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ON A VARIATIONAL PROBLEM INVOLVING CRITICAL SOBOLEV GROWTH IN DIMENSION FOUR

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Abstract. In this paper we consider the following nonlinear elliptic problem: $-\Delta u = Ku^3$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$, where K is a positive function and Ω is a bounded domain of R^4. We prove a version of the Morse lemma at infinity for this problem, which allows us to describe the critical points at infinity of the associated variational problem. Using a topological argument, we are able to prove an existence result.

1. Introduction

Let Ω be a bounded and regular domain of $\mathbb{R}^n$, $n \geq 3$. We consider the following nonlinear elliptic problem:

$$(P) \qquad \begin{cases} -\Delta u &= \frac{n-2}{4(n-1)} K(x) u^{(n+2)/(n-2)}, \quad u > 0 \quad \text{in} \quad \Omega \\ u &= 0 \qquad\qquad \text{on} \quad \partial\Omega, \end{cases}$$

where K is a C^3, positive function on $\overline{\Omega}$.

The interest in this equation comes from its resemblance to the scalar curvature problem in differential geometry, which consists in finding suitable conditions on a given function K defined on M such that K is the scalar curvature for a metric $\tilde{g}$ conformally equivalent to g, where (M, g) is an n-dimensional Riemannian manifold without boundary.

The special nature of problem (P) appears when we consider it from the variational viewpoint. The Euler functional associated to (P) does not satisfy the Palais-Smale condition. This means that there exist noncompact sequences along which the functional is bounded and its gradient goes to zero. This is due to the noncompactness of the embedding $H_0^1(\Omega) \hookrightarrow L^{2n/(n-2)}(\Omega)$. In the case of manifolds without boundary, this problem has been widely studied in various works (see for example the monographs [3] and [17] and

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the references therein). A phenomenon appears in dimension $n \geq 5$, due to the fact that the self-interaction of the functions failing the Palais-Smale condition dominates the interaction of two masses of those functions [6], while in dimension 3 the reverse happens [8]. In dimension 4, we have a balance phenomenon; that is, the self-interaction and the interaction are of the same size [11].

In contrast with manifolds, there are few known results of (P). In dimension greater than 4, E. Hebey [15] studied the case where Ω is a ball and $K = K(|x|)$; see also [16] for the dimension 4.

In this paper, we will focus our attention on the case where $n = 4$ and we will characterize the critical points at infinity of the associated variational problem. We will use this characterization in order to give some existence results. The same problem was studied in [11] but on a compact Riemannian manifold without boundary. The new fact here comes from the boundary since the regular part of the Green's function goes to infinity, and therefore it dominates the ΔK (see the expansion of J below). The difficulty here is to control the flow lines near the boundary. In [3], T. Aubin showed that any concentration point in $\partial\Omega$ of a sequence of subcritical solutions has to satisfy $\partial K/\partial\nu \geq 0$ (see Proposition 6.44). Thanks to (H_2), we can prove in our situation that the boundary does not make any contribution to the existence of a critical point at infinity. For this effect, near the boundary, we extend the vector field defined in [13], which allows us to control the minimal distance to the boundary (along the flow lines generated by this vector field, the minimal distance to the boundary is an increasing function if it is small enough (see Theorem 3.1 below)). Furthermore, if all the points are in a compact set of Ω, the regular part of the Green's function becomes bounded and we can extend the result of [11] in our situation.

In order to state our results, we need to introduce some notation, recall some known facts, and state the assumptions that we are using in our results. This problem has a variational structure. The related functional is

$$J(u) = \Big(\int_\Omega Ku^4 \Big)^{-1/2}$$

defined on $\Sigma = \{u \in H_0^1(\Omega) : |u|^2_{H_0^1} := \int_\Omega |\nabla u|^2 = 1\}$. As we said, the Palais-Smale condition fails for J on $\Sigma^+ = \{u \in \Sigma : u \geq 0\}$. Its failure has been analyzed throughout the works of [14], [18], and [20]. This analysis is carried out here with virtually no change. It has led to the characterization of the sequences failing the Palais-Smale condition. In order to describe this characterization, we need to introduce some notation.

Let, for $a \in \Omega$ and $\lambda > 0$ given,

$$\delta_{(a,\lambda)}(x) = \sqrt{8}\frac{\lambda}{1+\lambda^2|x-a|^2}; \tag{1.1}$$

$\delta_{(a,\lambda)}$ is the family of solutions of the Yamabe problem on $\mathbb{R}^4$.
Let P be the projection from $H^1(\Omega)$ onto $H_0^1(\Omega)$; that is, $u = Pf$ is the solution of

$$\Delta u = \Delta f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega.$$

Let, for $\varepsilon > 0$ and $p \in N^*$,

$$\begin{aligned} V(p,\varepsilon) = \Big\{ & u \in \Sigma : \exists\, (a_1,\dots,a_p) \in \Omega^p\,,\, \exists\,(\lambda_1,\dots,\lambda_p) \in ((\varepsilon^{-1},+\infty))^p, \\ & \exists\,(\alpha_1,\dots,\alpha_p) \in ((0,+\infty))^p \text{ such that } \lambda_i d(a_i,\partial\Omega) > \varepsilon^{-1}, \\ & \Big| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \Big|_{H_0^1} < \frac{\varepsilon}{p}, \Big| \frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} - 1 \Big| < \varepsilon,\, \varepsilon_{ij} < \varepsilon \Big\}, \end{aligned} \tag{1.2}$$

where, for $i \neq j$, $\varepsilon_{ij}^{-1} = (\frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i\lambda_j|a_i - a_j|^2)$. The failure of the Palais-Smale condition can be described as follows.

Proposition 1.1. *Assume that J has no critical point in Σ^+. Let $(u_k) \in \Sigma^+$ be a sequence such that $(\partial J(u_k))$ tends to zero and $(J(u_k))$ is bounded. Then, after possibly having extracted a subsequence, there exists $p \in N^*$, a sequence (ε_k), ε_k tending to zero, such that $u_k \in V(p,\varepsilon_k)$. (Here and in the sequel, ∂J denotes the gradient of J.)*

Thinking of these sequences which do not satisfy the Palais-Smale condition as "critical points," a natural idea is to try to find suitable parameters in order to complete a Morse lemma at infinity. For manifolds without boundary, this program has been completed in [6] and [11]. We would here extend the proof of the existence of a Morse lemma at infinity to the case of Dirichlet boundary conditions. We introduce the following parametrization of the set $V(p,\varepsilon)$. If a function u belongs to $V(p,\varepsilon)$ then, for $\varepsilon > 0$ small enough, the minimization problem

$$\min \Big\{ \Big| u - \sum_{i=1}^p \alpha_i P\delta_{(a_i,\lambda_i)} \Big|_{H_0^1}, \alpha_i > 0, a_i \in \Omega, \lambda_i > 0 \Big\}$$

has a unique solution, up to permutation (see Proposition 7 of [7]). Therefore, for $\varepsilon > 0$ sufficiently small, any u in $V(p,\varepsilon)$ can be uniquely written as

$$u \;=\; \sum_{i=1}^{p} \alpha_i P\delta_{(a_i,\lambda_i)} + v,$$

where v satisfies the conditions

$$(V_0) \qquad \begin{cases} (v, P\delta_{(a_i,\lambda_i)})_{H_0^1(\Omega)} = (v, \frac{\partial}{\partial\lambda_i} P\delta_{(a_i,\lambda_i)})_{H_0^1(\Omega)} = 0, \\ (v, \frac{\partial}{\partial a_i} P\delta_{(a_i,\lambda_i)})_{H_0^1(\Omega)} = 0 \end{cases} \tag{1.3}$$

and the α_i's satisfy

$$\frac{\alpha_i^2 K(a_i)}{\alpha_j^2 K(a_j)} \in [1-\varepsilon, 1+\varepsilon] \quad \forall i, \quad \forall j.$$

We denote by G the Green's function of the Laplacian with Dirichlet boundary condition on Ω and by H its regular part; that is,

$$\begin{array}{rcl} G(x,y) & = & |x-y|^{2-n} - H(x,y) \quad \text{for} \quad (x,y) \in \Omega^2, \\ \Delta_x H & = & 0 \;\text{ in }\; \Omega^2, \quad G = 0 \quad \text{on} \quad \partial(\Omega^2). \end{array}$$

For $q \in N^*$ and $x = (x_1, \ldots, x_q) \in \Omega^q$, such that $x_i \neq x_j \; \forall i \neq j$, we denote by $M(x) = (m_{ij})_{1\leq i,j\leq q}$ the matrix defined by

$$\begin{aligned} m_{ii} &= -\frac{1}{3}\frac{\Delta K(x_i)}{K(x_i)^2} + 8\frac{H(x_i,x_i)}{K(x_i)}, \\ m_{ij} &= -8\frac{G(x_i,x_j)}{(K(x_i)K(x_j))^{1/2}} \quad \text{for} \;\; i \neq j. \end{aligned} \tag{1.4}$$

In this paper we will assume the following:
(H_1) K has only nondegenerate critical points $y_1,\ldots,y_m$ such that

$$-\frac{1}{3}\frac{\Delta K(y_i)}{K(y_i)} + 8H(y_i,y_i) \neq 0 \quad \forall\, i = 1, \ldots, m.$$

(H_2) For each $x \in \partial\Omega$

$$\frac{\partial K(x)}{\partial \nu} < 0,$$

where ν is the outward normal to Ω.

Since Ω is bounded, the boundary of Ω is a compact set, and therefore there exists a positive constant c such that

$$\frac{\partial K}{\partial \nu} < -c < 0 \quad \text{ on } \partial\Omega.$$

For the sake of simplicity, we assume that $y_1,\dots,y_{m_1}$ are all the critical points of K with $-\frac{1}{3}\Delta K(y_i)/K(y_i)+8H(y_i,y_i)>0$.
(H_3) For any s-tuplet $\tau_s=(i_1,\dots,i_s)\in\{1,\dots,m_1\}^s$ with $i_j\neq i_q$ for $j\neq q$, $M(\tau_s)=M((y_{i_1},\dots,y_{i_s}))$ (defined by (1.4)) is nondegenerate.

Our main results are the following.

