

EXPONENTIALLY ACCURATE BALANCE DYNAMICS

D. WIROSOETISNO

School of Mathematics, University of Edinburgh, Edinburgh EH9 3JZ, UK

(Submitted by: Roger Temam)

Abstract. By explicitly bounding the growth of terms in a singular perturbation expansion with a small parameter ε , we show that it is possible to find a solution that satisfies a balance relation (which defines the slow manifold) up to an error that scales exponentially in ε as $\varepsilon \rightarrow 0$. This is first done for a generic finite-dimensional dynamical system with polynomial nonlinearity, followed by a continuous fluid case. In addition, for the finite-dimensional system, we show that, properly initialized, the solution of the full model stays within an exponential distance to that of the balance equation (i.e., evolution on the slow manifold) over a timescale of order one (independent of ε).

1. INTRODUCTION

An important factor that shapes atmospheric dynamics is the timescale separation between the fast inertial-gravity waves and the slow vortical dynamics, the ratio of whose timescales we denote by a small parameter ε . The observed large-scale dynamics of the atmosphere takes place mainly over the slow timescale, with the fast components being weak. Two of the earliest dynamical theories of atmospheric circulations, namely geostrophic balance and quasi-geostrophic dynamics, are based on this observation (see [7] for a historical survey). More systematically, these can be seen to arise as the zeroth and first order approximations in a perturbation expansion (cf. [19]). Naturally, one would like to do better, and an obvious route would be to compute higher-order approximations, generically known as balance models (or balance equations).

There are many different ways to construct balance equations perturbatively, depending on the choice of variables and perturbation expansions (or iterative methods) used. A common feature of these methods is the postulate that there exists a time-independent relation between some of the variables.

Accepted for publication: October 2003.

AMS Subject Classifications: 34E15, 76M45, 86A10.

Present address: The Institute for Scientific Computing and Applied Mathematics, Rawles Hall, Indiana University, Bloomington, IN 47405-7106, USA.

Equivalently, one postulates that the dynamics take place on an invariant “slow manifold,” of lower dimensionality than the full phase space, on which no gravity waves are present. These ideas date back at least to Charney (1948), and were formulated more or less in their present form by Leith [11] and Lorenz [12]; see also [16] and [10].

Balance equations are useful because they have been shown to be good approximations to the parent model, either in terms of pointwise accuracy or in terms of their qualitative behavior. Moreover, going to higher orders has also been shown to improve the accuracy, as done by Lorenz [13] for a lower-order model, and by, e.g., Allen [2] and Ford *et al.* [9] for more realistic atmospheric models. On a closely related front, the various practical methods devised to suppress fast gravity-wave oscillations from numerical weather-forecast models ([14] and [3]) have also been effective in suppressing the fast “unbalanced” oscillations for some time; these are in fact equivalent to the perturbation expansion discussed in the present paper, as shown in [20].

A natural question to ask is therefore: is there a limit to the accuracy of balanced equations (or to the suppression of fast oscillations), and if so what is it? As pointed out by Lorenz [12], there are two separate issues here: one is the existence of the object one is attempting to compute (an exact balance relation, or an invariant slow manifold), the other is the convergence of the method used to compute it (the asymptotic approximation).

After some years of controversy in the atmospheric science community, it has now been generally accepted that an invariant slow manifold does not exist in general. Warn [23] argued that the inevitable spectral overlap between chaotic slow dynamics and the fast dynamics generically implies the presence of inertial-gravity wave activity and, thus, the nonexistence of exact balance. Numerical evidence of generation of gravity waves by “quiet” vortical flows have been found for lower-order models ([25] and [13]); analogous results for continuous systems were given in [8]. We note also an interesting study by Daley [6] where he found that the forecast error is less sensitive to errors in the gravity-wave component of the dynamics than in the vortical part; in other words, the vortical dynamics are “more chaotic” than the gravity waves.

Analytically, it has been shown ([4] and [5]) that a lower-dimensional invariant manifold cannot exist for every value of the slow variable. Vanneste and Yavneh [22] have recently given an explicit example where gravity waves

are generated by vortical flow in a continuous fluid model, the three-dimensional Boussinesq equations; using asymptotic analysis, the residual gravity-wave activity in the Lorenz–Krishnamurthy model is estimated in [21]. In all these works, the gravity-wave amplitude decays rapidly, faster than any power of ε , as $\varepsilon \rightarrow 0$.

These works established in specific cases that there is a finite, although weak, gravity-wave emission by vortical flows, no matter how well balanced they initially are; in other words, they settled the first issue by proving the nonexistence of an invariant slow manifold. In this paper, we show that, even within the limitations of asymptotic expansions (cf. the second issue above), it is possible to obtain upper bounds on gravity-wave emission that also decay exponentially as $\varepsilon \rightarrow 0$, and that these bounds hold over a timescale independent of ε . Phrased differently, we show that it is possible to define a manifold which is slow and invariant up to an exponentially small error (defined below), and that solutions starting on this manifold stay close to it over a timescale independent of ε . The present work also differs from the ones cited earlier in that the bounds obtained here are valid in general, provided that the variables are bounded (in some appropriate norm).

Following Lorenz, in the first part of this paper we develop the main ideas in the context of simple ordinary differential equations. In Section 4 we show how this works for a particular continuous system. For concreteness, let us adopt as our parent model the dynamical system

$$\frac{dx}{dt} + \frac{1}{\varepsilon} Lx = F(x, y), \tag{1.1a}$$

$$\frac{dy}{dt} = G(x, y), \tag{1.1b}$$

where $x \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$; L is a skew-Hermitian, nonsingular $p \times p$ matrix; and F and G are p - and q -vector-valued polynomial functions of their arguments. One may think of this as the (finite-dimensional truncation of the) primitive, three-dimensional Boussinesq, or shallow-water equations and their analogs; the timescale separation parameter ε will typically be some combination of the Rossby and Froude numbers. The properties of the “inertia–gravity wave” matrix L imply, among other things, that the fast variable x undergoes undamped rapid oscillations in the limit $\varepsilon \rightarrow 0$.

In this paper, we shall use the method described in [24]. Following them, the slow solutions of (1.1) is obtained by taking as ansatz

$$x = U(y; \varepsilon), \tag{1.2}$$

namely that the fast variable x is slaved to the slow variable y . The function U is determined by substituting (1.2) into (1.1a) and using (1.1b), giving

$$U'(y; \varepsilon) \cdot G(U(y; \varepsilon), y) + \frac{1}{\varepsilon} LU(y; \varepsilon) - F(U(y; \varepsilon), y) = 0, \quad (1.3)$$

where the dot denotes inner multiplication¹. This is the “superbalance” equation of Lorenz [12]. The evolution of the slow variable y is governed by the “balance equation”

$$\frac{dy}{dt} = G(U(y; \varepsilon), y). \quad (1.4)$$

In other words: For a given $U(y; \varepsilon)$, the slaving ansatz (1.2) defines a manifold in phase space. The superbalance relation (1.3) states the invariance of this manifold, namely, that solutions that start on it stay on it. The dynamics on this manifold are given by the (exact) balance equation (1.4). It is shown in [26] that the manifold obtained in this way is slow, in the sense that it is free of fast oscillations.

