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WEAK AND STRONG MAXIMUM PRINCIPLES FOR SEMICONTINUOUS FUNCTIONS

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Abstract. A local differential criterion is shown to control maximum principles for many known classes of functions satisfying such a principle. In spite of its differential nature, this criterion requires only the upper semicontinuity of the function of interest. It is satisfied by the solutions of nonlinear second order elliptic problems incorporating various types of degeneracy, for which it yields generalizations of many known results. The approach developed here also reveals that the difference between a strong and a weak principle for a given function is in fact a matter of invariance or noninvariance, for that function, of the criterion under changes of variable. The invariance property can often be translated in more familiar terms, which however completely conceal its actual nature.

1. Introduction

Functions satisfying a weak or strong maximum principle include not only the solutions of second order elliptic inequalities but also seemingly unrelated classes, such as convex functions or functions with an everywhere nonvanishing derivative, or the maximum of any two or more such functions. This raises the question whether all these functions share a common property that implies the maximum principle. This question seems to have been one of those behind the work of Crandall and Ishii [4], where the classical properties of local maxima of C^2 functions are extended to the upper semicontinuous case by arguments inspired by the concept of viscosity solution. However, the results of [4] do not include a criterion to decide whether a given upper semicontinuous function u satisfies $\max_{\partial\Omega} u = \max_{\overline{\Omega}} u$ whenever Ω is a bounded open subset of $\mathbb{R}^N$ with $\overline{\Omega}$ contained in the domain of u.

In this paper, we show that such a criterion exists and that it is expressed by a local second order differential inequality, which remains meaningful without any differentiability or even continuity assumption. Once again, ideas from viscosity solutions are needed to eliminate such assumptions, but

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not via properties of local extrema. Both strong and weak forms of the principle exist and most classes of functions known to obey a maximum principle also satisfy the criteria of this paper.

For technical reasons (such as occasional reference to subdifferentials), it will be more convenient to consider *minimum* principles. Below, we explain the basic idea through a casual discussion of the possible extensions of the weak minimum principle $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$ for a smooth function u on some bounded open subset $\Omega \subset \mathbb{R}^N$, continuous on $\overline{\Omega}$ and satisfying $\Delta u \leq 0$ in Ω.

The classical proof that $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$ carries over to the case when $\Delta u(x) \leq 0$ is only required at the points where $D^2u(x) \geq 0$ since no minimum can be achieved at the interior point x when $D^2u(x) \not\geq 0$. (Here and throughout the paper, the notation $M \geq 0$ is used to indicate that the symmetric $N \times N$ matrix M is positive semidefinite and the inequality $M' \geq M$ between two such matrices means that $M' - M \geq 0$.) Since also $\Delta u(x) \geq 0$ when $D^2u(x) \geq 0$, the minimum principle $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$ thus holds if

$$\{x \in \Omega, D^2u(x) \geq 0\} \Rightarrow \Delta u(x) = 0. \tag{1.1}$$

Next, a key remark is that, more generally, $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$ can be obtained under the assumption generalizing (1.1)

$$\{x \in \Omega, D^2u(x) \geq 0\} \Rightarrow \Delta u(x) \leq \gamma |Du(x)|.$$

The proof remains very simple and the condition can even be localized, that is, replaced by

$$\{x \in \Omega_{x_0}, D^2u(x) \geq 0\} \Rightarrow \Delta u(x) \leq \gamma_{x_0} |Du(x)|, \tag{1.2}$$

for every $x_0 \in \Omega$, where Ω_{x_0} is some neighborhood of x_0 and $\gamma_{x_0} \geq 0$ is a constant.

Since only the points x with $D^2u(x) \geq 0$ are involved in (1.2), the Laplace operator could be replaced by any purely second order elliptic linear differential operator with continuous coefficients without affecting condition (1.2). This only changes γ_{x_0}. Thus, while Δ is convenient to use in (1.2), that condition accounts for a more intrinsic property of u, in which Δ does not really play a special role.

Of course, (1.2) does not make sense unless u is twice differentiable, but this can be relaxed as follows. First, the points x at which u cannot achieve a local minimum are actually irrelevant in (1.2). More precisely, only those points x such that $u - \varphi$ achieves a local minimum at x for some C^2 *convex* function φ are important (*points of convexity*, as we shall call them) because it is easy to check that those points are exactly the points where the local subdifferential $\partial_{loc}u(x)$ of u (see Section 2; the definition of $\partial_{loc}u$ is the expected one) is not empty. If (1.2) is replaced by

$$\{x \in \Omega_{x_0}, \partial_{loc}u(x) \neq \emptyset\} \Rightarrow \Delta u(x) \leq \gamma_{x_0}|Du(x)|, \tag{1.3}$$

for every $x_0 \in \Omega$, not only a more general condition is obtained, but no differentiability of u is required to make sense of the left-hand side.

To eliminate completely the need for derivatives of u, it suffices to notice that with φ as above, the hypothesis that $u-\varphi$ achieves a local minimum at x (henceforth summarized by saying that φ is *subtangent to u at x*) implies $\Delta\varphi(x) \leq \Delta u(x)$ and also $D\varphi(x) = Du(x)$. Thus, when (1.3) holds, then it is also true that

$$\{x \in \Omega_{x_0}, \partial_{loc}u(x) \neq \emptyset\} \Rightarrow \Delta \varphi(x) \leq \gamma_{x_0}|D\varphi(x)|, \tag{1.4}$$

for every C^2 convex function φ subtangent to u at x.

Clearly, (1.4) is a local condition near each point $x_0 \in \Omega$ (weaker than (1.1), (1.2) or (1.3)) and it requires no smoothness or even continuity of u, but now the question is whether it suffices to ensure that $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$. That this is the case if u is *lower semicontinuous* is the first (weak) minimum principle of this paper, except that a condition even weaker than (1.4) is used.

The idea of substituting φ for u above has a strong flavor of the "viscosity solution" concept in elliptic PDEs. Aside from technical differences, such as the choice of the test functions, which here need only be *convex*, a conceptually more important one is that the viscosity approach is not used to find "generalized" solutions but to transfer to test functions a criterion that cannot be directly formulated for non-differentiable and indeed possibly discontinuous functions.

In the next section, we give the formal definition of the points of convexity of a function u and introduce the related concept of *point of descent*, essentially characterized by (1.4). The weak minimum principle for lower semicontinuous functions is Theorem 3.1, where the local criterion is simply that each point of the domain should be a point of descent of the function of interest. For instance, this criterion is satisfied by concave functions, C^1 functions with a nonvanishing derivative and by the minimum of any two or more such functions.

The minimum principle criterion of this paper is well suited to the comparison of solutions[1] of fully nonlinear elliptic equations $F(x, u, Du, D^2u) = 0$. In such problems, the main virtue of the criterion is its local nature, which

[1]Here, "solution" loosely refers to solutions, subsolutions and supersolutions alike; the correct terminology is used in Section 4.

makes it much easier to identify properties of F leading to the principle behind the comparison issue. Since this application is intended to illustrate the use of the criterion, we have confined attention to the comparison between viscosity and classical solutions, which does not involve any heavy PDE machinery.

Within this framework, the comparison results of Section 4 are valid in a greater generality than provided by other methods. No continuity of F with respect to any variable is needed, only a mild form of ellipticity is required and in fact various cases of degenerate ellipticity can be accommodated (Section 5). Linear equations with the classical local boundedness assumptions fit in this setting. While there is no hope that pairs of viscosity solutions can be compared with that much generality about F, nevertheless these results should serve as a useful benchmark.

Convex functions lose their convexity under nonlinear changes of variables, hence the concept of point of convexity is highly sensitive to the such changes. Remarkably, in spite of their close relationship to points of convexity, points of descent are often oblivious to reparametrization. As we show in Section 6, if all the points of the domain are points of descent and if this property is not affected by changes of variable (*absolute points of descent*), then a *strong* minimum principle is true instead of only a weak one. Once again, most of the classical examples (concave functions, etc.) satisfy the required condition. We are not aware that weak and strong principles have been related via diffeomorphism invariance in other approaches.

Strong minimum principles for the viscosity solutions of second order elliptic PDEs are derived in Section 7. In these problems, it is easy to translate the diffeomorphism invariance in more familiar terms, but the invariance property then becomes unrecognizable. The natural principles are those ruling out a *strictly negative* interior minimum and they remain valid in great generality. Stronger principles that also exclude a zero interior minimum involve more restrictive additional assumptions (Corollary 7.1), even in the linear case (Example 7.1).

Throughout this paper, $|\cdot|$ denotes the euclidian norm on $\mathbb{R}^N$ as well as the norm induced by the euclidian norm on $\mathbb{R}^{N\times N}$. Thus, $|A|^2 = \rho(A^\top A)$ (spectral radius) for all $A \in \mathbb{R}^{N\times N}$ and $|A| = \rho(A)$ if A is symmetric.

2. Points of convexity and points of descent

We begin with a precise definition of the points of convexity of a function.

Definition 2.1. Let $u : \Omega \to \mathbb{R}$ be a given function and let $x \in \Omega$.

(*i*) If $B_x \subset \Omega$ is an open ball centered at x and $\varphi : B_x \to \mathbb{R}$ is such that $u - \varphi$ has a minimum at x in B_x, we shall say that φ is subtangent to u at x.

(*ii*) The point x is a point of convexity of u if there is a *convex* function subtangent to u at x. The set of points of convexity of u in Ω will be denoted by $C_u^+(\Omega)$. A point of concavity of u is a point of convexity of $-u$ and the set of points of concavity of u will be denoted by $C_u^-(\Omega)$.

The local subdifferential of a function $u : \Omega \to \mathbb{R}$ at a point $x \in \Omega$ is the set

$$\partial_{loc} u(x) = \{g \in \mathbb{R}^N : \exists B_x \subset \Omega, \quad u(y) \geq u(x) + g \cdot (y - x), \forall y \in B_x\},$$

where B_x denotes some open ball centered at x. Note that $\partial_{loc} u(x)$ is usually smaller than the Fréchet subdifferential $\partial_- u(x)$ used in fuzzy calculus (see for instance [7] and the references therein). The following simple proposition characterizes $C_u^+(\Omega)$ in terms of $\partial_{loc} u$.

Proposition 2.1. *Let $u : \Omega \to \mathbb{R}$ be a function. The following properties hold.*

(*i*) $C_u^+(\Omega) = \{x \in \Omega : \partial_{loc} u(x) \neq \emptyset\}$. *More precisely, if $x \in \Omega$ and $g \in \partial_{loc} u(x)$, then $\varphi(y) = g \cdot y$ is subtangent to u at x, while if φ is convex and subtangent to u at x, then $\partial\varphi(x) \subset \partial_{loc} u(x)$.*

(*ii*) *If u is (locally*[2]*) convex, then*[3] $C_u^+(\Omega) = \Omega$.

(*iii*) *If $x \in C_u^+(\Omega) \cap C_u^-(\Omega)$, then u is affine near x. In particular, on some neighborhood of x and to within an additive constant, the only convex function subtangent to u at x is u itself.*

Proof. (i) It is obvious that if $\partial_{loc} u(x) \neq \emptyset$, then $\varphi(y) = g \cdot y$ is subtangent to u at x for every $g \in \partial_{loc} u(x)$. If now φ is convex and subtangent to u at x, let B_x be as in Definition 2.1. If $g \in \partial\varphi(x)$ (nonempty), then

$$\varphi(y) \geq \varphi(x) + g \cdot (y - x) \quad \text{for} \quad y \in B_x.$$

Thus,

$$\varphi(y) + u(x) - \varphi(x) \geq u(x) + g \cdot (y - x)$$

for every $y \in B_x$, and since $u(x) - \varphi(x) \leq u(y) - \varphi(y)$, we get

$$u(y) \geq u(x) + g \cdot (y - x) \quad \text{for } y \in B_x.$$

This means that $g \in \partial_{loc} u(x)$. This proves (i).

(ii) follows from (i) and $\partial u(x) \neq \emptyset$ when u is convex.

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[2] In case Ω is not convex.

[3] The converse is not true in general (see Examples 2.3 and 2.4), but it is true if u is C^2.

(iii) By (i), the hypothesis means that there are vectors $g, h \in \mathbb{R}^N$ such that

$$u(y) \geq u(x) + g \cdot (y - x) \quad \text{and} \quad -u(y) \geq -u(x) + h \cdot (y - x)$$

for all y in some ball B_x with center x. Hence, $(g + h) \cdot (y - x) \leq 0$ for all $y \in B_x$, which implies $h = -g$. But then, $u(y) = u(x) + g \cdot (y - x)$ for all $y \in B_x$, i.e., u is affine near x. If so, $u - \varphi$ is concave near x for every convex φ subtangent to u at x and since it achieves its minimum value at x (interior point of B_x), it must be constant. Thus, $\varphi = u$ to within an additive constant. □

Proposition 2.1 (i) shows that if $x \in C_u^+(\Omega)$, there is a C^2 convex (even linear) function φ subtangent to u at x. This is implicitly used in the next definition.

Definition 2.2. Let $u : \Omega \to \mathbb{R}$ be a given function. We shall call the point $x_0 \in \Omega$ a point of descent of u if there are an open neighborhood Ω_{x_0} of x_0 in Ω and constants $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ such that

$$\{|D\varphi(x)| \leq \beta_{x_0} \quad \text{and} \quad D^2\varphi(x) \leq \beta_{x_0} I\} \Rightarrow \Delta\varphi(x) \leq \gamma_{x_0} |D\varphi(x)|, \tag{2.1}$$

for every $x \in \Omega_{x_0} \cap C_u^+(\Omega) = C_u^+(\Omega_{x_0})$ and every C^2 convex function φ subtangent to u at x.

From Definition 2.2, the set of points of descent of u is an open subset of Ω. Note that a point of descent x_0 need not be a point of convexity. The terminology is meant to suggest that the function u achieves values less than or equal to $u(x_0)$ in any pointed neighborhood of x_0, although the definition contains more than this simple feature [4]. This property is not intuitively obvious from Definition 2.2, even when $N = 1$ and u is C^2 (the case $u'(x_0) \neq 0$ is trivial; if $u'(x_0) = 0$, see Proposition 2.5 (ii) below). When u is lower semicontinuous, the minimum principle of the next section will provide a full justification (Corollary 3.1), but otherwise this interpretation may be incorrect (Example 2.5).

The following proposition gives simple criteria to decide whether a point x_0 is or is not a point of descent. These criteria do not exhaust all the possible situations. Part (i) shows that when u is C^2, points of descent can sometimes be identified without recourse to the auxiliary test functions φ.

Proposition 2.2. *Let $u : \Omega \to \mathbb{R}$ and $x_0 \in \Omega$ be given and let Ω_{x_0} denote some neighborhood of x_0 in Ω.*

[4]Which does not suffice for the validity of a weak minimum principle; a convex function achieving its minimum value at more than one point gives a counterexample.

(i) If u is C^2 and there is a constant $\gamma_{x_0} \geq 0$ such that

$$\Delta u(x) \leq \gamma_{x_0} |Du(x)|, \tag{2.2}$$

for every $x \in C_u^+(\Omega_{x_0})$, then x_0 is a point of descent of u.

(ii) If $0 \notin \overline{\cup_{x \in C_u^+(\Omega_{x_0})} \partial_{loc} u(x)}$, then x_0 is a point of descent of u.

(iii) If u is C^1 and $Du(x_0) \neq 0$, then x_0 is a point of descent of u.

(iv) If $C_u^+(\Omega_{x_0}) = \emptyset$, then x_0 is a point of descent of u.

(v) If u is C^2, $Du(x_0) = 0$ and $D^2u(x_0) > 0$, then x_0 is not a point of descent of u.

(vi) If u is a Morse function (i.e., u is C^2 and $Du(x) = 0 \Rightarrow \det D^2u(x) \neq 0$), then x_0 is a point of descent of u if and only if it is not a (strict) local minimum of u.

