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GLOBAL EXISTENCE CONDITIONS FOR A NONLOCAL PROBLEM ARISING IN STATISTICAL MECHANICS

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Abstract. We consider the evolution of the density and temperature of a three-dimensional cloud of self-interacting particles. This phenomenon is modeled by a parabolic equation for the density distribution combining temperature-dependent diffusion and convection driven by the gradient of the gravitational potential. This equation is coupled with Poisson's equation for the potential generated by the density distribution. The system preserves mass by imposing a zero-flux boundary condition. Finally the temperature is fixed by energy conservation; that is, the sum of kinetic energy (temperature) and gravitational energy remains constant in time. This model is thermodynamically consistent, obeying the first and the second laws of thermodynamics. We prove local existence and uniqueness of weak solutions for the system, using a Schauder fixed-point theorem. In addition, we give sufficient conditions for global-in-time existence and blow-up for radially symmetric solutions. We do this using a comparison principle for an equation for the accumulated radial mass.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded, open set satisfying $\sup_{x\in\Omega} |x| = 1$. In Ω we consider the parabolic-elliptic system

$$n_t = \operatorname{div}\{\Theta(t)\nabla n + n\nabla\phi\} \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{1.1}$$

$$\Delta\phi = n \quad \text{in} \quad \Omega\times\mathbb{R}^+, \tag{1.2}$$

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combined with the energy relation

$$E = \kappa\Theta(t) + \int_\Omega n\phi\, dx \qquad \text{in} \quad \mathbb{R}^+, \tag{1.3}$$

where $E \in \mathbb{R}$ and $\kappa > 0$ are given parameters. At the boundary $\partial\Omega \in C^{1+\alpha}$ $(\alpha > 0)$ we prescribe

$$(\Theta(t)\nabla n + n\nabla\phi) \cdot \vec{\nu} = 0 \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+, \tag{1.4}$$

$$\phi = 0 \qquad\qquad \text{in} \quad \partial\Omega \times \mathbb{R}^+, \tag{1.5}$$

where $\vec{\nu}$ denotes the exterior normal vector on $\partial\Omega$. At $t = 0$ we have the initial condition

$$n(x, 0) = n_0(x) \quad \text{in} \quad \Omega, \tag{1.6}$$

satisfying

$$\int_\Omega n_0\, dx = 1, \quad \text{and} \quad n_0(x) \geq 0 \quad \text{in} \quad \Omega. \tag{1.7}$$

This set of equations defines Problem **P** for the unknowns n, ϕ, and Θ. The underlying model is discussed in Section 2, as well as some known properties of the system.

The purpose of the paper is to demonstrate local existence for Problem **P** and to give sufficient conditions on E, κ, and n_0 for global existence. Local existence is shown in Section 3. The proof uses a Schauder fixed-point theorem and a careful construction of an invariant set to avoid degenerate diffusion in (1.1). It requires $n_0 \in L^p(\Omega)$ for $p > \frac{3}{2}$, implying that $n \in L^\infty_{loc}((0, T]; L^\infty(\Omega))$ for some $T \in (0, \infty)$. Hence we can allow for certain singular initial data which result in solutions that are locally bounded in $(0, T]$. Let

$$T^* = \sup\{T > 0 : \text{Problem } \mathbf{P} \text{ has a solution in } (0, T]\}.$$

If $T^* = \infty$, the solution is defined globally, and if $T^* < \infty$ we have at least $\lim_{t\to T^*} \|n(t)\|_{L^q(\Omega)} = \infty$ for each $q > \frac{3}{2}$. For Problem **P** the optimal $L^p(\Omega)$ space seems to be $p = \frac{3}{2}$, since there exists a singular stationary solution in the radial case belonging to $L^q(\Omega) \setminus L^{3/2}(\Omega)$, with $q < \frac{3}{2}$. This solution is given in Section 2.3. Uniqueness is proven for $n_0 \in L^p(\Omega)$ with $p \geq 2$.

Problem **P** was recently studied in [21]: local existence and uniqueness were obtained for $p > 3$. Although the result in [21] is proved only for $\kappa = 3$, the method seems applicable to any $\kappa > 0$.

In Section 4, we consider an auxiliary problem in which we drop the energy relation (1.3) and treat $\Theta(t)$ as a given function. This provides insight and bounds which we need in order to prove our main result about global existence. In Section 5, we first give the following result about blow-up:

Theorem 1.1. *Let $\Omega = B_1(0)$ be the unit ball in $\mathbb{R}^3$. If $\kappa > 6$ and $E < \frac{1}{4\pi}$, then $T^* < \infty$.*

Unfortunately the proof of Theorem 1.1 does not give any insight into the structure of the blow-up. We refer to [14] for a description of blow-up using mainly numerical results. This issue will be considered by us in a future publication. See [16] for a further discussion.

Before stating the global existence result we note from (1.3) at $t = 0$, that instead of prescribing E and n_0 one could equivalently prescribe $\Theta_0 := \Theta(0)$ and n_0. In fact it seems more natural to consider Θ_0 and n_0 as initial values. In view of the physical interpretation of the model we consider $\Theta_0 > 0$. With this in mind we have

Theorem 1.2. *Let $\Omega = B_1(0)$, and assume that solutions of Problem* **P** *are radially symmetric. If the pair $\langle n_0, \Theta_0\rangle$ satisfies one of the following conditions,*

(i) *$n_0 \in L^\infty(\Omega)$ and Θ_0 is sufficiently large;*
(ii) *$n_0 \log n_0 \in L^1(\Omega)$, and there exists $B > 0$, such that*

$$\|n_0\|_{L^1(B_r(0))} \leq (1 + B)\frac{r^3}{r^2 + B} \quad \textit{for} \quad r \in [0, 1],$$

and

$$\Theta_0 \geq \frac{1}{8\pi}(1 + B) \exp\Big(\frac{2}{\kappa}\Big[\int_\Omega n_0 \log n_0\, dx - \log\big(\frac{3}{4\pi}\big)\Big]\Big);$$

(iii) *$n_0 \equiv \frac{3}{4\pi}$, and $\Theta_0 > \frac{1}{8\pi}$,*

then $T^* = \infty$ (global existence).

Remark 1.3. (i) Due to the parabolic regularity, $n_0 \in L^\infty(\Omega)$ is not so restrictive. (ii) Condition (iii) is a special case of (ii). (iii) The condition on n_0 in (ii) implies a bound on the Morrey norm of exponent $3/2$, since $\|n_0\|_{M^{3/2}(\Omega)} = \sup_{x\in\mathbb{R}^3,\, 0\leq r\leq 1} r^{-1}\|n_0\|_{L^1(\Omega\cap B_r(x))}$. In [3], the space $M^{3/2}(\Omega)$ was suggested as the natural space to prove existence.

The proof of Theorem 1.2 contains two essential steps. To extend the local solution we first need a uniform bound from below on Θ. To achieve this we use a Lyapunov functional associated with Problem **P**, the so-called Boltzmann entropy (2.8). This functional provides a uniform lower bound on Θ, which depends only on the initial data and κ. If Θ_0 is positive, then Θ remains positive in the whole existence interval, including the blow-up time.

In the second step we construct a control on n. Here we use the radial symmetry which allows us to transform equations (1.1) and (1.2) into a single equation, still containing Θ as unknown. It has the crucial property that an ordered pair of given Θ's results in an ordered pair of solutions. As a comparison function we now use the solution of (1.1)–(1.2) with a suitably chosen fixed Θ. Under certain hypotheses this auxiliary problem has a global solution which provides the control on n. The different conditions in Theorem 1.2 are closely related to global existence conditions for the auxiliary problem.

2. Preliminaries

2.1. **Model issues.** Problem **P** was first derived for collisionless systems such as galaxies. The underlying argument is that rapid fluctuations of the gravitational field during the early stage of *violent relaxation* plays the same role as collisions, although the time scales involved for collisionless systems are smaller than for collisional systems (Brownian motion). The process of violent relaxation is considered in [15], and further results and interpretations can be found in [14].

Problem **P** also describes the evolution of density and temperature of a self-attracting cluster of Brownian particles in a bounded three-dimensional region. During the evolution mass and energy are conserved. A detailed derivation and discussion on the physical assumptions can be found in [4, 7, 23] and the references therein. Below we present a brief summary.

Suppose a cluster of particles is contained in a bounded region $\Omega \subset \mathbb{R}^3$. The spatial particle density n satisfies the mass-balance equation

$$n_t = \operatorname{div}\Big\{\frac{1}{\beta}(k\Theta\nabla n + n\nabla\phi)\Big\} \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{2.1}$$

where $\beta > 0$ is the friction coefficient, k the Boltzmann constant and Θ the temperature of the system. To ensure that the cluster of particles preserves mass we impose zero mass flux along the boundary; i.e.,

$$(k\Theta\nabla n + n\nabla\phi)\cdot\vec{\nu} = 0 \quad \text{on} \quad \partial\Omega. \tag{2.2}$$

This implies

$$\int_\Omega n(x,t)\,dx = \mathit{constant} = M \quad \text{for all} \quad t > 0,$$

where M is the total particle mass of the system, specified by the initial condition.

The function ϕ in (2.1) is the gravitational potential. It satisfies

$$\Delta\phi = 4\pi G\, n \quad \text{in} \quad \Omega \times \mathbb{R}^+, \tag{2.3}$$

with

$$\phi = -\frac{GM}{R} \quad \text{on} \quad \partial\Omega \times \mathbb{R}^+. \tag{2.4}$$

Here G is the gravitational constant and $R := \max_{x\in\Omega} |x|$. Note that we have chosen as boundary condition the gravitational potential of a mass M centered at the origin of a ball of radius R, as was introduced in [14]. An alternative way to define the potential, physically more relevant, is to consider the convolution of the fundamental solution of the Laplacian in $\mathbb{R}^3$ with the density; see [9]. Both definitions coincide when the domain is a ball. For general domains the Dirichlet condition (2.4) is somewhat artificial, but we hope that it will provide insight for the study of more realistic boundary conditions; see the discussion in [4].