As discussed above, no such $U(y; \varepsilon)$ likely exists in general. Nevertheless, the concepts of slaving relation and of slow manifold are still useful. This paper is devoted to “asymptotically accurate” solutions of (1.3), that is, to finding a $U_{\text{app}}(y; \varepsilon)$ that, when substituted into (1.3), results in a small but nonzero remainder on the right-hand side. In the next section, we show that this remainder can be made to scale as $\exp(-\kappa/\varepsilon^{d'})$ as $\varepsilon \rightarrow 0$ for finite-dimensional systems. In Section 3, it is shown that solutions of the balance model thus constructed stay exponentially close to solutions of the full parent model (for balanced initial conditions) over a timescale independent of ε . Finally, in Section 4, we show how the method of Section 2 may be generalized to a continuous fluid model. The continuous generalization of the finite-time result of Section 3 is mathematically more involved and is deferred to a future work. Logically, the main developments of Sections 3 and 4 depend on Section 2, but are mutually independent.

2. OPTIMAL TRUNCATION

Let us consider a series solution of (1.3),

$$U(y; \varepsilon) = U_0(y) + \varepsilon U_1(y) + \varepsilon^2 U_2(y) + \dots. \quad (2.1)$$

The aim of this section is to show that, by truncating this expansion optimally, it is possible to satisfy its defining equation (1.3) up to a remainder which decays faster than any power of ε as $\varepsilon \rightarrow 0$.

¹Thus, in components, the first term of (1.3) is $(U' \cdot G(U, y))_i = (\partial U_i / \partial y_j) G_j(U, y)$.

Upon substituting (2.1) in (1.3), we find at leading order,

$$LU_0 = 0, \quad (2.2)$$

which implies that $U_0 = 0$ since L is nonsingular. Due to analyticity of F and G at $x = 0$ (which holds in all models of interest here), and equating terms of like powers in ε , we Taylor-expand F about $x = 0$,

$$F(U, y) = F(0, y) + \varepsilon F'(0, y) \cdot U_1 + \varepsilon^2 [F''(0, y) \cdot U_2 + \frac{1}{2} F''(0, y) \cdot U_1 \otimes U_1] + \dots, \quad (2.3)$$

and do similarly² for G , giving, at the next order,

$$LU_1 = F(0, y). \quad (2.4)$$

To compute the higher-order U_n , we write the expansion (2.3) in the form,

$$F(U, y) = \sum_{j=0}^{\infty} \sum_{|m|=n} \frac{\varepsilon^n}{j!} F^{(j)}(0, y) \cdot U_{m_1} \otimes \dots \otimes U_{m_j}, \quad (2.5)$$

where $|m| := m_1 + \dots + m_j$ and $F^{(j)} := (\nabla_x)^j F$ and similarly³ for $G^{(j)}$. Equating terms of like orders gives, after some computation, the recursion relation,

$$\begin{aligned} LU_n &= \sum_{|m|=n-1} \frac{1}{j!} F^{(j)}(0, y) \cdot U_{m_1} \otimes \dots \otimes U_{m_j} \\ &\quad - \sum_{|k|+m=n-1} \frac{1}{j!} U'_m \cdot G^{(j)}(0, y) \cdot U_{k_1} \otimes \dots \otimes U_{k_j}. \end{aligned} \quad (2.6)$$

Noting that, for a given order n , the right-hand side contains only U_m of lower orders, one can solve for U_n order-by-order.

To minimize clutter, we will henceforth assume that F and G are quadratic in (x, y) ,

$$\begin{aligned} (F(x, y))_i &= f_{ijk}^{(xx)} x_j x_k + f_{ijk}^{(xy)} x_j y_k + f_{ijk}^{(yy)} y_j y_k + f_{ij}^{(x)} x_j + f_{ij}^{(y)} y_j, \\ (G(x, y))_i &= g_{ijk}^{(xx)} x_j x_k + g_{ijk}^{(xy)} x_j y_k + g_{ijk}^{(yy)} y_j y_k + g_{ij}^{(x)} x_j + g_{ij}^{(y)} y_j. \end{aligned} \quad (2.7)$$

²Notation: in components, $(F'' \cdot U_1 \otimes U_1)_i := \sum_{jk} (\partial^2 F_i / \partial x_j \partial x_k) U_{1,j} U_{1,k}$ and similarly in (2.5) below.

³We note that prime on U denotes the derivative with respect to y , while prime on F or G denotes the derivative with respect to x .

We note that most fluid models have (a formulation with) only quadratic nonlinearities, so this assumption is quite realistic. In this case, (2.6) reads

$$\begin{aligned} LU_n &= F' \cdot U_{n-1} + \frac{1}{2} \sum_{m=1}^{n-2} F'' \cdot U_m \otimes U_{n-m-1} \\ &- U'_{n-1} \cdot G - \sum_{m=1}^{n-2} U'_m \cdot G' \cdot U_{n-m-1} - \frac{1}{2} \sum_{k+m \leq n-2} U'_m \cdot G'' \cdot U_k \otimes U_{n-m-k-1}, \end{aligned} \quad (2.8)$$

where it is understood that F , G , and their derivatives are evaluated at $(0, y)$, while U_n and its derivatives have argument y . Now since F and G are both polynomials, all the U_n will be as well. Moreover, one has $\deg U_n \leq n+1$ as one can show by induction using (2.8) and the fact that $\deg U_0 = 0 \leq 1$. Therefore, let us write

$$U_n(y) = \sum_{m \leq n} \sum_i a_{i_1 \dots i_{n+1}} y_{i_1} \cdots y_{i_{n+1}}. \quad (2.9)$$

a. Bounding $U_n(y; \varepsilon)$. We first fix a convention: For vector objects such as y and $U_n(y; \varepsilon)$, we use the L^1 norm, i.e.,

$$|y| := |y_1| + \cdots + |y_q|, \quad |U_n| := |U_{n,1}| + \cdots + |U_{n,p}|,$$

and similarly for higher-rank objects such as U'_n or G'' , i.e.,

$$|U'_n| := \sum_{jk} |\partial U_{n,j} / \partial y_k|, \quad |F''| := \sum_{ijk} |a_{ijk}|,$$

and so on.

