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$C^{1,\beta}$ REGULARITY OF VISCOSITY SOLUTIONS VIA A CONTINUOUS-DEPENDENCE RESULT

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Abstract. In this article, we are interested in the existence, uniqueness, and regularity of solutions of fully nonlinear parabolic equations, with initial data u_0, in the whole space $\mathbb{R}^N$. Our main result is the existence of a strictly subquadratic solution with a local $C^{1,\beta}$ regularity with respect to the space variable, assuming $C^{1,\alpha}$ regularity on u_0 and local uniform ellipticity of the equation. Our proof relies on a result of N. Zhu which shows the local $C^{1,\beta}$ regularity of the solution provided it is Lipschitz continuous and Hölder continuous in t, with an exponent $\gamma > \frac{1}{2}$; we obtain this last property through a continuous-dependence result. Then we investigate further regularity for the solution using results of L. Wang.

1. Introduction

In this paper, we are interested in the regularity of viscosity solutions of fully nonlinear parabolic equations, namely,

$$u_t - F(D^2u, Du, u, x, t) = f(x,t) \qquad \text{in } \mathbb{R}^N \times (0,T], \tag{1.1}$$

$$u(x,0) = u_0(x) \qquad \text{in } \mathbb{R}^N, \tag{1.2}$$

where F is a continuous, real-valued function defined on $\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times [0,T]$, $\mathcal{S}^N$ being the space of $N \times N$ symmetric matrices equipped with usual ordering, and f is a continuous, real-valued function defined on $\mathbb{R}^N \times [0,T]$. We denote by Du and D^2u the gradient and the Hessian of u in x respectively.

Our main motivation concerns quasilinear-type equations, i.e., equations like

$$u_t - Tr(A(x,t,Du)D^2u) - H(x,t,u,Du) = f(x,t) \quad \text{in } \mathbb{R}^N \times (0,T]\ , \tag{1.3}$$

where A is a function taking values in the set of nonnegative symmetric matrices and $H \in C(\mathbb{R}^N \times [0,T] \times \mathbb{R} \times \mathbb{R}^N)$. Indeed, for such equations, the

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dependence of A in Du is a nontrivial difficulty in the obtaining of regularity results, and, to the best of our knowledge, this problem remains unsolved at this level of generality.

The contribution of this paper is to show the existence of smooth (or rather smooth) viscosity solutions of (1.1)–(1.2), namely, solutions which are locally $C^{1,\beta}$ or $C^{2,\beta}$ in x, the regularity properties in t being then either just a consequence of the regularity properties in x or proved simultaneously.

The cornerstone of our proof is a local regularity result of N. Zhu [15] who, essentially, shows that, if a solution in a bounded domain is Lipschitz continuous in x and locally γ-Hölder continuous in t, with $\gamma > \frac{1}{2}$, then it is locally $C^{1,\beta}$ in x, for some β depending on γ.

In order to use this result, our strategy is to show the existence of a viscosity solution of (1.1)–(1.2) which satisfies suitable properties, namely, which is locally Lipschitz continuous in x and γ-Hölder continuous in t, with $\gamma > \frac{1}{2}$. The advantage of this approach is to provide solutions of the initial-value problem with a certain regularity for a rather general class of fully nonlinear equations, in particular equations of the form (1.3); we are also able to deal with unbounded solutions, in fact, solutions with a strictly subquadratic growth. Unfortunately, this approach is global and requires $C^{1,\alpha}$ initial data u_0.

As we mentioned above, there are not so many regularity results concerning quasilinear equations like (1.3) in the literature, even in the case of stationary equations, the difficulty being the dependence in Du of A. In [13], N.S. Trudinger deals with equations of the type $F(D^2u, Du, u, x) = 0$, in a bounded domain of $\mathbb{R}^N$. He proves local $C^{1,\beta}$ regularity results for continuous viscosity solutions under quite restrictive conditions on F which exclude quasilinear cases. In [5], L.A. Caffelli and X. Cabre consider equations of the type $F(D^2u, x) = f(x)$, in a bounded domain of $\mathbb{R}^N$. They develop a theory based on the Pucci's extremal operators and on Harnack inequalities for a certain class of solutions. Again, it seems difficult to extend their theory to the case of quasilinear equations.

The results of L.A. Caffarelli and X. Cabre are generalized to the parabolic case by L. Wang in [16] and [17], who considers the case of general equations of the type $u_t - F(D^2u, Du, u, x, t) = f(x, t)$ in a bounded domain of $\mathbb{R}^{N+1}$. To do so, he introduces more general Pucci's operators and shows Harnack inequalities for a new class of solutions. However, his results do not seem to be applicable to quasilinear equations.

A general local regularity result for a continuous viscosity solution of a fully nonlinear elliptic equation of the type $F(D^2u, Du, u, x) = 0$ is given by C. Yatzhe in [14]. His methods are very different from the ones used by N.S. Trudinger or L.A. Caffarelli and X. Cabre. Everything is based on the use of the matricial lemma, proved by M.G. Crandall, H. Ishii, and P.L. Lions in [6] (Theorem 8.3). Then, in [15], N. Zhu extends the results of C. Yatzhe to the case of fully nonlinear parabolic equations. He considers parabolic equations in a bounded domain of $\mathbb{R}^{N+1}$. As we already mentioned above, he shows that if u is Lipschitz continuous in x and γ-Hölder continuous in t, in this bounded domain, with $\gamma > \frac{1}{2}$, then it is locally $C^{1,\beta}$ in x. This result is very interesting, because the conditions made on F are weak, but it is difficult to use, in view of the regularity in t imposed on u.

Indeed, it is not very difficult to show that a viscosity solution which is locally Lipschitz continuous in x, is also locally $\frac{1}{2}$-Hölder continuous in t (see G. Barles, S. Biton, and O. Ley [3]). But to obtain a better Hölder exponent is far more difficult, and this is the main problem we face and solve here.

In order to prove the existence and uniqueness of continuous solutions of (1.1)–(1.2), together with their local Lipschitz continuity in x and local Hölder continuity in t, the key steps are on one hand a comparison result for viscosity solutions and on the other hand a continuous-dependence result, which are both valid for strictly subquadratic viscosity solutions of (1.1)–(1.2). In particular, by assuming that $u_0 \in C^{1,\alpha}(\mathbb{R}^N)$ with $0 < \alpha < 1$, we show that the solution u satisfies, for every $x \in \mathbb{R}^N$ and $h \in [0,T]$, the following inequality,

$$|u(x,h) - u_0(x)| \leq C(1+|x|^2)h^{\frac{1+\alpha}{2}} ,$$

and the continuous-dependence result allows us to obtain an analogous estimate for $|u(x,t+h) - u(x,t)|$, thus proving the local Hölder continuity in t with a suitable exponent. To get the continuous-dependence result, we have been inspired by the methods used by E.R. Jakobsen and K.H. Karlsen in [12]. Nevertheless, a key difference between our respective results is that their solutions are supposed to be bounded, whereas ours are only supposed to have a strictly subquadratic growth.

The proof of the continuous-dependence result being unavoidably a global one is the reason why we consider an equation defined in $\mathbb{R}^N \times [0,T]$. For the local Lipschitz continuity of a solution, we refer also to G. Barles [1] and C. Yatzhe [14], their method being largely inspired of H. Ishii and P.L. Lions [9]. Their method is a local one, and it requires strictly uniform ellipticity of the function F, whereas in our case it can be degenerate. For another approach in a specific case, see G. Barles and P.E. Souganidis [2].

Now we note that, if F does not depend on Du, then we can consider solutions having polynomial growth. By assuming that u_0 satisfies, for every

$x, y \in \mathbb{R}^N$ and $t \in [0, T]$, the following inequality,

$$|Du_0(x) - Du_0(y)| \leq C(1 + |x|^p + |y|^p)|x - y|^\alpha,$$

where $p \in \mathbb{N}$ and $0 < \alpha \leq 1$, it is easy to show that the solution u of (1.1)–(1.2), which is unique among the functions with polynomial growth of order $p + 2$, is locally Lipschitz in x and locally Hölder in t. But if F does not depend on Du, L. Wang in [17] proves $C^{1,\beta}$ regularity for a solution, and in this case his results are local ones. The fact that F depends on Du obliges us to make restrictive growing conditions on the solution. Perhaps we should have considered solutions with a little bit better growth than a strictly subquadratic one, a comparison result existing for such solutions (see G. Barles, S. Biton, M. Bourgoing, and O. Ley [4]), but we have not checked if the continuous-dependence result is then valid. The fact of considering functions with a strictly subquadratic growth simplifies considerably the proof, which is already complicated enough.

Once we know that the solution u is locally $C^{1,\beta}$ in x, further regularity can be obtained by assuming also that F is convex (or concave) in D^2u, through the results of L. Wang [17].

In order to give an idea of the type of assumptions which are required, we now give them in detail in the case of (1.3). We refer to the Appendix for a proof of their equivalence with the assumptions given in Sections 2, 3, 4, and 5.

We denote by a, the nonnegative symmetric matrix such that $A = aa^T$. To obtain the local $C^{1,\beta}$ regularity of the solutions, we have to assume the following conditions on a, A, and H: there exists $C > 0$ and $\frac{1}{2} < \gamma < 1$ such that, for every $x, y, p, q \in \mathbb{R}^N$, $r \in \mathbb{R}$, $s, t \in [0, T]$,

$$||a(x, t, p) - a(y, s, q)|| \leq C\big(|x - y| + |p - q| + (1 + |x|)|s - t|^\gamma\big),$$
$$||a(x, t, p|| \leq C(1 + |x|), \qquad |H(0, r, 0, 0)| \leq C|r|,$$
$$|H(p, r, x, t) - H(q, r, y, t)| \leq C\big((1 + |x|)|p - q| + (1 + |p|)|x - y|\big),$$
$$|H(p, r, x, t) - H(p, r, x, s)| \leq C\big(1 + |x| + |p|\big)|s - t|^\gamma.$$

We also suppose that, for every $p, x \in \mathbb{R}^N$, $t \in [0, T]$, $r, z \in R$, with $r \leq z$, we have $H(p, r, x, t) \geq H(p, z, x, t)$, and that A satisfies, for every $R > 0$, there exists $\lambda_R > 0$ such that, for every $x \in B(0, R)$, $p \in \mathbb{R}^N$, and $t \in [0, T]$, $\lambda_R Id \leq A(x, t, p)$. Furthermore, to prove the local C^2 regularity of the solutions in x and their C^1 regularity in t, with locally Hölder-continuous derivatives, we have to add the following assumption on H: for every $R > 0$, there exists $C_R > 0$ such that, for every $x, p \in B(0, R)$, $r, z \in \mathbb{R}$, and

$t \in [0, T]$, we have

$$|H(p, r, x, t) - H(p, z, x, t)| \leq C_R |r - z|.$$

Under these assumptions, we prove, in the case of quasilinear equations, the existence of a smooth viscosity solution, and thus of a classical one. It is therefore interesting to compare this result with the ones obtained by the classical theory for parabolic equations, described in [10] then in [11]. Of course, this is a completely different approach. In our case, we first prove the existence (and uniqueness) of a continuous viscosity solution, and then we show its smoothness. In the classical approach, using a priori estimates, the existence of a solution in a space of smooth functions is obtained directly by using fixed-point theorems. Almost all the existence results concern the solvability of first boundary-value problems in bounded domains. Only a few results deal with Cauchy problems in the $\mathbb{R}^N$ case: see, for example, Theorem 8.1 in [10], but it requires the function u_0 to be bounded over $\mathbb{R}^N$, locally $C^{2,\alpha}$ in x, and the coefficients appearing in the equation to be very regular, in particular differentiable in each variable. Therefore our results, even in the quasilinear case (which is clearly more studied that the fully nonlinear one) seems rather different from the existing ones.

This article is organized as follows. In Section 2, we state and prove the key results of this article, namely the comparison and continuous-dependence results. In the third one, we deal with existence and uniqueness of a solution u of (1.1)–(1.2). In Section 4, we prove the local Lipschitz continuity in x and then the local Hölder continuity in t of the solution. Then, in the fifth section, we show how to use the result of N. Zhu [15] to obtain the local $C^{1,\beta}$ regularity and then of L. Wang [17] to get the further regularity of the solution. Finally, in the Appendix, we prove some auxiliary lemmas and propositions, useful to the demonstration of the main theorems and propositions.

2. A comparison–continuous-dependence result for strictly subquadratic solutions

As we mentioned in the Introduction, our article is based on a comparison result and a continuous dependence one, for strictly subquadratic growing solutions. A real-valued, continuous function f defined in $\mathbb{R}^N$ is said to have a strictly subquadratic (respectively strictly sublinear) growth, if it satisfies

$$\frac{|f(x)|}{1 + |x|^2} \longrightarrow 0 \quad \text{as} \quad |x| \to +\infty \quad \left(\text{respectively } \frac{|f(x)|}{1 + |x|} \longrightarrow 0 \quad \text{as} \quad |x| \to +\infty\right).$$

$\mathcal{E}_{ssq}$ is the subset of the real-valued, continuous functions defined in $\mathbb{R}^N$, with strictly subquadratic growth. For every $\delta > 0$ and $f \in \mathcal{E}_{ssq}$, we denote by $\bar{C}_\delta(f)$ a positive constant such that, for every $x \in \mathbb{R}^N$, we have

$$|f(x)| \leq \bar{C}_\delta + \delta|x|^2. \tag{2.1}$$

For I a subset of $\mathbb{R}^+$, we define $\mathcal{E}_{ssq}(I)$ as follows. A real-valued function g defined in $\mathbb{R}^N \times I$ lies in $\mathcal{E}_{ssq}(I)$ if for every $t \in I$, the function $g_t : x \mapsto g(x,t)$ lies in $\mathcal{E}_{ssq}$, and if for every $\delta > 0$, $\underset{t\in I}{\mathrm{Sup}}\bar{C}_\delta(g_t) < +\infty$. Then we set $\bar{C}_\delta(g) = \underset{t\in I}{\mathrm{Sup}}\bar{C}_\delta(g_t)$. $\mathcal{E}^L_{ssq}$ is the subset of the real-valued, locally Lipschitz functions defined in $\mathbb{R}^N$, whose gradient has a strictly sublinear growth. For every $\delta > 0$ and $f \in \mathcal{E}^L_{ssq}$, we denote by $C_\delta(f)$ a positive constant such that, for every $x, y \in \mathbb{R}^N$, we have

$$|f(x) - f(y)| \leq (C_\delta(f) + \delta|x| + \delta|y|)\ |x - y|. \tag{2.2}$$

We define , the same way as previously, the subset $\mathcal{E}^L_{ssq}(I)$ and set, for every $g \in \mathcal{E}^L_{ssq}(I)$ and $\delta > 0$, $C_\delta(g) = \underset{t\in I}{\mathrm{Sup}} C_\delta(g_t)$.