Theorem 1.2. *Under assumptions (H_1), (H_2), and (H_3), let $\varepsilon>0$ sufficiently small be given; for each $u=\sum_{i=1}^p\alpha_iP\delta_{(a_i,\lambda_i)}+v$ in $V(p,\varepsilon)$, after a change of variables $(a_i,\lambda_i,v)\to(a_i',\lambda_i',V)$, where V belongs to a neighborhood of zero in a fixed Hilbert space, one can write*

$$J(\sum_{i=1}^p\alpha_iP\delta_{(a_i,\lambda_i)}+v)=J(\sum_{i=1}^p\alpha_iP\delta_{(a_i',\lambda_i')})+|V|^2.$$

Furthermore, if each a_i belongs to a neighborhood of a critical point y_{j_i} such that $j_i\neq j_k$ for $i\neq k$ and $M(Y)$ is positive definite, where $Y=(y_{j_1},\dots,y_{j_p})$, then we can find another change of variables $(a_i,\lambda_i)\to(a_i',\lambda_i')$ such that, for η a fixed small constant,

$$J\Big(\sum_{i=1}^p\alpha_iP\delta_{(a_i,\lambda_i)}\Big)=\Psi(\alpha,a',\Lambda')$$
$$:=\frac{S_4^{1/2}\sum_{i=1}^p\alpha_i^2}{(\sum_{i=1}^p\alpha_i^4K(a_i'))^{1/2}}\big(1+(c_1'-\eta)^T\Lambda'M(Y)\Lambda'\big),$$

where $\alpha=(\alpha_1,\dots,\alpha_p)$, $a'=(a_1',\dots,a_p')$, ${}^T\Lambda'=(1/\lambda_1',\dots,1/\lambda_p')$, c_1' is a positive constant, and $S_4=\int_{\mathbb{R}^4}\delta_{(o,1)}^4$.

The next result will characterize the critical points at infinity. Recall that the critical points at infinity are the orbits of the gradient flow of J which remain in $V(p,\varepsilon(s))$, where $\varepsilon(s)$ is a given function such that $\varepsilon(s)$ goes to zero when s goes to $+\infty$ (see [5]). Furthermore, its Morse index will be defined as the Morse index of $\Psi(\alpha,a',\Lambda')+|V|^2$ (see Theorem 1.2).

Theorem 1.3. *Under assumptions (H_1), (H_2), and (H_3), the only critical points at infinity in $V(p,\varepsilon)$ correspond to $\sum_{i=1}^pK(y_{j_i})^{-1/2}P\delta_{(y_{j_i},\infty)}$, where $p\in N^*$ and the points y_{j_i} are critical points of K satisfying $M(y_{j_1},\dots,y_{j_p})$ is positive definite. Such a critical point at infinity has a Morse index equal to $5p-1-\sum_{i=1}^pk_{j_i}$, where $k_{j_i}=index(K,y_{j_i})$.*

Theorem 1.4. *Under assumptions* (H_1), (H_2), *and* (H_3), *if*

$$1 \neq \sum_{s=1}^{m_1} \sum_{\tau_s=(i_1,\dots,i_s)/M(\tau_s)>0} (-1)^{5s-1-\sum_{j=1}^{s} k_{i_j}},$$

where $k_{i_j} = index(K, y_{i_j})$, *then (P) has a solution.*

Next, we state another kind of result for problem (P). Our approach follows closely the idea developed in Aubin-Bahri [4]. For this effect, we will suppose that the following assumptions hold:
(H_4) K has only nondegenerate critical points y_0, y_1,…,y_m such that

$$K(y_0) \geq K(y_1) \geq \cdots \geq K(y_l) > K(y_{l+1}) \geq \cdots \geq K(y_m)$$

with

$$-\frac{1}{3}\frac{\Delta K(y_i)}{K(y_i)} + 8H(y_i, y_i) > 0 \quad \text{for } i \leq l \quad \text{and}$$
$$-\frac{1}{3}\frac{\Delta K(y_i)}{K(y_i)} + 8H(y_i, y_i) < 0 \quad \text{for } i > l.$$

Let Z be a pseudo gradient of K of Morse-Smale type (that is, the intersection of the unstable and stable manifolds of critical points of K are transverse). Set

$$X = \overline{\bigcup_{0\leq i\leq l} W_s(y_i)},$$

where $W_s(y)$ is the stable manifold of y for Z.
(H_5) Assume that X is not contractible and denote by k the dimension of the first nontrivial reduce homological group.
(H_6) There exists a positive constant $\overline{c} < K(y_l)$ such that X is contractible in $K^{\overline{c}} = \{x \in \Omega : K(x) \geq \overline{c}\}$.

We then have the following result:

Theorem 1.5. *Under the assumptions* (H_2), (H_4), (H_5), *and* (H_6), *if there exists a constant* c_0 *independent of* K *such that* $K(y_0)/\overline{c} \leq 1 + c_0$, *then (P) has a solution of an augmented Morse index* $\geq k$.

The remainder of this paper is organized as follows: In Section 2, we give the expansions of the functional J and its gradient. In Section 3 we construct a pseudo gradient W, which will be useful in the proof of Theorem 1.2. In Section 4 we give the proofs of our results. Some technical estimates, which are useful in Sections 2 and 3, are given in the Appendix.

2. Expansions of the functional and its gradient

In this section, we will give the expansion of $J(u)$, $\big(\partial J(u), \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\big)$, and $\big(\partial J(u), \lambda_i\frac{\partial P\delta_i}{\partial \lambda_i}\big)$, where u belongs to $V(p,\varepsilon)$ and δ_i denotes $\delta_{(a_i,\lambda_i)}$.

Proposition 2.1. [5] *There exists $\varepsilon_0 > 0$ such that for any $u = \sum_{i=1}^p \alpha_i P\delta_i + v \in V(p,\varepsilon)$, $\varepsilon < \varepsilon_0$, v satisfying (V_0), we have*

$$\begin{aligned} J(u) =& \frac{S_4^{1/2}\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^4 K(a_i))^{1/2}}\Big[1 + \frac{2w_3}{S_4\sum_{i=1}^p K(a_i)^{-1}}\left(\sum_{i=1}^p \frac{-\Delta K(a_i)}{3\lambda_i^2 K(a_i)^2}\right.\\ &\left. +8\sum_{i=1}^p \frac{H(a_i,a_i)}{\lambda_i^2 K(a_i)} - \sum_{i\neq j}\frac{8}{(K(a_i)K(a_j))^{\frac{1}{2}}}\Big(\varepsilon_{ij} - \frac{H(a_i,a_j)}{\lambda_i\lambda_j}\Big)\right) - f(v)\\ &+ \frac{1}{\sum_{i=1}^p \alpha_i^2 S_4}Q(v,v) + o\left(\sum \varepsilon_{kr} + \sum\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^2} + |v|^2_{H_0^1}\right)\Big] \end{aligned}$$

where w_3 is the volume of the sphere S^3, $S_4 = \int_{R^4}\delta^4_{(o,1)}$,

$$Q(v,v) = |v|^2_{H_0^1} - \frac{3\sum_{i=1}^p \alpha_i^2}{\sum_{i=1}^p \alpha_i^4 K(a_i)}\int_\Omega K\big(\sum_{i=1}^p \alpha_i P\delta_i\big)^2 v^2,$$

$$f(v) = \frac{2}{\sum_{i=1}^p \alpha_i^4 K(a_i) S_4}\int_\Omega K\big(\sum_{i=1}^p \alpha_i P\delta_i\big)^3 v.$$

Furthermore, if each a_i is near a critical point (which we will denote y_i) and the y_i's are different, this expansion becomes

$$\begin{aligned} J(u) =& \frac{S_4^{\frac{1}{2}}\sum_{i=1}^p \alpha_i^2}{(\sum_{i=1}^p \alpha_i^4 K(a_i))^{1/2}}\Big[1 + \frac{2w_3}{S_4\sum_{i=1}^p K(y_i)^{-1}}\\ &\times\Big(\sum_{i=1}^p\Big(\frac{-\Delta K(y_i)}{3K(y_i)^2} + 8\frac{H(y_i,y_i)}{K(y_i)}\Big)\frac{1}{\lambda_i^2} - \sum_{i\neq j}\frac{8G(y_i,y_j)}{(K(y_i)K(y_j))^{1/2}}\frac{1}{\lambda_i\lambda_j}\Big)\\ &- f(v) + \frac{1}{S_4\sum\alpha_i^2}Q(v,v) + o\Big(\sum\frac{1}{\lambda_i^2} + \sum\varepsilon_{ij} + |v|^2_{H_0^1}\Big)\Big]. \end{aligned}$$

A natural improvement of Proposition 2.1 is obtained by taking care of the v-part, in order to show that it can be neglected with respect to the concentration phenomenon. For this purpose, let us observe that

$$\int K(\sum_{i=1}^p \alpha_i P\delta_i)^2 v^2 = \sum_{i=1}^p \alpha_i^2 K(a_i)\int P\delta_i^2 v^2 + o(|v|^2_{H_0^1}).$$

Using the fact that $\alpha_i^2K(a_i)/(\alpha_j^2K(a_j)) = 1+o(1)$, we have that the function $Q(v,v)$ is equal to

$$|v|^2_{H^1_0} - 3\sum_{i=1}^{p}\int P\delta_i^2v^2 + o(|v|^2_{H^1_0}).$$

Therefore, $Q(v,v)$ is a positive-definite quadratic form; see Proposition 3.1 of [5] (up to minor modification to take into account the projection P). There follows this proposition:

Proposition 2.2. [5] *There exists a C^1 map which, to each (α,a,λ) satisfying $\sum_{i=1}^{p}\alpha_iP\delta_{(a_i,\lambda_i)} \in V(p,\varepsilon)$, with ε small enough, associates $\overline{v}=\overline{v}(\alpha,a,\lambda)$ such that $\overline{v}$ is unique, minimizes $J(\sum_{i=1}^{p}\alpha_iP\delta_{(a_i,\lambda_i)}+v)$ with respect to v satisfying (V_0), and satisfies the following estimate:*

$$|\overline{v}|_{H^1_0} \le c|f| \le c\sum_{i=1}^{p}\Big(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_id_i)^2} + \sum_{i\ne j}\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/2}\Big).$$

Proposition 2.3. *For $u=\sum_{i=1}^{p}\alpha_iP\delta_i \in V(p,\varepsilon)$, we have the following expansion:*

$$\begin{aligned}\big(\partial J(u),\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) =&4w_3J(u)\Big[\frac{\alpha_i}{3}\frac{\Delta K(a_i)}{K(a_i)\lambda_i^2}(1+o(1)) - 8\alpha_i\frac{H(a_i,a_i)}{\lambda_i^2}\big(1+o(1)\big)\\ &- 8\sum_{j\ne i}\alpha_j\Big(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{H(a_i,a_j)}{\lambda_i\lambda_j}\Big)\big(1+o(1)\big) + R_2\Big],\end{aligned}$$

where w_3 is the volume of the sphere S^3 and R_2 satisfies

$$R_2 = O\Big(\sum_{k=1}^{p}\frac{\log(\lambda_kd_k)}{(\lambda_kd_k)^4} + \sum_{k\ne r}\varepsilon_{kr}^{7/4}(\log\varepsilon_{kr}^{-1})^{7/8} + \sum_{j=1}^{p}\frac{1}{\lambda_j^{7/3}}\Big).$$

Proof. We have

$$\partial J(u) = 2J(u)\big[u + J(u)^2\Delta^{-1}\big(Ku^3\big)\big]. \tag{2.1}$$

Thus

$$\begin{aligned}&\big(\partial J(u),\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) \\ &= 2J(u)\Big[\sum\alpha_j\big(P\delta_j,\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) - J(u)^2\int K\big(\sum\alpha_jP\delta_j\big)^3\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\Big].\end{aligned} \tag{2.2}$$

