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ON THE SOLUTIONS OF THE RENORMALIZED EQUATIONS AT ALL ORDERS

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Abstract. The renormalization (or averaging) procedure is often used to construct approximate solutions in evolutionary problems with multiple timescales arising from a small parameter ε. We show in this paper that the leading-order approximation shares two important properties of the original system, namely energy conservation in the inviscid case and dissipation rate (coercivity) in the forced–dissipative case. This implies the boundedness of the solutions of the renormalized (approximate) equation. In the dissipative case, we also investigate the higher-order renormalized equations, pursuing [16]: in particular, we show for sufficiently small ε that the solutions of these equations are bounded and that the dissipativity property of the original system carries over in a modified form. This is shown by a simple estimate based on the above leading-order result, and, alternatively, by a "shadowing" argument.

1. Introduction

The renormalization method, also known as the method of averaging, is used in many applications to treat problems with multiple timescales. These timescales typically arise from a small parameter ε in the evolution equation. Although the method applies to a more general class of problems (see, e.g., [4], [1], [11]), in this paper we shall restrict ourselves to the case where one has rapid oscillations—this corresponds to the case where the leading-order term in the equation of motion is linear and antisymmetric.

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In fluid dynamical applications, this case arises, for example, in the small Rossby number limit of geophysical flows ([12], [17], [2], [8], [14]) or in the small Mach number limit ([5], [13], [6], [11]).

The central idea is to express the rapidly oscillating solution by an "algebraic" (a functional) relation involving the time explicitly and some slowly-varying variables, and to account for the latter by a system of differential equations. The latter, known simply as the averaged or renormalized equation, evolves over a slower timescale than the original equation, making it easier to identify important long-range interactions or to integrate numerically.

Although originally intended to provide a quantitatively accurate approximation to the true solution for a finite time (see, e.g., [16]), this method has also been used to construct approximate models which are then used in their own right to study the qualitative behavior of the system over timescales well beyond the formal validity of the original approximation. In geophysical applications, the former use would correspond to short-range (weather) forecast, while the latter use would correspond to long-range (climate) dynamics.

For the latter purpose, it is important that the approximate models share as much of the properties of the original model as possible. For example, when the original model conserves energy in its inviscid form, one would like the approximate model to do the same. When the approximate model is derived in an *ad hoc* fashion, this conservation property may have to be added by hand, for instance, by slightly modifying the equations of motion as when one derives the meteorological primitive equation from the compressible Euler equations (cf. e.g., [9], pp. 17–18).

One of the purposes of this paper is to show that this is not necessary when one uses the averaging/renormalization method: The leading-order approximate model obtained using the renormalization procedure has the same energy-conservation and dissipativity properties as the original model. This no longer holds for higher-order approximations (although the nonconservation would arise from higher-order terms, which is thus small). Another purpose of this article is to show that, in spite of this, one can nevertheless show that the higher-order solutions remain bounded for all time and, like the solution of the original equation, eventually fall into an absorbing set for sufficiently small values of the parameter ε. We derive these results with two different methods. One, based on the algebraic property of the first-order renormalized system and presented in Section 3, is close to PDEs and very simple. The other method, which is a little more involved but less dependent on the properties of the original system, is based on the so-called "shadowing" argument recently introduced in the study of dynamical systems. This shadowing argument is not commonly used in PDE theory and we believe that it can be useful for other purposes (or applications) so we decided to include it here.

We recall that the existence of an absorbing set is usually considered, for dynamical systems, as the "mathematical expression" of dissipation [7], [3]. Hence, although dissipation does not carry over in the original sense [see (1.2), (1.8) for the original equation (1.7)] beyond the leading-order renormalized equation [cf. Theorem 2], it is present in a modified sense in the renormalized equations at all orders.

This paper is organized as follows: In the rest of this section, we set up the problem and introduce some notation. The averaging procedure for leading order and higher orders is described in Section 2, recalling in part results from [16]. As mentioned above, results on the properties of leading-order and higher-order renormalized systems are then derived: in Section 3, based on a direct analysis of the algebraic properties of the leading-order equation, and in Section 4 using a "shadowing" technique. These results are illustrated with a simple example in Section 5.

We consider the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{\varepsilon}Lu + Au + B(u) = f, \qquad u(0) = u_0, \tag{1.1}$$

where $u \in \mathbb{R}^d$, L is a real antisymmetric matrix (therefore its eigenvectors form a basis), A is positive-definite,

$$(Au, u) \geq \lambda |u|^2, \qquad \forall u \in \mathbb{R}^d, \tag{1.2}$$

with $\lambda > 0$, and $f = f(t)$ is given with $|f|_\infty := \text{ess sup}_{0\leq t\leq\infty} |f(t)|$ finite (although we will also consider the inviscid case $A = 0$, $f = 0$). Here

$$B(u) = \sum_{j=1}^{r} B^{(j)}(u, \ldots, u),$$

where $B^{(j)}$ has j arguments, all u in this case; this notation will be particularly convenient in Section 3 below. Each $B^{(j)}(u, \ldots, u)$ is j-linear and satisfies the orthogonality property,

$$(B^{(j)}(u, \ldots, u), u) = 0, \qquad \text{for all } u. \tag{1.3}$$

In what follows we shall often refer to $B(u)$ as the "nonlinear" part, although strictly speaking it may contain a linear term $B^{(1)}(u)$. The scalar product and norm in $\mathbb{R}^d$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$. We also introduce the Hermitian scalar product on $\mathbb{C}^d$, denoted by $(\cdot,\cdot)_{\mathbb{C}}$,

$$(\alpha + \mathrm{i}\beta, \gamma + \mathrm{i}\delta)_{\mathbb{C}} := (\alpha,\gamma) + (\beta,\delta) + \mathrm{i}\big[((\beta,\gamma) - \alpha,\delta)\big].$$

When confusion may arise, we will denote the real scalar product by $(\cdot,\cdot)_{\mathbb{R}}$ and the real norm by $|\cdot|_{\mathbb{R}}$.

In the inviscid case $A = 0$, $f = 0$, the antisymmetry of L and (1.3) can be seen to imply the energy conservation property,

$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 = 0 \qquad \Rightarrow \qquad |u(t)|^2 = |u(0)|^2, \tag{1.4}$$

for all t. In the general case when $A \neq 0$, $f \neq 0$, in place of (1.4), the solution can be bounded for all t as follows: We use the following energy estimate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 = \Big(\frac{\mathrm{d}u}{\mathrm{d}t}, u\Big) = -(Au,u) + (f,u) \le -\lambda|u|^2 + \frac{\lambda}{2}|u|^2 + \frac{1}{2\lambda}|f|_\infty^2,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}|u|^2 + \lambda|u|^2 \le \frac{1}{\lambda}|f|_\infty^2.$$

The last inequality gives

$$|u(t)|^2 \le \mathrm{e}^{-\lambda t}|u_0|^2 + \frac{|f|_\infty^2}{\lambda^2}\big(1 - \mathrm{e}^{-\lambda t}\big), \tag{1.5}$$

thus showing that $u(t)$ exists and is bounded for all $t \ge 0$.

We now introduce the fast time $s = t/\varepsilon$ and define $v(s) := \mathrm{e}^{sL}u(\varepsilon s)$. Denoting

$$F(v,s) := -\mathrm{e}^{sL}A\mathrm{e}^{-sL}v - \mathrm{e}^{sL}B(\mathrm{e}^{-sL}v) + \mathrm{e}^{sL}f, \tag{1.6}$$

the equation of motion (1.1) can be written as

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \varepsilon F(v,s), \qquad\qquad v(0) = u_0. \tag{1.7}$$

Since L has purely imaginary eigenvalues, $\mathrm{e}^{\pm sL}$ has norm one and the bound (1.5) implies also

$$|v(s)|^2 \le \mathrm{e}^{-\lambda\varepsilon s}|u_0|^2 + \frac{|f|_\infty^2}{\lambda^2}\big(1 - \mathrm{e}^{-\lambda\varepsilon s}\big) \qquad\qquad \text{for } s \ge 0. \tag{1.8}$$

In other words, this shows that, for any $\rho > |f|_\infty/\lambda$, the ball $\{v : |v| \le \rho\}$ is absorbing in the entire phase space. In particular, every trajectory $v(s)$ will enter it after some time and will remain in it thereafter, with the time of entry into the absorbing set depending boundedly on $|u_0|$.