From (2.9) and these definitions, the following inequalities follow:

$$|U_n(y)| \leq (\sum |a_{i_1 \dots i_{n+1}}|) (1 + |y|)^{n+1}, \quad (2.10a)$$

$$|U'_n(y)| \leq (n+1) c (\sum |a_{i_1 \dots i_{n+1}}|) (1 + |y|)^n, \quad (2.10b)$$

where the constant c is independent of n . We now introduce positive numbers C_n , to be determined shortly, that majorizes the coefficient of the right-hand side of (2.10a), viz.,

$$\sum |a_{i_1 \dots i_{n+1}}| \leq C_n. \quad (2.11)$$

Demanding that the C_n grow at least exponentially, or,

$$C_n C_m \leq C_{n+m}, \quad (2.12)$$

we compute from (2.8), where the first inequality holds for any $m = 1, \dots, n-2$, et.,

$$\begin{aligned} |U_n(y)| &\leq c(|F'| |U_{n-1}| + n|F''| |U_m| |U_{n-m-1}| \\ &+ |U'_{n-1}| |G| + n|U'_m| |G'| |U_{n-m-1}| + n^2 |U'_m| |G''| |U_k| |U_{n-m-k-1}|) \end{aligned}$$

$$\begin{aligned} &\leq [cC_{n-1} + ncC_{n-1} + (n+1)cC_{n-1} \\ &\quad + cn(n+1)C_{n-1} + cn^2(n+1)C_{n-1}](1+|y|)^{n+1} \\ &\leq n^3c_oC_{n-1}(1+|y|)^{n+1}, \end{aligned} \tag{2.13}$$

where here and for the rest of this paper we have adopted the following convention: c denotes a constant which depends on F and G but not on n , and which may not be the same each time the symbol is used. Taking $C_0 = 1$, we can therefore choose

$$C_n = c_o^n(n!)^3. \tag{2.14}$$

Note that this choice is consistent with the assumption (2.12) above.

We now fix $\varepsilon \leq 1$ and look for a truncation level $n_*(\varepsilon)$ that minimizes the product $\varepsilon^n C_n(1+|y|)^{n+1}$. Treating n as continuous, we set

$$\begin{aligned} 0 &= \frac{d}{dn} \log[\varepsilon^n(1+|y|)^n c_o^n (n!)^3] \\ &\simeq \frac{d}{dn} [n \log \varepsilon + n \log(1+|y|) + n \log c_o + 3n \log n - 3n] \end{aligned} \tag{2.15}$$

(where we have used Stirling’s approximation for the factorial—we will often do this without mention below), giving

$$n_*(\varepsilon) \simeq (\varepsilon c_o(1+|y|))^{-1/3}. \tag{2.16}$$

b. Bound on the remainder. Now consider the $n_*(\varepsilon)$ -th order (asymptotic) solution to the superbalance equation (1.3),

$$U_*(Y; \varepsilon) := \sum_{n=1}^{n_*} \varepsilon^n U_n(Y), \tag{2.17}$$

where the U_n have been computed using (2.8) and $n_*(\varepsilon)$ given by (2.16). For notational convenience we write $U_k \equiv 0$ for $k > n_*(\varepsilon)$. Note that we have written Y in place of y ; this notation will be useful in the next section. Denote by $R(Y; \varepsilon)$ the remainder when $U_*(Y, \varepsilon)$ is substituted into the superbalance equation (1.3), viz.,

$$\begin{aligned} R(Y; \varepsilon) &:= F(U_*, Y) - (U_*)'(Y) \cdot G(U_*, Y) - \frac{1}{\varepsilon} L U_*(Y) \\ &= \varepsilon^{n_*} F' \cdot U_{n_*} - \varepsilon^{n_*} U'_{n_*} \cdot G + \sum_{k+l \geq n_*} \varepsilon^{k+l} F'' \cdot U_k \otimes U_l \\ &\quad - \sum_{k+m \geq n_*} \varepsilon^{k+m} U'_m \cdot G' \cdot U_k - \sum_{k+l+m \geq n_*} \varepsilon^{k+l+m} U'_m \cdot G'' \cdot U_k \otimes U_l. \end{aligned} \tag{2.18}$$

Here all the U_k are evaluated at Y , and the derivatives of F and G which are denoted by primes are evaluated at $(0, Y)$, as before.

We now show that our choice (2.16) for $n_*(\varepsilon)$ implies that $R(Y(t); \varepsilon)$ is exponentially small in ε . First, let us note that

$$\begin{aligned} \varepsilon^{n_*} |U_{n_*}| &\leq \varepsilon^{n_*} C_{n_*} (1 + |Y|)^{n_*+1} \\ &= \varepsilon^{n_*} c_o^{n_*} (n_*!)^3 (1 + |Y|)^{n_*+1} = (n_*! n_*^{-n_*})^3 (1 + |Y|) \\ &\simeq e^{-3n_*} (1 + |Y|) = (1 + |Y|) \exp[-3(\varepsilon c_o (1 + |Y|))^{-1/3}]. \end{aligned}$$

Now the first and second terms in (2.18) are bounded as

$$\begin{aligned} \varepsilon^{n_*} |F' \cdot U_{n_*}| &\leq c(1 + |Y|) e^{-3n_*}, \\ \varepsilon^{n_*} |U'_{n_*} \cdot G| &\leq c(1 + |Y|) n_* e^{-3n_*}. \end{aligned} \tag{2.19}$$

To bound the next term, we first write

$$\varepsilon^{k+l} |F'' \cdot U_k \otimes U_l| \leq c |\varepsilon^k U_k| \cdot |\varepsilon^l U_l| \leq c (\varepsilon^k C_k (1 + |Y|)^{k+1}) (\varepsilon^l C_l (1 + |Y|)^{l+1}).$$

Since $\varepsilon^k C_k (1 + |Y|)^{k+1}$ is decreasing for $k \leq n_*$, the largest terms in the sum are those with the smallest k and l , which are typically of the order $\varepsilon^{n_*} (C_{n_*/2})^2 (1 + |Y|)^{n_*+2}$. Substituting the expression for C_n from (2.14) and remembering that the sum consists of order n_*^2 terms, we have

$$\begin{aligned} \left| \sum_{k+l \geq n_*} F'' \cdot U_k \otimes U_l \right| &\leq c n_*^2 \varepsilon^{n_*} (C_{n_*/2})^2 (1 + |Y|)^{n_*+2} \\ &\leq c n_*^2 \varepsilon^{n_*} C_{n_*} (1 + |Y|)^{n_*+2} \leq c (1 + |Y|)^2 n_*^2 e^{-3n_*}, \end{aligned} \tag{2.20}$$

where in the last line we have used (2.12). Similarly, the next sum is bounded by $c(1 + |Y|)^2 n_*^3 e^{-3n_*}$. Finally, the last sum is bounded as

$$\begin{aligned} \sum_{k+l+m \geq n_*} \varepsilon^{k+l+m} |U'_m \cdot G'' \cdot U_k \otimes U_l| &\leq c n_*^4 (C_{n_*/3})^3 (1 + |Y|)^{n_*+2} \\ &\leq c n_*^4 e^{-3n_*} (1 + |Y|)^2. \end{aligned} \tag{2.21}$$

Putting all these together, we have the estimate

$$\begin{aligned} |R(Y; \varepsilon)| &\leq c n_*^4 e^{-3n_*} (1 + |Y|)^2 \\ &\leq C_o (1 + |Y|)^{-2} \varepsilon^{-4/3} \exp(-\kappa \varepsilon^{-1/3} (1 + |Y|)^{-1/3}). \end{aligned} \tag{2.22}$$