Proof. (i) If $x \in C_u^+(\Omega_{x_0})$ and φ is C^2 convex and subtangent to u at x, then $D^2\varphi(x) \leq D^2u(x)$ and in particular $\partial_{ii}\varphi(x) \leq \partial_{ii}u(x)$ for $1 \leq i \leq N$. Thus, $\Delta\varphi(x) \leq \Delta u(x)$. Also, $D\varphi(x) = Du(x)$, so that $\Delta\varphi(x) \leq \gamma_{x_0}|D\varphi(x)|$ by (2.2). Thus, (2.1) holds with any choice of $\beta_{x_0} > 0$.

(ii) If $x \in C_u^+(\Omega_{x_0})$ and φ is C^1 convex and subtangent to u at x, then $D\varphi(x) \in \partial_{loc}u(x)$ by Proposition 2.1 (i). The hypothesis thus shows that $|D\varphi(x)| \geq a$ for some constant $a > 0$, every $x \in C_u^+(\Omega_{x_0})$ and every C^1 convex function φ subtangent to u at x. As a result, (2.1) holds with any $\gamma_{x_0} \geq 0$ upon choosing $\beta_{x_0} < a$ (hence $|D\varphi(x)| < \beta_{x_0}$ never happens).

(iii) If $x \in C_u^+(\Omega_{x_0})$, then $\partial_{loc}u(x) \neq \emptyset$ by Proposition 2.1 (i) and it is easily checked that the differentiability of u at x implies $\partial_{loc}u(x) = \{Du(x)\}$. The result thus follows from (ii) and the continuity of Du after possibly shrinking Ω_{x_0}.

(iv) is trivial from Definition 2.2.

(v) By contradiction, assume that $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ exist such that (2.1) holds. Since $D^2u(x_0) > 0$, we have $D^2u(x_0) > \varepsilon I$ for $\varepsilon > 0$ small enough. Let $\varphi_\varepsilon(y) = \frac{\varepsilon}{2}|y - x_0|^2$, so that $D(u - \varphi_\varepsilon)(x_0) = 0$ and $D^2(u - \varphi_\varepsilon)(x_0) = D^2u(x_0) - \varepsilon I > 0$, whence x_0 is a local minimum of $u - \varphi_\varepsilon$. Thus, φ_ε is subtangent to u at x_0 and $x_0 \in C_u^+(\Omega)$ since φ_ε is convex. It then follows from $D\varphi_\varepsilon(x_0) = 0$ and $D^2\varphi_\varepsilon(x_0) = \varepsilon I$ that (2.1) must hold with $x = x_0$ and $\varphi = \varphi_\varepsilon$ when $\varepsilon < \beta_{x_0}$, which is impossible since $\Delta\varphi_\varepsilon(x_0) = N\varepsilon > 0$.

(vi) If x_0 is a local minimum of u, then $Du(x_0) = 0$ and $D^2u(x_0) > 0$ since u is a Morse function. Hence, x_0 is not a point of descent of u by (v). Conversely, if x_0 is not a local minimum of u, then either $Du(x_0) \neq 0$ and x_0 is a point of descent of u by (iii), or $Du(x_0) = 0$ and $D^2u(x_0)$ has $k \geq 1$ negative and $N - k$ positive eigenvalues since u is a Morse function. If so,

$D^2u(x)$ has k negative eigenvalues for x in some neighborhood Ω_{x_0} of x_0 and hence $C_u^+(\Omega_{x_0}) = \emptyset$. Thus, x_0 is a point of descent of u by (iv). □

By Proposition 2.2 (iii), functions defined on Ω for which every point of Ω is a point of descent generalize C^1 functions with a nowhere vanishing derivative. They also generalize concave functions:

Proposition 2.3. *If $u : \Omega \to \mathbb{R}$ is locally concave, then every point of Ω is a point of descent of u.*

Proof. If $x \in C_u^+(\Omega)$, then $x \in C_u^+(\Omega) \cap C_u^-(\Omega)$ by Proposition 2.1 (ii) for $-u$. Then, Proposition 2.1 (iii) shows that u is affine in a neighborhood of x and that a convex function φ subtangent to u at x has the form $\varphi = u + c$ with $c \in \mathbb{R}$ on that neighborhood and hence is affine. Now, for affine φ, (2.1) holds trivially with any $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$. □

The next result shows that the "min" operation preserves points of descent.

Proposition 2.4. *If $x_0 \in \Omega$ is a point of descent of two functions $u : \Omega \to \mathbb{R}$ and $v : \Omega \to \mathbb{R}$, then x_0 is a point of descent of $\min(u, v)$.*

Proof. Let $x_0 \in \Omega$. Since x_0 is a point of descent of u, there are an open neighborhood Ω_{x_0} and constants $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ such that (2.1) holds for every C^2 convex function φ subtangent to u at $x \in C_u^+(\Omega_{x_0})$. The exact same property holds with v instead of u. Thus, we may assume that the same neighborhood Ω_{x_0} and constants $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ are chosen for both u and v.

With the above notation, let now $x \in C^+_{\min(u,v)}(\Omega_{x_0})$ and let φ be subtangent to $\min(u, v)$ at x, so that

$$\min(u, v)(y) - \varphi(y) \geq 0 = \min(u, v)(x) - \varphi(x)$$

for y in some open ball B_x with center x. Since $\min(u, v)(x) = u(x)$ or $v(x)$, assume that $\min(u, v)(x) = u(x)$ with no loss of generality. Then,

$$u(y) - \varphi(y) \geq \min(u, v)(y) - \varphi(y) \geq \min(u, v)(x) - \varphi(x) = u(x) - \varphi(x)$$

for all $y \in B_x$. Thus, x is a local minimum of $u - \varphi$, so that $x \in C_u^+(\Omega)$ and φ is subtangent to u at x. Therefore, (2.1) holds if φ is C^2 and convex. □

The following proposition clarifies the concept of point of descent when $N = 1$ and u is C^2 and will be convenient to discuss some examples.

Proposition 2.5. *Suppose that $N = 1$, J is an open interval and let $u \in C^2(J)$.*

(*i*) *The point* $x_0 \in J$ *is a point of descent of* u *if and only if there are an open interval* $J_{x_0} \ni x_0$ *and a constant* $\gamma_{x_0} \geq 0$ *such that*

$$u''(x) \leq \gamma_{x_0}|u'(x)|, \qquad \forall x \in J_{x_0}. \tag{2.3}$$

(*ii*) *If* $x_0 \in J$ *is a point of descent of* u *and* $u'(x_0) = 0$, *then* u *has a local maximum at* x_0. *Furthermore, if this maximum is not strict, then* u *is constant in some nontrivial interval with endpoint* x_0.

Proof. (i) It follows from Proposition 2.2 (i) that the condition is sufficient. Conversely, let $J_{x_0} = \Omega_{x_0}, \beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ be given by Definition 2.2. With no loss of generality, assume that $u''(x) \leq k$ for every $x \in J_{x_0}$, where $k \geq \beta_{x_0} > 0$ is a constant. There are two cases.

Case 1: $u'(x_0) \neq 0$. Then, after shrinking J_{x_0} if necessary, there is $a > 0$ such that $|u'(x)| \geq a$ and therefore (2.3) holds with γ_{x_0} replaced by k/a.

Case 2: $u'(x_0) = 0$. Then, after shrinking J_{x_0} if necessary, we may assume that $|u'(x)| \leq \beta_{x_0}$ for every $x \in J_{x_0}$. If also $u''(x) \leq 0$, then (2.3) holds trivially and continues to hold with γ_{x_0} replaced by $k\gamma_{x_0}/\beta_{x_0}$.

If $u''(x) > 0$, then u is convex near x and subtangent to itself at x. Also, by Proposition 2.1 (i), the linear function $\varphi(y) = u'(x)y$ is subtangent to u at x. Thus, for every $0 \leq \varepsilon \leq 1$, the function $\varphi_\varepsilon = \varepsilon u + (1-\varepsilon)\varphi$ is C^2 convex near x and subtangent to u at x. Since $\varphi''_\varepsilon = \varepsilon u''$, the condition $\varphi''_\varepsilon \leq \beta_{x_0} I$ is satisfied with $\varepsilon = \beta_{x_0}/k \leq 1$, and since $\varphi'_\varepsilon(x) = u'(x)$ we also have $|\varphi'_\varepsilon(x)| \leq \beta_{x_0}$. Thus, $\varphi''_\varepsilon(x) \leq \gamma_{x_0}|\varphi'_\varepsilon(x)|$ by (2.1). This is just $u''(x) \leq \frac{\gamma_{x_0}}{\varepsilon}|u'(x)| = \frac{k\gamma_{x_0}}{\beta_{x_0}}|u'(x)|$, so that (2.3) holds with γ_{x_0} replaced by $k\gamma_{x_0}/\beta_{x_0}$. In summary, (2.3) holds for every $x \in J_{x_0}$ with γ_{x_0} replaced by $k\gamma_{x_0}/\beta_{x_0}$.

(ii) Let $J_{x_0} = (x_0 - \delta, x_0 + \delta)$ and let $\gamma_{x_0} \geq 0$ be given by part (i) above. Thus, $v = u'$ satisfies the inequality $v' \leq \gamma_{x_0}|v|$ in J_{x_0} and $v(x_0) = 0$. We claim that $v \leq 0$ in $(x_0, x_0 + \delta)$. Otherwise, let $x \in (x_0, x_0 + \delta)$ be such that $v(x) > 0$. Then, $v > 0$ in some maximal subinterval of J_{x_0} of the form $(x_1, x]$ with $x_1 < x$. Necessarily, $x_1 \geq x_0$ since $v(x_0) = 0$, and v (continuous) must vanish at x_1. Therefore, $v(x_1) = 0$ and $v' \leq \gamma_{x_0} v$ in $[x_1, x]$ since $v \geq 0$ on that interval. But then it follows from Gronwall's lemma that $v = 0$ on $[x_1, x]$, which is a contradiction. This proves that $v \leq 0$ in $(x_0, x_0 + \delta)$. That $v \geq 0$ in $(x_0 - \delta, x_0)$ follows by similar arguments, and since $v = u'$, this amounts to saying that u has a local maximum at x_0.

If the maximum is not strict, then there is $x'_0 \in J_{x_0} \backslash \{x_0\}$ such that $u(x'_0) = u(x_0)$. Suppose for instance $x'_0 > x_0$. From the above, $v = u' \leq 0$ in (x_0, x'_0) and hence u' must be 0 on that interval, so that u is constant on $[x_0, x'_0]$. A similar conclusion holds if $x'_0 < x_0$. $\square$

As we shall see later (Corollary 6.3), the following result implies the validity of a strong minimum principle without any further assumptions when $N = 1$.

Lemma 2.1. *Suppose that $J \subset \mathbb{R}$ is an open interval* (*bounded or unbounded*) *and let $u : J \to \mathbb{R}$. If $x_0 \in J$ is a point of descent of u, there are an open interval $J_{x_0} \subset J$ and constants $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ such that $\varphi''(x) \leq \gamma_{x_0}|\varphi'(x)|$ for every $x \in J_{x_0}$ and every C^2 function φ subtangent to u at x (*convex or not*), satisfying the conditions $|\varphi'(x)| \leq \beta_{x_0}$ and $|\varphi''(x)| \leq \beta_{x_0}$.*

Proof. Choose $J_{x_0} \subset J, \beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ consistent with the definition of x_0 as a point of descent. Next, let φ be any C^2 function subtangent to u at $x \in J_{x_0}$ satisfying $|\varphi'(x)| \leq \beta_{x_0}$and $|\varphi''(x)| \leq \beta_{x_0}$. If $\varphi''(x) \leq 0$, the inequality $\varphi''(x) \leq \gamma_{x_0}|\varphi'(x)|$ holds. If $\varphi''(x) > 0$, then φ is convex near x. Since φ is subtangent to u at x, we have $x \in C_u^+(J_{x_0})$ and the inequality $\varphi''(x) \leq \gamma_{x_0}|\varphi'(x)|$ follows from the choice of $J_{x_0}, \beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$. □

We conclude this section with some examples when $N = 1$, meant to illustrate the concept of point of convexity and point of descent in the simplest way possible.

Example 2.1. Let $\alpha \geq 1$ and let $u(x) = |x|^\alpha$. Then, $C_u^+(\mathbb{R}) = \mathbb{R}$ by Proposition 2.1 (ii) and every point $x_0 \neq 0$ is a point of descent by Proposition 2.2 (iii). The point $x_0 = 0$ is not a point of descent. This follows from Proposition 2.5 (ii) if $\alpha \geq 2$ (so that u is C^2). If $1 \leq \alpha < 2$, observe that $\varphi_\varepsilon(x) = \varepsilon x^2$ is subtangent to u at 0 for every $\varepsilon > 0$. Since $0 \in C_u^+(\mathbb{R})$ and $\varphi_\varepsilon'(0) = 0$, $\varphi_\varepsilon''(0) = 2\varepsilon$, the inequality $\varphi_\varepsilon''(0) \leq \gamma_0|\varphi_\varepsilon'(0)|$ should hold for some $\gamma_0 \geq 0$ and $\varepsilon > 0$ small enough if 0 were a point of descent, which obviously is not the case.

Example 2.2. Let $0 < \alpha < 1$ and let $u(x) = |x|^\alpha$. Now, $C_u^+(\mathbb{R}) = \{0\}$, for if $x \neq 0$, then u is concave near x, whence $x \in C_u^-(\mathbb{R})$ by Proposition 2.1 (ii) for $-u$, and hence $x \notin C_u^+(\mathbb{R})$ by Proposition 2.1 (iii) since u is not affine near x. That $0 \in C_u^+(\mathbb{R})$ follows from $\partial_{loc}u(0) \neq \emptyset$ and Proposition 2.1 (i). The point 0 is not a point of descent of u. To see this, argue as in the case $1 \leq \alpha < 2$ of Example 2.1. All the other points are points of descent of u since they have neighborhoods deprived of points of convexity (Proposition 2.2 (iv)).

Example 2.3. Let $u(x) = x$ if $x \leq 0$ and $u(x) = 2x + a$ with $a \geq 0$ if $x > 0$, so that u is lower semicontinuous but not differentiable and even discontinuous if $a > 0$. Every $x \in \mathbb{R}$ is a point of convexity and a point of descent of u. The first statement follows from $\partial_{loc}u(x) \neq \emptyset$ (Proposition 2.1 (i)) and the second from Proposition 2.2 (ii) since $\cup_{x\in\mathbb{R}}\partial_{loc}u(x) = [1,\infty)$ if $a > 0$ and $\cup_{x\in\mathbb{R}}\partial_{loc}u(x) = [1,2]$ if $a = 0$. Note that u is not convex if $a > 0$.

Example 2.4. Let u be the characteristic function of the interval $(-1,1)$, so that u is lower semicontinuous and $C_u^+(\mathbb{R}) = \mathbb{R}$. Every $x \neq \mp 1$ is trivially a point of descent (only constant functions are subtangent to u at x). Perhaps surprisingly in light of our earlier interpretation, ∓ 1 are *not* points of descent of u : For every $\varepsilon > 0$, the function $\varphi_\varepsilon(x) = \varepsilon(x-1)^2 + \varepsilon^2(x-1)$ is C^2 convex and subtangent to u at 1. Since $\varphi_\varepsilon'(1) = \varepsilon^2$ and $\varphi_\varepsilon''(1) = 2\varepsilon$, both requirements $|\varphi_\varepsilon'(1)| \leq \beta_{x_0}$ and $\varphi_\varepsilon''(1) \leq \beta_{x_0}$ hold for $\varepsilon > 0$ small enough irrespective of $\beta_{x_0} > 0$, but not $\varphi_\varepsilon''(1) \leq \gamma_{x_0}|\varphi_\varepsilon'(1)|$ as $\varepsilon \to 0$. Thus, 1 is not a point of descent and neither is -1 by similar arguments. This is actually consistent with the validity of a strong minimum principle alluded to earlier, which would otherwise be violated. If $N > 1$, the situation is different: Every point of $\mathbb{R}^N$ is a point of descent of the characteristic function of the open unit ball (Example 3.1).