Observe that

$$\Big(\sum_{j=1}^{p}\alpha_j P\delta_j\Big)^3=\sum_{j=1}^{p}\big(\alpha_j P\delta_j\big)^3+3\sum_{j\neq i}\big(\alpha_i P\delta_i\big)^2\alpha_j P\delta_j \tag{2.3}$$

$$+\,O\Big[\sum_{j\neq i}\Big(\alpha_i P\delta_i\Big)\Big(\alpha_j P\delta_j\Big)^2\Big]+O\Big[\sum_{k\neq j,j\neq i}\Big(\alpha_j P\delta_j\Big)^2\Big(\alpha_k P\delta_k\Big)\Big].$$

Combining (2.2), (2.3), Lemmas 5.5,...,5.9 and the facts that $|\lambda_i\partial P\delta_i/\partial\lambda_i|\leq c\delta_i$, $P\delta_k\leq\delta_k$, and $J(u)^2\alpha_j^2K(a_j)=1+o(1)\ \forall\ j=1,\dots,p$, Proposition 2.3 follows. □

Proposition 2.4. *For* $u=\sum_{i=1}^{p}\alpha_i P\delta_i\in V(p,\varepsilon)$, *we have the following expansion:*

$$\Big(\partial J(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big)=16J(u)w_3\Big[-c\alpha_i^3J(u)^2\frac{\nabla K(a_i)}{\lambda_i}(1+o(1))$$
$$+\frac{c'\alpha_i}{\lambda_i^3}\frac{\partial H(a_i,a_i)}{\partial a_i}\big(1+o(1)\big)+O\Big(\frac{1}{\lambda_i^2}+\frac{\log(\lambda_i d_i)}{(\lambda_i d_i)^4}+\sum_{j\neq i}\varepsilon_{ij}\Big)\Big].$$

We can ameliorate this expansion, and we obtain

$$\Big(\partial J(u),\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\Big)=16J(u)w_3\Big[-c\alpha_i^3J(u)^2\frac{\nabla K(a_i)}{\lambda_i}(1+o(1))$$
$$+\frac{c'\alpha_i}{\lambda_i^3}\frac{\partial H(a_i,a_i)}{\partial a_i}\big(1+o(1)\big)+2c'\sum_{j\neq i}\alpha_j\Big(\frac{1}{\lambda_i}\frac{\partial\varepsilon_{ij}}{\partial a_i}-\frac{1}{\lambda_i^2\lambda_j}\frac{\partial H(a_i,a_j)}{\partial a_i}\Big)$$
$$\times\Big(1-J(u)^2(\alpha_j^2K(a_j)+\alpha_i^2K(a_i))\Big)+O(\frac{1}{\lambda_i^2})\Big]+R_3,$$

where R_3 *satisfies*

$$R_3=O\Big(\sum_k\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^4}+\frac{1}{\lambda_k^{\frac{7}{3}}}+\sum_{k\neq j}\varepsilon_{kj}^{\frac{7}{4}}(\log\varepsilon_{kj}^{-1})^{\frac{7}{8}}+\sum_{j\neq i}\lambda_j|a_i-a_j|\varepsilon_{ij}^{\frac{5}{2}}\Big). \tag{2.4}$$

Proof. As in the proof of Proposition 2.3 , we get (2.2) but with $\lambda_i\partial P\delta_i/\partial\lambda_i$ changed by $\lambda_i^{-1}\partial P\delta_i/\partial a_i$. Thus, using lemmas 5.10,...,5.14, the proposition follows. □

3. Construction of a pseudo gradient

In this section we construct a pseudo gradient W near infinity as in Proposition A.2 of [6], Lemma 3.3 of [11], and Theorem 4.1 of [13]. In the general case, to decrease the functional J, we use $-\partial J$ as pseudo gradient. In this paper, we need other claims to be satisfied. For this effect, we will define another vector field which decreases J and satisfies other properties (see Theorem 3.1 below).

Observe that for $u = \sum_{i=1}^p \alpha_i P\delta_i + v \in V(p,\varepsilon)$, using Proposition 2.2, after a change of variables, we can write

$$J(u) = J(\sum_{i=1}^p \alpha_i P\delta_i + \overline{v}) + |V|^2.$$

Therefore, we need to define a vector field in $\{\sum_{i=1}^p \alpha_i P\delta_i + \overline{v} \in V(p,\varepsilon)\}$. The difficulty here comes from the boundary. We need a new technical point showing the following:

1) Near the boundary, the regular part of the Green's function goes to infinity, and therefore it dominates ΔK. As in [13], we want to move the points along the inward normal to the boundary. Another term is added in the expansion of the gradient of J which is $\nabla K(a_i)/\lambda_i$ (see Proposition 2.4). Using (H_2), this term allows us to have a better lower bound for the pseudo gradient.

2) We can construct W so that, on decreasing flow lines, the minimal distance to the boundary only increases if it is small enough.

3) If all the points are in a compact set of Ω, the regular part of the Green's function becomes bounded, and therefore our problem becomes as in [11].

The second property requires a careful study of the behavior of the Green's function and its regular part near the boundary.

We begin by the following main result.

Theorem 3.1. *There exists a pseudo gradient W so that the following holds. There is a constant $c>0$ independent of $u = \sum_{i=1}^p \alpha_i P\delta_i$ in $V(p,\varepsilon)$ so that*

i) $$\big(-\partial J(u), W\big) \geq c\Big(\sum_i \frac{1}{\lambda_i^2} + \sum_i \frac{1}{(\lambda_i d_i)^3} + \sum_{k\neq r} \varepsilon_{kr}^{3/2} + \sum_i \frac{|\nabla K(a_i)|}{\lambda_i}\Big)$$

ii)
$$\big(-\partial J(u+\overline{v}), W + \frac{\partial \overline{v}}{\partial(\alpha,a,\lambda)}(W)\big) \geq c\sum_i \Big(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^3} + \sum_{k\neq i} \varepsilon_{ki}^{\frac{3}{2}} + \frac{|\nabla K(a_i)|}{\lambda_i}\Big)$$

iii) $|W|$ *is bounded.*

iv) W *satisfies the Palais-Smale condition away from the critical points at infinity.*

Moreover, if we denote by $u(t) = \sum_{i=1}^{p} \alpha_i(t) P\delta_{(a_i(t),\lambda_i(t))}$ *the solution of* $\partial u(t)/\partial t = W(u(t))$, $u(0) = u$, *we have*

v) The minimal distance to the boundary, $d_i(t) = d(a_i(t), \partial\Omega)$, *only increases if it is small enough.*

vi) The $\lambda_i(t)$*'s are bounded if* $(a_1(t), \dots, a_p(t))$ *is away from a neighborhood of* $(y_{j_1}, \dots, y_{j_p})$ *where the* y_{j_i}*'s are critical points of* K *satisfying* $j_i \neq j_k$ *for* $i \neq k$ *and* $M(y_{j_1}, \dots, y_{j_p})$ *is positive definite.*

Before giving the proof of Theorem 3.1, we need to state two results which deal with two specific cases of Theorem 3.1. The proofs of these results will be given later. Let $\eta > 0$ be a constant small enough such that

(1) $\eta < (1/4) \min d(y_i, y_j)$ for $i \neq j$ where the y_k's are the critical points of K.

(2) $\eta < (1/4) \min d(y_i, \partial\Omega)$.

(3) $|\frac{\Delta K(a)}{3K(a)} - 8H(a,a)| > c > 0$ for any $a \in \cup B(y_i, 2\eta)$ (it is possible using (H_1)).

We then have the following propositions.

Proposition 3.2. *In* $V_1(p,\varepsilon) := \{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p,\varepsilon) : \forall\, i \ a_i \notin \cup_k B(y_k, \eta)\}$, *there exists a pseudo gradient* W_1 *so that the following holds. There is a constant* $c > 0$ *independent of* $u = \sum_{i=1}^{p} \alpha_i P\delta_i$ *in* $V_1(p,\varepsilon)$ *so that*

$$\big(-\partial J(u), W_1\big) \geq c\Big(\sum_i \frac{1}{\lambda_i} + \sum_i \frac{1}{(\lambda_i d_i)^3} + \sum_{k\neq r} \varepsilon_{kr}^{3/2}\Big).$$

The W_1 *is bounded. Moreover, using the notation of Theorem* 3.1, *the* $\lambda_i(t)$*'s are decreasing functions and the minimal distance to the boundary,* $d_i(t) = d(a_i(t), \partial\Omega)$, *is an increasing function if it is small enough.*

Proposition 3.3. *In* $V_2(p,\varepsilon) := \{u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V(p,\varepsilon) : \forall\, i \ a_i \in \cup_k B(y_k, 2\eta)\}$, *there exists a pseudo gradient* W_2 *so that the following holds. There is a constant* $c > 0$ *independent of* $u = \sum_{i=1}^{p} \alpha_i P\delta_i$ *in* $V_2(p,\varepsilon)$ *so that*

$$\big(-\partial J(u), W_2\big) \geq c\Big(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_i \frac{1}{\lambda_i^2} + \sum_{k\neq r} \varepsilon_{kr}\Big).$$

The W_2 *is bounded. Furthermore, using the notation of Theorem* 3.1, *the* $\lambda_i(t)$*'s are bounded if* $(a_1(t), \dots, a_p(t))$ *is away from a neighborhood of* $(y_{j_1},$

$\dots, y_{j_p})$ where the y_{j_i}'s are critical points of K satisfying $j_i \neq j_k$ for $i \neq k$ and $M(y_{j_1}, \dots, y_{j_p})$ is positive definite.

Proof of Theorem 3.1. We divide the set $\{1, \dots, p\}$ into two sets. The first contains the indices of the points far away from the critical points of K, and the other contains the indices of the points near the critical points of K. Let us define

$$\begin{aligned} B' &= \{i/a_i \notin \cup B(y_k, 2\eta)\}, \\ B &= B' \cup \{j \notin B' : \exists\, (i_1, \dots, i_r) \text{ such that } i_r \in B',\ i_1 = j, \\ &\qquad |a_{i_{k-1}} - a_{i_k}| < \frac{\eta}{p} \forall k \leq r\}. \end{aligned}$$

We observe that

1) For each $j \notin B$ we have $a_j \in \cup B(y_k, 2\eta)$.

2) The advantage of B is that if $i \in B$ and $j \notin B$, then $|a_i - a_j| \geq \eta/p$.