We summarize our result so far in the following

Lemma 2.1. *For the system (1.1) with F and G of the form (2.7), one can find a $U_*(Y; \varepsilon)$ which satisfies the superbalance equation (1.3) up to an exponentially small remainder,*

$$R(Y; \varepsilon) := F(U_*, Y) - (U_*)'(Y) \cdot G(U_*, Y) - \frac{1}{\varepsilon} L U_*(Y) \tag{2.23}$$

with

$$|R(Y; \varepsilon)| \leq C_o \varepsilon^{-4/3} \exp(-\kappa \varepsilon^{-1/3} (1 + |Y|)^{-1/3}). \tag{2.24}$$

We have thus shown that it is possible to find a slaving relation $x = U_*(Y; \varepsilon)$ that satisfies the defining relation (1.3) up to an exponentially small remainder $R(Y; \varepsilon)$. Since we can add to $U_*(Y; \varepsilon)$ any function of Y which is exponentially small in ε and still satisfies (2.24) (possibly with a different C_o) it follows that the slaving relation $x = U_*(Y; \varepsilon)$ can be defined up to an exponential accuracy. Put differently, our result shows that one can define an exponentially thin region (called “the fuzzy manifold” by some authors) in which the fast (“gravity-wave”) activity can be made exponentially weak. We note that the counterexamples mentioned in the Introduction showed that this result cannot be improved in general, although the constants and the exponent $1/3$ may not be optimal.

A couple of additional remarks are in order. First, the reader may have noticed that, through (2.11), the result quickly becomes worse as the number of variables increases and it becomes meaningless in the limit of infinite-dimensional systems. However, as will be clear from the development in Section 4, it is possible in some cases to construct a norm that, by weighting different degrees of freedom differently, makes the result effectively independent of the number of degrees of freedom.

Second, our result also depends on the degree d of the x -nonlinearity of the original system, with the bound growing as $\exp(-\kappa/\varepsilon^{1/(d+1)})$. Again, this bound becomes useless as $d \rightarrow \infty$. Using more delicate estimates, in some cases (such as F and G complex analytic), it may be possible to make the bound independent of d , but we shall not pursue this here for simplicity in Section 4 below.

3. VALIDITY FOR FINITE TIME

In the previous section, we have taken the slow variable y as given, effectively fixed. In this section we consider the full dynamics, and obtain a bound on the error—the difference between the solution of the full model and that of the balance equation. It turns out that this error is also exponentially small over timescales of order one.

First, we need to bound the solution of the high-order balance equation. Let $Y(\cdot)$ be the solution of

$$\frac{dY}{dt} = G(U_*(Y; \varepsilon), Y). \quad (3.1)$$

The theory of ordinary differential equations tell us that the solution of (3.1) exists as long as the right-hand side remains bounded. Therefore let us take T_1 such that $|Y(t)| \leq 2|Y(0)|$ for $0 \leq t \leq T_1$. In order to estimate the right-hand side, we need a bound on $|U_*(Y; \varepsilon)|$. This is obtained using the fact we have proved that $\varepsilon^n |U_n|$ is bounded by a decreasing sequence for $n \leq n_*$:

$$\begin{aligned} \max_{|Y| \leq 2|Y(0)|} |U_*(Y; \varepsilon)| &\leq \max_{|Y| \leq 2|Y(0)|} \left| \sum_{n=1}^{n_*} \varepsilon^n U_n(Y) \right| \\ &\leq n_* \max_{|Y| \leq 2|Y(0)|} (\varepsilon C_1 (1 + |Y|)^2) \leq c_b (|Y(0)|) \varepsilon^{2/3}, \end{aligned} \quad (3.2)$$

where we have substituted $n_* = c\varepsilon^{-1/3}(1 + |Y|)^{-1/3}$ (cf. (2.16)) to obtain the last inequality. This means that $|U_*(Y; \varepsilon)|$ can be bounded by a constant of order one (independent of ε) if ε is sufficiently small, similarly for $|G(U_*(Y; \varepsilon), Y)|$. Therefore, T_1 can be taken independently of ε —in other words, the solution of our “optimal order” balance equation exists at least for a time T_1 which is independent of ε . We note that this bound is not optimal: in many practical cases it can be shown that the timescale of existence is qualitatively much longer (i.e., $T_1 \rightarrow \infty$ as $\varepsilon \rightarrow 0$), but the present estimate suffices for our purpose here.

Now let $x = U_*(Y; \varepsilon) + x'$ and $y = Y + y'$, where $Y(t)$ is the solution of (3.1). For initial conditions, we take $(x', y') = 0$, meaning that we start on the slow manifold $(U_*(y; \varepsilon), y)$. Substituting into (1.1) and Taylor-expanding about $(U_*(Y), Y)$, we find that the “deviation” $x'(t)$ and $y'(t)$ satisfy

$$\begin{aligned} \frac{dy'}{dt} &= G(x' + U_*, y' + Y) - G(U_*, Y) \\ &= G_x(U_*, Y) \cdot x' + \frac{1}{2} G_{xx} \cdot x' \otimes x' + G_{xy} \cdot x' \otimes y' \\ &\quad + \frac{1}{2} G_{yy} \cdot y' \otimes y' + G_y \cdot y', \\ \frac{dx'}{dt} + \frac{1}{\varepsilon} Lx' &= R(Y; \varepsilon) + F_x(U_*, Y) \cdot x' \\ &\quad + \frac{1}{2} F_{xx} \cdot x' \otimes x' + F_{xy} \cdot x' \otimes y' + \frac{1}{2} F_{yy} \cdot y' \otimes y' + F_y \cdot y', \end{aligned} \quad (3.3)$$

where all the derivatives of F and G which are denoted by subscripts are evaluated at $(U_*(Y), Y)$; this is to be distinguished from F' , G'' , etc., in the

previous section, which are evaluated at $(0, Y)$. The rest of this section is devoted to finding a time T such that $|x'(t)| + |y'(t)| \leq \varepsilon^{-4/3} \exp(-\kappa/\varepsilon^{1/3})$ for $0 \leq t \leq T$ using a standard Gronwall-type argument. For conciseness, we allow the constants to depend on $|Y(0)|$ from now on.

