Advances in Differential Equations Volume 9, Numbers 1-2, January-February 2004, Pages 221–239

ON THE LOCATION OF CONCENTRATION POINTS FOR SINGULARLY PERTURBED ELLIPTIC EQUATIONS

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Abstract. By exploiting a variational identity of Pohožaev-Pucci-Serrin type for solutions of class C^1, we get some necessary conditions for locating the peak-points of a class of singularly perturbed quasilinear elliptic problems in divergence form. More precisely, we show that the points where the concentration occurs, in general, must belong to what we call the set of weak-concentration points. Finally, in the semilinear case, we provide a new necessary condition which involves the Clarke subdifferential of the ground-state function.

1. Introduction

Let $\varepsilon > 0$, $n \geq 3$, and $1 < p < n$. In this paper we consider the following class of singularly perturbed quasilinear elliptic problems in divergence form:

$$\begin{cases} -\varepsilon^p \mathrm{div}\,(\alpha(x)\nabla\beta(\nabla u)) + V(x)u^{p-1} = K(x)f(u) & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{P_ε}$$

We assume that the functions α, V, $K\colon \mathbb{R}^n \to \mathbb{R}$ are positive, of class C^1 with bounded derivatives and $\alpha, K \in L^\infty(\mathbb{R}^n)$. Moreover, let

$$\inf_{x\in\mathbb{R}^n} \alpha(x) > 0 \quad \text{and} \quad \inf_{x\in\mathbb{R}^n} V(x) > 0.$$

Accepted for publication: November 2003.
AMS Subject Classifications: 35J40; 58E05.
The first author was supported by the MIUR national research project "Variational Methods and Nonlinear Differential Equations." The second author was supported by the MIUR national research project "Variational and Topological Methods in the Study of Nonlinear Phenomena" and by the Istituto Nazionale di Alta Matematica "F. Severi" (INdAM).

The function $\beta\colon \mathbb{R}^n \to \mathbb{R}$ is of class C^1, strictly convex, and positively homogeneous of degree p; namely, $\beta(\lambda\xi) = \lambda^p\beta(\xi)$ for every $\lambda > 0$ and $\xi \in \mathbb{R}^n$. Moreover, there exist $\nu > 0$ and $c_1, c_2 > 0$ such that

$$\nu|\xi|^p \le \beta(\xi) \le c_1|\xi|^p, \tag{1.1}$$

$$|\nabla\beta(\xi)| \le c_2|\xi|^{p-1}, \tag{1.2}$$

for every $\xi \in \mathbb{R}^n$. The nonlinearity $f : \mathbb{R}^+ \to \mathbb{R}$ is of class C^1 and such that

$$\lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = 0 \quad \text{and} \quad \lim_{s\to +\infty} \frac{f(s)}{s^{q-1}} = 0,$$

for some $p < q < p^*$, with $p^* = np/(n-p)$. Moreover, $0 < \vartheta F(s) \le f(s)s$, for every $s > 0$, for some $\vartheta > p$, where we have set $F(s) = \int_0^s f(t)\,dt$, $s \in \mathbb{R}^+$.

Let us define the space $W_V(\mathbb{R}^n)$ by setting

$$W_V(\mathbb{R}^n) := \Big\{u \in W^{1,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^p\,dx < \infty\Big\},$$

endowed with the natural norm $\|u\|^p_{W_V} = \int_{\mathbb{R}^n} |\nabla u|^p\,dx + \int_{\mathbb{R}^n} V(x)|u|^p\,dx$. For $p = 2$, we write $H_V(\mathbb{R}^n)$ in place of $W_V(\mathbb{R}^n)$. Under the previous assumptions, if $K \equiv 1$, it has been recently proved in [12] (see also [24]) that if for some compact subset $\Lambda \subset \mathbb{R}^n$ we have

$$V(z_0) = \min_\Lambda V < \min_{z\in\partial\Lambda} V(z) \quad \text{and} \quad \alpha(z_0) = \min_{z\in\Lambda} \alpha(z),$$

then, for every ε sufficiently small, there exists a solution $u_\varepsilon \in W_V(\mathbb{R}^n)$ of (P_ε) which has a maximum point $z_\varepsilon \in \Lambda$, with

$$\lim_{\varepsilon\to 0} V(z_\varepsilon) = \min_\Lambda V \quad \text{and} \quad \lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega\setminus B_\rho(z_\varepsilon))} = 0, \quad \text{for every } \rho > 0.$$

In the semilinear case, the construction of solutions concentrating at critical points (or minima) of the potential $V(x)$ or other finite-dimensional driven functions has been deeply investigated in the last decade, and also stronger results can be found in the literature (see e.g. [1, 6, 8, 9, 10, 11, 17, 21, 26] and references therein).

The goal of this paper is to establish some *necessary conditions* for a sequence of solutions (u_{ε_h}) of (P_ε) to concentrate around a given point $z_0 \in \mathbb{R}^n$, in the sense of Definition 2.8. If $\beta(\xi) = \xi$, we will prove (see Theorem 3.6) that if z_0 is a concentration point for a sequence $(u_{\varepsilon_h}) \subset H_V(\mathbb{R}^n)$ of solutions of the problem, then there exists a locally Lipschitz function $\Sigma : \mathbb{R}^n \to \mathbb{R}$, the ground-state function, which has, under suitable assumptions, a critical point in the sense of the Clarke subdifferential at z_0; that is, $0 \in \partial\Sigma(z_0)$. Under more stringent assumptions, it turns out that Σ admits all the directional derivatives at z_0 and $\nabla\Sigma(z_0) = 0$. In the general case, as a first necessary condition, the gradient vectors $\nabla\alpha(z_0)$, $\nabla V(z_0)$, and $\nabla K(z_0)$ must be linearly dependent. Moreover, in Theorem 2.6 (see also Theorem 2.11), we show that the concentration points for problem (P_ε) must belong to a set $\mathfrak{C}$ (which has a variational structure) that we call the set of weak-concentration points (see Definition 2.1). To the authors' knowledge, this kind of necessary conditions in terms of generalized gradients seem to be new. Quite interestingly, the lack of uniqueness (up to translations) for the limiting problem (namely the rescaled problem with frozen coefficients)

$$\begin{cases} -\alpha(z)\mathrm{div}\,(\nabla\beta(\nabla u)) + V(z)u^{p-1} = K(z)f(u) & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \end{cases} \tag{P_z}$$

induces a lack of regularity for Σ. Some conditions ensuring uniqueness of solutions for (P_z) can be found in [5, 23]. For instance, for $1 < p \leq 2$, $\beta(\xi) = |\xi|^{p-2}\xi$, and $f(u) = u^{q-1}$ with $p < q < p^*$, we have uniqueness and Σ admits all the directional derivatives.