Now, we write u as

$$u = u_1 + u_2, \quad \text{where} \quad u_1 = \sum_{i\in B} \alpha_i P\delta_i \quad \text{and} \quad u_2 = \sum_{i \notin B} \alpha_i P\delta_i.$$

Observe that $u_1 \in V_1(card(B), \varepsilon)$. We then use the previous constructions as in Proposition 3.2 to u_1, which means we apply the previous constructions to the subpack of functions $u = \sum_{i=1}^{card(B)} \alpha_i P\delta_i$ forgetting the indices $j \notin B$. Let $W_1(u_1)$ be the vector field thus defined. The same argument can be repeated for u_2, which is in $V_2(p - card(B), \varepsilon)$, and we will denote by $W_2(u_2)$ the vector field thus defined. Define W as $W(u) = W_1(u_1) + W_2(u_2)$. Thus, we have

$$\begin{aligned} \big(-\partial J(u), W\big) &= \big(-\partial J(u), W_1 + W_2\big) \\ &\geq c \sum_{i\in B} \Big(\frac{1}{\lambda_i} + \frac{1}{(\lambda_i d_i)^3} + \sum_{k\in B} \varepsilon_{ki}^{3/2} + O(\sum_{k\notin B} \varepsilon_{ki})\Big) \\ &\quad + c \sum_{i\notin B} \Big(\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{\lambda_i^2} + \sum_{k\notin B} \varepsilon_{ki} + O(\sum_{k\in B} \varepsilon_{ki})\Big). \end{aligned} \tag{3.1}$$

Observe that, for $i \in B$ and $k \notin B$, $|a_i - a_k| \geq c$; thus, $\varepsilon_{ki} = o(\lambda_i^{-1})$. So claim i) of the theorem follows.

The proof of claim ii) is similar to the proof of Lemma 3.3 of [11] and Appendix 2 of [6]. The other conditions are satisfied by the definition of the vector field W. Thus our result follows. □

Next we will prove some technical results needed for the proof of Proposition 3.2.

For $u = \sum_{i=1}^p \alpha_i P\delta_i \in V_1(p,\varepsilon)$, we introduce the following condition: for $i \in \{1,\dots,p\}$

$$\frac{1}{2^{p+1}} \sum_{k\neq i} \varepsilon_{ki} \leq \sum_{j=1}^p \frac{H(a_i,a_j)}{\lambda_i\lambda_j}. \tag{3.2}$$

We divide the set $\{1,\dots,p\}$ into $T_1 \cup T_2$, where

$$T_1 = \{i \text{ such that } i \text{ satisfies (3.2)}\} \quad \text{and}$$

$$T_2 = \{i \text{ such that } i \text{ does not satisfy (3.2)}\}.$$

In T_2 , we order the λ_i's: $\lambda_{i_1} \leq \lambda_{i_2} \leq \cdots \leq \lambda_{i_s}$.

Lemma 3.4. *There exists a vector field X_1 defined on the variables λ_j, for $j \in T_2$, as follows:*

$$X_1 = -\sum_{k=1}^s 2^{k-1}\alpha_{i_k}\lambda_{i_k}\frac{\partial P\delta_{i_k}}{\partial\lambda_{i_k}} + m_2 \sum_{i\in T_2} \frac{\alpha_i}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\frac{\nabla K(a_i)}{|\nabla K(a_i)|},$$

where m_2 is a small constant. This vector field satisfies

$$\Big(-\partial J(u), X_1\Big) \geq c \sum_{i\in T_2}\Big(\sum_{j\neq i}\varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + \frac{1}{\lambda_i}\Big) + R_2,$$

where R_2 is defined in Proposition 2.3.

Proof. Using Proposition 2.3, we derive that

$$\begin{aligned} &\big(-\partial J(u), -\sum_{k=1}^s 2^{k-1}\alpha_{i_k}\lambda_{i_k}\frac{\partial P\delta_{i_k}}{\partial\lambda_{i_k}}\big) \\ &= 32 w_3 J(u)\sum_{k=1}^s\Big[-\sum_{j\neq i_k} 2^{k-1}\alpha_j\alpha_{i_k}\lambda_{i_k}\frac{\partial\varepsilon_{j i_k}}{\partial\lambda_{i_k}}(1+o(1)) \\ &-\sum_{j=1}^p 2^{k-1}\alpha_j\alpha_{i_k}\frac{H(a_j,a_{i_k})}{\lambda_j\lambda_{i_k}}(1+o(1)) + O(\frac{1}{\lambda_{i_k}^2}) + R_2\Big]. \end{aligned} \tag{3.3}$$

Observe that

$$-\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} = \varepsilon_{ij}\Big(1 - 2\frac{\lambda_j}{\lambda_i}\varepsilon_{ij}\Big). \tag{3.4}$$

Thus, if $\lambda_i \geq \lambda_j$

$$-2\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} - \lambda_j \frac{\partial \varepsilon_{ij}}{\partial \lambda_j} \geq -\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} \geq \frac{1}{2}\varepsilon_{ij}, \tag{3.5}$$

and for $j \in T_1$ and $i \neq j$,

$$\frac{\lambda_j}{\lambda_i}\varepsilon_{ij} = o(1). \tag{3.6}$$

Indeed, if $\frac{1}{2}d_j \leq d_i \leq 2d_j$, we use the fact that j satisfies (3.2) and $H(a_j, a_k) \leq (d_j d_k)^{-1}$ (see Lemma 5.17), and in the other case we use the inequality $|a_i - a_j| \geq \frac{1}{2}\max(d_i, d_j)$. Thus

$$\begin{aligned}&\big(-\partial J(u), -\sum_{k=1}^{s} 2^{k-1}\alpha_{i_k}\lambda_{i_k}\frac{\partial P\delta_{i_k}}{\partial \lambda_{i_k}}\big)\\ &\geq c\sum_{i\in T_2}\Big(\sum_{j\neq i}\varepsilon_{ij} - 2^p \sum_{j=1}^{p}\frac{H(a_i, a_j)}{\lambda_i\lambda_j} + O(\frac{1}{\lambda_i^2})\Big) + R_2.\end{aligned}$$

Since $i \in T_2$ and $H(a_i, a_i) \geq c d_i^{-2}$ (see Lemma 5.17), we get

$$\begin{aligned}&\big(-\partial J(u), -\sum_{k=1}^{s} 2^{k-1}\alpha_{i_k}\lambda_{i_k}\frac{\partial P\delta_{i_k}}{\partial \lambda_{i_k}}\big)\\ &\geq c\sum_{i\in T_2}\Big(\sum_{j\neq i}\varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} + O(\frac{1}{\lambda_i^2})\Big) + R_2.\end{aligned} \tag{3.7}$$

Using Proposition 2.4, we derive that

$$\begin{aligned}&\big(-\partial J(u), \sum_{i\in T_2}\frac{\alpha_i}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\frac{\nabla K(a_i)}{|\nabla K(a_i)|}\big)\\ &\geq c\sum_{i\in T_2}\frac{|\nabla K(a_i)|}{\lambda_i} + O\Big(\frac{1}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^3} + \sum_{j\neq i}\varepsilon_{ij}\Big).\end{aligned} \tag{3.8}$$

Using (H_2) and the fact that a_i is away from the critical points of K, we derive $|\nabla K(a_i)| > c$. Combining (3.7) and (3.8), and using the fact that m_2 is small, the lemma follows. □

We order all the $\lambda_i d_i$: $\lambda_1 d_1 \leq \lambda_2 d_2 \leq \cdots \leq \lambda_p d_p$. Let us define

$$I_1 = \{1\} \cup \{i \text{ such that } \forall\ k \leq i,\ c_1\lambda_k d_k \leq \lambda_{k-1}d_{k-1} \leq \lambda_k d_k\},$$

where c_1 is a constant chosen small enough.

Corollary 3.5. *If $T_2 \cap I_1 \neq \emptyset$, the vector field*
$$X_1' = X_1 + m_2 \sum_{i \in T_1} \frac{\alpha_i}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|}$$
(*where m_2 is a small constant*) *satisfies the conditions of Proposition* 3.2.

Proof. The lower bound of the estimate of Lemma 3.4 is limited to the indices of T_2; thus, if $T_2 \cap I_1 \neq \emptyset$, we can make the term $(\lambda_1 d_1)^{-2}$ appear in this lower bound, and therefore we can also make all the $(\lambda_i d_i)^{-2}$ appear in this formula. For $i \in T_1$ (that is, i satisfies (3.2)), we have
$$\frac{1}{2^{p+1}} \sum_{j \neq i} \varepsilon_{ij} \leq \sum_{j=1}^{p} \frac{H(a_i, a_j)}{\lambda_i \lambda_j} \leq \frac{p}{(\lambda_1 d_1)^2}. \tag{3.9}$$
Therefore, from $(\lambda_1 d_1)^{-2}$, we can make the term $\sum_{i \in T_1} \varepsilon_{ij}$ appear in this lower bound. Thus,
$$\Big(-\partial J(u), X_1 \Big) \geq c \sum_{i=1}^{p} \Big(\sum_{j \neq i} \varepsilon_{ij} + \frac{1}{(\lambda_i d_i)^2} \Big) + \sum_{i \in T_2} \frac{1}{\lambda_i} + R_2. \tag{3.10}$$
Using Proposition 2.4, we derive that (since $|\nabla K(a_i)| > c$)
$$\big(-\partial J(u), \sum_{i \in T_1} \frac{\alpha_i}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|}\big) \geq c \sum_{i \in T_1} \Big(\frac{1}{\lambda_i} + O\big(\frac{1}{(\lambda_i d_i)^3} + \sum_{j \neq i} \varepsilon_{ij}\big) \Big). \tag{3.11}$$
Combining (3.10) and (3.11) and using the fact that m_2 is small, the estimate of Proposition 3.2 follows.

For the other claims, observe that, along the flow lines generated by X_1', the derivative of the variables λ_i and a_i satisfies
$$\frac{\dot{\lambda}_{i_k}}{\lambda_{i_k}} = -2^{k-1} \text{ for } i_k \in T_2, \quad \dot{\lambda}_i = 0 \text{ for } i \notin T_2, \quad \lambda_i \dot{a}_i = m_2 \frac{\nabla K(a_i)}{|\nabla K(a_i)|}.$$
This implies that, along the flow lines, the λ_i's are decreasing functions. Furthermore, using (H_2), the distance $d_i = d(a_i, \partial\Omega)$ is an increasing function along the flow lines if it is small enough. The corollary follows. □

Now, let us define, for c_2 a fixed small constant
$$L = \{ j \in T_1 \text{ such that } \exists\, i \in T_1 \text{ such that } c_2 \max(d_i, d_j) \geq |a_i - a_j| \} \tag{3.12}$$
For $i \in L$, we denote i_0 the index such that
$$c_2 \max(d_i, d_{i_0}) \geq |a_i - a_{i_0}|. \tag{3.13}$$

Lemma 3.6. *If there exist two indices i and i_0 in T_1 and satisfying* (3.13), *then* (*we can assume that $\lambda_i \geq \lambda_{i_0}$*) *there exists a vector field X_3, defined on the variable λ_i as follows:*

$$X_3 = -\alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i}.$$

This vector field satisfies

$$\big(-\partial J(u), X_3\big) \geq c\big(\varepsilon_{ii_0} + \frac{1}{(\lambda_i d_i)^2}\big) + R_2 + O\big(\sum_{k\in T_2} \varepsilon_{ik}\big).$$