In general the temperature varies in space and time. It satisfies an energy-balance equation containing thermal diffusion, heat convection and a term due to gravitational effects [7, Equation (1.4)]. This results in the so-called Streater model. However, the integrated energy balance does not contain the thermal diffusivity [7, Equation (2.1)]. Furthermore, we expect that a large thermal diffusivity will result in a temperature which is nearly constant in space. Taking this limit in the integrated energy balance, one finds

$$E = \frac{\kappa M}{2}\Theta(t) + \frac{1}{2}\int_\Omega n\phi\, dx \quad \text{in} \quad \mathbb{R}^+, \tag{2.5}$$

where E denotes the total energy of the system and κ the specific heat of the particles. If the cluster resembles an ideal gas we have $\kappa = 3k$.

Regarding the initial data for the system (2.1)–(2.5), there are two ways to proceed. If the energy E is given, it suffices to specify only the initial density

$$n(x, 0) = n_0(x) \geq 0 \quad \text{for} \quad x \in \Omega. \tag{2.6}$$

Equivalently we can specify both initial density and temperature

$$\Theta(0) = \Theta_0 > 0.$$

Now E is fixed by (2.5) at $t = 0$.

If the temperature is constant in time as well we drop the energy balance (2.5) and obtain the isothermal model. This model also arises in the context of polytropic stars and the biological phenomena of chemotaxis. The corresponding mathematical problem has received considerable attention in the past years because of its rich structure. Blow-up in the form of singular solutions and gravitational collapse can occur, as well as global existence. To our knowledge there is no full description of these phenomena in $\mathbb{R}^3$. The reason is that in contrast to the two-dimensional case, global existence in $\mathbb{R}^3$ depends not only on the parameters of the problem, but also on the shape of the initial density profile. A detailed discussion and references are given in [13], [17], and [18]. The isothermal model, however, plays a crucial role in the analysis presented in this paper.

Since Problem **P** has an additional equation, one expects that conservation of energy will act as a selection principle to favor global existence. This has been demonstrated in [22] for the two-dimensional case: the energy balance implies that temperature increases whenever density concentrates near a point. This in turn has a smoothing effect (through (2.1)) on the density profile, preventing blow-up from happening. Theorem 1.1 tells us that this general observation is not true in $\mathbb{R}^3$.

2.2. **Nondimensionalization.** We put equations (2.1)–(2.5) in dimensionless form by setting

$$\tilde{x} = \frac{1}{R}x, \quad \tilde{n} = \frac{R^3}{M}n, \quad \tilde{\phi} = \frac{R}{4\pi GM}(\phi + \frac{GM}{R}), \tag{2.7}$$

and

$$\tilde{\Theta} = \frac{kR}{4\pi GM}\Theta, \quad \tilde{t} = \frac{4\pi GM}{\beta R^3}t.$$

Introducing $\tilde{E} = \frac{R}{2\pi GM^2}(E + \frac{1}{2}\frac{GM^2}{R})$ and $\tilde{\kappa} = \frac{1}{k}\kappa$, and dropping the tildes, results in Problem **P**.

2.3. **Lyapunov functional and stationary solutions.** If a triple $\langle n, \phi, \Theta\rangle$ solves Problem **P**, then it is easy to check that

$$W(t) = \int_\Omega n \log n \, dx - \frac{\kappa}{2}\log\Big(E - \int_\Omega n\phi \, dx\Big) \quad \text{on} \quad \mathbb{R}^+ \tag{2.8}$$

satisfies

$$\frac{d}{dt}W(t) = -\int_\Omega \frac{|\Theta(t)\nabla n + n\nabla\phi|^2}{\Theta(t)n}dx, \quad \text{for all} \quad t > 0. \tag{2.9}$$

Hence W is a Lyapunov functional for Problem **P**, sometimes called the Boltzmann entropy; see [21] and [15] for the original reference.

One consequence of (2.9) is the following. Let $\langle n_s, \phi_s, \Theta_s\rangle$ denote a stationary solution of Problem **P**. Then (2.9) implies

$$\Theta_s\nabla n_s + n_s\nabla\phi_s \equiv 0 \quad \text{in} \quad \Omega.$$

Introducing the scaled potential $\psi := \frac{\phi_s}{\Theta_s}$ we observe that $n_s \equiv \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx}$, where ψ satisfies

$$(S) \quad \begin{cases} \Delta\psi = \dfrac{1}{\Theta_s}\dfrac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx} & \text{in} \quad \Omega, \\ \psi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The corresponding energy relation takes the form

$$E\frac{1}{(\Theta_s)^2} = \kappa\frac{1}{\Theta_s} - \int_\Omega |\nabla\psi|^2 dx. \tag{2.10}$$

Problem **S** has only one singular radially symmetric solution [12], the Chandrasekhar solution

$$n_s = U := \frac{1}{4\pi}\frac{1}{|x|^2}, \tag{2.11}$$

provided $\Theta_s = \frac{1}{8\pi}$. It satisfies (2.10) for $E = \frac{(\kappa-2)}{8\pi}$. Observe that $U \in L^q(\Omega) \setminus L^{3/2}(\Omega)$, with $q < \frac{3}{2}$. If this solution is attained by Problem **P** for $t \uparrow T^* < \infty$, we have a blow-up without concentration of mass at the origin.

For completeness we recall a result [4, Proposition 5.6] for bounded, radially symmetric solutions of Problem **S** and (2.10).

Theorem 2.1. *Let $\Omega = B_1(0)$. For any $\kappa > 0$, there exists $E_\kappa \in \mathbb{R}$ such that*

(i) *If $E > E_\kappa$ there exist bounded, negative solutions;*
(ii) *If $E < E_\kappa$ there are no nontrivial, bounded, negative solutions.*

This observation is originally due to Antonov [1] as a result of a computational approach. He also showed that stationary solutions are local maxima or saddle points of an entropy and there is no global entropy maximum.

Theorem 2.1 is still open for general domains [4]. This is related to the nontrivial nature of the set of singular solutions [12].

2.4. **Radially symmetric solutions.** Our main theorem about global existence is stated in terms of radially symmetric solutions. Radial symmetry in Problem **P** not only reduces the spatial dimension, but also allows us to combine equations (1.1) and (1.2) into a single equation for the accumulated mass:

$$Q(r,t) := \int_{B_r(0)} n(x,t)dx \quad \text{for} \quad r \in (0,1] \quad \text{and} \quad t \in \mathbb{R}^+.$$

This is shown in [6]. Redefining $t := \frac{3}{4\pi}t$ and introducing $\vartheta := 12\pi\Theta$, we obtain in terms of $Q(y,t) := Q(r,t)$, with $y = r^3$, the equation

$$Q_t = y^{4/3}\vartheta(t)Q_{yy} + QQ_y \quad \text{for} \quad y \in (0,1) \quad \text{and} \quad t \in \mathbb{R}^+. \tag{2.12}$$

To transform the energy relation (1.3), we first note that (1.2) and (1.5) give $\int n\phi\, dx = -\int |\nabla\phi|^2 dx$. Further, radial symmetry and (1.2) imply $4\pi r^2\partial_r\phi = Q(r,t)$. Finally we introduce $\mathcal{E} := 12\pi E$, to get in terms of $Q(y,t)$

$$\mathcal{E} = \kappa\vartheta(t) - \int_0^1 \frac{Q^2}{y^{4/3}}\, dy \quad \text{for} \quad t \in \mathbb{R}^+. \tag{2.13}$$

The boundary conditions for Q are

$$Q(0,t) = 0, \quad Q(1,t) = 1, \quad \text{for} \quad t \in \mathbb{R}^+, \tag{2.14}$$

and the initial condition becomes

$$Q(y,0) = Q_0(y) := \frac{4\pi}{3}\int_0^y n_0(y^{1/3})dy \quad \text{for} \quad 0 \le y \le 1. \tag{2.15}$$

Equations (2.12)–(2.15) define Problem $\mathcal{Q}$. Note that

$$\vartheta(t) = \mathit{constant} = \frac{3}{2} \quad \text{and} \quad Q(y,t) = y^{1/3}$$

satisfy equation (2.12) and boundary conditions (2.14). The energy relation (2.13) is satisfied for $\mathcal{E} = \frac{3}{2}(\kappa - 2)$. This is the transformed Chandrasekhar solution (2.11).

3. Well-posedness for problem **P**

Before we give a formal solution definition for Problem **P** we observe that ϕ is known in terms of n by the boundary-value problem (1.2) and (1.5). Therefore we denote a solution by $\langle n, \Theta\rangle$ instead of the triple $\langle n, \phi, \Theta\rangle$.

We call $\langle n, \Theta\rangle$ a weak solution of Problem **P** if, for some $T > 0$,

(i) $n \in L^2\big(0,T; H^1(\Omega)\big)$ and $n_t \in L^2\big(0,T;(H^1(\Omega))'\big)$;
(ii) $\Theta \in C([0,T])$ and $\Theta(t) > 0$ for $t \in [0,T]$;
(iii) the triple $\langle n, \phi, \Theta\rangle$, where $\phi \in C\big([0,T]; H_0^1(\Omega)\big)$, solves the boundary-value problem (1.2) and (1.5), satisfies (1.1) in the weak sense and (1.3) for all $t \in [0,T]$;
(iv) $n(\cdot, 0) = n_0 \ge 0$ almost everywhere in Ω.

Remark 3.1. The regularity in (i) implies $n \in C\big([0,T]; L^2(\Omega)\big)$ [24, p. 260]. Therefore Θ and ϕ are continuous in time in the sense of (ii) and (iii) respectively, and the initial value of n can be prescribed.

3.1. **Local existence.** Let $R_T := \Omega \times (0,T]$ for arbitrarily chosen $T > 0$.

The first result asserts local existence for Problem **P**.