Example 2.5. Let u denote the function on $[-1,1]$ defined by $u(x) = |x| - \frac{1}{n+1}$ for $|x| \in (\frac{1}{n+1}, \frac{1}{n}], n \in \mathbb{N}$, and $u(0) = 0$. This function is not *lsc* at the points $\mp\frac{1}{n}, n \geq 2$. It is clear from Proposition 2.1 (i) that $C_u^+((-1,1)) = (-1,1)\backslash\{\mp\frac{1}{n} : n \geq 2\}$. Every point of $(-1,1)$ is a point of descent of u : Since u is affine near each point $x \neq 0$ of $C_u^+((-1,1))$, the only convex function subtangent to u at x is u itself (to within an additive constant, and on some neighborhood of x) by Proposition 2.1 (iii), so that (2.1) holds trivially with any $\beta_{x_0} > 0$ and any $\gamma_{x_0} \geq 0$. The same thing is true at $x = 0$, because the only convex functions subtangent to u at 0 are constant (note that u takes on arbitrarily small positive values near each point $\mp\frac{1}{n}$). However, 0 is a strict minimum of u, so that the interpretation of the points of descent given earlier is incorrect in this case. If u is replaced by its lower semicontinuous envelope $\tilde{u}(x) = |x| - \frac{1}{n+1}$ for $|x| \in [\frac{1}{n+1}, \frac{1}{n}), n \in \mathbb{N}, \tilde{u}(0) = 0$, then $C_{\tilde{u}}^+((-1,1)) = (-1,1)$ but neither 0 nor the points $\mp\frac{1}{n}$ are points of descent any more.

3. The weak minimum principle for lower semicontinuous functions

The weak minimum principle for lower semicontinuous functions follows by a variant of the proof of the corresponding classical principle for the solutions of linear elliptic inequalities:

Theorem 3.1. *Suppose that* Ω *is bounded. If* $u : \overline{\Omega} \to \mathbb{R}$ *is lower semicontinuous and every point of* Ω *is a point of descent of* u*, then,* $\min_{\partial\Omega} u = \min_{\overline{\Omega}} u$.

Proof. By contradiction, suppose otherwise, so that the set $K = \{x \in \Omega : u(x) \leq u(y), \forall y \in \overline{\Omega}\}$ is a compact subset of Ω. Cover K with finitely many open neighborhoods Ω_i such that for each index i there are $\beta_i > 0$ and $\gamma_i \geq 0$ with the property that if $x \in C_u^+(\Omega_i)$ and φ is C^2 convex and subtangent to u at x, then $|D\varphi(x)| \leq \beta_i$ and $D^2\varphi(x) \leq \beta_i I$ together imply $\Delta\varphi(x) \leq \gamma_i|D\varphi(x)|$. Let $\omega = \cup_i \Omega_i \supset K, \beta = \min_i \beta_i > 0$ and $\gamma = \max_i \gamma_i \geq 0$, so that for every $x \in C_u^+(\omega)$ and every C^2 convex function φ subtangent to u at x, we have

$$\{|D\varphi(x)| \leq \beta \text{ and } D^2\varphi(x) \leq \beta I\} \Rightarrow \Delta\varphi(x) \leq \gamma|D\varphi(x)|. \tag{3.1}$$

Now, let $t > \max(\gamma, 1)$ be chosen once and for all. For every $\varepsilon > 0$, set $\varphi_\varepsilon(y) = \varepsilon e^{-ty_1}$ (with $y = (y_1, ..., y_N)$). Since $0 < \varphi_\varepsilon \leq \varepsilon e^{-td}$ in $\overline{\Omega}$, where $d = \min_{\overline{\Omega}} y_1$, it follows from the definition of K that if $\varepsilon > 0$ is small enough, then $u - \varphi_\varepsilon$ achieves its minimum on $\overline{\Omega}$ at some point $x_\varepsilon \in \omega$. (Otherwise, a point $x \notin \omega$ would be obtained where u achieves its minimum value in $\overline{\Omega}$, which contradicts the fact that all the minimizers of u are in $K \subset \omega$.)

Because φ_ε is convex and subtangent to u at x_ε, it follows that $x_\varepsilon \in C_u^+(\omega)$. In addition, φ_ε is C^2, $|D\varphi_\varepsilon(x_\varepsilon)| = \varepsilon te^{-tx_{\varepsilon 1}}$ and the matrix $D^2\varphi_\varepsilon(x_\varepsilon)$ is diagonal with first entry $\varepsilon t^2 e^{-tx_{\varepsilon 1}}$ and all other entries 0. Thus, $|D\varphi_\varepsilon(x_\varepsilon)| \leq \varepsilon te^{-td}$ and $D^2\varphi_\varepsilon(x) \leq \varepsilon t^2 e^{-tx_{\varepsilon 1}} I \leq \varepsilon t^2 e^{-td} I$. If $\varepsilon > 0$ is small enough, we have (recall $t > 1$) $\varepsilon te^{-td} < \varepsilon t^2 e^{-td} < \beta$ and hence, by (3.1), the inequality $\Delta\varphi_\varepsilon(x_\varepsilon) \leq \gamma|D\varphi_\varepsilon(x_\varepsilon)|$ must hold. However, this is not the case since $\Delta\varphi_\varepsilon(x_\varepsilon) = \varepsilon t^2 e^{-tx_{\varepsilon 1}}, |D\varphi_\varepsilon(x_\varepsilon)| = t\varepsilon e^{-tx_{\varepsilon 1}}$ and $t > \gamma$. This completes the proof. □

Example 2.3 of the previous section with $a > 0$ gives a discontinuous function for which Theorem 3.1 is valid and Examples 2.1 and 2.2 show that the theorem may break down if even one point of the domain is not a point of descent. By Example 2.5, the lower semicontinuity of u cannot be dropped either.

The converse of Theorem 3.1 is false: If $\min_{\partial\omega} u = \min_{\overline{\omega}} u$ for every open subset $\omega \subset \Omega$, it is not true that every point of Ω must be a point of descent of u. Example 2.4 gives a (discontinuous) counterexample. A smooth one is $u(x) = x^3$ when $N = 1$, for which 0 is not a point of descent by Proposition 2.5 (ii). However, if $\min_{\partial\omega} u = \min_{\overline{\omega}} u$ for every open subset $\omega \subset \Omega$ *and* u is a Morse function, then every point of Ω is a point of descent of u by

Proposition 2.2 (vi). Since "almost[5]" every C^2 function is a Morse function, the criterion of Theorem 3.1 is thus close to being optimal, at least in the C^2 case.

The following corollary justifies the interpretation of the points of descent given in the previous section.

Corollary 3.1. *Let $u : \Omega \to \mathbb{R}$ be lower semicontinuous. If $x_0 \in \Omega$ is a point of descent of u, every pointed neighborhood of x_0 in Ω contains a point x such that $u(x) \le u(x_0)$.*

Proof. Otherwise, there is an open ball B_{x_0} with $\overline{B}_{x_0} \subset \Omega$ such that $u(x) > u(x_0)$ for every $x \in \overline{B}_{x_0} \backslash \{x_0\}$. From the remark that the set of points of descent of u is open, we may also assume that every point of B_{x_0} is a point of descent of u. Now, $\min_{\overline{B}_{x_0}} u = u(x_0)$ and $\min_{\partial B_{x_0}} u > u(x_0)$ since u is *lsc* on ∂B_{x_0} and hence achieves its minimum value, which contradicts Theorem 3.1. □

Remark 3.1. There are several ways to generalize the concept of point of descent with no prejudice to Theorem 3.1 or Corollary 3.1. For instance, it may be decided that every point x_0 such that $0 \notin \partial_{loc} u(x_0)$ is a point of descent of u while Definition 2.2 is used if $0 \in \partial_{loc} u(x_0)$. If so, more functions have the property that every point of their domain is a point of descent (such as $u(x) = x^3$ when $N = 1$), but the set of points of descent is no longer open ($u(x) = x \sin \frac{1}{x}$ gives a counterexample) and there are difficulties with Proposition 2.4. Which definition is used does not seem to have an impact on the PDE problems discussed later. More useful generalizations are introduced in Section 5.

Our next example shows that the hypotheses of Theorem 3.1 do not yield a strong maximum principle when $N > 1$.

Example 3.1. In $\mathbb{R}^N, N > 1$, let u be the characteristic function of the open unit ball B. Clearly, u is *lsc* and $C_u^+(\mathbb{R}^N) = \mathbb{R}^N$. If $x \notin \partial B$, the only convex functions subtangent to u at x are constant in the vicinity of x by Proposition 2.1 (iii). Actually, the same thing is true if $x \in \partial B$ as we now show. Suppose that φ is C^2, convex on some open ball B_x centered at x and that $u - \varphi$ achieves its minimum value at x. After changing φ by a constant, assume that $\varphi(x) = u(x)(= 0)$. Since also $u = 0$ in $B_x \cap (\mathbb{R}^N \backslash B)$, it follows that $\varphi \le 0$ in $B_x \cap (\mathbb{R}^N \backslash B)$. Since φ is convex, the set $\{y \in B_x : \varphi(y) \le 0\}$ is convex and hence contains the convex hull of $B_x \cap (\mathbb{R}^N \backslash B)$. It is readily checked that this convex hull is a neighborhood of x (because $N > 1$).

[5]Relative to the C^2 Whitney topology.

Thus, φ has a local maximum at x and, being convex, is constant near x. This proves that the convex functions subtangent to u at x are constant irrespective of $x \in \mathbb{R}^N$, and then (2.1) holds trivially. Thus, every point of $\mathbb{R}^N$ is a point of descent of u (compare with Example 2.4). Although u is not constant on $\mathbb{R}^N$, it achieves its minimum value at interior points. Therefore, a strong minimum principle does not hold. Smooth variants of this example are easily found.

4. Comparison of classical and viscosity solutions

In what follows, $\mathcal{S}_N$ denotes the space of $N \times N$ real symmetric matrices and Tr the (matrix) trace operator. As before, Ω is an open subset of $\mathbb{R}^N$ and now $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \to \mathbb{R}$ is a given function.

A function $u : \Omega \to \mathbb{R}$ is called a viscosity supersolution of $F = 0$ if

$$F(x, u(x), D\psi(x), D^2\psi(x)) \leq 0,$$

whenever $x \in \Omega$ and ψ is a C^2 function such that $u - \psi$ has a local minimum at x (subtangent to u at x in our terminology, but ψ need not be convex here). Viscosity subsolutions $v : \Omega \to \mathbb{R}$ are defined analogously by

$$F(x, v(x), D\psi(x), D^2\psi(x)) \geq 0,$$

whenever ψ is a C^2 function such that $v - \psi$ has a local maximum at x. For further details, we refer to Crandall, Ishii and Lions [5].

For the function F, we introduce the *generalized ellipticity* condition

(**GE**) There are functions $\lambda > 0, \beta > 0$ and $\gamma \geq 0$ on $\Omega \times \mathbb{R} \times (\mathbb{R}^N \times \mathcal{S}_N)^2$ such that

$$\begin{aligned} &F(x, z, p', M') - F(x, z, p, M) \\ &\quad \geq \lambda(x, z, p, M, p', M') \left(Tr(M' - M) - \gamma(x, z, p, M, p', M')\right) |p' - p|, \end{aligned}$$

whenever $|p' - p| \leq \beta(x, z, p, M, p', M')$ and $M' - M \geq 0$.

We emphasize that, in (**GE**), γ does appear with the negative sign. Next, we need a *local boundedness* condition complementing (**GE**):

(**LB**) For every $(x_0, z_0, p_0, M_0) \in \Omega \times \mathbb{R} \times (\mathbb{R}^N \times \mathcal{S}_N)$, there are an open neighborhood $\mathcal{V}_0$ of $(x_0, z_0, p_0, M_0, p_0, M_0)$ in $\Omega \times \mathbb{R} \times (\mathbb{R}^N \times \mathcal{S}_N)^2$ and constants $B_0 > 0$ and $C_0 \geq 0$ such that $\beta(x, z, p, M, p', M') \geq B_0$ and $\gamma(x, z, p, M, p', M') \leq C_0$ for every $(x, z, p, M, p', M') \in \mathcal{V}_0$.

The condition (**LB**) is simply the local boundedness of β^{-1} and γ near points where the arguments (p, M) and (p', M') are the same. No continuity is assumed of F or λ, β, γ. The last condition is just *monotonicity* in the z variable:

(**M**) $z \in \mathbb{R} \mapsto F(x,z,p,M)$ is nonincreasing for every $(x,p,M) \in \Omega \times \mathbb{R}^N \times \mathcal{S}_N$.

Remark 4.1. The condition (**GE**) implies $F(x,z,p,M') - F(x,z,p,M) \geq 0$ when $M' - M \geq 0$. Thus, the classical subsolutions (or supersolutions) of $F = 0$ are also C^2 viscosity subsolutions (or supersolutions).

Example 4.1. (F independent of p) Simplifications occur when F does not depend upon p. The condition (**GE**) becomes
(**GE'**) $F(x,z,M') - F(x,z,M) > 0$,
for every $x \in \Omega, z \in \mathbb{R}, M, M' \in \mathcal{S}_N$ with $M' - M \geq 0$ and $M' \neq M$ and (**LB**) is vacuous. Indeed, if (**GE'**) holds, so does (**GE**) with any $\beta > 0, \gamma = 0$ and $\lambda(x,z,M,M') = \frac{F(x,z,M') - F(x,z,M)}{Tr(M'-M)}$ if $M' - M \geq 0$ and $M' \neq M$ and (say) $\lambda(x,z,M,M) = 1$. Conversely, since $TrM' > TrM$ if $M' - M \geq 0$ and $M' \neq M$, (**GE**) implies (**GE'**). Thus, the assumptions (**GE**), (**LB**) and (**M**) reduce to (**GE'**) and (**M**).

The application of Theorem 3.1 yields the following minimum principle for the viscosity supersolutions of $F = 0$.

Lemma 4.1. *Assume that Ω is bounded and that $F(x,0,0,0) \geq 0$ for every $x \in \Omega$. If (**GE**), (**LB**) and (**M**) hold and $u : \overline{\Omega} \to \mathbb{R}$ is a lower semicontinuous viscosity supersolution of $F = 0$, then* $\max_{\partial\Omega} u_- = \max_{\overline{\Omega}} u_-$.

Proof. The function $G = F - F(x,0,0,0)$ satisfies (**GE**), (**LB**) and (**M**) and the assumption $F(x,0,0,0) \geq 0$ shows that u is a lower semicontinuous viscosity supersolution of $G = 0$. Also, 0 is a classical solution of $G = 0$ and hence also a viscosity supersolution by Remark 4.1. As a result, $w = -u_- = \inf(u,0)$ is *lsc* on $\overline{\Omega}$ and a viscosity supersolution of $G = 0$, and $w = 0$ on $\partial\Omega$. As we shall see below, every point of Ω is a point of descent of w, so that Theorem 3.1 implies that $\min_{\overline{\Omega}} w = \min_{\partial\Omega} w = 0$, which is the relation $\max_{\partial\Omega} u_- = \max_{\overline{\Omega}} u_-$ of the theorem.

