_{\varepsilon,y_\varepsilon}^{\frac{N}{N-2}\frac{8}{N+2}}\right)^{\frac{N+2}{2N}}.$$

If $N = 5$, then

$$\zeta_\varepsilon \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^{N\left(1-\frac{4}{N+2}\right)\frac{N+2}{2N}} \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^{\frac{N-2}{2}} = C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^{\frac{3}{2}}.$$

If $N = 6$, then

$$\zeta_\varepsilon \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^2|\log\varepsilon|^{\frac{2}{3}}.$$

Finally, if $N \geq 7$, then

$$\zeta_\varepsilon \leq C|\eta_\varepsilon|_{2^*}|\bar{\varphi}_\varepsilon|\varepsilon^2.$$

In all three cases, (3.31) implies that $\zeta_\varepsilon \to 0$. □

Remark 3.10. If in the previous lemma, instead of assuming $\int|\nabla\varphi_\varepsilon|^2 \to 0$, we assume that $\int|\nabla\varphi_\varepsilon|^2$ is bounded, then we can still conclude $\bar{\varphi}_\varepsilon\varepsilon^{\frac{s}{2}} \to 0$, $\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\bar{\varphi}_\varepsilon^2 \to 0$ and $\int U_{\varepsilon,y_\varepsilon}^{2^*-2}\bar{\varphi}_\varepsilon\eta_\varepsilon \to 0$, as $\varepsilon \to 0$.

Proof of Lemma 3.8. We basically adapt the argument of the proof of Lemma 3.3 of [2] (and Lemma 5.8 of Z.Q. Wang in [21]), modified according to Lemma 3.9 and Remark 3.10. The value of k_0 in Lemma 3.3 of [2] is equal to N. The proof is by induction. We first consider $i = 1$. By the Rayleigh quotient, $\mu_{1,\varepsilon}$ is given by

$$\begin{aligned}\mu_{1,\varepsilon} &= \inf\Big\{|\nabla u|_2^2 + \nu_\varepsilon \int U_{\varepsilon,y_\varepsilon}^{qt-2}u^2 : \int U_{\varepsilon,y_\varepsilon}^{2^*-2}u^2 = 1\Big\}\\ &= \inf\Big\{\int_{\Omega_\varepsilon}|\nabla v|^2 + \varepsilon^{2-s}\nu_\varepsilon\int_{\Omega_\varepsilon}U^{qt-2}v^2 : \int_{\Omega_\varepsilon}U^{2^*-2}v^2 = 1\Big\}.\end{aligned}$$

To estimate $\mu_{1,\varepsilon}$ from above, we choose $v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ defined by

$$v_\varepsilon = U\Big/\Big(\textstyle\int_{\Omega_\varepsilon} U^{2^*}\Big)^{\frac{1}{2}}.$$

From the assumption $\varepsilon^{2-s}\nu_\varepsilon \to 0$ as $\varepsilon \to 0$, we get

$$\mu_{1,\varepsilon} \le \int_{\Omega_\varepsilon}|\nabla v_\varepsilon|^2 + \varepsilon^{2-s}\nu_\varepsilon\int_{\Omega_\varepsilon}U^{qt-2}v_\varepsilon^2 \to \frac{\int_{\mathbb{R}^N_+}|\nabla U|^2}{\int_{\mathbb{R}^N_+}U^{2^*}} = \mu_1 = 1,$$

as $\varepsilon \to 0$. Hence $\limsup_{\varepsilon\to 0}\mu_{1,\varepsilon} \le \mu_1$. Up to a subsequence, which we still denote by ε, $\lim_{\varepsilon\to 0}\mu_{1,\varepsilon} = \hat\mu_1 \le \mu_1$. The functions $\varphi_{1,\varepsilon}$ satisfy

$$\mu_{1,\varepsilon} = |\nabla\varphi_{1,\varepsilon}|_2^2 + \nu_\varepsilon\int U_{\varepsilon,y_\varepsilon}^{qt-2}\varphi_{1,\varepsilon}^2,$$

so $|\nabla\varphi_{1,\varepsilon}|_2$ is bounded and $\int U_{\varepsilon,y_\varepsilon}^{qt-2}\varphi_{1,\varepsilon}^2 \to 0$, as $\varepsilon \to 0$. If $\hat\mu_1$ were equal to 0, then $|\nabla\varphi_{1,\varepsilon}|_2 \to 0$ and $\int U_{\varepsilon,y_\varepsilon}^{qt-2}\varphi_{1,\varepsilon}^2 \to 0$, as $\varepsilon \to 0$. Lemma 3.9 would imply $1 = \int U_{\varepsilon,y_\varepsilon}^{2^*-2}\varphi_{1,\varepsilon}^2 \to 0$, as $\varepsilon \to 0$, a contradiction. So $\hat\mu_1 \ne 0$ and $|\nabla\varphi_{1,\varepsilon}|_2 \not\to 0$ as $\varepsilon \to 0$.

The functions $\tilde\varphi_{1,\varepsilon}$ satisfy

$$\begin{cases} -\Delta\tilde\varphi_{1,\varepsilon} + \varepsilon^{2-s}\nu_\varepsilon U^{qt-2}\tilde\varphi_{1,\varepsilon} = \mu_{1,\varepsilon}U^{2^*-2}\tilde\varphi_{1,\varepsilon} & \text{in } \Omega_\varepsilon,\\ \tilde\varphi_{1,\varepsilon} > 0 & \text{in } \Omega_\varepsilon,\\ \frac{\partial\tilde\varphi_{1,\varepsilon}}{\partial\nu} = 0 & \text{on } \partial\Omega_\varepsilon,\\ \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde\varphi_{1,\varepsilon}^2 = 1. \end{cases}$$

By the hypothesis $\lim_{\varepsilon\to 0}\varepsilon^{2-s}\nu_\varepsilon \to 0$, and elliptic regularity theory [4], $\tilde\varphi_{1,\varepsilon} \to \hat\varphi_1$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N_+)$, as $\varepsilon \to 0$, where $\hat\varphi_1$ satisfies (3.26) with $\mu = \hat\mu_1$. We conclude that $\hat\mu_1 = \mu_1$ and $\hat\varphi_1 = \tilde\varphi_1$.