2. The renormalization procedure

In going from (1.1) to (1.7), we have managed to eliminate the leading-order rapidly oscillating behavior of the solution $u = u(t)$, and the right-hand side of (1.7) is now of order ε. However, $F(v, s)$ explicitly contains rapidly oscillating terms, which can be difficult to handle for numerical schemes, etc. To obtain (approximate) solutions which are better behaved, we adopt an averaging ansatz in which the true solution $v(s)$ is approximated as

$$v(s) \simeq R(\bar{v}(s), s; \varepsilon) = \bar{v}(s) + \varepsilon R_1(\bar{v}, s) + \varepsilon^2 R_2(\bar{v}, s) + \cdots , \tag{2.1}$$

where the renormalized variable $\bar{v}(s)$ evolves according to the so-called renormalized equation

$$\frac{\mathrm{d}\bar{v}}{\mathrm{d}s} = W(\bar{v}; \varepsilon) = \varepsilon W_1(\bar{v}) + \varepsilon^2 W_2(\bar{v}) + \cdots . \tag{2.2}$$

Here $R(\bar{v}, s; \varepsilon)$ and $W(\bar{v}; \varepsilon)$ are formal series, which are not convergent in general. However, the partial sums still give us useful information: in particular, it is shown in [16] that truncating the series at order N gives an $\mathcal{O}(\varepsilon^N)$ approximation to the true solution; more details are given below.

Substituting (2.1) and (2.2) into (1.7), we obtain the (formal) relation

$$\frac{\partial R}{\partial \bar{v}}(\bar{v}, s; \varepsilon) \cdot W(\bar{v}; \varepsilon) + \frac{\partial R}{\partial s}(\bar{v}, s; \varepsilon) = \varepsilon F(R(\bar{v}, s; \varepsilon), s), \tag{2.3}$$

which is to be solved for R and W following the procedure we now describe (see [16], [14] for more details). For this purpose, we introduce the following notation. First, for a function $G(v, s)$ of the form

$$G(v, s) = \sum_j \mathrm{e}^{\mathrm{i}\nu_j s} g_j(v),$$

where $g_j(v)$ a polynomial in v, we define

$$\begin{aligned} \{G(v)\}_{\mathrm{r}} &:= \sum_{\nu_j = 0} g_j(v), \\ \{G(v, s)\}_{\mathrm{n}} &:= \sum_{\nu_j \neq 0} \mathrm{e}^{\mathrm{i}\nu_j s} g_j(v), \\ \{G(v, s)\}_{\mathrm{np}} &:= \sum_{\nu_j \neq 0} \frac{1}{\mathrm{i}\nu_j} \mathrm{e}^{\mathrm{i}\nu_j s} g_j(v). \end{aligned} \tag{2.4}$$

Here $\{\cdot\}_{\rm r}$, $\{\cdot\}_{\rm n}$ and $\{\cdot\}_{\rm p}$ are the resonant, nonresonant and primitive operators, respectively. By definition, we have

$$F(v,s) = \{F(v)\}_{\rm r} + \{F(v,s)\}_{\rm n} = \{F(v)\}_{\rm r} + \frac{\partial}{\partial s}\{F(v,s)\}_{\rm np}. \tag{2.5}$$

Substituting the formal expansions (2.1) and (2.2) into (2.3), and Taylor-expanding $F(R,s)$ about $R(\bar v,s;\varepsilon)=\bar v$ as

$$F(R(\bar v,s;\varepsilon),s) \simeq F(\bar v,s) + \varepsilon F'(\bar v,s)\cdot R_1(\bar v,s) + \cdots,$$

we find at leading order,

$$W_1(\bar v) + \frac{\partial R_1}{\partial s}(\bar v,s) = F(\bar v,s). \tag{2.6}$$

Taking the resonant part of this equation gives us (we note that $\partial R_1/\partial s$ is nonresonant)

$$W_1(\bar v) = \{F(\bar v)\}_{\rm r}; \tag{2.7}$$

integrating the remaining (nonresonant) part, we find

$$R_1(\bar v,s) = \{F(\bar v,s)\}_{\rm np}. \tag{2.8}$$

At the next order, we find

$$W_2(\bar v) + \frac{\partial R_2}{\partial s}(\bar v,s) = -\frac{\partial R_1}{\partial \bar v}(\bar v,s)\cdot W_1(\bar v) + F'(\bar v,s)\cdot R_1(\bar v,s), \tag{2.9}$$

where $F'(\bar v,s) := \partial F(\bar v,s)/\partial \bar v$. Substituting W_1 and R_1 we found earlier, and taking the resonant part and taking the primitive of the nonresonant part, we find

$$\begin{aligned} W_2(\bar v) &= \{F'(\bar v)\cdot\{F(\bar v)\}_{\rm np}\}_{\rm r} \\ R_2(\bar v,s) &= \{F'(\bar v,s)\cdot\{F(\bar v,s)\}_{\rm np}\}_{\rm np} - \{F'(\bar v,s)\}_{\rm npp}\cdot\{F(\bar v)\}_{\rm r}. \end{aligned} \tag{2.10}$$

This (formal) procedure can be carried out to higher orders, and in Section 4 we will be working with the Nth-order approximation v^N, defined by

$$\begin{aligned} v^N(s) &\simeq R^N(\bar v^N,s;\varepsilon) := \bar v^N + \varepsilon R_1(\bar v^N,s) + \cdots + \varepsilon^N R_N(\bar v^N,s) \\ \frac{\mathrm{d}\bar v^N}{\mathrm{d}s} &= W^N(\bar v^N;\varepsilon) := \varepsilon W_1(\bar v^N) + \cdots + \varepsilon^N W_N(\bar v^N), \end{aligned} \tag{2.11}$$

where N is arbitrary but fixed and finite. From the construction, we conclude that R^N and W^N are polynomial functions of $\bar v^N$ and that R^N is analytic in s.

These are approximations to the solution $v(s)$ of the original system (1.7) in the following sense (cf. [16]): The difference between $v(s)$ and its Nth-order approximation $v^N(s) = R^N(\bar{v}^N(s), s; \varepsilon)$ satisfies

$$|v(s) - R^N(\bar{v}^N(s), s; \varepsilon)| \leq \varepsilon^N \tag{2.12}$$

for $0 \leq \varepsilon s \leq T$, where T depends on $v(0)$ and N but is independent of ε. We could, but there is no need to, introduce an absolute constant in the right-hand side of (2.12) (see [16]).

3. An orthogonality result

In the first part of this section we return to equations (1.6)–(1.7) and prove an orthogonality result for the leading-order renormalized system (2.1)–(2.2) with $N = 1$. In the second part of this section, this result will be used to prove special properties of higher-order renormalized systems, namely boundedness and the existence of absorbing balls for sufficiently small ε.

3.1. **An algebraic property.** We will show that the leading-order renormalized system, which we shall write as [cf. (2.7)]

$$\frac{1}{\varepsilon}\frac{\mathrm{d}\bar{v}^1}{\mathrm{d}s} = W_1(\bar{v}^1) = \{F(\bar{v}^1)\}_{\mathrm{r}} =: -A_{\mathrm{r}}\bar{v}^1 - B_{\mathrm{r}}(\bar{v}^1) + f_{\mathrm{r}}, \tag{3.1}$$

inherits two important properties of the parent system (1.6)–(1.7), namely the orthogonality of B (1.3) and the positive-definiteness of A (1.2). This leads to an energy estimate parallel to (1.8) which, besides its intrinsic interest, guarantees that the solution of (3.1) remains bounded for all $s > 0$.