Theorem 3.2. *Let $E \in \mathbb{R}$, $\kappa > 0$, and let $n_0 \in L^2(\Omega)$ be such that $\Theta(0) = \Theta_0 > 0$. Then there exists a weak solution $\langle n, \Theta\rangle$ of Problem* **P** *with $T = T(\|n_0\|_{L^2(\Omega)}, \Omega, \Theta_0) > 0$. It satisfies $n \geq 0$ in R_T and $n \in L^\infty_{loc}\big((0,T]; L^\infty(\Omega)\big)$.*

Proof. The proof uses a Schauder fixed-point theorem [25, Corollary 9.7]. For any fixed $T > 0$, let

$$X = \big\{v \in L^2\big(0,T; H^1(\Omega)\big) \quad \text{with} \quad v_t \in L^2\big(0,T;(H^1(\Omega))'\big)\big\}$$

and let $F\colon X \to C([0,T])$ be defined by

$$F(v)(t) = \frac{1}{\kappa}\big[\|v(t)\|^2_{H^{-1}(\Omega)} + E\big] \text{ for any } t \in [0,T], \text{ and for all } v \in X. \quad (3.1)$$

This map is clearly well-defined.

Observe that v and v_t belong to $L^2\big(0,T;H^{-1}(\Omega)\big)$. Note that $F(n)(t)$ is the temperature $\Theta(t)$ whenever n is the solution of Problem **P**.

Next define $N\colon X \to X$, with $u = N(v)$ satisfying

$$\left.\begin{array}{l} u_t = \operatorname{div}(F(v)(t)\nabla u + u\nabla\phi) \\ \Delta\phi = u \end{array}\right\} \quad \text{in} \quad R_T \qquad (3.2)$$

$$\left.\begin{array}{l} \phi = 0 \\ (F(v)(t)\nabla u + u\nabla\phi)\cdot\vec{\nu} = 0 \end{array}\right\} \quad \text{on} \quad \partial\Omega\times[0,T], \qquad (3.3)$$

$$u(x,0) = n_0(x) \quad \text{for} \quad x \in \Omega. \qquad (3.4)$$

For given $v \in X$, this problem is essentially Problem **P** with prescribed temperature. As we point out in Remark 4.1, we have local existence and uniqueness provided F remains positively bounded from below. Under this condition the operator N is well-defined.

To apply the fixed-point theorem, we need to prove that there exists $\mathcal{C} \subset X$, with $\mathcal{C}$ convex, bounded, and closed in $(X, \|\cdot\|)$, such that

(i) $N(\mathcal{C}) \subset \mathcal{C}$;
(ii) N is weakly–weakly sequentially continuous in X.

For any $v \in \mathcal{C}$, the operator N has to be well-defined. Thus in addition to (i) and (ii) we need

(iii) there exists $F_0 = F_0(\mathcal{C})$ such that $F(v)(t) \geq F_0 > 0$ for all $t \in [0,T]$ and for all $v \in \mathcal{C}$.

We show below that

$$\mathcal{C} = \Big\{v \in X : v(0) = n_0, \quad \|v\|_{L^2(0,T;L^2(\Omega))} \leq RT^{1/2},$$

$$\|\nabla v\|_{L^2(0,T;L^2(\Omega))} \le R', \quad \text{and} \quad \|v_t\|_{L^2(0,T;(H^1(\Omega))')} \le R'' \},$$

for suitably chosen constants R, R', and R'', and for T sufficiently small. In fact $R = 2\|n_0\|_{L^2(\Omega)}$, $R' = 2\|n_0\|_{L^2(\Omega)}/\Theta_0^{1/2}$, and $R'' = 2\|n_0\|_{L^2(\Omega)}\Theta_0^{1/2} + 4C\|n_0\|^2_{L^2(\Omega)}/\Theta_0^{1/2}$, where $C = C(\Omega)$ is a positive constant. Clearly $\mathcal{C}$ is convex, bounded, and closed in X. Note that $\mathcal{C}$ is not empty: the solution of the heat equation with initial value n_0 and diffusion coefficient $\Theta_0/4$ satisfies $\|\nabla n\|_{L^2(0,T;L^2(\Omega))} \le \sqrt{2}\|n_0\|_{L^2(\Omega)}/\Theta_0^{1/2}$ and $\|n_t\|_{L^2(0,T;(H^1(\Omega))')} \le \frac{1}{2\sqrt{2}}\|n_0\|_{L^2(\Omega)}\Theta_0^{1/2}$. Hence $n \in \mathcal{C}$ for $T > 0$.

We first show (iii). Differentiating expression (3.1), and applying Cauchy-Schwarz and the continuous injections $(H^1(\Omega))' \hookrightarrow H^{-1}(\Omega)$ and $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, yields the estimate

$$\kappa|F(v)_t(t)| \le 2\|v(t)\|_{H^{-1}(\Omega)}\|v_t(t)\|_{H^{-1}(\Omega)} \le C\|v(t)\|_{L^2(\Omega)}\|v_t(t)\|_{(H^1(\Omega))'} \tag{3.5}$$

almost everywhere in $[0,T]$, where $C = C(\Omega) > 0$. Integration now gives

$$|F(v)(t) - F(n_0)| \le \frac{C}{\kappa}R''RT^{1/2} \quad \text{for all} \quad t \in [0,T]. \tag{3.6}$$

Hence, $F(v)(t) \ge F(n_0) - \frac{C}{\kappa}R''RT^{1/2} = \Theta_0 - \frac{C}{\kappa}R''RT^{1/2}$ for $0 < t \le T$. If we now choose $F_0 = \Theta_0/2$ and T^* such that $\frac{C}{\kappa}R''R(T^*)^{1/2} = \Theta_0/2$, we have established (iii) for all $0 < t \le T \le T^*$.

Next we verify (i) for a suitable $T \le T^*$. Starting point is inequality (4.3) with $\Theta = F(v)(t)$ and $v \in \mathcal{C}$. It follows that the solution of (3.2)–(3.4) satisfies

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)} + F(v)(t)\|\nabla u\|^2_{L^2(\Omega)} \le \|u\|_{L^3(\Omega)}\|\nabla\phi\|_{L^6(\Omega)}\|\nabla u\|_{L^2(\Omega)}.$$

Since (4.4) holds for any $\Theta > 0$, we use it with $\Theta = F_0$ to find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\|u(t)\|^2_{L^2(\Omega)}) + (F(v)(t) - \frac{F_0}{2})\|\nabla u\|^2_{L^2(\Omega)} \\ \le \frac{C}{F_0^3}(\|u(t)\|^2_{L^2(\Omega)})^3 + \frac{F_0}{2}\|u(t)\|^2_{L^2(\Omega)}\end{aligned} \tag{3.7}$$

for $0 \le t \le T$ and for some $C = C(\Omega) > 0$.

Since $v \in \mathcal{C}$ and consequently $F(v)(t) \ge F_0$, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|u\|^2_{L^2(\Omega)}) + \frac{F_0}{2}\|\nabla u\|^2_{L^2(\Omega)} \le \frac{C}{F_0^3}(\|u(t)\|^2_{L^2(\Omega)})^3 + \frac{F_0}{2}\|u(t)\|^2_{L^2(\Omega)} \tag{3.8}$$

in $[0,T]$. This inequality implies some useful bounds. Disregarding the gradient in the left-hand side of (3.8) gives a differential inequality in terms of

$\|u(t)\|^2_{L^2(\Omega)}$. It follows that there exists $T_0 = T_0(\Theta_0, \Omega, \|n_0\|_{L^2(\Omega)})$ such that u is well-defined in R_{T_0} and satisfies

$$\sup_{t\in[0,T_0]} \|u(t)\|^2_{L^2(\Omega)} \leq (2\|n_0\|_{L^2(\Omega)})^2, \tag{3.9}$$

and thus

$$\|u\|_{L^2(0,T_0;L^2(\Omega))} \leq 2\|n_0\|_{L^2(\Omega)} T_0^{1/2}.$$

Integrating (3.8) and using (3.9) gives

$$\begin{aligned} \|\nabla u\|_{L^2(0,T_0;L^2(\Omega))} &\leq CT_0^{1/2} + \|n_0\|_{L^2(\Omega)}/(F_0)^{1/2} \\ &= CT_0^{1/2} + \sqrt{2}\|n_0\|_{L^2(\Omega)}/\Theta_0^{1/2} \end{aligned} \tag{3.10}$$

for a positive constant $C = C(\Theta_0, \Omega, \|n_0\|_{L^2(\Omega)})$. Note that (3.9) and (3.10) imply $u \in L^\infty(0, T_0; L^2(\Omega))$ and $u \in L^2(0, T_0; H^1(\Omega))$. To show that $u \in \mathcal{C}$, for sufficiently small T, it remains to prove the bound on u_t. With $\xi \in L^2(0, T; H^1(\Omega))$, we have from (3.2)

$$\int_0^T \langle u_t, \xi\rangle\, dt = -\int_0^T F(v)(t) \int_\Omega \nabla u \nabla \xi \, dx\, dt + \int_0^T \int_\Omega u \nabla\phi\nabla\xi\, dx\, dt, \tag{3.11}$$

where $\langle\cdot,\cdot\rangle$ denotes the pairing between $(H^1(\Omega))'$, and $H^1(\Omega)$. To estimate the right-hand side we first note that $F(v) \in L^\infty(0, T^*)$. Indeed, from (3.6) we deduce

$$F_0 < F(v)(t) \leq \frac{C}{\kappa} RR'' T^{1/2} + \Theta_0, \quad \text{for} \quad 0 \leq t \leq T \leq T^*. \tag{3.12}$$

Next we use (6.3) and interpolation inequality (6.2) from the appendix. This gives

$$\begin{aligned} \Big|\int_\Omega u\nabla\phi\nabla\xi\, dx\Big| &\leq \|u\|_{L^3(\Omega)} \|\nabla\phi\|_{L^6(\Omega)} \|\nabla\xi\|_{L^2(\Omega)} \\ &\leq C_s^{1/2} C_I \|u\|_{H^1(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla\xi\|_{L^2(\Omega)}. \end{aligned}$$

Finally, we combine this expression with (3.9), (3.10), and (3.12), and obtain after some manipulation

$$\begin{aligned} \int_0^T |\langle u_t, \xi\rangle|\, dt \leq \Big\{ & C(T^{1/2} + T) + \sqrt{2}\|n_0\|_{L^2(\Omega)} \Theta_0^{1/2} \\ & + 2\sqrt{2} C_s^{1/2} C_I \|n_0\|^2_{L^2}/\Theta_0^{1/2} \Big\} \|\xi\|_{L^2(0,T,H^1(\Omega))} \end{aligned}$$

for some $C = C(\Omega, \|n_0\|_{L^2(\Omega)}, \Theta_0)$. Taking now $T_1 < T_0 < T^*$ sufficiently small we obtain that $u \in \mathcal{C}$ for $0 \leq T \leq T_1$, and consequently $N(\mathcal{C}) \subset \mathcal{C}$.