We will now prove (3.28) in case $i = 1$, i.e.,

$$\int_{\Omega_\varepsilon} U^{2^*-2}(\tilde\varphi_{1,\varepsilon} - \tilde\varphi_1)^2 = \int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\varphi_{1,\varepsilon} - \varsigma_{1,\varepsilon})^2 \to 0, \text{ as } \varepsilon \to 0,$$

where $\varsigma_{1,\varepsilon}(\cdot) = \varepsilon^{-\frac{N-2}{2}}\tilde{\varphi}_1(\frac{\cdot\, - y_\varepsilon}{\varepsilon})$. The function $\tilde{\varphi}_1$ belongs to $L^{2^*}(\mathbb{R}^N)$ and $\varsigma_{1,\varepsilon} \rightharpoonup 0$ in $H^1(\Omega)$. We denote the averages of $\varphi_{1,\varepsilon}$ and $\varsigma_{1,\varepsilon}$, in Ω, by $\bar{\varphi}_{1,\varepsilon}$ and $\bar{\varsigma}_{1,\varepsilon}$, respectively. By Poincaré's inequality, we can write $\varphi_{1,\varepsilon} = \bar{\varphi}_{1,\varepsilon} + \eta_{1,\varepsilon}$ and $\varsigma_{1,\varepsilon} = \bar{\varsigma}_{1,\varepsilon} + \zeta_{1,\varepsilon}$ with $|\eta_{1,\varepsilon}|_{2^*}$ and $|\zeta_{1,\varepsilon}|_{2^*}$ uniformly bounded, as $\varepsilon \to 0$. Moreover,

$$\int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\varphi_{1,\varepsilon} - \varsigma_{1,\varepsilon})^2 = \int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\bar{\varphi}_{1,\varepsilon} - \bar{\varsigma}_{1,\varepsilon})^2$$
$$+ 2\int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\bar{\varphi}_{1,\varepsilon} - \bar{\varsigma}_{1,\varepsilon})(\eta_{1,\varepsilon} - \zeta_{1,\varepsilon}) + \int U_{\varepsilon,y_\varepsilon}^{2^*-2}(\eta_{1,\varepsilon} - \zeta_{1,\varepsilon})^2.$$

The first two terms on the right hand side converge to 0 as $\varepsilon \to 0$, due to Remark 3.10. But

$$\tilde{\eta}_{1,\varepsilon} = \tilde{\varphi}_{1,\varepsilon} - \tilde{\bar{\varphi}}_{1,\varepsilon} = \tilde{\varphi}_{1,\varepsilon} - \varepsilon^{\frac{N-2}{2}}\bar{\varphi}_{1,\varepsilon} = \tilde{\varphi}_{1,\varepsilon} - \varepsilon^{\frac{N-2-s}{2}}(\varepsilon^{\frac{s}{2}}\bar{\varphi}_{1,\varepsilon})$$

and $\tilde{\zeta}_{1,\varepsilon} = \tilde{\varphi}_1 - \varepsilon^{\frac{N-2}{2}}\bar{\varsigma}_{1,\varepsilon}$. These equalities and, again, Remark 3.10 show that $\tilde{\eta}_{1,\varepsilon} \to \tilde{\zeta}_{1,\varepsilon}$ in $C^2_{\mathrm{loc}}(\mathbb{R}^N_+)$, as $\varepsilon \to 0$. We conclude that the term

$$\int_{\Omega_\varepsilon} U^{2^*-2}(\tilde{\eta}_{1,\varepsilon} - \tilde{\zeta}_{1,\varepsilon})^2$$

also converges to 0 as $\varepsilon \to 0$ (see (3.42) and (3.43) in the proof of Lemma 3.3 of [2]). This proves (3.28) for $i = 1$.

Now assume that (3.27) and (3.28) hold for $1 \le i \le L-1$. To estimate $\mu_{L,\varepsilon}$ from above we choose $v_\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ defined by $v_\varepsilon = \tilde{\varphi}_L$. Let $v_\varepsilon = \sum_{i=1}^\infty a_{i,\varepsilon}\tilde{\varphi}_{i,\varepsilon}$. Clearly,

$$\sum_{i=1}^\infty \mu_{i,\varepsilon}a_{i,\varepsilon}^2 = \int_{\Omega_\varepsilon}|\nabla v_\varepsilon|^2 + \varepsilon^{2-s}\nu_\varepsilon\int_{\Omega_\varepsilon}U^{qt-2}v_\varepsilon^2 \to \mu_L \tag{3.32}$$

and

$$\sum_{i=1}^{L-1}\mu_{i,\varepsilon}a_{i,\varepsilon}^2 + \mu_{L,\varepsilon}\sum_{i=L}^\infty a_{i,\varepsilon}^2 \le \sum_{i=1}^\infty \mu_{i,\varepsilon}a_{i,\varepsilon}^2. \tag{3.33}$$

We claim that $a_{i,\varepsilon} \to 0$ for $1 \le i \le L-1$ and $\sum_{i=L}^\infty a_{i,\varepsilon}^2 \to 1$, as $\varepsilon \to 0$. Indeed,

$$a_{i,\varepsilon} = \int_{\Omega_\varepsilon}U^{2^*-2}v_\varepsilon\tilde{\varphi}_{i,\varepsilon} = \int_{\Omega_\varepsilon}U^{2^*-2}v_\varepsilon\tilde{\varphi}_i + \int_{\Omega_\varepsilon}U^{2^*-2}v_\varepsilon(\tilde{\varphi}_{i,\varepsilon} - \tilde{\varphi}_i).$$