We stress that some necessary conditions for the location of concentration points were previously obtained by Ambrosetti et al. in [1] and by Wang and Zeng in [26, 27] in the case $p = 2$ and $\beta(\xi) = \xi$. Their approach is based on a repeated use of the divergence theorem. With respect to those papers we prove our main results by means of a locally Lipschitz variant of the celebrated Pucci-Serrin variational identity [19]. In our possibly degenerate setting, classical C^2 solutions might not exist, the highest general regularity class being $C^{1,\beta}$ (see [25]). Therefore, the classical identity is not applicable in our framework. However, it has been recently shown in [7] that, under minimal regularity assumptions, the identity holds for locally Lipschitz solutions (see Theorem 2.5), provided that the operator (β, in our case) is strictly convex in the gradient, which, from our viewpoint, is a very natural requirement.

This identity has also turned out to be useful in characterizing the exact energy level of the least-energy solutions of the problem (P_z). Indeed, in [12, Theorem 3.2] it was proved that (P_z) admits a least-energy solution $u_z \in W^{1,p}(\mathbb{R}^n)$ having the mountain-pass energy level. This is precisely the motivation that led us to define the ground-state function Σ also in a degenerate setting.

2. The quasilinear case

The aim of this section is the study of some necessary conditions for the concentration of the solutions at a point z_0 to occur, in the quasilinear framework.

2.1. **Some preliminary definitions and properties.** If z is fixed in $\mathbb{R}^n$, we consider the limiting functional $I_z : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$,

$$I_z(u) := \alpha(z) \int_{\mathbb{R}^n} \beta(\nabla u)\, dx + \frac{V(z)}{p} \int_{\mathbb{R}^n} |u|^p\, dx - K(z) \int_{\mathbb{R}^n} F(u)\, dx.$$

It follows from our assumptions on β and f that I_z is a C^1 functional and its critical points are solutions of the limiting problem (P_z). We define the minimax value c_z for I_z by setting

$$c_z := \inf_{\gamma \in \mathcal{P}_z} \sup_{t \in [0,1]} I_z(\gamma(t)), \tag{2.1}$$
$$\mathcal{P}_z := \Big\{ \gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \ \gamma(0) = 0, \ \ I_z(\gamma(1)) < 0 \Big\}.$$

Throughout the rest of the paper, we will denote by $G(z)$ the set of all the nontrivial solutions, up to translations, of the limiting problem (P_z) (the set of bound-states). Under our assumptions on f, $G(z) \neq \emptyset$ for every $z \in \mathbb{R}^n$. Finally, $\cdot$ will always stand for the usual inner product of $\mathbb{R}^n$.

We now introduce two functions $\partial\Gamma^-$ and $\partial\Gamma^+$ that will be useful in the sequel.

Definition 2.1. *For every $z, w \in \mathbb{R}^n$ we define $\partial\Gamma^-(z;w)$ and $\partial\Gamma^+(z;w)$ by setting*

$$\partial\Gamma^-(z;w) := \sup_{v \in G(z)} \nabla_z I_z(v) \cdot w, \qquad \partial\Gamma^+(z;w) := \inf_{v \in G(z)} \nabla_z I_z(v) \cdot w,$$

where ∇_z denotes the gradient with respect to z. Explicitly, for every $z, w \in \mathbb{R}^n$,

$$\begin{aligned}
\partial\Gamma^-(z;w) = \sup_{v \in G(z)} \Big[&\nabla\alpha(z) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v)\, dx \\
&+ \nabla V(z) \cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p}\, dx - \nabla K(z) \cdot w \int_{\mathbb{R}^n} F(v)\, dx \Big], \\
\partial\Gamma^+(z;w) = \inf_{v \in G(z)} \Big[&\nabla\alpha(z) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v)\, dx \\
&+ \nabla V(z) \cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p}\, dx - \nabla K(z) \cdot w \int_{\mathbb{R}^n} F(v)\, dx \Big].
\end{aligned}$$

Finally, we define a set $\mathfrak{C} \subset \mathbb{R}^n$ by

$$\mathfrak{C} := \big\{ z \in \mathbb{R}^n : \ \ \partial\Gamma^-(z,w) \geq 0 \text{ and } \partial\Gamma^+(z,w) \leq 0, \text{ for every } w \in \mathbb{R}^n \big\}.$$

We say that $\mathfrak{C}$ is the set of weak-concentration points for problem (P_ε).

The motivations that lead us to introduce the functions $\partial\Gamma^-$ and $\partial\Gamma^+$, and the set of weak-concentration points, will be clear in the course of the investigation.

For the sake of completeness, we recall the following:

Definition 2.2. *We define the* ground-state function $\Sigma : \mathbb{R}^n \to \mathbb{R}$ *by setting*

$$\Sigma(z) := \min_{u \in G(z)} I_z(u), \quad \textit{for every } z \in \mathbb{R}^n.$$

We now collect a few useful properties of the function Σ.

Lemma 2.3. *Assume that*

$$\textit{the map} \;\; s \in \mathbb{R}^+ \mapsto \frac{f(s)}{s^{p-1}} \;\; \textit{is increasing.} \tag{2.2}$$

Then, the following facts hold:

(i) *the map Σ is well defined and continuous, and*

$$\Sigma(z) = c_z, \quad \textit{for every } z \in \mathbb{R}^n;$$

(ii) *the map Σ can be written as*

$$\Sigma(z) = \inf_{u \in W^{1,p}(\mathbb{R}^n)\setminus\{0\}} \max_{\vartheta \geq 0} I_z(\vartheta u) = \inf_{u \in \mathcal{N}_z} I_z(u), \quad \textit{for every } z \in \mathbb{R}^n,$$

where $\mathcal{N}_z$ is the Nehari manifold, defined as

$$\mathcal{N}_z := \Big\{ u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\} : \; I_z'(u)[u] = 0 \Big\}.$$

Proof. To prove (ii), it suffices to argue as in [18, Proposition 2.5]. We now come to assertion (i). By [12, Theorem 3.2], for every $z \in \mathbb{R}^n$, problem (P_z) admits a solution $v_z \in W^{1,p}(\mathbb{R}^n)$, $v_z \neq 0$, such that $I_z(v_z) = \Sigma(z) = c_z$, where c_z is defined as in (2.1). The continuity of Σ then follows from the continuity of the map $z \mapsto c_z$, which we now prove directly using an argument envisaged by Rabinowitz [21]. For $\alpha, V, K \in \mathbb{R}$, define the functional $I_{\alpha,V,K} : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$ by

$$I_{\alpha,V,K}(u) := \alpha \int_{\mathbb{R}^n} \beta(\nabla u)\, dx + \frac{V}{p} \int_{\mathbb{R}^n} |u|^p \, dx - K \int_{\mathbb{R}^n} F(u)\, dx.$$