Next we show (ii): i.e., we claim that $v_k \in \mathcal{C}$ and $v_k \rightharpoonup v$ in X imply $N(v_k) \rightharpoonup N(v)$ in X. For any such sequence v_k, define $u_k := N(v_k) \in \mathcal{C}$. Using the weak compactness of $\mathcal{C}$ we extract a subsequence $u_{k'} \in \mathcal{C}$ such that $u_{k'} \rightharpoonup u^*$ in X. We show below that $u^* = N(v)$, which proves the assertion. Since $v_{k'} \rightharpoonup v$ and $u_{k'} \rightharpoonup u^*$ in $\mathcal{C}$, we obtain by Aubin's lemma [20, p. 58] for a subsequence, denoted again by k', $v_{k'} \to v$ and $u_{k'} \to u^*$ in $L^2\big(0,T;L^2(\Omega)\big)$. We use this in (3.11) for $u_{k'}, v_{k'}$, and $\phi_{k'}$. Since $\Delta\phi_{k'} = u_{k'}$, we have $\phi_{k'} \to \phi^*$ in $L^2\big(0,T;H^2(\Omega)\big)$ satisfying $\Delta\phi^* = u^*$. Moreover, as $k' \to \infty$,

$$\begin{aligned} u_{k'_t} &\rightharpoonup u^*_t \quad \text{in} \quad L^2\big(0,T,(H^1(\Omega))'\big), \\ \nabla\phi_{k'} &\to \nabla\phi^* \quad \text{in} \quad L^2\big(0,T,H^1(\Omega)\big), \\ \nabla u_{k'} &\rightharpoonup \nabla u^* \quad \text{in} \quad L^2\big(0,T;L^2(\Omega)\big). \end{aligned}$$

Now suppose $F(v_{k'}) \to F(v)$ in $C([0,T])$. Then letting $k' \to \infty$ in (3.11) we obtain a solution u^* of problem (3.2)–(3.4) for the temperature $F(v)(t)$. By uniqueness we have $u^* = N(v)$.

It remains to show that $F(v_{k'}) \to F(v)$ in $C([0,T])$. In view of the continuous injection $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$, we find from (3.1)

$$\begin{aligned} &\kappa|F(v_{k'})(t) - F(v)(t)| \\ &\le C(\Omega)\left[\|v_{k'}(t)\|_{L^2(\Omega)} + \|v(t)\|_{L^2(\Omega)}\right]\|v_{k'}(t) - v(t)\|_{L^2(\Omega)}. \end{aligned}$$

This implies directly $F(v_{k'}) \to F(v)$ in $L^1([0,T])$. Writing (3.5) for the difference $F(v_{k'}) - F(v)$, using the continuous injection $(H^1(\Omega))' \hookrightarrow H^{-1}(\Omega)$, and integrating the result gives

$$\begin{aligned} &\kappa\int_0^T |F(v_{k'})_t(t) - F(v)_t(t)|dt \le \int_0^T |(v(t), v_{k'_t}(t) - v_t(t))_{H^{-1}}|\,dt \\ &+ \|v - v_{k'}\|_{L^2(0,T;L^2(\Omega))}\|(v_{k'})_t\|_{L^2(0,T;(H^1(\Omega))')}. \end{aligned}$$

Since $v_{k'_t} \rightharpoonup v_t$, in $L^2(0,T,H^{-1}(\Omega))$, we obtain $F(v_{k'}) \to F(v)$ in $W^{1,1}([0,T])$. This concludes the proof of (ii) and establishes local existence for Problem **P**.

The boundedness of n follows from [5, Theorem 2] and $n \ge 0$ almost everywhere in R_T is essentially demonstrated in [8].

Remark 3.3. Let $n_0 \in L^p(\Omega)$ with $p > 3$, and let $\Theta(0) = \Theta_0 > 0$. Then Problem **P** has a local solution satisfying $n \in L^\infty\big(0,T;L^p(\Omega)\big)$ and $n^{p/2} \in L^2\big(0,T;H^1(\Omega)\big)$. The proof is almost identical to the proof of Theorem 3.2.

3.2. **Uniqueness.** Uniqueness is stated for an equivalent formulation of Problem **P** in which we replace t by $\tau = \int_0^t \Theta(t)dt$. This transformation affects only equation (1.1), which now becomes

$$n_\tau = \operatorname{div}\Big\{\nabla n + \frac{n}{\Theta(\tau)}\nabla\phi\Big\} \quad \text{in} \quad R_{\hat{T}}, \tag{3.13}$$

where $\hat{T} = \int_0^T \Theta(t)dt$. The problem stated in terms of x and τ is denoted by Problem $\mathbf{P}_e$. Without proof we remark that $\langle n = n(x,t), \Theta = \Theta(t)\rangle$ solves Problem **P** if and only if $\langle n = n(x,\tau), \Theta = \Theta(\tau)\rangle$ solves Problem $\mathbf{P}_e$. This is due to the strict positivity of Θ in the existence interval.

Theorem 3.4. *If $n_0 \in L^2(\Omega)$ and $\Theta_0 > 0$, then Problem $\mathbf{P}_e$ has at most one solution $\langle n, \Theta\rangle$.*

Proof. We use a uniqueness result of Biler and Nadzieja [8], who considered the problem

$$n_\tau = \operatorname{div}\big(\nabla n + nX(n)\big) \qquad \text{in} \qquad R_T, \tag{3.14}$$

$$\big(\nabla n + nX(n)\big)\cdot\vec{\nu} = 0 \qquad \text{on} \qquad \partial\Omega\times[0,T], \tag{3.15}$$

$$n(\cdot,0) = n_0 \qquad \text{in} \qquad \Omega, \tag{3.16}$$

where X is a general, nonlocal, vector-field operator in $\mathbb{R}^3$. For this problem uniqueness in $L^2(\Omega)$ was proved in [8, Theorem 1 (i)] under the following condition: there exists $C > 0$ such that

$$(U) \qquad \|X(u) - X(v)\|_{L^6(\Omega)} \le C\|u - v\|_{L^2(\Omega)}$$

for all $u, v \in L^2(\Omega)$.

Note that the constant C in (U) does not depend on the choice of $u, v \in L^2(\Omega)$. In our case $X(n) = \frac{\nabla\phi}{\Theta(\tau)}$. Below, in Lemma 3.5, we show that again (U) holds but with C depending on both the norms $\|u\|_{L^2(\Omega)}$ and $\|v\|_{L^2(\Omega)}$. Now suppose that Problem $\mathbf{P}_e$ admits two solutions $\langle n_1, \Theta_1\rangle$ and $\langle n_2, \Theta_2\rangle$ in some interval $[0,T]$. From the solution definition we know that both $\|n_1(t)\|_{L^2(\Omega)}$ and $\|n_2(t)\|_{L^2(\Omega)}$ are uniformly bounded in $[0,T]$. Therefore, (U) is satisfied for the two solutions $n_1(t)$ and $n_2(t)$, with $0 \le t \le T$, for an appropriately chosen constant C. As a consequence we can apply the result of [8]. This proves the theorem.

Lemma 3.5. *Suppose there exist $\delta > 0$ and $u, v \in L^2(\Omega)$ such that*

$$\min\Big\{\Theta_u := E + \int_\Omega |\nabla\phi_u|^2\,dx,\ \Theta_v := E + \int_\Omega |\nabla\phi_v|^2\,dx\Big\} \ge \delta > 0,$$

where

$$\Delta\phi_u = u, \qquad \Delta\phi_v = v \qquad \textit{in} \qquad \Omega, \tag{3.17}$$

$$\phi_u = \phi_v = 0 \qquad on \qquad \partial\Omega. \tag{3.18}$$

Then

$$\Big\|\nabla\Big(\frac{\phi_u}{\Theta_u} - \frac{\phi_v}{\Theta_v}\Big)\Big\|_{L^6(\Omega)} \le C\|u-v\|_{L^2(\Omega)},$$

where $C = C(\delta, \|u\|_{L^2(\Omega)}, \|v\|_{L^2(\Omega)})$.

Proof. Using $\|\nabla(\phi_u - \phi_v)\|_{L^6(\Omega)} \le C_I\|u-v\|_{L^2(\Omega)}$ we estimate

$$\begin{aligned}
&\Big\|\nabla\Big(\frac{\phi_u}{\Theta_u} - \frac{\phi_v}{\Theta_v}\Big)\Big\|_{L^6(\Omega)} = \Big\|\nabla\Big(\frac{\phi_u}{\Theta_u} - \frac{\phi_u}{\Theta_v} + \frac{\phi_u}{\Theta_v} - \frac{\phi_v}{\Theta_v}\Big)\Big\|_{L^6(\Omega)} \\
&\le \frac{1}{\Theta_u\Theta_v}\|\nabla\phi_u\|_{L^6(\Omega)}|\Theta_u - \Theta_v| + \frac{C_I}{\Theta_v}\|u-v\|_{L^2(\Omega)} \\
&\le \frac{C_I}{\Theta_v}\Big\{\frac{\|u\|_{L^2(\Omega)}}{\Theta_u}\Big|\int_\Omega \big(|\nabla\phi_u|^2 - |\nabla\phi_v|^2\big)\,dx\Big| + \|u-v\|_{L^2(\Omega)}\Big\}.
\end{aligned}$$

Since

$$\begin{aligned}
\Big|\int_\Omega \big(|\nabla\phi_u|^2 - |\nabla\phi_v|^2\big)\,dx\Big| &\le \|\nabla(\phi_u+\phi_v)\|_{L^2(\Omega)}\|\nabla(\phi_u-\phi_v)\|_{L^2(\Omega)}, \\
&\le C\big(\|u\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}\big)\|u-v\|_{L^2(\Omega)}
\end{aligned}$$

for some $C > 0$, we obtain the assertion. □

Remark 3.6. Note that for $n_0 \in L^p(\Omega)$ with $p > 2$, we can apply the above theorem and obtain uniqueness of solutions for such initial data. Thus the local solution given in Remark 3.3 of Theorem 3.2 for $n_0 \in L^p(\Omega)$ with $p > 3$ is unique.