As $\varepsilon \to 0$, for $1 \le i \le L-1$, the first term on the right hand side approaches $\int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_L\tilde{\varphi}_i = 0$ and the second one is bounded by

$$\Big(\int_{\Omega_\varepsilon} U^{2^*-2}v_\varepsilon^2\Big)^{\frac{1}{2}}\Big(\int_{\Omega_\varepsilon} U^{2^*-2}(\tilde{\varphi}_{i,\varepsilon}-\tilde{\varphi}_i)^2\Big)^{\frac{1}{2}} \to 0.$$

Moreover,

$$\sum_{i=1}^{\infty} a_{i,\varepsilon}^2 = \int_{\Omega_\varepsilon} U^{2^*-2}v_\varepsilon^2 \to \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_L^2 = 1.$$

This proves our claim.

Combining (3.32), (3.33) and the previous claim, we have $\limsup_{\varepsilon\to 0}\mu_{L,\varepsilon} \le \mu_L$. Up to a subsequence, which we still denote by ε, $\lim_{\varepsilon\to 0}\mu_{L,\varepsilon} = \hat{\mu}_L \le \mu_L$. The value of $\mu_{L,\varepsilon}$ is

$$\mu_{L,\varepsilon} = \inf\Big\{\int_{\Omega_\varepsilon}|\nabla v|^2 + \varepsilon^{2-s}\nu_\varepsilon\int_{\Omega_\varepsilon} U^{qt-2}v^2 : \int_{\Omega_\varepsilon} U^{2^*-2}v^2 = 1,$$
$$\int_{\Omega_\varepsilon} U^{2^*-2}\tilde{\varphi}_{i,\varepsilon}v = 0 \text{ for } 1 \le i \le L-1\Big\}.$$

We repeat part of the argument given for $i = 1$ and conclude that $\tilde{\varphi}_{L,\varepsilon} \to \hat{\varphi}_L$ in $C^2_{\rm loc}(\mathbb{R}^N_+)$, as $\varepsilon \to 0$, where $\hat{\varphi}_L$ satisfies (3.26) with $\mu = \hat{\mu}_L$. The space consisting in the completion with norm $|\nabla\cdot|_{L^2(\mathbb{R}^N)}$ of the smooth functions with compact support in $\mathbb{R}^N$ is dense in $L^2(U^{2^*-2}dx)$. Therefore, the Gagliardo-Nirenberg-Sobolev inequality implies that $\hat{\varphi}_L \in L^{2^*}(\mathbb{R}^N)$. So we can also conclude

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} U^{2^*-2}(\tilde{\varphi}_{L,\varepsilon}-\hat{\varphi}_L)^2 = 0. \qquad (3.34)$$

To prove that $\hat{\varphi}_L = \tilde{\varphi}_L$ and $\hat{\mu}_L = \mu_L$, we show that $\hat{\varphi}_L$ is orthogonal to $\tilde{\varphi}_i$ for $1 \le i \le L-1$. So let $1 \le i \le L-1$. Then

$$0 = \int U_{\varepsilon,y_\varepsilon}^{2^*-2}\varphi_{i,\varepsilon}\varphi_{L,\varepsilon} = \int_{\Omega_\varepsilon} U^{2^*-2}\tilde{\varphi}_{i,\varepsilon}\tilde{\varphi}_{L,\varepsilon} \to \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_i\hat{\varphi}_L,$$

as $\varepsilon \to 0$, as the difference $\int_{\Omega_\varepsilon} U^{2^*-2}\tilde{\varphi}_{i,\varepsilon}\tilde{\varphi}_{L,\varepsilon} - \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_i\hat{\varphi}_L$, approaches zero, as $\varepsilon \to 0$, because of (3.28) for $i = i$ and (3.34). Indeed,

$$\int_{\Omega_\varepsilon} U^{2^*-2}\tilde{\varphi}_{i,\varepsilon}\tilde{\varphi}_{L,\varepsilon} - \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_i\hat{\varphi}_L = \int_{\Omega_\varepsilon} U^{2^*-2}\tilde{\varphi}_i\hat{\varphi}_L - \int_{\mathbb{R}^N_+} U^{2^*-2}\tilde{\varphi}_i\hat{\varphi}_L$$
$$+ \int_{\Omega_\varepsilon} U^{2^*-2}(\tilde{\varphi}_{i,\varepsilon}-\tilde{\varphi}_i)\hat{\varphi}_L + \int_{\Omega_\varepsilon} U^{2^*-2}\tilde{\varphi}_{i,\varepsilon}(\tilde{\varphi}_{L,\varepsilon}-\hat{\varphi}_L).$$

Using Lemma 3.8 and the proof of Lemma 3.4 of [2], we deduce:

Lemma 3.11. *Suppose* $y_\varepsilon \in \partial\Omega$, $\lim_{\varepsilon\to 0} y_\varepsilon = y_0$ *and* $\lim_{\varepsilon\to 0} \varepsilon^{2-s}\nu_\varepsilon = 0$. *There exists a constant* $\gamma_1 > 0$ *such that, for sufficiently small* ε,

$$|\nabla w|_2^2 + \nu_\varepsilon \int U_{\varepsilon,y_\varepsilon}^{qt-2} w^2 \geq (2^* - 1 + \gamma_1) \int U_{\varepsilon,y_\varepsilon}^{2^*-2} w^2 + O(\varepsilon^2 ||w||^2)$$

for w *orthogonal to* $T_{1,\varepsilon,y_\varepsilon}(\mathcal{M})$.