Remark 2.1. It is clear that $\mathcal{E}^L_{ssq} \subset \mathcal{E}_{ssq}$.

We consider the following structure conditions on the function F.

(H1) $r \mapsto F(X,p,r,x,t)$ is nonincreasing in $\mathbb{R}$, for every $X \in \mathcal{S}^N$, $p, x \in \mathbb{R}^N$, and $t \in [0,T]$.

(H2) There exists $C_1 > 0$ such that, if $X, Y \in \mathcal{S}^N$ and $\lambda_1, \lambda_2 \geq 0$ satisfy

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \lambda_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \lambda_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}, \tag{2.3}$$

then for every $p, q, x, y \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $s, t \in [0,T]$,

$$\begin{aligned} &F(X,p,r,x,t) - F(Y,q,r,y,t) \leq \lambda_1 C_1\big(|x-y|^2 + |p-q|^2\big) \\ &+ \lambda_2 C_1\left(1 + |x|^2 + |y|^2\right) + C_1(1+|p|)|x-y| + C_1(1+|y|)|p-q|. \end{aligned} \tag{2.4}$$

(H3) There exists $C_2 > 0$ and $0 \leq \alpha_1, \alpha_2 \leq 1$ such that, if $X, Y \in \mathcal{M}^N$ and $\lambda_1, \lambda_2 > 0$ satisfy (2.3), then, for every $p, x \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $t, s \in [0,T]$, we have

$$\begin{aligned} F(X,p,r,x,t) - F(Y,p,r,x,s) \leq &\lambda_1 C_2(1+|x|^2)|s-t|^{2\alpha_1} + \lambda_2 C_2\left(1+|x|^2\right) \\ &+ C_2(1+|x|+|p|)|s-t|^{\alpha_2}. \end{aligned} \tag{2.5}$$

Our main result is the following theorem, which is a key point to get the existence and the regularity of a solution u of (1.1)–(1.2), in $\mathcal{E}_{ssq}([0,T])$.

Theorem 2.1. *Let u be an upper-semicontinuous viscosity subsolution and v be a lower-semicontinuous viscosity supersolution of the equation* (1.1). *Assume that u and v satisfy the initial condition*

$$u(\cdot,0) \le w(\cdot) \le v(\cdot,0) \quad in \quad \mathbb{R}^N,$$

where $w \in \mathcal{E}^L_{ssq}$. Suppose that u and $v \in \mathcal{E}_{ssq}([0,T])$ and that $f \in \mathcal{E}^L_{ssq}([0,T])$. (For every $\delta > 0$, we set $\tilde{C}_\delta = \max(1, C_\delta(w), C_\delta(f))$.)
1) *A comparison result: if F satisfies conditions $(H1)$ and $(H2)$, then there exists $\delta_0 > 0$ and $C_\ell > 0$ depending only on C_1 and T, such that, for every $\delta \le \delta_0$, $t \in [0,T]$, and $x, y \in \mathbb{R}^N$ with $|x-y| \le 1$, we have*

$$u(x,t) - v(y,t) \le C_\ell(\tilde{C}_\delta + \delta|x| + \delta|y|)|x-y|. \tag{2.6}$$

2) *A continuous-dependence result: assume that F satisfies conditions $(H1)$, $(H2)$, and $(H3)$ and that there exists $\tilde{C} > 0$, $0 < h_0 \le T$, and $\tilde{\alpha} \le 1$, such that for every $x \in \mathbb{R}^N$, $0 \le h \le h_0$, and $s, t \in [0,T]$,*

$$|f(x,t) - f(x,s)| \le \tilde{C}(1+|x|^2)|s-t|^{\tilde{\alpha}} \tag{2.7}$$
$$u(x,h) - u(x,0) \le \tilde{C}(1+|x|^2)h^{\tilde{\alpha}} \tag{2.8}$$
$$v(x,0) - v(x,-h) \le \tilde{C}(1+|x|^2)h^{\tilde{\alpha}}. \tag{2.9}$$

Then, setting $\bar{\alpha} = \min(\alpha_1, \alpha_2, \tilde{\alpha})$, there exists $\delta_1 > 0$, depending only on C_1, C_2, and T, such that, for every $0 < \beta < \bar{\alpha}$, for every $x \in \mathbb{R}^N$ and $s, t \in [0,T]$, with $|s-t| \le h_1$, we have

$$u(x,t) - v(x,s) \le C_\beta |s-t|^\beta \ (\tilde{C}^2_{\delta_1} + |x|^2), \tag{2.10}$$

where $C_\beta > 0$ depends on C_1, C_2, β, and T, and $0 < h_1 \le h_0$ depends on C_1, C_2, $\tilde{C}$, β, $\bar{\alpha}$, T, and h_0.

Proof of Theorem 2.1. 1. Preliminaries. For $0 \le h \le T$, $0 < \varepsilon, \delta \le 1$, and $K > 0$, we consider the function defined in $\mathbb{R}^N \times \mathbb{R}^N \times [0, T-h]$ by $\Psi(x,y,t) = u(x,t+h) - v(y,t) - \Phi(x,y,t)$, where

$$\Phi(x,y,t) = \big(\frac{1}{2\varepsilon}|x-y|^2 + 2\varepsilon(\tilde{C}^2_\delta + \delta^2|x|^2 + \delta^2|y|^2)\big)\exp(Kt).$$

The function Ψ is an upper-semicontinuous, locally bounded function which satisfies

$$\operatorname*{Sup}_{t\in[0,T-h]} \Psi(x,y,t) \to -\infty \quad \text{as} \quad |x| + |y| \to +\infty.$$

Indeed, since u and v are in $\mathcal{E}_{ssq}([0,T])$, then for every $x, y \in \mathbb{R}^N$ and $t \in [0, T-h]$

$$\Psi(x,y,t) \le u(x,t+h) - v(y,t) - 2\varepsilon(\tilde{C}^2_\delta + \delta^2|x|^2 + \delta^2|y|^2)$$

$$\leq (\bar{C}_{\varepsilon\delta^2}(u) + \bar{C}_{\varepsilon\delta^2}(v) - 2\varepsilon\tilde{C}_\delta^2) - \varepsilon\delta^2(|x|^2 + |y|^2).$$

This implies that the function Ψ achieves its maximum in $\mathbb{R}^N \times \mathbb{R}^N \times [0, T-h]$ and that the function $\Psi(\cdot,\cdot,0)$ achieves its own in $\mathbb{R}^N \times \mathbb{R}^N$. We set

$$E_0 = \operatorname*{Max}_{\mathbb{R}^N\times\mathbb{R}^N} \Psi(x,y,0), \tag{2.11}$$

$$\mathcal{S} = \operatorname*{Max}_{\mathbb{R}^N\times\mathbb{R}^N\times[0,T-h]} \Psi(x,y,t). \tag{2.12}$$

In fact, the constants E_0 and $\mathcal{S}$ depend on h, δ, ε, and K. To be precise, we can set, for any $0 \leq h \leq T$, $0 < \delta, \varepsilon \leq 1$, and $K > 0$, $\mathcal{S} = \mathcal{S}(h, \delta, \varepsilon, K)$ and $E_0 = E_0(h, \delta, \varepsilon, K)$. When there is no ambiguity, we will denote them by E_0 and $\mathcal{S}$. We have the following result.

2. A technical lemma.

Lemma 2.1. *There exist some positive constants K_1 and δ_0 such that*

$$\mathcal{S}(0, \delta, \varepsilon, K_1) \leq 0, \quad \text{for all} \quad 0 < \delta \leq \delta_0,\ 0 < \varepsilon \leq 1. \tag{2.13}$$

There exists some positive constants K_2, δ_1, and $0 < h_1 \leq h_0$ such that, for any $0 < \beta < \bar{\alpha}$,

$$\mathcal{S}(h, \delta_1, \frac{h^\beta}{T^\beta}, K_2) \leq 0, \quad \text{for all} \quad 0 < h \leq h_1. \tag{2.14}$$

The constant K_1 depends on C_1; K_2 depends on C_1 and C_2. $\delta_0 \leq 1$ depends on C_1 and T; $\delta_1 \leq 1$ depends on C_1, C_2, and T; and h_1 depends on C_1, C_2, $\tilde{C}$, β, $\bar{\alpha}$, and h_0.

We postpone the proof of this lemma to the end of the section. From now on, we suppose that inequalities (2.13) and (2.14) are true and end the proof of the theorem.

3. Proof of the comparison result. Inequality (2.13) implies that, for every $x, y \in \mathbb{R}^N$, $t \in [0, T-h]$, $0 < \delta \leq \delta_0$, and $0 < \varepsilon \leq 1$, the following inequality holds:

$$u(x,t) - v(y,t) \leq \big(\frac{1}{2\varepsilon}|x-y|^2 + 2\varepsilon(\tilde{C}_\delta^2 + \delta^2|x|^2 + \delta^2|y|^2)\big)\exp(K_1 T). \tag{2.15}$$

Now, for fixed $x, y \in \mathbb{R}^N$, with $0 < |x-y| \leq 1$, for $0 < \delta \leq \delta_0$, we can choose $0 < \varepsilon = \frac{|x-y|}{\tilde{C}_\delta + \delta|x| + \delta|y|} \leq 1$ (using that $\tilde{C}_\delta \geq 1$), and we obtain (2.6) with $C_\ell = 3\exp(K_1 T)$.

4. Proof of the continuous-dependence result. We consider $0 < \beta < \bar{\alpha}$. Inequality (2.14) implies that, for every $x \in \mathbb{R}^N$, $0 < h \leq h_1$, and $t \in$

$[0, T-h]$, the following inequality holds:

$$u(x,t+h) - v(x,t) \leq 2\frac{h^\beta}{T^\beta}(\tilde{C}^2_{\delta_1} + 2\delta_1^2|x|^2)\exp(K_2T)$$
$$\leq h^\beta \frac{4\exp(K_2T)}{T^\beta}(\tilde{C}^2_{\delta_1} + |x|^2) \quad \text{as} \quad \delta_1 \leq 1. \tag{2.16}$$

Changing $u(x,t+h)-v(y,t)$ into $u(x,t)-v(y,t+h)$ in the definition of Ψ and using inequality (2.9) instead of (2.8), we prove identically that inequality (2.16) holds, for $u(x,t)-v(x,t+h)$ instead of $u(x,t+h)-v(x,t)$. Therefore, we get (2.10), with $C_\beta = \frac{4\exp(K_2T)}{T^\beta}$.

5. It only remains for us to prove Lemma 2.1, and to do it, we need another technical one.

Lemma 2.2. *For every $0 < \delta, \varepsilon \leq 1$ and $K > 0$, we have*

$$E_0(0, \delta, \varepsilon, K) \leq 0. \tag{2.17}$$

For every $0 < h \leq h_0$, $K > 0$, and $0 < \delta, \varepsilon \leq 1$, with $\varepsilon\delta^2 \geq \tilde{C}h^{\tilde{\alpha}}$, we have

$$E_0(h, \delta, \varepsilon, K) \leq 0. \tag{2.18}$$

We first admit this lemma and conclude the proof of Lemma 2.1.

1. We argue by contradiction assuming that $\mathcal{S} > 0$. Lemma 2.2 shows that if (h, ε, δ) satisfies one of the following conditions,

$$h = 0 \qquad \text{and} \quad 0 < \varepsilon, \delta \leq 1, \tag{2.19}$$
$$0 < h \leq h_0 \qquad \text{and} \quad 0 < \varepsilon, \delta \leq 1 \quad \text{with} \quad \varepsilon\delta^2 \geq \tilde{C}h^{\tilde{\alpha}}, \tag{2.20}$$

then $E_0 \leq 0$.

2. We suppose that (h, δ, ε) satisfy (2.20). As by hypothesis $\mathcal{S} > 0$, it implies that Ψ achieves its maximum at a point $(\bar{x}, \bar{y}, \bar{t})$ with $\bar{t} > 0$. Let us set $p_x = D_x\Phi(\bar{x}, \bar{y}, \bar{t})$ and $q_y = D_y\Phi(\bar{x}, \bar{y}, \bar{t})$.