Let us set

$$c(\alpha, V, K) := \inf_{\gamma \in \mathcal{P}_{\alpha,V,K}} \max_{t \in [0,1]} I_{\alpha,V,K}(\gamma(t)),$$

$$\mathcal{P}_{\alpha,V,K} := \Big\{ \gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \; \gamma(0) = 0, \; I_{\alpha,V,K}(\gamma(1)) < 0 \Big\}.$$

Claim: For every $(\alpha, V, K) \in \mathbb{R}^3$ we have

$$\lim_{\eta\to 0} c(\alpha+\eta, V+\eta, K-\eta) = c(\alpha, V, K).$$

We first observe that a simple adaptation of the argument of [21, Lemma 3.17] yields

$$\alpha_1 > \alpha_2,\ V_1 > V_2,\ K_1 < K_2 \implies c(\alpha_1, V_1, K_1) \geq c(\alpha_2, V_2, K_2). \tag{2.3}$$

The proof of the claim will be accomplished indirectly. By virtue of (2.3), we get

$$\lim_{\eta\to 0^-} c(\alpha+\eta, V+\eta, K-\eta) := c^- \leq c(\alpha, V, K).$$

Suppose that $c^- < c(\alpha, V, K)$. For the sake of brevity, we define

$$J_\eta(u) := I_{\alpha+\eta, V+\eta, K-\eta}(u).$$

Let $\eta_h \to 0^-$ as $h \to \infty$, and $\delta_j \to 0^+$ as $j \to \infty$. For each $h \in \mathbb{N}$, by assertion (ii), there is a sequence (u_{hj}) in $W^{1,p}(\mathbb{R}^n)$, $u_{hj} \neq 0$, such that

$$\alpha \int_{\mathbb{R}^n} \beta(\nabla u_{hj})\, dx + V \int_{\mathbb{R}^n} |u_{hj}|^p\, dx = 1 \tag{2.4}$$

and

$$\max_{\vartheta\geq 0} J_{\eta_h}(\vartheta u_{hj}) \leq c(\alpha+\eta_h, V+\eta_h, K-\eta_h) + \delta_j. \tag{2.5}$$

Notice that we can choose the sequence (u_{hj}) satisfying (2.4), since the position

$$u \mapsto \alpha \int_{\mathbb{R}^n} \beta(\nabla u)\, dx + V \int_{\mathbb{R}^n} |u|^p\, dx$$

defines on $W^{1,p}(\mathbb{R}^n)$ a norm equivalent to the natural one, as follows from (1.1). Take now $h = j$ and set $u_h = u_{hh}$. Hence, in view of (2.5), we have

$$\begin{aligned}
c(\alpha, V, K) &\leq \max_{\vartheta\geq 0} I_{\alpha,V,K}(\vartheta u_h) = I_{\alpha,V,K}(\phi(u_h)u_h) \\
&= J_{\eta_h}(\phi(u_h)u_h) - \eta_h \phi(u_h)^p \int_{\mathbb{R}^n} \frac{|u_h|^p}{p}\, dx - \eta_h \phi(u_h)^p \int_{\mathbb{R}^n} \beta(\nabla u_h)\, dx \\
&\quad - \eta_h \int_{\mathbb{R}^n} F(\phi(u_h)u_h)\, dx \\
&\leq \max_{\vartheta\geq 0} J_{\eta_h}(\vartheta u_h) - \eta_h \phi(u_h)^p \int_{\mathbb{R}^n} \frac{|u_h|^p}{p}\, dx - \eta_h \phi(u_h)^p \int_{\mathbb{R}^n} \beta(\nabla u_h)\, dx \\
&\quad - \eta_h \int_{\mathbb{R}^n} F(\phi(u_h)u_h)\, dx
\end{aligned}$$

$$\begin{aligned}
&\leq c(\alpha+\eta_h, V+\eta_h, K-\eta_h)+\delta_h-\eta_h\phi(u_h)^p\int_{\mathbb{R}^n}\frac{|u_h|^p}{p}\,dx\\
&\quad-\eta_h\phi(u_h)^p\int_{\mathbb{R}^n}\beta(\nabla u_h)\,dx+\eta_h\int_{\mathbb{R}^n}F(\phi(u_h)u_h)\,dx\\
&\leq c^-+\delta_h-\eta_h\phi(u_h)^p\int_{\mathbb{R}^n}\frac{|u_h|^p}{p}\,dx-\eta_h\phi(u_h)^p\int_{\mathbb{R}^n}\beta(\nabla u_h)\,dx\\
&\quad-\eta_h\int_{\mathbb{R}^n}F(\phi(u_h)u_h)\,dx.
\end{aligned}$$

At this point, one can show exactly as in [21, pp. 281–282] that there exists a constant $C>0$ such that $\phi(u_h)\leq C$, for every $h\in\mathbb{N}$ sufficiently large. Therefore, recalling the properties of F and the Sobolev embedding, the above chain of inequalities contradicts $c^-<c(\alpha,V,K)$, at least for every $h\in\mathbb{N}$ large enough. We conclude that $c^-<c(\alpha,V,K)$ is impossible. In a completely similar fashion one can prove that the inequality

$$c(\alpha,V,K)<\lim_{\eta\to 0^+}c(\alpha+\eta,V+\eta,K-\eta)$$

leads to a contradiction. Therefore the claim is proved.

Let now (z_h) be a sequence in $\mathbb{R}^n$ such that $z_h\to z$ as $h\to\infty$. Observe that, given $\eta>0$, for large $h\in\mathbb{N}$, we have

$$\begin{aligned}
V(z)+\eta&\geq V(z)+|V(z_h)-V(z)|\\
&\geq V(z)\geq V(z)-|V(z_h)-V(z)|\geq V(z)-\eta,
\end{aligned}$$

and similar relations hold for α and K. Therefore the continuity of $z\mapsto c_z$ follows from the previous claim, applied with $\alpha=\alpha(z)$, $V=V(z)$, and $K=K(z)$. This completes the proof of assertion (i). □

Remark 2.4. As we have already pointed out in the introduction, we believe that the lack of regularity of the ground-state map Σ is essentially inherited by the lack of uniqueness assumptions on the limiting equation (P_z). From this viewpoint, in the degenerate case $p\neq 2$, the problem of establishing the regularity of Σ seems quite a difficult matter. On the contrary, if $p=2$ and, for instance, $\beta(\xi)=\xi$, it is known that Σ is always at least locally Lipschitz continuous (cf. Lemma 3.1). If, additionally, $f(u)$ is exactly the power u^{p-1} (in which case equation (P_z) has in fact a unique solution [3]), then Σ is smooth and it also admits an explicit representation formula (see Remark 3.2).