We now show that every point $x_0 \in \Omega$ is a point of descent of w. By (**LB**) for F with $z_0 = 0, p_0 = 0$ and $M_0 = 0$, there are an open neighborhood Ω_{x_0} of x_0 and constants $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ such that $\beta(x,0,0,0,p',M') \geq \beta_{x_0}$ and $\gamma(x,0,0,0,p',M') \leq \gamma_{x_0}$ whenever $x \in \Omega_{x_0}$ and $|p'|$ and $|M'|$ are small enough. After shrinking $\beta_{x_0} > 0$ if necessary, the smallness of $|p'|$ and $|M'|$ amounts to $|p'| \leq \beta_{x_0}$ and $|M'| \leq \beta_{x_0}$. We may thus assume with no loss of generality that

$$\{|p'| \leq \beta_{x_0}, 0 \leq M' \leq \beta_{x_0} I\} \Rightarrow \begin{cases} \beta(x,0,0,0,p',M') \geq \beta_{x_0}, \\ \gamma(x,0,0,0,p',M') \leq \gamma_{x_0}. \end{cases} \tag{4.1}$$

Let $x \in C_w^+(\Omega_{x_0})$ and let φ be a C^2 convex function subtangent to w at x (Definition 2.1). Since $w - \varphi$ has a local minimum at x and w is a supersolution of $G = 0$, we have

$$F(x, w(x), D\varphi(x), D^2\varphi(x)) - F(x, 0, 0, 0) \leq 0,$$

while

$$F(x, 0, D\varphi(x), D^2\varphi(x)) \leq F(x, w(x), D\varphi(x), D^2\varphi(x))$$

by (**M**) and $w(x) \leq 0$. Hence,

$$F(x, 0, D\varphi(x), D^2\varphi(x)) - F(x, 0, 0, 0) \leq 0. \tag{4.2}$$

Above, $D^2\varphi(x) \geq 0$ since φ is convex. Suppose that, in addition, $|D\varphi(x)| \leq \beta_{x_0}$ and $D^2\varphi(x) \leq \beta_{x_0} I$. From (4.1) with $p' = D\varphi(x)$ and $M' = D^2\varphi(x)$, we infer that

$$\beta(x, 0, 0, 0, D\varphi(x), D^2\varphi(x)) \geq \beta_{x_0} \tag{4.3}$$

and

$$\gamma(x, 0, 0, 0, D\varphi(x), D^2\varphi(x)) \leq \gamma_{x_0}. \tag{4.4}$$

Since we assumed $|D\varphi(x)| \leq \beta_{x_0}$, (4.3) implies

$$|D\varphi(x)| \leq \beta(x, 0, 0, 0, D\varphi(x), D^2\varphi(x)),$$

so that we may now use (**GE**) with the left-hand side of (4.2). Because $\lambda > 0$ in (**GE**), this yields

$$\Delta\varphi(x) \leq \gamma(x, 0, 0, 0, D\varphi(x), D^2\varphi(x))|D\varphi(x)|.$$

Then, by (4.4),

$$\Delta\varphi(x) \leq \gamma_{x_0}|D\varphi(x)|, \tag{4.5}$$

which means that x_0 is a point of descent of w. □

Remark 4.2. The proof of Lemma 4.1 uses only (**GE**) and (**LB**) with $(p, M) = (0, 0)$, and $F(x, z, p, M) \geq F(x, 0, p, M)$ for $z \leq 0$ instead of (**M**). However, the full assumptions are needed to use Lemma 4.1 in the setting of Theorem 4.1 below.

Theorem 4.1. *Assume that Ω is bounded. If (**GE**), (**LB**) and (**M**) hold and $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u : \overline{\Omega} \to \mathbb{R}$ are a classical subsolution and a lower semicontinuous viscosity supersolution of $F = 0$, respectively, then*

$$\max_{\partial\Omega}(u - v)_- = \max_{\overline{\Omega}}(u - v)_-.$$

*If F is independent of p, then (**GE'**) and (**M**) suffice.*

Proof. With

$$G(x,z,p,M) = F(x, v(x) + z, Dv(x) + p, D^2v(x) + M) \\ - F(x, v(x), Dv(x), D^2v(x)),$$

a routine verification shows that $G(x, 0, 0, 0) = 0$ and that G satisfies the hypotheses (**GE**), (**LB**) and (**M**). Furthermore**,** $w = u - v$ is lower semicontinuous on $\overline{\Omega}$ and a viscosity supersolution of $G = 0$. By Lemma 4.1, $\max_{\partial\Omega} w_- = \max_{\overline{\Omega}} w_-$ The statement when F is independent of p follows from the earlier remark that (**GE'**) implies (**GE**) and (**LB**) in that case. □

Corollary 4.1. *Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u \in C^0(\overline{\Omega})$ be a classical and a viscosity solution of $F = 0$, respectively. If (**GE**), (**LB**) and (**M**) hold and if $u = v$ on $\partial\Omega$, then $u = v$ in $\overline{\Omega}$. If F is independent of p, (**GE'**) and (**M**) suffice.*

Proof. That $u \geq v$ follows from Theorem 4.1. To get $u \leq v$, use Theorem 4.1 with u and v replaced by $-u$ and $-v$, respectively, and F replaced by $G(x,z,p,M) = -F(x,-z,-p,-M)$. It is trivial to check that G satisfies (**GE**) (resp. (**GE'**)), (**LB**) and (**M**) whenever the same thing is true of F. □

Remark 4.3. By Remark 4.1, Theorem 4.1 and Corollary 4.1 are valid when u is a classical (super)solution as well.

Naturally, one of the conditions (**GE**), (**LB**) or (**M**) breaks down in the various counterexamples to Theorem 4.1 or Corollary 4.1 discussed in the literature. Examples when (**M**) fails are self-evident. With $F(p,M) = TrM + |p|^s$ and $s \in (0,1)$, a counterexample to Corollary 4.1 is given in [2]. Since (**M**) is not an issue, (**GE**) and (**LB**) cannot both hold, i.e., functions λ, β and γ cannot be found with the desired properties. In contrast, (**GE**), (**LB**) and (**M**) hold if $s \geq 1$ (take $\lambda = \beta = 1, \gamma(p,p') = s\max(|p'|^{s-1}, |p|^{s-1})$), whence $v = 0$ is the only continuous viscosity solution of $\Delta v + |Dv|^s = 0$ vanishing on $\partial\Omega$ (any bounded Ω) by Corollary 4.1. This also follows from [1, Example 6] and, for $s = 1$, from [9, Theorem 3].

Although Theorem 4.1 and Corollary 4.1 give only a comparison of viscosity (super)solutions against classical (sub)solutions, their hypotheses are rather weak. The assumptions (**GE**), (**LB**) and (**M**) cover the linear case under the classical conditions: If $F(x,z,p,M) = Tr(A(x)M) + b(x)\cdot p + c(x)z$ with $A(x) \in \mathcal{S}_N$, they amount to assuming that $A(x) > 0, (TrA^{-1}(x))b(x)$ is locally bounded (that is, $|b(x)|/\lambda(x)$ is locally bounded, where $\lambda(x)$ is the smallest eigenvalue of $A(x)$) and $c(x) \leq 0$. Here, we may take $\beta = 1$ and $\gamma = \gamma(x) = |b(x)|/\lambda(x)$ and we used the standard fact that $TrC \leq TrB^{-1}TrBC$

if $B, C \in \mathcal{S}_N$ and $B > 0, C \geq 0$. See also [6, Theorem 3.3, p. 33] for classical solutions and Ramaswamy and Ramaswamy [11] for the viscosity case (and continuous coefficients). In both these works, degenerate problems are considered as well. We shall turn to this issue in the next section.

In the nonlinear case, Theorem 4.1 and Corollary 4.1 seem to be known only under more restrictive assumptions about F, even for the comparison of classical solutions: Locally uniform Lipschitz continuity with respect to (z, p, M) and local boundedness of $D_p F/\lambda$ as in Gilbarg and Trudinger [6, Theorem 17.5, p. 446], uniform continuity with respect to p and various other conditions as in Jensen, Lions and Souganidis [8], or uniform ellipticity, continuity (and more) in x and Lipschitz continuity in p as in Trudinger [12]. Bardi and Da Lio [1] assume that F is lower semicontinuous with respect to all the variables and proper, plus some special scaling features. In Kawhol and Kutev [9], the uniform ellipticity and Lipschitz continuity are replaced by assumptions about some moduli of continuity (which imply the uniform continuity in (p, M)). On the other hand, comparison results for two viscosity solutions are given in [8] and [12].

5. Generalizations and degenerate elliptic problems

The proof of Theorem 3.1 reveals that the definition of points of descent can be generalized without affecting the validity of the weak minimum principle. More precisely, let $L(x) \in \mathcal{S}_N$ be a matrix function defined for $x \in \Omega$ and suppose that there are a unit vector $\ell \in \mathbb{R}^N$ and a constant $\alpha > 0$ independent of x such that

$$(L(x)\ell) \cdot \ell \geq \alpha, \quad \forall x \in \Omega. \tag{5.1}$$

Then, we may define a point of descent x_0 of $u : \Omega \to \mathbb{R}$ along L (figuratively speaking) by the condition that there are an open neighborhood Ω_{x_0} of x_0 in Ω and two constants $\beta_{x_0} > 0$ and $\gamma_{x_0} \geq 0$ such that

$$\{|D\varphi(x)| \leq \beta_{x_0} \text{ and } D^2\varphi(x) \leq \beta_{x_0} I\} \Rightarrow Tr\left(L(x)D^2\varphi(x)\right) \leq \gamma_{x_0}|D\varphi(x)|, \tag{5.2}$$

for every $x \in C_u^+(\Omega_{x_0})$ and every C^2 convex function φ subtangent to u at x.

With this definition, a simple verification shows that the proof of Theorem 3.1 goes through upon replacing the function φ_ε of that proof by $\varphi_\varepsilon(y) = \varepsilon e^{ty\cdot\ell}$ with t large enough and $\varepsilon > 0$ small enough. Indeed, $D\varphi_\varepsilon(y) = t\varphi_\varepsilon(y)\ell$, whence $|D\varphi_\varepsilon(y)| = t\varphi_\varepsilon(y)$ and $D^2\varphi_\varepsilon(y) = t^2\varphi_\varepsilon(y)\ell \otimes \ell$, so that

$$Tr\left(L(x)D^2\varphi(x)\right) = t^2\varphi_\varepsilon(y)Tr(L(x)\ell \otimes \ell) = t^2\varphi_\varepsilon(y)(L(x)\ell) \cdot \ell \geq \alpha t^2\varphi_\varepsilon(y)$$

by (5.1). If ℓ depends upon x in (5.1), then finding a suitable φ_ε may be more demanding. However, if $\ell(x) = \frac{DV(x)}{|DV(x)|}$ for some C^2 convex "potential" V with nowhere vanishing DV, then $\varphi_\varepsilon(y) = \varepsilon e^{tV(y)}$ works (for large enough t and small enough ε). The convexity of V ensures the convexity of φ_ε.

Remark 5.1. The case of a vanishing DV can also be considered. The existence of a constant $\delta_{x_0} > 0$ such that $Tr\left(L(x)D^2V(x)\right) \geq \delta_{x_0} > 0$ if $x \in \Omega_{x_0}$ and $DV(x) = 0$ is then needed in (5.2). One such example is given by $V(y) = y_1^2$ and $L(y) = L = (1, ..., 0) \otimes (1, ..., 0)$. Details are routine and left to the reader.

Definition 2.2 corresponds to $L(x) = I$, so that (5.1) holds with any unit vector ℓ and $\alpha = 1$. The proof of Theorem 3.1 uses $\ell = (-1, 0, ..., 0)$. As noted in the Introduction, an elliptic L with continuous coefficients does not change the definition of points of descent in Section 2, but a degenerate L does. Without further assumptions about L, some of the results in Section 2 are no longer valid, but the new definition makes it possible to incorporate some very degenerate problems in Section 4. It suffices to modify the condition (**GE**) in the natural way:

$(\mathbf{GE})_L$ There are functions $\lambda > 0, \beta > 0$ and $\gamma \geq 0$ on $\Omega\times\mathbb{R}\times(\mathbb{R}^N\times\mathcal{S}_N)^2$such that

$$F(x, z, p', M') - F(x, z, p, M)$$
$$\geq \lambda(x, z, p, M, p', M')\left(Tr\left(L(x)(M' - M)\right) - \gamma(x, z, p, M, p', M')|p' - p|\right),$$

whenever $|p' - p| \leq \beta(x, z, p, M, p', M')$ and $M' - M \geq 0$.

No modification of (**LB**) or (**M**) is needed. Although $L(x)$ need not be positive semidefinite, recall that the condition

$$F(x, z, p, M') - F(x, z, p, M) \geq 0$$

must hold for the classical (super/sub) solutions of

$$F(x, u(x)Du(x)D^2u(x)) = 0$$

to be viscosity (super/sub)solutions as well. Thus, this requirement should be added if $L(x)$ is not positive semidefinite in $(\mathbf{GE})_L$.

As an example, consider once again linear problems, that is,

$$F(x, z, p, M) = Tr(A(x)M) + b(x) \cdot p + c(x)z$$

with $A(x) \in \mathcal{S}_N$ positive semidefinite. Suppose that there is a unit vector $\ell \in \mathbb{R}^N$ such that $(A(x)\ell) \cdot \ell > 0$ for every $x \in \Omega$. Then $(\mathbf{GE})_L$ holds with $\beta > 0$ arbitrary, $L(x) = \frac{1}{(A(x)\ell)\cdot\ell}A(x), \lambda(x) = (A(x)\ell) \cdot \ell$ and $\gamma(x) = \frac{|b(x)|}{(A(x)\ell)\cdot\ell}$.

The condition (**LB**) thus reduces to the local boundedness of $\frac{|b(x)|}{(A(x)\ell)\cdot\ell}$. This (almost) corresponds to the most general hypotheses for the weak minimum principle for classical solutions given in [6, p. 33], except that the local boundedness of $\frac{|b(x)|}{(A(x)\ell)\cdot\ell}$ is replaced by the local boundedness of $\frac{b(x)\cdot\ell}{(A(x)\ell)\cdot\ell}$. This can also be obtained here, but Theorem 4.1 must then be specifically proved in the linear case to exploit fully the linear structure. The setting of Remark 5.1 gives other types of examples, such as $x_1^2\partial_{11}u + x_1\partial_1 u - u$.

The hypotheses $(\mathbf{GE})_L$, (**LB**) and (**M**) are well suited to problems of the form $F(x,z,p,M) = G(M) + H(x,z,p)$, with possibly degenerate G, also treated in [8] under more technical conditions. A very simple nonlinear example of this form is given by $F(p,M) = M_{11} - |p_2| - 1$, which is $\partial_{11}u - |\partial_2 u| - 1 = 0$ in [1, Example 7]. Here, $\ell = (1,0,...,0), L(x) = \ell\otimes\ell$ (independent of x) and $\lambda = \beta = \gamma = 1$ work.

While many degenerate elliptic problems fall within the class discussed above, in others the degeneracy occurs because λ in (**GE**) vanishes at some points. Remark 5.1 already covers some problems of that sort. If λ vanishes due to variables other than x, the approach may have to be modified substantially. Rather than giving a tedious list of conditions, we prefer to discuss an example. Consider

$$F(x,z,p,M) = Tr(A(p)M) - c(x)z, \tag{5.3}$$

with $A(p)\in\mathcal{S}_N$, $A(p) > 0$ if $p \neq 0$, $A(0) = 0$, A continuous in $\mathbb{R}^N$ and locally Lipschitz continuous in $\mathbb{R}^N\backslash\{0\}$ and $c(x) > 0$ (in particular, (**M**) holds). This includes m-Laplacians. While (5.3) was chosen for simplicity, it will be clear that many generalizations are possible: $A(p)$ need not be symmetric (only $A(p) > 0$ for $p \neq 0$ is needed since $Tr(A(p)M) = Tr\left(\frac{A(p)+A(p)^\top}{2}M\right)$ if $M\in\mathcal{S}_N$), $A(p)$ can be replaced by $A(x,p)$ and $c(x)z$ by $c(x,z,p)$, etc. First,

$$\begin{aligned}&F(x,z,p',M') - F(x,z,p,M)\\ &= Tr\left(A(p')(M'-M)\right) + Tr\left((A(p')-A(p))M\right)\\ &\geq Tr(A(p')(M'-M)) - N|A(p')-A(p)|\,|M|.\end{aligned}$$

Next, by writing

$$\begin{aligned}&F(x,z,p',M') - F(x,z,p,M)\\ &= Tr\left(A(p)(M'-M)\right) + Tr\left((A(p')-A(p))M'\right)\end{aligned}$$

we also get

$$F(x,z,p',M') - F(x,z,p,M) \geq Tr\left(A(p)(M'-M)\right) - N|A(p')-A(p)|\,|M'|.$$

Thus,

$$F(x,z,p',M') - F(x,z,p,M) \geq \frac{1}{2}Tr((A(p') + A(p))(M' - M)) - \frac{N}{2}|A(p') - A(p)|(|M'| + |M|).$$

With $\sigma(p)$ denoting the smallest eigenvalue of $A(p)$ and with

$$\lambda(p,p') = \frac{\sigma(p) + \sigma(p')}{2N}, \qquad \gamma(p,M,p',M') = \begin{cases} \frac{N^2|A(p')-A(p)|}{|p'-p|(\sigma(p')+\sigma(p))}(|M'| + |M|) \text{ if } p \neq p', \\ 0 \text{ if } p = p', \end{cases} \tag{5.4}$$

we have that λ vanishes only at $(0,0)$ (since $A(p) > 0$ if $p \neq 0$) and that

$$F(x,z,p',M') - F(x,z,p,M) \geq \lambda(p,p')\left(Tr(M' - M) - \gamma(p,M,p',M')|p' - p|\right) \tag{5.5}$$

for all (x,z,p,M,p',M') such that $M' - M \geq 0$.