From the matricial lemma (see Theorem 8.3 in [6]), using that u and v are respectively a subsolution and a supersolution of (1.1), we have the following result. For all $\nu > 0$, there exists $(a, b) \in \mathbb{R}^2$ and $X, Y \in \mathcal{S}^N$, such that

$$(a, p_x, X) \in \bar{\mathcal{P}}^{2,+}(u)(\bar{x}, \bar{t} + h),$$
$$(b, q_y, -Y) \in \bar{\mathcal{P}}^{2,+}(-v)(\bar{y}, \bar{t}) = -\bar{\mathcal{P}}^{2,-}(v)(\bar{y}, \bar{t}),$$
$$a + b \geq \frac{\partial\Phi}{\partial t}(\bar{x}, \bar{y}, \bar{t}) = K\Phi(\bar{x}, \bar{y}, \bar{t}),$$

and

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \mathcal{C} + \nu\mathcal{C}^2, \quad \text{where} \quad \mathcal{C} = D^2_{xy}\Phi(\bar{x}, \bar{y}, \bar{t}). \tag{2.21}$$

As u and v are respectively a subsolution and a supersolution of (1.1), we have

$$K\Phi(\bar{x},\bar{y},\bar{t}) = a+b \le F(X,p_x,u(\bar{x},\bar{t}+h),\bar{x},\bar{t}) - F(Y,-q_y,v(\bar{y},\bar{t}),\bar{y},\bar{t}) + f(\bar{x},\bar{t}+h) - f(\bar{y},\bar{t}). \tag{2.22}$$

Since we have supposed that $\mathcal{S} > 0$, then $u(\bar{x},\bar{t}+h) - v(\bar{y},\bar{t}) > 0$, and using (H2), we obtain

$$F(X,p_x,u(\bar{x},\bar{t}+h),\bar{x},\bar{t}+h) \le F(X,p_x,v(\bar{y},\bar{t}),\bar{x},\bar{t}+h). \tag{2.23}$$

Next, easy computations yield

$$\mathcal{C} = \frac{1}{\varepsilon}\exp(K\bar{t})\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + 4\varepsilon\delta^2\exp(K\bar{t})\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix},$$

and using (2.21) together with the conditions (H2) and (H3) on F, we get

$$F(X,p_x,v(\bar{y},\bar{t}),\bar{x},\bar{t}+h) - F(Y,-q_y,v(\bar{y},\bar{t}),\bar{y},\bar{t}) \le \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + C_\nu, \tag{2.24}$$

where C_ν is a positive constant, which converges to zero when ν goes to zero, and

$$\begin{aligned}
\mathcal{A}_1 &= \frac{C}{\varepsilon}\exp(K\bar{t})\big(|\bar{x}-\bar{y}|^2 + |p_x+q_y|^2 + (1+|\bar{y}|^2)h^{2\alpha_1}\big),\\
\mathcal{A}_2 &= C(1+|p_x|)|\bar{x}-\bar{y}| + C(1+|\bar{y}|)|p_x+q_y|,\\
\mathcal{A}_3 &= 4\varepsilon\delta^2 C\exp(K\bar{t})(1+|\bar{x}|^2+|\bar{y}|^2) + C(1+|\bar{y}|+|q_y|)h^{\alpha_2},
\end{aligned}$$

where $C = \max(C_1, C_2)$. Combining (2.22), (2.23), and (2.24), and letting ν go to zero, we get

$$K\Phi(\bar{x},\bar{y},\bar{t}) \quad\le\quad \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4, \tag{2.25}$$

where $\mathcal{A}_4 = f(\bar{x},\bar{t}+h) - f(\bar{y},\bar{t})$. In order to estimate the previous quantities, we need the following inequalities:

$$|p_x+q_y| \le 4\varepsilon\delta^2\exp(K\bar{t})(|\bar{x}|+|\bar{y}|), \tag{2.26}$$

$$|p_x| \le \frac{1}{\varepsilon}\exp(K\bar{t})|\bar{x}-\bar{y}| + 4\varepsilon\delta^2\exp(K\bar{t})|\bar{x}|, \tag{2.27}$$

$$|q_y| \le \frac{1}{\varepsilon}\exp(K\bar{t})|\bar{x}-\bar{y}| + 4\varepsilon\delta^2\exp(K\bar{t})|\bar{y}|, \tag{2.28}$$

and the following Young's inequalities for positive constants λ_1, λ_2, and λ_3 to be chosen later:

$$|\bar{y}| \le \frac{\lambda_1|\bar{y}|^2}{2} + \frac{1}{2\lambda_1}, \tag{2.29}$$

$$|\bar{x}-\bar{y}| \leq \frac{\lambda_2}{2} + \frac{|\bar{x}-\bar{y}|^2}{2\lambda_2}, \tag{2.30}$$

$$|\bar{x}||\bar{x}-\bar{y}| \leq \frac{\lambda_3|\bar{x}|^2}{2} + \frac{|\bar{x}-\bar{y}|^2}{2\lambda_3}. \tag{2.31}$$

3. Estimate of $\mathcal{A}_1$.

$$\begin{aligned}\mathcal{A}_1 &\leq \frac{C}{\varepsilon}\exp(K\bar{t})\big(|\bar{x}-\bar{y}|^2 + 32\varepsilon^2\delta^4\exp(2K\bar{t})(|\bar{x}|^2+|\bar{y}|^2) + (1+|\bar{y}|^2)h^{2\alpha_1}\big)\\ &\leq C\big(2+\frac{h^{2\alpha_1}}{2\varepsilon^2\delta^2}\big)\Phi(\bar{x},\bar{y},\bar{t}) + 16C\delta^2\exp(2K\bar{t})\Phi(\bar{x},\bar{y},\bar{t})\\ &\leq C\Phi(\bar{x},\bar{y},\bar{t})\big(2+\frac{h^{2\alpha_1}}{2\varepsilon^2\delta^2} + 16\delta^2\exp(2KT)\big),\end{aligned}$$

using that $\tilde{C}_\delta \geq \delta$.

4. Estimate of $\mathcal{A}_2$.

$$\begin{aligned}&\mathcal{A}_2 \leq C\big(1+\frac{1}{\varepsilon}\exp(K\bar{t})|\bar{x}-\bar{y}| + 4\varepsilon\delta^2\exp(K\bar{t})|\bar{x}|\big)|\bar{x}-\bar{y}|\\ &\quad + 4C\varepsilon\delta^2\exp(K\bar{t})(1+|\bar{y}|)(|\bar{x}|+|\bar{y}|)\\ &\leq \frac{C\varepsilon}{2} + 8C\varepsilon^3\delta^4|\bar{x}|^2\exp(K\bar{t}) + 4C\Phi(\bar{x},\bar{y},\bar{t}) + 8C\varepsilon\delta^2\exp(K\bar{t})(1+|\bar{x}|^2+|\bar{y}|^2)\\ &\leq 13C\ \Phi(\bar{x},\bar{y},\bar{t}),\end{aligned}$$

using (2.30) with $\lambda_2=\varepsilon$, (2.31) with $\lambda_3 = 4\varepsilon^2\delta^2$, and also that $0<\varepsilon,\delta\leq 1$ and $\tilde{C}_\delta \geq 1$.

5. Estimate of $\mathcal{A}_3$.

$$\begin{aligned}\mathcal{A}_3 &\leq 2C\Phi(\bar{x},\bar{y},\bar{t}) + Ch^{\alpha_2}\big(1+|\bar{y}| + \frac{1}{\varepsilon}\exp(K\bar{t})|\bar{x}-\bar{y}| + 4\varepsilon\delta^2|\bar{y}|\exp(K\bar{t})\big)\\ &\leq 2C\Phi(\bar{x},\bar{y},\bar{t}) + Ch^{\alpha_2}\big(1+\frac{1}{4\delta^2} + \frac{1}{2}\exp(K\bar{t}) + 4\varepsilon^2\delta^2\exp(K\bar{t}) + \frac{\Phi(\bar{x},\bar{y},\bar{t})}{\varepsilon}\big)\\ &\leq C(2+\frac{h^{\alpha_2}}{\varepsilon})\Phi(\bar{x},\bar{y},\bar{t}) + 6C\frac{h^{\alpha_2}}{\delta^2}\exp(K\bar{t}) \leq C(2+4\frac{h^{\alpha_2}}{\varepsilon\delta^2})\Phi(\bar{x},\bar{y},\bar{t}),\end{aligned}$$

using (2.29) with $\lambda_1 = 2\delta^2$, then $\lambda_1 = \frac{1}{2\varepsilon}$, (2.30) with $\lambda_2=\varepsilon$, and using that $0<\varepsilon,\delta\leq 1$ and $\tilde{C}_\delta \geq 1$.

6. Estimate of $\mathcal{A}_4$. As f $\in \mathcal{E}^L_{ssl}([0,T])$ and satisfies (2.7), we have

$$\begin{aligned}\mathcal{A}_4 &= f(\bar{x},\bar{t}+h) - f(\bar{y},\bar{t}) \leq \tilde{C}(1+|\bar{x}|^2)h^{\tilde{\alpha}} + (C_\delta(f)+\delta|\bar{x}|+\delta|\bar{y}|)\ |\bar{x}-\bar{y}|\\ &\leq \tilde{C}(1+|\bar{x}|^2)h^{\tilde{\alpha}} + \frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} + \frac{\varepsilon}{2}(C_\delta(f)+\delta|\bar{x}|+\delta|\bar{y}|)^2\end{aligned}$$

$$\le \tilde{C}(1+|\bar{x}|^2)h^{\tilde{\alpha}} + \frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} + \varepsilon(C_\delta(f)^2 + \delta^2|\bar{x}|^2 + \delta^2|\bar{y}|^2)$$
$$\le (\tilde{C}\frac{h^{\tilde{\alpha}}}{2\varepsilon\delta^2} + 1)\ \Phi(\bar{x},\bar{y},\bar{t}),$$

using (2.30) with $\lambda_2 = \varepsilon(C_\delta(f) + \delta|\bar{x}| + \delta|\bar{y}|)$, then that $0 < \varepsilon, \delta \le 1$ and $\tilde{C}_\delta \ge \max(\delta, C_\delta(f))$.

7. Inequality (2.25) and the above estimates on $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, and $\mathcal{A}_4$ imply that

$$K\Phi(\bar{x},\bar{y},\bar{t}) \le \bar{C}\Big(1 + \frac{\tilde{C}h^{\tilde{\alpha}} + h^{\alpha_2}}{\varepsilon\delta^2} + \frac{h^{2\alpha_1}}{\varepsilon^2\delta^2} + \delta^2\exp(2KT)\Big)\Phi(\bar{x},\bar{y},\bar{t}),$$

where $\bar{C}$ is a positive constant depending only on C_1 and C_2. Thus, choosing $K = 2\bar{C}$ and using that $\Phi(\bar{x},\bar{y},\bar{t}) > 0$, we have proved that, for every (h,δ,ε) satisfying (2.20), the following inequality holds:

$$0 \le \bar{C}(a_1 + a_2 + a_3 + a_4 - 1), \tag{2.32}$$

where

$$a_1 = \frac{\tilde{C}h^{\tilde{\alpha}}}{\varepsilon\delta^2},\ a_2 = \frac{h^{\alpha_2}}{\varepsilon\delta^2},\ a_3 = \frac{h^{2\alpha_1}}{\varepsilon^2\delta^2},\ a_4 = \delta^2\exp(4\bar{C}T). \tag{2.33}$$

In the particular case where $h = 0$, without assuming condition (H3) on F, we can show identically that there exists $\bar{C}_0 > 0$ depending only on C_1 such that, for every $K > 0$ and $0 < \varepsilon, \delta \le 1$, we have

$$K \le \bar{C}_0(1 + \delta^2\exp(2KT)),$$

and thus, choosing $K = 2\bar{C}_0$, we get

$$0 \le \bar{C}_0(\delta^2\exp(4\bar{C}_0T) - 1). \tag{2.34}$$

8. End of the proof of Lemma 2.1. (i) We first consider the case where $h = 0$. Setting $\delta_0 = \frac{\exp(-2\bar{C}_0T)}{2}$, we get, for every $0 < \delta \le \delta_0$ and $0 < \varepsilon \le 1$, a contradiction in (2.34). That shows the first part of Lemma 2.1.

(ii) We now want to prove the second part of this lemma. Letting $0 < h \le h_0$, we note that if $a_1 \le 1$, then (h,δ,ε) satisfies (2.20), and therefore inequality (2.32) holds. We choose $\delta_1 = \frac{\exp(-2\bar{C}T)}{2}$ (and then $a_4 = \frac{1}{4}$), $0 < \beta < \bar{\alpha}$, and $0 < \varepsilon = \frac{h^\beta}{T^\beta} \le 1$. It is easy to show that if $h > 0$ is small enough, then $a_1 + a_2 + a_3 < \frac{3}{4}$. Indeed,

$$a_1 = \frac{\tilde{C}T^\beta}{\delta^2}h^{\tilde{\alpha}-\beta}, \qquad a_2 = \frac{T^\beta}{\delta^2}h^{\alpha_2-\beta}, \qquad a_3 = \frac{T^{2\beta}}{\delta^2}h^{2(\alpha_1-\beta)}, \tag{2.35}$$

and, as the numbers $\tilde{\alpha} - \beta$, $\alpha_2 - \beta$, and $\alpha_1 - \beta$ are positive (recall that $0 < \beta < \bar{\alpha}$), there exists $0 < h_1 \le h_0$ depending on $\tilde{C}$, β, $\bar{\alpha}$, h_0, and T, such that, if $0 < h \leq h_1$, then $a_1 + a_2 + a_3 < \frac{3}{4}$. That implies in particular that $a_1 \leq 1$, and we obtain a contradiction in (2.32), which ends the proof of Lemma 2.1.

We now give the proof of Lemma 2.2.

Proof of Lemma 2.2. Using the inequality (2.9) and the fact that $w \in \mathcal{E}_{ssq}^L$, we have, for every $0 \leq h \leq h_0$, $x, y \in \mathbb{R}^N$, $0 < \varepsilon, \delta \leq 1$, and $K > 0$,

$$u(x,h) - v(y,0) - \Phi(x,y,0) \leq u(x,h) - u(x,0) + w(x) - w(y) - \Phi(x,y,0)$$
$$\leq \tilde{C}(1+|x|^2)h^{\tilde{\alpha}} + (C_\delta(w) + \delta|x| + \delta|y|)|x-y| - \Phi(x,y,0). \tag{2.36}$$

But, thanks to Young's inequality and then to a convexity inequality, we have, for every $x, y \in \mathbb{R}^N$,

$$(C_\delta(w) + \delta|x| + \delta|y|)|x-y| \leq \frac{|x-y|^2}{2\varepsilon} + \varepsilon(C_\delta(w)^2 + \delta^2|x|^2 + \delta^2|y|^2). \tag{2.37}$$

Thus, for every $x, y \in \mathbb{R}^N$, we get

$$\begin{aligned} &u(x,h) - v(y,0) - \Phi(x,y,0) \\ &\leq \tilde{C}(1+|x|^2)h^{\tilde{\alpha}} - \varepsilon(\tilde{C}_\delta^2 + \delta^2|x|^2) \quad \text{as } \tilde{C}_\delta \geq C_\delta(w) \\ &\leq (\tilde{C}\frac{h^{\tilde{\alpha}}}{\delta^2} - \varepsilon)\ (\tilde{C}_\delta^2 + \delta^2|x|^2) \quad \text{as } \tilde{C}_\delta \geq \delta. \end{aligned} \tag{2.38}$$

In the particular case where h=0, we notice that inequality (2.8) is not necessary to get that E_0 is nonpositive, for every $0 < \delta, \varepsilon \leq 1$, and the proof of Lemma 2.2 is complete.