Proof. Using Proposition 2.3, we derive (3.3) (for $s = 1$). The terms with $j \in T_2$ can be seen like $O(\varepsilon_{ij})$. We are interesting now in the indices $j \in T_1$. Observe that, for $i, k \in T_1$, we have

$$\begin{aligned}\varepsilon_{ik} &= \frac{1}{\lambda_i\lambda_k|a_i - a_k|^2}\big(1 + O(\frac{1}{\lambda_i^2|a_i - a_k|^2} + \frac{1}{\lambda_k^2|a_i - a_k|^2})\big)\\ &= \frac{1}{\lambda_i\lambda_k|a_i - a_k|^2} + O(\frac{1}{(\lambda_1 d_1)^4} + \varepsilon_{ik}^2).\end{aligned} \tag{3.14}$$

Indeed, if $2d_i \leq d_k$ or $d_i \geq 2d_k$, we have $2|a_i - a_k| \geq \max(d_i, d_k)$, and the result follows. In the other case, i.e., $\frac{1}{2}d_k \leq d_i \leq 2d_k$, using that $i, k \in T_1$, we have, as in (3.6),

$$\varepsilon_{ik} = o\big((\frac{\lambda_i}{\lambda_k}) + (\frac{\lambda_k}{\lambda_i})\big),$$

and therefore

$$\begin{aligned}\frac{1}{\lambda_k^2|a_i - a_k|^2} &= \frac{\lambda_i}{\lambda_k}\frac{1}{\lambda_i\lambda_k|a_i - a_k|^2} \leq c\frac{\lambda_i}{\lambda_k}\varepsilon_{ik} \leq c\frac{\lambda_i}{\lambda_k}\frac{1}{(\lambda_i d_i)(\lambda_1 d_1)}\\ &= O\big(\frac{1}{(\lambda_1 d_1)^2}\big),\end{aligned} \tag{3.15}$$

and we use the same argument for $(\lambda_i|a_i - a_k|)^{-2}$. Using (3.4), (3.14), and (3.15), we obtain

$$\sum_{j\in T_1}\big(-\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} - \frac{H(a_i, a_j)}{\lambda_i\lambda_j}\big) = \sum_{j\in T_1}\frac{G(a_i, a_j)}{\lambda_i\lambda_j} + O\big(\frac{1}{(\lambda_1 d_1)^4} + \sum_{k\neq r}\varepsilon_{kr}^2\big). \tag{3.16}$$

Using the fact that the Green's function is positive and that $H(a_i, a_i) \leq d_i^{-2}$ (see Lemma 5.17), we have that

$$\big(\partial J(u), \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) \geq c\Big(-\frac{1}{(\lambda_i d_i)^2} - \frac{C}{\lambda_i^2} + \frac{G(a_i, a_{i_0})}{\lambda_i\lambda_{i_0}}\Big) + O\big(R_2 + \sum_{k\in T_2}\varepsilon_{ik}\big), \tag{3.17}$$

where i and i_0 are the indices satisfying (3.13). Using (3.13) and the fact that $\lambda_i \geq \lambda_{i_0}$, we have that

$$\frac{C}{\lambda_i^2} + \frac{1}{(\lambda_i d_i)^2} + \frac{H(a_i, a_{i_0})}{\lambda_i \lambda_{i_0}} \leq \frac{c_2^2}{\lambda_i \lambda_{i_0} |a_i - a_{i_0}|^2}. \tag{3.18}$$

Using (3.14), (3.17), and (3.18), the lemma follows. □

Corollary 3.7. *Assume that $T_2 \cap I_1 = \emptyset$ and there are two indices i and i_0 in I_1 satisfying* (3.13). *Thus, for M_3 a fixed large constant and m_4 a small constant, the vector field*

$$X_3 + M_3 X_1 + m_4 \sum_{i \in T_1} \frac{\alpha_i}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|}$$

satisfies the conditions of Proposition 3.2.

Proof. Since i belongs to I_1 and the term $(\lambda_i d_i)^{-2}$ appears in the lower bound of the estimate of Lemma 3.6, as in the proof of Corollary 3.5, we can make all the $(\lambda_k d_k)^{-2}$ and $\sum_{k \in T_1} \varepsilon_{kr}$ appear in this lower bound. Thus, using Lemma 3.4, Lemma 3.6, and (3.11), since M_3 is a large constant and m_4 is small, we obtain the estimate of Proposition 3.2. Furthermore, as in the proof of Corollary 3.5, the variables λ_i and a_i satisfy the claims of Proposition 3.2. Thus the corollary follows. □

For d_0 a fixed small constant, we introduce the set $I_1' = \{i \in I_1 : d_i < d_0\}$. Let $\lambda' = \inf\{\lambda_i : i \in I_1'\}$. We order the λ_i's for $i \in I_1' : \lambda_{i_1} = \lambda' \leq \cdots \leq \lambda_{i_s}$. We define, for $\overline{C}$ a large constant,

$$I_1'' = \{i_1\} \cup \{i_j \in I_1' : \lambda_{i_k} \leq \overline{C} \lambda_{i_{k-1}} \forall\, k \leq j\}.$$

Lemma 3.8. *If $T_2 \cap I_1 = \emptyset$ and, for each i and j in I_1,* (3.13) *is not satisfied, then there exists two vector fields defined as follows:*

$$X_4 = \sum_{i \in I_1''} \alpha_i \frac{\partial P\delta_i}{\partial a_i} (-\frac{n_i}{\lambda_{j_0}}),$$

$$X_4' = \sum_{i \in T_1 \setminus I_1'} \alpha_i \Big(- M_5 \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} + \frac{1}{\lambda_i} \frac{\partial P\delta_i}{\partial a_i} \frac{\nabla K(a_i)}{|\nabla K(a_i)|} \Big),$$

where $\lambda_{j_0} = \max\{\lambda_i : i \in I_1''\}$, n_i is the outward normal to

$$\partial\Omega_{d_i} = \{x \in \Omega \textit{ such that } d(x, \partial\Omega) = d_i\}$$

at a_i, and M_5 is a large constant. The vector fields satisfy

$$\big(-\partial J(u), X_4 + X_4'\big) \geq c \sum_{i\in T_1} \frac{1}{\lambda_i} + \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^3} + O\big(\sum_{i\in T_1, j\in T_2\cup L_i} \varepsilon_{ij}\big) + R_2,$$

where R_2 is defined in Proposition 2.3 *and*

$$L_i = \{j \in L \textit{ such that } i \textit{ and } j \textit{ satisfy } (3.13)\}.$$

Proof. For $i \in T_1 \setminus I_1'$, we have $d_i \geq d_0$. Therefore, the function H and its derivative are bounded. We have also $|\nabla K(a_i)| \geq c$. Thus, using Proposition 2.4 and 2.3 we derive that

$$\begin{aligned}&\big(-\partial J(u), X_4'\big)\\ &\geq c \sum_{i\in T_1\setminus I_1'} \Big[\big(\frac{1}{\lambda_i} + O(\sum_{j\neq i} \varepsilon_{ij})\big) + M_5\big(\sum_{j\in T_1} \varepsilon_{ij} + R_2 + O(\sum_{j=1}^{p} \frac{1}{\lambda_i\lambda_j} + \sum_{j\in T_2} \varepsilon_{ij})\big)\Big]\\ &\geq c \sum_{i\in T_1\setminus I_1'} \Big(\frac{1}{\lambda_i} + \sum_{j\in T_1} \varepsilon_{ij} + R_2 + O(\sum_{j\in T_2} \varepsilon_{ij})\Big). \qquad (3.19)\end{aligned}$$

Using Proposition 2.4, we derive that

$$\begin{aligned}\big(-\partial J(u), X_4\big) &= \frac{c}{\lambda_{j_0}} J(u) \sum_{i\in I_1''} \Big[- c\alpha_i \frac{\partial K(a_i)}{\partial n_i}(1+o(1))\\ &\quad + O(\frac{1}{\lambda_i}) + \frac{\alpha_i^2}{\lambda_i^2}\frac{\partial H(a_i,a_i)}{\partial n_i}(1+o(1))\\ &+ 2\sum_{j\in T_1\setminus L_i} \alpha_i\alpha_j\Big(\frac{\partial\varepsilon_{ij}}{\partial n_i} - \frac{\partial H(a_i,a_j)}{\partial n_i}\frac{1}{(\lambda_i\lambda_j)}\Big)\Big(1 - J(u)^2(\alpha_i^2 K(a_i) + \alpha_j^2 K(a_j))\Big)\\ &+ O\big(\lambda_i \sum_{j\in T_2\cup L_i} \varepsilon_{ij} + \sum_{j\in T_1\setminus L_i} \lambda_i\lambda_j |a_i - a_j| \varepsilon_{ij}^{5/2}\big)\Big] + R_2, \qquad (3.20)\end{aligned}$$

where R_2 is defined in Proposition 2.3. For $i \in I_1''$, we have (see Lemma 5.17)

$$\frac{\partial H(a_i,a_i)}{\partial n_i} = \frac{1}{2d_i^3}(1+o(1)). \qquad (3.21)$$

For $i \in I_1''$ and $j \in T_1 \setminus (I_1 \cup L_i)$, using Lemma 5.17 and the fact that $c_2 \max(d_i, d_j) \leq |a_i - a_j|$, we have that

$$\frac{1}{(\lambda_i\lambda_j)}\frac{\partial H}{\partial n_i}(a_i,a_j) \leq \frac{cH(a_i,a_j)}{(\lambda_i\lambda_j)d_i} \leq \frac{c}{d_i(\lambda_i d_i \lambda_j d_j)} = o\Big(\frac{1}{d_i(\lambda_i d_i)^2}\Big) \qquad (3.22)$$

$$|\frac{\partial\varepsilon_{ij}}{\partial a_i}| = 2\lambda_i\lambda_j|a_i - a_j|\varepsilon_{ij}^2 \leq \frac{c}{(\lambda_i\lambda_j)}\frac{1}{|a_i - a_j|^3} \tag{3.23}$$
$$= O\Big(\frac{1}{c_2^3(\lambda_i d_i \lambda_j d_j)d_i}\Big) = O\Big(\frac{c_1}{c_2^3(\lambda_i d_i)^2 d_i}\Big) = o\Big(\frac{1}{d_i(\lambda_i d_i)^2}\Big)$$

if we choose c_1 and c_2 so that $c_1 = o(c_2^3)$.

