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# STABILITY FOR A SYSTEM OF WAVE EQUATIONS OF KIRCHHOFF WITH COUPLED NONLINEAR AND BOUNDARY CONDITIONS OF MEMORY TYPE

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**Abstract.** In this paper, we consider a system of two wave equations of Kirchhoff with coupled nonlinear and memory conditions at the boundary, and we study the asymptotic behavior of the corresponding solutions. We prove that the energy decays with the same rate of decay of the relaxation functions; that is, the energy decays exponentially when the relaxation functions decay exponentially and polynomially when the relaxation functions decay polynomially.

## 1. Introduction

The main purpose of this work is to study the asymptotic behavior of the solutions of a system of two nonlinear wave equations of Kirchhoff type with coupled nonlinear and boundary conditions of memory type. To formalize this problem let us take $\Omega$ an open, bounded set of $\mathbb{R}^n$ with smooth boundary $\Gamma$, and let us assume that $\Gamma$ can be divided in two nonempty parts $\Gamma = \Gamma_0 \cup \Gamma_1$ with $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$. Let us denote by $\nu(x)$ the unit normal vector at $x \in \Gamma$ outside of $\Omega$, and let us consider the following initial boundary value problem:

$$u_{tt} - M(||\nabla u||_2^2 + ||\nabla v||_2^2)\Delta u - \Delta u_t + f(u - v) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.1}$$

$$v_{tt} - M(||\nabla u||_2^2 + ||\nabla v||_2^2)\Delta v - \Delta v_t - f(u - v) = 0 \quad \text{in} \quad \Omega \times (0, \infty), \tag{1.2}$$

$$u = v = 0 \quad \text{on} \quad \Gamma_0 \times (0, \infty), \tag{1.3}$$

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$$u+\int_0^t g_1(t-s)(M(||\nabla u(s)||_2^2+||\nabla v(s)||_2^2)\frac{\partial u}{\partial \nu}(s)+\frac{\partial u_t}{\partial \nu}(s))ds=0 \quad \text{on} \quad \Gamma_1\times(0,\infty), \tag{1.4}$$

$$v+\int_0^t g_2(t-s)(M(||\nabla u(s)||_2^2+||\nabla v(s)||_2^2)\frac{\partial v}{\partial \nu}(s)+\frac{\partial v_t}{\partial \nu}(s))ds=0 \quad \text{on} \quad \Gamma_1\times(0,\infty), \tag{1.5}$$

$$(u(0,x),v(0,x))=(u_0(x),v_0(x)), \quad (u_t(0,x),v_t(0,x))=(u_1(x),v_1(x)) \text{ in } \Omega. \tag{1.6}$$

Here, $u$ and $v$ are the transverse displacements, the relaxation functions $g_i$ are positive and nondecreasing, and the functions $f\in C^1(\mathbb{R})$ satisfy

$$f(s)s\geq 0 \quad \forall s\in\mathbb{R}.$$

Additionally, we suppose that $f$ is superlinear; that is

$$f(s)s\geq(2+\delta)F(s), \quad F(z)=\int_0^z f(s)ds \quad \forall s\in\mathbb{R},$$

for some $\delta>0$ with the following growth conditions:

$$|f(x)-f(y)|\leq c(1+|x|^{\rho-1}+|y|^{\rho-1})|x-y|, \quad \forall x,y\in\mathbb{R},$$

for some $c>0$ and $\rho\geq 1$ such that $(n-2)\rho\leq n$. We shall assume that the functions $M\in C^1([0,\infty))$ satisfy

$$M(\lambda)\geq m_0>0, \quad M(\lambda)\lambda\geq\widehat{M}(\lambda), \quad \forall\lambda\geq 0, \tag{1.7}$$

where $\widehat{M}(\lambda)=\int_0^\lambda M(s)ds$. The integral equations (1.4)–(1.5) describe the memory effects which can be caused, for example, by the interaction with another viscoelastic element. We refer to [11] for the physical motivation of this model. Also, we shall assume that there exists $x_0\in\mathbb{R}^n$ such that

$$\Gamma_0=\{x\in\Gamma \;:\; \nu(x)\cdot(x-x_0)\leq 0\},$$
$$\Gamma_1=\{x\in\Gamma \;:\; \nu(x)\cdot(x-x_0)>0\}.$$

As a example of a set $\Omega$ satisfying those properties, we can consider the domain shown in Figure 1.

Let us denote by $m(x)=x-x_0$. Note that the compactness of $\Gamma_1$ implies that there exists a small positive constant $\delta_0$ such that

$$0<\delta_0\leq m(x)\cdot\nu(x), \quad \forall x\in\Gamma_1. \tag{1.8}$$

The existence of global solutions and exponential decay to the problem (1.1), (1.3) with $\partial\Omega=\Gamma_0$ has been investigated by many authors (see, e.g., [4, 5, 8, 9, 14, 15, 16, 18] ). There exists a large body of literature regarding viscoelastic problems with the memory term acting in the domain or in the

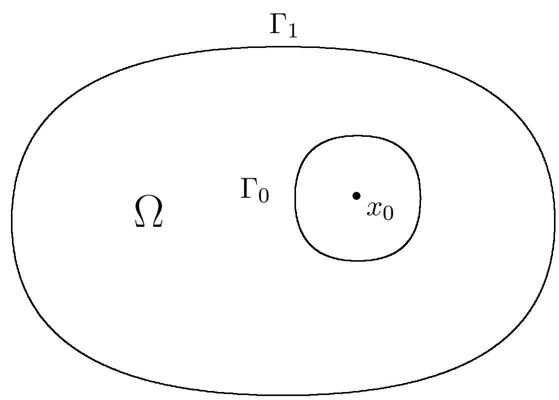


Figure 1

boundary. Among the numerous works in this direction, we can cite Rivera [10] and M.L. Santos [21, 22]. Cavalcanti et al. [6, 7] studied the existence and uniform decay of strong solutions of wave equation (1.1) without strong damping term $\Delta u_t$ with nonlinear boundary damping and memory source term when $M(s) = 1$ and $f(s) = 0$. Park and Bae [19] studied the existence and uniform decay of strong solutions of the coupled wave equations (1.1)–(1.2) with nonlinear boundary damping and memory source term and $M(s) = 1 + s$. It is important to emphasize that in [19] they obtained only uniform decay of strong solutions of the coupled wave equations (1.1)–(1.2) without the coupledness nonlinear $f$. Dissipative coupled systems of the wave equations with $M(s) = 1$ and $f(s) = \alpha s$ were studied by some authors, as for example [1, 2, 3, 12]. We shall briefly describe some of them. In [12], Komornik and Rao studied a linear system of two compactly coupled wave equations with boundary frictional damping in both equations. They showed the existence, regularity, and stability of the corresponding solutions. The stability results obtained in [12] were extended by Aassila [1] for a coupled system with weak frictional damping at the infinity. In a second work, Aassila [2] removes the dissipation of the one equation and shows the strong asymptotic stability or the nonuniform stability for some particular cases depending on the coupling constant. Another similar coupled system with boundary frictional damping on only one of the equations was studied by Alabau [3]; she shows the polynomial decay of the corresponding strong solutions when the speed of wave propagation of both the equations are the same. It seems to us that there is no result concerning the asymptotic stability of solutions for the system (1.1)–(1.6) where the coupledness is nonlinear. So to fill this gap we study here this topic.

The main result in this paper is to show that the solutions of the coupled system (1.1)–(1.6) decays uniformly in time with the same rate of decay as the relaxation functions. More precisely, denoting by $k_1$ and $k_2$ the resolvent kernels of $-g_1'/g_1(0)$ and $-g_2'/g_2(0)$ respectively, we show that the solution decays exponentially to zero provided $k_1$ and $k_2$ decay exponentially to zero. When the resolvent kernels $k_1$ and $k_2$ decay polynomially, we show that the corresponding solution also decays polynomially to zero. The method used is based on the construction of a suitable Lyapunov functional $\mathcal{L}$ satisfying

$$\frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t) + c_2 e^{-\gamma t} \quad \text{or} \quad \frac{d}{dt}\mathcal{L}(t) \le -c_1\mathcal{L}(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\alpha+1}}$$

for some positive constants $c_1, c_2, \gamma$, and $\alpha$. Note that, because of condition (1.3) the solution of the system (1.1)–(1.6) must belong to the following space: $V := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$. The notations we use in this paper are standard and can be found in Lion's book [13]. In the sequel, by $c$ (sometime $c_1, c_2, \dots$) we denote various positive constants which do not depend on $t$ or on the initial data. The organization of this paper is as follows. In section 2 we establish the existence and uniqueness of strong solutions for the system (1.1)–(1.6). In section 3 we prove the uniform rate of exponential decay. Finally in section 4 we prove the uniform rate of polynomial decay.