3. Existence and uniqueness results of solutions with strictly subquadratic growth.

We consider the following additional structure conditions on F.

(H4) There exists $\Lambda_0 > 0$ such that, for every $X, Y \in \mathcal{S}^N$, $Y \geq 0$, $x, p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $t \in [0,T]$,

$$0 \leq F(X+Y,p,r,x,t) - F(X,p,r,x,t) \leq \Lambda_0(1+|x|^2)||Y||. \tag{3.1}$$

Theorem 3.1. *Assume that F satisfies* (H1), (H2), *and* (H4), *and suppose that $u_0 \in \mathcal{E}_{ssq}^L$ and $f \in \mathcal{E}_{ssq}^L([0,T])$. Then there exists a unique continuous viscosity solution u of* (1.1)–(1.2) *in $\mathcal{E}_{ssq}([0,T])$.*

Proof of Theorem 3.1. The existence of a discontinuous viscosity solution of (1.1)–(1.2) follows from Perron's method. We give the main steps of this method in our precise case. For a complete description, we refer to [8].

In the sequel, we set $\tilde{C}_\delta = \max(\bar{C}_\delta(f), \bar{C}_\delta(u_0), \frac{1}{\delta}, \delta)$, for every $\delta > 0$ (recall (2.1)). For positive constants δ and K fixed, we consider the function defined in $\mathbb{R}^N \times [0,T]$ by $g_\delta(x,t) = (\tilde{C}_\delta + \delta|x|^2)\exp(Kt)$. We first notice that, as $\tilde{C}_\delta \geq \bar{C}_\delta(u_0)$, then for every $x \in \mathbb{R}^N$ and $t \in [0,T]$, we have

$$-g_\delta(x,0) \leq u_0(x) \leq g_\delta(x,0). \tag{3.2}$$

Let us prove now that, for K large enough (independent of δ), for any $\delta > 0$, g_δ is a supersolution of (1.1). At first, the condition (H4) on F implies, for every $X \in \mathcal{S}^N$, $x, p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $t \in [0,T]$, the following inequality:

$$|F(X,p,r,x,t) - F(0,p,r,x,t)| \leq \Lambda_0(1+|x|^2)||X||. \tag{3.3}$$

Then Lemma 6.1-1 (see the Appendix) shows that, for every $x, p \in \mathbb{R}^N$ and $t \in [0,T]$, we have

$$\begin{aligned} |F(0,p,0,x,t)| &\leq |F(0,0,0,0,t)| + 2C_1(1+|x|)(1+|p|) \\ &\leq C_T + 2C_1(1+|x|)(1+|p|), \end{aligned} \tag{3.4}$$

where $C_T = \max_{t\in[0,T]} |F(0,0,0,0,t)|$. Now, using condition (H1) on F, as $g_\delta > 0$, then inequalities (3.3) and (3.4), and making easy computations, we obtain

$$\begin{aligned} F(D^2 g_\delta, Dg_\delta, g_\delta, x, t) &\leq \Lambda_0(1+|x|^2)||D^2 g_\delta|| + F(0, Dg_\delta, 0, x, t) \\ &\leq \big(2\delta\Lambda_0(1+|x|^2) + 2C_1(1+|x|)(1+2\delta|x|) + C_T\big)\exp(Kt). \end{aligned}$$

And, using that $\tilde{C}_\delta \geq \max(\delta, \frac{1}{\delta})$, which implies that $\tilde{C}_\delta \geq \max(1, \delta, \frac{1}{\delta})$, we get

$$\begin{aligned} &F(D^2 g_\delta, Dg_\delta, g_\delta, x, t) \\ &\leq 2\Lambda_0 g_\delta(x,t) + 2C_1(1+|x|+2\delta|x|)\exp(Kt) + (4C_1 + C_T)g_\delta(x,t) \\ &\leq (2\Lambda_0 + 4C_1 + C_T)g_\delta(x,t) + 2C_1(1 + \frac{1}{2\delta} + \delta + \frac{3}{2}\delta|x|^2)\exp(Kt) \\ &\leq (2\Lambda_0 + 10C_1 + C_T)g_\delta(x,t). \end{aligned}$$

Now, as the function f lies in $\mathcal{E}_{ssq}([0,T])$ and as $\tilde{C}_\delta \geq \bar{C}_\delta(f)$, we finally have, for every $x \in \mathbb{R}^N$ and $t \in [0,T]$,

$$F(D^2 g_\delta, Dg_\delta, g_\delta, x, t) + f(x,t) \leq (2\Lambda_0 + 10C_1 + C_T + 1)g_\delta(x,t). \tag{3.5}$$

But, since $\frac{\partial g_\delta}{\partial t} = K g_\delta$, by choosing $K = 2\Lambda_0 + 10C_1 + C_T + 1$, the inequality (3.5) implies that g_δ is a supersolution of (1.1). Similar arguments show that, for this choice of K, $-g_\delta$ is a subsolution of (1.1). We set $g_1(x,t) = \operatorname{Inf}_{\delta>0} g_\delta(x,t)$, for $(x,t) \in \mathbb{R}^N \times [0,T]$. Classical arguments show that g_1 is a lower-semicontinuous viscosity supersolution of (1.1), and we prove similarly that $-g_1$ is an upper-semicontinuous subsolution of the same equation. Moreover, the inequality (3.2) implies that, for every $x \in \mathbb{R}^N$, we have

$$-g_1(x,0) \leq u_0(x) \leq g_1(x,0). \tag{3.6}$$

We define the set $\mathcal{S}$ of all viscosity subsolutions w of (1.1) satisfying $w(x,t) \leq g_1(x,t)$ in $\mathbb{R}^N \times [0,T]$ and the following relaxed initial condition:

$$\min\{w_t - F(D_x^2 w, D_x w, w, x, 0), w^*(x,0) - u_0(x)\} \leq 0. \tag{3.7}$$

$\mathcal{S}$ is a nonempty set, since obviously $-g_1 \in \mathcal{S}$; we set, for every $x \in \mathbb{R}^N$ and $t \in [0,T]$,

$$u(x,t) = \operatorname*{Sup}_{w \in \mathcal{S}} w(x,t).$$

It is clear that u satisfies, for every $x \in \mathbb{R}^N$, $t \in [0,T]$, and $\delta > 0$,

$$|u(x,t)| \leq g_1(x,t) \leq \left(\tilde{C}_\delta + \delta|x|^2\right)\exp(KT). \tag{3.8}$$

Classical arguments show that u is a discontinuous solution of (1.1) satisfying (3.7) and also, for every $x \in \mathbb{R}^N$ and $t \in [0,T]$, the following relaxed initial condition:

$$\max\{w_t - F(D^2 w, Dw, w, x, 0), w_*(x,0) - u_0(x)\} \geq 0. \tag{3.9}$$

And it is well known that u fulfills the initial data in the classical sense (see, for example, Lemma 3.1 in [7]). Then to prove the continuity of this solution and the uniqueness of such a solution in $\mathcal{E}_{ssq}([0,T])$, we need the following proposition, which is a direct consequence of Theorem 2.1-1.

Proposition 3.1. *Assume* (H1) *and* (H2). *Let* u_1 *be an upper-semicontinous viscosity subsolution and* v_1 *be a lower-semicontinuous viscosity supersolution of* (1.1). *Assume that* u_1 *and* v_1 *satisfy the initial condition*

$$u_1(\cdot,0) \leq w \leq v_1(\cdot,0) \quad in \quad \mathbb{R}^N,$$

with $w \in \mathcal{E}_{ssq}^L$. *If* $u_1, v_1 \in \mathcal{E}_{ssq}([0,T])$ *and* $f \in \mathcal{E}_{ssq}^L([0,T])$, *then we have, for every* $x \in \mathbb{R}^N$ *and* $t \in [0,T]$, $u_1(x,t) \leq v_1(x,t)$.

Let us conclude the proof of Theorem 3.1. As the function u satisfies (3.8), then $u \in \mathcal{E}_{ssq}([0,T])$. It is then clear, using Proposition 3.1 (with $w = u_0$), that u is continuous and that it is the unique solution of (1.1)–(1.2), in $\mathcal{E}_{ssq}([0,T])$.

4. Regularity of the solution in x and t : Local Lipschitz continuity in x and local Hölder continuity in t

In this section, we still denote by u the unique solution of (1.1)–(1.2) in $\mathcal{E}_{ssq}([0,t])$. We set, for every $\delta > 0$, $\tilde{C}_\delta = \max(1, C_\delta(u_0), C_\delta(f))$. The following result is an immediate consequence of Theorem 2.1-1.

Theorem 4.1. (Local Lipschitz continuity in space) *Assume that F satisfies* (H1), (H2), *and* (H4), $u_0 \in \mathcal{E}^L_{ssq}$, *and* u *and* f *are in* $\mathcal{E}_{ssq}([0,T])$. *Then there exist positive constants* δ_0 *and* C_ℓ, *depending on* C_1 *and* T, *such that, for every* $0 < \delta \le \delta_0$, $t \in [0,T]$, *and* $x, y \in \mathbb{R}^N$, *with* $|x-y| \le 1$,

$$|u(x,t) - u(y,t)| \le C_\ell\ (\tilde{C}_\delta + \delta|x| + \delta|y|)|x-y|. \tag{4.1}$$

Remark 4.1. Condition (H4) is assumed in order to have existence of a solution in $\mathcal{E}_{ssq}([0,T])$, using Theorem 3.1. If we already know that there exists a solution of (1.1)–(1.2) in $\mathcal{E}_{ssq}([0,T])$, it is not necessary to get Theorem 4.1.

Now we want to prove that u is locally Hölder continuous in time. To this end, we have to suppose further regularity on the initial data u_0 and to add some other structure conditions on F and f.

(H5) There exists $C_3 > 0$ such that, $r \in \mathbb{R}$,

$$|F(0,0,r,0,0)| \le C_3|r|. \tag{4.2}$$

(Hf) There exists $C_f > 0$ and $0 < \alpha_f \le 1$, such that, for every $x \in \mathbb{R}^N$ and $s, t \in [0,T]$, the following inequality holds:

$$|f(x,t) - f(x,s)| \le C_f|s-t|^{\alpha_f}(1+|x|^2). \tag{4.3}$$

We set $\bar{\alpha} = \min(\alpha_1, \alpha_2, \alpha_f, \frac{1+\alpha}{2})$. In the sequel, a constant is said to depend on u_0 if it depends on $|u_0(0)|$, $|Du_0(0)|$, $|Du_0|_{0,\alpha}$, and α; it is said to depend on F if it depends on N, T, Λ_0, C_1, C_2, and C_3.

Using Theorem 2.1-2, we can show the following theorem.

Theorem 4.2. (Local Hölder continuity in time) *We suppose that F satisfies conditions* (H1)–(H5), *and* $f \in \mathcal{E}^L_{ssq}([0,T])$ *satisfies* (Hf). *We assume that* $u_0 \in C^{1,\alpha}(\mathbb{R}^N)$, *with* $0 < \alpha < 1$. *Let* u *be the unique viscosity solution of* (1.1)–(1.2) *in* $\mathcal{E}_{ssq}([0,T])$. *There exists* $0 < \bar{h} \le T$ *such that, for every* $0 < \beta < \bar{\alpha}$, *for every* $x \in \mathbb{R}^N$, $s, t \in [0,T]$ *with* $|s-t| \le \bar{h}$, *we have*

$$|u(x,t) - u(x,s)| \le \bar{C}_\beta\big(1+|x|^2\big)|s-t|^\beta, \tag{4.4}$$

where $\bar{C}_\beta > 0$ *depends on* u_0, F, β, C_f, $\bar{C}_1(f)$, *and* $C_{\delta_0}(f)$ *(where* δ_0 *is a constant depending only on* u_0*), and* $\bar{h} > 0$ *depends on* u_0, F, β, $\bar{\alpha}$, *and* $\bar{C}_1(f)$.

Proof of Theorem 4.2. 1. At first, using the remark below, it is clear that the functions F, f, and u_0 satisfy the required conditions to use Theorem 3.1. Therefore, there exists a unique, continuous viscosity solution u of (1.1)–(1.2) in $\mathcal{E}_{ssq}([0,T])$.

Remark 4.2. It is easy to show that, if $w \in C^{1,\alpha}(\mathbb{R}^N)$ with $0 < \alpha < 1$, then $w \in \mathcal{E}_{ssq}^L$, and moreover, for every $\delta > 0$, the constants $C_\delta(w)$ and $\bar{C}_\delta(w)$ can be chosen depending only on δ, $|w(0)|$, $|Dw(0)|$, $|Dw|_{0,\alpha}$, and on α.

2. The following proposition gives an estimate of $|u(x,t) - u_0(x)|$.

Proposition 4.1. *Assume* (H1) *and* (H5). *Let u be a continuous viscosity solution of* (1.1)–(1.2), *with $u_0 \in C^{1,\alpha}(\mathbb{R}^N)$, where $0 < \alpha < 1$.*

If u and $f \in \mathcal{E}_{ssq}([0,T])$, then there exists $\bar{C}_3 > 0$, $0 < T_0 \leq T$, such that, for every $x \in \mathbb{R}^N$ and $t \in [0, T_0]$,

$$|u(x,t) - u_0(x)| \leq \bar{C}_3 \; t^{\frac{1+\alpha}{2}} \; (1 + |x|^2). \tag{4.5}$$

The constants $\bar{C}_3$ and T_0 depend only on u_0, F, and $\bar{C}_1(f)$.

We postpone the proof of this proposition to the end of this section, and end the proof of Theorem 4.2.

3. We are now in position to use Theorem 2.1-2. Indeed, Proposition 4.1 and condition (Hf) show that f and u satisfy inequalities (2.7), (2.8), and (2.9), with $\tilde{C}_1 = \max(C_f, \bar{C}_3)$, $\tilde{\alpha} = \min(\alpha_f, \frac{1+\alpha}{2})$, and $h_0 = T_0$. Therefore, applying Theorem 2.1-2, with $w = u_0$, we get (4.4), with $\bar{C}_\beta = C_\beta \tilde{C}_{\delta_0}^2$. This ends the proof of Theorem 4.2.

It only remains for us to prove Proposition 4.1.