Remark 3.7. A slight modification of the above argument directly gives uniqueness for $n_0 \in L^p(\Omega)$ with $p > 3$ and $\Theta_0 > 0$. Again following [8, Theorem 1 (ii)], we need to show

$$(U') \qquad \|X(u) - X(v)\|_{L^\infty(\Omega)} \le C\|u-v\|_{L^p(\Omega)} \quad (p > 3).$$

With $X(n) = \frac{\nabla\phi}{\Theta(\tau)}$, inequality (U') results from inequality (6.3).

The next theorem extends the local existence result for $n_0 \in L^p(\Omega)$ with $p > \frac{3}{2}$. To do this we modify our definition of weak solution for $1 < p < 2$. We replace (i) by $n \in L^\infty(0,T;L^p(\Omega))$ and $n_t \in L^p\big(0,T;(W^{1,p'}(\Omega))'\big)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. We show that $n \in C\big([0,T];L^p(\Omega)\big)$, implying that n_0 can be prescribed. In fact as $n \in L^\infty\big(0,T;L^p(\Omega)\big)$ and $n \in C\big([0,T],(W^{1,p'}(\Omega))'\big)$ (since $n \in L^p\big(0,T;(W^{1,p'}(\Omega))'\big)$ and $n_t \in L^p\big(0,T;(W^{1,p'}(\Omega))'\big)$), we use the injection $L^p(\Omega) \hookrightarrow (W^{1,p'}(\Omega))'$, and follow the argument in [20, p. 23] to conclude.

Theorem 3.8. *Let $n_0 \in L^p(\Omega)$ with $p > 3/2$, and $\Theta(0) = \Theta_0 > 0$. Then there exists $T = T(\Omega, \|n_0\|_{L^p(\Omega)}, \Theta_0) > 0$, and a weak solution $\langle n, \Theta\rangle$ of Problem* **P**. *This solution satisfies $n^{p/2} \in L^2\big(0,T; H^1(\Omega)\big)$, and furthermore $n \in L^\infty_{loc}\big(0,T]; L^\infty(\Omega)\big)$.*

Proof. Due to Remarks 3.3 and 3.6, we need only to demonstrate local existence for $n_0 \in L^p(\Omega)$ with $p \in (3/2, 3)$. We follow the proof of [8, Theorem 1 (iii)] and approximate $n_0 \in L^p(\Omega)$ by functions in $L^{p_*}(\Omega)$ with $p_* > 3$. Thus let $\{n_{0\epsilon}\} \subset L^{p_*}(\Omega)$ satisfy $\|n_{0\epsilon} - n_0\|_{L^p(\Omega)} \to 0$ as $\epsilon \to 0$.

For each $\epsilon > 0$ we consider Problem **P** with initial data $\langle n_{0\epsilon}, \Theta_0\rangle$. By Remark 3.6 there exists a solution $\langle n_\epsilon, \Theta_\epsilon\rangle$, with $\Theta_\epsilon \geq \Theta_0/2$, in some interval $[0, T_\epsilon]$. Next we use the estimate [8, equation (10)]

$$\|n_\epsilon(t)\|^p_{L^p(\Omega)} + \int_0^t \big|\nabla |n_\epsilon|^{p/2}\big|^2\, d\tau \leq \exp\Big(C\int_0^t \Big\|\frac{\nabla\phi_\epsilon}{\Theta_\epsilon}\Big\|^{\frac{2q}{q-3}}_{L^q(\Omega)}\, d\tau\Big)\|n_{0\epsilon}\|^p_{L^p(\Omega)} \tag{3.19}$$

for almost every $t \in [0, T_\epsilon]$. Here $p \in (3/2, 3)$, $1/q = 1/p - 1/3$, and $C = C(\Omega, p)$. Note that $q > 3$. Further, using $\|\nabla\phi_\epsilon\|_{L^q(\Omega)} \leq C\|n_\epsilon\|_{L^p(\Omega)}$ and the uniform lower bound on Θ_ϵ, we obtain

$$\|n_\epsilon(t)\|^p_{L^p(\Omega)} \leq C\exp\Big(\frac{2}{\Theta_0}\int_0^t \big\|n_\epsilon(\tau)\big\|^{\frac{2q}{q-3}}_{L^p(\Omega)}\, d\tau\Big) \text{ for almost every } \ t \in [0, T_\epsilon],$$

where $C = C(\bar\epsilon) > 0$ and $0 < \epsilon \leq \bar\epsilon$. Since C does not depend on ϵ we have that $T_\epsilon = T$ and

$$\|n_\epsilon(t)\|_{L^p(\Omega)} \leq C \text{ and } \Theta_\epsilon(t) \geq \Theta_0/2 \text{ for almost every } t \in [0, T] \tag{3.20}$$

and for all $0 < \epsilon \leq \bar\epsilon$. Using this and (3.19), we deduce

$$\|n_\epsilon(t)^{p/2}\|_{H^1(\Omega)} \leq C \text{ and } \Theta_\epsilon(t) \geq \Theta_0/2 \text{ for almost every } t \in [0, T] \tag{3.21}$$

and for all $0 < \epsilon \leq \bar\epsilon$.

Next we separate the demonstration into two cases: $p < 2$ and $p > 2$.

We begin with $p < 2$. Under this condition, we have

$$\|\nabla n\|_{L^p(\Omega)} \leq C(\Omega, p)\|\nabla n^{p/2}\|_{L^2(\Omega)}\|n\|^{(2-p)/2}_{L^p(\Omega)} \quad \text{for} \quad n^{p/2} \in H^1(\Omega).$$

Combining this with (3.20) and (3.21), since $p < 2$, we obtain

$$\|\nabla n_\epsilon\|_{L^p(0,T; L^p(\Omega))} \leq C \quad \text{for all} \quad 0 < \epsilon \leq \bar\epsilon. \tag{3.22}$$

Consequently, we can check that $\|n_{\epsilon t}\|_{L^p\left(0,T,(W^{1,p'}(\Omega))'\right)} \leq C$ for all $0 < \epsilon \leq \bar\epsilon$.

Now using a compactness theorem [20, p. 141], with $L^p(\Omega) \hookrightarrow (W^{1,p'}(\Omega))'$, we find for a subsequence $\epsilon \to 0$,

$$n_\epsilon \to n \qquad \text{in} \qquad L^p\big(0, T; L^p(\Omega)\big). \tag{3.23}$$

Now using standard arguments and the above estimates, we get as $\epsilon \to 0$

$$n_{\epsilon t} \rightharpoonup n_t \quad \text{in} \quad L^p\big(0,T,(W^{1,p'}(\Omega))'\big), \tag{3.24}$$

$$\nabla\phi_\epsilon \to \nabla\phi \quad \text{in} \quad L^p\big(0,T,W^{1,p}(\Omega)\big), \tag{3.25}$$

$$\nabla n_\epsilon \rightharpoonup \nabla n \quad \text{in} \quad L^p\big(0,T;L^p(\Omega)\big). \tag{3.26}$$

To conclude, it suffices to prove $\Theta_\epsilon \to \Theta$ in $C([0,T])$. This follows from showing that

$$\kappa\int_0^T |\Theta_{\epsilon t} - \Theta_t|\,dt \le 2\int_0^T \Big|\int_\Omega n_\epsilon\phi_{\epsilon t} - n\phi_t\,dx\Big|\,dt \to 0 \quad \text{as} \quad \epsilon \to 0. \tag{3.27}$$

We obtain this using [25, Proposition 23.9 (d)], combining (i) strong convergence of $n_\epsilon \to n$ in $L^{p'}\big(0,T;L^p(\Omega)\big)$ with (ii) weak convergence of $\phi_{\epsilon t} \rightharpoonup \phi_t$ in $L^p\big(0,T;L^{p'}(\Omega)\big)$. In fact, (i) is consequence of (3.20) and (3.23), and (ii) follows using (3.24) along with the estimate $\|\phi_{\epsilon t}\|_{p'} \le C\|\phi_{\epsilon t}\|_{W^{1,p}(\Omega)} \le C\|n_{\epsilon t}\|_{(W^{1,p'}(\Omega))'}$, where we have used $p > \frac{3}{2}$.

Now we take the limit $\epsilon \to 0$ to conclude that n satisfies Problem **P**.

For $p > 2$, we use that $n_{0\epsilon} \in L^2(\Omega)$, and in particular (3.20) implies

$$\|n_\epsilon(t)\|_{L^2(\Omega)} \le C \quad \text{and} \quad \Theta_\epsilon(t) \ge \Theta_0/2 \quad \text{for all} \quad t \in [0,T].$$

We follow the proof of Theorem 3.2, to find $\|n_{\epsilon t}\|_{L^2(0,T,(H^1(\Omega))')} \le C$. With this we may apply again the compactness theorem [20, p. 141], now with $L^p(\Omega) \hookrightarrow (H^1(\Omega))'$, since $p > 2$, and obtain $n_\epsilon \to n$ in $L^p\big(0,T,L^p(\Omega)\big)$. Finally, we show (3.27) using $p = p' = 2$ and obtain $\Theta_\epsilon \to \Theta$ in $C([0,T])$, which concludes the proof of the theorem. □

3.3. **Radially symmetric solutions.** In Section 2.4 we introduced Problem $\mathcal{Q}$ describing radially symmetric solutions of Problem **P** in the unit ball. In this paper we do not prove existence for Problem $\mathcal{Q}$. Instead we shall assume that if $\Omega = B_1(0)$ and if n_0 is radially symmetric, then the corresponding weak solution is radially symmetric. By standard regularity theory weak solutions of Problem **P** satisfy equations (1.1)–(1.3) and boundary conditions (1.4)–(1.5) in a classical sense. With this in mind we introduce for Problem $\mathcal{Q}$ the following solution definition.