For $i \in I_1''$ and $j \in I_1' \setminus (I_1'' \cup L_i)$, using also Lemma 5.17, and the fact that λ_i and λ_j are not of the same order, we derive that

$$\frac{1}{(\lambda_i\lambda_j)}\frac{\partial H}{\partial n_i}(a_i, a_j) \leq \frac{cH(a_i, a_j)}{(\lambda_i\lambda_j)d_i} \leq \frac{c}{d_i^3(\lambda_i\lambda_j)} = o\Big(\frac{1}{d_i(\lambda_i d_i)^2}\Big). \tag{3.24}$$

Furthermore, as in (3.23), we prove that $|\frac{\partial\varepsilon_{ij}}{\partial a_i}| = o\big(\frac{1}{d_i(\lambda_i d_i)^2}\big)$ if we choose $\overline{C}$ so that $c_2^3\overline{C}$ is very large. For i and j in I_1'' , if d_i/d_j, d_j/d_i, and $|a_i - a_j|/d_i$ are bounded, we use the estimate of $(\partial H/\partial n_i)(a_i, a_j)$ provided in Appendix B and proved in the Appendix of [9]; we derive $(\partial H/\partial n_i)(a_i, a_j) > 0$. In the other case, we have

$$\frac{\partial H}{\partial n_i}(a_i, a_j) \leq \frac{H(a_i, a_j)}{d_i} \leq \frac{1}{d_i \max(d_i, d_j, |a_i - a_j|)^2} = o(\frac{1}{(d_i d_j)^{\frac{3}{2}}}), \tag{3.25}$$

and therefore

$$\frac{1}{\lambda_i\lambda_j}\frac{\partial H}{\partial n_i}(a_i, a_j) = o\Big(\frac{1}{d_i(\lambda_i d_i)^2} + \frac{1}{d_j(\lambda_j d_j)^2}\Big). \tag{3.26}$$

For i and j belonging to I_1'', $n_i - n_j = O(|a_i - a_j|)$; therefore,

$$\frac{\partial\varepsilon_{ij}}{\partial a_i}n_i + \frac{\partial\varepsilon_{ij}}{\partial a_j}n_j = \lambda_i\lambda_j(a_i - a_j)\varepsilon_{ij}^2(n_j - n_i) = O(\varepsilon_{ij}). \tag{3.27}$$

For $i \in I_1''$ and $j \in I_1 \setminus I_1'$, we claim that

$$\frac{\partial H(a_i, a_j)}{\partial n_i}\frac{1}{\lambda_i\lambda_j} - \frac{\partial\varepsilon_{ij}}{\partial a_i}n_i = -\frac{\partial G(a_i, a_j)}{\partial n_i}\frac{1}{\lambda_i\lambda_j} + o\Big(\frac{\lambda_i}{(\lambda_1 d_1)^5}\Big). \tag{3.28}$$

Indeed, since $T_2 \cap I_1 = \emptyset$, i and j belong to T_1. Using (3.14), (3.15), and the fact that (3.13) is not satisfied, we have that

$$\frac{\partial\varepsilon_{ij}}{\partial a_i} = 2(a_j - a_i)\lambda_i\lambda_j\varepsilon_{ij}^2 = \frac{2(a_j - a_i)}{\lambda_i\lambda_j|a_i - a_j|^4}\Big(1 + O\big(\frac{1}{(\lambda_1 d_1)^2}\big)\Big). \tag{3.29}$$

Therefore,

$$\frac{\partial\varepsilon_{ij}}{\partial a_i} = \frac{1}{\lambda_i\lambda_j}\frac{\partial}{\partial a_i}\Big(\frac{1}{|a_i - a_j|^2}\Big) + O\Big(\frac{\lambda_i}{c_2^3(\lambda_1 d_1)^5}\Big), \tag{3.30}$$

and our claim follows.

Observe that for $j \in T_1 \setminus L_i$, we have $|a_i - a_j| \geq c_2 d_i$, and then $\lambda_i |a_i - a_j| \to \infty$. Thus

$$\lambda_j |a_i - a_j| \varepsilon_{ij}^{5/2} \leq \frac{1}{\lambda_i |a_i - a_j|} \varepsilon_{ij}^{3/2} = o(\varepsilon_{ij}^{3/2}). \tag{3.31}$$

Thus, using (3.21),...,(3.31) and the fact that $\partial K(a_i)/\partial n_i \leq -c$, (3.20) becomes

$$\Big(-\partial J(u), X_4 \Big) \geq \frac{c}{\lambda_{j_0}} \sum_{i \in I_1''} \Big[c + \frac{1}{(\lambda_i d_i)^2 d_i} - \sum_{j \in I_1 \setminus I_1'} \frac{1}{(\lambda_i \lambda_j)} \frac{\partial G(a_i, a_j)}{\partial n_i} + O\Big(\sum_{j \in I_1''} \varepsilon_{ij} \Big) + O\Big(\sum_{j \in T_2 \cup L_i} \lambda_i \varepsilon_{ij} \Big) + o\Big(\sum_{j \in T_1 \setminus L_i} \lambda_i \varepsilon_{ij}^{3/2} \Big) \Big] + R_2.$$

Observe that, for $i \in I_1''$ and $j \in I_1 \setminus I_1'$ we have $d_i \leq d_j$; then using Lemma 5.18, $-\partial G(a_i, a_j)/\partial n_i > 0$. For $i, j \in I_1''$, (3.13) is not satisfied. Thus $\varepsilon_{ij} = O((\lambda_i d_i)^{-2} + (\lambda_j d_j)^{-2})$. Since d_i and d_j are small, we derive that $\varepsilon_{ij} = o\big((\lambda_i d_i)^{-2} d_i^{-1} + (\lambda_j d_j)^{-2} d_j^{-1}\big)$. Recall that j_0 belongs to I_1''. Since $\lambda_{j_0} d_{j_0}$ and $\lambda_1 d_1$ are of the same order, we can make all the $(\lambda_i d_i)^{-3}$ appear in the lower bound. Thus

$$\Big(-\partial J(u), X_4 \Big) \geq \sum_{i \in I_1'} \frac{c}{\lambda_i} + \sum_{i=1}^{p} \frac{1}{(\lambda_i d_i)^3} + O\Big(\sum_{i \in I_1'', j \in T_2 \cup L_i} \varepsilon_{ij} \Big) + R_2. \tag{3.32}$$

Using (3.19) and (3.32), the lemma follows. □

Lemma 3.9. *There exists a vector field X_5 satisfying*

$$\big(-\partial J(u), X_5 \big) \geq c \sum_{i \in T_1, j \in L_i} \varepsilon_{ij} + O\big(\sum_{i \in T_1, j \in T_2} \varepsilon_{ij} \big) + R_2.$$

Proof. For $i \in T_1$ and $j \in L_i$, using Lemma 3.6, we can find a vector field $X_3(i,j)$ depending on the indices i and j and satisfying the estimate of Lemma 3.6. The vector field X_5 will be defined as follows:

$$X_5 = \sum_{i \in T_1, j \in L_i} X_3(i,j).$$

The result follows. □

Corollary 3.10. *If $T_2 \cap I_1 = \emptyset$ and, for each i and j belonging to I_1,* (3.13) *is not satisfied, then for M_6 and M_7 two fixed large constants, the vector field $X_4 + X_4' + M_6 X_1 + M_7 X_5$ satisfies the conditions of Proposition* 3.2.

Proof. For M_6 and M_7 two large constants, by Lemmas 3.4, 3.8, and 3.9, we derive that

$$\big(-\partial J(u), X_4 + X_4' + M_6 X_1 + M_7 X_5\big) \geq c \sum_{i=1}^{p} \Big(\frac{1}{(\lambda_i d_i)^3} + \frac{1}{\lambda_i} + \sum_{j \in T_2} \varepsilon_{ij} \Big) + R_2. \tag{3.33}$$

As in the proof of Corollary 3.5, we can make $\sum_{i\in T_1} \varepsilon_{ij}^{3/2}$ appear in the lower bound of (3.33). Therefore the corollary follows. □

Now, we are ready to prove Propositions 3.2 and 3.3.

Proof of Proposition 3.2 The vector field W_1 required in Proposition 3.2 will be defined by a convex combination of the vector fields defined by Corollaries 3.5, 3.7, and 3.10. □

Proof of Proposition 3.3 We divide the set $V_2(p, \varepsilon)$ into four sets:

$$V_1(p,\varepsilon,\eta) = \{u/a_i \in B(y_{j_i}, 2\eta),\ j_i \neq j_k \text{ if } i \neq k \text{ and } \rho(j_1, \dots, j_p) > 0\}$$

$$V_2(p,\varepsilon,\eta) = \{u/a_i \in B(y_{j_i}, 2\eta),\ j_i \neq j_k \text{ if } i \neq k, \quad \frac{-\Delta K(y_{j_i})}{3K(y_{j_i})} + 8H(y_{j_i}, y_{j_i}) > 0,\ \rho(j_1, \dots, j_p) < 0\}$$

$$V_3(p,\varepsilon,\eta) = \{u/a_i \in B(y_{j_i}, 2\eta),\ j_i \neq j_k \text{ if } i \neq k,\ \exists j_l \text{ such that } \frac{-\Delta K(y_{j_l})}{3K(y_{j_l})} + 8H(y_{j_l}, y_{j_l}) < 0\}$$

$$V_4(p,\varepsilon,\eta) = \{u/a_i \in B(y_{j_i}, 2\eta),\ \exists i \neq k \text{ such that } j_i = j_k\},$$

where ρ is the least eigenvalue of the matrix M defined in (1.4).

We will define the pseudo gradient depending on the sets V_i to which u belongs.

1. If $u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V_1(p, \varepsilon, \eta)$, we have for any $i \neq j$, $|a_i - a_j| > c$, and therefore

$$\varepsilon_{ij} = \frac{1}{\lambda_i \lambda_j |a_i - a_j|^2}(1 + o(1)).$$

Thus

$$\lambda_i \frac{\partial \varepsilon_{ij}}{\partial \lambda_i} = -\varepsilon_{ij}(1 + o(1)) = -\frac{1}{\lambda_i \lambda_j |a_i - a_j|^2}(1 + o(1)).$$

Observe that, since $u \in V(p, \varepsilon)$, we have $\alpha_i^2 J(u)^2 K(a_i) = 1 + o(1)$. Thus, Proposition 2.3 becomes

$$\langle \partial J(u), \alpha_i \lambda_i \frac{\partial P\delta_i}{\partial \lambda_i} \rangle = \frac{4w_3}{J(u)} \Big[\frac{1}{3} \frac{\Delta K(a_i)}{K(a_i)^2 \lambda_i^2} - 8 \frac{H(a_i, a_i)}{K(a_i)\lambda_i^2} \tag{3.34}$$

$$+ 8\sum_{j\neq i}\frac{G(a_i,a_j)}{(K(a_i)K(a_j))^{\frac{1}{2}}}\frac{1}{\lambda_i\lambda_j} + o(\sum_k \frac{1}{\lambda_k^2})\Big].$$

We define Z_1 by $Z_1 = \sum_{i=1}^p \alpha_i\lambda_i\partial P\delta_i/\partial\lambda_i$. Thus we derive that

$$\langle -\partial J(u), Z_1\rangle = \frac{4w_3}{J(u)}{}^T\Lambda M\Lambda + o(\sum_k \frac{1}{\lambda_k^2}) \geq c\sum_k \frac{1}{\lambda_k^2} \geq c\sum_k \frac{1}{\lambda_k^2} + c\sum_{i\neq j}\varepsilon_{ij}, \tag{3.35}$$

where M is the matrix defined by (1.4) and $\Lambda =^T (1/\lambda_1,\dots,1/\lambda_p)$.