Most of the work is done by the following algebraic lemma.

Lemma 1. *Let L be real antisymmetric and $B^{(n)}(u^1, \ldots, u^n)$ a real n-linear form with the orthogonality property,*

$$(B^{(n)}(u, \ldots, u), u)_{\mathbb{R}} = 0, \qquad \text{for all } u \in \mathbb{R}^d, \tag{3.2}$$

where $(\cdot, \cdot)_{\mathbb{R}}$ denotes the real inner product. Then the resonant part of $B^{(n)}$ is also orthogonal:

$$(B_{\mathrm{r}}^{(n)}(\bar{v}, \ldots, \bar{v}), \bar{v})_{\mathbb{R}} = 0, \qquad \text{for all } \bar{v} \in \mathbb{R}^d. \tag{3.3}$$

Proof. We first recall a few (elementary) facts about the eigenspaces of L in order to set up notation: Denote the eigenvectors of L (which has complex entries in general) by ϕ_k,

$$L\phi_k = \mu_k \phi_k. \tag{3.4}$$

Taking the complex inner product of (3.4) with ϕ_l gives us

$$(\phi_l, L\phi_k)_{\mathbb{C}} = (\phi_l, \mu_k \phi_k)_{\mathbb{C}} = \mu_k^*(\phi_l, \phi_k)_{\mathbb{C}},$$

where the μ_k^* is the complex conjugate of μ_k. On the other hand, using the fact that L is real and antisymmetric gives

$$(-L\phi_l, \phi_k)_{\mathbb{C}} = -(\mu_l\phi_l, \phi_k)_{\mathbb{C}} = -\mu_l(\phi_l, \phi_k)_{\mathbb{C}}.$$

Taking $l = k$, we find that $\mu_k^* = -\mu_k$; in other words, the eigenvalues of L are imaginary and we write $\mu_k =: \mathrm{i}\omega_k$. Taking the complex conjugate (component-wise) of (3.4) gives ($L^* = L$)

$$L\phi_k^* = -\mathrm{i}\omega_k\phi_k^*,$$

which tells us that, if ϕ_k is an eigenvector of L with eigenvalue $\mathrm{i}\omega_k$, ϕ_k^* is also an eigenvector of L, with eigenvalue $-\mathrm{i}\omega_k$. We write $\phi_{-k} := \phi_k^*$ and $\omega_{-k} := -\omega_k$. Here $k = 0$ may label a space of dimension ≥ 1 (the kernel of L); this will not cause any difficulty for what follows.

Next we expand u in (1.1) in eigenvectors of L:

$$u = \textstyle\sum_k u_k\phi_k. \tag{3.5}$$

The fact that u is real implies that $u_{-k} = u_k^*$ (and $u = \sum_k u_k^*\phi_{-k}$; see below). For u real, (3.2) is equivalent to

$$(B^{(n)}(u, \dots, u), u)_{\mathbb{C}} = 0. \tag{3.6}$$

For clarity, we provide full details of the proof for the case $j = 2$ in what follows; how this carries over to the general case will be described at the end of the proof. We thus compute

$$\begin{aligned}(B^{(2)}(u, u), u)_{\mathbb{C}} &= \big(B^{(2)}\big(\textstyle\sum_j u_j\phi_j, \sum_k u_k\phi_k\big), \sum_l u_l\phi_l\big)_{\mathbb{C}} \\ &= \big(B^{(2)}\big(\textstyle\sum_j u_j\phi_j, \sum_k u_k\phi_k\big), \sum_l u_l^*\phi_{-l}\big)_{\mathbb{C}} \\ &= \textstyle\sum_{jkl} u_ju_ku_l\big(B^{(2)}(\phi_j, \phi_k), \phi_{-l}\big)_{\mathbb{C}}.\end{aligned}$$

Denoting $b_{jkl} := (B^{(2)}(\phi_j, \phi_k), \phi_l)_{\mathbb{C}}$, we have

$$\begin{aligned}&(B^{(2)}(u, u), u)_{\mathbb{C}} = \textstyle\sum_{jkl} u_ju_ku_lb_{jk-l} \\ &= \tfrac{1}{6}\textstyle\sum_{jkl} u_ju_ku_l\big[b_{jk-l} + b_{lj-k} + b_{kl-j} + b_{kj-l} + b_{jl-k} + b_{lk-j}\big].\end{aligned} \tag{3.7}$$

We now show that the orthogonality property (3.6) for any u real implies that the square bracket in (3.7) vanishes for any $\{j, k, l\}$. First, we note that (3.6) implies by (real) differentiation that

$$\begin{aligned}&(B^{(2)}(u, v), w)_{\mathbb{C}} + (B^{(2)}(w, u), v)_{\mathbb{C}} + (B^{(2)}(v, w), u)_{\mathbb{C}} \\ &\quad + (B^{(2)}(v, u), w)_{\mathbb{C}} + (B^{(2)}(u, w), v)_{\mathbb{C}} + (B^{(2)}(w, v), u)_{\mathbb{C}} = 0,\end{aligned} \tag{3.8}$$

for u, v, w real. Now take $u = \alpha_j\phi_j + \alpha_j^*\phi_{-j}$, $v = \alpha_k\phi_k + \alpha_k^*\phi_{-k}$, $w = \alpha_l\phi_l + \alpha_l^*\phi_{-l}$ (which are real). Regrouping terms, we find

$$\begin{aligned} 0 &= \alpha_j\alpha_k\alpha_l\big[b_{jk-l} + b_{lj-k} + b_{kl-j} + b_{jl-k} + b_{kj-l} + b_{lk-j}\big] \\ &\quad + \alpha_j\alpha_k\alpha_l^*\big[b_{jkl} + b_{-lj-k} + b_{k-l-j} + b_{kjl} + b_{j-l-k} + b_{-lk-j}\big] \\ &\quad + \cdots + \alpha_j^*\alpha_k^*\alpha_l^*\big[b_{-j-kl} + b_{-l-jk} + b_{-k-lj} + b_{-k-jl} + b_{-j-lk} + b_{-l-kj}\big], \end{aligned} \tag{3.9}$$

where the five terms not written explicitly can be determined by the symmetry of the indices. This equation is of the form

$$\alpha_j\alpha_k\alpha_l S_{jkl} + \alpha_j\alpha_k\alpha_l^* S_{jk-l} + \cdots + \alpha_j^*\alpha_k^*\alpha_l^* S_{-j-k-l} = 0, \tag{3.10}$$

where S_{jkl} denotes the first bracket in (3.9), etc.

We claim that if (3.10) holds for any α_j, α_k, α_l, then necessarily all S-terms vanish, $S_{jkl} = S_{jk-l} = \cdots = S_{-j-k-l} = 0$. To prove this, we alternately take $\alpha_m = 1$ and i, for $m = j, k, l$, generating $2^3 = 8$ independent equations, each having eight S-terms. The first two equations, for $\alpha_j = \alpha_k = \alpha_l = 1$, and for $\alpha_j = \alpha_k = 1$, $\alpha_l = \mathrm{i}$, are

$$\begin{aligned} S_{jkl} + S_{jk-l} + S_{j-kl} + S_{j-k-l} + S_{-jkl} + S_{-jk-l} + S_{-j-kl} + S_{-j-k-l} &= 0, \\ \mathrm{i}S_{jkl} - \mathrm{i}S_{jk-l} + \mathrm{i}S_{j-kl} - \mathrm{i}S_{j-k-l} + \mathrm{i}S_{-jkl} - \mathrm{i}S_{-jk-l} + \mathrm{i}S_{-j-kl} - \mathrm{i}S_{-j-k-l} &= 0. \end{aligned}$$

The coefficient matrix for all eight equations, which we denote by M_2, can be computed using the recursion relation (3.14) below. Using the relation (3.15) below, one computes that $\det M_2 = -(2\mathrm{i})^{12} \neq 0$, implying that (3.10) has no nontrivial solution, or in other words, $S_{jkl} = \cdots = S_{-j-k-l} = 0$. Thus, we have

$$b_{jk-l} + b_{kj-l} + b_{lj-k} + b_{jl-k} + b_{kl-j} + b_{lk-j} = 0. \tag{3.11}$$

This shows that each term in the sum (3.7) vanishes separately. It is straightforward to show the converse: (3.2) follows from (3.11).