Let now $\mathcal{L}:\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ be a function of class C^1 such that

$$\text{the function }\ \xi\mapsto\mathcal{L}(x,s,\xi)\ \text{ is strictly convex,}$$

for every $(x,s) \in \mathbb{R}^n \times \mathbb{R}$, and let $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$.

Next, we recall a Pucci-Serrin variational identity for locally Lipschitz continuous solutions of a general class of Euler equations, recently proved in [7]. As we have already remarked in the introduction, the classical identity [19] is not applicable here, since it requires the C^2 regularity of the solutions, while the maximal regularity for degenerate equations is $C^{1,\beta}$ (see e.g. [25]).

Theorem 2.5. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz solution of*

$$-\mathrm{div}\,(\partial_\xi \mathcal{L}(x,u,\nabla u)) + \partial_s \mathcal{L}(x,u,\nabla u) = \varphi \quad \text{in } \mathcal{D}'(\mathbb{R}^n).$$

Then,

$$\sum_{i,j=1}^n \int_{\mathbb{R}^n} \partial_i h^j \partial_{\xi_i} \mathcal{L}(x,u,\nabla u)\partial_j u\,dx - \int_{\mathbb{R}^n} \big[(\mathrm{div}\,h)\,\mathcal{L}(x,u,\nabla u) + h\cdot \partial_x \mathcal{L}(x,u,\nabla u)\big]\,dx = \int_{\mathbb{R}^n} (h\cdot\nabla u)\varphi\,dx, \tag{2.6}$$

for every $h \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$.

2.2. **Necessary conditions for locating peak-points.** We now state and prove the main results of this section.

Theorem 2.6. *Let $z_0 \in \mathbb{R}^n$ and assume that (u_{ε_h}) is a sequence of solutions of problem (P_ε) such that*

$$u_{\varepsilon_h} = v_0\Big(\frac{\cdot - z_0}{\varepsilon_h}\Big) + o(1), \quad \text{strongly in } W_V(\mathbb{R}^n), \tag{2.7}$$

for some $v_0 \in W_V(\mathbb{R}^n)\setminus\{0\}$. Then, the following facts hold:

(a) *the vectors $\nabla\alpha(z_0)$, $\nabla V(z_0)$, and $\nabla K(z_0)$ are linearly dependent;*
(b) *$z_0 \in \mathfrak{C}$; that is, z_0 is a weak-concentration point for (P_ε);*
(c) *if $G(z_0) = \{v_0\}$, then all the partial derivatives of Σ at z_0 exist and*

$$\nabla\Sigma(z_0) = 0;$$

that is, z_0 is a critical point of Σ.

Proof. We write u_h in place of u_{ε_h}, and we define

$$v_h(x) := u_h(z_0 + \varepsilon_h x). \tag{2.8}$$

Therefore, v_h satisfies the rescaled equation

$$-\mathrm{div}\,(\alpha(z_0+\varepsilon x)\nabla\beta(\nabla v_h)) + V(z_0+\varepsilon x)v_h^{p-1} = K(z_0+\varepsilon x)f(v_h) \quad \text{in } \mathbb{R}^n.$$

By (2.7), we have $v_h \to v_0$ strongly in $W_V(\mathbb{R}^n)$. We now prove that $v_h \to v_0$ in the C^1 sense over the compact sets of $\mathbb{R}^n$ and that v_0 is a nontrivial positive solution of the equation

$$-\alpha(z_0)\mathrm{div}(\nabla\beta(\nabla v)) + V(z_0)v^{p-1} = K(z_0)f(v) \quad \text{in } \mathbb{R}^n. \tag{2.9}$$

Let us set

$$d_h(x) := \begin{cases} V(z_0+\varepsilon_h x) - K(z_0+\varepsilon_h x)\frac{f(v_h(x))}{v_h^{p-1}(x)} & \text{if } v_h(x) \neq 0 \\ 0 & \text{if } v_h(x) = 0, \end{cases}$$

$$A(x,s,\xi) := \alpha(z_0+\varepsilon_h x)\nabla\beta(\xi), \qquad B(x,s,\xi) := d_h(x)s^{p-1},$$

for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^+$ and $\xi \in \mathbb{R}^n$. Taking into account (1.2) and the strict convexity of β, we get

$$A(x,s,\xi)\cdot\xi \geq \nu|\xi|^p \quad \text{and} \quad |A(x,s,\xi)| \leq c_2|\xi|^{p-1}.$$

Notice that, in view of the growth assumptions on f, there exists $\delta > 0$ sufficiently small such that $d_h \in L^{n/(p-\delta)}(B_{2\rho})$ for every $\rho > 0$ and

$$S = \sup_{h\in\mathbb{N}} \|d_h\|_{L^{n/(p-\delta)}(B_{2\rho})} \leq D_\rho\Big(1 + \sup_{h\in\mathbb{N}} \|v_h\|_{L^{p^*}(B_{2\rho})}\Big) < \infty,$$

for some $D_\rho > 0$. Since we have $\mathrm{div}(A(x,v_h,\nabla v_h)) = B(x,v_h,\nabla v_h)$ for every $h \in \mathbb{N}$, by exploiting [22, Theorem 1] there exists a radius $\rho > 0$ and a positive constant $M = M(\nu, c_2, S\rho^\delta)$ such that

$$\sup_{h\in\mathbb{N}} \max_{x\in B_\rho} |v_h(x)| \leq M(2\rho)^{-N/p} \sup_{h\in\mathbb{N}} \|v_h\|_{L^p(B_{2\rho})} < \infty,$$

so that (v_h) is uniformly bounded in B_ρ. Then, by virtue of [22, Theorem 8], up to a subsequence (v_h) converges uniformly to v_0 in a small neighborhood of zero. Similarly one shows that $v_h \to v_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore, it is easily seen that v_0 is a nontrivial positive solution of (2.9); that is, $v_0 \in G(z_0)$. Since the map β is strictly convex, we can use Theorem 2.5 by choosing in (2.6) $\varphi = 0$ and

$$\mathcal{L}(x,s,\xi) := \alpha(z_0+\varepsilon_h x)\beta(\xi) + V(z_0+\varepsilon_h x)\frac{s^p}{p} - K(z_0+\varepsilon_h x)F(s),$$

$$h(x) = h_{\varepsilon,k}(x) := (\underbrace{0,\dots,0}_{k-1}, T(\varepsilon x), \underbrace{0,\dots,0}_{n-k}), \quad \text{for } \varepsilon > 0 \text{ and } k = 1,\dots,n,$$

for every $x \in \mathbb{R}^n$, $s \in \mathbb{R}^+$, and $\xi \in \mathbb{R}^n$, the function $T \in C^1_c(\mathbb{R}^n)$ being chosen so that $T(x) = 1$ for $|x| \leq 1$ and $T(x) = 0$ for $|x| \geq 2$. In particular, $h_{\varepsilon,k} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$ and

$$\partial_i h^j_{\varepsilon,k}(x) = \varepsilon\partial_i T(\varepsilon x)\delta_{kj}, \quad \text{for every } x \in \mathbb{R}^n,\ \varepsilon > 0, \text{ and } i,\ j, \text{ and } k.$$