We take $Y(t)$ as given and integrate (3.3b) with $x'(0) = 0$ to get

$$\begin{aligned} e^{tL/\varepsilon} x'(t) &= \int_0^t e^{sL/\varepsilon} \{ R(s) + F_x(s) \cdot x'(s) + \frac{1}{2} F_{xx}(s) \cdot x'(s) \otimes x'(s) \\ &\quad + F_{xy}(s) \cdot x'(s) \otimes y'(s) + \frac{1}{2} F_{yy}(s) \cdot y'(s) \otimes y'(s) + F_y(s) \cdot y'(s) \} ds; \end{aligned} \quad (3.4)$$

with a slight abuse of notation we have written

$$F_x(t) := F_x(U_*(Y(t); \varepsilon), Y(t)),$$

etc. Remembering that L is skew-Hermitian, this implies the inequality (because of our L^1 norm, the basis of x must be chosen such that L is diagonal; we assume that this has been done)

$$\begin{aligned} |x'(t)| &\leq \left| \int_0^t e^{sL/\varepsilon} R(s) ds \right| + \int_0^t \{ |F_x(s)| |x'(s)| + \frac{1}{2} |F_{xx}(s)| |x'(s)|^2 \\ &\quad + |F_{xy}(s)| |x'(s)| |y'(s)| + |F_y(s)| |y'(s)| + \frac{1}{2} |F_{yy}(s)| |y'(s)|^2 \} ds. \end{aligned} \quad (3.5)$$

Here the norm for matrices has been defined in the usual way, $|F_{xx}| := \sum_{ij} |\partial^2 F / \partial x_i \partial x_j|$. Similarly, we have for $|y'(t)|$,

$$\begin{aligned} |y'(t)| &\leq \int_0^t \{ |G_x(s)| |x'(s)| + \frac{1}{2} |G_{xx}(s)| |x'(s)|^2 + |G_{xy}(s)| |x'(s)| |y'(s)| \\ &\quad + |G_y(s)| |y'(s)| + \frac{1}{2} |G_{yy}(s)| |y'(s)|^2 \} ds. \end{aligned} \quad (3.6)$$

We work in the time interval $0 \leq t \leq T_1$. It follows from the boundedness of $Y(t)$ in this interval that the constants

$$\begin{aligned} c_R &:= \max_{0 \leq t \leq T_1} \left| \int_0^t e^{sL/\varepsilon} R(s) ds \right| = \max_{0 \leq t \leq T_1} \int_0^t |R(s)| ds, \\ c'_1 &:= \max_{0 \leq t \leq T_1} (|F_x(t)| + |F_y(t)| + |G_x(t)| + |G_y(t)|), \\ c_2 &:= \max_{0 \leq t \leq T_1} (|F_{xx}(t)| + |F_{yy}(t)| + |F_{xy}(t)| + |G_{xx}(t)| + |G_{yy}(t)| + |G_{xy}(t)|), \end{aligned} \quad (3.7)$$

all exist. Here c'_1 and c_2 are assumed to be of order one (i.e., independent of ε) and

$$c_R = T_1 C_o \varepsilon^{-4/3} \exp(-\kappa/\varepsilon^{1/3}). \quad (3.8)$$

Let $u(t) := |x'(t)| + |y'(t)|$. Adding (3.5) and (3.6), we find after some manipulation

$$u(t) \leq c_R + \int_0^t [c'_1 u(s) + c_2 u(s)^2] ds. \quad (3.9)$$

We look for a time $T \leq T_1$ such that $u(t) \leq \varepsilon^{-4/3} \exp(-\kappa/\varepsilon^{1/3})$ for $0 \leq t \leq T$. Within this time interval, we can drop the quadratic term in the integrand by replacing c'_1 by $c_1 := c'_1 + c_2 \varepsilon^{-8/3} \exp(-2\kappa/\varepsilon^{1/3})$, which is essentially independent of ε when the latter is small, leaving us with

$$u(t) \leq c_R + c_1 \int_0^t u(s) ds. \quad (3.10)$$

Writing this inequality as

$$\frac{d}{dt} \left[e^{-c_1 t} \int_0^t u(s) ds \right] \leq c_R e^{-c_1 t},$$

and taking the integral of both sides from 0 to t , gives us

$$u(t) \leq c_R \exp(c_1 t). \quad (3.11)$$

Equating the right-hand side at $t=T$ with the desired bound $\varepsilon^{-4/3} \exp(-\kappa/\varepsilon^{1/3})$ and using (3.8), we find

$$T = \log(c_3 T_1) / c_1. \quad (3.12)$$

We summarize our result for ODEs in the following

Theorem 3.1. *Let $(x(\cdot), y(\cdot))$ be the solution of the system (1.1), let $U_*(Y; \varepsilon)$ be that in Lemma 2.1, and let $(U_*(Y(t); \varepsilon), Y(t))$ be the solution of the balanced system (3.1). We take balanced initial conditions, $x(0) = U_*(y(0); \varepsilon)$ and $y(0) = Y(0)$. Then $(x(t), y(t))$ and $(U_*(Y(t); \varepsilon), Y(t))$ stay exponentially close to each other,*

$$|x(t) - U_*(Y(t); \varepsilon)| + |y(t) - Y(t)| \leq \varepsilon^{-4/3} \exp(-\kappa/\varepsilon^{1/3}), \quad (3.13)$$

for a timescale of order one, that is, for $0 \leq t \leq T$ and for $\varepsilon \leq \varepsilon_0$ with ε_0 sufficiently small.

From (3.11), it is clear that c_1 determines the timescale of validity of the bound, while c_R determines how small the error can be, initially and at the timescale of $1/c_1$. We note that this timescale is likely to be qualitatively optimal (i.e., independent of ε): if the physical system is chaotic (as is almost always the case in any realistic model), most solutions will diverge within several eddy turnaround times regardless of the Rossby or Froude number.

One may argue that a better question to pose is whether a properly initialized solution will stay (very) close to the slow manifold over longer times. In other words, instead of the error $(x'(t), y'(t))$, the object of interest should be $w(t) := x(t) - U_*(y(t), t; \varepsilon)$, where $(x(t), y(t))$ is the solution of the parent model. The computation in this case is very similar to the foregoing (and thus is not shown explicitly), leading to an estimate analogous to (3.11),

$$w(t) \leq c'_R \exp(c'_1 t), \tag{3.14}$$

where $c'_R \sim \mathcal{O}(c_R)$ and c'_1 , like c_1 , is independent of ε in general. The behaviour of $w(t)$ is thus “no better” than that of $(x'(t), y'(t))$: it remains exponentially small for timescales of order unity, but nothing can be said beyond that. This result can be improved if $c_1 \sim \mathcal{O}(\varepsilon)$, say, or smaller by some “accident,” but we are aware of no obvious example of a geophysical fluid model where this is the case. We have chosen to present the computation for the error $(x'(t), y'(t))$ instead of the “imbalance” $w(t)$ since the result obtained is stronger for essentially the same amount of computation.