## 2. Existence and regularity

In this section we shall study the existence and regularity of solutions for the coupled system (1.1)–(1.6). First, we shall use equations (1.4)–(1.5) to estimate the terms $M(||\nabla u(t)||_2^2 + ||\nabla v(t)||_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}$ and $M(||\nabla u(t)||_2^2 + ||\nabla v(t)||_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}$. Denoting by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds,$$

the convolution product operator and differentiating the equations (1.4) and (1.5) we arrive at the following Volterra equations:

$$\begin{aligned} &M(||\nabla u(t)||_2^2 + ||\nabla v(t)||_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} \\ &+ \frac{1}{g_1(0)} g_1' * (M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}) = -\frac{1}{g_1(0)} u_t, \end{aligned}$$

$$M(\|\nabla u(t)\|_2^2 + \|\nabla v(t)\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}$$
$$+\frac{1}{g_2(0)} g_2' * (M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}) = -\frac{1}{g_2(0)} v_t.$$

Applying the Volterra's inverse operator, we get

$$\begin{aligned} M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} &= -\frac{1}{g_1(0)}\{u_t + k_1 * u_t\}, \\ M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} &= -\frac{1}{g_2(0)}\{v_t + k_2 * v_t\}, \end{aligned}$$

where the resolvent kernels satisfy

$$k_i + \frac{1}{g_i(0)} g_i' * k_i = -\frac{1}{g_i(0)} g_i' \quad \text{for} \quad i = 1, 2.$$

Denoting by $\tau_1 = \frac{1}{g_1(0)}$ and $\tau_2 = \frac{1}{g_2(0)}$ we obtain

$$M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} = -\tau_1\{u_t + k_1(0)u - k_1(t)u_0 + k_1' * u\} \tag{2.1}$$

$$M(\|\nabla u\|_2^2 + \|\nabla v\|_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} = -\tau_2\{v_t + k_2(0)v - k_2(t)v_0 + k_2' * v\}. \tag{2.2}$$

Reciprocally, taking initial data such that $u_0 = v_0 = 0$ on $\Gamma_1$, the identities (2.1)–(2.2) imply (1.4)–(1.5). Since we are interested in relaxation functions of exponential or polynomial type and the identities (2.1)–(2.2) involve the resolvent kernels $k_i$, we want to know whether $k_i$ has the same property or not. The following lemma answers this question. Let $h$ be a relaxation function and $k$ its resolvent kernel; that is,

$$k(t) - k * h(t) = h(t). \tag{2.3}$$

**Lemma 2.1.** *If $h$ is a positive continuous function, then $k$ also is a positive continuous function. Moreover,*

(1) *If there exist positive constants $c_0$ and $\gamma$ with $c_0 < \gamma$ such that $h(t) \le c_0 e^{-\gamma t}$, then the function $k$ satisfies $k(t) \le \frac{c_0(\gamma-\epsilon)}{\gamma-\epsilon-c_0} e^{-\epsilon t}$, for all $0 < \epsilon < \gamma - c_0$.*

(2) *Given $p > 1$, let us denote by*

$$c_p := \sup_{t\in\mathbb{R}^+} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p}\, ds.$$

*If there exists a positive constant $c_0$ with $c_0 c_p < 1$ such that $h(t) \leq c_0(1+t)^{-p}$, then the function $k$ satisfies $k(t) \leq \frac{c_0}{1-c_0c_p}(1+t)^{-p}$.*

**Proof.** Note that $k(0) = h(0) > 0$. Now, we take $t_0 = \inf\{t \in \mathbb{R}^+ : k(t) = 0\}$, so $k(t) > 0$ for all $t \in [0, t_0)$. If $t_0 \in \mathbb{R}^+$, from equation (2.3) we get that $-k * h(t_0) = h(t_0)$, but this is contradictory. Therefore, $k(t) > 0$ for all $t \in \mathbb{R}_0^+$. Now, let us fix $\epsilon$, such that $0 < \epsilon < \gamma - c_0$ and denote by

$$k_\epsilon(t) := e^{\epsilon t} k(t), \quad h_\epsilon(t) := e^{\epsilon t} h(t).$$

Multiplying equation (2.3) by $e^{\epsilon t}$ we get $k_\epsilon(t) = h_\epsilon(t) + k_\epsilon * h_\epsilon(t)$; hence,

$$\begin{aligned}\sup_{s\in[0,t]} k_\epsilon(s) &\leq \sup_{s\in[0,t]} h_\epsilon(s) + \Big(\int_0^\infty c_0 e^{(\epsilon-\gamma)s}\, ds\Big) \sup_{s\in[0,t]} k_\epsilon(s)\\ &\leq c_0 + \frac{c_0}{(\gamma-\epsilon)} \sup_{s\in[0,t]} k_\epsilon(s).\end{aligned}$$

Therefore, $k_\epsilon(t) \leq \frac{c_0(\gamma-\epsilon)}{\gamma-\epsilon-c_0}$, which implies our first assertion. To show the second part let us consider the following notations:

$$k_p(t) := (1+t)^p k(t), \quad h_p(t) := (1+t)^p h(t).$$

Multiplying equation (2.3) by $(1+t)^p$ we get

$$k_p(t) = h_p(t) + \int_0^t k_p(t-s)(1+t-s)^{-p}(1+t)^p h(s)\, ds;$$

hence,

$$\sup_{s\in[0,t]} k_p(s) \leq \sup_{s\in[0,t]} h_p(s) + c_0c_p \sup_{s\in[0,t]} k_p(s) \leq c_0 + c_0c_p \sup_{s\in[0,t]} k_p(s).$$

Therefore, $k_p(t) \leq \frac{c_0}{1-c_0c_p}$, which proves our second assertion. □

**Remark.** The finiteness of the constant $c_p$ can be found in [20, Lemma 7.4].

Due to this lemma, in the remainder of this paper, it suffices for our purpose to utilize the identities (2.1) and (2.2) instead of the boundary conditions (1.4) and (1.5). Let us denote by

$$(g\Box\varphi)(t) := \int_0^t g(t-s)|\varphi(t)-\varphi(s)|^2 ds.$$

The following lemma states an important property of the convolution operator.

**Lemma 2.2.** *For $g, \varphi \in C^1([0,\infty):\mathbb{R})$ we have*

$$(g*\varphi)\varphi_t = -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g'\Box\varphi - \frac{1}{2}\frac{d}{dt}\Big[g\Box\varphi - \Big(\int_0^t g(s)ds\Big)|\varphi|^2\Big].$$

The proof of this lemma follows by differentiating the term $g\Box\varphi$.

The first-order energy of coupled system (1.1)–(1.6) is given by

$$\begin{aligned} E(t) := E(t,u,v) &= \frac{1}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}\int_\Omega |v_t|^2 dx \\ &+ \frac{1}{2}\widehat{M}(||\nabla u||_2^2 + ||\nabla v||_2^2) + \int_\Omega F(u-v)dx + \frac{\tau_1}{2}k_1(t)\int_{\Gamma_1}|u|^2 d\Gamma_1 \\ &- \frac{\tau_1}{2}\int_{\Gamma_1} k_1'\Box u d\Gamma_1 + \frac{\tau_2}{2}k_2(t)\int_{\Gamma_1}|v|^2 d\Gamma_1 - \frac{\tau_2}{2}\int_{\Gamma_1} k_2'\Box v d\Gamma_1. \end{aligned}$$

The well-posedness of system (1.1)–(1.6) is given by the following theorem.