Let $D_T = (0,1) \times (0,T]$. A pair $\langle Q,\vartheta\rangle$ solves Problem $\mathcal{Q}$ if, for some $T > 0$,

(i) $Q \in C^{2,1}(D_T) \cap C(\overline{D_T})$, and $\vartheta \in C([0,T])$;
(ii) $\langle Q,\vartheta\rangle$ satisfies equations (2.12)–(2.15);
(iii) $Q_y \ge 0$ in D_T and $\vartheta > 0$ in $[0,T]$.

Clearly radial solutions of Problem **P** with $n_0 \in L^p(B_1(0))$, $p > \frac{3}{2}$, satisfy this definition. This follows directly from the identity

$$Q_y(y,t) = \frac{4\pi}{3} n(y^{1/3},t) \quad \text{for} \quad (y,t) \in D_T. \tag{3.28}$$

4. Prescribed temperature problem

In this section we study Problem **P** with prescribed temperature $\Theta(t)$ satisfying

$$\Theta\colon [0,T] \to \mathbb{R} \text{ such that } \Theta \in C([0,T]) \text{ and } \Theta(t) > \delta > 0 \text{ for } t \in [0,T]. \tag{4.1}$$

Thus we drop the energy relation (1.3) and assume that Θ in (1.1) and (1.4) is given and satisfies (4.1). We denote this modified problem by $\mathbf{P}^*$. Clearly, if Problem $\mathbf{P}^*$ has a radially symmetric solution, then the corresponding formulation in terms of the accumulated mass Q and temperature ϑ, which we denote by $\mathcal{Q}^*$, has a classical solution according to the definition given in Section 3.3.

We first recall some recent results of Biler and Nadzieja [10, 8], related to local existence for Problem $\mathbf{P}^*$ and global existence for Problem $\mathcal{Q}^*$.

Remark 4.1.

(i) Let $n_0 \in L^2(\Omega)$, and let Θ satisfy (4.1). Then there exists $T = T(\Omega, \|n_0\|_{L^2(\Omega)}, \delta) > 0$ so that Problem $\mathbf{P}^*$ has a unique weak solution in $[0,T]$ which satisfies $n \in L^\infty_{loc}\big((0,T], L^\infty(\Omega)\big)$. Proof: see [8, Theorem 1 (i)].

(ii) Let $n_0 \in L^p(\Omega)$ with $p > \frac{3}{2}$ and let Θ satisfy (4.1). Then there exist $T = T(\Omega, \|n_0\|_{L^p(\Omega)}, \delta) > 0$ so that Problem $\mathbf{P}^*$ has a weak solution in $[0,T]$ satisfying $n \in L^\infty\big(0,T,L^p(\Omega)\big)$ and $n^{p/2} \in L^2\big(0,T,H^1(\Omega)\big)$. For $p > 3$ the solution is unique. Proof: see [8, Theorem 1 (ii) and (iii)].

(iii) If for some $B > 0$, $Q_0(y) \le y\frac{1+B}{y^{2/3}+B}$ for $0 \le y \le 1$ and $\vartheta(t) = constant \ge \frac{3}{2}(1+B)$, then Problem $\mathcal{Q}^*$ has a global, classical solution satisfying $Q(y,t) \le y\frac{1+B}{y^{2/3}+B}$ for $(y,t) \in \overline{D_\infty}$. Proof: see [10, Theorem 1 (iii)].

In the remainder of this section we present some new results related to Problem $\mathbf{P}^*$ with *constant* temperature $\Theta > 0$. We first extend a global existence result of [9].

Theorem 4.2. *For a given domain Ω, there exist positive constants, α_1, α_2, A, B, and C with $\alpha_2 \ge \alpha_1$ so that if the constant temperature Θ and the* *initial condition n_0 satisfy*

$$\Theta \geq \alpha_1 \quad \textit{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq A + \frac{B}{\Theta^4}$$

or

$$\Theta \geq \alpha_2 \quad \textit{and} \quad \|n_0\|^2_{L^2(\Omega)} \leq C(\Theta^2 - \Theta),$$

then Problem $\mathbf{P}^*$ has a global (weak) solution for which the $L^2(\Omega)$ norm is uniformly bounded in time.

Proof. Integrating (1.1) we obtain the expression

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{L^2(\Omega)} + \Theta\|\nabla n\|^2_{L^2(\Omega)} = -\int_\Omega n\nabla n\nabla\phi\, dx. \tag{4.2}$$

As in the proof of [9, Theorem 2 (iii)] we estimate

$$\frac{1}{2}\frac{d}{dt}\|n\|^2_{L^2(\Omega)} + \Theta\|\nabla n\|^2_{L^2(\Omega)} \leq \|n\|_{L^3(\Omega)}\|\nabla\phi\|_{L^6(\Omega)}\|\nabla n\|_{L^2(\Omega)}. \tag{4.3}$$

The aim is to obtain a differential inequality for $\|n(t)\|^2_{L^2(\Omega)}$. From the appendix we first use (6.2) and then (6.3) with $r = 6$ and $p = 2$. This gives

$$\|n\|_{L^3(\Omega)}\|\nabla\phi\|_{L^6(\Omega)}\|\nabla n\|_{L^2(\Omega)} \leq \frac{\Theta}{2}\|n\|^2_{H^1(\Omega)} + \frac{C_1}{\Theta^3}(\|n\|^2_{L^2(\Omega)})^3. \tag{4.4}$$

Further, we use (6.1) with $q = 2$ and $p = 1$ to obtain

$$2\Theta\|n\|^2_{L^2(\Omega)} \leq \frac{\Theta}{2}\|n\|^2_{H^1(\Omega)} + C_2\Theta\|n\|^2_{L^1(\Omega)}. \tag{4.5}$$

The combination (4.3)–(4.5) eliminates the gradient term. Since $\|n\|_{L^1(\Omega)} = 1$, we are left with an inequality of the form

$$\frac{d}{dt}w \leq p_\Theta(w) := \frac{C_1}{\Theta^3}w^3 - \Theta w + C_2\Theta \quad \text{for} \quad t > 0 \tag{4.6}$$

with $w(t) := \|n(t)\|^2_{L^2(\Omega)}$. Here C_1 and C_2 are positive constants depending only on Ω. The assertions of the theorem now follow from particular properties of (4.6).

First observe that if $\Theta > \alpha_1 := \frac{3^{3/4}}{2^{1/2}}(C_2 C_1^{1/2})^{1/2}$, then $p_\Theta(w) = 0$ has two positive real roots $w_* < w^*$ and $p_\Theta(w) < 0$ for $w_* < w < w^*$. If $\Theta = \alpha_1$, these roots coincide. A simple calculation shows that

$$w_0 = C_2 + \frac{C_2^3 C_1}{\Theta^4}$$

satisfies $0 < w_0 < w_*$ for all $\Theta \geq \alpha_1$. Since $p_\Theta(w) > 0$ for $0 \leq w < w_*$, we deduce that $w(t)$, with $w(0) \leq w_0$, satisfies $w(t) \leq w_*$ for all $t \geq 0$. This proves the first assertion.

Next consider $w^0 := C_1^{-1/2}(\Theta^2 - \Theta)$. Then $p_\Theta(w^0) \le 0$ provided

$$\Theta \ge \alpha_2 = \Big(\Big(\frac{3 + C_2 C_1^{1/2}}{4}\Big)^2 - \frac{1}{2}\Big)^{1/2} + \frac{3 + C_2 C_1^{1/2}}{4}.$$

Clearly $\alpha_2 \ge \alpha_1$, since $p_\Theta(w) > 0$ for $\Theta < \alpha_1$ and for all $w > 0$. As before we have that $w(t)$, with $w(0) \le w^0$, satisfies $w(t) \le w^0 \le w^*$ for all $t \ge 0$. This proves the second assertion. □

In a similar fashion global existence results are obtained in $L^p(\Omega)$ for $p > 3/2$. Instead of (4.6) one now finds

$$\frac{d}{dt} w \le \frac{C_1}{\Theta^\beta} w^\beta - \Theta w + C_2 \Theta \quad \text{for} \quad t > 0 \tag{4.7}$$

with $w(t) := \|n(t)\|^p_{L^p(\Omega)}$. Here

$$\beta = \begin{cases} \frac{2p-1}{2p-3} & \text{for} \quad 3/2 < p < 3 \\ \frac{p+2}{p} & \text{for} \quad p > 3. \end{cases}$$

Inequality (4.7) implies the following result.

Theorem 4.3. *For a given domain Ω, there exist positive constants, β_1, β_2, $\bar{A}$, $\bar{B}$, and $\bar{C}$ with $\beta_2 \ge \beta_1$ so that if the constant temperature Θ and the initial condition n_0 satisfy*

$$\Theta \ge \beta_1 \quad \textit{and} \quad \|n_0\|^p_{L^p(\Omega)} \le \bar{A} + \frac{\bar{B}}{\Theta^{\beta+1}}$$

or

$$\Theta \ge \beta_2 \quad \textit{and} \quad \|n_0\|^p_{L^p(\Omega)} \le \bar{C}(\Theta^\gamma - \Theta), \quad \textit{with} \quad \gamma = \frac{\beta+1}{\beta-1},$$

then Problem $\mathbf{P}^*$ has a global (weak) solution for which the $L^p(\Omega)$ norm is uniformly bounded in time.

5. Global existence for problem $\mathcal{Q}$

In this section we study radially symmetric solutions of Problem $\mathbf{P}$. We will use the classical formulation in terms of Problem $\mathcal{Q}$. Before we prove the global existence results, we first demonstrate the blow-up result Theorem 1.1.