4. A FLUID EXAMPLE

We now show how the development of Section 2 may be carried through for the “weak-wave” model [17], which can be regarded as a simplified form of the shallow-water equations.

a. Model and formulation. With B the rotational Froude number, Ro the Rossby number and $\varepsilon = \text{Ro} B / \sqrt{1 + B^2}$, the system reads (cf. (3.7) in [17]),

$$\begin{aligned} \frac{\partial q}{\partial t} &= -\partial(\psi, q), \\ \frac{\partial \delta}{\partial t} + \frac{1}{\varepsilon} \left(\frac{\Delta - B^2}{1 + B^2} \right) \eta &= \frac{b}{\varepsilon} q, \\ \frac{\partial \eta}{\partial t} + \frac{1}{\varepsilon} \delta &= 0, \end{aligned} \tag{4.1}$$

where $b := B / \sqrt{1 + B^2}$ and $\Delta \psi = q + b\eta$. We work in the domain $\Omega = \mathbb{T}^2 = [0, \ell]^2$ with $\ell < 2\pi$ and periodic boundary conditions. Note that here Δ is the Laplacian, ∂ the Jacobian and $\delta := \partial_x u + \partial_y v$ the divergence. To put (4.1) in the form (1.1), we subtract the geostrophic component from the (disturbance) height η ; viz., let

$$h := \eta - b \left(\frac{\Delta - B^2}{1 + B^2} \right)^{-1} q, \tag{4.2}$$

and work with dependent variables (q, δ, h) . In the following we take B to be of order unity and rescale the variables to absorb any factors involving B . (As pointed out in [8] and [9], in the limit $B \rightarrow 0$ the gravity-wave emission scales as B^4 ; this is essentially due to the absence of timescale separation.) With these, (4.1) reads

$$\begin{aligned} \frac{\partial q}{\partial t} &= \partial(q, \psi), \\ \frac{\partial \delta}{\partial t} + \frac{1}{\varepsilon}(\Delta - 1)h &= 0, \\ \frac{\partial h}{\partial t} + \frac{1}{\varepsilon}\delta &= -(\Delta - 1)^{-1}\partial(q, \psi), \end{aligned} \quad (4.3)$$

where $\Delta\psi = [1 + (\Delta - 1)^{-1}]q + h$; here and in what follows, Δ^{-1} and $(\Delta - 1)^{-1}$ are uniquely defined to have zero mean over Ω . (4.3) is the analogue of (1.1), with q being the slow variable y , and (δ, h) being the fast variable x . Furthermore,

$$L = \begin{pmatrix} 0 & (\Delta - 1) \\ 1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} F^\delta(q, \delta, h) \\ F^h(q, \delta, h) \end{pmatrix} = \begin{pmatrix} 0 \\ -(\Delta - 1)^{-1}\partial(q, \psi) \end{pmatrix}, \quad (4.4)$$

and $G(q, \delta, h) = \partial(q, \psi)$. Note that here L is an invertible linear operator, and F and G are nonlinear operators which happen to be independent of δ .

Since F , G , and U (below) are now nonlinear operators, we will need the Fréchet derivative: For a nonlinear operator K acting on (v, \dots) , we define (cf. [18], Section 5.2)

$$(\mathbf{D}_v K)w := \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} K(v + \varepsilon w, \dots) \quad (4.5)$$

for any arbitrary function w . Thus, $\mathbf{D}_v K$ is a linear operator whose coefficients may depend on the arguments of K . For example, with the above $G(q, \delta, h)$ we have

$$\begin{aligned} (\mathbf{D}_q G)w &= \partial(w, \psi) + \partial(q, \Delta^{-1}[1 + (\Delta - 1)^{-1}]w), \\ (\mathbf{D}_h G)w &= \partial(q, \Delta^{-1}w). \end{aligned} \quad (4.6)$$

As in the finite-dimensional case, we look for slow solutions of (4.3) with the aid of the slaving ansatz [cf. (1.2)],

$$\delta = U^\delta(q; \varepsilon), \quad h = U^h(q; \varepsilon), \quad (4.7)$$

where here $U(q; \varepsilon) = (U^\delta, U^h)$ are nonlinear operators acting on q . Lorenz's superbalance equation (1.3) then reads

$$(\mathbf{D}_q U)G(q, U) + \frac{1}{\varepsilon}LU = F(q, U). \quad (4.8)$$

Expanding $U(q; \varepsilon)$ in powers of ε as in (2.1) and substituting into (4.8), we find at $\mathcal{O}(\varepsilon)$,

$$\begin{aligned} (\Delta - 1)U_1^h(q) &= F^\delta(q, 0, 0) = 0, \\ U_1^\delta(q) &= F^h(q, 0, 0) = -(\Delta - 1)^{-1}\partial(q, \psi_0), \end{aligned} \quad (4.9)$$

where $\Delta\psi_0 := [1 + (\Delta - 1)^{-1}]q$. For $n > 1$, we substitute the expansion of $U(q; \varepsilon)$ and the explicit forms of F , G , and L into (4.8) and, after some computation, we obtain the recursion relations [cf. (2.8)],

$$\begin{aligned} (\Delta - 1)U_n^h &= (\mathbf{D}_q U_{n-1}^\delta)\partial(\psi_0, q) - \sum_{m=1}^{n-2} (\mathbf{D}_q U_m^\delta)\partial(q, \Delta^{-1}U_{n-m-1}^h), \\ U_n^\delta &= -(\Delta - 1)^{-1}\partial(q, \Delta^{-1}U_{n-1}^h) \\ &\quad - (\mathbf{D}_q U_{n-1}^h)\partial(q, \psi_0) - \sum_{m=1}^{n-2} (\mathbf{D}_q U_m^h)\partial(q, \Delta^{-1}U_{n-m-1}^h). \end{aligned} \quad (4.10)$$

b. A norm for all n . Let $|\cdot|_s$ be the usual Sobolev norm on $H^s(\Omega)$ (cf., e.g., [1]). We recall the following properties:

$$\begin{aligned} |u|_s &\leq |u|_r && \text{when } s \leq r, \\ |\nabla u|_s &\leq c|u|_{s+1}, \\ |\Delta u|_s &\leq c|u|_{s+2}, \\ |u + v|_s &\leq |u|_s + |v|_s, \end{aligned} \quad (4.11)$$

where, as before, c denotes a constant which may not be the same each time the symbol is used. From the theory of elliptic partial differential equations, we have

$$|(\Delta - a)^{-1}u|_s \leq c|u|_{s-2} \quad (4.12)$$

whenever $-\lambda_0 < a(x, y)$ everywhere in Ω , with $\lambda_0 = 2\pi/\ell$ the smallest eigenvalue of the Laplacian. We will also need the following (Banach-algebra) property of $|\cdot|_s$: For $s > 1$,

$$|uv|_s \leq c|u|_s|v|_s, \quad \forall u, v \in H^s(\Omega). \quad (4.13)$$

For our simple domain, this can be proved by considering the Fourier representation of u and v (cf., e.g., [15], appendix 2.2).