Proof of Proposition 4.1. For every $0 < \varepsilon \leq 1$, we set $u_0^\varepsilon = u_0 * \rho_\varepsilon$, with $\rho_\varepsilon(\cdot) = \frac{1}{\varepsilon^N}\rho(\frac{\cdot}{\varepsilon})$. ρ is a real-valued function defined in $\mathbb{R}^N$, symmetric, nonnegative, smooth, with $\mathrm{Supp}(\rho) \subset B(0,1)$ and $\int_{\mathbb{R}^N} \rho(x)dx = 1$.
It is easy to show, using in the first inequality the symmetry of ρ, that for every $x \in \mathbb{R}^N$ and $0 < \varepsilon \leq 1$, we have

$$|u_0^\varepsilon(x) - u_0(x)| \leq C_5 \; \varepsilon^{1+\alpha}, \tag{4.6}$$

$$|Du_0^\varepsilon(x)| \leq C_6 \; (1 + |x|^\alpha), \tag{4.7}$$

where $C_5 = |Du_0|_{0,\alpha}$ and $C_6 = \max(|Du_0(0)| + 1, |Du_0|_{0,\alpha})$.

Now we want to have an estimate on $||D^2 u_0^\varepsilon||$. For every $1 \leq i,j \leq N$, $0 < \varepsilon \leq 1$, and $x \in \mathbb{R}^N$, the following equalities hold:

$$\frac{\partial^2 u_0^\varepsilon}{\partial x_i \partial x_j}(x) \;\; = \;\; \frac{\partial u_0}{\partial x_i} * \frac{\partial \rho_\varepsilon}{\partial x_j}(x) = \int_{\mathbb{R}^N} \frac{\partial u_0}{\partial x_i}(x-y) \; \frac{\partial \rho_\varepsilon}{\partial x_j}(y) \; dy.$$

But, as the function ρ_ε is symmetric, $\displaystyle\int_{\mathbb{R}^N} \frac{\partial \rho_\varepsilon}{\partial x_j}(y)\, dy = 0$, and thus

$$
\begin{aligned}
|\frac{\partial^2 u_0^\varepsilon}{\partial x_i \partial x_j}| &= \int_{\mathbb{R}^N} \left(\frac{\partial u_0}{\partial x_i}(x-y) - \frac{\partial u_0}{\partial x_i}(x) \right) \frac{\partial \rho_\varepsilon}{\partial x_j}(y)\, dy \\
&\leq |Du_0|_{0,\alpha}\, \varepsilon^\alpha \int_{B(0,\varepsilon)} |\frac{\partial \rho_\varepsilon}{\partial x_j}(y)| dy \leq \frac{|Du_0|_{0,\alpha}}{\varepsilon^{1-\alpha}} \int_{B(0,1)} |\nabla \rho(y)|\, dy.
\end{aligned}
$$

This implies that

$$
||D^2 u_0^\varepsilon|| \leq \frac{C_7}{\varepsilon^{1-\alpha}}, \tag{4.8}
$$

where $C_7 = N|Du_0|_{0,\alpha} \int_{\mathbb{R}^N} |\nabla \rho(x)| dx$. For every $0 < \nu, \varepsilon \leq 1$, we consider the smooth functions, defined in $\mathbb{R}^N \times [0,T]$, by

$$
\begin{aligned}
v^+(x,t) &= u_{0\varepsilon}(x) + C_5 \varepsilon^{1+\alpha} + \frac{t}{T_0} \frac{(1+|x|^2)}{\varepsilon^{1-\alpha}} + \nu t + 3\nu(1+|x|^2), \\
v^-(x,t) &= u_{0\varepsilon}(x) - C_5 \varepsilon^{1+\alpha} - \frac{t}{T_0} \frac{(1+|x|^2)}{\varepsilon^{1-\alpha}} - \nu t - 3\nu(1+|x|^2).
\end{aligned} \tag{4.9}
$$

We have the following lemma.

Lemma 4.1. *There exists $0 < T_0 \leq T$, depending on u_0, F, T, and $\bar{C}_1(f)$, such that, for every $x \in \mathbb{R}^N$ and $t \in [0, T_0]$, the following inequalities hold:*

$$
\frac{\partial v^+}{\partial t}(x,t) - F(D^2 v^+, Dv^+, u_0, x, t) - f(x,t) \geq \nu. \tag{4.10}
$$

$$
\frac{\partial v^-}{\partial t}(x,t) - F(D^2 v^-, Dv^-, u_0, x, t) - f(x,t) \leq -\nu. \tag{4.11}
$$

We suppose that the preceding lemma is true and go on with the proof of Proposition 4.1. The function $u - v^+$ is a continuous function defined in $\mathbb{R}^N \times [0,T]$, and it satisfies

$$
\operatorname*{Sup}_{t\in[0,T]} (u - v^+)(x,t) \to -\infty \quad \text{as} \quad |x| \to \infty.
$$

Indeed, as $u_0 \in \mathcal{E}_{ssq}$ and $u \in \mathcal{E}_{ssq}([0,T])$, we show, using inequality (4.6), that for every $x \in \mathbb{R}^N$ and $t \in [0,T]$, we have

$$
\begin{aligned}
& u(x,t) - v^+(x,t) \\
& \leq |u(x,t)| + |u_0(x)| + |u_0(x) - u_{0\varepsilon}(x)| - C_5 \varepsilon^{1+\alpha} - 3\nu(1+|x|^2) \\
& \leq \bar{C}_\nu(u) + \bar{C}_\nu(u_0) + 2\nu(1+|x|^2) - 3\nu(1+|x|^2) \\
& \leq \bar{C}_\nu(u) + \bar{C}_\nu(u_0) - \nu|x|^2.
\end{aligned} \tag{4.12}
$$

Therefore, $u - v^+$ achieves its maximum over $\mathbb{R}^N \times [0, T_0]$ at a point (x_0, t_0), and we denote by M the value of this maximum. We want to prove that $M \leq 0$; to do it, assume for the sake of contradiction that $M > 0$. At first, inequality (4.6) implies that, for every $x \in \mathbb{R}^N$, $u_0(x) - v^+(x, 0) \leq 0$, and thus, as $M > 0$ by hypothesis, that shows that $t_0 > 0$. Still using inequality (4.6) and the fact that $M > 0$, then $u(x_0, t_0) > u_0(x_0)$. Condition (H1) on F combined with (4.10) imply that

$$\frac{\partial v^+}{\partial t} - F(D^2 v^+, Dv^+, u, x_0, t_0) - f(x_0, t_0) \geq \nu.$$

This contradicts the fact that u is a subsolution of (1.1), and therefore $M \leq 0$. We show similarly that $u - v^-$ achieves its minimum, of nonnegative value, in $\mathbb{R}^N \times [0, T]$. Finally, we get, for every $x \in \mathbb{R}^N$ and $t \in [0, T_0]$,

$$v^-(x, t) \leq u(x, t) \leq v^+(x, t).$$

And, using again (4.6), we have , for every $x \in \mathbb{R}^N$, $t \in [0, T_0]$, and $0 < \varepsilon, \nu \leq 1$, the following inequality:

$$|u(x, t) - u_0(x)| \leq 2C_5 \varepsilon^{1+\alpha} + \frac{t}{T_0} \frac{1 + |x|^2}{\varepsilon^{1-\alpha}} + \nu t + 3\nu(1 + |x|^2).$$

For $0 < t \leq T_0$, we choose $\varepsilon = \sqrt{\frac{t}{T_0}}$ and let ν go to zero, which proves inequality (4.5), with $\bar{C}_3 = \frac{1+2C_5}{T_0^{\frac{1+\alpha}{2}}}$.

We just have to prove Lemma 4.1. First, Lemma 6.1 (see the Appendix), then condition (H5) implies that, for every $p, x \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $t \in [0, T]$, we have

$$\begin{aligned} |F(0, p, r, x, t)| &\leq 2C_1(1 + |x|)(1 + |p|) + C_2 T^{\alpha_2} + |F(0, 0, r, 0, 0)| \\ &\leq \bar{C}\big(|r| + (1 + |x|)(1 + |p|)\big), \end{aligned} \tag{4.13}$$

where $\bar{C} = \max(C_3, 2C_1, C_2 T, C_2)$ depends only on F. Then combining inequalities (3.3) and (4.13), we get for every $x \in \mathbb{R}^N$ and $t \in [0, T_0]$

$$\begin{aligned} &F(D^2 v^+, Dv^+, u_0(x), x, t) \\ &\leq \Lambda_0(1 + |x|^2)||D^2 v^+(x, t)|| + F(0, Dv^+, u_0(x), x, t) \\ &\leq \Lambda_0(1 + |x|^2)||D^2 v^+(x, t)|| + \bar{C}|u_0(x)| + \bar{C}(1 + |x|)(1 + |Dv^+(x, t)|). \end{aligned} \tag{4.14}$$

Now, Remark 4.2 shows that $|u_0(x)| \leq \bar{C}_1(1 + |x|^2)$, where $\bar{C}_1$ is a constant depending only on u_0. Then, using inequalities (4.7), (4.8), and (4.14), we obtain

$$F(D^2v^+, Dv^+, u_0, x, t) \leq C \; \frac{1+|x|^2}{\varepsilon^{1-\alpha}},$$

where C is a positive constant depending on u_0 and on F.
Now, using that $f \in \mathcal{E}_{ssq}([0,T])$, we get, for every $x \in \mathbb{R}^N$ and $t \in [0,T_0]$,

$$\begin{aligned} &\frac{\partial v^+}{\partial t} - F(D^2v^+, D_x v^+, u_0, x, t) - f(x,t) \\ &\geq \nu + \frac{1}{T} \; \frac{1+|x|^2}{\varepsilon^{1-\alpha}} - \frac{C}{\varepsilon^{1-\alpha}}(1+|x|^2) - \tilde{C}_1(1+|x|^2) \\ &\geq \nu + \big(\frac{1}{T} - C - \tilde{C}_1\big) \; \frac{1+|x|^2}{\varepsilon^{1-\alpha}}, \end{aligned} \tag{4.15}$$

where $\tilde{C}_1 = \max(1, \bar{C}_1(f))$. Setting $T_0 = \min(T, \frac{1}{C+\tilde{C}_1})$, we get (4.10). Inequality (4.11) is identically obtained. That ends the proof of Lemma 4.1.

5. Further regularity in x and in t

In the sequel, a constant is said to depend on f if it depends on C_f, $\bar{C}_1(f)$, and $C_{\delta_0}(f)$ (where δ_0 is defined in Theorem 4.2) and on α_f.

5.1. **Local $C^{1,\gamma}$ regularity in space.** Thanks to Theorems 3.1, 4.1, and 4.2, adding a condition on F, we are in position to apply the theorem proved by N. Zhu in [15], to get local $C^{1,\gamma}$ regularity in space for u. The condition we add to F is a local uniform ellipticity one:
(H6) For every $R > 0$, there exists $\lambda_R > 0$ such that, for every $X, Y \in \mathcal{S}^N$, with $Y \geq 0$, for every $x \in B(0,R)$, $p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $t \in [0,T]$, we have

$$\lambda_R ||Y|| \leq F(X+Y, p, r, x, t) - F(X, p, r, x, t). \tag{5.1}$$

We recall that $\bar{\alpha} = \min(\alpha_1, \alpha_2, \alpha_f, \frac{1+\alpha}{2})$. We have the following theorem.

Theorem 5.1. *Assume that F satisfies conditions* (H1)–(H6), $f \in \mathcal{E}^L_{ssq}([0,T])$ *satisfies* (Hf), *and* $u_0 \in C^{1,\alpha}(\mathbb{R}^N)$, *with* $0 < \alpha < 1$. *Moreover, assume that* $\bar{\alpha} > \frac{1}{2}$. *Then the unique solution* u *of* (1.1)–(1.2) *in* $\mathcal{E}_{ssq}([0,T])$ *is locally* $C^{1,\gamma}$ *in* x. *More precisely, it satisfies, for every* $R > 0$ *and* $0 < \nu < T$, *there exists* $C > 0$, $0 < \gamma < 1$, *such that, for every* $x, y \in B(0,R)$ *and* $t \in [\nu, T]$, *we have*

$$|Du(x,t) - Du(y,t)| \leq C|x-y|^\gamma, \tag{5.2}$$

where C *and* γ *depend on* R, ν, u_0, F, f, *and* $\bar{\alpha}$. *To be more precise we set* $C = C(R,\nu)$ *and* $\gamma = \gamma(R,\nu)$ *when it is needed.*

Proof of Theorem 5.1. We give the main steps to get this result.

1. We recall the theorem proved by N. Zhu in [15]. For every $R > 0$, we denote by B_R the open ball of center 0 and radius R in $\mathbb{R}^N$. Let $R_0, T_0 > 0$ and let G be a continuous, real-valued function defined on $\mathcal{S}^N \times \mathbb{R}^N \times \mathbb{R} \times B_{R_0} \times [0, T_0]$. Assume that v is a viscosity solution of

$$v_t - G(D^2v, Dv, v, x, t) = 0 \quad \text{in} \quad B_{R_0} \times [0, T_0]. \tag{5.3}$$

Theorem 5.2. (N. Zhu [15]) *Suppose that there exist $M_0 > 0$, $M_1 > 0$, and $\beta > \dfrac{1}{2}$ such that v satisfies, for every $x, y \in B_{R_0}$ and $s, t \in [0, T_0]$,*

$$|v(x,t)| \leq M_0, \tag{5.4}$$

$$|v(x,t) - v(y,s)| \leq M_1\big(|x-y| + |s-t|^\beta\big). \tag{5.5}$$

Then $v \in C^1(B_{R_0}) \times (0, T_0])$ and, for every $0 < \nu_0 < min(T_0, R_0)$, there exists $C > 0$ and $0 < \gamma < 1$ such that, for every $x, y \in B_{R_0 - \nu_0}$ and $t \in [\nu_0, T_0]$, we have

$$|Dv(x,t) - Dv(y,t)| \leq C|x-y|^\gamma. \tag{5.6}$$

The constants C and γ depend on the structure conditions on G and on R_0 and ν_0.