**Theorem 2.1.** *Let $k_i \in C^2(\mathbb{R}^+)$ be such that $k_i, -k_i', k_i'' \geq 0$ for $i = 1, 2$. If $(u_0, v_0) \in (H^2(\Omega)\cap V)^2$ and $(u_1, v_1) \in (H^2(\Omega)\cap V)^2$ satisfy the compatibility conditions*

$$M(||\nabla u_0||_2^2 + ||\nabla v_0||_2^2)\frac{\partial u_0}{\partial \nu} + \frac{\partial u_1}{\partial \nu} + \tau_1 u_1 = 0 \quad on \quad \Gamma_1, \tag{2.4}$$

$$M(||\nabla u_0||_2^2 + ||\nabla v_0||_2^2)\frac{\partial v_0}{\partial \nu} + \frac{\partial v_1}{\partial \nu} + \tau_2 v_1 = 0 \quad on \quad \Gamma_1, \tag{2.5}$$

*then there exists only one solution $(u, v)$ of the coupled system (1.1)–(1.6) satisfying*

$$\begin{gathered} u, v \in L^\infty(0, T : V), \quad u_t, v_t \in L^\infty(0, T : V), \quad u_{tt}, v_{tt} \in L^\infty(0, T : L^2(\Omega)) \\ \Delta u, \Delta v \in L^\infty(0, T : L^2(\Omega)), \quad \Delta u_t, \Delta v_t \in L^2(0, T : L^2(\Omega)). \end{gathered}$$

**Proof.** The main idea is to use the Galerkin method. To do this let us take a basis $\{w_j\}_{j\in\mathbb{N}}$ to $V$ which is orthonormal in $L^2(\Omega)$, and we represent by $V_m$ the subspace of $V$ generated by the first $m$ vectors. Standard results on ordinary differential equations guarantee that there exists only one local solution

$$(u^m(t), v^m(t)) := \sum_{j=1}^m (g_{j,m}(t), h_{j,m}(t))w_j$$

of the approximate system,

$$\begin{aligned} &\int_\Omega u_{tt}^m w dx + M(||\nabla u^m(t)||_2^2 + ||\nabla v^m(t)||_2^2)\int_\Omega \nabla u^m\cdot\nabla w dx + \int_\Omega \nabla u_t^m \cdot \nabla w dx \\ &+ \int_\Omega f(u^m - v^m) w dx = -\tau_1 \int_{\Gamma_1}\{u_t^m + k_1(0)u^m - k_1(t)u^m(0) + k_1' * u^m\} w d\Gamma_1, \end{aligned} \tag{2.6}$$

$$\int_\Omega v_{tt}^m w dx + M(||\nabla u^m(t)||_2^2 + ||\nabla v^m(t)||_2^2)\int_\Omega \nabla v^m\cdot\nabla w dx + \int_\Omega \nabla v_t^m \cdot \nabla w dx$$

$$-\int_\Omega f(u^m - v^m) w dx = -\tau_2 \int_{\Gamma_1} \{v_t^m + k_2(0)v^m - k_2(t)v^m(0) + k_2' * v^m\} w d\Gamma_1, \tag{2.7}$$

for all $w \in V_m$ with the initial data

$$(u^m(0), v^m(0)) = (u_0, v_0), \quad (u_t^m(0), v_t^m(0)) = (u_1, v_1).$$

The extension of these solutions to the whole interval $[0, T]$, $0 < T < \infty$, is a consequence of the first estimate which we are going to prove below.

**A priori estimate I**. Replacing $w$ by $u_m'(t)$ in (2.6) and $v_m'(t)$ in (2.7), respectively, and then adding the results we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (|u_t^m|^2 + |v_t^m|^2) dx + \frac{1}{2}\widehat{M}(||\nabla u^m||_2^2 + ||\nabla v^m||_2^2) \\
&+ \int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2) dx + \int_\Omega F(u - v) dx \\
&= -\tau_1 \int_{\Gamma_1} \{u_t^m + k_1(0)u^m - k_1(t)u^m(0) + k_1' * u^m\} u_t^m d\Gamma_1 \\
&-\tau_2 \int_{\Gamma_1} \{v_t^m + k_2(0)v^m - k_2(t)v^m(0) + k_2' * v^m\} v_t^m d\Gamma_1.
\end{aligned} \tag{2.8}$$

Using Lemma 2.2, we have

$$\begin{aligned}
\int_{\Gamma_1} k_1' * u^m u_t^m d\Gamma_1 &= -\frac{1}{2} k_1'(t) \int_{\Gamma_1} |u^m|^2 d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1} k_1'' \Box u^m d\Gamma_1 \\
&\quad - \frac{1}{2}\frac{d}{dt}\int_{\Gamma_1} [k_1' \Box u^m - (\int_0^t k_1'(s) ds)|u^m|^2] d\Gamma_1, \\
\int_{\Gamma_1} k_2' * v^m v_t^m d\Gamma_1 &= -\frac{1}{2} k_2'(t) \int_{\Gamma_1} |v^m|^2 d\Gamma_1 + \frac{1}{2}\int_{\Gamma_1} k_2'' \Box v^m d\Gamma_1 \\
&\quad - \frac{1}{2}\frac{d}{dt}\int_{\Gamma_1} [k_2' \Box v^m - (\int_0^t k_2'(s) ds)|v^m|^2] d\Gamma_1.
\end{aligned}$$

Substituting the above equalities into (2.8), we conclude

$$\frac{d}{dt} E(t, u^m, v^m) \le c E(0, u^m, v^m).$$

Integrating it over $[0, t]$ and taking into account the definition of the initial data of $(u^m, v^m)$ we conclude that

$$E(t, u^m, v^m) \le c, \quad \forall t \in [0, T], \quad \forall m \in \mathbb{N}. \tag{2.9}$$

**A priori estimate II.** First, let us estimate the initial data $u_{tt}^m(0)$ and $v_{tt}^m(0)$ in the $L^2$-norm. Letting $t \to 0^+$ in the equations (2.6)–(2.7), replacing

$w$ by $u''_m(0)$ and $v''_m(0)$, respectively, and using the compatibility conditions (2.4)–(2.5), we get

$$||u^m_{tt}(0)||_2^2 = M(||\nabla u_0||_2^2 + ||\nabla v_0||_2^2)\int_\Omega \Delta u_0 u^m_{tt}(0)dx + \int_\Omega \Delta u_1 u^m_{tt}(0)dx - \int_\Omega f(u_0 - v_0)u^m_{tt}(0)dx,$$

$$||v^m_{tt}(0)||_2^2 = M(||\nabla u_0||_2^2 + ||\nabla v_0||_2^2)\int_\Omega \Delta v_0 v^m_{tt}(0)dx + \int_\Omega \Delta v_1 v^m_{tt}(0)dx + \int_\Omega f(u_0 - v_0)v^m_{tt}(0)dx.$$

Since $(u_0, v_0) \in [H^2(\Omega)]^2$, the growth hypothesis for the function $f$ together with the Sobolev imbedding imply that $f(u_0 - v_0) \in L^2(\Omega)$. Hence

$$||u^m_{tt}(0)||_2 + ||v^m_{tt}(0)||_2 \le c_1, \quad \forall m \in \mathbb{N}. \tag{2.10}$$

Differentiating the equations (2.6)–(2.7) with respect to time, replacing $w$ by $u''_m(t)$ and $v''_m(t)$, respectively, and summing the results we arrive at