We begin with an explicit computation. For the rest of this section, let $s \geq 2$ (but it is otherwise arbitrary). We also assume that $|q|_s \leq \infty$ for all (finite) s , without making any assumption about its behaviour as $s \rightarrow \infty$. From (4.9), we have that $|U_1^h|_s = 0$ and, using properties (4.11)–(4.13),

$$\begin{aligned} |U_1^\delta|_s &= |(\Delta-1)^{-1}\partial(q, \psi_0)|_s \leq c|\partial(q, \psi_0)|_{s-2} \\ &\leq c|q|_{s-1}|q|_{s-3} \leq c|q|_{s-1}^2. \end{aligned} \quad (4.14)$$

Some more similar computations show that

$$|U_2^h|_s \leq c(1 + |q|_{s-3})^3 \quad \text{and} \quad |U_2^\delta|_s = 0, \quad (4.15a)$$

$$|U_3^h|_s = 0 \quad \text{and} \quad |U_3^\delta|_s \leq c(1 + |q|_{s-1})^4. \quad (4.15b)$$

We note that for $n \geq 4$, it is no longer true that one of the U_n is zero.

We now show that one can indeed find a suitable norm to bound U_n for all n . First, let us note that for $K(q)$ a “polynomial” operator on q , such as the $U_n(q)$ constructed using (4.9) and (4.10), we have the property that

$$|K(q)|_s \leq (1 + |q|_{s+r})^{n+1} \Rightarrow |(D_q K)w|_s \leq c'(1 + |q|_{s+r})^n |w|_{s+r} \quad (4.16)$$

whenever $r \geq 0$ and $|w|_r \leq \infty$ for all r . Here and elsewhere, c' denotes a generic constant like c , but which may depend on n . Note that we need not know the precise form of $K(q)$; this is where the boundedness of $|q|_s$ for all s is essential.

We proceed by induction using (4.10). Suppose that at order $n \geq 2$ we have

$$|U_n^\delta|_s \leq C_n(1 + |q|_{s+1})^{n+1}, \quad |U_n^h|_s \leq C_n(1 + |q|_s)^{n+1}, \quad (4.17)$$

where C_n may depend on $s \geq 2$, but not on q . From (4.15), this holds for $n = 2$. We compute at order $n + 1$,

$$\begin{aligned} |(\Delta-1)U_{n+1}^h|_s &\leq |(D_q U_n^\delta)\partial(q, \psi_0)|_s + \sum_m |(D_q U_m^\delta)\partial(q, \Delta^{-1}U_{n-m-1}^h)|_s \\ &\leq c'(1 + |q|_{s+1})^n |\partial(q, \psi_0)|_{s+1} + c' \sum_m (1 + |q|_{s+1})^m |\partial(q, \Delta^{-1}U_{n-m-1}^h)|_{s+1} \\ &\leq c'(1 + |q|_{s+1})^n (1 + |q|_{s+2})^2 + c' \sum_m (1 + |q|_{s+1})^m (1 + |q|_{s+2}) |U_{n-m-1}^h|_s \\ &\leq c'(1 + |q|_{s+2})^{n+2}, \\ \Rightarrow |U_{n+1}^h|_s &\leq c'(1 + |q|_s)^{n+2}, \end{aligned} \quad (4.18)$$

which proves that (4.17b) holds for all n . Similarly,

$$\begin{aligned}
 |U_{n+1}^\delta|_s &\leq c' |\partial(q, \Delta^{-1}U_n^h)|_{s-2} + |(\mathbb{D}_q U_n^h) \partial(q, \psi_0)|_s \\
 &\quad + c' \sum_m |(\mathbb{D}_q U_m^h) \partial(q, \Delta^{-1}U_{n-m-1}^h)|_s \\
 &\leq c' (1 + |q|_{s-1}) |U_n^h|_{s-3} + c' (1 + |q|_s)^n |\partial(q, \psi_0)|_s \\
 &\quad + c' \sum_m (1 + |q|_s)^m (1 + |q|_{s+1}) |U_{n-m-1}^h|_{s-1} \\
 &\leq c' (1 + |q|_{s+1})^{n+2},
 \end{aligned} \tag{4.19}$$

thus establishing (4.17a) for all n .

c. Bounds on the constant and the remainder. Having taken care of the norm, it remains to show that the constant C_n in (4.17) does not grow too rapidly as $n \rightarrow \infty$. We do this as in the finite-dimensional case.

From their construction (cf. (4.10)), it is clear that U_n^δ and U_n^h are “polynomial” operators of degree $n + 1$ acting on q . Therefore, the analogs of (2.10b) here are

$$\begin{aligned}
 |(\mathbb{D}_q U_n^\delta) w|_s &\leq (n + 1) c_s C_n (1 + |q|_{s+1})^n |w|_{s+1}, \\
 |(\mathbb{D}_q U_n^h) w|_s &\leq (n + 1) c_s C_n (1 + |q|_s)^n |w|_s,
 \end{aligned} \tag{4.20}$$

where the constant c_s is independent of n . Mirroring Section 2, from (4.10) we compute (cf. (4.18)–(4.19))

$$\begin{aligned}
 |(\Delta - 1)U_{n+1}^h|_s &\leq |(\mathbb{D}_q U_n^\delta) \partial(q, \psi_0)|_s + \sum_m |(\mathbb{D}_q U_m^\delta) \partial(q, \Delta^{-1}U_{n-m-1}^h)|_s \\
 &\leq c n C_n (1 + |q|_{s+1})^n (1 + |q|_{s+2})^2 \\
 &\quad + c \sum_m m C_m (1 + |q|_{s+1})^m (1 + |q|_{s+1}) C_{n-m-1} (1 + |q|_s)^{n-m}.
 \end{aligned} \tag{4.21}$$

Taking as in (2.12) C_n that grow at least exponentially,

$$C_n C_m \leq C_{n+m}, \tag{4.22}$$

we have from the last line of (4.21),

$$|U_{n+1}^h|_s \leq c n^2 C_n |q|_s^{n+2}. \tag{4.23}$$

Similarly for $|U_{n+1}^\delta|_s$,

$$\begin{aligned}
 |U_{n+1}^\delta|_s &\leq c (1 + |q|_{s-1}) C_n (1 + |q|_{s-3})^n + c n C_n (1 + |q|_s)^n (1 + |q|_{s+1})^2 \\
 &\quad + c \sum_m m C_m (1 + |q|_s)^m (1 + |q|_{s+1}) C_{n-m-1} (1 + |q|_{s-1})^{n-m} \\
 &\leq c n^2 C_n (1 + |q|_{s+1})^{n+2}.
 \end{aligned} \tag{4.24}$$

(4.23) and (4.24) then imply that we can take

$$C_n = c_w^n (n!)^2, \quad (4.25)$$

for a fixed constant c_w , giving the optimal truncation $n_* \simeq [\varepsilon c_w (1 + |q|_{s+1})]^{-\frac{1}{2}}$ (cf. (2.15)–(2.16)). The rest follows as in Section 2.b, which does not depend on the system being finite-dimensional.