Now, Let $v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ and $u \in C^0(\overline{\Omega})$ be a classical subsolution and a viscosity supersolution of $F = 0$, respectively and set $w = u - v$. Then, w is a C^0 supersolution of $G = 0$ in Ω, where

$$G(x,z,p,M) = F(x, v(x) + z, Dv(x) + p, D^2v(x) + M) - F(x, v(x), Dv(x), D^2v(x)). \tag{5.6}$$

In what follows, we set $\Omega^- = \{x \in \Omega : w(x) < 0\}$. To show that every point of Ω^- is a point of descent of w, we need the technical

Lemma 5.1. *Let $x_0 \in \Omega^-$ be given. There is an open neighborhood $\Omega^-_{x_0}$ of x_0 in Ω^- and a constant $\alpha_{x_0} > 0$ such that, for every $x \in C^+_w(\Omega^-_{x_0})$ and every C^2convex function φ subtangent to w at x satisfying $D^2\varphi(x) \leq I$, the inequality*

$$|Dv(x)| + |Dv(x) + D\varphi(x)| \geq \alpha_{x_0} \tag{5.7}$$

holds.

Proof. The result is clear if $Dv(x_0) \neq 0$. If $Dv(x_0) = 0$, choose $\Omega^-_{x_0} \subset \Omega^-$ such that

$$cw + Tr(A(Dv)D^2v) \leq -\varepsilon < 0 \text{ on } \Omega^-_{x_0}. \tag{5.8}$$

This is possible since the left-hand side is strictly negative at x_0 (recall $A(0) = 0$, $c > 0, x_0 \in \Omega^-$ and w is continuous). Let $x \in C^+_w(\Omega^-_{x_0})$ and let φ be a C^2 convex function subtangent to w at x. Since $w - \varphi$ has a local minimum at x and w is a viscosity supersolution of $G = 0$, we have $G(x, w(x), D\varphi(x), D^2\varphi(x)) \leq 0$. By (5.3), (5.6) and (5.8) this means

$$Tr\left(A(Dv(x) + D\varphi(x))(D^2v(x) + D^2\varphi(x))\right) \leq c(x)w(x) + Tr\left(A(Dv(x))D^2v(x)\right) \leq -\varepsilon. \quad (5.9)$$

By contradiction, if $\alpha_{x_0} > 0$ does not exist after shrinking $\Omega^-_{x_0}$, there is a sequence $x_n \in C^+_w(\Omega^-_{x_0})$ with $x_n \to x_0$ and corresponding C^2 convex functions φ_n subtangent to w at x_n such that $(0 \leq) D^2\varphi_n(x_n) \leq I$ and $|Dv(x_n)| + |Dv(x_n) + D\varphi_n(x_n)| \to 0$. Hence,

$$Tr\left(A(Dv(x_n) + D\varphi_n(x_n))(D^2v(x_n) + D^2\varphi_n(x_n))\right) \to 0$$

since $A(0) = 0$ and the sequence $D^2v(x_n) + D^2\varphi_n(x_n)$ is bounded (due to $0 \leq D^2\varphi_n(x_n) \leq I$). But this contradicts (5.9) for $x = x_n$ and $\varphi = \varphi_n$ if n is large enough. □

Let $x_0, \Omega^-_{x_0}, x$ and φ be as in Lemma 5.1. Since λ vanishes only at $(0, 0)$, we have

$$\lambda(Dv(x), Dv(x) + D\varphi(x)) > 0, \quad (5.10)$$

by (5.7). Once again, $G(x, w(x), D\varphi(x), D^2\varphi(x)) \leq 0$ and since G satisfies (**M**) and $w(x) < 0$, it follows that $G(x, 0, D\varphi(x), D^2\varphi(x)) \leq 0$. Observe also that $G(x, 0, 0, 0) = 0$. Then, by (5.5), (5.6) and (5.10),

$$\Delta\varphi(x) \leq \gamma(Dv(x), D^2v(x), Dv(x) + D\varphi(x), D^2v(x) + D^2\varphi(x))|D\varphi(x)|. \quad (5.11)$$

Now, let $K_{x_0} > 0$ be a Lipschitz constant for A in the vicinity of $Dv(x_0)$. Since Dv is continuous, $\Omega^-_{x_0}$ may be shrunk further so that if $|p| \leq \beta_{x_0}$ with $\beta_{x_0} > 0$ small enough, then

$$|A(Dv(x) + p) - A(Dv(x))| \leq K_{x_0}|p|$$

for every $x \in \Omega^-_{x_0}$. In particular, if $|D\varphi(x)| \leq \beta_{x_0}$, the formula for γ in (5.4) yields

$$\gamma(Dv(x), D^2v(x), Dv(x) + D\varphi(x), D^2v(x) + D^2\varphi(x)) \leq \frac{N^2K_{x_0}\left(|D^2v(x)| + |D^2v(x) + D^2\varphi(x)|\right)}{\sigma(Dv(x)) + \sigma(Dv(x) + D\varphi(x))}.$$

By Lemma 5.1, the denominator is bounded away from 0 ($\sigma(p)$ is continuous in p and vanishes only at 0) and the numerator is bounded since v is C^2 and $0 \leq D^2\varphi(x) \leq I$ by hypothesis. Thus,

$$\gamma(Dv(x), D^2v(x), Dv(x) + D\varphi(x), D^2v(x) + D^2\varphi(x)) \leq \gamma_{x_0},$$

where $\gamma_{x_0} \geq 0$ is a constant. Then, $\Delta\varphi(x) \leq \gamma_{x_0}|D\varphi(x)|$ by (5.11). This holds for every $x \in C_w^+(\Omega_{x_0}^-)$ and every C^2 convex function φ subtangent to w at x such that $|D\varphi(x)| \leq \beta_{x_0}$ and $D^2\varphi(x) \leq I$.

After replacing β_{x_0} by $\min(1, \beta_{x_0})$, we infer that $\Delta\varphi(x) \leq \gamma_{x_0}|D\varphi(x)|$ for every $x \in C_w^+(\Omega_{x_0}^-)$ and every C^2 convex function φ subtangent to w at x such that $|D\varphi(x)| \leq \beta_{x_0}$ and $D^2\varphi(x) \leq \beta_{x_0} I$. This shows that every point of Ω^- is a point of descent for w (Definition 2.2).

If $\Omega^- \neq \emptyset$, it follows from the above and Theorem 3.1 (assuming Ω is bounded) that w achieves its minimum value in $\overline{\Omega^-}$, and hence in $\overline{\Omega}$, at some point $x_* \in \partial\Omega^-$. Since $w = 0$ in $\Omega \cap \partial\Omega^-$, the point x^* must lie on $\partial\Omega$, which yields $\max_{\partial\Omega}(u-v)_- = \max_{\overline{\Omega}}(u-v)_-$. This relation remains trivially true if $\Omega^- = \emptyset$.

Thus, Theorem 4.1 is still valid for *continuous* u in spite of the elliptic degeneracy. The continuity of u is not needed in [1], where scaling properties available only in restricted forms of (5.3) are assumed. The remark that the loss of ellipticity at $(p, p') = (0, 0)$ is inconsequential is also made by Pucci and Serrin [10, Lemma 5] in the context of $C^1(\Omega) \cap C^0(\overline{\Omega})$ distribution solutions of

$$\nabla \cdot (a(|Du|)Du) - B(x, u, Du) \leq 0$$

with scalar a (so that $A(p) = a(p)I$ in (5.3)).

6. Absolute points of descent and strong minimum principle

In this section, we show that whether a given lower semicontinuous function satisfies a strong minimum principle depends upon the invariance (for that function) of the concept of point of descent under changes of variable. That changes of variable have something to do with the problem is plausible since the property of minimum boundary value, unlike the definition of points of descent, is unaffected by changes of coordinates. This leads to the following definition.

Definition 6.1. Let $u : \Omega \to \mathbb{R}$ be a given function. We shall say that the point $x_0 \in \Omega$ is an absolute point of descent of u if, given any open subset $\omega \subset \Omega$ containing x_0 and any C^2 diffeomorphism H from some open subset $\widetilde{\omega} \subset \mathbb{R}^N$ onto ω, the point $\tilde{x}_0 = H^{-1}(x_0)$ is a point of descent of $u \circ H$.

Absolute points of descent are preserved by the "min" operation. This follows by a straightforward variant of the proof of Proposition 2.4 . Definition 6.1 is less restrictive than it may appear. A first criterion is as follows.

Proposition 6.1. *Let $u : \Omega \to \mathbb{R}$ and $x_0 \in \Omega$ be given. Suppose that there are an open neighborhood Ω_{x_0} of x_0 in Ω and constants $\beta_{x_0} > 0$ and*

$\gamma_{x_0} \geq 0$ *such that* $\Delta\varphi(x) \leq \gamma_{x_0}|D\varphi(x)|$ *for every* $x \in \Omega_{x_0}$ *and every* C^2 *function* φ *subtangent to* u *at* x (convex or not) *such that* $|D\varphi(x)| \leq \beta_{x_0}$ *and* $|D^2\varphi(x)| \leq \beta_{x_0}$. *Then,* x_0 *is an absolute point of descent of* u.

Proof. We use the notation of Definition 6.1. We must show that there are an open neighborhood $\widetilde{\omega}_{\tilde{x}_0}$ of $\tilde{x}_0$ in $\widetilde{\omega}$ and constants $\tilde{\beta}_{\tilde{x}_0} > 0$ and $\tilde{\gamma}_{\tilde{x}_0} \geq 0$ such that $\Delta\tilde{\varphi}(\tilde{x}) \leq \tilde{\gamma}_{\tilde{x}_0}|D\tilde{\varphi}(\tilde{x})|$ whenever $\tilde{x} \in \widetilde{\omega}_{\tilde{x}_0}$ and $\tilde{\varphi}$ is a C^2 convex function subtangent to $\tilde{u} = u \circ H$ at $\tilde{x}$ [6] satisfying $|D\tilde{\varphi}(\tilde{x})| \leq \tilde{\beta}_{\tilde{x}_0}$ and $D^2\tilde{\varphi}(\tilde{x}) \leq \tilde{\beta}_{\tilde{x}_0}I$. For simplicity, we use the notation $x = H(\tilde{x})$, so that $\tilde{x} = H^{-1}(x)$, and set

$$\varphi = \tilde{\varphi} \circ H^{-1}. \tag{6.1}$$

It is obvious that φ is subtangent to u at x. By the chain rule,

$$D\varphi(x) = DH^{-1}(x)^\top D\tilde{\varphi}(\tilde{x}), \tag{6.2}$$

$$D\tilde{\varphi}(\tilde{x}) = DH(\tilde{x})^\top D\varphi(x), \tag{6.3}$$

$$D^2\varphi(x) = DH^{-1}(x)^\top D^2\tilde{\varphi}(\tilde{x})DH^{-1}(x) + D^2H^{-1}(x) \cdot D\tilde{\varphi}(\tilde{x}), \tag{6.4}$$

where

$$D^2H^{-1}(x) \cdot D\tilde{\varphi}(\tilde{x}) = \sum_{k=1}^{N} \partial_k\tilde{\varphi}(\tilde{x})D^2(H^{-1})_k(x)$$

and, with a similar notation,

$$D^2\tilde{\varphi}(\tilde{x}) = DH(\tilde{x})^\top D^2\varphi(x)DH(\tilde{x}) + D^2H(\tilde{x}) \cdot D\varphi(x). \tag{6.5}$$

By substitution of (6.3) into (6.4), we find

$$D^2\varphi(x) - D^2H^{-1}(x) \cdot DH(\tilde{x})^\top D\varphi(x) = DH^{-1}(x)^\top D^2\tilde{\varphi}(\tilde{x})DH^{-1}(x),$$

whence, since $\tilde{\varphi}$ is convex,

$$D^2\varphi(x) - D^2H^{-1}(x) \cdot DH(\tilde{x})^\top D\varphi(x) \geq 0. \tag{6.6}$$

Now, by (6.5),

$$\Delta\tilde{\varphi}(\tilde{x}) = Tr DH(\tilde{x})^\top D^2\varphi(x)DH(\tilde{x}) + Tr\left(D^2H(\tilde{x}) \cdot D\varphi(x)\right),$$

which may be rewritten as

$$\begin{aligned}\Delta\tilde{\varphi}(\tilde{x}) = {} & Tr\left(DH(\tilde{x})^\top \left(D^2\varphi(x) - D^2H^{-1}(x) \cdot DH(\tilde{x})^\top D\varphi(x)\right) DH(\tilde{x})\right)\\ & + Tr DH(\tilde{x})^\top \left(D^2H^{-1}(x) \cdot DH(\tilde{x})^\top D\varphi(x)\right) DH(\tilde{x})\\ & + Tr\left(D^2H(\tilde{x}) \cdot D\varphi(x)\right).\end{aligned} \tag{6.7}$$

[6] So that $\tilde{x} \in C^+_{\tilde{u}}(\tilde{\omega})$.