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |u^m_{tt}|^2dx + \int_\Omega |v^m_{tt}|^2dx\right\} + \int_\Omega |\nabla u^m_{tt}|^2dx + \int_\Omega |\nabla v^m_{tt}|^2dx\\
&= -M(||\nabla u^m||_2^2 + ||\nabla v^m||_2^2)\left\{\int_\Omega \nabla u^m_t \cdot \nabla u^m_{tt}dx + \int_\Omega \nabla v^m_t \cdot \nabla v^m_{tt}dx\right\}\\
&+ 2M'(||\nabla u^m||_2^2 + ||\nabla v^m||_2^2)\left\{\int_\Omega \nabla u^m \nabla u^m_t dx + \int_\Omega \nabla v^m \nabla v^m_t dx\right\}\\
&\times \left\{\int_\Omega \nabla u^m \cdot \nabla u^m_{tt}dx + \int_\Omega \nabla v^m \cdot \nabla v^m_{tt}dx\right\} - \int_\Omega f'(u^m - v^m)(u^m_t - v^m_t)u^m_{tt}dx\\
&+ \int_\Omega f'(u^m - v^m)(u^m_t - v^m_t)v^m_{tt}dx - \tau_1\int_{\Gamma_1} |u^m_{tt}|^2dx - \tau_1\int_{\Gamma_1} k_1(0)u^m_t u^m_{tt}d\Gamma_1\\
&+ \tau_1 k_1'(t)\int_{\Gamma_1} u^m(0)u^m_{tt}d\Gamma_1 - \tau_1\int_{\Gamma_1}(k_1' * u^m)_t u^m_{tt}d\Gamma_1 - \tau_2\int_{\Gamma_1}|v^m_{tt}|^2dx\\
&- \tau_2\int_{\Gamma_1} k_2(0)v^m_t v^m_{tt}d\Gamma_1 + \tau_2 k_2'(t)\int_{\Gamma_1} v^m(0)v^m_{tt}d\Gamma_1 - \tau_2\int_{\Gamma_1}(k_2' * v^m)_t v^m_{tt}d\Gamma_1.
\end{aligned}$$

Noting that

$$(k_1' * u^m)_t = k_1'(t)u^m_0 + \int_0^t k_1'(t-s)u^m_t(\cdot, s)ds,$$

$$(k_2' * v^m)_t = k_2'(t)v^m_0 + \int_0^t k_2'(t-s)v^m_t(\cdot, s)ds$$

and using Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega |u_{tt}^m|^2dx+\int_\Omega |v_{tt}^m|^2dx+\tau_1k_1(t)\int_{\Gamma_1}|u_t^m|^2d\Gamma_1-\tau_1\int_{\Gamma_1}k_1'\Box u_t^m d\Gamma_1\right.\\
&\left.-\tau_2k_2(t)\int_{\Gamma_1}|v_t^m|^2d\Gamma_1-\tau_2\int_{\Gamma_1}k_2'\Box v_t^m d\Gamma_1\right\}+\int_\Omega|\nabla u_{tt}^m|^2dx+\int_\Omega|\nabla v_{tt}^m|^2dx\\
&=-M(||\nabla u^m||_2^2+||\nabla v^m||_2^2)\left\{\int_\Omega \nabla u_t^m\cdot\nabla u_{tt}^m dx+\int_\Omega\nabla v_t^m\cdot\nabla v_{tt}^m dx\right\}\\
&+2M'(||\nabla u^m||_2^2+||\nabla v^m||_2^2)\left\{\int_\Omega u^m u_t^m dx+\int_\Omega v^m v_t^m dx\right\} \qquad (2.11)\\
&\times\left\{\int_\Omega\nabla u^m\cdot\nabla u_{tt}^m dx+\int_\Omega\nabla v^m\cdot\nabla v_{tt}^m dx\right\}-\int_\Omega f'(u^m-v^m)(u_t^m-v_t^m)u_{tt}^m dx\\
&+\int_\Omega f'(u^m-v^m)(u_t^m-v_t^m)v_{tt}^m dx-\tau_1\int_{\Gamma_1}|u_{tt}^m|^2dx-\tau_1k_1'(t)\int_{\Gamma_1}u^m(0)u_{tt}^m d\Gamma_1\\
&+\frac{\tau_1}{2}k_1'(t)\int_{\Gamma_1}|u_t^m|^2d\Gamma_1-\frac{\tau_1}{2}\int_{\Gamma_1}k_1''\Box u_t^m d\Gamma_1-\tau_2\int_{\Gamma_1}|v_{tt}^m|^2dx\\
&-\tau_2k_2'(t)\int_{\Gamma_1}v^m(0)v_{tt}^m d\Gamma_1+\frac{\tau_2}{2}k_2'(t)\int_{\Gamma_1}|v_t^m|^2d\Gamma_1-\frac{\tau_2}{2}\int_{\Gamma_1}k_2''\Box v_t^m d\Gamma_1.
\end{aligned}$$

Let us take $p_n=\frac{2n}{n-2}$. From the growth condition of the function $f$ and from the Sobolev imbedding, we have

$$\begin{aligned}
&\int_\Omega f'(u^m-v^m)u_t^m u_{tt}^m dx\le c\int_\Omega(1+2|u^m-v^m|^{\rho-1})|u_t^m||u_{tt}^m|dx\\
&\le c\left[\int_\Omega(1+2|u^m-v^m|^{\rho-1})^n dx\right]^{\frac{1}{n}}\left[\int_\Omega|u_t^m|^{p_n}dx\right]^{\frac{1}{p_n}}\left[\int_\Omega|u_{tt}^m|^2dx\right]^{\frac{1}{2}}\\
&\le c\left[\int_\Omega(1+|\nabla(u^m-v^m)|^2)dx\right]^{\frac{\rho-1}{2}}\left[\int_\Omega|\nabla u_t^m|^2dx\right]^{\frac{1}{2}}\left[\int_\Omega|u_{tt}^m|^2dx\right]^{\frac{1}{2}}.
\end{aligned}$$

Taking into account the first estimate (2.8), we conclude that

$$\begin{aligned}
\int_\Omega f'(u^m-v^m)u_t^m u_{tt}^m dx&\le c\left[\int_\Omega|\nabla u_t^m|^2dx\right]^{\frac{1}{2}}\left[\int_\Omega|u_{tt}^m|^2dx\right]^{\frac{1}{2}}\\
&\le c\left\{\int_\Omega|\nabla u_t^m|^2dx+\int_\Omega|u_{tt}^m|^2dx\right\}. \qquad (2.12)
\end{aligned}$$

Similarly, we get

$$\int_\Omega f'(u^m-v^m)v_t^m u_{tt}^m dx\le c\left\{\int_\Omega|\nabla v_t^m|^2dx+\int_\Omega|u_{tt}^m|^2dx\right\}, \qquad (2.13)$$

$$\int_\Omega f'(u^m - v^m)u_t^m v_{tt}^m dx \le c\left\{\int_\Omega |\nabla u_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\right\}, \tag{2.14}$$

$$\int_\Omega f'(u^m - v^m)v_t^m v_{tt}^m dx \le c\left\{\int_\Omega |\nabla v_t^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\right\}. \tag{2.15}$$

Note that Young's inequality, the first estimate and the hypothesis on $M$ give us

$$\begin{aligned} &M(||\nabla u^m||_2^2 + ||\nabla v^m||_2^2)\left\{\int_\Omega \nabla u_t^m \cdot \nabla u_{tt}^m dx + \int_\Omega \nabla v_t^m \cdot \nabla v_{tt}^m dx\right\} \\ &\le c\Big\{\int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2)dx + \frac{1}{4}\int_\Omega |\nabla u_{tt}^m|^2 dx + \frac{1}{4}\int_\Omega |\nabla v_{tt}^m|^2 dx\Big\} \end{aligned} \tag{2.16}$$

and similarly

$$\begin{aligned} &2M'(||\nabla u^m||_2^2 + ||\nabla v^m||_2^2)\left\{\int_\Omega u^m u_t^m dx + \int_\Omega v^m v_t^m dx\right\} \\ &\times\left\{\int_\Omega \nabla u^m \cdot \nabla u_{tt}^m dx + \int_\Omega \nabla v^m \cdot \nabla v_{tt}^m dx\right\} \\ &\le c\Big\{\int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2)dx + \frac{1}{4}\int_\Omega |\nabla u_{tt}^m|^2 dx + \frac{1}{4}\int_\Omega |\nabla v_{tt}^m|^2 dx\Big\}. \end{aligned} \tag{2.17}$$

Using the trace theorem and combining the inequalities (2.12)–(2.17) with the identity (2.11), we arrive at

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big\{\int_\Omega |u_{tt}^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx + \tau_1 k_1(t)\int_{\Gamma_1} |u_t^m|^2 d\Gamma_1 - \tau_1\int_{\Gamma_1} k_1'\Box u_t^m d\Gamma_1 \\ &+ \tau_2 k_2(t)\int_{\Gamma_1} |v_t^m|^2 d\Gamma_1 - \tau_2\int_{\Gamma_1} k_2'\Box v_t^m d\Gamma_1\Big\} + \frac{1}{2}\int_\Omega |\nabla u_{tt}^m|^2 dx + \frac{1}{2}\int_\Omega |\nabla v_{tt}^m|^2 dx \\ &\le \frac{\tau_1 c}{2}\int_{\Gamma_1} |u_0|^2 d\Gamma_1 + \frac{\tau_2 c}{2}\int_{\Gamma_1} |v_0|^2 d\Gamma_1 + c\int_\Omega (|\nabla u_t^m|^2 + |\nabla v_t^m|^2)dx \\ &+ c\Big\{\int_\Omega |u_{tt}^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx\Big\}. \end{aligned}$$

Integrating with respect to time and applying Gronwall's inequality we conclude that

$$\int_\Omega |u_{tt}^m|^2 dx + \int_\Omega |v_{tt}^m|^2 dx + \int_0^t \int_\Omega |\nabla u_{tt}^m|^2 dx + \int_0^t \int_\Omega |\nabla v_{tt}^m|^2 dx \le c, \tag{2.18}$$

$\forall m \in \mathbb{N}$, $\forall t \in [0, T]$.