Returning to (1.1), with $B(u) = B^{(2)}(u,u)$, the bilinear form $B_{\mathrm{r}}^{(2)}(\cdot,\cdot)$ in the renormalized equation (3.1) is the resonant part of $B^{(2)}(\cdot,\cdot)$, defined by its coefficients as

$$(b_{\mathrm{r}})_{jkl} := \begin{cases} b_{jkl} & \text{when } \omega_l = \omega_j + \omega_k \\ 0 & \text{otherwise.} \end{cases} \tag{3.12}$$

Now in (3.7), the condition for b_{jk-l} to be resonant is that $\omega_j + \omega_k + \omega_l = 0$. It can now be seen that if b_{jk-l} is resonant, the other terms in the square bracket in (3.7) are resonant as well. The sum in (3.7) therefore splits into a resonant and nonresonant parts, giving us

$$(B_{\mathrm{r}}^{(2)}(u,u), u)_{\mathbb{C}} = 0 \tag{3.13}$$

for any real u (in which case the index $\mathbb{C}$ in (3.13) is redundant).

Following the above construction, when $B^{(n)}$ is n-linear the square bracket in the analogue of (3.7) will contain $(n+1)!$ terms; similarly, there will be 2^{n+1} terms in the analogue of (3.9), each of whose coefficients consisting of $(n+1)!$ terms corresponding to those in the analogue of (3.7). Taking the α_m alternately 1 and i as before, the $2^{n+1}\times 2^{n+1}$ coefficient matrix can be computed by the relations

$$M_0 := \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix} \qquad \text{and} \qquad M_{n+1} := \begin{pmatrix} M_n & M_n \\ \mathrm{i}M_n & -\mathrm{i}M_n \end{pmatrix}. \tag{3.14}$$

The determinants of these matrices satisfy the recursion relation,

$$\begin{aligned} \det|M_{n+1}| &= \det\begin{vmatrix} M_n & M_n \\ \mathrm{i}M_n & -\mathrm{i}M_n \end{vmatrix} = \det\begin{vmatrix} M_n & M_n \\ 2\mathrm{i}M_n & 0 \end{vmatrix} \\ &= \det\begin{vmatrix} 0 & M_n \\ 2\mathrm{i}M_n & 0 \end{vmatrix} = -(2\mathrm{i})^{2^{n+1}}\bigl(\det|M_n|\bigr)^2, \end{aligned} \tag{3.15}$$

with $\det|M_0| = -2\mathrm{i}$. They are thus nonzero for all $n \ge 1$, implying that the analogue of (3.10) has no nontrivial solutions. This in turn implies that

$$(B_{\mathrm{r}}^{(n)}(u,\dots,u),u)_{\mathbb{C}} = 0 \tag{3.16}$$

for all real u and for all $n \ge 1$. The lemma is thus proved. □

3.2. **The first-order renormalized equation.** We now return to the renormalized equation (3.1) and study the other terms. Here [cf. (2.7)], A_{r} is the resonant part of A, and f_{r} the resonant part of f. As before, we work in the eigenbasis of L. We then have

$$(f_{\mathrm{r}})_k = \begin{cases} f_k & \text{if } \mu_k = 0 \\ 0 & \text{otherwise;} \end{cases} \tag{3.17}$$

this implies that $|f_{\mathrm{r}}| \le |f|$. Similarly, the matrix A_{r} (in the eigenbasis of L) is formed from A by

$$(A_{\mathrm{r}})_{jk} = \begin{cases} A_{jk} & \text{if } \mu_j = \mu_k \\ 0 & \text{otherwise.} \end{cases} \tag{3.18}$$

It is therefore block-diagonal, with each block corresponding to a distinct value of μ_k and its size equal to the multiplicity. Since each block can be regarded as the restriction of A to a subspace (i.e., an eigenspace of L), it is also positive definite, with a constant not less than the smallest eigenvalue of A, namely λ [cf. (1.2) above]. We therefore have

$$(A_{\mathrm{r}}\bar v^1, \bar v^1) \ge \lambda_{\mathrm{r}}|\bar v^1|^2, \qquad \text{with } \lambda_{\mathrm{r}} \ge \lambda. \tag{3.19}$$

Combined with Lemma 1, this implies [cf. the derivation of (1.8)]

Theorem 2. *The solution of* (3.1) *satisfies the energy estimate,*

$$|\bar{v}^1(s)|^2 \leq \mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}|\bar{v}^1(0)|^2 + \frac{|f_\mathrm{r}|^2_\infty}{\lambda_\mathrm{r}^2}\big(1 - \mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}\big), \tag{3.20}$$

with $|f_\mathrm{r}|_\infty \leq |f|_\infty$ *and* $\lambda_\mathrm{r} \geq \lambda$.

Thus, for any $\rho_\mathrm{r} > |f_\mathrm{r}|_\infty/\lambda_\mathrm{r}$, the ball $\{\bar{v}^1 : |\bar{v}^1| \leq \rho_\mathrm{r}\}$ is absorbing in the entire phase space. The functions $v(s)$ and $\bar{v}^1(s)$ living in the same phase space, the fact that $\rho_\mathrm{r} \leq \rho$ shows that the absorbing ball of the first-order renormalized system (3.1) is contained within that of the original system (1.7). It also strengthens Proposition 2 of [16], which provides only boundedness of $\bar{v}(s)$ for finite time. We note that, since the approximate solution

$$v^1(s) = \bar{v}^1(s) + \varepsilon\{F(\bar{v}^1, s)\}_\mathrm{np}$$

is determined from $\bar{v}^1(s)$ by an algebraic relation which is a polynomial function of $\bar{v}^1$ whose coefficients contain complex exponentials $\mathrm{e}^{\mathrm{i}\nu s}$ with $\nu \in \mathbb{R}$, $v^1(s)$ is uniformly bounded for all $s \geq 0$.

3.3. **Higher-order renormalized systems.** It turns out that higher-order renormalized equations also have an absorbing ball, although it only absorbs a part of the phase space whose size increases as ε decreases to 0, as shown by the following argument:

We consider the Nth-order renormalized evolution equation (2.11b), where W^N is constructed using the recursive procedure of Section 2,

$$\frac{\mathrm{d}\bar{v}^N}{\mathrm{d}s} = W^N(\bar{v}^N;\varepsilon) =: -\varepsilon A_\mathrm{r}\bar{v}^N - \varepsilon B_\mathrm{r}(\bar{v}^N) + \varepsilon f_\mathrm{r} + \varepsilon^2 W_+^N(\bar{v}^N;\varepsilon), \tag{3.21}$$

with initial data $\bar{v}^N(0) = v_0$. For notational brevity, we fix N and drop the superscript from $\bar{v}^N$, W^N and W_+^N in the remainder of this section.