By (6.6), the first term in the right-hand side of (6.7) is majorized by

$$Tr(DH(\tilde{x})DH(\tilde{x})^\top)Tr\left(D^2\varphi(x) - D^2H^{-1}(x)\cdot DH(\tilde{x})^\top D\varphi(x)\right)$$
$$= -Tr(DH(\tilde{x})DH(\tilde{x})^\top)Tr\left(D^2H^{-1}(x)\cdot DH(\tilde{x})^\top D\varphi(x)\right)$$
$$+ Tr(DH(\tilde{x})DH(\tilde{x})^\top)\Delta\varphi(x)$$

and hence by

$$Tr(DH(\tilde{x})DH(\tilde{x})^\top)\Delta\varphi(x)$$
$$+ NTr(DH(\tilde{x})DH(\tilde{x})^\top)|D^2H^{-1}(x)|\,|DH(\tilde{x})|\,|D\varphi(x)|,$$

where we used $TrAB = TrBA, |TrA| \le N|A|, |A| = |A^\top|$ and $TrAB \le TrATrB$ for $A, B \in \mathcal{S}_N$ with $A \ge 0, B \ge 0$. A similar procedure provides

$$Tr\left(DH(\tilde{x})^\top(D^2H^{-1}(x)\cdot DH(\tilde{x})^\top D\varphi(x))DH(\tilde{x})\right) + Tr\left(D^2H(\tilde{x})\cdot D\varphi(x)\right)$$
$$\le N\left(|DH(\tilde{x})|^3|D^2H^{-1}(x)| + |D^2H(\tilde{x})|\right)|D\varphi(x)|.$$

As a result, from (6.7) and the above,

$$\Delta\tilde{\varphi}(\tilde{x}) \le Tr(DH(\tilde{x})DH(\tilde{x})^\top)\Delta\varphi(x) + k(\tilde{x})|D\varphi(x)|, \tag{6.8}$$

with (making the substitution $x = H(\tilde{x})$)

$$k(\tilde{x}) = N\left(Tr(DH(\tilde{x})DH(\tilde{x})^\top) + |DH(\tilde{x})|^2\right)|D^2H^{-1}(H(\tilde{x}))|\,|DH(\tilde{x})|$$
$$+ N|D^2H(\tilde{x})| \ge 0. \tag{6.9}$$

We now make a choice for the neighborhood $\widetilde{\omega}_{\tilde{x}_0}$ and the constant $\tilde{\beta}_{\tilde{x}_0}$. Since H is a diffeomorphism, we may choose $\widetilde{\omega}_{\tilde{x}_0}$ such that $H(\widetilde{\omega}_{\tilde{x}_0}) \subset \Omega_{x_0}$ and that the derivatives up to order 2 of H and H^{-1} are bounded in $\widetilde{\omega}_{\tilde{x}_0}$ and $H(\widetilde{\omega}_{\tilde{x}_0})$, respectively. This implies that, in (6.8), both $Tr(DH(\tilde{x})DH(\tilde{x})^\top)$ and $k(\tilde{x})$ (see (6.9)) are bounded and also, by (6.2) and (6.4), that $\tilde{\beta}_{\tilde{x}_0} > 0$ can be found such that (recall that $\tilde{\varphi}$ is convex, hence $D^2\tilde{\varphi}(\tilde{x}) \ge 0$)

$$\{\tilde{x} \in \widetilde{\omega}_{\tilde{x}_0}, |D\tilde{\varphi}(\tilde{x})| \le \tilde{\beta}_{\tilde{x}_0} \text{ and } D^2\tilde{\varphi}(\tilde{x}) \le \tilde{\beta}_{\tilde{x}_0}I\} \Rightarrow$$
$$\{|D\varphi(x)| \le \beta_{x_0} \text{ and } |D^2\varphi(x)| \le \beta_{x_0}\}.$$

Thus, from the hypothesis of the proposition,

$$\{\tilde{x} \in \widetilde{\omega}_{\tilde{x}_0}, |D\tilde{\varphi}(\tilde{x})| \le \tilde{\beta}_{\tilde{x}_0} \text{ and } D^2\tilde{\varphi}(\tilde{x}) \le \tilde{\beta}_{\tilde{x}_0}I\} \Rightarrow \Delta\varphi(x) \le \gamma_{x_0}|D\varphi(x)|,$$

whence, by (6.8),

$$\{\tilde{x} \in \widetilde{\omega}_{\tilde{x}_0}, |D\tilde{\varphi}(\tilde{x})| \le \tilde{\beta}_{\tilde{x}_0} \text{ and } D^2\tilde{\varphi}(\tilde{x}) \le \tilde{\beta}_{\tilde{x}_0}I\} \Rightarrow$$

$$\Delta\tilde{\varphi}(\tilde{x}) \leq \Big(\gamma_{x_0} Tr(DH(\tilde{x})DH(\tilde{x})^\top) + k(\tilde{x})\Big)\,|D\varphi(x)|. \quad (6.10)$$

From the previous remark that $Tr(DH(\tilde{x})DH(\tilde{x})^\top)$ and $k(\tilde{x})$ are uniformly bounded for $\tilde{x} \in \widetilde{\omega}_{\tilde{x}_0}$, we further obtain that $\Delta\tilde{\varphi}(\tilde{x}) \leq C(\gamma_{x_0} + 1)|D\varphi(x)|$ in (6.10), where $C > 0$ is a constant. Finally, going back to (6.2) and since $|DH^{-1}|$ is bounded in $H(\widetilde{\omega}_{\tilde{x}_0})$, we get $\Delta\tilde{\varphi}(\tilde{x}) \leq C(\gamma_{x_0}+1)|D\tilde{\varphi}(\tilde{x})|$ in (6.10), after modifying $C > 0$ accordingly. This means that $\tilde{\gamma}_{\tilde{x}_0} = C(\gamma_{x_0} + 1)$ works. □

The assumption that $\tilde{\varphi}$ is convex is essential to the above proof: Otherwise, (6.6) does not hold and the proof breaks down. If u is C^2, the condition of Proposition 6.1 holds (for instance) if $Du(x_0) \neq 0$ or if $Du(x_0) = 0$ and $D^2u(x_0) \not\geq 0$ (in particular, if $\Delta u(x_0) < 0$). Together with Lemma 2.1, Proposition 6.1 yields

Corollary 6.1. *If $N = 1$, every point of descent is an absolute point of descent.*

That Corollary 6.1 is not true when $N > 1$ can be seen on Example 3.1: By locally straightening out the boundary ∂B, the transformed function $\tilde{u}$ exhibits properties similar to those of the one-dimensional Example 2.4 and its points of discontinuity are no longer points of descent. (Alternatively, Theorem 6.1 below gives an immediate indirect argument.)

Functions whose domains consist of absolute point of descent include once again C^1 functions with a nowhere vanishing derivative, or, more generally, functions satisfying condition (ii) of Proposition 2.2 (use Proposition 6.1), concave functions (if u is concave and φ is C^2 and subtangent to u at x, then φ is also subtangent to an affine function at x, so that $D^2\varphi(x) \leq 0$ and hence $\Delta\varphi(x) \leq 0$; then use Proposition 6.1) and the minimum of two or more such functions.

If "point of descent" is replaced by "absolute point of descent" in Theorem 3.1, then the weak minimum principle becomes a strong one. This is Theorem 6.1 below, whose proof is somewhat of a hybrid between that of Theorem 3.1 and the classical proof of strong principles. Here, the annulus construction is a bit more elaborate.

Theorem 6.1. *Let Ω be connected, possibly unbounded. If $u : \Omega \to \mathbb{R}$ is lower semicontinuous*[7] *and every point of Ω is an absolute point of descent of u, then u cannot achieve a minimum value at any point of Ω unless it is constant.*

[7] The function u need not be defined on $\partial\Omega$ for this result.

Proof. Assuming, by contradiction, that u achieves its minimum value m at a point of Ω but that u is not equal to m in Ω, we can find (by the standard argument) an open ball $B \subset\subset \Omega$ such that $u > m$ in B while $u = m$ at some point of ∂B. By replacing B by a smaller ball $B_0 \subset B$ with a different center and radius $R_0 > 0$, we may then ensure that $u > m$ in B_0 while $u = m$ at exactly *one* point $x_* \in \partial B_0$. Then, after removing a closed ball with same center and radius $\rho_0 < R_0$, we obtain an open annulus $\mathcal{A}_0$ with $\overline{\mathcal{A}}_0 \subset \Omega$ such that $u > m$ in $\overline{\mathcal{A}}_0$ except at the single point x_* of its outer boundary, where $u(x_*) = m$.

After an affine change of coordinates, we may assume that $\mathcal{A}_0 = \{x \in \Omega : \rho_0 < |x| < R_0\}$ and that $x_* = (R_0, 0, ..., 0)$. Let $\mathcal{A}'$ be a larger annulus with the same inner radius ρ_0 and with outer radius $R' > R_0$. Choose R' so that $\mathcal{A}' \subset \Omega$, which is possible since $\overline{\mathcal{A}}_0 \subset \Omega$. Denote by $\mathcal{A}'_+$ the (open) half-annulus $\mathcal{A}' \cap \{x_1 > 0\}$. Clearly, $\mathcal{A}'_+$ is smoothly diffeomorphic to the rectangle $\widetilde{\mathcal{R}'} = (\rho_0, R') \times \widetilde{\mathcal{Q}} \subset \mathbb{R}^N$ where $\widetilde{\mathcal{Q}} = (-1,1)^{N-1}$, via a diffeomorphism $H : \widetilde{\mathcal{R}'} \to \mathcal{A}'_+$ extending to a homeomorphism from $\overline{\widetilde{\mathcal{R}'}}$ to $\overline{\mathcal{A}'}_+$ (hence mapping $\partial\widetilde{\mathcal{R}'}$ to $\partial\mathcal{A}'_+$) and such that

$$H(R_0, 0) = x_* \text{ and } |H(r, q)| = r. \tag{6.11}$$

If $N = 1$, then $H(r) = r$ works (no change of variable is needed). If $N = 2$, a formula for H is $H(r,q) = (r\cos\frac{\pi q}{2}, r\sin\frac{\pi q}{2})$. If $N > 2$, use spherical coordinates and next a chart to transform the unit open half-sphere into $(-1,1)^{N-1}$(this is why a half-annulus is being used; an annulus involves the full unit sphere which cannot be transformed into a rectangle).

Choose $\rho_0 < \rho < R_0$ and consider the compact subsets

$$\widetilde{K} = [\rho, \frac{R' + R_0}{2}] \times [-\frac{1}{2}, \frac{1}{2}]^{N-1} \subset \widetilde{\mathcal{R}'}, \qquad K = H(\widetilde{K}) \subset \mathcal{A}'_+. \tag{6.12}$$

Since every point of Ω (hence of $\mathcal{A}'_+ \subset \Omega$) is an absolute point of descent of u, every point of $\widetilde{\mathcal{R}'}$ is a point of descent of $\tilde{u} = u \circ H$. Hence, we can cover $\widetilde{K}$ with finitely many open neighborhoods $\widetilde{\omega}_i$ such that for each index i there are $\tilde{\beta}_i > 0$ and $\tilde{\gamma}_i \geq 0$ with the property that if $\tilde{x} \in C^+_{\tilde{u}}(\widetilde{\omega}_i)$ and $\tilde{\varphi}$ is C^2 convex and subtangent to $\tilde{u}$ at $\tilde{x}$, then $|D\tilde{\varphi}(\tilde{x})| \leq \tilde{\beta}_i$ and $D^2\tilde{\varphi}(\tilde{x}) \leq \tilde{\beta}_i I$ together imply $\Delta\tilde{\varphi}(\tilde{x}) \leq \tilde{\gamma}_i |D\tilde{\varphi}(\tilde{x})|$. Let $\widetilde{\omega} = \cup_i \widetilde{\omega}_i \supset \widetilde{K}, \tilde{\beta} = \min_i \tilde{\beta}_i > 0$ and $\tilde{\gamma} = \max_i \tilde{\gamma}_i \geq 0$, so that

$$\forall \tilde{x} \in C^+_{\tilde{u}}(\widetilde{\omega}), \quad \{|D\tilde{\varphi}(\tilde{x})| \leq \tilde{\beta} \text{ and } D^2\tilde{\varphi}(\tilde{x}) \leq \tilde{\beta} I\} \Rightarrow \Delta\tilde{\varphi}(\tilde{x}) \leq \tilde{\gamma}|D\tilde{\varphi}(\tilde{x})|, \tag{6.13}$$

whenever $\tilde{\varphi}$ is a C^2 convex function subtangent to $\tilde{u}$ at $\tilde{x}$.

We now choose another annulus $\mathcal{A}$ "intermediate" between $\mathcal{A}_0$ and $\mathcal{A}'$, with the same inner radius ρ_0 and outer radius R satisfying $R_0 < R < \frac{R'+R_0}{2} < R'$. We denote by $\mathcal{A}_+$ the corresponding (open) half annulus. By decreasing R towards R_0, we may ensure that all the minimizers of u on $\overline{\mathcal{A}}_+$ lie in the interior of K (see (6.12)). Indeed, as $R \downarrow R_0$, these minimizers exist since u is *lsc* in $\overline{\mathcal{A}}_+ \subset \Omega$ and must tend to the unique minimizer $x_* \in \overline{\mathcal{A}}_{0+}$ of u, while $x_* = H(R_0, 0)$ is an interior point of K since $(R_0, 0)$ is an interior point of $\widetilde{K}$.

The half-annulus $\mathcal{A}_+$ is the image $H(\widetilde{\mathcal{R}})$ of the rectangle $\widetilde{\mathcal{R}} = (\rho_0, R) \times \widetilde{\mathcal{Q}} \subset \widetilde{\mathcal{R}'}$ and $\overline{\mathcal{A}}_+ = H(\overline{\widetilde{\mathcal{R}}})$. The boundary $\partial\mathcal{A}_+$ consists of the "flat" part $\overline{\mathcal{A}}_+ \cap \{x_1 = 0\}$ and the two (closed) half-spheres $S_+^{\rho_0}$ and S_+^R. From (6.11), it follows that $S_+^R = H(\{R\} \times [-1,1]^{N-1}), S_+^{\rho_0} = H(\{\rho_0\} \times [-1,1]^{N-1})$ and that $\overline{\mathcal{A}}_+ \cap \{x_1 = 0\} = H([\rho_0, R] \times \partial\widetilde{\mathcal{Q}})$. In particular, since $\widetilde{K}$ does not intersect $\{\rho_0\} \times [-1,1]^{N-1} \cup [\rho_0, R] \times \partial\widetilde{\mathcal{Q}}$ (see (6.12)), it follows that K does not intersect $\overline{\mathcal{A}}_+ \cap \{x_1 = 0\} \cup S_+^{\rho_0}$. Together with the earlier remark that the minimizers of u on $\overline{\mathcal{A}}_+$ are in (the interior of) K, this implies

$$u(x) \geq m_+ > m, \qquad \forall x \in \overline{\mathcal{A}}_+ \cap \left(\{x_1 = 0\} \cup S_+^{\rho_0}\right). \tag{6.14}$$

At this stage, choose $t > \max(\gamma, 1)$. Given any $0 < \varepsilon < m_+ - m$ with m_+ as in (6.14), set $\varphi_\varepsilon(x) = \varepsilon e^{-t|x|}$. If $x \in S_+^R$, we have

$$\begin{aligned} u(x) - \varphi_\varepsilon(x) &= u(x) - \varepsilon e^{-tR} \geq m - \varepsilon e^{-tR} \\ &= u(x_*) - \varepsilon e^{-tR} > u(x_*) - \varepsilon e^{-tR_0} = u(x_*) - \varphi_\varepsilon(x_*). \end{aligned}$$

On the other hand, if $x \in \overline{\mathcal{A}}_+ \cap \{x_1 = 0\} \cup S_+^{\rho_0}$, then, by (6.14) and since $0 < \varphi_\varepsilon < \varepsilon$,

$$u(x) - \varphi_\varepsilon(x) > u(x) - \varepsilon \geq m_+ - \varepsilon > m = u(x_*) > u(x_*) - \varphi_\varepsilon(x_*).$$

As a result, $u(x_*) - \varphi_\varepsilon(x_*)$ is strictly less than any value achieved by $u - \varphi_\varepsilon$ on all of $\partial\mathcal{A}_+$ and therefore $u - \varphi_\varepsilon$ achieves its minimum value on $\overline{\mathcal{A}}_+$ only at points of $\mathcal{A}_+$.

We claim that, in addition, K contains all the minimizers of $u - \varphi_\varepsilon$ on $\overline{\mathcal{A}}_+$ if $\varepsilon > 0$ is small enough. (Once again, the existence of such minimizers follows from the lower semicontinuity of u in $\overline{\mathcal{A}}_+ \subset \Omega$.) Otherwise, by considering a sequence $\varepsilon_n \to 0$, an immediate contradiction arises with the *interior* of K containing all the minimizers of u on $\overline{\mathcal{A}}_+$. In terms of the functions $\tilde{u} = u \circ H$ and $\tilde{\varphi}_\varepsilon = \varphi_\varepsilon \circ H$, the properties of $u - \varphi_\varepsilon$ obtained so far mean that the minimum of $\tilde{u} - \tilde{\varphi}_\varepsilon$ on $\overline{\widetilde{\mathcal{R}}}$ is achieved only at points of $\widetilde{K} \cap \widetilde{\mathcal{R}}$.