**A priori estimate III.** Replacing $w$ by $-\Delta u_t^m$ in (2.6) and by $-\Delta v_t^m$ in (2.7), respectively, and then using the Green's formula and adding the result, we have

$$-\int_\Omega u_{tt}^m \Delta u_t^m dx - \int_\Omega v_{tt}^m \Delta v_t^m dx + M(||\nabla u^m||_2^2 + ||\nabla v^m||_2^2)\Big\{\int_\Omega \Delta u^m \Delta u_t^m dx$$
$$+ \int_\Omega \Delta v^m \Delta v_t^m dx + \int_\Omega |\Delta u_t^m|^2 dx + \int_\Omega |\Delta v_t^m|^2 dx\Big\}$$
$$= -\int_\Omega f(u^m - v^m)\Delta u_t^m dx + \int_\Omega f(u^m - v^m)\Delta v_t^m dx.$$

Using similar arguments as in (2.18) we conclude

$$||\Delta u^m||_2^2 + \int_0^T ||\Delta u_t^m(t)||_2^2 dt + ||\Delta v^m||_2^2 + \int_0^T ||\Delta v_t^m(t)||_2^2 dt \leq c, \quad (2.19)$$

$\forall m \in \mathbb{N}$, $\forall t \in [0,T]$. Now, from the estimates (2.9), (2.18), and (2.19) and of the Lions-Aubin's compactness theorem we can pass to the limit in (2.6)–(2.7). The rest of the proof is a matter of routine. □

## 3. Exponential decay

In this section we shall study the asymptotic behavior of the solutions of system (1.1)–(1.6) when the resolvent kernels $k_1$ and $k_2$ are exponentially decreasing; that is, there exist positive constants $b_1$, $b_2$ such that

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i(t), \quad k_i''(t) \geq -b_2 k_i'(t) \quad \text{for} \quad i = 1,2. \quad (3.1)$$

Note that this conditions implies that

$$k_i(t) \leq k_i(0)e^{-b_1 t} \quad \text{for} \quad i = 1,2.$$

Our point of departure will be to establish some inequalities for the strong solution of coupled system (1.1)–(1.6).

**Lemma 3.1.** *Any strong solution* $(u,v)$ *of the system* (1.1)–(1.6) *satisfies*

$$\frac{d}{dt}E(t) \leq -\frac{\tau_1}{2}\int_{\Gamma_1}(|u_t|^2 + k_1''\Box u - k_1'(t)|u|^2 - k_1^2(t)|u_0|^2)d\Gamma_1$$
$$-\frac{\tau_2}{2}\int_{\Gamma_1}(|v_t|^2 + k_2''\Box v - k_2'(t)|v|^2 - k_2^2(t)|v_0|^2)d\Gamma_1 - \int_\Omega(|\nabla u_t|^2 + |\nabla v_t|^2)dx.$$

**Proof.** Multiplying the equation (1.1) by $u_t$ and integrating by parts over $\Omega$ we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u_t|^2 dx + \frac{1}{2}M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{d}{dt}\int_\Omega |\nabla u|^2 dx + \int_\Omega f(u-v)u_t dx$$
$$+ \int_\Omega |\nabla u_t|^2 dx = \int_{\Gamma_1}\{M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu}\}u_t d\Gamma_1.$$

Similarly, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |v_t|^2dx + \frac{1}{2}M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{d}{dt}\int_\Omega |\nabla v|^2dx - \int_\Omega f(u-v)v_tdx$$
$$+ \int_\Omega |\nabla v_t|^2dx = \int_{\Gamma_1}\{M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu}\}v_td\Gamma_1.$$

Summing the above identities, substituting the boundary terms by (2.1)–(2.2) and using Lemma 2.2, our conclusion follows. $\square$

Let us consider the following binary operator:

$$(k\diamond\varphi)(t) := \int_0^t k(t-s)(\varphi(t)-\varphi(s))ds.$$

Then applying the Schwarz inequality for $0 \le \mu \le 1$ we have

$$|(k\diamond\varphi)(t)|^2 \le \left[\int_0^t |k(s)|^{2(1-\mu)}ds\right](|k|^{2\mu}\square\varphi)(t). \tag{3.2}$$

Let us introduce the following functionals:

$$\mathcal{N}(t) := \int_\Omega (|u_t|^2 + |v_t|^2 + F(u-v))dx + \widehat{M}(||\nabla u||_2^2 + ||\nabla v||_2^2),$$
$$\psi(t) = \int_\Omega \Big\{m\cdot\nabla u + \Big(\frac{n}{2}-\theta\Big)u\Big\}u_tdx + \int_\Omega \Big\{m\cdot\nabla v + \Big(\frac{n}{2}-\theta\Big)v\Big\}v_tdx,$$

where $\theta$ is a small positive constant. The following lemma plays an important role in the construction of the Lyapunov functional.

**Lemma 3.2.** *For any strong solution of the system* (1.1)–(1.6)*, we get*

$$\frac{d}{dt}\psi(t) \le -\frac{\theta}{2}\mathcal{N}(t) + c\int_{\Gamma_1}(|u_t|^2 + |k_1(t)u|^2 + |k_1'\diamond u|^2 + |k_1(t)u_0|^2)d\Gamma_1$$
$$+ c\int_{\Gamma_1}(|v_t|^2 + |k_2(t)v|^2 + |k_2'\diamond v|^2 + |k_2(t)v_0|^2)d\Gamma_1 + c_\epsilon\int_\Omega(|\nabla u_t|^2 + |\nabla v_t|^2)dx,$$

*for some positive constants $c$ and $\epsilon$.*

**Proof.** From equation (1.1) we obtain

$$\frac{d}{dt}\int_\Omega u_t\Big\{m\cdot\nabla u + \Big(\frac{n}{2}-\theta\Big)u\Big\}dx = \int_\Omega u_tm\cdot\nabla u_tdx + \Big(\frac{n}{2}-\theta\Big)\int_\Omega |u_t|^2dx$$
$$+ \int_\Omega m\cdot\nabla u(M(||\nabla u||_2^2 + ||\nabla v||_2^2)\Delta u + \Delta u_t - f(u-v))dx$$
$$+ (\frac{n}{2}-\theta)\int_\Omega u(M(||\nabla u||_2^2 + ||\nabla v||_2^2)\Delta u + \Delta u_t - f(u-v))dx.$$