Taking the inner product of (3.21) with $\bar{v}$, we find

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}|\bar{v}(s)|^2 &= \Big(\frac{\mathrm{d}\bar{v}}{\mathrm{d}s}, \bar{v}\Big) \\
&= -\varepsilon(B_\mathrm{r}(\bar{v}), \bar{v}) - \varepsilon(A_\mathrm{r}\bar{v}, \bar{v}) + \varepsilon(f_\mathrm{r}, \bar{v}) + \varepsilon^2(W_+(\bar{v}), \bar{v}) \\
&\leq -\varepsilon\lambda_\mathrm{r}|\bar{v}|^2 + \frac{\varepsilon}{2\lambda_\mathrm{r}}|f_\mathrm{r}|^2_\infty + \varepsilon\frac{\lambda_\mathrm{r}}{2}|\bar{v}|^2 + \varepsilon^2(W_+(\bar{v}), \bar{v})
\end{aligned}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}s}\Big(\mathrm{e}^{\lambda_\mathrm{r}\varepsilon s}|\bar{v}(s)|^2\Big) \leq \mathrm{e}^{\lambda_\mathrm{r}\varepsilon s}\Big(\varepsilon\frac{|f_\mathrm{r}|^2_\infty}{\lambda_\mathrm{r}} + 2\varepsilon^2(W_+(\bar{v}), \bar{v})\Big). \tag{3.22}$$

Integrating this, we find

$$|\bar v(s)|^2 \le \mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}|\bar v(0)|^2 + \frac{|f_\mathrm{r}|_\infty^2}{\lambda_\mathrm{r}^2}(1-\mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}) + \mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}\int_0^s 2\varepsilon^2\,\mathrm{e}^{\lambda_\mathrm{r}\varepsilon s'}\,(W_+(\bar v(s')),\bar v(s'))\,\mathrm{d}s'. \tag{3.23}$$

We fix $\alpha \ge 2$ and take ε sufficiently small such that

$$2\varepsilon\,|(W_+(\bar v),\bar v)| \le |f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r} \qquad \text{for all } |\bar v|^2 \le 2\alpha\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2. \tag{3.24}$$

We denote the largest value of ε satisfying (3.24) by ε_α; since $|(W_+(\bar v),\bar v)|$ is bounded by a polynomial in $\bar v$, $\varepsilon_\alpha > 0$. (The dependence of W_+ on ε is harmless: since $W_+(\bar v;\varepsilon)$ is a polynomial in the parameter ε, assuming that $\varepsilon \le 1$ we simply write $\sup_{\varepsilon\le 1}|(W_+(\bar v;\varepsilon),\bar v)|$ in place of $|(W_+(\bar v),\bar v)|$ in the above.)

Now suppose that $\varepsilon \le \varepsilon_\alpha$ and that the initial condition satisfies $|\bar v(0)|^2 \le \alpha\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2$. Then as long as $|\bar v(s)|^2 \le 2\alpha\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2$, we have using (3.24)

$$\begin{aligned} |\bar v(s)|^2 &\le \mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}|v_0|^2 + \frac{|f_\mathrm{r}|_\infty^2}{\lambda_\mathrm{r}^2}\big(1-\mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}\big) + \frac{|f_\mathrm{r}|_\infty^2}{\lambda_\mathrm{r}}\mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s}\int_0^s \varepsilon\,\mathrm{e}^{\lambda_\mathrm{r}\varepsilon s'}\,\mathrm{d}s' \\ &\le \frac{|f_\mathrm{r}|_\infty^2}{\lambda_\mathrm{r}^2}\big[\alpha\mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s} + 2(1-\mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s})\big] \\ &= \frac{|f_\mathrm{r}|_\infty^2}{\lambda_\mathrm{r}^2}\big[(\alpha-2)\mathrm{e}^{-\lambda_\mathrm{r}\varepsilon s} + 2\big]. \end{aligned} \tag{3.25}$$

This gives us a bound on $|\bar v(s)|^2$ which decreases monotonically from $\alpha|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2$ at $s = 0$, so the estimate (3.25) is satisfied for all time $s \ge 0$. In other words, for $\varepsilon \le \varepsilon_\alpha$, the ball $\{\bar v : |\bar v|^2 \le \alpha\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2\}$ is invariant. Moreover, the bound on the right-hand side of (3.25) converges to $2\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2$ as $s \to \infty$, so any ball $\{\bar v : |\bar v|^2 \le \rho'\}$ with $2\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2 < \rho' < \alpha\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2$ is absorbing in the set $\{\bar v : |\bar v|^2 \le \alpha\,|f_\mathrm{r}|_\infty^2/\lambda_\mathrm{r}^2\}$ whenever $\varepsilon \le \varepsilon_\alpha$.

Finally, we remark that no use has been made of the fact that $W_+(\bar v)$ is the higher-order part of a renormalized equation, so the above argument actually carries through for any polynomial $W_+(\bar v)$.

4. A shadowing argument

In this section, we use a different technique to obtain results which are parallel to those of the previous section. The results here are slightly weaker in the sense that we do not have an invariant set as proved in the previous section, but they are more general since only weaker assumptions on the structure of the equation are needed.

We consider a system in the form of (1.7),

$$\frac{\mathrm{d}v}{\mathrm{d}s} = \varepsilon F(v,s), \qquad\qquad v(0) = u_0, \tag{4.1}$$

where $F(v,s)$ is a polynomial in v and depends on s by exponentials $\mathrm{e}^{\mathrm{i}\nu s}$ with $\nu \in \mathbb{R}$. We assume that solutions of (4.1) are defined for all $s \geq 0$, and a bound of the form (1.8),

$$|v(s)|^2 \leq \mathrm{e}^{-\lambda\varepsilon s}|u_0|^2 + \frac{|f|_\infty^2}{\lambda^2}\bigl(1 - \mathrm{e}^{-\lambda\varepsilon s}\bigr), \qquad\qquad \text{for } s \geq 0, \tag{4.2}$$

holds for the solutions.

Let us consider the Nth-order renormalized evolution equation, (2.11b), which we rewrite here:

$$\frac{\mathrm{d}\bar{v}^N}{\mathrm{d}s} = W^N(\bar{v}^N;\varepsilon), \qquad\qquad \bar{v}^N(0) = v_0, \tag{4.3}$$

where $W^N(\bar{v}^N)$ is constructed from the original system (4.1) using the recursive procedure described in Section 2. From the definitions (2.11b), (2.5) and (2.7), we recall that $W^N(\bar{v}^N;\varepsilon) = \varepsilon W_1(\bar{v}^N) + \cdots + \varepsilon^N W^N(\bar{v}^N)$, $W_1(\bar{v}^N) = \{F(\bar{v}^N)\}_{\mathrm{r}}$ and $F(v,s) = \{F(v)\}_{\mathrm{r}} + \{F(v,s)\}_{\mathrm{n}}$. Here and henceforth N is arbitrary but fixed, and we shall often omit the superscript N from $\bar{v}^N$ and write $\bar{v}$ for the solution of (4.3). The following result then follows from the bound (4.2) on the solution of the original system (4.1):

Theorem 3. *Consider the system* (4.3) *above, which is the Nth-order renormalized equation for* (4.1), *and assume that* (4.2) *holds. Then there exists $\varepsilon_* = \varepsilon_*(F,N)$, and for $0 < \varepsilon \leq \varepsilon_*$, there exists an open ball $\mathcal{U}_\varepsilon^N$ such that all solutions $\bar{v}^N(s)$ of* (4.3) *with $\bar{v}^N(0) \in \mathcal{U}_\varepsilon^N$ remain bounded for all $s \geq 0$; the radius of $\mathcal{U}_\varepsilon^N$ grows to infinity as $\varepsilon \to 0$. Furthermore, the ball $\mathcal{B}_0 := \{\bar{v}^N : |\bar{v}^N| \leq 10\,|f|_\infty/\lambda\}$ is absorbing in $\mathcal{U}_\varepsilon^N$, and the semigroup $\mathcal{S}_\varepsilon^N$ defined by* (4.3) *possesses an attractor $\mathcal{A}_\varepsilon^N \subset \mathcal{B}_0$ which attracts $\mathcal{U}_\varepsilon^N$.*