By definition, $\tilde{\varphi}_\varepsilon(r,q) = \varphi_\varepsilon(H(r,q)) = \varepsilon e^{-t|H(r,q)|}$ and $|H(r,q)| = r$ from (6.11). Thus, $\tilde{\varphi}_\varepsilon(r,q) = \varepsilon e^{-tr}$ for every $(r,q) \in \widetilde{\mathcal{R}'}$ and in particular $(r,q) \in \widetilde{\mathcal{R}}$. As a result, $\tilde{\varphi}_\varepsilon$ is convex[8], so that a point $\tilde{x} = (r,q) \in \widetilde{\mathcal{R}}$ where $\tilde{u} - \tilde{\varphi}_\varepsilon$ achieves its minimum value is a point of convexity of $\tilde{u}$ (because $\widetilde{\mathcal{R}}$ is open). Since also $\tilde{x} \in \widetilde{K} \subset \widetilde{\omega}$, it follows from (6.13) that $\Delta\tilde{\varphi}_\varepsilon(\tilde{x}) \leq \tilde{\gamma}|D\tilde{\varphi}_\varepsilon(\tilde{x})|$ must hold if the conditions $|D\tilde{\varphi}_\varepsilon(\tilde{x})| \leq \beta$ and $D^2\tilde{\varphi}_\varepsilon(\tilde{x}) \leq \beta I$ are met. They are met if also $\varepsilon \leq \frac{\beta e^{t\rho_0}}{t^2}$ (recall $t > 1$) because $r \geq \rho_0, |D\tilde{\varphi}_\varepsilon(\tilde{x})| = \varepsilon t e^{-tr}$ and $D^2\tilde{\varphi}_\varepsilon(\tilde{x})$ is the $N \times N$ matrix with first entry $\varepsilon t^2 e^{-tr}$ and all other entries 0. But $\Delta\tilde{\varphi}_\varepsilon(\tilde{x}) \leq \tilde{\gamma}|D\tilde{\varphi}_\varepsilon(\tilde{x})|$ is just $\varepsilon t^2 e^{-tr} \leq \varepsilon\gamma t e^{-tr}$, i.e., $t \leq \gamma$, which contradicts the choice $t > \gamma$. This completes the proof. □

The (trivial) analog of Corollary 3.1 is

Corollary 6.2. *Let $u : \Omega \to \mathbb{R}$ be lower semicontinuous. If $x_0 \in \Omega$ is an absolute point of descent of u, every neighborhood of x_0 in Ω contains a point x such that $u(x) < u(x_0)$.*

When $N = 1$, Corollary 6.1 and Theorem 6.1 provide

Corollary 6.3. *Let $J \subset \mathbb{R}$ be an interval, possibly unbounded. If $u : J \to \mathbb{R}$ is lower semicontinuous and every point of J is a point of descent of u, then u does not achieve a minimum value in J unless it is constant.*

7. Strong minimum principle for viscosity solutions

We return to the elliptic problems (and the notation) of Section 4. Let $u : \Omega \to \mathbb{R}$ be a viscosity supersolution of $F = 0, \omega \subset \Omega$ an open subset and $x \in \omega$. If φ is a C^2 function subtangent to u at x, then

$$F(x, u(x), D\varphi(x), D^2\varphi(x)) \leq 0. \tag{7.1}$$

Now, let H denote a C^2 diffeomorphism from some open subset $\widetilde{\omega} \subset \mathbb{R}^N$ onto ω and set $x = H(\tilde{x}), \tilde{u} = u \circ H$ and $\tilde{\varphi} = \varphi \circ H$. The derivatives $D\varphi(x)$ and $D^2\varphi(x)$ are given in terms of $D\tilde{\varphi}(\tilde{x}), D^2\tilde{\varphi}(\tilde{x})$, respectively, by the formulas (6.2) and (6.4), which we shall here write in the condensed form

$$D\varphi(x) = U^\top(\tilde{x})D\tilde{\varphi}(\tilde{x}), \quad D^2\varphi(x) = U^\top(\tilde{x})D^2\tilde{\varphi}(\tilde{x})U(\tilde{x}) + T(\tilde{x}) \cdot D\tilde{\varphi}(\tilde{x}), \tag{7.2}$$

where

$$U(\tilde{x}) = DH^{-1}(H(\tilde{x})) \qquad \text{and} \qquad T(\tilde{x}) = D^2H^{-1}(H(\tilde{x})). \tag{7.3}$$

[8]In contrast with φ_ε, at least if $N > 1$, and this property is the only purpose of changing variables.

Above, $U(\tilde{x})$ is identified with an $N \times N$ matrix and $T(\tilde{x})$ with

$$(D^2(H^{-1})_1(H(\tilde{x})), \ldots, D^2(H^{-1})_N(H(\tilde{x}))) \in (\mathcal{S}_N)^N$$

and for $T \in (\mathcal{S}_N)^N$ and $p \in \mathbb{R}^N$ the "dot" product $T \cdot p$ is $\sum_{k=1}^N p_k T_k \in \mathcal{S}_N$. With this notation, (7.1) becomes

$$F(H(\tilde{x}), \tilde{u}(\tilde{x}), U^\top(\tilde{x}) D\tilde{\varphi}(\tilde{x}), U^\top(\tilde{x}) D^2\tilde{\varphi}(\tilde{x}) U(\tilde{x}) + T(\tilde{x}) \cdot D\tilde{\varphi}(\tilde{x})) \leq 0.$$

This inequality can be rewritten as

$$\widetilde{F}(\tilde{x}, \tilde{u}(\tilde{x}), D\tilde{\varphi}(\tilde{x}), D^2\tilde{\varphi}(\tilde{x})) \leq 0, \tag{7.4}$$

where

$$\widetilde{F}(\tilde{x}, \tilde{z}, \tilde{p}, \widetilde{M}) = F(H(\tilde{x}), \tilde{z}, U^\top(\tilde{x})\tilde{p}, U^\top(\tilde{x})\widetilde{M}U(\tilde{x}) + T(\tilde{x}) \cdot \tilde{p}). \tag{7.5}$$

Evidently, $\tilde{\varphi}$ is C^2 and subtangent to $\tilde{u}$ at $\tilde{x}$. Conversely, any such $\tilde{\varphi}$ yields $\varphi = \tilde{\varphi} \circ H^{-1}$ of class C^2 and subtangent to u at x. Since $\tilde{x} \in \widetilde{\omega}$ is arbitrary, (7.4) means that $\tilde{u}$ is a supersolution of $\widetilde{F} = 0$ in $\widetilde{\omega}$. In particular, from the proof of Lemma 4.1, all the points of $\widetilde{\omega}$ are points of descent of $-\tilde{u}_-$ if $F(x, 0, 0, 0) \geq 0$ for every $x \in \Omega$ (hence $\widetilde{F}(\tilde{x}, 0, 0, 0) \geq 0$ in $\widetilde{\omega}$) and $\widetilde{F}$ satisfies (**GE**), (**LB**) and (**M**) of Section 4. If this is true for every choice of ω and H, it follows from Definition 6.1 and the relation $-\tilde{u}_- = (-u_-) \circ H$ that every point of Ω is an absolute point of descent of $-u_-$. If so, the strong minimum principle of Theorem 6.1 asserts that if also u is lower semicontinuous in Ω, so that the same thing is true of $-u_-$, then $-u_-$ cannot achieve a minimum value in Ω unless it is constant. But if $-u_-$ is constant and strictly negative, then $u = -u_-$ and hence u is constant. Thus, u cannot achieve a strictly negative minimum in Ω unless it is constant.

The assumption that $\widetilde{F}$ satisfies (**GE**), (**LB**) and (**M**) for all possible choices of ω and diffeomorphism H is not as restrictive as one might think, as we shall check on several general examples later on. In particular, it is obvious that condition (**M**) holds for every $\widetilde{F}$ if and only if it holds for F. Next, U and T depend continuously upon $\tilde{x}$ and $U(\tilde{x})$ is invertible, which motivates the "absolute" assumption (**A**) below. For simplicity, we have dropped the "$\sim$" notation in $\widetilde{\omega}$ and in the $\tilde{x}, \tilde{z}, \tilde{p}$ and $\widetilde{M}$ variables and $\mathcal{GL}_N$ denotes the set of invertible $N \times N$ matrices.

(**A**) F satisfies condition (**M**) and for every open subset $\omega \subset \mathbb{R}^N$ and every continuous field $(H, U, T) : \omega \to \Omega \times \mathcal{GL}_N \times (\mathcal{S}_N)^N$, the mapping

$$\begin{aligned}(x, z, p, M) \in \omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \mapsto \widetilde{F}(x, z, p, M)\\ = F(H(x), z, U^\top(x)p, U^\top(x)MU(x) + T(x) \cdot p)\end{aligned} \tag{7.6}$$

satisfies the conditions (**GE**) and (**LB**).

Before giving examples, we summarize the previous considerations in the form of the following strong minimum principle.

Theorem 7.1. *Let Ω be an arbitrary open connected subset of $\mathbb{R}^N$ and assume that $F(x,0,0,0) \geq 0$ for every $x \in \Omega$ and that (**A**) holds. Let $u : \Omega \to \mathbb{R}$ be a lower semicontinuous viscosity supersolution of $F = 0$. If u achieves a strictly negative minimum at some point of Ω, then u is constant. If also F is independent of z, then u cannot achieve any minimum in Ω without being constant.*

Of course, the statement when F is independent of z (hence (**M**) holds) follows trivially from the first part with u replaced by $u - c$ where $c > 0$ is an arbitrary constant since $u - c$ is also a supersolution of $F = 0$ in this case.

Remark 7.1. Since Theorem 7.1 is a strong form of Lemma 4.1, the comment in Remark 4.2 can be repeated, to the effect that (**GE**) and (**LB**) for $\widetilde{F}$ in (7.6) need only hold with $(p, M) = (0, 0)$ and that $F(x, z, p, M) \geq F(x, 0, p, M)$ for $z \leq 0$ suffices instead of (**M**). For the applications similar to Theorem 4.1 and involving a supersolution and a subsolution, the condition (**A**) must hold as stated.

Theorem 7.1 does not say that u cannot achieve an interior *nonnegative* minimum. This is why no contradiction arises with the existence of nonnegative solutions u with compact support of $\Delta u - u|u|^{-s} \leq 0$ when $s \in (0, 1)$ (see Bénilan, Brézis and Crandall [3]) or other examples of this sort ([2], [13]).

The principles that also rule out the possibility of a zero interior minimum require significantly stronger assumptions. Roughly speaking, these assumptions must ensure that if u is a supersolution, then other supersolutions $v < u$ exists, so that a contradiction can be obtained via Theorem 7.1. A special case that easily fits in the framework of this paper is given below.

Corollary 7.1. *Given a positive strictly decreasing C^2 function Φ on $(0, \infty)$ and $x_0 \in \Omega$, let $\varphi_{x_0}(x) = \Phi(|x - x_0|)$.*

(*i*) *In addition to the assumptions of Theorem* 7.1 (*see also Remark* 7.1), *suppose that for every $\rho_0 > 0$ sufficiently small and every $x_0 \in \Omega$ there is $\varepsilon_0 > 0$ such that*

$$F(x, z - \varepsilon\varphi_{x_0}(x), p - \varepsilon D\varphi_{x_0}(x), M - \varepsilon D^2\varphi_{x_0}(x)) \leq F(x, z, p, M), \quad (7.7)$$

for every $(x, z, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$ with $\rho_0 \leq |x - x_0| \leq 2\rho_0$ and $z \geq 0$ and for every $0 < \varepsilon < \varepsilon_0$. Let $u : \Omega \to \mathbb{R}$ be a lower semicontinuous viscosity *supersolution of $F = 0$. If u achieves a nonpositive minimum at some point of Ω, then u is constant.*

(ii) If (7.7) holds only for every $(x, z, p, M) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N$ with $\rho_0 \leq |x - x_0| \leq 2\rho_0$ and $0 \leq z < z_0$ (resp. $0 \leq z + |p| < z_0$) with $z_0 > 0$ independent of x_0, then the conclusion of part (i) remains true for continuous (resp. C^1) u.

Proof. (i) By Theorem 7.1, u cannot achieve a strictly negative minimum without being constant. Assume now that u achieves the minimum value $m = 0$ in Ω, so that $u \geq 0$ in Ω. If u is not identically 0, a simplified variant of the annulus construction in the proof of Theorem 6.1 (with x_0 the center of the annuli, the outer radii of $\mathcal{A}_0$ and $\mathcal{A}$ chosen less than $2\rho_0$ and $\mathcal{A}'$ not needed) shows that if $\varepsilon > 0$ is small enough, then $u - \varepsilon\varphi_{x_0}$ achieves a negative minimum at some (interior) point[9] of $\mathcal{A}$. But it is clear from (7.7) that $u - \varepsilon\varphi_{x_0}$ is a viscosity supersolution of $F = 0$ in $\mathcal{A}$, so that the existence of a negative interior minimum contradicts Theorem 7.1 since $u - \varepsilon\varphi_{x_0}$ cannot be constant for two different values of ε.

(ii) Proceed as in (i) above, just noticing that the continuity of u ensures that the set $\{x \in \Omega : u(x) < z_0\}$ is open, hence can be assumed to contain the annulus $\mathcal{A}$ with no loss of generality (recall that only $u \geq 0$ in Ω has to be considered). In the C^1 case, use the set $\{x \in \Omega : u(x) + |Du(x)| < z_0\}$. □

Part (i) of Corollary 7.1 is relevant in linear problems (see Example 7.1). A simple nonlinear problem, with multiple variants, when part (ii) applies with $\Phi(r) = e^{-r}$ (or e^{-r^2}) is given by $F(z, M) = TrM - z|z|^s$ with $s \geq 0$ (but not $s < 0$). More elaborate examples can be found, but checking (7.7) is technical and we shall not attempt to compare Corollary 7.1 with similar principles in the literature.

Example 7.1. (Linear problems) Let $F(x, z, p, M) = Tr(A(x)M) + b(x) \cdot p + c(x)z$, with $A(x) \in \mathcal{S}_N, A(x) > 0, b(x) \in \mathbb{R}^N$ and $c(x) \in \mathbb{R}$, so that (**M**) amounts to $c \leq 0$. We denote by $\sigma(x)(> 0)$ and $\Sigma(x)$ the smallest and largest eigenvalues of $A(x)$, respectively and discuss the technicalities in some detail since similar arguments will be used in the other examples. With the notation of condition (**A**), assume that $M' - M \geq 0$. Then,

$$\widetilde{F}(x, z, p', M') - \widetilde{F}(x, z, p, M) = Tr\left(A(H(x))U^\top(x)(M' - M)U(x)\right) + Tr\left(A(H(x))T(x) \cdot (p' - p)\right) + b(H(x)) \cdot U^\top(x)(p' - p).$$

[9]Half-annuli are not needed in this argument.

By using $TrBC = TrCB, |TrB| \leq N|B|, |B^\top| = |B|, TrBC \leq TrBTrC$ if $B \geq 0, C \geq 0$ and $TrC \leq TrB^{-1}TrBC$ if $B > 0, C \geq 0$, we find

$$\begin{aligned}
&\widetilde{F}(x,z,p',M') - \widetilde{F}(x,z,p,M) = Tr\left(U(x)A(H(x))U^\top(x)(M'-M)\right) \\
&+ Tr\left(A(H(x))T(x)\cdot(p'-p)\right) + b(H(x))\cdot U^\top(x)(p'-p) \\
&\geq \frac{Tr(M'-M)}{Tr\left(U^{-\top}(x)A^{-1}(H(x))U^{-1}(x)\right)} \\
&- \left[N|A(H(x))|\,|T(x)| + |b(H(x))|\,|U(x)|\right]\,|p'-p|.
\end{aligned}$$

Next,

$$\begin{aligned}
&Tr\left(U^{-\top}(x)A^{-1}(H(x))U^{-1}(x)\right) = Tr\left(A^{-1}(H(x))U^{-1}(x)U^{-\top}(x)\right) \\
&\leq TrA^{-1}(H(x))Tr(U^{-1}(x)U^{-\top}(x)) \leq \frac{NTr(U^{-1}(x)U^{-\top}(x))}{\sigma(H(x))}
\end{aligned}$$

and $|A(H(x))| = \Sigma(H(x))$, whence

$$\begin{aligned}
\widetilde{F}(x,z,p',M') - \widetilde{F}(x,z,p,M) \geq{}& \frac{\sigma(H(x))}{NTr(U^{-1}(x)U^{-\top}(x))}Tr(M'-M) \\
&- \left[N\Sigma(H(x))|T(x)| + |b(H(x))|\,|U(x)|\right]\,|p'-p|.
\end{aligned}$$

This is (**GE**) for $\widetilde{F}$ with β arbitrary and

$$\begin{cases}
\lambda(x) = \frac{\sigma(H(x))}{NTr(U^{-1}(x)U^{-\top}(x))} > 0, \\
\gamma(x) = NTr(U^{-1}(x)U^{-\top}(x))\left(N\frac{\Sigma(H(x))}{\sigma(H(x))}|T(x)| + \frac{|b(H(x))|}{\sigma(H(x))}|U(x)|\right) \geq 0.
\end{cases}$$

Since H, U and T are continuous, γ is locally bounded if and only if $\frac{\Sigma}{\sigma}$ and $\frac{|b|}{\sigma}$ are locally bounded, so that (**LB**) holds for $\widetilde{F}$ under classical conditions (see [6, p. 35]), except that the boundedness of $\frac{|c|}{\sigma}$ is not required. Thus, Theorem 7.1 is valid without this assumption. However, if $\frac{|c|}{\sigma}$ is bounded, it is easily checked that the stronger principle of Corollary 7.1 (i) is available: With $\Phi(r) = e^{-tr^2}$ and $t > 0$ large enough, this verification is exactly what is done in [6, p. 34] (see also [11]).