Then, it follows from (2.6) that

$$
\begin{aligned}
0 = &\sum_{i=1}^{n} \int_{\mathbb{R}^n} \varepsilon \partial_i T(\varepsilon x) \alpha(z_0 + \varepsilon_h x) \partial_{\xi_i} \beta(\nabla v_h) \partial_k v_h \, dx \\
&- \int_{\mathbb{R}^n} \varepsilon \partial_k T(\varepsilon x) \Big[\alpha(z_0 + \varepsilon_h x) \beta(\nabla v_h) + V(z_0 + \varepsilon_h x) \frac{v_h^p}{p} \\
&\qquad - K(z_0 + \varepsilon_h x) F(v_h) \Big] \, dx \\
&- \int_{\mathbb{R}^n} \varepsilon_h T(\varepsilon x) \Big[\frac{\partial \alpha}{\partial x_k}(z_0 + \varepsilon_h x) \beta(\nabla v_h) + \frac{\partial V}{\partial x_k}(z_0 + \varepsilon_h x) \frac{v_h^p}{p} \\
&\qquad - \frac{\partial K}{\partial x_k}(z_0 + \varepsilon_h x) F(v_h) \Big] \, dx
\end{aligned}
$$

for every $\varepsilon > 0$, $h \in \mathbb{N}$, and $k = 1, \ldots, n$. Since the sequence (v_h) is bounded in $W_V(\mathbb{R}^n)$, by (1.1), (1.2), and the boundedness of α and K, we have

$$
\begin{aligned}
&\Big| \sum_{i=1}^{n} \int_{\mathbb{R}^n} \partial_i T(\varepsilon x) \alpha(z_0 + \varepsilon_h x) \partial_{\xi_i} \beta(\nabla v_h) \partial_k v_h \, dx \Big| \leq C, \\
&\Big| \int_{\mathbb{R}^n} \partial_k T(\varepsilon x) \Big[\alpha(z_0 + \varepsilon_h x) \beta(\nabla v_h) + V(z_0 + \varepsilon_h x) \frac{v_h^p}{p} \\
&\qquad\qquad - K(z_0 + \varepsilon_h x) F(v_h) \Big] dx \Big| \leq C',
\end{aligned}
$$

for some positive constants C and C'. Therefore, letting first $\varepsilon \to 0$ yields

$$
\begin{aligned}
\int_{\mathbb{R}^n} \Big[\frac{\partial \alpha}{\partial x_k}(z_0 + \varepsilon_h x) \beta(\nabla v_h) &+ \frac{\partial V}{\partial x_k}(z_0 + \varepsilon_h x) \frac{v_h^p}{p} \\
&- \frac{\partial K}{\partial x_k}(z_0 + \varepsilon_h x) F(v_h) \Big] \, dx = 0,
\end{aligned}
\tag{2.10}
$$

for every $h \in \mathbb{N}$ and $k = 1, \ldots, n$. Letting now $h \to \infty$, by (2.7), we find

$$
\frac{\partial \alpha}{\partial x_k}(z_0) \int_{\mathbb{R}^n} \beta(\nabla v_0) \, dx + \frac{\partial V}{\partial x_k}(z_0) \int_{\mathbb{R}^n} \frac{v_0^p}{p} \, dx - \frac{\partial K}{\partial x_k}(z_0) \int_{\mathbb{R}^n} F(v_0) \, dx = 0,
$$

for every $k = 1, \ldots, n$, which yields

$$
\nabla \alpha(z_0) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v_0) \, dx + \nabla V(z_0) \cdot w \int_{\mathbb{R}^n} \frac{v_0^p}{p} \, dx = \nabla K(z_0) \cdot w \int_{\mathbb{R}^n} F(v_0) \, dx,
$$

for every $w \in \mathbb{R}^n$. Then, since $v_0 \not\equiv 0$, assertion (a) immediately follows. Moreover, since $v_0 \in G(z_0)$, by the definition of $\partial\Gamma^-$, we obtain

$$\begin{aligned}\partial\Gamma^-(z_0; w) = \sup_{v\in G(z_0)} \Big[&\nabla\alpha(z_0)\cdot w \int_{\mathbb{R}^n} \beta(\nabla v)\,dx \\ &+ \nabla V(z_0)\cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p}\,dx - \nabla K(z_0)\cdot w \int_{\mathbb{R}^n} F(v)\,dx\Big] \\ \geq\ &\nabla\alpha(z_0)\cdot w \int_{\mathbb{R}^n} \beta(\nabla v_0)\,dx \\ &+ \nabla V(z_0)\cdot w \int_{\mathbb{R}^n} \frac{v_0^p}{p}\,dx - \nabla K(z_0)\cdot w \int_{\mathbb{R}^n} F(v_0)\,dx = 0,\end{aligned}$$

for every $w \in \mathbb{R}^n$. Analogously, by the definition of $\partial\Gamma^+$, we have

$$\begin{aligned}\partial\Gamma^+(z_0; w) = \inf_{v\in G(z_0)} \Big[&\nabla\alpha(z_0)\cdot w \int_{\mathbb{R}^n} \beta(\nabla v)\,dx \\ &+ \nabla V(z_0)\cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p}\,dx - \nabla K(z_0)\cdot w \int_{\mathbb{R}^n} F(v)\,dx\Big] \\ \leq\ &\nabla\alpha(z_0)\cdot w \int_{\mathbb{R}^n} \beta(\nabla v_0)\,dx \\ &+ \nabla V(z_0)\cdot w \int_{\mathbb{R}^n} \frac{v_0^p}{p}\,dx - \nabla K(z_0)\cdot w \int_{\mathbb{R}^n} F(v_0)\,dx = 0,\end{aligned}$$