Theorem 5.1. *Let $\kappa > 6$ and $\mathcal{E} < 3$. Then $T^* < \infty$.*

Proof. Suppose Problem $\mathcal{Q}$ has a global solution $Q = Q(y,t)$ and $\vartheta = \vartheta(t) \in (0,\infty)$ for all $t > 0$. Setting

$$w_\epsilon(t) := \int_\epsilon^1 Q(y,t) y^{-1/3}\, dy \quad \text{for all} \quad t > 0,$$

a multiplication of equation (2.12) by $y^{-1/3}$ and an integration in $(t_1,t_2)\times(\epsilon,1)$ for some arbitrary $t_2>t_1>0$ gives

$$\begin{aligned}
&w_\epsilon(t_2)-w_\epsilon(t_1)\\
&=\int_{t_1}^{t_2}\vartheta(t)Q_y(1,t)\,dt-\epsilon\int_{t_1}^{t_2}\vartheta(t)Q_y(\epsilon,t)\,dt-\int_{t_1}^{t_2}\vartheta(t)\,dt+\int_{t_1}^{t_2}\vartheta(t)Q(\epsilon,t)\,dt\\
&-\frac{1}{2}\int_{t_1}^{t_2}Q(\epsilon,t)^2\epsilon^{-1/3}\,dt+\frac{1}{2}(t_2-t_1)+\frac{\kappa}{6}\int_{t_1}^{t_2}\vartheta(t)\,dt-\frac{\mathcal{E}}{6}(t_2-t_1)\\
&+\frac{1}{6}\int_{t_1}^{t_2}\int_0^\epsilon Q^2(y,t)y^{-4/3}\,dy\,dt\quad\text{for all}\quad t>0. \qquad (5.1)
\end{aligned}$$

Since $\int_{t_1}^{t_2}\vartheta(t)\,dt<\infty$, we obtain from (2.13)

$$\int_{t_1}^{t_2}\int_0^1 Q(y,t)^2y^{-4/3}\,dy\,dt<\infty. \qquad (5.2)$$

This estimate implies that there exists a sequence $\epsilon_n\downarrow 0$ along with

$$\lim_{\epsilon_n\downarrow 0}\int_{t_1}^{t_2}Q(\epsilon_n,t)^2\epsilon_n^{-1/3}\,dt=0.$$

In fact, arguing by contradiction, assume that there exists $C>0$ such that $\int_{t_1}^{t_2}Q(y,t)^2y^{-1/3}\,dt>C$ for all $y\in[0,1]$. This implies that

$$\int_{t_1}^{t_2}Q(y,t)^2y^{-4/3}\,dt>Cy^{-1}\notin L^1(0,1),$$

which contradicts (5.2). Using this and $Q_y(1,t)\geq 0$ in (5.1), we find in the limit $\epsilon_n\downarrow 0$

$$w_0(t_2)-w_0(t_1)\geq\Big[\frac{1}{2}-\frac{\mathcal{E}}{6}\Big](t_2-t_1)+\int_{t_1}^{t_2}\Big(\frac{\kappa}{6}-1\Big)\vartheta(t)\,dt$$

for all $t_2>t_1>0$. The parameter choice implies that $\frac{dw_0(t)}{dt}\geq\delta>0$ for all $t>0$. This contradicts $w_0\leq\frac{3}{2}$ (implied by $Q\leq 1$ in $\overline{D_\infty}$). □

Next we turn to global existence. The proof uses a comparison principle for the Q equation (2.12) with respect to given ordered temperatures, and the fact that temperature is positively bounded from below. The results are stated in terms of an equivalent formulation, as in (3.13), in which we replace t by τ; i.e.,

$$Q_\tau = y^{4/3}Q_{yy}+\frac{1}{\vartheta(\tau)}QQ_y \quad\text{in}\quad D_T \qquad (5.3)$$

$$Q(0,\tau) = 0,\qquad Q(1,\tau)=1\qquad\text{for}\quad \tau\in[0,T], \qquad (5.4)$$

$$Q(y,0) \quad = \quad Q_0(y) \qquad\qquad \text{for} \quad y \in [0,1], \tag{5.5}$$

and the energy relation

$$\mathcal{E} \quad = \quad \kappa\vartheta(\tau) - \int_0^1 \frac{Q^2}{y^{4/3}}\,dy \qquad \text{for} \quad \tau \in [0,T]. \tag{5.6}$$

We first consider (5.3)–(5.5) for given ordered temperatures and ordered initial data.

Proposition 5.2. *Let $i = 1,2$. Suppose Q_i solves (5.3)–(5.5) in D_{T_i} subject to $Q_0 = Q_{0i}$, and given $\vartheta = \vartheta_i$, satisfying (4.1) in $[0,T_i]$. Let $T = \min\{T_1,T_2\}$. If*

$$\vartheta_1 \le \vartheta_2 \quad \textit{in} \quad [0,T], \quad \textit{and} \quad Q_{01} \ge Q_{02} \quad \textit{in} \quad (0,1),$$

and if there exists $K > 0$ such that either

$$0 \le Q_{1_y} \le K \quad \textit{or} \quad 0 \le Q_{2_y} \le K \quad \textit{in} \quad \overline{D_T},$$

then

$$Q_1 \ge Q_2 \quad \textit{in} \quad D_T.$$

Proof. Suppose $0 \le Q_{2_y} \le K$. Since

$$Q_{2_\tau} = y^{4/3}Q_{2_{yy}} + \frac{1}{\vartheta_1(\tau)}Q_2Q_{2_y} + \Big(\frac{1}{\vartheta_2(\tau)} - \frac{1}{\vartheta_1(\tau)}\Big)Q_2Q_{2_y} \quad \text{in} \quad D_T,$$

it follows from $\vartheta_1 \le \vartheta_2$ and $Q_{2y} \ge 0$ that

$$Q_{2_\tau} \le y^{4/3}Q_{2_{yy}} + \frac{1}{\vartheta_1(\tau)}Q_2Q_{2_y} \quad \text{in} \quad D_T.$$

This inequality and the boundedness of Q_{2_y} allows us to use [19, Theorem 3.2], which shows that Q_2 is a subsolution for the Q equation with ϑ_1. □

Next we use the Boltzmann entropy (2.8) in terms of $Q = Q(y,\tau)$ to establish a positive lower bound on ϑ (see [21] for a similar estimate).

Proposition 5.3. *Let $\langle Q,\vartheta\rangle$ be a solution of problem (5.3)–(5.6). Suppose $\vartheta(0) = \frac{1}{\kappa}\Big(\mathcal{E} + \int_0^1 \frac{Q_0^2(y)}{y^{4/3}}\,dy\Big) > 0$. Then*

$$\vartheta(\tau) \ge \lambda\vartheta(0) \textit{ for } \tau > 0, \textit{ with } \lambda = \exp\Big(-\frac{2}{\kappa}\int_0^1 Q_{0_y}\log Q_{0_y}\,dy\Big). \tag{5.7}$$

Proof. Rewriting (2.8) results in

$$W(\tau) := \int_0^1 Q_y \log Q_y\,dy - \frac{\kappa}{2}\log\Big(\mathcal{E} + \int_0^1 \frac{Q^2}{y^{4/3}}\,dy\Big) \quad \text{for} \quad \tau > 0,$$

and differentiation gives (see also (2.9))

$$\frac{dW(\tau)}{d\tau} = -\int_0^1 \frac{Q_\tau^2}{y^{4/3}Q_y}\,dy \le 0 \quad \text{for} \quad \tau > 0.$$

Hence $W(\tau)$ is decreasing in τ. As a consequence

$$W(\tau) = \int_0^1 Q_y \log Q_y\,dy - \frac{\kappa}{2}\log\Big(\mathcal{E} + \int_0^1 \frac{Q^2}{y^{4/3}}\,dy\Big) \le$$
$$\le W(0) = \int_0^1 Q_{0_y} \log Q_{0_y}\,dy - \frac{\kappa}{2}\log\Big(\mathcal{E} + \int_0^1 \frac{Q_0^2}{y^{4/3}}\,dy\Big) \quad \text{for} \quad \tau > 0.$$

Here we use Jensen's inequality to estimate

$$\int_0^1 Q_y \log Q_y\,dy \ge \Big(\int_0^1 Q_y\,dy\Big)\log\Big(\int_0^1 Q_y\,dy\Big) = 0,$$

from which lower bound (5.7) directly follows. □

Note that whenever ϑ is bounded away from zero, blow-up in problem (5.3)–(5.6) can occur only at the boundary $y = 0$. This is a direct consequence of classical regularity theory, which implies that Q is smooth away from $y = 0$. Blow-up manifests itself through singular behaviour of Q_y as (y, τ) approaches the point $(0, T^*)$. This corresponds to unbounded density at the origin of the radially symmetric solution of Problem **P**. Below we use the comparison argument (Proposition 5.2) to control the behaviour of $Q_y(0, \tau)$. We show that this implies a uniform bound on $\|Q_y(\tau)\|_{L^2(0,1)}$ and thus on $\|n(\tau)\|_{L^2(B_1(0))}$ for all $0 \le \tau < T^*$. Global existence for $Q = Q(y, \tau)$ is a consequence of Theorem 3.2. The results translate in a straightforward manner to the assertions of Theorem 1.2

Theorem 5.4. *Let* $Q_0\colon [0,1] \mapsto [0,1]$ *be nondecreasing,* $Q_{0_y} \in L^\infty(0,1)$, *and* $Q_0(0) = 0$, $Q(1) = 1$. *Let* $\vartheta(0) = \vartheta_0 > 0$. *If either*

(i) ϑ_0 *is sufficiently large, or*
(ii) *there exists* $B > 0$ *such that*

$$\vartheta_0 \ge \frac{3}{2}\frac{(1+B)}{\lambda} \quad \textit{and} \quad Q_0(y) \le \frac{y(1+B)}{y^{2/3}+B},$$

with $\lambda = \exp\big(-\frac{2}{\kappa}\int_0^1 Q_{0_y}\log Q_{0_y}dy\big)$,

then Problem (5.3)–(5.6) *has a global solution* $\langle Q, \vartheta\rangle$ *in the sense of Section* 3.3. *Moreover, there exist constants* $L > 0$ *and* $\vartheta^* > 0$ *such that* $\vartheta(\tau) \le \vartheta^*$ *and* $\|Q_y(\tau)\|_{L^2(0,1)} \le \|Q_{0_y}\|_{L^2(0,1)}\exp(L\tau)$ *for all* $\tau > 0$. *If (ii) is satisfied* *we have in addition*

$$Q(y,\tau) \le \frac{y(1+B)}{y^{2/3}+B} \quad \textit{for all} \quad (y,\tau) \in \overline{D_\infty}.$$

Proof. First we consider the auxiliary problem

$$(\mathbf{AP}) \quad \begin{cases} \bar{Q}_\tau &= y^{4/3}\bar{Q}_{yy} + \frac{1}{A}\bar{Q}\bar{Q}_y \qquad \text{in} \quad D_\infty, \\ \bar{Q}(0,\tau) &= 0, \qquad \bar{Q}(1,\tau) = 1 \quad \text{for} \quad \tau > 0, \\ \bar{Q}(y,0) &= Q_0(y) \qquad\qquad \text{for} \quad y \in [0,1], \end{cases} \tag{5.8}$$

where $A > 0$ and where Q_0 satisfies the conditions of the theorem.