4. Proof of the main theorem

In this section we prove Theorem 2.2 and give one more lower bound for α_0, in addition to one in Remark 2.3.

Assume the positive functions $u_k = C_k U_{\varepsilon_k,y_k} + w_k$, satisfy (3.4), (3.21), (3.22), (3.23) and (3.25).

We start by collecting some useful estimates. For brevity, we shall write $U_k := U_{\varepsilon_k,y_k}$.

Estimate for $\int U_k w_k$*:* From Lemma 4.1 of [10],

$$\Big|\int U_k w_k\Big| \leq \begin{cases} O\big(\varepsilon_k^{\frac{3}{2}} ||w_k||\big) & \text{if } N = 5, \\ O\big(\varepsilon_k^2 |\log \varepsilon_k|^{\frac{2}{3}} ||w_k||\big) & \text{if } N = 6, \\ O\big(\varepsilon_k^2 ||w_k||\big) & \text{if } N \geq 7. \end{cases} \tag{4.1}$$

Estimate for $\int U_k^{2^*-1} w_k$*:* From [2], equations (3.15), for $N \geq 5$,

$$\int U_k^{2^*-1} w_k = O(\varepsilon_k ||w_k||). \tag{4.2}$$

Estimate for $\int U_k^{q-1}|w_k|$*:* Since $(q-1)\frac{2N}{N+2} \geq (\frac{2N}{N-1} - 1)\frac{2N}{N+2} = \frac{N+1}{N-1}\frac{2N}{N+2} > \frac{N}{N-2}$ (for $N \geq 4$),

$$\begin{aligned} \int U_k^{q-1}|w_k| &\leq |w_k|_{2^*}\Big(\int U_k^{(q-1)\frac{2N}{N+2}}\Big)^{\frac{N+2}{2N}} \leq C|w_k|_{2^*}\varepsilon_k^{N\left(1-\frac{q-1}{2^*-1}\right)\frac{2^*-1}{2^*}} \\ &= C|w_k|_{2^*}\varepsilon_k^{\frac{N-2}{2}(2^*-q)} = O(\varepsilon_k^{1/t}||w_k||). \end{aligned} \tag{4.3}$$

Estimate for $\int U_k^{qt-2} w_k^2$*:* If $2^\flat \leq q < 2^\#$, then $(qt-2)\frac{N}{2} < \frac{N}{N-2}$. From (3.29) in the proof of Lemma 3.9,

$$\int U_k^{qt-2}|w_k|^2 \leq C|w_k|_{2^*}^2 \varepsilon_k^s. \tag{4.4}$$

If $q = 2^{\#}$, from (3.30) in the proof of Lemma 3.9,

$$\int U_k^{qt-2}|w_k|^2 \leq C|w_k|_{2^*}^2 \varepsilon_k |\log \varepsilon_k|^{\frac{2}{N}}. \tag{4.5}$$

Estimate for $\int U_k^{2^*-2} w_k^2$*:*

$$\int U_k^{2^*-2} w_k^2 = O(||w_k||^2). \tag{4.6}$$

Now, we will obtain a lower bound for $\Psi_{\alpha_k}(u_k)$. Let $v_k = u_k/C_k = U_k + \tilde{w}_k = U_k + w_k/C_k$. Because of (3.22), the sequence (v_k) satisfies (3.4) and the sequence $(\tilde{w}_k)$ satisfies (3.25). Of course, $d(v_k, M)$ is achieved by U_k. Because Ψ_α is homogeneous of degree zero, $\Psi_{\alpha_k}(u_k) = \Psi_{\alpha_k}(v_k)$. We will compute $\Psi_{\alpha_k}(v_k)$ but we will still call v_k by u_k, and $\tilde{w}_k$ by w_k.

The value of $\Psi_{\alpha_k}(u_k)$ is the sum of $\beta(u_k)$ and $\alpha_k\beta(u_k)\delta(u_k)$. As in [10], we can obtain the following lower bound for $\beta(u_k)$:

$$\beta(u_k) \geq \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \Big[\gamma_2 ||w_k||^2 - (2^*-1)\int U_k^{2^*-2} w_k^2\Big] + o(\varepsilon_k),$$

for any fixed number $\gamma_2 < 1$.

We also wish to obtain a lower bound for

$$\alpha_k \beta(u_k)\delta(u_k) = \alpha_k \frac{||u_k||^s}{|u_k|_{2^*}^{2+2^*s/2}} |u|_q^{qt}. \tag{4.7}$$

We obtain a lower bound for $||u_k||^s$ from

$$||u_k||^2 = ||U_k||^2 + 2\Big(\int \nabla U_k \cdot \nabla w_k + a\int U_k w_k\Big) + ||w_k||^2.$$

Using (2.17) and (2.38) in Adimurthi and Mancini [1], (3.24) and (4.1),

$$||u_k||^2 = \frac{S^{\frac{N}{2}}}{2} + O(\varepsilon_k) + O(||w_k||^2).$$

This implies that

$$||u_k||^s \geq \Big(\frac{S^{\frac{N}{2}}}{2}\Big)^{\frac{s}{2}} + O(\varepsilon_k) + O(||w_k||^2).$$

We obtain a lower bound for $|u_k|_{2^*}^{-(2+2^*s/2)}$ from

$$|u_k|_{2^*}^{2^*} = |U_k|_{2^*}^{2^*} + 2^* \int U_k^{2^*-1} w_k + \tfrac{2^*(2^*-1)}{2} \int U_k^{2^*-2} w_k^2 + O(||w_k||^r)$$