Performing an integration by parts and using Young's inequality, we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_t\Big\{m\cdot\nabla u+\Big(\frac{n}{2}-\theta\Big)u\Big\}dx \le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|u_t|^2 d\Gamma_1-\theta\int_\Omega |u_t|^2dx\\
&+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial \nu})\Big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\Big\}\,d\Gamma_1\\
&-(1-\theta)M(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega|\nabla u|^2dx\\
&+\epsilon cM(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega|\nabla u|^2dx+c_\epsilon\int_\Omega|\nabla u_t|^2dx\\
&-(\frac{n}{2}-\theta)\int_\Omega f(u-v)udx-\int_\Omega(m\cdot\nabla u)f(u-v)dx\\
&-\frac{1}{2}M(||\nabla u||_2^2+||\nabla v||_2^2)\int_{\Gamma_1}m\cdot\nu|\nabla u|^2d\Gamma_1,
\end{aligned}$$

where $\epsilon$ is a positive constant. Similarly, using equation (1.2) instead of (1.1) we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega v_t\Big\{m\cdot\nabla v+\Big(\frac{n}{2}-\theta\Big)v\Big\}dx \le \frac{1}{2}\int_{\Gamma_1} m\cdot\nu|v_t|^2 d\Gamma_1-\theta\int_\Omega |v_t|^2dx\\
&+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial \nu})\Big\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\Big\}\,d\Gamma_1\\
&-(1-\theta)M(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega|\nabla v|^2dx\\
&+\epsilon cM(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega|\nabla v|^2dx+c_\epsilon\int_\Omega|\nabla v_t|^2dx\\
&+(\frac{n}{2}-\theta)\int_\Omega f(u-v)vdx+\int_\Omega(m\cdot\nabla v)f(u-v)dx\\
&-\frac{1}{2}M(||\nabla u||_2^2+||\nabla v||_2^2)\int_{\Gamma_1}m\cdot\nu|\nabla v|^2d\Gamma_1.
\end{aligned}$$

Summing these two last inequalities, we have

$$\begin{aligned}
\frac{d}{dt}\psi(t)\le{}&\frac{1}{2}\int_{\Gamma_1}m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1-\theta\int_\Omega(|u_t|^2+|v_t|^2)dx\\
&+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial \nu})\Big\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\Big\}\,d\Gamma_1\\
&-(1-\theta)M(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx
\end{aligned}$$

$$
\begin{aligned}
&+\epsilon c M(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx\\
&+c_\epsilon\int_\Omega(|\nabla u_t|^2|\nabla v_t|^2)dx-(\frac{n}{2}-\theta)\int_\Omega f(u-v)(u-v)dx\\
&+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\left\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\right\}d\Gamma_1\\
&-\int_\Omega(m\cdot\nabla(u-v))f(u-v)dx\\
&-\frac{1}{2}M(||\nabla u||_2^2+||\nabla v||_2^2)\int_{\Gamma_1}m\cdot\nu(|\nabla u|^2+|\nabla v|^2)d\Gamma_1. \qquad (3.3)
\end{aligned}
$$

Using the Green's formula, we obtain

$$
-\int_\Omega(m\cdot\nabla(u-v))f(u-v)dx=-\int_\Gamma m\cdot\nu F(u-v)d\Gamma+n\int_\Omega F(u-v)dx.
$$

Noting that $u=v=0$ on $\Gamma_0$, $m\cdot\nu>0$ on $\Gamma_1$ and $F$ is a nonnegative function, we have

$$
-\int_\Omega(m\cdot\nabla(u-v))f(u-v)dx\le n\int_\Omega F(u-v)dx.
$$

Substituting the above inequality into (3.3) and taking into account that $f$ is superlinear, we arrive at

$$
\begin{aligned}
&\frac{d}{dt}\psi(t)\le\frac{1}{2}\int_{\Gamma_1}m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1-\theta\int_\Omega(|u_t|^2+|v_t|^2)dx\\
&+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\left\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\right\}d\Gamma_1\\
&+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\left\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\right\}d\Gamma_1\\
&-(1-\theta)M(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega(|\nabla u|^2+|\nabla v|^2)dx\\
&+\epsilon c M(||\nabla u||_2^2+||\nabla v||_2^2)\int_\Omega(|\nabla u|^2dx+|\nabla v|^2)dx\\
&+c_\epsilon\int_\Omega(|\nabla u_t|^2+|\nabla v_t|^2)dx-(\frac{n\delta}{2}-\theta(2+\delta))\int_\Omega F(u-v)dx\\
&-\frac{1}{2}M(||\nabla u||_2^2+||\nabla v||_2^2)\int_{\Gamma_1}m\cdot\nu|\nabla u|^2d\Gamma_1\\
&-\frac{1}{2}M(||\nabla u||_2^2+||\nabla v||_2^2)\int_{\Gamma_1}m\cdot\nu|\nabla v|^2d\Gamma_1.
\end{aligned}
$$

Taking $\theta$ and $\epsilon$ small enough, we obtain

$$\frac{d}{dt}\psi(t) \leq -\theta\mathcal{N}(t) + \frac{1}{2}\int_{\Gamma_1} m\cdot\nu(|u_t|^2+|v_t|^2)d\Gamma_1 + c_\epsilon\int_\Omega(|\nabla u_t|^2+|\nabla v_t|^2)dx$$
$$-\frac{1}{2}M(||\nabla u||_2^2+||\nabla v||_2^2)\int_{\Gamma_1} m\cdot\nu(|\nabla u|^2+|\nabla v|^2)d\Gamma_1$$
$$+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\left\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\right\}d\Gamma_1$$
$$+\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\left\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\right\}d\Gamma_1. \tag{3.4}$$

Now, we analyze some boundary term of the above inequality. Applying Young's and Poincaré's inequalities we have, for $\epsilon_1 > 0$,

$$\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})\left\{m\cdot\nabla u+(\frac{n}{2}-\theta)u\right\}d\Gamma_1$$
$$\leq \epsilon_1\int_{\Gamma_1}\left\{|m\cdot\nabla u|^2+\left(\frac{n}{2}-\theta\right)^2|u|^2\right\}d\Gamma_1$$
$$+c_{\epsilon_1}\int_{\Gamma_1}|(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})|^2d\Gamma_1$$
$$\leq \epsilon_1 c\Big\{\int_{\Gamma_1} m\cdot\nu|\nabla u|^2d\Gamma_1+\mathcal{N}(t)\Big\}$$
$$+c_{\epsilon_1}\int_{\Gamma_1}|(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial u}{\partial\nu}+\frac{\partial u_t}{\partial\nu})|^2d\Gamma_1.$$

Similarly, we obtain

$$\int_{\Gamma_1}(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})\left\{m\cdot\nabla v+(\frac{n}{2}-\theta)v\right\}d\Gamma_1$$
$$\leq \epsilon_1\int_{\Gamma_1}\left\{|m\cdot\nabla v|^2+\left(\frac{n}{2}-\theta\right)^2|v|^2\right\}d\Gamma_1$$
$$+c_{\epsilon_1}\int_{\Gamma_1}|(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})|^2d\Gamma_1$$
$$\leq \epsilon_1 c\Big\{\int_{\Gamma_1} m\cdot\nu|\nabla v|^2d\Gamma_1+\mathcal{N}(t)\Big\}$$
$$+c_{\epsilon_1}\int_{\Gamma_1}|(M(||\nabla u||_2^2+||\nabla v||_2^2)\frac{\partial v}{\partial\nu}+\frac{\partial v_t}{\partial\nu})|^2d\Gamma_1.$$

Substituting the two above inequalities into (3.4), choosing $\epsilon_1$ small enough and taking into account that the boundary conditions (2.1)–(2.2) can be written as

$$M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{\partial u}{\partial \nu} + \frac{\partial u_t}{\partial \nu} = -\tau_1\{u_t + k_1(t)u - k_1' \diamond u - k_1(t)u_0\},$$
$$M(||\nabla u||_2^2 + ||\nabla v||_2^2)\frac{\partial v}{\partial \nu} + \frac{\partial v_t}{\partial \nu} = -\tau_2\{v_t + k_2(t)v - k_2' \diamond v - k_2(t)v_0\},$$

our conclusion follows. □

To show that the energy decays exponentially we shall need the following lemma.

**Lemma 3.3.** *Let $f$ be a real positive function of class $C^1$. If there exist positive constants $\gamma_0$, $\gamma_1$, and $c_0$ such that*

$$f'(t) \le -\gamma_0 f(t) + c_0 e^{-\gamma_1 t},$$

*then there exist positive constants $\gamma$ and $c$ such that*

$$f(t) \le (f(0) + c)e^{-\gamma t}.$$

**Proof.** See e.g. [22].

Finally, we shall show the main result of this section.