By Theorem 4.2 and Remark 4.1 (iii), we have the following: if either

(H_1) $A \ge 12\pi\alpha_2$ and $\|Q_{0_y}\|^2_{L^2(0,1)} \le \frac{4\pi C}{3}(A^2 - A)$, or

(H_2) $A = \frac{3}{2}(1+B)$ and $Q_0(y) \le \frac{y(1+B)}{y^{2/3}+B}$ for $0 \le y \le 1$, and for some $B > 0$,

then Problem **AP** has a global solution $\bar{Q}\colon \overline{D_\infty} \mapsto [0,1]$. Since $Q_{0_y} \in L^\infty(0,1)$, the regularity theory of [6, Theorem 2] gives

$$\|\bar{Q}_y\|_{L^\infty(0,1)} \in L^\infty_{loc}([0,\infty)). \tag{5.9}$$

The conditions on Q_0 and ϑ_0 guarantee that Problem (5.3)–(5.6) has a classical solution in D_T for some T. Now suppose

$$T^* = \sup\{T > 0 : \text{solution of Problem (5.3)–(5.6) exists in } D_T\} < \infty. \tag{5.10}$$

Fix $A > 0$ such that (H_1) is satisfied, and choose $\vartheta_0 \ge \frac{A}{\lambda}$. By (5.7), we have

$$\vartheta(\tau) \ge A \quad \text{for all} \quad \tau \in [0, T^*), \tag{5.11}$$

and by Proposition 5.2 and (5.9), we find

$$Q(y,\tau) \le \bar{Q}(y,\tau) \le Ky \quad \text{for} \quad (y,\tau) \in [0,1] \times [0,T^*) \tag{5.12}$$

for some $K > 0$. Below we show that this implies a uniform bound on $\|Q_y(\cdot)\|_{L^2(\Omega)}$ in $[0,T^*)$. Multiplying (5.3) by $Q_\tau / y^{4/3}$ gives

$$\frac{1}{y^{4/3}}Q_\tau^2 = Q_\tau Q_{yy} + \frac{1}{\vartheta(\tau)y^{4/3}}Q_\tau Q Q_y \quad \text{in} \quad [0,1]\times(0,T^*). \tag{5.13}$$

Using (5.11) and (5.12), the second term on the right can be estimated by

$$\frac{1}{\vartheta(\tau)y^{4/3}}Q_\tau Q Q_y \le \frac{1}{y^{4/3}}Q_\tau^2 + \frac{1}{4\vartheta^2(\tau)y^{4/3}}Q^2Q_y^2 \le \frac{1}{y^{4/3}}Q_\tau^2 + \frac{K^2}{4A^2}Q^2Q_y^2.$$

Using this in (5.13) and integrating the results give

$$\frac{d}{d\tau}\|Q_y(\tau)\|^2_{L^2(0,1)} \le \frac{K^2}{2A^2}\|Q_y(\tau)\|^2_{L^2(0,1)} \quad \text{for all} \quad 0 \le \tau < T^*.$$

Hence

$$\|Q_y(\tau)\|_{L^2(0,1)} \leq \|Q_{0_y}\|_{L^2(0,1)} \exp\Big(\Big(\frac{M}{2A}\Big)^2 T^*\Big)$$

and

$$A \leq \vartheta(\tau) \leq \frac{1}{\kappa}\Big(\mathcal{E} + \int_0^1 \frac{\bar{Q}^2(y,\tau)}{y^{4/3}}\,dy\Big) \leq \frac{1}{\kappa}\Big(\mathcal{E} + \frac{3}{5}K^2\Big)$$

for all $0 \leq \tau < T^*$. This allows us to use Theorem 3.2 at T^{*-}, which contradicts (5.10). The uniform upper bound in the temperature follows from the observation

$$\bar{Q}(y,\tau) = \int_0^y \bar{Q}_y(y,\tau)\,dy \leq y^{1/2}\|\bar{Q}_y(\tau)\|_{L^2(0,1)},$$

implying

$$\int_0^1 \frac{\bar{Q}^2(y,\tau)}{y^{4/3}}\,dy \leq \frac{3}{2}\|\bar{Q}_y(\tau)\|^2_{L^2(0,1)},$$

and thus last expression is uniformly bounded if A satisfies (H_1) (Theorem 4.2).

If (ii) holds, global existence follows in a identical way. Again (5.11) and (5.12) hold, yielding the same bounds on $\|Q_y(\tau)\|_{L^2(0,1)}$ and $\Theta(\tau)$. The pointwise bound on Q in D_∞ results from the fact that $y(1+B)/(y^{2/3}+B)$ is a supersolution for Problem **AP** if A and Q_0 satisfy (H_2). Take for instance $K = \frac{B+1}{B}$ in (5.12). The corresponding temperature bound is a direct consequence. □

As a special case of Theorem 5.4 (ii) we have

Corollary 5.5. *If $Q_0(y) = y$, and $\vartheta_0 > \frac{3}{2}$, then problem* (5.3)–(5.6) *has a global $\langle Q, \vartheta\rangle$ solution and $\vartheta_0 \leq \vartheta(\tau) < \vartheta_0 + \frac{12}{5\kappa}$ for all $\tau \geq 0$.*

Proof. Since $\lambda = 1$, we can select a sufficiently small $B > 0$ such that Theorem 5.4 (ii) holds. The pointwise bound on Q implies $Q(y,\tau) \leq y^{1/3}$ for all $(y,\tau) \in \overline{D_\infty}$. Since

$$\vartheta(\tau) = \vartheta_0 + \int_0^1 \frac{Q^2(y,\tau) - Q_0^2(y)}{y^{4/3}}\,dy,$$

the upper bound is immediate. □

6. Appendix: Inequalities

For completeness we give in this appendix some inequalities which are used at various places in the paper.

Let Ω be a bounded, open subset of $\mathbb{R}^N$ with a $C^{1+\alpha}(\alpha > 0)$ boundary.

First interpolation inequality. Let $N > 2$, $r \le \frac{2N}{N-2}$, and let $p \le q \le r$ satisfy $\frac{1}{q} = \frac{\alpha}{p} + \frac{(1-\alpha)}{r}$ for some $\alpha \in (0,1)$. Then

$$\|n\|_{L^q(\Omega)} \le C_s^{1-\alpha} \|n\|_{H^1(\Omega)}^{1-\alpha} \|n\|_{L^p(\Omega)}^{\alpha} \quad \text{for all} \quad n \in H^1(\Omega) \cap L^p(\Omega). \tag{6.1}$$

Proof. Use the Sobolev inequality $\|n\|_{L^r(\Omega)} \le C_s \|n\|_{H^1(\Omega)}$ for $N > 2$ and $r \le \frac{2N}{N-2}$, and the interpolation inequality $\|n\|_{L^q(\Omega)} \le \|n\|_{L^p(\Omega)}^{\alpha} \|n\|_{L^r(\Omega)}^{1-\alpha}$. □

Second interpolation inequality. Let $N = 3$. Then

$$\|n\|_{L^3(\Omega)} \le C_s^{1/2} \|n\|_{H^1(\Omega)}^{1/2} \|n\|_{L^2(\Omega)}^{1/2} \quad \text{for all} \quad n \in H^1(\Omega). \tag{6.2}$$

Proof. Take $p = 2$, $q = 3$, $r = \frac{2N}{N-2} = 6$, and $\alpha = 1/2$ in (6.1). □

Poisson's equation and L^p norms. Let $n \in L^p(\Omega)$, $p > \frac{N}{2}$, and let ϕ satisfy (1.2) and (1.5). Then

$$\begin{cases} \|\nabla\phi\|_{L^r(\Omega)} \le C_I \|n\|_{L^p(\Omega)} & \text{for} \quad 1 < r \le \frac{pN}{N-p} \quad \text{and} \quad \frac{N}{2} < p < N, \\ \|\nabla\phi\|_{L^\infty(\Omega)} \le C_I \|n\|_{L^p(\Omega)} & \text{for} \quad p > N, \end{cases} \tag{6.3}$$

where the constant C_I depends on Ω and p.

Proof. Since ϕ satisfies (1.2) with (1.5), we use the representation by the Green's function to obtain $\|\phi\|_{L^p(\Omega)} \le \|\Delta\phi\|_{L^p(\Omega)}$ for $N > 2$ and $p > N/2$. If $p < N$ we combine this with the Sobolev inequality $\|\nabla\phi\|_{L^r(\Omega)} \le C(\|\Delta\phi\|_{L^p(\Omega)} + \|\phi\|_{L^p(\Omega)})$ for $r \le pN/(N-p)$ to obtain the desired inequality. If $p > N$ we proceed similarly. □

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