(see [2]), where $r=\min\{2^*,3\}$, i.e., $r=3$ if $N=5$, and $r=2^*$ if $N>5$. Using (2.18) in Adimurthi and Mancini [1], (4.2), (4.6) and $(1+z)^{-\eta}\geq 1-\eta z$, for $\eta>0$ and $z\geq -1$, we deduce

$$|u_k|_{2^*}^{-(2+2^*s/2)}\geq\left(\frac{S^{\frac{N}{2}}}{2}\right)^{-\frac{s}{2}-\frac{2}{2^*}}+O(\varepsilon_k)+O(||w_k||^2).$$

For the product we obtain the lower bound

$$\frac{||u_k||^s}{|u_k|_{2^*}^{2+2^*s/2}}\geq 2^{\frac{N-2}{N}}S^{\frac{2-N}{2}}+O(\varepsilon_k)+O(||w_k||^2)=A_1+A_{2,k}+A_{3,k}. \quad (4.8)$$

To estimate $|u_k|_q^{qt}$ we use:

Lemma 4.1. *Suppose* $2^\flat\leq q\leq 2^\#$ *and* t *is given by* (2.2). *For* $x\geq -1$,

$$(1+x)^q\geq 1+\frac{qt}{2}|x|^{\frac{2}{t}}-q|x|.$$

Proof. From (2.3) if follows that $\frac{2}{t}\leq q$, as $s\geq 0$. We will consider separately the cases $x>0$ and $x<0$, since the inequality is obviously true if $x=0$. For $x\geq -1$, $x\neq 0$, define $f(x)=(1+x)^q-1-\frac{qt}{2}|x|^{\frac{2}{t}}+q|x|$. Then

$$f'(x)=q(1+x)^{q-1}-q|x|^{\frac{2}{t}-1}\operatorname{sign}x+q\operatorname{sign}x.$$

If $x>0$, then $f'(x)>q(1+x)^{q-1}-qx^{\frac{2}{t}-1}>0$. If $-1\leq x<0$, then

$$f'(x)=q(1+x)^{q-1}+qx^{\frac{2}{t}-1}-q\leq 0,$$

as $(1+x)^{q-1}+x^{\frac{2}{t}-1}\leq 1$, since both $x\mapsto(1+x)^{q-1}$ and $x\mapsto x^{\frac{2}{t}-1}$ are convex. □

As a consequence of Lemma 4.1,

$$|u_k|_q^{qt}\geq\left(|U_k|_q^q+\tfrac{qt}{2}\int U_k^{q-\frac{2}{t}}|w_k|^{\frac{2}{t}}-\eta_k\right)^t,$$

with

$$\eta_k:=\min\left\{q\int U_k^{q-1}|w_k|,\ |U_k|_q^q+\tfrac{qt}{2}\int U_k^{q-\frac{2}{t}}|w_k|^{\frac{2}{t}}\right\}.$$

We now use the fact that $(1-x)^\eta\geq 1-x$ for $0\leq x\leq 1$ and $0<\eta\leq 1$ to write

$$|u_k|_q^{qt}\geq\left(|U_k|_q^q+\tfrac{qt}{2}\int U_k^{q-\frac{2}{t}}|w_k|^{\frac{2}{t}}\right)^t-q|U_k|_q^{q(t-1)}\int U_k^{q-1}|w_k|.$$

Estimates (4.3), (4.11) and the Hölder inequality yield

$$\begin{aligned}|u_k|_q^{qt} &\geq \left(|U_k|_q^q + \tfrac{qt}{2}\int U_k^{q-\frac{2}{t}}|w_k|^{\frac{2}{t}}\right)^t - C\varepsilon_k^{(t-1)\frac{1}{t}}\varepsilon_k^{1/t}||w_k|| \\ &= \left(|U_k|_q^q + \tfrac{qt}{2}\int U_k^{q-\frac{2}{t}}|w_k|^{\frac{2}{t}}\right)^t + O(\varepsilon_k)||w_k|| \\ &= \tfrac{1}{2^{1-t}}|U_k|_q^{qt} + \tfrac{1}{2^{1-t}}\tfrac{q^tt^t}{2^t}\left(\int U_k^{q-\frac{2}{t}}|w_k|^{\frac{2}{t}}\right)^t + O(\varepsilon_k)||w_k|| \\ &\geq \tfrac{1}{2^{1-t}}|U_k|_q^{qt} + \tfrac{1}{|\Omega|^{1-t}}\tfrac{q^tt^t}{2}\int U_k^{qt-2}w_k^2 + O(\varepsilon_k)||w_k||.\end{aligned}$$

Using (4.11) again,

$$\begin{aligned}\alpha_k|u_k|_q^{qt} &\geq \tfrac{B(q,N)^t}{2}\alpha_k\varepsilon_k + \tfrac{1}{|\Omega|^{1-t}}\tfrac{q^tt^t}{2}\alpha_k\int U_k^{qt-2}w_k^2 + o(\alpha_k\varepsilon_k) \\ &= B_{1,k} + B_{2,k} + B_{3,k}.\end{aligned} \tag{4.9}$$

The next step is to substitute (4.8) and (4.9) in (4.7). We notice that $(A_1 + A_{2,k} + A_{3,k})B_{3,k} = o(\alpha_k\varepsilon_k)$ and $(A_{2,k} + A_{3,k})B_{1,k} = o(\alpha_k\varepsilon_k)$. The term $A_{2,k}B_{2,k}$ is also $o(\alpha_k\varepsilon_k)$. In fact, if $2^\flat \leq q < 2^\#$, by (4.4),

$$A_{2,k}B_{2,k} = O(\alpha_k\varepsilon_k^{s+1}||w_k||^2) = o(\alpha_k\varepsilon_k).$$

If $q = 2^\#$, by (4.5),

$$A_{2,k}B_{2,k} = O(\alpha_k\varepsilon_k^2|\log\varepsilon_k|^{\frac{2}{N}}||w_k||^2) = o(\alpha_k\varepsilon_k).$$