We define also

$$Z_a = \sum_{i=1}^p \varphi\left(\lambda_i|\nabla K(a_i)|\right)\frac{\alpha_i}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\frac{\nabla K(a_i)}{|\nabla K(a_i)|} \quad \text{and} \quad X_1 = CZ_1 + Z_a,$$

where C is a large constant and φ is a C^∞ function which satisfies $0 \leq \varphi \leq 1$, $\varphi(t) = 0$ if $t \leq 1$ and $\varphi(t) = 1$ if $t \geq 2$.

Using Proposition 2.4 and (3.35), we derive that

$$\langle -\partial J(u), X_1\rangle \geq c\Big(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\Big). \tag{3.36}$$

2. If $u = \sum_{i=1}^p \alpha_i P\delta_i \in V_2(p,\varepsilon,\eta)$. Let ρ be the least eigenvalue of M. Then there exists an eigenvector e associated to ρ such that $||e|| = 1$ with $e_i > 0$ for all i (see [9] and [11]). Let $\gamma > 0$ such that for any $x \in B(e,\gamma) = \{y \in S^{p-1} : ||y-e|| \leq \gamma\}$ we have ${}^T xMx < (1/2)\rho$. Two cases may occur.

Case 1. $|\Lambda|^{-1}\Lambda \in B(e,\gamma)$. In this case we define $X_2' = -CZ_1 + Z_a$, where C is a large constant. As in (3.35) and (3.36), we derive that

$$\langle -\partial J(u), X_2'\rangle \geq c\Big(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\Big). \tag{3.37}$$

Case 2. $|\Lambda|^{-1}\Lambda \notin B(e,\gamma)$. As in [11], we move the vector Λ to the vector $|\Lambda|e$ on the sphere of radius $|\Lambda|$. We obtain the vector field Z_2 defined by

$$Z_2 = -\sum_{i=1}^p |\Lambda|\alpha_i\lambda_i^2\frac{\partial P\delta_i}{\partial\lambda_i}\Big[\frac{|\Lambda|e_i - \Lambda_i}{||y(0)||} - \frac{y_i(0)}{||y(0)||^3}(y(0), |\Lambda|e - \Lambda)\Big],$$

where $y(t) = (1-t)\Lambda + t|\Lambda|e$. Define $\Lambda(t) = y(t)/||y(t)||$. Using Proposition 2.3, as in [11], we derive that

$$\langle -\partial J(u), Z_2\rangle = -c|\Lambda|^2\frac{\partial}{\partial t}({}^T\Lambda(t)M\Lambda(t)) + o(\sum \frac{1}{\lambda_k^2}).$$

Observe that

$$^T\Lambda(t)M\Lambda(t) = \rho + \frac{(1-t)^2}{||y(t)||^2}(^T\Lambda M\Lambda - \rho||\Lambda||^2).$$

Thus, we obtain $(\partial/\partial t)(^T\Lambda(t)M\Lambda(t)) < -c$. Therefore, for $X_2'' = CZ_2 + Z_a$, we obtain

$$\langle -\partial J(u), X_2''\rangle \geq c\Big(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

The vector field X_2 defined on $V_2(p,\varepsilon,\eta)$ will be a convex combination of X_2' and X_2''.

3. If $u = \sum_{i=1}^p \alpha_i P\delta_i \in V_3(p,\varepsilon,\eta)$. We can assume without loss of generality that $1,\dots,q$ are the indices which satisfy $-\Delta K(a_i)/(3K(a_i)) + 8H(a_i,a_i) < 0$. Let $I = \{i : \lambda_i \leq (1/10)\inf_{k=1,\dots,q}\lambda_k\}$. Let M_I be the matrix defined by the points $(y_{j_i})_{i\in I}$ (as in (1.4)) and ρ_I be the least eigenvalue of M_I. Define

$$Z_3 = -\sum_{i=1}^q \alpha_i\lambda_i \frac{\partial P\delta_i}{\partial\lambda_i}.$$

Then, since $|a_i - a_j| > c$, using (3.34), we derive that

$$\langle -\partial J(u), Z_3\rangle \geq c\sum_{k=1}^q\Big(\frac{1}{\lambda_k^2} + \sum_{i\neq k}\frac{G(a_i,a_k)}{\lambda_i\lambda_k}\Big) + R_2 \geq c\sum_{k\notin I}\Big(\frac{1}{\lambda_k^2} + \sum_{i\neq k}\varepsilon_{ik}\Big) + R_2. \tag{3.38}$$

If $I \neq \varnothing$, then the lower bound becomes limited to those indices such that $k \notin I$. We have to add another vector field. If the matrix M_I is positive definite, we define $Z_3' = X_1(\sum_{i\in I}\alpha_i P\delta_i)$, which means the action of X_1 but using only the indices of I. In the other case, that is, when the matrix M_I is not positive definite, we define $Z_3' = X_2(\sum_{i\in I}\alpha_i P\delta_i)$. In both cases we have

$$\langle -\partial J(u), Z_3'\rangle \geq c\sum_{k\in I}\Big(\frac{1}{\lambda_k^2} + \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i\neq k, i\in I}\varepsilon_{ik}\Big) - c\sum_{k\in I, i\notin I}\varepsilon_{ik}. \tag{3.39}$$

Now, we define $X_3 = CZ_3 + Z_3' + m_4 Z_a$, where C is a large constant and m_4 is a small constant. Using (3.38), (3.39), and Proposition 2.4, we derive that

$$\langle -\partial J(u), X_3\rangle \geq c\Big(\sum_i \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_k \frac{1}{\lambda_k^2} + \sum_{i\neq j}\varepsilon_{ij}\Big).$$

4. If $u = \sum_{i=1}^{p} \alpha_i P\delta_i \in V_4(p,\varepsilon,\eta)$. Let $B_i = \{j : a_j \in B(y_i,\eta)\}$. In this case, there is at least one B_i which contains at least two indices. Without loss of generality, we can assume that $1,\dots,q$ are the indices such that the set B_i contains at least two indices. For $i \le q$ we will decrease the λ_j for $j \in B_i$ with different speed. For this fact, let χ be a smooth cut-off function such that $\chi \ge 0$, $\chi = 0$ if $t \le \gamma'$, and $\chi = 1$ if $t \ge 1$, where γ' is a small constant. Set $\overline{\chi}(\lambda_j) = \sum_{i\ne j, i,j\in B_k} \chi(\lambda_j/\lambda_i)$. Define

$$Z_4 = -\sum_{k=1}^{q}\sum_{j\in B_k} \alpha_j \overline{\chi}(\lambda_j)\lambda_j \frac{\partial P\delta_j}{\partial \lambda_j}.$$

Using Proposition 2.3, we derive that

$$\begin{aligned}\langle -\partial J(u), Z_4\rangle =& 2J(u)\sum_{k=1}^{q}\sum_{j\in B_k}\alpha_j\overline{\chi}(\lambda_j)\Big[-c\sum_{i\ne j}\alpha_i(\lambda_j\frac{\partial\varepsilon_{ij}}{\partial\lambda_j}+\frac{H(a_i,a_j)}{\lambda_i\lambda_j})\\ &+O(\frac{1}{\lambda_j^2})+o(\sum_r\frac{1}{\lambda_r^2}+\sum_{i\ne r}\varepsilon_{ir})\Big].\end{aligned}$$

For $j \in B_k$, with $k \le q$, if $\overline{\chi}(\lambda_j) \ne 0$, then there exists $i \in B_k$ such that $\lambda_j^{-2} = o(\varepsilon_{ij})$ (for η small enough).

Furthermore, for $j \in B_k$, if $i \notin B_k$ or $i \in B_k$ with λ_i and λ_j of the same order, i.e., $\gamma' < \lambda_i/\lambda_j < 1/\gamma'$, then we have $-\lambda_r\partial\varepsilon_{ij}/\partial\lambda_r = \varepsilon_{ij}(1+o(1))$, for $r = i, j$. In the case where $i \in B_k$ with (assuming that $\lambda_i < \lambda_j$) $\lambda_i/\lambda_j < \gamma'$, we have $\overline{\chi}(\lambda_j) - \overline{\chi}(\lambda_i) \ge 1$. Thus

$$-\overline{\chi}(\lambda_j)\lambda_j\partial\varepsilon_{ij}/\partial\lambda_j - \overline{\chi}(\lambda_i)\lambda_i\partial\varepsilon_{ij}/\partial\lambda_i \ge -\lambda_j\partial\varepsilon_{ij}/\partial\lambda_j = \varepsilon_{ij}(1+o(1)).$$

Thus we derive that

$$\langle -\partial J(u), Z_4\rangle \ge c\sum_{k=1}^{q}\sum_{j\in B_k,\overline{\chi}(\lambda_j)\ne 0}\big(\frac{1}{\lambda_j^2}+\sum_{i\ne j}\varepsilon_{ij}\big)+o(\sum_r\frac{1}{\lambda_r^2}+\sum_{i\ne r}\varepsilon_{ir}). \quad (3.40)$$

The lower bound does not contain all the indices. We need to add some terms. Let

$$\lambda_{i_0} = \inf\{\lambda_i,\, i = 1,\dots,p\}. \quad (3.41)$$

Two cases may occur:

Case a: If there exists j such that $\overline{\chi}(\lambda_j) \ne 0$ and $\lambda_{i_0}/\lambda_j > \gamma'$, then we can make appear in the lower bound $1/\lambda_{i_0}^2$, and therefore all the $1/\lambda_i^2$ and the ε_{ik}. Thus we can define $X_4 = CZ_4 + Z_a$, where C is a large constant.