for every $w \in \mathbb{R}^n$. Therefore $z_0 \in \mathfrak{C}$ and assertion (b) is proved. If $G(z_0) = \{v_0\}$, then clearly Σ admits all the directional derivatives at z_0 and

$$\frac{\partial\Sigma}{\partial w}(z_0) = \partial\Gamma^-(z_0; w) = \partial\Gamma^+(z_0; w) = 0, \quad \text{for every } w \in \mathbb{R}^n,$$

by virtue of (b). This proves assertion (c). □

The strong convergence required by (2.7) allows us to take the limit as $h \to \infty$ in equation (2.10). In the semilinear case one can construct uniform exponential barriers for the family (v_h), and therefore the strong convergence of (v_h) follows easily from the Lebesgue convergence theorem (see [18, 26, 27]). The well-known loss of regularity for solutions of quasilinear equations is usually an obstruction to this kind of argument. However, if the solutions belong to a suitable space, then a pointwise concentration suffices (see Corollary 2.9).

Remark 2.7. We wish to point out that Theorem 2.6 holds true also for the more general class of quasilinear equations

$$-\varepsilon^p \mathrm{div}\,(\alpha(x)\partial_\xi \beta(u,\nabla u)) + \varepsilon^p \alpha(x)\partial_s \beta(u,\nabla u) + V(x)u^{p-1} = K(x)f(u),$$

under suitable assumptions on $\partial_\xi \beta(s,\xi)$ and $\partial_s \beta(s,\xi)$ (see [12]). On the other hand, although the ground-state function Σ can be defined exactly as in Definition 2.2 and $\Sigma(z) = c_z$ (cf. [12, Theorem 3.2]), the presence of u itself in the function β makes the problems of the regularity of Σ and of the decay at infinity for the rescaled family of solutions very complicated, even in the nondegenerate case $p = 2$.

Definition 2.8. *Let $z_0 \in \mathbb{R}^n$. We say that a sequence (u_{ε_h}) of solutions of (P_ε) concentrates at z_0 if $u_{\varepsilon_h}(z_0) \geq \ell > 0$ for some $\ell > 0$ and for every $\eta > 0$ there exist $\varrho > 0$ and $h_0 \in \mathbb{N}$ such that*

$$u_{\varepsilon_h}(x) \leq \eta, \quad \text{for every } h \geq h_0 \text{ and } |x - z_0| \geq \varepsilon_h \varrho.$$

This is precisely the notion of concentration adopted in [26, 27].

Corollary 2.9. *Let (u_{ε_h}) be a family of solutions of (P_ε) which concentrates at a point $z_0 \in \mathbb{R}^n$. Suppose that, for every $h \in \mathbb{N}$ sufficiently large,*

$$u_{\varepsilon_h} \in C^1_d(\mathbb{R}^n) \cap W^{2,n}(\mathbb{R}^n),$$

where

$$C^1_d(\mathbb{R}^n) := \Big\{ u \in C^1(\mathbb{R}^n) : \lim_{|x|\to\infty} u(x) = 0 \text{ and } \lim_{|x|\to\infty} \nabla u(x) = 0 \Big\}.$$

Then, all the conclusions of Theorem 2.6 hold true.

Proof. If $u_{\varepsilon_h} \in C^1_d(\mathbb{R}^n) \cap W^{2,n}(\mathbb{R}^n)$, then one can apply the results contained in [20] to show that the rescaled sequence v_{ε_h} decays exponentially fast at infinity, uniformly with respect to h, together with all its partial derivatives. Hence we can pass to the limit in equation (2.10), and complete the proof as in Theorem 2.6. □

For the particular but important case $\alpha(x) = 1$, $\beta(\xi) = |\xi|^{p-2}\xi$, and $f(s) = s^{q-1}$, $p < q < p^*$, we can still prove a fast-decay at infinity for the solutions.

Lemma 2.10. *Let (u_{ε_h}) be a sequence of solutions of the problem*

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x)u^{p-1} = K(x)u^{q-1} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \end{cases}$$

which concentrates at $z_0 \in \mathbb{R}^n$. Then, if we set

$$v_h(x) := u_{\varepsilon_h}(z_0 + \varepsilon_h x),$$

for each $\eta > 0$ there exist $R_\eta, C_\eta > 0$ independent of h such that

$$|v_h(x)| \leq C_\eta \exp\Big\{ -\Big(\frac{\eta}{p-1}\Big)^{1/p} |x| \Big\},$$

for every $|x| \geq R_\eta$ and every $h \in \mathbb{N}$.

Proof. For every $h \in \mathbb{N}$, v_h satisfies the equation

$$-\Delta_p v_h + V(z_0 + \varepsilon_h x) v_h^{p-1} = K(z_0 + \varepsilon_h x) v_h^{q-1} \quad \text{in } \mathbb{R}^n.$$

Since (u_{ε_h}) is a concentrating sequence, it results that

$$\lim_{|x|\to\infty} v_h(x) = 0, \quad \text{uniformly in } h \in \mathbb{N}.$$

Then, setting $\inf_{x\in\mathbb{R}^n} V(x) = V_0$, given $\eta > 0$ there exists a positive constant R_η independent of h such that

$$V(z_0 + \varepsilon_h x) v_h^{p-1}(x) - K(z_0 + \varepsilon_h x) v_h^{q-1}(x) \geq (V_0 - \eta) v_h^{p-1}(x),$$

for every $|x| \geq R_\eta$. It follows that the inequality

$$-\mathrm{div}\,(|\nabla v_h|^{p-2} \nabla v_h) + (V_0 - \eta) v_h^{p-1} \leq 0 \tag{2.11}$$

holds true for every $h \in \mathbb{N}$ and $|x| \geq R_\eta$. Define now the function

$$\Phi(x) := C_\eta \exp\Big\{ -\Big(\frac{V_0 - \eta}{p-1}\Big)^{1/p} |x| \Big\},$$

where $C_\eta := \exp\big\{\big(\frac{V_0-\eta}{p-1}\big)^{1/p} R_\eta\big\} \max_{|x|=R_\eta} v_h(x)$. Notice that, since v_h is uniformly bounded, we can assume that C_η is independent of h. Now, exactly the same computations of [14, Theorem 2.8] entail

$$-\mathrm{div}\,(|\nabla \Phi|^{p-2} \nabla \Phi) + (V_0 - \eta) \Phi^{p-1} \geq 0. \tag{2.12}$$