So,

$$\begin{aligned}\alpha_k\beta(u_k)\delta(u_k) &\geq 2^{\frac{N-2}{N}}S^{\frac{2-N}{2}}\tfrac{B(q,N)^t}{2}\alpha_k\varepsilon_k + 2^{\frac{N-2}{N}}S^{\frac{2-N}{2}}\tfrac{1}{|\Omega|^{1-t}}\tfrac{q^tt^t}{2}\alpha_k\int U_k^{qt-2}w_k^2 \\ &\quad + O\left(||w_k||^2\right)\alpha_k\int U_k^{qt-2}w_k^2 + o(\alpha_k\varepsilon_k) \\ &\geq 2^{-\frac{2}{N}}S^{\frac{2-N}{2}}\left[B(q,N)^t\alpha_k\varepsilon_k + \gamma_2\tfrac{1}{|\Omega|^{1-t}}q^tt^t\alpha_k\int U_k^{qt-2}w_k^2\right] + o(\alpha_k\varepsilon_k),\end{aligned}$$

for any fixed number $\gamma_2 < 1$. This is our lower bound for $\alpha_k\beta(u_k)\delta(u_k)$.

Combining the lower bounds for $\beta(u_k)$ and for $\alpha_k\beta(u_k)\delta(u_k)$,

$$\begin{aligned}\Psi_{\alpha_k}(u_k) &\geq \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{-\frac{2}{N}}S^{\frac{2-N}{2}}B(q,N)^t\alpha_k\varepsilon_k \\ &\quad + 2^{\frac{N-2}{N}}S^{\frac{2-N}{2}}\left[\gamma_2||w_k||^2 + \gamma_2\tfrac{1}{|\Omega|^{1-t}}\tfrac{q^tt^t}{2}\alpha_k\int U_k^{qt-2}w_k^2\right.\end{aligned}$$

$$- (2^* - 1)\int U_k^{2^*-2} w_k^2\Big] + o(\alpha_k\varepsilon_k).$$

From Lemma 3.11, the term inside the square parenthesis is greater than

$$\Big[\Big(\gamma_2 - \frac{(2^*-1)}{(2^*-1)+\gamma_1}\Big)\Big(||w_k||^2 + \tfrac{1}{|\Omega|^{1-t}}\tfrac{q^t t^t}{2}\alpha_k \int U_k^{qt-2} w_k^2\Big) + o(\varepsilon_k)\Big].$$

We choose $\gamma_2 \geq \frac{(2^*-1)}{(2^*-1)+\gamma_1}$. As a consequence, this term is greater than $o(\varepsilon_k)$. Hence,

$$\Psi_{\alpha_k}(u_k) \geq \frac{|\nabla U_k|_2^2}{|U_k|_{2^*}^2} + 2^{-\frac{2}{N}} S^{\frac{2-N}{2}} B(q,N)^t \alpha_k \varepsilon_k + o(\alpha_k\varepsilon_k).$$

Recall, from Adimurthi and Mancini [1], that for $N \geq 5$ and $y \in \partial\Omega$,

$$\frac{|\nabla U_{\varepsilon,y}|_2^2}{|U_{\varepsilon,y}|_{2^*}^2} = \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}} S A(N) H(y)\varepsilon + O(\varepsilon^2), \tag{4.10}$$

with

$$A(N) = \tfrac{N-1}{N}\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{N-3}{2}\right)}{\Gamma\left(\frac{N-2}{2}\right)}$$

and $H(y)$ the mean curvature of $\partial\Omega$ at y with respect to the unit outward normal. Therefore,

$$\begin{aligned}\Psi_{\alpha_k}(u_k) &\geq \frac{S}{2^{\frac{2}{N}}} + 2^{-\frac{2}{N}} S^{\frac{2-N}{2}} B(q,N)^t \alpha_k\varepsilon_k\Big[1 - 2S^{\frac{N}{2}}\frac{A(N)}{B(q,N)^t} H(y_k)\frac{1}{\alpha_k} + o(1)\Big]\\ &> \frac{S}{2^{\frac{2}{N}}},\end{aligned}$$

for large k.

Remark 4.2. If in the argument above, instead of using the inequality $(x+y)^t \geq \frac{1}{2^{1-t}}x^t + \frac{1}{2^{1-t}}y^t$, we use $(x+y)^t \geq (1-\varsigma)^{1-t}x^t + \varsigma^{1-t}y^t$, for x, $y > 0$ and ς such that $0 < \varsigma < 1$, then we obtain the following lower bound for $\Psi_{\alpha_k}(u_k)$:

$$\frac{S}{2^{\frac{2}{N}}} + \frac{2^{\frac{N-2}{N}}}{S^{\frac{N-2}{2}}}\frac{(1-\varsigma)^{1-t}B(q,N)^t}{2^t}\alpha_k\varepsilon_k\left[1 - \tfrac{2^t}{(1-\varsigma)^{1-t}} S^{\frac{N}{2}}\tfrac{A(N)}{B(q,N)^t}\tfrac{H(y_k)}{\alpha_k} + o(1)\right].$$

Proof of Theorem 2.2. So assume α_0, in (3.2), is $+\infty$. Choose a sequence $\alpha_k \to +\infty$ as $k \to +\infty$ and denote by u_k a positive minimizer for Ψ_{α_k} satisfying $(2.9)_{\alpha_k}$. From Lemmas 3.4 and 3.6, the conditions (3.4), (3.21), (3.22), (3.23) and (3.25) hold. Hence, $S_{\alpha_k} = \Psi_{\alpha_k}(u_k) > \frac{S}{2^{\frac{2}{N}}}$ for large k, which is impossible. Therefore, α_0 is finite. Remarks 3.1 and 3.3 imply Theorem 2.2. □

We give one more lower bound for α_0, in addition to one in Remark 2.3.