2. We want to apply Theorem 5.2 to get Theorem 5.1. Let u be the viscosity solution of (1.1)–(1.2). We take $R_0 = R+1$, $T_0 = T$, $\nu_0 = \min(1, \nu)$, and $G(X, p, r, x, t) = F(X, p, r, x, t) - f(x, t)$, for every $X \in \mathcal{S}^N$, $x, p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $t \in [0, T]$. It is easy to see, using the following technical lemma, that if F satisfies conditions (H1)–(H6) and if $f \in \mathcal{E}^L_{ssq}([0, T])$ with (Hf), then G satisfies the required conditions, in $B_{R+1} \times [0, T]$, to use Theorem 5.2 (see [15], to have precisely the conditions imposed on the function G). To this end, we just have to prove that, if F satisfies conditions (H2) and (H3), then it is locally Lipschitz in x and in p, and locally Hölder in t, which implies that G has the same properties, as f is supposed to be locally Lipschitz in x and locally Hölder in t. This is a direct consequence of Lemma 6.1 (see the Appendix).

3. Now we want to estimate the constants M_0 and M_1 such that u satisfies (5.4) and (5.5), in $B(0, R+1) \times [0, T]$.

(i) It is clear that u satisfies (5.4) with $M_0 = M_R = \max\limits_{B_{R+1} \times [0,T]} |u(x,t)|$. By (3.8), with $\delta = 1$, we know that for every $R > 0$,

$$M_R \leq \tilde{C}_1(1 + (R+1)^2)\exp(KT), \tag{5.7}$$

where $K > 0$ depends on C_1 and on $C_T = \max_{t\in[0,T]} |F(0,0,0,0,t)|$ and $\tilde{C}_1 = \max(\bar{C}_1(f), \bar{C}_1(u_0), 1)$. Now Remark 4.2 implies that $\bar{C}_1(u_0)$ depends only on u_0, and Lemma 6.1 (see the Appendix) shows that

$$|F(0,0,0,0,t)| \le |F(0,0,0,0,0)| + C_2 T^{\alpha_2},$$

which implies that $C_T \le |F(0,0,0,0,0)| + C_2 \max(1,T)$. Then (5.7) and the preceding remarks yield, for every $R > 0$,

$$M_R \le \bar{C}(1 + R^2), \tag{5.8}$$

where $\bar{C} > 0$ depends on u_0, F, and f.

(ii) To get (5.5), we need to have a precise Lipschitz bound in x and Hölder bound in t for the function u in $B_{R+1} \times [0,T]$. As $\alpha > \frac{1}{2}$ by hypothesis, we can choose $\beta = \frac{\bar{\alpha}}{2} + \frac{1}{4} < \bar{\alpha}$ in Theorem 4.2 so that $\frac{1}{2} < \beta$. Then Theorems 4.1 and 4.2 with this choice of β show (5.5), for every $x, y \in B_{R+1}$ and $s,t \in [0,T]$, with $|x-y| \le 1$ and $|s-t| \le \bar{h}$, with M_1 and $\bar{h}$ depending only on R, F, u_0, f, and $\bar{\alpha}$. Now, if $x, y \in B_{R+1}$ and $s,t \in [0,T]$, with $|x-y| \ge 1$ or $|s-t| \ge \bar{h}$, then (5.8) implies that

$$|u(x,t) - u(y,s)| \le 2\bar{C}(1+R^2)\big(|x-y| + \frac{|s-t|^\beta}{\bar{h}^\beta}\big),$$

where $\bar{C}$ depends only on F, u_0, and f. Finally, we have shown that u satisfies (5.5), in $B^2_{R+1} \times [0,T]^2$, with β depending only on $\bar{\alpha}$ and M_1 depending only on R, F, u_0, f, and $\bar{\alpha}$.

4. We can now apply Theorem 5.2, and we get Theorem 5.1.

5.2. **C^2 regularity in x and C^1 regularity in t.** First, we need to introduce some notation. For $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, and $r > 0$, we set $Q_r(0,0) = B(0,r) \times (-r^2, 0]$ and $Q_r(x,t) = (x,t) + Q_r(0,0)$. If U is an open set of $\mathbb{R}^N$, I is an interval of $\mathbb{R}$, $0 < \gamma < 1$, and $k \in \mathbb{N}^*$, we denote by $C^k_x(U \times I)$ (respectively $C^k_t(U \times I)$), the subset of real-valued functions defined in $U \times I$, which are C^k in x (respectively C^k in t). We denote by $\tilde{C}^{2,\gamma}(U \times I)$, the space of the functions which have second derivatives in x and first in t, both γ-Hölder in x and $\frac{\gamma}{2}$-Hölder in t.

To get further regularity for a solution u, we are going to use Theorems 1.1 and 4.13 proved by L. Wang in [17]. In order to do it, we first need to suppose that F is convex with respect to the variable $X \in \mathcal{S}^N$ and that it satisfies moreover the following condition.

(H7) For every $R > 0$, there exists $C_R > 0$ such that, for every $X \in \mathcal{S}^N$, $x \in B(0,R)$, $r, z \in \mathbb{R}$, and $t \in [0,T]$, we have

$$|F(X,p,r,x,t) - F(X,p,z,x,t)| \le C_R\big(1 + ||X||\big)|r - z|. \tag{5.9}$$

In the sequel, we say that a constant depends on data if it depends on u_0, F, f, and $\bar{\alpha}$. We say that it depends on data(R) if it depends on R, on data, and on λ_m and C_m (see (5.1) and (5.9)), for any m depending on data and on R.

Theorems 3.1, 4.1, and 5.1, and Theorems 1.1 and 4.13 in [17], yield the following result.

Theorem 5.3. *Suppose that F is convex with respect to the variable $X \in \mathcal{S}^N$. Assume that F satisfies conditions* (H1)–(H7), *$f \in \mathcal{E}^L_{ssq}([0,T])$, with* (Hf), *and $u_0 \in C^{1,\alpha}(\mathbb{R}^N)$, with $0 < \alpha < 1$. Suppose moreover that $\bar{\alpha} > \frac{1}{2}$. Then the unique solution u of* (1.1)–(1.2), *in $\mathcal{E}_{ssq}([0,T])$, lies in $C^2_x(\mathbb{R}^N \times (0,T]) \cap C^1_t(\mathbb{R}^N \times (0,T])$. Moreover, u satisfies the following: for every $R > 0$ and $0 < \nu \leq T$, there exist $C > 0$ and $0 < \gamma < 1$ depending on data(R) and r such that, for every $x, y \in B(0,R)$ and $s, t \in [\nu, T]$, we have*

$$|D^2u(x,t) - D^2u(y,s)| \leq C\big(|x-y|^\gamma + |s-t|^{\frac{\gamma}{2}}\big),$$
$$|\frac{\partial u}{\partial t}(x,t) - \frac{\partial u}{\partial t}(y,s)| \leq C\big(|x-y|^\gamma + |s-t|^{\frac{\gamma}{2}}\big). \tag{5.10}$$

Proof of Theorem 5.3. 1. As $u \in C^1_x(\mathbb{R}^N \times (0,T])$, it is clear that u is a viscosity solution of

$$u_t - G(D^2u, x, t) = f(x,t) \quad \text{in} \quad \mathbb{R}^N \times (0,T],$$

where $G(X,x,t) = F(X, Du(x,t), u(x,t), x, t)$, for every $X \in \mathcal{S}^N$, $x \in \mathbb{R}^N$, and $t \in [0,T]$.

2. In order to apply Theorem 1.1 in [17], we perform a scaling, to get a solution in Q_1. For $0 < R$, $0 < r_0 < \min(\frac{T}{2}, 1)$, $x_0 \in B(0,R)$, and $t_0 \in [2r_0, T]$, we consider the function v, defined for every $(x,t) \in Q_1$ by

$$v(x,t) = \frac{1}{r_0} u(x_0 + \sqrt{r_0}x, t_0 + r_0 t).$$

It is clear that v is a viscosity solution of

$$v_t - K(D^2v, x, t) = g(x,t) \quad \text{in} \quad Q_1,$$

where $K(M,x,t) = G(M, x_0 + \sqrt{r_0}x, t_0 + r_0 t)$ and $g(x,t) = f(x_0 + \sqrt{r_0}x, t_0 + r_0 t)$, for every $(x,t) \in Q_1$. It is easy to show that K is uniformly elliptic, in Q_1, with ellipticity constants λ_{R+1} and $3\Lambda_0(1+R^2)$.

3. We study the following function Θ, which is defined for every $(x,t) \in Q_1$ by

$$\Theta(x,t) = \operatorname*{Sup}_{M \in \mathcal{S}^N} \frac{|K(M,x,t) - K(M,0,0)|}{|M|+1}.$$

We have the following proposition, whose proof is given in the Appendix.

Proposition 5.1. *There exist $C > 0$ and $0 < \gamma < 1$ depending on data(R) and r_0 such that, for every $(x,t) \in Q_1$,*

$$\Theta(x,t) \leq C\big(|x|^\gamma + |t|^\gamma\big). \tag{5.11}$$

4. On an other hand, the function g satisfies, for every $(x,t) \in Q_1$

$$|g(x,t) - g(0,0)| \leq C\big(|x| + |t|^{\alpha_f}\big),$$

where $C > 0$ depends on f, R, and r_0, and thus on data(R) and r_0. This property is an immediate consequence of the fact that $f \in \mathcal{E}^L_{ssq}([0,T])$ and satisfies (Hf).

5. For every $M \in \mathcal{S}^N$ and $d \in \mathbb{R}$ such that $K(M,0,0) = d$, the solution v of $v_t - K(D^2v + M, 0, 0) = d$ in Q_1 is in $\tilde{C}^{2,\delta}(Q_{\frac{1}{2}})$, where δ is a positive constant depending on R, r_0, λ_{R+1}, and Δ_0, and thus on data(R) and r_0. This is an immediate consequence of Theorem 4.13 in [17], as $X \mapsto K(X + M, 0, 0) - d$ is convex and uniformly elliptic, with ellipticity constants λ_{R+1} and $3\Lambda_0(1 + R^2)$.

6. We can now apply Theorem 1.1 in [17], and we obtain the following result. There exists $0 < r_1$, $\nu < 1$, $C > 0$, and a paraboloid P_{x_0,t_0}, denoted by P to simplify, defined for every $(x,t) \in Q_1$ by

$$P(x,t) = a_0 t + \frac{1}{2}x^T B_0 x + c_0^T x + d_0,$$

such that, for every $r \leq r_1$,

$$\begin{aligned} ||P - v||_{\infty,Q_r} &\leq C||v||_{\infty,Q_1}\frac{r^{2+\nu}}{r_1^{2+\nu}}, \\ r_1|c_0| + r_1^2(|a_0| + ||B_0||) &\leq C||v||_{\infty,Q_1}. \end{aligned} \tag{5.12}$$

The constants C, r_1, and ν depend on data(R) and r_0, but not on the choice of $x_0 \in B(0,R)$ and $t_0 \in [2r_0, T]$.

7. To conclude, we use the following proposition, whose proof is again given in the Appendix.

Proposition 5.2. *Let w be a bounded function defined in $\mathbb{R}^{N+1}$ and I an open set of $\mathbb{R}$. Suppose that there exist some positive constants C, $\tilde{r}$, and $\nu < 1$ such that for every $(x,t) \in B(0,R) \times I$, there exists a paraboloid $P_{x,t}$, denoted by P to simplify, with, for every $r \leq \tilde{r}$,*

$$\begin{aligned} ||w - P||_{\infty,Q_r(x,t)} &\leq Cr^{\nu+2}, \qquad (5.13) \\ |D_xP(x,t)| + |P_t(x,t)| + ||D_x^2P(x,t)|| &\leq C. \qquad (5.14) \end{aligned}$$

Then w lies in $(C_x^2 \cap C_t^1)(B(0,R)\times I)$, and there exists a positive constant $\tilde{C}$ depending only on $\tilde{r}$ such that, for every $(x,t),(y,s)\in B(0,R)\times I$,

$$|w_t(x,t)-w_t(y,s)| \le \tilde{C}C\big[|x-y|^\nu + |t-s|^{\frac{\nu}{2}}\big], \tag{5.15}$$

$$||D_x^2 w(x,t) - D_x^2 w(y,s)|| \le \tilde{C}C\big[|x-y|^\nu + |t-s|^{\frac{\nu}{2}}\big]. \tag{5.16}$$

8. End of the proof of Theorem 5.3. For $x_0\in B(0,R)$, $t_0\in[2r_0,T]$, and $(x,t)\in Q_1$, we set $Q_{x_0,t_0}(x,t) = r_0 P_{x_0,t_0}(\frac{x-x_0}{\sqrt{r_0}}, \frac{s-t_0}{r_0})$. We set $Q = Q_{x_0,t_0}$ to simplify. It is easy to show, thanks to (5.12), that for all $r\le r_1$

$$||u-Q||_{\infty,Q_r(x_0,t_0)} \le \bar{C}r^{2+\nu}, \tag{5.17}$$

$$|D_xQ(x_0,t_0)| + |Q_t(x_0,t_0)| + ||D_x^2Q(x_0,t_0)|| \le \bar{C}||u||_{\infty,Q_r(x_0,t_0)} \le \bar{C}M_{R+1},$$

where $\bar{C}$ depends on the constants C, r_1, and ν defined in step 6, and thus only on data(R) and r_0. $M_{R+1} = \max_{B_{R+1}\times[0,T]}|u(x,t)|$ depends only on R and u_0 (using (5.8)). Now we use Proposition 5.2, with

$$w(x,t) = \begin{cases} u(x,t) & \text{in} \quad \mathbb{R}^N\times[0,t] \\ 0 & \text{otherwise,}\end{cases}$$

$I=[2r_0,T]$, $\tilde{r}=r_1$, and $P_{x,t}=Q_{x,t}$, and we get the theorem, by taking $2r_0=\nu$ in (5.3).

6. Appendix

In the particular case where F is given by $F(X,p,r,x,t) = Tr(A(x,t,p)X) + H(p,r,x,t)$, we prove the equivalence between the conditions (H1) to (H8) on F and these on the functions A and H, given in the Introduction and recalled below in detail.