Testing inequalities (2.11) and (2.12) with $\phi = (v_h - \Phi)^+$ on $\{|x| \geq R_\eta\}$ yields

$$\int_{\{|x|\geq R_\eta\}\cap\{v_h > \Phi\}} \big(|\nabla v_h|^{p-2} \nabla v_h \cdot \nabla(v_h - \Phi) + (V_0 - \eta) v_h^{p-1} (v_h - \Phi)\big) dx \leq 0,$$

$$\int_{\{|x|\geq R_\eta\}\cap\{v_h > \Phi\}} \big(|\nabla \Phi|^{p-2} \nabla \Phi \cdot \nabla(v_h - \Phi) + (V_0 - \eta) \Phi^{p-1} (v_h - \Phi)\big) dx \geq 0.$$

By subtracting the previous inequalities and taking into account that

$$\sum_{i=1}^{n}(|\xi|^{p-2}\xi_i - |\zeta|^{p-2}\zeta_i)(\xi_i - \zeta_i) > 0, \quad \text{for every } \xi, \zeta \in \mathbb{R}^n,\ \xi \neq \zeta,$$

we get

$$\int_{\{|x|\geq R_\eta\}\cap\{v_h>\Phi\}} (v_h^{p-1} - \Phi^{p-1})(v_h - \Phi)\,dx \leq 0.$$

Since v_h and Φ are continuous functions, it has to be that

$$\{|x| \geq R_\eta\} \cap \{v_h > \Phi\} = \emptyset, \quad \text{for every } h \in \mathbb{N},$$

which implies the assertion. □

Theorem 2.11. *Let (u_{ε_h}) be a sequence of solutions of the problem*

$$\begin{cases} -\varepsilon^p \Delta_p u + V(x)u^{p-1} = K(x)u^{q-1} & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \end{cases} \tag{2.13}$$

which concentrates at $z_0 \in \mathbb{R}^n$. Then, the following facts hold:

(a) *the vectors $\nabla V(z_0)$ and $\nabla K(z_0)$ are proportional;*
(b) *$z_0 \in \mathfrak{C}$; that is, z_0 is a weak-concentration point for* (2.13);
(c) *if $1 < p \leq 2$, then all the partial derivatives of Σ at z_0 exist and $\nabla\Sigma(z_0) = 0$; that is, z_0 is a critical point of Σ.*

Proof. By virtue of Lemma 2.10 we can pass to the limit in equation (2.10) and get assertions (a) and (b) as in Theorem 2.6. If $1 < p \leq 2$, by combining the results of [5, 15] and [23], for every $z \in \mathbb{R}^n$, problem (P_z) admits a unique positive C^1 solution (up to translations) such that $u(x) \to 0$ as $|x| \to \infty$. Then $G(z_0) = \{v_0\}$ and assertion (c) follows by the corresponding assertion in Theorem 2.6. □

3. The semilinear case

The main goal of this section is that of getting, in the particular case $\beta(\xi) = \xi$, namely semilinear equations, a more accurate version of Theorem 2.6 involving the Clarke subdifferential of the ground-state function Σ. We wish to stress that we have in mind the case when f is *not* simply the power nonlinearity u^{p-1} (cf. Remark 3.2).

For $z \in \mathbb{R}^n$ fixed, we consider the limiting functional $I_z : H^1(\mathbb{R}^n) \to \mathbb{R}$,

$$I_z(u) := \alpha(z)\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \frac{V(z)}{p}\int_{\mathbb{R}^n} |u|^p\,dx - K(z)\int_{\mathbb{R}^n} F(u)\,dx,$$

whose critical points are of course solutions of (P_z). The minimax levels c_z of I_z are defined according to (2.1). Throughout the rest of this section, we will denote by $S(z)$ the set of all the nontrivial solutions of (P_z) corresponding to the energy level $\Sigma(z)$ (the set of ground-states). It is known that $S(z) \neq \emptyset$ for every $z \in \mathbb{R}^n$ (see [2]).

As the next lemma shows, in this particular situation, the function Σ has further regularity properties (and in some cases it relates to the maps $\partial\Gamma^-$ and $\partial\Gamma^+$).

Lemma 3.1. *If $p = 2$ and condition (2.2) holds, then the following facts hold:*

(i) *Σ is locally Lipschitz;*
(ii) *the directional derivatives from the left and the right of Σ at z along w, $\big(\frac{\partial\Sigma}{\partial w}\big)^-(z)$ and $\big(\frac{\partial\Sigma}{\partial w}\big)^+(z)$ respectively, exist at every point $z \in \mathbb{R}^n$, and it holds that*

$$\Big(\frac{\partial\Sigma}{\partial w}\Big)^-(z) = \sup_{v\in S(z)} \nabla_z I_z(v)\cdot w,$$

$$\Big(\frac{\partial\Sigma}{\partial w}\Big)^+(z) = \inf_{v\in S(z)} \nabla_z I_z(v)\cdot w,$$

for every $z, w \in \mathbb{R}^n$. In particular, if $G(z) = S(z)$, we have

$$\partial\Gamma^-(z;w) = \Big(\frac{\partial\Sigma}{\partial w}\Big)^-(z) \quad \text{and} \quad \partial\Gamma^+(z;w) = \Big(\frac{\partial\Sigma}{\partial w}\Big)^+(z), \tag{3.1}$$

for every $w \in \mathbb{R}^n$.

Proof. By the results of [27], Σ is a locally Lipschitz map. We remark here that, since z acts as a parameter, the functional I_z is invariant under orthogonal change of variables. Therefore, without loss of generality, to get the formulas for the left and right directional derivatives of Σ, it suffices to show that

$$\Big(\frac{\partial\Sigma}{\partial z_i}\Big)^-(z) = \sup_{v\in S(z)} \Big[\frac{\partial\alpha}{\partial z_i}(z)\int_{\mathbb{R}^n}\frac{|\nabla v|^2}{2} + \frac{\partial V}{\partial z_i}(z)\int_{\mathbb{R}^n}\frac{|v|^p}{p} - \frac{\partial K}{\partial z_i}(z)\int_{\mathbb{R}^n}F(v)\Big],$$

$$\Big(\frac{\partial\Sigma}{\partial z_i}\Big)^+(z) = \inf_{v\in S(z)} \Big[\frac{\partial\alpha}{\partial z_i}(z)\int_{\mathbb{R}^n}\frac{|\nabla v|^2}{2} + \frac{\partial V}{\partial z_i}(z)\int_{\mathbb{R}^n}\frac{|v|^p}{p} - \frac{\partial K}{\partial z_i}(z)\int_{\mathbb{R}^n}F(v)\Big],$$

for every $z \in \mathbb{R}^n$ and $i = 1, \dots, n$. These can be obtained arguing as in [18, 27]. Finally, formulas (3.1) follow by the definition of $\partial\Gamma^+(z;w)$ and $\partial\Gamma^-(z;w)$. $\square$

Remark 3.2. Assume that $p = 2$, K is bounded from below away from zero, and $f(u) = u^{q-1}$, where $2 < q < 2^*$. Then Σ is smooth and it can be given an explicit form (cf. [18, Remark 2.1]): there exists $C_q > 0$ such that

$$\Sigma(z) = C_q \Big[\frac{V(z)}{K(z)}\Big]^{\frac{q}{q-2}-\frac{n}{2}} \sqrt{\alpha(z)K(z)}, \quad \text{for every } z \in \mathbb{R}^n.$$

Let us now recall from [4] two definitions that will be useful in the sequel.