Lemma 4.3. *The value α_0 has the lower bound*

$$\alpha_0 \geq 2^t S^{\frac{N}{2}} \frac{A(N)}{B(q,N)^t} \max_{\partial\Omega} H.$$

Proof. Suppose $\alpha < 2^t S^{\frac{N}{2}} \frac{A(N)}{B(q,N)^t} \max_{\partial\Omega} H$. Choose $P \in \partial\Omega$ such that $H(P) = \max_{\partial\Omega} H$. From (4.10),

$$\beta(U_{\varepsilon,P}) = \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}} S A(N) H(P)\varepsilon + o(\varepsilon).$$

From (2.17) and (2.38) in Adimurthi and Mancini [1],

$$||U_{\varepsilon,P}||^s \leq \left(\frac{S^{\frac{N}{2}}}{2}\right)^{\frac{s}{2}} + O(\varepsilon)$$

and, from (2.18) in Adimurthi and Mancini [1],

$$|U_{\varepsilon,P}|_{2^*}^{-(2+2^*s/2)} \leq \left(\frac{S^{\frac{N}{2}}}{2}\right)^{-(\frac{s}{2}+\frac{2}{2^*})} + O(\varepsilon).$$

Together,

$$\frac{||U_{\varepsilon,P}||^s}{|U_{\varepsilon,P}|_{2^*}^{2+2^*s/2}} \leq 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} + O(\varepsilon).$$

Moreover, from (4.11),

$$|U_{\varepsilon,P}|_q^{qt} \leq \frac{B(q,N)^t}{2^t}\varepsilon + O(\varepsilon^2).$$

Combining the last two estimates,

$$(\beta\delta)(U_{\varepsilon,P}) \leq 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \frac{B(q,N)^t}{2^t}\varepsilon + o(\varepsilon).$$

We can estimate S_α from above by

$$\begin{aligned} S_\alpha &\leq \Psi_\alpha(U_{\varepsilon,P}) \\ &\leq \frac{S}{2^{\frac{2}{N}}} - 2^{\frac{N-2}{N}} S^{\frac{2-N}{2}} \frac{B(q,N)^t}{2^t} \alpha\varepsilon \left[2^t S^{\frac{N}{2}} \frac{A(N)}{B(q,N)^t} H(P) \frac{1}{\alpha} - 1 + o(1)\right] \end{aligned}$$

as $\varepsilon \to 0$. Since we are assuming $\alpha < 2^t S^{\frac{N}{2}} \frac{A(N)}{B(q,N)^t} H(P)$, the value of S_α satisfies $S_\alpha < \frac{S}{2^{\frac{2}{N}}}$. This proves the lemma. □

APPENDIX: THE ESTIMATE FOR $|U_{\varepsilon,y}|_q^q$ FOR $y \in \partial\Omega$

In this Appendix we prove that if $y \in \partial\Omega$, then

$$|U_{\varepsilon,y}|_q^q = \frac{B(q,N)}{2}\varepsilon^{(2^*-q)\frac{N-2}{2}} + O(\varepsilon^{1+1/t}) = \frac{B(q,N)}{2}\varepsilon^{1/t} + O(\varepsilon^{1+1/t}) \tag{4.11}$$

with

$$B(q,N) = \int_{\mathbb{R}^N} U^q = \pi^{\frac{N}{2}}[N(N-2)]^{\frac{N}{2}} \frac{\Gamma\left(\frac{N-2}{2}q - \frac{N}{2}\right)}{\Gamma\left(\frac{N-2}{2}q\right)},$$

by adapting an estimate due to Adimurthi and Mancini [1] ($U_{\varepsilon,y}$ is defined in (2.6)). By a change of coordinates we can assume that $y = 0$, $B_R(0) \cap \Omega = \{(x', x_N) \in B_R(0) : x_N > \rho(x')\}$ and $B_R(0) \cap \partial\Omega = \{(x', x_N) \in B_R(0) : x_N = \rho(x')\}$, for some $R > 0$, where $x' = (x_1, \ldots, x_{N-1})$,

$$\rho(x') = \sum_{i=1}^{N-1} \lambda_i x_i^2 + O(|x'|^3),$$

$\lambda_i \in \mathbb{R}$, $1 \le i \le N-1$.

Let $U_\varepsilon := U_{\varepsilon,0}$ and $\Sigma := \{(x', x_N) \in B_R(0) : 0 < x_N < \rho(x')\}$. Then

$$|U_\varepsilon|_q^q = \frac{1}{2}\int_{B_R(0)} U_\varepsilon^q - \int_\Sigma U_\varepsilon^q + \int_{B_R^C(0)\cap\Omega} U_\varepsilon^q \tag{4.12}$$

if all the λ_i's are positive. If all the λ_i's are negative, then the minus sign on the right hand side turns into a plus sign. Henceforth, we will assume all the λ_i's are positive. The final estimate for $|U_\varepsilon|_q^q$ will hold no matter what the sign of the λ_i's, for it holds when the λ_i's all have the same sign.