The proof of Theorem 3 makes use of (4.2), regardless of how it is obtained, and of a "shadowing" argument. Most of the work is done in the following:

Lemma 4. *Assume that*

$$|v_0| \geq 8\,|f|_\infty/\lambda, \tag{4.4}$$

in (4.3), *and let $T_\lambda := \lambda^{-1}\log 32$. Then we can find an $\varepsilon_0 = \varepsilon_0(|v_0|;F,N)$, monotonically decreasing in $|v_0|$, such that whenever $0 < \varepsilon \leq \varepsilon_0$, the following bounds hold:*

$$|\bar{v}(s)| \leq |v_0| + 2\,|f|_\infty/\lambda, \qquad 0 \leq s \leq T_\lambda/\varepsilon, \tag{4.5}$$

$$|\bar{v}(T_\lambda/\varepsilon)| \leq \tfrac{1}{2}\,|v_0|. \tag{4.6}$$

Proof. We first show that, with $u_0 = v_0$, the solution of (4.3) stays close to that of (4.1) for some time. For this it is convenient to decompose W^N into its leading-order and higher-order parts,

$$W^N(\bar{v};\varepsilon) =: \varepsilon W_1(\bar{v}) + \varepsilon^2 W_+(\bar{v};\varepsilon),$$

where the dependence of W_+ on N is not indicated explicitly.

Let $w(s) := \bar{v}(s) - v(s)$; by our initial conditions, $w(0) = 0$. Then

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \frac{\mathrm{d}\bar{v}}{\mathrm{d}s} - \frac{\mathrm{d}v}{\mathrm{d}s} = \varepsilon^2 W_+(\bar{v};\varepsilon) + \varepsilon W_1(\bar{v}) - \varepsilon F(v,s). \tag{4.7}$$

Recalling from (2.7) that $W_1(\bar{v}) = \{F(\bar{v})\}_{\mathrm{r}}$ and using (2.5) for the last term, we can write the last equation as

$$\frac{\mathrm{d}w}{\mathrm{d}s} = \varepsilon^2 W_+(v+w;\varepsilon) + \varepsilon\{F(v+w)\}_{\mathrm{r}} - \varepsilon\{F(v)\}_{\mathrm{r}} - \varepsilon\{F(v,s)\}_{\mathrm{n}}.$$

Integrating this equation, recalling that $w(0) = 0$, we find

$$\begin{aligned} w(s) = \int_0^s \big[&\varepsilon^2 W_+(v(s') + w(s');\varepsilon) + \varepsilon\{F(v(s') + w(s'))\}_{\mathrm{r}} - \varepsilon\{F(v(s'))\}_{\mathrm{r}} \\ &- \varepsilon\{F(v(s'),s')\}_{\mathrm{n}}\big]\,\mathrm{d}s'. \end{aligned} \tag{4.8}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}s'}\{F(v(s'),s')\}_{\mathrm{np}} = \{F'(v(s'),s')\}_{\mathrm{np}} \cdot \frac{\mathrm{d}v}{\mathrm{d}s'}(s') + \{F(v(s'),s')\}_{\mathrm{n}}\,,$$

the last term in (4.8) is equal to

$$\begin{aligned} -\varepsilon\int_0^s \{F(v(s'),s')\}_{\mathrm{n}}\,\mathrm{d}s' = -\varepsilon\{F(v(s),s)\}_{\mathrm{np}} &+ \varepsilon\{F(v(0),0)\}_{\mathrm{np}} \\ &+ \varepsilon^2\int_0^s \{F'(v(s'),s')\}_{\mathrm{np}} \cdot F(v(s'),s')\,\mathrm{d}s'. \end{aligned}$$

Since $F(v,s)$ is a polynomial in v, we recall from its construction in Section 2 that $W(v)$ is also a polynomial in v. We have therefore the following bounds:

$$\begin{aligned} &\big|\{F(v+w)\}_{\mathrm{r}} - \{F(v)\}_{\mathrm{r}}\big| \le C_1(|v|)|w|, && \forall\,|w| \le 2\,|f|_\infty/\lambda, \\ &|W_+(v+w;\varepsilon)| \le C_+(|v|), && \forall\,0 < \varepsilon \le 1,\ \forall\,|w| \le 2\,|f|_\infty/\lambda. \end{aligned} \tag{4.9}$$

Since $\{F(v)\}_{\mathrm{r}}$ and $W_+(v)$ are polynomials in their arguments, we can take $C_1(\cdot)$ and $C_+(\cdot)$ to be polynomials with positive coefficients; similarly, we will take $C_+(\cdot)$, $C''(\cdot)$ and $C_{\mathrm{np}}(\cdot)$ appearing below to be polynomials with positive coefficients. Moreover, we take $C_1(0) > 0$. Noting that (4.2) and (4.4) imply that $v(s)$ is bounded as

$$|v(s)| \le |v_0|, \qquad \text{for all } s \ge 0, \tag{4.10}$$

we can replace $C_1(|v(s)|)$ and $C_+(|v(s)|)$ by $C_1(|v_0|)$ and $C_+(|v_0|)$. Similarly, noting that $F(v,s)$ depends on s through complex exponentials $\mathrm{e}^{\mathrm{i}\nu s}$ with $\nu \in \mathbb{R}$, we can write

$$\begin{aligned}\big|\{F(v(s),s)\}_{\mathrm{np}} - \{F(v(0),0)\}_{\mathrm{np}}\big| &\le C_{\mathrm{np}}(|v_0|)\\ \big|\{F'(v(s),s)\}_{\mathrm{np}} \cdot F(v(s),s)\big| &\le C''(|v_0|)\end{aligned}$$

for all $s \ge 0$, and let $C'(|v_0|) := C_+(|v_0|) + C''(|v_0|)$. Taking the absolute values in (4.8) and using these bounds, we find

$$\begin{aligned}|w(s)| &\le \varepsilon\, C_{\mathrm{np}}(|v_0|) + \int_0^s \big[\, \varepsilon^2\, C'(|v_0|) + \varepsilon\, C_1(|v_0|)\, |w(s')|\, \big]\, \mathrm{d}s'\\ &\le \varepsilon\, [C_{\mathrm{np}}(|v_0|) + \varepsilon s\, C'(|v_0|)] + \varepsilon\, C_1(|v_0|) \int_0^s |w(s')|\, \mathrm{d}s'.\end{aligned} \tag{4.11}$$

With the aid of

$$z(s) := \int_0^s |w(s')|\, \mathrm{d}s',$$

we can integrate the last inequality and we find

$$\begin{aligned}|w(s)| &\le \varepsilon\, C_{\mathrm{np}}(|v_0|)\, \mathrm{e}^{\varepsilon s\, C_1(|v_0|)} + \varepsilon\, (C'(|v_0|)/C_1(|v_0|))\big(\mathrm{e}^{\varepsilon s\, C_1(|v_0|)} - 1\big)\\ &\le \varepsilon\, C(|v_0|)\, \mathrm{e}^{\varepsilon s\, C_1(|v_0|)}.\end{aligned} \tag{4.12}$$

Using (4.2) and (4.12), we can now bound $|\bar{v}(s)|$ with the following "shadowing" argument. Since $w = \bar{v} - v$, we have

$$|\bar{v}(s)| = |v(s) + w(s)| \le |v(s)| + |w(s)|. \tag{4.13}$$

Using the bound (4.2) and the hypothesis (4.4), we compute

$$|v(s)|^2 \le |v_0|^2 \Big[\frac{1}{64} + \big(1 - \frac{1}{64}\big)\mathrm{e}^{-\lambda\varepsilon s}\Big] \le \frac{1}{16}\, |v_0|^2, \qquad \text{for } s \ge T_\lambda/\varepsilon. \tag{4.14}$$

From (4.4) and (4.12),

$$|w(s)| \le \frac{2\, |f|_\infty}{\lambda} \le \frac{|v_0|}{4} \tag{4.15}$$

for all $0 \le s \le T_\lambda/\varepsilon$ whenever

$$0 < \varepsilon \le \varepsilon_0(|v_0|) := \frac{2\, |f|_\infty}{\lambda\, C(|v_0|)} \exp(-C_1(|v_0|)\, T_\lambda). \tag{4.16}$$

We note that $\varepsilon_0(\cdot)$ is a decreasing function of its argument.