Proposition 6.1. (The quasilinear case) *We set $A=aa^T$, where a is a nonnegative symmetric matrix, and we consider the following assumptions on the function a, A, and H: there exist $C>0$ and $\frac{1}{2}<\gamma<1$ such that, for every $x,y,p,q\in\mathbb{R}^N$, $r\in\mathbb{R}$, and $s,t\in[0,T]$, we have*

$$||a(x,t,p)|| \le (1+|x|) \tag{6.1}$$

$$||a(x,t,p)-a(y,t,q)|| \le C\big(|x-y|+|p-q|\big) \tag{6.2}$$

$$||a(x,t,p)-a(x,s,q)|| \le C\big(1+|x|\big)|s-t|^\gamma \tag{6.3}$$

$$|H(0,r,0,0)| \le C|r| \tag{6.4}$$

$$|H(p,r,x,t)-H(q,r,y,t)| \le C(1+|x|)|p-q| + C(1+|p|)|x-y| \tag{6.5}$$

$$|H(p,r,x,t)-H(p,r,x,s)| \le C(1+|x|+|p|)|s-t|^\gamma. \tag{6.6}$$

For every $p, x \in \mathbb{R}^N$, $t \in [0,T]$, *and* $r, z \in R$, *with* $r \leq z$,

$$H(p,r,x,t) \geq H(p,z,x,t). \tag{6.7}$$

The function A *satisfies, for every* $R > 0$, *there exists* $\lambda_R > 0$ *such that, for every* $x \in B(0,R)$, $p \in \mathbb{R}^N$, *and* $t \in [0,T]$,

$$\lambda_R Id \leq A(x,t,p). \tag{6.8}$$

H *satisfies, for every* $R > 0$, *there exists* $C_R > 0$ *such that, for every* $x, p \in B(0,R)$, $r, z \in \mathbb{R}$, *and* $t \in [0,T]$,

$$|H(p,r,x,t) - H(p,z,x,t)| \leq C_R|r-z|. \tag{6.9}$$

Proof. It is easy to show the following:

(i) (6.1) implies (H4), with $\Delta_0 = 2NC^2$.
(ii) (6.4) implies (H5), with $C_3 = C$.
(iii) (6.8) implies (H6), with the same constant λ_R.
(iv) (6.9) implies (H7), with the same constant C_R.
(v) (6.7) implies (H1).

We are now going to prove that if a and H satisfy (6.1), (6.2), (6.3), (6.5), and (6.6), then F satisfies the conditions (H2) and (H3). To this end, consider $X, Y \in \mathcal{S}^N$ and $\lambda_1, \lambda_2 \geq 0$, such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \lambda_1 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \lambda_2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \tag{6.10}$$

Inequality (6.10) implies that, for every $v, w \in \mathbb{R}^N$, the following inequality holds:

$$(Xv,v) - (Yw,w) \leq \lambda_1|v-w|^2 + \lambda_2\big(|v|^2 + |w|^2\big). \tag{6.11}$$

Let $(e_i)_{1\leq i\leq N}$ be an orthonormal basis of $\mathbb{R}^N$. For $x, y \in \mathbb{R}^N$, $s, t \in [0,T]$, and $p, q \in \mathbb{R}^N$, we set $p_i = a(x,t,p)e_i$ and $q_i = a(y,s,q)e_i$. For every $1 \leq i \leq N$, using (6.11) with $v = p_i$ and $w = q_i$, we show that

$$(Xp_i,p_i) - (Yq_i,q_i) \leq \lambda_1|p_i - q_i|^2 + \lambda_2(|p_i|^2 + |q_i|^2).$$

Then summing and using that $Tr(BC) = Tr(CB)$, for every $C, B \in \mathcal{M}^N$, we obtain

$$Tr(A(x,t,p)X) - Tr(A(y,s,q)Y) \leq N\lambda_1||a_x - a_y||^2 + N\lambda_2\big(||a_x||^2 + ||a_y||^2\big), \tag{6.12}$$

where $a_x = a(x,t,p)$ and $a_y = a(y,s,q)$. Then using (6.2), (6.3), (6.5), and (6.6), it is clear that F satisfies conditions (H2) and (H3), with $C_1 = C_2 = 4NC^2 + C$ and $\alpha_1 = \alpha_2 = \gamma$.

We now prove a technical lemma, which is used in the different preceding sections.

Lemma 6.1. 1) *Assume that* F *satisfies* (H2); *then it satisfies, for every* $X \in \mathcal{S}^N$, $p, q, x, y \in \mathbb{R}^N$, $r \in \mathbb{R}$, *and* $t \in [0, T]$,

$$|F(X,p,r,x,t) - F(X,q,r,y,t)| \le 2C_1||X||(1+|x|+|y|)\big(|x-y|+|p-q|\big) + C_1(1+|p|)|x-y| + C_1(1+|y|)|p-q|. \quad (6.13)$$

2) *Assume that* F *satisfies* (H2) *and* (H3); *then it satisfies, for every* $X \in \mathcal{S}^N$, $x, p \in \mathbb{R}^N$, $r \in \mathbb{R}$, *and* $s, t \in [0, T]$,

$$|F(X,p,r,x,t) - F(X,p,r,x,s)| \le 2C_2||X||(1+|x|)|s-t|^{\alpha_1} + C_2(1+|x|+|p|)|s-t|^{\alpha_2}. \quad (6.14)$$

Proof. This lemma relies on the following observation. For every $X \in \mathcal{S}^N$ and $\nu > 0$, we have

$$\begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \le \frac{||X||}{\nu^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + ||X||\nu^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}. \quad (6.15)$$

Indeed, for $X \in \mathcal{S}^N$, $p, q \in \mathbb{R}^N$, and $\nu > 0$, the following inequalities hold:

$$\begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} X & 0 \\ 0 & -X \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \big(X(p-q), p+q\big)$$
$$\le ||X|||p-q||p+q| \le \frac{||X||}{\nu^2}|p-q|^2 + ||X||\nu^2\big(|p|^2+|q|^2\big). \quad (6.16)$$

1) Assume that $x, y, p, q \in \mathbb{R}^N$, with $|x-y|+|p-q| > 0$, and $r \in \mathbb{R}$ and $t \in [0, T]$; then (H2) and (6.15), with $\nu = (|x-y|+|p-q|)(1+|x|+|y|)$, imply (6.13).

2) Assume that $x, p \in \mathbb{R}^N$, $r \in \mathbb{R}$, and $s, t \in [0, T]$, with $|s-t| > 0$; then (H3) and (6.15), with $\nu = |s-t|^{\alpha_1}(1+|x|)$, imply (6.14).

Next we demonstrate Proposition 5.1, which is useful in proving Theorem 5.3.

Proof of Proposition 5.1. 1. To prove this proposition, we first show this technical lemma, which proves Hölder regularity in t for Du.

Lemma 6.2. *Let* $0 < T_1, R_1$ *and* $w \in C^1(B(0, 2R_1) \times [0, T_1])$ *satisfying, for every* $x, y \in B(0, 2R_1)$ *and* $t, s \in [0, T_1]$,

$$|w(x,t) - w(y,s)| \le C\big(|x-y| + |s-t|^\beta\big) \quad (6.17)$$
$$|Dw(x,t) - Dw(y,t)| \le C|x-y|^\gamma, \quad (6.18)$$

where $C > 0$ and $0 < \beta, \gamma < 1$. Then for every $x \in B(0, R_1)$ and $t, s \in [0, T_1]$, we have

$$|Dw(x,t) - Dw(x,s)| \leq \tilde{C}|s-t|^{\frac{\beta\gamma}{1+\gamma}}, \tag{6.19}$$

where $\tilde{C}$ is a positive constant depending only on R_1 and C.

Proof. For $r > 0$, we set $B_r = B(0, r)$. For $t, t+h$ in $[0, T_1)$, with $0 < h < 1$, we consider the function, defined for every $x \in B(0, 2R_1)$ by

$$v(x) = w(x, t+h) - w(x,t).$$

Let $\delta > 0$ be small, and denote by $0 < \lambda < 1$ the constant such that

$$1 = \lambda\delta + (1-\lambda)(1+\gamma).$$

We first show that, for every $x \in B(0, R_1)$, we have

$$|Dv(x)| \leq C_{R_1}|v|^{\lambda}_{0,\delta,B_{2R}}\big(|v|_{0,\delta,B_{2R_1}} + |Dv|_{0,\gamma,B_{2R_1}}\big)^{1-\lambda}, \tag{6.20}$$

where $C_{R_1} = \max(1 + R_1, 1 + \frac{1}{R_1})$.

In the sequel, we consider $x_0 \in B(0, R_1)$ fixed, with $|Dv(x_0)| > 0$. For every $y \in B_{2R_1}$, using Taylor's formula, we have

$$\begin{aligned} |\langle Dv(x_0), x_0 - y\rangle| &\leq |v(x_0) - v(y)| + |Dv|_{0,\gamma,B_{2R_1}}|x_0 - y|^{1+\gamma} \\ &\leq |v|_{0,\delta,B_{2R_1}}|x_0 - y|^{\delta} + |Dv|_{0,\gamma,B_{2R_1}}|x_0 - y|^{1+\gamma}. \end{aligned}$$

Then, for every $0 < \varepsilon \leq 1$, choosing $y = x_0 + \varepsilon R_1 \frac{Dv(x_0)}{|Dv(x_0)|} \in B(0, 2R_1)$ in the preceding inequality, we obtain

$$|Dv(x_0)| \leq |v|_{0,\delta,B_{2R_1}}(\varepsilon R_1)^{\delta-1} + |Dv|_{0,\gamma,B_{R_1}}(\varepsilon R_1)^{\gamma}. \tag{6.21}$$

We consider two cases.
(i) If $|Dv|_{0,\gamma,B_{2R_1}} \leq |v|_{0,\delta,B_{2R_1}}$, choosing $\varepsilon = 1$, we get

$$|Dv(x_0)| \leq \Big((R_1)^{\delta-1} + (R_1)^{\gamma}\Big)|v|_{0,\delta,B_{2R_1}} \leq C_{R_1}|v|_{0,\delta,B_{2R_1}}. \tag{6.22}$$

(ii) If $|Dv|_{0,\gamma,B_{2R_1}} > |v|_{0,\delta,B_{2R_1}}$, choosing $\varepsilon = \left(\frac{|v|_{0,\delta,B_{2R_1}}}{|Dv|_{0,\gamma,B_{2R_1}}}\right)^{1+\gamma-\delta}$, as $1 = \lambda\delta + (1-\lambda)(1+\gamma)$, we get

$$\begin{aligned} &|Dv(x_0)| \leq C_{R_1}\varepsilon^{-1}\varepsilon^{\delta}|v|_{0,\delta,B_{2R_1}} \qquad (6.23) \\ &= C_{R_1}\varepsilon^{-1}\big(\varepsilon^{\delta}|v|_{0,\delta,B_{2R_1}}\big)^{\lambda}\big(\varepsilon^{1+\gamma}|Dv|_{0,\gamma,B_{2R_1}}\big)^{1-\lambda} = C_{R_1}|v|^{\lambda}_{0,\delta,B_{2R_1}}|Dv|^{1-\lambda}_{0,\gamma,B_{2R_1}}. \end{aligned}$$

Therefore, we have proved inequality (6.20). We now want to show (6.19), using this inequality. To do so, we have to estimate $|v|_{0,\delta,B_{2R_1}}$ and $|Dv|_{0,\gamma,B_{2R_1}}$.

(i) An estimate of $|Dv|_{0,\gamma,B_{2R_1}}$: it is easy to show thanks to inequality (6.18) that

$$|Dv|_{0,\gamma,B_{2R_1}} \leq 2C. \tag{6.24}$$

(ii) An estimate of $|v|_{0,\delta,B_{2R_1}}$: inequality (6.17) implies that, for every $x, y \in B_{2R_1}$, with $x \neq y$, the following inequalities hold:

$$\frac{|v(x)-v(y)|}{|x-y|^\delta} = \frac{|w(x,t+h)-w(x,t)-w(y,t+h)+w(y,t)|}{|x-y|^\delta}$$
$$\leq 2C \min\big(|x-y|^{1-\delta}, \frac{h^\beta}{|x-y|^\delta}\big).$$

Let $a > 0$; if $|x-y| \leq a$, then $|x-y|^{1-\delta} \leq a^{1-\delta}$, whereas if $|x-y| \geq a$, we have $\dfrac{h^\beta}{|x-y|^\delta} \leq \dfrac{h^\beta}{a^\delta}$. Therefore, in any case,

$$\min(|x-y|^{1-\delta}, \frac{h^\beta}{|x-y|^\delta}) \leq a^{1-\delta} + \frac{h^\beta}{a^\delta},$$

and choosing $a = h^\beta$, we get

$$|v|_{0,\delta,B_{2R_1}} \leq 4Ch^{\beta(1-\delta)}. \tag{6.25}$$

Combining inequalities (6.20), (6.24), and (6.25) we have, for every $x \in B_{R_1}$,

$$|Dw(x,t+h) - Dw(x,t)| = |Dv(x)|$$
$$\leq C_{R_1}(4C)^\lambda h^{\beta(1-\gamma)\lambda}\big(4Ch^{\beta(1_\gamma)} + 2C\big)^{1-\lambda} \leq \bar{C}C_{R_1}h^{\frac{\beta\gamma(1-\delta)}{1+\gamma-\delta}}, \tag{6.26}$$

where $\bar{C} = 6\max(C,1)$, using that $0 < h < 1$ and the expression of λ. As C_{R_1} and $\bar{C}$ are independent of $\delta > 0$, we can let δ go to zero, and we obtain (6.43), with $\tilde{C} = C_{R_1}\bar{C}$.

We now return to the proof of Proposition 5.1.

2. Using Theorems (4.1), (4.2), and (5.1) along with the preceding lemma, it is easy to show that u satisfies, for every $x \in B(0, R+1)$ and $s, t \in [r_0, T]$,

$$|Du(x,t) - Du(x,s)| \leq C|s-t|^{\bar{\gamma}}, \tag{6.27}$$

where C and $\bar{\gamma}$ depend on F, f, u_0, R, r_0, $C(2(R+1), r_0)$, and $\gamma_{2(R+1),r_0}$, and thus only on data(R) and on r_0. ($C(2(R+1), r_0)$ and $\gamma(2(R+1), r_0)$ are the constants appearing in Theorem 5.1.)

3. Theorem 4.1 and in particular inequality (4.1) allows us to get a local estimate of $|Du|$. Precisely, for every $x \in B(0, R+1)$ and $t \in [0, T]$, the following inequality holds:

$$|Du(x,t)| \leq C(1+R), \tag{6.28}$$

where C depends on F, u_0, and f.