**Theorem 3.1.** *Let us take $(u_0, v_0) \in V^2$ and $(u_1, v_1) \in [L^2(\Omega)]^2$. If the resolvent kernels $k_1$ and $k_2$ satisfy* (3.1), *then there exist positive constants $\alpha_1$ and $\gamma_1$ such that $E(t) \le \alpha_1 e^{-\gamma_1 t} E(0)$, for all $t \ge 0$.*

**Proof.** We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in \big(H^2(\Omega) \cap V\big)^2$ and $(u_1, v_1) \in \big(H^2(\Omega) \cap V\big)^2$ satisfying the compatibility conditions (2.4)–(2.5). Our conclusion is followed by the standard density argument. Using hypothesis (3.1) in Lemma 3.1, we get

$$\frac{d}{dt}E(t) \le -\frac{\tau_1}{2}\int_{\Gamma_1}\Big(|u_t|^2 - b_2 k_1' \Box u + b_1 k_1(t)|u|^2 - |k_1(t)u_0|^2\Big)d\Gamma$$
$$-\frac{\tau_1}{2}\int_{\Gamma_1}\Big(|v_t|^2 - b_2 k_2' \Box v + b_1 k_2(t)|v|^2 - |k_2(t)v_0|^2\Big)d\Gamma - \int_\Omega (|\nabla u_t|^2 + |\nabla v_t|^2)dx.$$

On the other hand, applying inequality (3.2) with $\mu = 1/2$ in Lemma 3.2, we obtain

$$\frac{d}{dt}\psi(t) \le -\frac{\theta}{2}\mathcal{N}(t) + C\int_{\Gamma_1}\Big(|u_t|^2 + k_1(t)|u|^2 - k_1'\Box u + |k_1(t)u_0|^2\Big)d\Gamma$$
$$+ C\int_{\Gamma_1}\Big(|v_t|^2 + k_2(t)|v|^2 - k_2'\Box v + |k_2(t)v_0|^2\Big)d\Gamma + c_\epsilon\int_\Omega (|\nabla u_t|^2 + |\nabla v_t|^2)dx.$$

Let us introduce the Lyapunov functional

$$\mathcal{L}(t) := NE(t) + \psi(t), \tag{3.5}$$

with $N > 0$. Taking $N$ large enough, we see from the previous inequalities that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}E(t) + 2NR^2(t)E(0),$$

where $R(t) = k_1(t) + k_2(t)$. Moreover, using Young's inequality and taking $N$ large we find that

$$\frac{N}{2}E(t) \leq \mathcal{L}(t) \leq 2NE(t). \tag{3.6}$$

From this inequality we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}\mathcal{L}(t) + 2NR^2(t)E(0),$$

from which follows, in view of Lemma 3.3 and of the exponential decay of $k_1$, $k_2$, that

$$\mathcal{L}(t) \leq \{\mathcal{L}(0) + c\}e^{-\gamma_1 t},$$

for some positive constants $c, \gamma$. From the inequality (3.6) our conclusion follows. $\square$

## 4. Polynomial rate of decay

Here our attention will be focused on the uniform rate of decay when the resolvent kernels $k_1$ and $k_2$ decay polynomially like $(1+t)^{-p}$. In this case we will show that the solution also decays polynomially with the same rate. Therefore, we will assume that the resolvent kernels $k_1$ and $k_2$ satisfy

$$k_i(0) > 0, \quad k_i'(t) \leq -b_1 k_i(t)^{1+\frac{1}{p}}, \quad k_i''(t) \geq b_2[-k_i'(t)]^{1+\frac{1}{p+1}} \quad \text{for} \quad i = 1, 2, \tag{4.1}$$

for some $p > 1$ and some positive constants $b_1$ and $b_2$. The following lemmas will play an important role in the sequel.

**Lemma 4.1.** *Let $(u, v)$ be a solution of system (1.1)–(1.6), and let us denote by $(\phi_1, \phi_2) = (u, v)$. Then, for $p > 1$, $0 < r < 1$ and $t \geq 0$, we have*

$$\Big(\int_{\Gamma_1} |k_i'|\Box\phi_i d\Gamma_1\Big)^{\frac{1+(1-r)(p+1)}{(1-r)(p+1)}}$$
$$\leq 2\Big(\int_0^t |k_i'(s)|^r ds||\phi_i||^2_{L^\infty(0,t;L^2(\Gamma_1))}\Big)^{\frac{1}{(1-r)(p+1)}} \int_{\Gamma_1} |k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1$$

*while for $r = 0$, we get*

$$\Big(\int_{\Gamma 1} |k_i'|\Box\phi_i d\Gamma_1{}^{\frac{p+2}{p+1}}\Big)$$
$$\leq 2\Big(\int_0^t ||\phi_i(s,\cdot)||^2_{L^2(\Gamma_1)}ds + t||\phi_i(s,\cdot)||^2_{L^2(\Gamma_1)}\Big)^{p+1}\int_{\Gamma_1}|k_i'|^{1+\frac{1}{p+1}}\Box\phi_i d\Gamma_1,$$

*for $i = 1, 2$.*

**Proof.** See e.g. [21].

**Lemma 4.2.** *Let $f \geq 0$ be a differentiable function satisfying*

$$f'(t) \leq -\frac{c_1}{f(0)^{\frac{1}{\alpha}}}f(t)^{1+\frac{1}{\alpha}} + \frac{c_2}{(1+t)^{\beta}}f(0) \quad \textit{for} \quad t \geq 0,$$

*for some positive constants $c_1$, $c_2$, $\alpha$ and $\beta$ such that $\beta \geq \alpha + 1$. Then there exists a constant $c > 0$ such that*

$$f(t) \leq \frac{c}{(1+t)^{\alpha}}f(0) \quad \textit{for} \quad t \geq 0.$$

**Proof.** See e.g. [22].

**Theorem 4.1.** *Let us take $(u_0, v_0) \in V^2$ and $(u_1, v_1) \in [L^2(\Omega)]^2$. If the resolvent kernels $k_1$ and $k_2$ satisfy the conditions* (4.1), *then there exists a positive constant $c$ such that*

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0).$$

**Proof.** We shall prove this result for strong solutions, that is, for solutions with initial data $(u_0, v_0) \in \big(H^2(\Omega) \cap V\big)^2$ and $(u_1, v_1) \in \big(H^2(\Omega) \cap V\big)^2$ satisfying the compatibility conditions (2.4)–(2.5). We employ the standard density argument. We use some estimates of the previous section which are independent of the behavior of the resolvent kernels $k_1$ and $k_2$. We use the condition (4.1) to obtain

$$\frac{d}{dt}E(t) \leq -\frac{\tau_1}{2}\int_{\Gamma_1}\Big(|u_t|^2 + b_2[-k_1']^{1+\frac{1}{p+1}}\Box u + b_1 k_1^{1+\frac{1}{p}}(t)|u|^2 - |k_1(t)u_0|^2\Big)d\Gamma_1$$
$$-\frac{\tau_1}{2}\int_{\Gamma_1}\Big(|v_t|^2 + b_2[-k_2']^{1+\frac{1}{p+1}}\Box v + b_1 k_2^{1+\frac{1}{p}}(t)|v|^2 - |k_2(t)v_0|^2\Big)d\Gamma_1.$$

Applying inequality (3.2) with $\mu = \frac{p+2}{2(p+1)}$ and using hypothesis (4.1) we obtain the following estimates:

$$|k_1' \diamond u|^2 \leq c[-k_1']^{1+\frac{1}{p+1}}\Box u, \quad |k_2' \diamond v|^2 \leq c[-k_2']^{1+\frac{1}{p+1}}\Box v.$$

Using the above inequalities in Lemma 3.2, we see that

$$\frac{d}{dt}\psi(t) \leq -\frac{\theta}{2}\mathcal{N}(t) + c\int_{\Gamma_1}\Big(|u_t|^2 + k_1^{1+\frac{1}{p}}(t)|u|^2 + [-k_1']^{1+\frac{1}{p+1}}\Box u + |k_1(t)u_0|^2\Big)d\Gamma_1$$
$$+ c\int_{\Gamma_1}\Big(|v_t|^2 + k_2^{1+\frac{1}{p}}(t)|v|^2 + [-k_2']^{1+\frac{1}{p+1}}\Box v + |k_2(t)v_0|^2\Big)d\Gamma_1.$$