The inequalities (4.13), (4.14) and (4.15) together imply the desired bound (4.6); furthermore, (4.10), (4.13) and (4.15a) imply

$$|\bar v(s)| \le |v_0| + \varepsilon C(|v_0|)\mathrm{e}^{T_\lambda C_1(|v_0|)} \le |v_0| + \frac{2\,|f|_\infty}{\lambda}, \tag{4.17}$$

for $0 \le s \le T_\lambda/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0(|v_0|)$, whence follows (4.5). □

Proof of Theorem 3. We shall use repeated applications of Lemma 4.

Let v_0, the initial condition of (4.3), be fixed. If $|v_0| \ge 8\,|f|_\infty/\lambda$, by Lemma 4 we can find an $\varepsilon_0(|v_0|)$ such that $|\bar v(T_\lambda/\varepsilon)| \le \frac{1}{2}|v_0|$ whenever $\varepsilon \le \varepsilon_0$. If $|\bar v(T_\lambda/\varepsilon)| \ge 8\,|f|_\infty/\lambda$, we again consider the systems (4.4) and (4.1) with $\bar v(T_\lambda/\varepsilon)$ as the new initial data and apply Lemma 4. We repeat this until $|\bar v(kT_\lambda/\varepsilon)| \le 8\,|f|_\infty/\lambda$, with $k \in \mathbb{N}$, noting that since $\varepsilon_0(\cdot)$ is monotonically decreasing, the ε_0 for the initial v_0 will do for the subsequent applications (and T_λ is independent of ε and v_0). After k iterations, with k finite, $\bar v(kT_\lambda/\varepsilon)$ will be inside $\mathcal{B} = \{\bar v : |\bar v| \le 8\,|f|_\infty/\lambda\}$. Hence in all cases we arrive at the situation where the initial condition ($\bar v(0) = v_0$ or $\bar v(kT_\lambda/\varepsilon)$) is inside $\mathcal{B}$.

At this level of generality we are not able to show that $\mathcal{B}$ is invariant for (4.3), that is $|\bar v(0)| \le 8\,|f|_\infty/\lambda \;\Rightarrow\; |\bar v(s)| \le 8\,|f|_\infty/\lambda$ (which would imply boundedness of $\bar v(s)$ for all $s \ge 0$).

Assume however that $|\bar v(s)| < 8\,|f|_\infty/\lambda$ for some $s > 0$ and consider the first time s_*, if it exists, where $|\bar v(s_*)| = 8\,|f|_\infty/\lambda$. Taking this as our initial condition, Lemma 4 then tells us that for any $0 < \varepsilon \le \varepsilon_* := \varepsilon_0(8\,|f|_\infty/\lambda)$,

$$\begin{aligned} &|\bar v(s)| \le \frac{10\,|f|_\infty}{\lambda}, \qquad \text{for } s_* \le s \le s_* + T_\lambda/\varepsilon,\\ &|\bar v(s_* + T_\lambda/\varepsilon)| \le \frac{1}{2}|\bar v(s_*)| \le \frac{4\,|f|_\infty}{\lambda}, \end{aligned}$$

so that $\bar v(s_* + T_\lambda/\varepsilon)$ belongs again to $\mathcal{B}$ and the argument above can be repeated.

In conclusion, if $0 < \varepsilon \le \varepsilon_*$, all solutions of (4.3) which satisfy $\varepsilon_0(|v_0|) \le \varepsilon$ will enter $\mathcal{B}_0 : \{\bar v : |\bar v| \le 10\,|f|_\infty/\lambda\}$ in a finite time and remain in it. The set $\mathcal{U}_\varepsilon^N$ that is absorbed by $\mathcal{B}$ is $\mathcal{U}_\varepsilon^N = \{\bar v : \varepsilon_0(|\bar v|) \le \varepsilon\}$. Since $\varepsilon_0(|v_0|)$ is monotonically decreasing towards zero as $|v_0| \to \infty$, $\mathcal{U}_\varepsilon^N$ contains a ball centered at 0 of arbitrary radius R, provided ε is small enough, $\varepsilon \le \min(\varepsilon_*, \varepsilon_1(R))$.

The existence of a compact attractor $\mathcal{A}_\varepsilon^N$ for the semigroup $\mathcal{S}_\varepsilon^N$ defined by (4.3) follows from Theorem 1.1 in Chapter I of [15]. Since hypotheses (1.1) and (1.4) are easy, $W^N(\bar v^N)$ being a polynomial in $\bar v^N$, we need only to verify hypothesis (1.12) of [15], that is, the set $\cup_{s\ge s_0}\mathcal{S}_\varepsilon^N\mathcal{B}_0$ is bounded for some s_0. But the analysis above shows that any bounded set $\mathcal{D} \subset \mathcal{U}_\varepsilon^N$ enters

$\mathcal{B}_0$ after a finite time kT_λ/ε and that it remains in $\mathcal{B}_0$ for all time thereafter; hence the conclusion. $\square$

Remark 1. Note that for $N = 1$, and with a slight abuse of notation writing $\bar{v}^1 = \bar{v}^1(t)$, the renormalized equation can be written in the form

$$\frac{\mathrm{d}\bar{v}^1}{\mathrm{d}t} = W_1(\bar{v}^1),$$

in which ε appears nowhere. The conclusion of Theorem 3 must then be valid for any value of ε. In particular, considering the limit $\varepsilon \to 0$, we find that the radius of the set $\mathcal{U}_\varepsilon^N$ in which $\mathcal{B}_0$ is absorbing must be infinite; that is, $\mathcal{B}_0$ absorbs the whole phase space. This conclusion parallels the result of the first part of Section 3, but given (4.2), it makes no assumption on the structure of $W_1(\bar{v})$.

Remark 2. Theorem 3 strengthens Proposition 4 of [16], which states the boundedness of $\bar{v}(s)$ for finite time. As in the leading-order case, the approximate solution $v^N(s) := R^N(\bar{v}, s; \varepsilon)$ is also bounded for all s since it is related to $\bar{v}(s)$ by a bounded algebraic relation. It is important to note, however, that since we take a different trajectory in each step (i) or (ii) above, this does not mean that $v^N(s)$ is a good approximation to the true solution $v(s)$ for times s beyond what is given in Proposition 5 of [16]. Indeed, the timescale of accuracy is typically limited by the largest Lyapunov exponent in the system, which is (generically) independent of ε.

5. An example

To illustrate the developments in Sections 3 and 4, in this section we consider a simple example which consists of three variables $u = (\upsilon, \chi, \zeta)$:

$$\begin{aligned}
&\frac{\mathrm{d}\upsilon}{\mathrm{d}t} + \frac{1}{\varepsilon}(\chi + \zeta) = -\lambda_1 \upsilon + \chi^2 + \zeta^2 - \upsilon(a\chi + \zeta) + f,\\
&\frac{\mathrm{d}\chi}{\mathrm{d}t} - \frac{1}{\varepsilon}\upsilon = -\lambda_2 \chi + \upsilon(a\upsilon - \chi) + g,\\
&\frac{\mathrm{d}\zeta}{\mathrm{d}t} - \frac{1}{\varepsilon}\upsilon = -\lambda_3 \zeta + \upsilon(\upsilon - \zeta) + h,
\end{aligned} \tag{5.1}$$

where f, g and h are given functions of the slow time t and where a and λ_i are positive constants. Here the dissipation matrix A of (1.1) is $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$; it can be verified that the nonlinear part $B(u)$ is indeed orthogonal in the sense of (1.3). When $a = 1$, this system reduces to the Example 1 of [10] with our (υ, χ, ζ) corresponding to their (x, y, z).