The case $F = F(x,z,M)$ independent of p was considered in Example 4.1 because (**GE**) and (**LB**) reduce to the simple condition (**GE'**). The p-independence is not preserved in condition (**A**) above, since F is replaced by $\widetilde{F}(x,z,p,M) = F(H(x),z,U^\top(x)MU(x)+T(x)\cdot p)$, but some simplifications still occur. Two special cases are discussed in Examples 7.2 and 7.3.

Example 7.2. This generalizes Example 7.1 with $b = 0$. Let $F = F(x,z,M)$ be convex and differentiable (and therefore C^1) in M for all (x,z). Note that (**GE'**) requires $D_M F(x,z,M)W \geq 0$ for every $W \in \mathcal{S}_N, W \geq 0$. Suppose that, in fact, $D_M F(x,z,M)W > 0$ for every $W \in \mathcal{S}_N, W \geq 0, W \neq 0$. Then, $D_M F(x,z,M)W = Tr\Gamma(x,z,M)W$ for some $\Gamma(x,z,M) \in \mathcal{S}_N$ (the gradient of F with respect to M and the inner product $\langle B, C\rangle = TrBC$ on $\mathcal{S}_N$). The hypothesis that $Tr\Gamma(x,z,M)W > 0$ for every $W \in \mathcal{S}_N, W \geq 0, W \neq 0$, implies that $\Gamma(x,z,M) > 0$ (just let $W = e \otimes e$ with $e \in \mathbb{R}^N \backslash \{0\}$). In particular, $\Gamma(x,z,M)$ is invertible. Denote by $\sigma(x,z,M)(> 0)$ and $\Sigma(x,z,M)$ the smallest and largest eigenvalues of $\Gamma(x,z,M)$, respectively.

Let $M' - M \geq 0$ and with H, U and T from condition (**A**), set

$$\begin{aligned}
\widetilde{\Gamma}(x,z,p,M) &= \Gamma(H(x)x, z, U^\top(x)MU(x) + T(x)\cdot p),\\
\tilde{\sigma}(x,z,p,M) &= \sigma(H(x), z, U^\top(x)MU(x) + T(x)\cdot p),\\
\widetilde{\Sigma}(x,z,p,M) &= \Sigma(H(x), z, U^\top(x)MU(x) + T(x)\cdot p).
\end{aligned}$$

Then, by the convexity of F in M and arguments similar to those in Example 7.1,

$$\begin{aligned}
&\widetilde{F}(x,z,p',M') - \widetilde{F}(x,z,p,M)\\
&\geq Tr\left(\widetilde{\Gamma}(x,z,p,M)\left(U^\top(x)(M'-M)U(x) + T(x)\cdot(p'-p)\right)\right)\\
&\geq \lambda(x,z,p,M)\left(Tr(M'-M) - \gamma(x,z,p,M)|p'-p|\right),
\end{aligned}$$

with

$$\begin{cases}
\lambda(x,z,p,M) = \frac{\tilde{\sigma}(x,z,p,M)}{NTr(U^{-1}(x)U^{-\top}(x))} > 0,\\
\gamma(x,z,p,M) = N^2 Tr(U^{-1}(x)U^{-\top}(x))\frac{\tilde{\Sigma}(x,z,p,M)}{\tilde{\sigma}(x,z,p,M)}|T(x)| \geq 0.
\end{cases}$$

This is condition (**GE**) for $\widetilde{F}$, where β can be arbitrarily chosen. By the continuity of H, U and T, the function γ is locally bounded if $\frac{\Sigma}{\sigma}$ (hence $\frac{\tilde{\Sigma}}{\tilde{\sigma}}$) is locally bounded, for instance if $D_M F$ is continuous in (x,z,M). Then, (**LB**) holds for $\widetilde{F}$ and so does (**A**) if F satisfies the monotonicity condition (**M**). This example can be generalized when $F = F(x,z,p,M)$ also depends upon p and F is convex in (p,M) for all (x,z).

Example 7.3. Let $F = F(x,z,M)$ satisfy the ellipticity condition

$$F(x,z,M') - F(x,z,M) \geq \sigma(x,z,M,M')Tr(M'-M) \text{ if } M'-M \geq 0, \quad (7.8)$$

where $\sigma(x, z, M, M') > 0$, and the following Lipschitz continuity condition with respect to $M \in \mathcal{S}_N$ holds: Given (x_0, z_0, M_0), there is an open neighborhood $\mathcal{W}_0$ of (x_0, z_0, M_0, M_0) and a constant $C_0 > 0$ such that

$$|F(x,z,M') - F(x,z,M)| \le C_0|M' - M|, \qquad \forall (x,z,M,M') \in \mathcal{W}_0. \quad (7.9)$$

Below, we show that (**A**) holds if (**M**) does and σ in (7.8) is locally bounded away from 0. Here, it takes some work to get a suitable function β for a given $\widetilde{F}$. Cover the set $\{(x_0, z_0, M_0, M_0) : (x_0, z_0, M_0) \in \Omega \times \mathbb{R} \times \mathcal{S}_N\}$ with the neighborhoods $\mathcal{W}_0$, call $\mathcal{O}$ their union and consider a partition of unity $(\mathcal{O}_\alpha, \theta_\alpha)$ of $\mathcal{O}$ subordinate to a locally finite refinement of the covering by the $\mathcal{W}_0$. For each index α, choose $c_\alpha = C_0 > 0$ corresponding to some neighborhood $\mathcal{W}_0$ containing $\mathcal{O}_\alpha$ and set $c = \sum_\alpha c_\alpha \theta_\alpha$ (so that $c > 0$ is a continuous function of $(x, z, M, M') \in \mathcal{O}$). From the very construction of $\mathcal{O}$ and c, we have

$$(x,z,M,M') \in \mathcal{O} \Rightarrow |F(x,z,M') - F(x,z,M)| \le c(x,z,M,M')|M' - M|. \quad (7.10)$$

Since $\mathcal{O}$ is open, there are an open neighborhood $\mathcal{V}_0$ of every $(x_0, z_0, M_0) \in \Omega \times \mathbb{R} \times \mathcal{S}_N$ and a constant $D_0 > 0$ such that $(x, z, M) \in \mathcal{V}_0$ and $|M' - M| < D_0$ implies $(x, z, M, M') \in \mathcal{O}$. By a similar procedure as above, we obtain a continuous function $d > 0$ on $\Omega \times \mathbb{R} \times \mathcal{S}_N$ such that $|M' - M| \le d(x, z, M) \Rightarrow (x, z, M, M') \in \mathcal{O}$. Thus,

$$\begin{aligned} &|M' - M| < d(x,z,M) \\ &\qquad \Rightarrow |F(x,z,M') - F(x,z,M)| \le c(x,z,M,M')|M' - M|. \end{aligned} \quad (7.11)$$

With H, U and T from condition (**A**), set

$$\widetilde{\beta}(x,z,p,M) = \frac{d(H(x), z, U^\top(x)MU(x) + T(x)\cdot p)}{1 + |T(x)|}.$$

Since $\widetilde{\beta} > 0$ is continuous, it is locally bounded away from 0. Furthermore,

$$\begin{aligned} &|p' - p| \le \widetilde{\beta}(x,z,p,M) \Rightarrow \\ &|(U^\top(x)MU(x) + T(x)\cdot p') - (U^\top(x)MU(x) + T(x)\cdot p)| \\ &= |T(x)(p' - p)| \le |T(x)|\widetilde{\beta}(x,z,p,M) \\ &\le d(H(x), z, U^\top(x)MU(x) + T(x)\cdot p), \end{aligned}$$

and hence, by (7.11) and the definition of $\widetilde{F}$ in (**A**),

$$\begin{aligned} |p' - p| \le \widetilde{\beta}(x,z,p,M) \Rightarrow &|\widetilde{F}(x,z,p',M) - \widetilde{F}(x,z,p,M)| \\ &\le \tilde{c}(x,z,p,M,p')|T(x)|\,|p' - p|, \end{aligned} \quad (7.12)$$

where

$$\begin{aligned}&\tilde{c}(x,z,p,M,p')\\&=c(H(x),z,U^\top(x)MU(x)+T(x)\cdot p,U^\top(x)MU(x)+T(x)\cdot p').\end{aligned}$$

Now, assume $M'-M\geq 0$ and let $|p'-p|\leq\widetilde{\beta}(x,z,p,M)$. Write

$$\begin{aligned}&\widetilde{F}(x,z,p',M')-\widetilde{F}(x,z,p,M)\\&=\left(\widetilde{F}(x,z,p',M')-\widetilde{F}(x,z,p',M)\right)+\left(\widetilde{F}(x,z,p',M)-\widetilde{F}(x,z,p,M)\right).\end{aligned}$$

By using (7.8) for the first term and (7.12) for the second and with

$$\tilde{\sigma}(x,z,M,p')=\sigma(H(x),z,M+V(x)p',M+T(x)p'),$$

we find

$$\begin{aligned}&\widetilde{F}(x,z,p',M')-\widetilde{F}(x,z,p,M)\\&\geq\tilde{\sigma}(x,z,M,p')Tr\left(U^\top(x)(M'-M)U(x)\right)-\tilde{c}(x,z,p,M,p')|p'-p|\\&\geq\frac{\tilde{\sigma}(x,z,M,p')}{Tr(U^{-1}(x)U^{-\top}(x))}Tr(M'-M)-\tilde{c}(x,z,p,M,p')|p'-p|.\end{aligned}$$

This is condition (**GE**) for $\widetilde{F}$ with $\beta=\widetilde{\beta}$ defined earlier and

$$\begin{cases}\lambda(x,z,M,p')=\frac{\tilde{\sigma}(x,z,M,p')}{Tr(U^{-1}(x)U^{-\top}(x))}>0,\\ \gamma(x,z,p,M,p')=Tr(U^{-1}(x)U^{-\top}(x))\frac{\tilde{c}(x,z,p,M,p')|T(x)|}{\tilde{\sigma}(x,z,M,p')}\geq 0.\end{cases}$$

Since the functions H,U,T and $\tilde{c}$ are continuous, the condition (**LB**) for $\widetilde{F}$ depends only upon σ (and hence $\tilde{\sigma}$) being locally bounded away from 0. More generally, if (7.10) holds for some $c\geq 0$ (irrespective of (7.9)), then (**LB**) for $\widetilde{F}$ is satisfied if $\frac{c}{\sigma}$ is locally bounded. This gives another generalization of Example 7.1 with $b=0$.

Example 7.4. We revisit the example $F(x,z,p,M)=Tr(A(p)M)-c(x)z$, discussed in Section 5. First, assume $A(p)\in\mathcal{S}_N, A(p)>0$ (so that no degeneracy occurs), A locally Lipschitz continuous in $\mathbb{R}^N$ and $c(x)\geq 0$ (whence (**M**) holds). We denote by $\sigma(p)(>0)$ and $\Sigma(p)$ the smallest and largest eigenvalues of $A(p)$. Note that $\Sigma(p)=|A(p)|$. From the calculation at the beginning of Section 5,

$$\begin{aligned}&F(x,z,p',M')-F(x,z,p,M)\\&\geq\frac{1}{2}Tr\left((A(p')+A(p))(M'-M)\right)-\frac{1}{2}|A(p')-A(p)|(|M'|+|M|),\end{aligned}$$

for all values of the variables. Thus, with H, U and T from condition (**A**),

$$\begin{aligned}
&\widetilde{F}(x,z,p',M') - \widetilde{F}(x,z,p,M)\\
&\geq \frac{1}{2}Tr\left((A(U^\top(x)p') + A(U^\top(x)p))(U^\top(x)(M'-M)U(x)\right)\\
&-\frac{1}{2}\left(|A(U^\top(x)p')| + |A(U^\top(x)p)|\right)|U(x)|^2|T(x)|\,|p'-p|\\
&-\frac{1}{2}|A(p')-A(p)|(|M'|+|M|).
\end{aligned}$$

If now $M'-M\geq 0$, then

$$\begin{aligned}
&\frac{1}{2}Tr\left((A(U^\top(x)p') + A(U^\top(x)p))(U^\top(x)(M'-M)U(x)\right)\\
&\geq \frac{1}{2}\left(\frac{1}{TrA^{-1}(U^\top(x)p')} + \frac{1}{TrA^{-1}(U^\top(x)p)}\right)Tr(U^\top(x)(M'-M)U(x))\\
&\geq \frac{\sigma(U^\top(x)p') + \sigma(U^\top(x)p)}{2NTr(U^{-1}(x)U^{-\top}(x))}Tr(M'-M).
\end{aligned}$$

As a result, with

$$\begin{aligned}
&\lambda(x,p,p') = \frac{\sigma(U^\top(x)p') + \sigma(U^\top(x)p)}{2NTr(U^{-1}(x)U^{-\top}(x))} > 0,\\
&\gamma(x,p,M,p',M')\\
&= NTr(U^{-1}(x)U^{-\top}(x))\left[\frac{\Sigma(U^\top(x)p') + \Sigma(U^\top(x)p)}{\sigma(U^\top(x)p') + \sigma(U^\top(x)p)}|U(x)|^2|T(x)|\right.\\
&\left.+\frac{|A(U^\top(x)p') - A(U^\top(x)p)|}{|p'-p|}(|M'|+|M|)\right] \geq 0 \text{ if } p'\neq p,\\
&\gamma(x,p,M,p,M') = NTr(U^{-1}(x)U^{-\top}(x))\frac{\Sigma(U^\top(x)p)}{\sigma(U^\top(x)p)}|U(x)|^2|T(x)|,
\end{aligned}$$

we have

$$\begin{aligned}
&\widetilde{F}(x,z,p',M') - \widetilde{F}(x,z,p,M)\\
&\geq \lambda(x,p,p')\left(Tr(M'-M) - \gamma(x,p,M,p',M')|p'-p|\right).
\end{aligned}$$

Thus, the condition (**GE**) holds with an arbitrary β. Since σ does not vanish and σ and Σ are continuous (as well as U and T), the only issue regarding (**LB**) is the local boundedness of the term $\frac{|A(U^\top(x)p') - A(U^\top(x)p)|}{|p'-p|}$, which follows at once from the assumption that A is locally Lipschitz continuous and from the continuity of U.

In the problem of Section 5 when σ may vanish at 0, the local boundedness of γ above remains true away from the points $(x, 0, M, 0, M')$. Theorem 7.1 is not directly applicable, but, assuming $c > 0$, the arguments of Section 5 can be adapted in the obvious way: The use of Theorem 6.1 instead of Theorem 3.1 yields the strong minimum principle of Theorem 7.1 for the C^0 viscosity supersolutions of $F = 0$. The stronger principles in [1] or [10] (the latter in a different setting) similar to Corollary 7.1 assume that the problem has more structure than was assumed above.

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