4. For every $x_0 \in B(0,R)$, $t \in [2r_0, T]$, and $(x,t) \in Q_1$, it is clear that $x_0 + \sqrt{r_0}x \in B(0, R+1)$ and $t_0 + r_0 t \in [r_0, T]$. Now, using Lemma 6.1 and Theorems (4.1), (4.2), and (5.1), along with the inequalities (6.27) and (6.28), we get Proposition 5.1.

We end the Appendix with the proof of Proposition 5.2.

Proof of Proposition 5.2. 1. It is easy to show that (5.13) implies for every $x \in B(0,R)$, $t \in I$, and $(y,s) \in Q_{\tilde{r}}$, the following inequality:

$$|w(x+y,t+s) - P_{x,t}(x+y,t+s)| \le C\big(|y|^2 + |s|\big)^{1+\frac{\nu}{2}}. \tag{6.29}$$

To simplify, we denote $P_{x,t}$ by P, where there is no ambiguity. Using the preceding inequality, it is clear that $w(x,t) = P(x,t)$, and that w is differentiable in space with $D_x w(x,t) = D_x P(x,t)$ and left derivable in time with $w_t^-(x,t) = P_t(x,t)$, and then inequality (6.29) implies that, for every $x \in B(0,R)$, $t \in I$, and $(y,s) \in Q_{\tilde{r}}$, setting $p_x = D_x w(x,t)$ and $S_{x,t} = D_x^2 P(x,t)$,

$$\begin{aligned} w(x+y,t+s) = w(x,t) - \langle p_x, y\rangle - w_t^-(x,t)s - \tfrac{1}{2}\langle S_{x,t}y, y\rangle \\ + CO\big((|y|^2 + |s|)^{1+\frac{\nu}{2}}\big). \end{aligned} \tag{6.30}$$

2. We prove now that w is differentiable in t, with $w_t(x,t) = P_t(x,t)$. For $x \in B(0,R)$ and $t, t+h \in I$, with $0 \le 2h < \tilde{r}^2$, we have, using (6.30),

$$\begin{aligned} |w(x,t) - w(x,t+h) + hw_t^-(x,t)| \le Ch^{1+\frac{\nu}{2}}, \\ \text{and} \quad |w(x,t-h) - w(x,t+h) + 2hw_t^-(x,t)| \le Ch^{1+\frac{\nu}{2}}. \end{aligned} \tag{6.31}$$

That implies that, for every $x \in B(0,R)$ and $t, t+h \in I$, with $0 \le 2h < \tilde{r}^2$,

$$|w(x,t+h) + w(x,t-h) - 2w(x,t)| \le 3Ch^{1+\frac{\nu}{2}}, \tag{6.32}$$

and thus, for every $x \in B(0,R)$ and $t, t+h \in I$ with $0 \le 2h < \tilde{r}^2$, we obtain

$$\frac{w(x,t+h) - w(x,t)}{h} = \frac{w(x,t-h) - w(x,t)}{-h} + O(h^{1+\frac{\nu}{2}}).$$

Therefore, as I is open and w is left derivable in time in (x,t), w is derivable in time in (x,t), with $w_t(x,t) = \frac{\partial P_{x,t}}{\partial t}(x,t)$.

3. Inequality (6.30) and the preceding step imply that, for every $x \in B(0,R)$, $t \in I$, and $(y,s) \in Q_{\tilde{r}}$, we have

$$w(x+y,t+s) = Q_{x,t}(y,s) + \tfrac{1}{2}\langle S_{x,t}y, y\rangle + CO(|y|^2 + |s|)^{1+\frac{\nu}{2}}, \tag{6.33}$$

where

$$Q_{x,t}(y,s) \quad = \quad w(x,t) + \langle D_x w(x,t), y\rangle + w_t(x,t)s, \tag{6.34}$$

$$S_{x,t} \quad = \quad D_x^2 P_{x,t}(x,t). \tag{6.35}$$

4. Proof of inequality (5.15). (i) For $x \in B(0,R)$, $t, t+h \in I$, and $0 < h < \tilde{r}^2$, we have, using (6.34)

$$\begin{aligned} &h\big(w_t(x,t+h) - w_t(x,t)\big) \\ &= w(x,t+h) - Q_{x,t+h}(0,-h) + Q_{x,t}(0,-h) - w(x,t). \end{aligned} \tag{6.36}$$

Then (6.33) implies that

$$Q_{x,t}(0,-h) = w(x,t-h) + CO(h^{1+\frac{\nu}{2}})$$

and

$$Q_{x,t+h}(0,-h) = w(x,t) + CO(h^{1+\frac{\nu}{2}}).$$

Combining that with (6.36), we get

$$\begin{aligned} &h\big(w_t(x,t+h) - w_t(x,t)\big) \\ &= w(x,t-h) + w(x,t+h) - 2w(x,t) + CO(h^{1+\frac{\nu}{2}}), \end{aligned} \tag{6.37}$$

and using (6.32) we have, for every $x \in B(0,R)$ and $t, t+h \in I$, with $0 < 2h < \tilde{r}^2$,

$$|w_t(x,t) - w_t(x,t+h)| \le \tilde{C} C h^{\frac{\nu}{2}}. \tag{6.38}$$

(ii) For $x, x+y \in B(0,R)$, with $y \in B(0,\tilde{r})$, $t \in I$, and $0 < h < \tilde{r}^2$, we show, using (6.33) then (6.34), the following equalities:

$$\begin{aligned} w(x+y,t-h) &= Q_{x+y,t}(0,-h) + CO(h^{1+\frac{\nu}{2}}), \\ &= w(x+y,t) - h w_t(x+y,t) + CO(h^{1+\frac{\nu}{2}}). \end{aligned} \tag{6.39}$$

On the other hand, using again (6.33), we also have

$$\begin{aligned} w(x+y,t-h) &= Q_{x,t}(y,-h) + \tfrac{1}{2}\langle S_{x,t} y, y\rangle + O(|y|^2 + h)^{1+\frac{\nu}{2}}, \\ w(x+y,t) &= Q_{x,t}(y,0) + \tfrac{1}{2}\langle S_{x,t} y, y\rangle + O(|y|^2 + h)^{1+\frac{\nu}{2}}, \end{aligned}$$

which implies, with (6.34), that

$$\begin{aligned} w(x+y,t-h) - w(x+y,t) &= Q_{x,t}(y,-h) - Q_{x,t}(y,0) + CO(|y|^{2+\nu} + h^{1+\frac{\nu}{2}}) \\ &= -w_t(x,t)h + CO(|y|^{2+\nu} + h^{1+\frac{\nu}{2}}). \end{aligned} \tag{6.40}$$

Finally, (6.39) and (6.40) show that, for every $x, x+y \in B(0,R)$, with $0 < |y| < \tilde{r}$ and $t \in I$, $0 < h < \tilde{r}^2$, we have

$$h|w_t(x+y,t) - w_t(x,t)| \le CO(|y|^{2+\nu} + h^{1+\frac{\nu}{2}}).$$

Choosing $h = |y|^2$ we get, for every $x, x+y \in B(0,R)$, with $0 < |y| < \tilde{r}$,

$$|w_t(x+y,t) - w_t(x,t)| \le CO(|y|^{1+\nu}). \tag{6.41}$$

(iii) As by hypothesis $|P_t(x,t)| \leq C$, for every $x \in B(0,R)$ and $t \in I$, then as $w_t(x,t) = P_t(x,t)$, we have, for every $x, x+y \in B(0,R)$ and $t, t+h \in I$, with $|y| \geq \tilde{R}$ or $2h \geq \tilde{r}^2$,

$$|w_t(x+y,t+h) - w_t(x,t)| \leq 2C \leq 2C\Big(\frac{|y|^{1+\nu}}{\tilde{r}^{1+\nu}} + \frac{(2h)^{\frac{\nu}{2}}}{\tilde{r}^{\frac{\nu}{2}}}\Big). \tag{6.42}$$

(iv) Inequalities (6.38), (6.41), and (6.42) show (5.15), where $\tilde{C}$ depends only on $\tilde{r}$.

5. We prove that $w \in C^2_x(B(0,R) \times I)$ and satisfies (5.16).

(i) An estimate of $||S_{x,t} - S_{y,t}||$, for $|x-y|$ small. Because e is a unit vector of $\mathbb{R}^N$, $x \in B(0,R)$, $t \in I$, and $0 < \lambda < \tilde{r}$, (6.33) implies that

$$\begin{aligned} &w(x+\lambda e,t) + w(x-\lambda e,t) - 2w(x,t) \\ &= Q_{x,t}(\lambda e,0) + Q_{x,t}(-\lambda e,0) - 2w(x,t) + \lambda^2 \langle S_{x,t}e,e\rangle + CO(\lambda^{2+\nu}) \\ &= \lambda^2 \langle S_{x,t}e,e\rangle + CO(\lambda^{2+\nu}). \end{aligned} \tag{6.43}$$

Let $x, y \in B(0,R)$, with $0 < 2|x-y| < \tilde{r}$ and $t \in I$; we set $\lambda = |x-y|$, $\tau = \frac{x+y}{2} \in B(0,R)$, and $z = \frac{x-y}{2}$; we show, using (6.33), then (6.34), that

$$\begin{aligned} w(x+\lambda e,t) &+ w(x-\lambda e,t) - 2w(x,t) \\ &= Q_{\tau,t}(z+\lambda e,0) + Q_{\tau,t}(z-\lambda e,0) - 2Q_{\tau,t}(z,0) + 0(\lambda^{2+\nu}) \\ &= \lambda^2 \langle S_{\tau,t}e,e\rangle + O(\lambda^{2+\nu}), \end{aligned} \tag{6.44}$$

and we have the same inequality with x replaced by y. Inequalities (6.43) and (6.44) with x then y, show that

$$\langle S_{\tau,t}e,e\rangle = \langle S_{x,t}e,e\rangle + CO(\lambda^\nu) = \langle S_{y,t}e,e\rangle + CO(\lambda^\nu).$$

Therefore, for every $x, y \in B(0,R)$, with $0 \leq |x-y| < \dfrac{\tilde{r}}{2}$ and $t \in I$,

$$||S_{x,t} - S_{y,t}|| = CO(|x-y|^\nu). \tag{6.45}$$

(ii) We show that $w \in C^2_x(B(0,R) \times I)$ with $D^2_x w(x,t) = S_{x,t}$. Let $x, y \in B(0,R)$, with $0 < 2|x-y| < \tilde{r}$, $t \in I$, and e a unit vector of $\mathbb{R}^N$ and $0 < \lambda < \tilde{r}$; (6.33) then (6.34) imply that

$$\begin{aligned} &\lambda \langle D_x w(x,t) - D_x w(y,t), e\rangle \\ &= \big(Q_{x,t}(\lambda e,0) - w(x,t)\big) - \big(Q_{y,t}(\lambda e,0) - w(y,t)\big) \\ &= w(x+\lambda e,t) - w(x,t) - w(y+\lambda e,t) + w(y,t) \\ &\quad + \langle (S_{x,t} - S_{y,t})e,e\rangle + O(\lambda^{2+\nu}). \end{aligned} \tag{6.46}$$

We still use (6.33) at the point (τ, t), where $\tau = \frac{x+y}{2}$, with $\lambda = |x-y|$ and get

$$w(x+\lambda e, t) - w(x,t) + w(y,t) - w(y+\lambda e, t) = \lambda \langle S_{\tau,t}(x-y), e\rangle + O(\lambda^{2+\nu}). \tag{6.47}$$

(iii) Therefore, combining inequalities (6.45), (6.46), and (6.47), we have proved that, for every $x, y \in B(0,R)$ with $0 \le |x-y| < \frac{\tilde{r}}{2}$, and for every $t \in I$, the following inequality holds:

$$|D_x w(y,t) - D_x w(x,t) - S_{x,t}(y-x)| \le \tilde{C}|x-y|^{1+\nu}. \tag{6.48}$$

That shows that w is twice differentiable in space in (x,t), with $D_x^2 w(x,t) = S_{x,t}$.

(iv) An estimate of $||S_{x,t+h} - S_{x,t}||$, for h small. Let e be a unit vector of $\mathbb{R}^N$, $x \in B(0,R)$, $0 < h < \tilde{r}^2$, and $t, t+h \in I$; (6.43) shows that

$$h \langle (S_{x,t+h} - S_{x,t})e, e\rangle \quad = \quad g(\sqrt{h}e, 0) - g(\sqrt{h}e, -h), \tag{6.49}$$

where $g(y,s) = w(x+y, t+h+s) + w(x-y, t+h+s) - 2w(x, t+h+s)$. It is not difficult, using (6.33), to prove that, for every $(y,s) \in Q_{\tilde{r}}$,

$$g(y,s) = \langle S_{x,t+h}y, y\rangle + CO(|y|^{2+\nu} + |s|^{1+\frac{\nu}{2}}).$$

That implies that $g(\sqrt{h}e, 0) - g(\sqrt{h}e, -h) = CO(h^{\frac{\nu}{2}})$, and thus we have, for every $x \in B(0,R)$ and $t, t+h \in I$, with $0 < h < \tilde{r}^2$,

$$\langle (S_{x,t+h} - S_{x,t})e, e\rangle = CO(h^{\frac{\nu}{2}}). \tag{6.50}$$

(v) Using that, by hypothesis, $||S_{x,t}|| \le C$, for every $x \in B(0,R)$ and $t \in I$, then as $D_x^2 w(x,t) = S_{x,t}$, we have, for every $x, y \in B(0,R)$ and $t, t+h \in I$, with $2|x-y| \ge \tilde{r}$ or $h \ge \tilde{r}^2$,

$$||D_x^2 w(x,t) - D^2 w(y,s)|| \le 2C\Big(\frac{2^\nu |x-y|^\nu}{\tilde{r}^\nu} + \frac{h^{\frac{\nu}{2}}}{\tilde{r}^{\frac{\nu}{2}}}\Big). \tag{6.51}$$

(vi) Inequalities (6.45), (6.50), and (6.51) show (5.16), where $\tilde{C} > 0$ depends only on $\tilde{r}$. That ends the proof of Proposition 5.2.

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