To facilitate the renormalization procedure, we work in the fast time $s = t/\varepsilon$ and define the new variables [which are the analogues of $v(s)$ in (1.6)],

$$\begin{aligned} y &= \chi - \zeta \\ x &= \mathrm{e}^-[\sqrt{2}\upsilon + \mathrm{i}(\chi+\zeta)] \\ z &= \mathrm{e}^+[\sqrt{2}\upsilon - \mathrm{i}(\chi+\zeta)] = x^*, \end{aligned} \tag{5.2}$$

where $\mathrm{e}^+ := \exp(\sqrt{2}\mathrm{i}s)$ and $\mathrm{e}^- := \exp(-\sqrt{2}\mathrm{i}s)$; with a slight abuse of notation, it is understood henceforth that f, g and h are functions of εs. In these variables, (5.1) reads

$$\begin{aligned} \frac{1}{\varepsilon}\frac{\mathrm{d}y}{\mathrm{d}s} &= -\frac{1}{\sqrt{8}}(\mathrm{e}^+x + \mathrm{e}^-z)y + \frac{a-1}{8}\big(\mathrm{e}^+x + \mathrm{e}^-z\big)^2 - \Big(\frac{1}{\sqrt{8}} + \frac{a+1}{4\mathrm{i}}\Big)\mathrm{e}^+x^2 \\ &\quad + \Big(\frac{1}{\sqrt{8}} - \frac{a+1}{4\mathrm{i}}\Big)\mathrm{e}^-xz - \frac{\lambda_2+\lambda_3}{2}y + \frac{\lambda_3-\lambda_2}{4\mathrm{i}}(\mathrm{e}^+x - \mathrm{e}^-z) + g - h \\ \frac{1}{\varepsilon}\frac{\mathrm{d}x}{\mathrm{d}s} &= \frac{1}{\sqrt{2}}\mathrm{e}^-y^2 + \frac{1-a}{4}y\big(x + \mathrm{e}^{-2}z\big) - \frac{2\lambda_1+\lambda_2+\lambda_3}{4}x \\ &\quad - \frac{2\lambda_1-\lambda_2-\lambda_3}{4}\mathrm{e}^{-2}z + \frac{\lambda_2-\lambda_3}{2\mathrm{i}}\mathrm{e}^-y + \mathrm{e}^-(\sqrt{2}f + \mathrm{i}g + \mathrm{i}h), \end{aligned} \tag{5.3}$$

where $\mathrm{e}^{-2} := \exp(-2\sqrt{2}\mathrm{i}s)$ and $\mathrm{e}^{+2} := \exp(2\sqrt{2}\mathrm{i}s)$. The equation for z is just the complex conjugate of that for x and is therefore not shown explicitly.

Carrying out the renormalization procedure described in Section 2 to $\mathcal{O}(\varepsilon^2)$, we find

$$\begin{aligned} \frac{\mathrm{d}\bar{y}}{\mathrm{d}s} &= \varepsilon\Big[\frac{a-1}{4}\bar{x}\bar{z} - \frac{\lambda_2+\lambda_3}{2}\bar{y} + g - h\Big] \\ &\quad + \varepsilon^2\Big[\frac{a+1}{8}\bar{x}\bar{y}\bar{z} + \frac{\lambda_3-\lambda_2}{4}(2\bar{y}^2 + \bar{x}\bar{z} - 2f) + \frac{g+h}{2}\bar{y}\Big] \\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}s} &= \varepsilon\Big[\frac{1-a}{4}\bar{x}\bar{y} - \frac{2\lambda_1+\lambda_2+\lambda_3}{4}\bar{x}\Big] \\ &\quad + \varepsilon^2\Big[\Big(\frac{8+(a-1)^2}{32\sqrt{2}\mathrm{i}} - \frac{a+1}{8}\Big)\bar{x}\bar{y}^2 + \frac{16+8(a+1)^2-(a-1)^2}{64\sqrt{2}\mathrm{i}}\bar{x}^2\bar{z} \\ &\quad + \frac{4(\lambda_3-\lambda_2)^2 + (2\lambda_1-\lambda_2-\lambda_3)^2}{32\sqrt{2}\mathrm{i}}\bar{x} \\ &\quad + \Big(\frac{\lambda_2-\lambda_3}{4} - \frac{(2-2a)\lambda_1 - (5a+7)\lambda_2 + (7a+5)\lambda_3}{16\sqrt{2}\mathrm{i}}\Big)\bar{x}\bar{y} \end{aligned} \tag{5.4}$$

$$+\Big(\frac{g+h-(a+1)f}{4}+\frac{6f+(3a+3)(g+h)}{4\sqrt{2}\mathrm{i}}\Big)\bar{x}\Big]$$

again with the equation for $\bar{z}$ given by the complex conjugate of that for $\bar{x}$.

We note at order ε that the dissipation rate, given by the eigenvalues $(\lambda_2+\lambda_3)/2$ and $(2\lambda_1+\lambda_2+\lambda_3)/4$, are not less than that for the original system (5.1); this is consistent with the general property of the renormalized dissipation operator A_r proved earlier. Moreover, the forcing term is present only for the slow variable $\bar{y}$; this is a generic property of renormalized systems, arising from the fact that the forcing on fast variables averages out to zero at leading order.

At order ε^2, the previously absent ("fast") forcing terms, f and $g+h$, appear in products with the variables or the dissipation. Although the nonlinear part of this equation is not orthogonal [cf. (5.6) below], boundedness of its solution follows if ε is sufficiently small, as shown in Section 4; this latter property is not fully obvious from (5.4).

To study the role of the nonlinear terms, we consider the inviscid case, obtained by setting f, g, h and λ_i to 0:

$$\begin{aligned}\frac{\mathrm{d}\bar{y}}{\mathrm{d}s}&=\varepsilon\frac{a-1}{4}\,\bar{x}\bar{z}+\varepsilon^2\frac{a+1}{8}\,\bar{x}\bar{y}\bar{z}\\ \frac{\mathrm{d}\bar{x}}{\mathrm{d}s}&=\varepsilon\frac{1-a}{4}\,\bar{x}\bar{y}+\varepsilon^2\Big[\Big(\frac{a^2-2a+9}{32\sqrt{2}\mathrm{i}}-\frac{a+1}{8}\Big)\bar{x}\bar{y}^2+\frac{7a^2+18a+23}{64\sqrt{2}\mathrm{i}}\,\bar{x}^2\bar{z}\Big].\end{aligned} \tag{5.5}$$

We compute

$$\frac{\mathrm{d}}{\mathrm{d}s}\big[\bar{y}^2+|\bar{x}|^2\big]=\varepsilon\cdot 0+\varepsilon^2\,\frac{(a-1)^2}{16\sqrt{2}}\,\bar{y}^2|\bar{x}|^2\,. \tag{5.6}$$

That the order-ε contribution is zero is due to the orthogonality of the order-ε part in the renormalized equation (5.5), as shown for the general case in Section 3. When one goes to order ε^2, however, there is a nonzero contribution, so the energy is not conserved at that order.

In the special case $a=1$ treated in [10], there is no contribution from the nonlinear terms $B(u)$ at order ε, but as we see here this situation is not generic. At order ε^2, however, a nonlinear term appears. Curiously, the second-order renormalized system also happens to be orthogonal in its inviscid form.

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