Definition 3.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near a given point $z \in \mathbb{R}^n$. The generalized derivative of the function f at z along the direction $w \in \mathbb{R}^n$ is defined by*

$$f^0(z; w) := \limsup_{\substack{\xi \to z \\ \lambda \to 0+}} \frac{f(\xi + \lambda w) - f(\xi)}{\lambda}.$$

Definition 3.4. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near a given point $z \in \mathbb{R}^n$. The Clarke subdifferential (or generalized gradient) of f at z is defined by $\partial f(z) := \big\{\eta \in \mathbb{R}^n : \ f^0(z, w) \geq \eta \cdot w, \ \text{for every } w \in \mathbb{R}^n\big\}$.*

By [4, Proposition 2.3.1] we learn that

Proposition 3.5. *For every $z \in \mathbb{R}^n$, the set $\partial f(z)$ is nonempty and convex, and $\partial(-f)(z) = -\partial f(z)$.*

The next is the main result of this section.

Theorem 3.6. *Assume that (u_{ε_h}) is a sequence of solutions of the problem*

$$\begin{cases} -\varepsilon^2 \mathrm{div}\,(\alpha(x)\nabla u) + V(x)u = K(x)f(u) & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n \end{cases} \tag{3.2}$$

which concentrates at z_0. Then, the following facts hold:

(a) *the vectors $\nabla\alpha(z_0)$, $\nabla V(z_0)$, and $\nabla K(z_0)$ are linearly dependent;*
(b) *$z_0 \in \mathfrak{C}$; that is, z_0 is a weak-concentration point for (3.2);*
(c) *if either $G(z_0) = S(z_0)$ or*

$$\varepsilon_h^{-n} J_{\varepsilon_h}(u_{\varepsilon_h}) \to c_{z_0}, \tag{3.3}$$

where

$$J_\varepsilon(v) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} \alpha(x)|\nabla v|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^n} V(x)|v|^2\,dx - \int_{\mathbb{R}^n} K(x)F(v)\,dx, \tag{3.4}$$

we have $0 \in \partial\Sigma(z_0)$; that is, z_0 is a critical point of Σ in the sense of the Clarke subdifferential;

(d) *if* $S(z_0) = \{v_0\}$*, then all the partial derivatives of* Σ *at* z_0 *exist and*

$$\nabla\Sigma(z_0) = 0;$$

that is, z_0 *is a critical point of* Σ*.*

Proof. For problem (3.2) it is possible to prove the existence of uniform exponentially decaying barriers. Then we can pass to the limit in equation (2.10), to get assertions (a) and (b) as in Theorem 2.6. If $G(z_0) = S(z_0)$, by combining formulas (3.1) of Lemma 3.1 with (b) of Theorem 2.6, we have

$$\Big(\frac{\partial\Sigma}{\partial w}\Big)^-(z_0) \geq 0 \quad \text{and} \quad \Big(\frac{\partial\Sigma}{\partial w}\Big)^+(z_0) \leq 0, \tag{3.5}$$

for every $w \in \mathbb{R}^n$. In particular, it holds that

$$\Big(\frac{\partial(-\Sigma)}{\partial w}\Big)^+(z_0) \geq 0, \quad \text{for every } w \in \mathbb{R}^n.$$

Then, by the definition of $(-\Sigma)^0(z_0; w)$ we get

$$(-\Sigma)^0(z_0; w) \geq \Big(\frac{\partial(-\Sigma)}{\partial w}\Big)^+(z_0) \geq 0, \quad \text{for every } w \in \mathbb{R}^n.$$

By the definition of $\partial(-\Sigma)(z_0)$ we immediately get $0 \in \partial(-\Sigma)(z_0)$, which, together with Proposition 3.5, yields assertion (c). To prove the same conclusion when (3.3) holds, we simply remark that $c_{z_0} = \Sigma(z_0)$. Therefore, if v_0 is the limit of the sequence (v_h) defined in (2.8), then $v_0 \in S(z_0)$ because we can exploit again some exponential barrier to pass to the limit. As a consequence, arguing as in Theorem 2.6, it follows that inequalities (3.5) hold and we are reduced to the previous case. Finally, if $S(z_0) = \{v_0\}$, the map Σ admits all the directional derivatives at z_0 and, by virtue of (3.5) they are equal to zero, which proves (d). □

We would like to remark that a different definition of concentration has been used in [13]. We recall it here, suitably adapted to our purposes.

Definition 3.7. Assume that $u_\varepsilon \in C^2(\mathbb{R}^n)$ is a family of solutions of (3.2), and let J_ε be as in (3.4). Moreover, let $x_\varepsilon \in \mathbb{R}^n$ be such that $\max_{x\in\mathbb{R}^n} u_\varepsilon = u_\varepsilon(x_\varepsilon)$. We say that u_ε concentrates at $z_0 \in \mathbb{R}^n$ if the following facts hold:

(i) $\lim_{\varepsilon\to 0} x_\varepsilon = z_0$;

(ii) $\lim_{\varepsilon\to 0} \varepsilon^{-n} J_\varepsilon(u_\varepsilon) = c_{z_0}$.

It is not difficult to check that if (u_ε) is a sequence as in the above definition, then (u_ε) concentrates at z_0 in the sense of Definition 2.8, vanishing at an exponential rate away from z_0 (cf. [13, Lemma 4.2]). In particular, according to (c) of Theorem 3.6, we have $0 \in \partial\Sigma(z_0)$.

We finish the paper with an open problem. Assume that (u_h) is a sequence of solutions of problem (3.2). Suppose that these solutions concentrate at $z_0 \in \mathbb{R}^n$ and $S(z_0) = \{v_0\}$. Is it possible to prove that z_0 is a C^1-stable critical point of Σ, according to the definition of Yanyan Li [16]?

Acknowledgment. The authors are indebted to the anonymous referee for her/his careful reading of the manuscript and for valuable remarks and comments.

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