We will estimate each of the three terms on the right hand side of (4.12). For the third term we have

$$\begin{aligned}\int_{B_R^C(0)\cap\Omega} U_\varepsilon^q \le \int_{B_R^C(0)} U_\varepsilon^q &= O\Big(\varepsilon^{\frac{1}{t}} \int_{R/\varepsilon}^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}q}} dr\Big) \\ &= O\big(\varepsilon^{\frac{1}{t}} \times \varepsilon^{N-\frac{2}{t}}\big) = O\big(\varepsilon^{N-\frac{1}{t}}\big) = O\big(\varepsilon^{\frac{N-2}{2}q}\big).\end{aligned}$$

Using this estimate, for the first term on the right hand side of (4.12) we have

$$\begin{aligned}\frac{1}{2}\int_{B_R(0)} U_\varepsilon^q = \frac{1}{2}\int_{\mathbb{R}^N} U_\varepsilon^q + O\big(\varepsilon^{N-\frac{1}{t}}\big) &= \frac{1}{2}\varepsilon^{\frac{1}{t}}\int_{\mathbb{R}^N} U^q + O\big(\varepsilon^{N-\frac{1}{t}}\big) \\ &= \frac{1}{2}B(q,N)\varepsilon^{\frac{1}{t}} + O\big(\varepsilon^{N-\frac{1}{t}}\big),\end{aligned}$$

with

$$\begin{aligned} B(q,N) := \int_{\mathbb{R}^N} U^q &= [N(N-2)]^{\frac{N}{2}} \omega_N \int_0^{+\infty} \frac{r^{N-1}}{(1+r^2)^{\frac{N-2}{2}q}}\, dr \\ &= [N(N-2)]^{\frac{N}{2}} \omega_N \frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{N-2}{2}q - \frac{N}{2}\right)}{2\Gamma\left(\frac{N-2}{2}q\right)} \\ &= \pi^{\frac{N}{2}} [N(N-2)]^{\frac{N}{2}} \frac{\Gamma\left(\frac{N-2}{2}q - \frac{N}{2}\right)}{\Gamma\left(\frac{N-2}{2}q\right)}; \end{aligned}$$

in particular,

$$B(2^\flat, N) = \pi^{\frac{N}{2}} [N(N-2)]^{\frac{N}{2}} \frac{\Gamma(\frac{N(N-3)}{2(N-1)})}{\Gamma(\frac{N(N-2)}{N-1})}$$

$$B(2^\#, N) = \pi^{\frac{N}{2}} [N(N-2)]^{\frac{N}{2}} \frac{\Gamma(\frac{N-2}{2})}{\Gamma(N-1)}.$$

So we are left with the estimate of the second term on the right hand side of (4.12). Let $\sigma > 0$ be such that

$$L_\sigma := \{x \in \mathbb{R}^N : |x_i| < \sigma, 1 \le i \le N\} \subset B_{\frac{R}{4}}(0)$$

and define

$$\Delta_\sigma := \{x' : |x_i| < \sigma, 1 \le i \le N-1\}.$$

For the second term on the right hand side of (4.12),

$$\begin{aligned} \int_\Sigma U_\varepsilon^q = \int_{\Sigma \cap L_\sigma} U_\varepsilon^q + O(\varepsilon^{N-\frac{1}{t}}) &= \int_{\Delta_\sigma} \int_0^{\rho(x')} U_\varepsilon^q dx_N\, dx' + O(\varepsilon^{N-\frac{1}{t}}) \\ &= O\Big(\int_{\Delta_\sigma} \int_0^{\rho(x')} \frac{\varepsilon^{\frac{N-2}{2}q}}{(\varepsilon^2+|x|^2)^{\frac{N-2}{2}q}}\, dx_N\, dx' \Big) + O(\varepsilon^{N-\frac{1}{t}}); \end{aligned}$$

using the change of variables $\sqrt{\varepsilon^2+|x'|^2}\, y_N = x_N$,

$$= O\Big(\int_{\Delta_\sigma} \frac{\varepsilon^{\frac{N-2}{2}q}}{(\varepsilon^2+|x'|^2)^{\frac{N-2}{2}q-\frac{1}{2}}} \int_0^{\frac{\rho(x')}{\sqrt{\varepsilon^2+|x'|^2}}} \frac{1}{(1+y_N^2)^{\frac{N-2}{2}q}}\, dy_N\, dx' \Big) + O(\varepsilon^{N-\frac{1}{t}});$$

let $\kappa \ge 0$; $\int_0^s \frac{1}{(1+t^2)^\kappa}\, dt \le s$ for $s > 0$ and $\int_0^s \frac{1}{(1+t^2)^\kappa}\, dt = s - \frac{\kappa}{3}s^3 + \frac{\kappa(\kappa+1)}{10}s^5 - O(s^7)$ for small s; thus $\int_0^s \frac{1}{(1+t^2)^\kappa}\, dt = s + O(s^3)$ for all s and we can continue

$$= O\Big(\varepsilon^{\frac{N-2}{2}q} \int_{\Delta_\sigma} \frac{\sum \lambda_i x_i^2}{(\varepsilon^2+|x'|^2)^{\frac{N-2}{2}q}}\, dx' \Big)$$

$$+ O\Big(\varepsilon^{\frac{N-2}{2}q} \int_{\Delta_\sigma} \frac{|x'|^3}{(\varepsilon^2 + |x'|^2)^{\frac{N-2}{2}q}}\, dx'\Big) + O(\varepsilon^{N-\frac{1}{t}})$$
$$= O\Big(\varepsilon^{\frac{1}{t}+1} \int_{\frac{\Delta_\sigma}{\varepsilon}} \frac{|y'|^2}{(1 + |y'|^2)^{\frac{N-2}{2}q}}\, dy'\Big)$$
$$+ O\Big(\varepsilon^{\frac{1}{t}+2} \int_{\frac{\Delta_\sigma}{\varepsilon}} \frac{|y'|^3}{(1 + |y'|^2)^{\frac{N-2}{2}q}}\, dy'\Big) + O(\varepsilon^{N-\frac{1}{t}}) = O(\varepsilon^{\frac{1}{t}+1}).$$

Combining the estimates for the three terms on the right hand side of (4.12),

$$|U_\varepsilon|_q^q = \frac{1}{2} B(q,N)\varepsilon^{\frac{1}{t}} + O(\varepsilon^{\frac{1}{t}+1}).$$

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