Thus we have arrived at our main result:

Theorem 4.1. *Consider the system (4.3), and assume that $|q|_s < \infty$ for all s . Then one can find a $U_*(y; \varepsilon)$ that satisfies the equation defining its slow manifold (i.e., its superbalance equation), namely*

$$(\mathbf{D}_q U_*)G(q, U_*) + \frac{1}{\varepsilon} L U_* - F(q, U_*) =: R \quad (4.26)$$

(F , G , and L are defined in (4.4)), up to an exponentially small remainder,

$$|R|_s := |R^\delta|_s + |R^h|_s \leq (C/\varepsilon) \exp(-\kappa \varepsilon^{-1/2} (1 + |q|_{s+1})^{-1/2}) \quad (4.27)$$

as $\varepsilon \rightarrow 0$.

Put differently, the “fuzzy slow manifold” is also exponentially thin for this (infinite-dimensional) fluid model. As with the bound (2.24) in the finite-dimensional case, the exponent 1/2 in (4.27) is probably not optimal. However, in order to improve the result, one would likely have to do work which is more technically involved.

5. DISCUSSION

Combined with the explicit analytical and numerical computation of gravity-wave emission cited in the Introduction, the results obtained in this paper suggest that in the limit $\varepsilon \rightarrow 0$, the amplitude of such emission generically decays as $\exp(-\kappa/\varepsilon^\alpha)$. Although the values of α obtained for our models (1/2 and 1/3, respectively) are different from that ($\alpha = 1$) obtained in [22], it may be possible to reconcile these by more careful computation of our bounds or by considering more general flows in their work.

Also, it would be useful to obtain the analog of Section 4 for models such as the shallow-water or Boussinesq equations. The main problem here is to find a set of (slow and fast) variables such that (4.17) hold for all n . For this, one might have to consider formulations with nonpolynomial nonlinearity, e.g., with Ertel potential vorticity as variable. This would entail more complicated analysis than that presented in the present article. It should be kept in mind, however, that (4.17) likely does not hold for

arbitrary PDEs with fast and slow variables—in other words, models for which it holds are in some sense special. What this precisely means, and how, if indeed, it is related to known properties of the models, such as the material conservation of potential vorticity, is not clear at present and will be investigated in a future work.

Another issue we have relegated to future work is obtaining the analog of the finite-time result of Section 3 for continuous models. For this one would have to prove that the solution of the high-order balance equation, the analog of (3.1) for the appropriate model, is bounded in an appropriate norm over a timescale independent of ε . Showing this for a given PDE is a nontrivial task, and the problem is still open for many equations; doing so for an equation whose explicit form cannot (or is impractical to) be written down is presumably much harder.

Acknowledgments. This work was funded by an EPSRC research grant. Additional support was provided by the EPSRC Network “Geometric methods in geophysical fluid dynamics,” by a visitor’s stipend from the Twente Institute of Mechanics, and by NSF Grants DMS–0074334 and DMS–0305110, and the Research Fund of Indiana University. The author thanks K. Ngan and the anonymous referees for constructive comments.

REFERENCES

- [1] R.A. Adams, “Sobolev Spaces,” Academic Press, 1975.
- [2] J.S. Allen, *Iterated geostrophic intermediate models*, J. Phys. Ocean., 23 (1993), 2447–2461.
- [3] F. Baer and J.J. Tribbia, *On complete filtering of gravity modes through nonlinear initialization*, Mon. Wea. Rev., 105 (1977), 1536–1539.
- [4] O. Bokhove and T.G. Shepherd, *On Hamiltonian balanced dynamics and the slowest invariant manifold*, J. Atmos. Sci., 53 (1996), 276–297.
- [5] R. Camassa, *On the geometry of an atmospheric slow manifold*, Physica, D84 (1995), 357–397.
- [6] R. Daley, *Predictability experiments with a baroclinic model*, Atmos.-Ocean, 19 (1981), 77–89.
- [7] R. Daley, “Atmospheric Data Analysis,” Cambridge University Press, 1991.
- [8] R. Ford, *The instability of an axisymmetric vortex with monotonic potential vorticity in rotating shallow water*, J. Fluid Mech., 280 (1994), 303–334.
- [9] R. Ford, M.E. McIntyre, and W. Norton, *Balance and the slow quasi-manifold: some explicit results*, J. Atmos. Sci., 57 (2000), 1236–1254.
- [10] B.J. Hoskins, M.E. McIntyre, and A.W. Robertson, *On the use and significance of isentropic potential vorticity maps*, Quart. J. Roy. Met. Soc., 111 (1985), 877–936.
- [11] C.E. Leith, *Nonlinear normal mode initialization and quasi-geostrophic theory*, J. Atmos. Sci., 37 (1980), 958–968.

- [12] E.N. Lorenz, *Attractor sets and quasi-geostrophic equilibrium*, J. Atmos. Sci., 37 (1980), 1685–1699.
- [13] E.N. Lorenz, *On the existence of a slow manifold*, J. Atmos. Sci., 43 (1986), 1547–1557.
- [14] B. Machenhauer, *On the dynamics of gravity oscillations in a shallow water model, with applications to normal mode initialization*, Beitr. Phys. Atmos., 50 (1977), 253–271.
- [15] C. Marchioro and M. Pulvirenti, “Mathematical Theory of Incompressible Nonviscous Fluids,” Springer-Verlag, 1994.
- [16] J.C. McWilliams and P.R. Gent, *Intermediate models of planetary circulations in the atmosphere and oceans*, J. Atmos. Sci., 37 (1980), 1657–1678.
- [17] C. Nore and T.G. Shepherd, *A Hamiltonian weak-wave model for shallow-water flow*, Proc. R. Soc. Lond., A453 (1997), 563–580.
- [18] P.J. Olver, “Applications of Lie Groups to Differential Equations,” 2nd ed., Springer-Verlag, 1993.
- [19] J. Pedlosky, “Geophysical Fluid Dynamics,” 2nd ed., Springer-Verlag, 1987.
- [20] T.G. Shepherd, R.M. Temam, and D. Wirosoetisno, *A note on perturbation expansions: renormalisation, averaging, multiple scales and slaving principle*, article in preparation.
- [21] J. Vanneste, *Inertia-gravity-wave generation by balanced motion: revisiting the Lorenz-Krishnamurthy model*, J. Atmos. Sci., to appear.
- [22] J. Vanneste and I. Yavneh, *Exponentially small inertia-gravity waves and the breakdown of quasigeostrophic balance*, J. Atmos. Sci., to appear.
- [23] T. Warn, *Nonlinear balance and quasi-geostrophic sets*, Atmos.-Ocean, 35 (1997), 135–145. This paper was written in 1983 but not published. It was finally published in its original form in 1997 after having been referred to in print many times.
- [24] T. Warn, O. Bokhove, T.G. Shepherd, and G.K. Vallis, *Rossby number expansions, slaving principles, and balance dynamics*, Quart. J. Roy. Met. Soc., 121 (1995), 723–739.
- [25] T. Warn and R. Ménard, *Nonlinear balance and gravity-inertial wave saturation in a simple atmospheric model*, Tellus, 38A (1986), 285–294.
- [26] D. Wirosoetisno, T.G. Shepherd, and R.M. Temam, *Free gravity waves and balance dynamics*, J. Atmos. Sci., 59 (2002), 3382–3398.