Hence, taking $N$ large enough, we see that the Lyapunov functional defined in (3.5) satisfies

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{\theta}{2}\mathcal{N}(t) + 2NR^2(t)E(0)$$
$$- \frac{Nc_2}{2}\Big\{\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma_1 + \int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v d\Gamma_1\Big\}. \tag{4.2}$$

Let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. From (4.1) it follows that

$$\int_0^\infty |k_i'|^r \leq c\int_0^\infty \frac{1}{(1+t)^{r(p+1)}} < \infty \quad \text{for} \quad i = 1, 2.$$

Using this estimate in Lemma 4.1, we get

$$\int_{\Gamma_1}[-k_1']^{1+\frac{1}{p+1}}\Box u d\Gamma_1 \geq cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big(\int_{\Gamma_1}[-k_1']\Box u d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}, \tag{4.3}$$

$$\int_{\Gamma_1}[-k_2']^{1+\frac{1}{p+1}}\Box v\ d\Gamma_1 \geq cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big(\int_{\Gamma_1}[-k_2']\Box v d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}. \tag{4.4}$$

On the other hand, from the trace theorem it follows that

$$E(t)^{1+\frac{1}{(1-r)(p+1)}} \leq cE(0)^{\frac{1}{(1-r)(p+1)}}\mathcal{N}(t). \tag{4.5}$$

Substituting the estimates (4.3), (4.4), and (4.5) into the inequality (4.2), we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -cE(0)^{-\frac{1}{(1-r)(p+1)}}E(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0)$$
$$- cE(0)^{-\frac{1}{(1-r)(p+1)}}\Big\{\Big(\int_{\Gamma_1}[-k_1']\Box u d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}$$
$$+ \Big(\int_{\Gamma_1}[-k_2']\Box v d\Gamma_1\Big)^{1+\frac{1}{(1-r)(p+1)}}\Big\}.$$

Taking into account the inequality (3.6) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{(1-r)(p+1)}}}\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + 2NR^2(t)E(0),$$

for some $c > 0$. This and Lemma 4.2 imply that

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{(1-r)(p+1)}}\mathcal{L}(0).$$

Since $(1-r)(p+1) > 1$, we get for $t \geq 0$ the following bounds:

$$t\|u\|^2_{L^2(\Gamma_1)} + t\|v\|^2_{L^2(\Gamma_1)} \leq t\mathcal{L}(t) \quad < \quad \infty,$$
$$\int_0^t \left(\|u\|^2_{L^2(\Gamma_1)} + \|v\|^2_{L^2(\Gamma_1)}\right) ds \leq c\int_0^t \mathcal{L}(t)\, ds \quad < \quad \infty.$$

Using the above estimates in Lemma 4.1 with $r = 0$ we get

$$\int_{\Gamma_1} [-k_1']^{1+\frac{1}{p+1}} \Box u d\Gamma_1 \geq \frac{c}{E(0)^{\frac{1}{p+1}}}\left(\int_{\Gamma_1} [-k_1']\Box u d\Gamma\right)^{1+\frac{1}{p+1}},$$
$$\int_{\Gamma_1} [-k_2']^{1+\frac{1}{p+1}} \Box v d\Gamma_1 \geq \frac{c}{E(0)^{\frac{1}{p+1}}}\left(\int_{\Gamma_1} [-k_2']\Box v d\Gamma\right)^{1+\frac{1}{p+1}}.$$

Using these inequalities instead of (4.3)–(4.4) and reasoning in the same way as above we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\frac{c}{\mathcal{L}(0)^{\frac{1}{p+1}}}\mathcal{L}(t)^{1+\frac{1}{p+1}} + 2NR^2(t)E(0).$$

Applying Lemma 4.2 again, we obtain

$$\mathcal{L}(t) \leq \frac{c}{(1+t)^{p+1}}\mathcal{L}(0).$$

Finally, from (3.6) we conclude

$$E(t) \leq \frac{c}{(1+t)^{p+1}}E(0),$$

which completes the present proof. □

## 5. Final comments

The methods explored in this paper can be used to solve problems with the partial differential equations (5.1) and (5.2) below:

$$\begin{cases} u_{tt}+\Delta^2 u - M\big(\|\nabla u\|_2^2+\|\nabla v\|_2^2\big)\Delta u - \Delta u_t + f(u-v)=0 \\ \text{in } \Omega\times(0,\infty) \\ v_{tt}+\Delta^2 v - M\big(\|\nabla u\|_2^2+\|\nabla v\|_2^2\big)\Delta v - \Delta v_t - f(u-v)=0 \\ \text{in } \Omega\times(0,\infty) \\ u=v=\frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0, \\ \text{on } \Gamma_0\times(0,\infty) \\ \Delta u+\int_0^t g_1(t-s)(M\big(\|\nabla u(s)\|_2^2+\|\nabla v(s)\|_2^2\big)\frac{\partial u}{\partial\nu}(s)+\frac{\partial u_t}{\partial\nu}(s))ds=0 \\ \text{on } \Gamma_1\times(0,\infty) \\ \Delta v+\int_0^t g_2(t-s)(M\big(\|\nabla u(s)\|_2^2+\|\nabla v(s)\|_2^2\big)\frac{\partial v}{\partial\nu}(s)+\frac{\partial v_t}{\partial\nu}(s))ds=0 \\ \text{on } \Gamma_1\times(0,\infty) \\ (u(0,x),v(0,x))=(u_0(x),v_0(x)) \text{ in } \Omega \\ (u_t(0,x),v_t(0,x))=(u_1(x),v_1(x)) \text{ in } \Omega \end{cases} \tag{5.1}$$

$$\begin{cases} K_1(x,t)u_{tt} - M\big(\|\nabla u\|_2^2+\|\nabla v\|_2^2\big)\Delta u + K_3(x,t)u_t - \Delta u_t + f(u-v)=0, \\ \text{in } \Omega\times(0,\infty), \\ K_2(x,t)v_{tt} - M\big(\|\nabla u\|_2^2+\|\nabla v\|_2^2\big)\Delta v + K_4(x,t)v_t - \Delta v_t - f(u-v)=0, \\ \text{in } \Omega\times(0,\infty), \\ u=v=0 \text{ on } \Gamma_0\times(0,\infty), \\ u+\int_0^t g_1(t-s)(M\big(\|\nabla u(s)\|_2^2+\|\nabla v(s)\|_2^2\big)\frac{\partial u}{\partial\nu}(s)+\frac{\partial u_t}{\partial\nu}(s))ds=0, \\ \text{on } \Gamma_1\times(0,\infty), \\ v+\int_0^t g_2(t-s)(M\big(\|\nabla u(s)\|_2^2+\|\nabla v(s)\|_2^2\big)\frac{\partial v}{\partial\nu}(s)+\frac{\partial v_t}{\partial\nu}(s))ds=0, \\ \text{on } \Gamma_1\times(0,\infty), \\ (u(0,x),v(0,x))=(u_0(x),v_0(x)) \text{ in } \Omega, \\ (u_t(0,x),v_t(0,x))=(u_1(x),v_1(x)) \text{ in } \Omega \end{cases} \tag{5.2}$$

where $K_i(x,t)\geq 0$ and $K_j(x,0)\geq d>0$ almost everywhere in $\Omega\times(0,\infty)$ for all $i=1,2$ and $j=3,4$ respectively. With regard to the problems (5.1) and (5.2), it is important to observe, however, that as far as we are concerned, nonlinear memory terms acting in the boundary have never been considered in the literature. With regard to the problem (5.2) it is a system of equations degenerate nonlinear and nonlinear boundary feedback combined with a nonlinear memory source term which requires new arguments to overcome the difficulties. The authors of the paper already obtained the results of global existence, uniqueness and the declines exponential and polynomial respectively for the problems (5.1) and (5.2), and they meet in the final phase of two papers, which will shortly be submitted for publication. Now we would like to mention that the problem (1.1)–(1.6) as well as the corresponding problem for the systems (5.1) and (5.2) in a bounded domain with moving boundary are interesting open problems.

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