Case b: For any j satisfying $\overline{\chi}(\lambda_j) \neq 0$ we have $\lambda_{i_0}/\lambda_j \leq \gamma'$. In this case, we define

$$D = (\{i : \overline{\chi}(\lambda_i) = 0\} \cup (\cup_{k=1}^q B_k)^c) \cap \{i \text{ such that } \lambda_i/\lambda_{i_0} < 1/\gamma'\}.$$

It is easy to see that $\{i : \overline{\chi}(\lambda_i) = 0\}$ contains at most one index from each B_j for $1 \leq j \leq q$, and therefore for $i, r \in D$ such that $i \neq r$ we have $a_i \in B(y_{j_i}, \eta)$ and $a_r \in B(y_{j_r}, \eta)$ with $j_i \neq j_r$. Let

$$u_1 = \sum_{i \in D} \alpha_i P\delta_i.$$

u_1 has to satisfy one of the three above cases, that is, $u_1 \in V_i(card(D), \varepsilon, \eta)$ for $i = 1, 2$ or 3. Thus we can apply the associated vector field, which we will denote Z_4', and we have the following estimate:

$$\langle -\partial J(u), Z_4' \rangle \geq c\Big(\sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i \in D} \frac{1}{\lambda_i^2} + \sum_{i,j \in D} \varepsilon_{ij} \Big) + O(\sum_{k \in D, r \notin D} \varepsilon_{kr}) + o(\sum_{i \notin D} \frac{1}{\lambda_i^2}).$$

Observe that for $k \in D$ and $r \notin D$, we have either $r \in B^q := \{i : \overline{\chi}(\lambda_i) \neq 0\} \cap (\cup_{j=1}^q B_j)$ (in this case we have ε_{kr} in the lower bound of (3.40)) or $r \in (B^q)^c$ (in this case, since $r \notin D$, we have $\lambda_{i_0}/\lambda_r \leq \gamma'$). Furthermore, $a_j \in B(y_{i_j}, 2\eta)$ for $j = k, r$ and $i_k \neq i_r$. Therefore $|a_k - a_r| > c$. Thus,

$$\varepsilon_{kr} \leq \frac{c}{\lambda_k \lambda_r} \leq \frac{c\gamma'}{\lambda_k \lambda_{i_0}} = o(\varepsilon_{k i_0})$$

(γ' small). Since $i_0 \in D$ (i_0 is defined by (3.41)), we have that from $1/\lambda_{i_0}^2$, we can make appear in the lower bound all the $1/\lambda_i^2$ and ε_{ir} for $i, r \in (B^q)^c$ (since for those indices we have $|a_i - a_r| > c$). Thus we derive that

$$\langle -\partial J(u), Z_4' \rangle \geq c\Big(\sum_{i \in D} \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i,j \in (B^q)^c} \varepsilon_{ij} \Big) + O(\sum_{k \in D, r \in B^q} \varepsilon_{kr}). \tag{3.42}$$

Now, we define $X_4 = CZ_4 + Z_4' + m_5 Z_a$, where C is a large constant and m_5 is a small constant. We obtain

$$\langle -\partial J(u), X_4 \rangle \geq c\Big(\sum_{i=1}^p \frac{|\nabla K(a_i)|}{\lambda_i} + \sum_{i=1}^p \frac{1}{\lambda_i^2} + \sum_{i \neq j} \varepsilon_{ij} \Big). \tag{3.43}$$

The vector field W_2 will be as a convex combination of all above cases. It satisfies the estimate of Proposition 3.3. Moreover, in $V_3(p,\varepsilon,\eta)\cup V_4(p,\varepsilon,\eta)$, we have $\dot{\lambda}_{max}\leq 0$. In $V_2(p,\varepsilon,\eta)$, when $|\Lambda|^{-1}\Lambda\notin B(e,\gamma)$, since $|\dot{\lambda}_i/\lambda_i|\leq c$ and the time is bounded in this region, we can control λ_{max}. The proposition follows. □

4. Proofs of theorems

Proof of Theorem 1.2. Using Theorem 3.1, the proof is similar to an argument in Appendix 2 of [6] (see also Proposition 3.1 of [11]). □

Proof of Theorem 1.3. By Theorem 3.1, the only case where the λ_i are not bounded is when $M(y_{i_1},\dots,y_{i_p})$ is positive definite. In this case, using Theorem 1.2, the normal form of J allows us to split the variables, and it is easy to see that we have a critical point at infinity. The index of such a critical point at infinity is the same as the index of the critical point of

$$g(\alpha_1,\dots,\alpha_p,a_1,\dots,a_p):=\frac{\sum_{i=1}^p\alpha_i^2}{(\sum_{i=1}^p\alpha_i^4K(a_i))^{1/2}}.$$

In the variable α_i's we have a degenerated critical point $(\overline{\alpha}_1,\dots,\overline{\alpha}_p)$ which satisfies

$$\frac{\overline{\alpha}_i^2K(a_i)}{\overline{\alpha}_j^2K(a_j)}=1.$$

This critical point has a index equal to $p-1$ (since the function g is homogenous in the variables α_i and the critical point corresponds to a maximum), and we have

$$g(\overline{\alpha}_1,\dots,\overline{\alpha}_p,a_1,\dots,a_p)=(\sum_{i=1}^p\frac{1}{K(a_i)})^{1/2}(1-|X|^2),$$

where X belongs to $\mathbb{R}^{p-1}$. Thus the index is equal to

$$p-1+\sum_{i=1}^p(4-index(K,y_{j_i})=5p-1-\sum_{i=1}^p k_{j_i}.$$

The theorem follows. □

Proof of Theorem 1.4. We argue by contradiction. We assume that (P) has no solution. First, we will define a vector field X in Σ by a convex combination of $-\partial J$ (defined outside of $\cup_p V(p,\varepsilon/2)$) and the vector field W (defined on each $V(p,\varepsilon)$ by Theorem 3.1). Observe that the set Σ^+ is not invariant under the flow generated by X. For this effect, we introduce, for

$\varepsilon_0 > 0$ small enough, the set $V_{\varepsilon_0}(\Sigma^+) = \{u \in \Sigma : J(u)^2 |u^-|_{L^4} < \varepsilon_0\}$ for $\varepsilon_0 > 0$ small enough and satisfying $\varepsilon_0 > 2S_4\varepsilon/(\min K)$. Using Lemma 4.1 of [11], we have $V_{\varepsilon_0}(\Sigma^+)$ is invariant under the flow generated by X. For $\tau_s = (i_1, \ldots, i_s)$ such that $M(\tau_s)$ is positive definite, using Theorem 1.3, we have $\sum_{j=1}^s P\delta_{(y_{i_j},\infty)}$ is a critical point at infinity of index $5s - 1 - \sum_{j=1}^s k_{i_j}$, where $k_{i_j} = index(K, y_{i_j})$. Let $c(\tau_s)$ be the critical value associated to this critical point. (For simplicity, we will assume that all the critical values are different.) Let χ be the Euler-Poincare characteristic. Thus for $\theta > 0$ small enough (we choose θ so that the only critical value between $c(\tau_s) + \theta$ and $c(\tau_s) - \theta$ is $c(\tau_s)$), we have

$$\chi(J_{c(\tau_s)+\theta}) = \chi(J_{c(\tau_s)-\theta}) + (-1)^{5s-1-\sum_{j=1}^s k_{i_j}}.$$

Since $V_{\varepsilon_0}(\Sigma^+)$ is contractible, $\chi(V_{\varepsilon_0}(\Sigma^+)) = 1$. Thus

$$\chi(V_{\varepsilon_0}(\Sigma^+)) = \sum_{s=1}^{m_1} \sum_{\tau_s=(i_1,\ldots,i_s)/M(\tau_s)>0} (-1)^{5s-1-\sum_{j=1}^s k_{i_j}}$$

is a contradiction. Therefore, (P) has a solution $u_0 \in V_{\varepsilon_0}(\Sigma^+)$.

Proposition 4.1. *The solution u_0 found by the Theorem* 1.4 *is positive.*

Proof. We can write $u_0 = u_0^+ - u_0^-$. By definition of $V_{\varepsilon_0}(\Sigma^+)$ we have $J(u_0)^2 |u_0^-|_{L^4} < \varepsilon_0$. Multiplying the equation by the test function u_0^- and integrating, we obtain

$$|\nabla u_0^-|_{L^2}^2 \le c|u_0^-|_{L^4}^4 \le c'|\nabla u_0^-|_{L^2}^4.$$

Hence, either $u_0^- = 0$, or $|\nabla u_0^-|_{L^2} \ge c_0$ for some constant $c_0 > 0$, a contradiction with ε_0 small enough. Thus, $u_0^- = 0$. Therefore, $u_0 > 0$. □

Proof of Theorem 1.5. Our proof follows the algebraic topological arguments introduced in [4]. Arguing by contradiction, we suppose that J has no critical points. It follows from Theorem 1.3 that under the assumptions (H_2) and (H_4), the critical points at infinity of J under the level $c_1 = S_4^{1/2} K(y_l)^{-1/2} + \varepsilon$, for ε small enough, are in one-to-one correspondence with the critical points of K $y_0, y_1, \ldots, y_l$. One can see, using (H_2), that X is a compact set of Ω. The unstable manifold at infinity of such critical points at infinity, $W_u(y_0)_\infty, \ldots, W_u(y_l)_\infty$ can be described, using Theorem 1.2, as the product of $W_s(y_0), \ldots, W_s(y_l)$ (for a pseudo gradient of K) by $[A, +\infty)$ domain of the variable λ, for some positive number A large enough.

Since J has no critical points, it follows that $J_{c_1} = \{u \in \Sigma^+ : J(u) \leq c_1\}$ retracts by deformation on $X_\infty = \cup_{0\leq j\leq l} W_u(y_j)_\infty$ (see Sections 7 and 8 of [10]), which can be parametrized as we said before by $X \times [A, +\infty)$.

From another part, we have X_∞ is contractible in $J_{c_2+\varepsilon}$, where $c_2 = S_4^{1/2}/\bar{c}^{1/2}$. Indeed from (H_6), it follows that there exists a contraction $h : [0,1] \times X \to K^{\bar{c}}$, h continuous such that for any $a \in X$, $h(0,a) = a$ and $h(1,a) = a_0$ a point of X. Such a contraction gives rise to the following contraction $\tilde{h} : X_\infty \to \Sigma^+$ defined by

$$[0,1] \times X \times [A, +\infty) \ni (t, a_1, \lambda_1) \longmapsto P\delta_{(h(t,a_1),\lambda)} + \bar{v} \in \Sigma^+.$$

For $t = 0$, $P\delta_{(h(0,a_1),\lambda_1)} + \bar{v} = P\delta_{a_1,\lambda_1} + \bar{v} \in X_\infty$. $\tilde{h}$ is continuous and $\tilde{h}(1, a_1, \lambda_1) = P\delta_{a_0,\lambda_1} + \bar{v}$; hence, our claim follows.

Now, using Proposition 2.1 with $p = 1$, we deduce that

$$J(P\delta_{h(t,a_1),\lambda_1} + \bar{v}) \sim S_4^{1/2}(K(h(t,a_1)))^{-1/2}\left(1 + O(A^{-2})\right),$$

where $K(h(t,a_1)) \geq \bar{c}$ by construction. Therefore such a contraction is performed under $c_2 + \varepsilon$, for A large enough, so X_∞ is contractible in $J_{c_2+\varepsilon}$. In addition, choosing c_0 small enough, between $c_2 + \varepsilon$ and c_1 there is no critical point at infinity; then $J_{c_2+\varepsilon}$ retracts by deformation on J_{c_1}, which retracts by deformation on X_∞; therefore X_∞ is contractible, leading to the contractibility of X, which is in contradiction with our assumption. Hence, J has a critical point.

It remains to prove that such a critical point has an augmented Morse index $\geq k$. Arguing by contradiction, we may assume that the Morse index of the critical point is $\leq k-1$.

Perturbing, if necessary J, we may assume that all the critical points of J are nondegenerate and have their Morse index $\leq k-1$. Such critical points do not change the homological group in dimension k of level sets of J. Since X_∞ defines a homological class in dimension k which is nontrivial in J_{c_1}, but trivial in $J_{c_2+\varepsilon}$, our result follows. □

5. APPENDIX

In this Appendix, we collect the estimates of the different integral quantities which occur in the paper. These estimates were originally introduced by Bahri [5] and Bahri-Coron [7]. For the proof, we refer the interested reader to [5], [7], and [19]. In this Appendix, we suppose that $\lambda_i d_i$ is large enough and ε_{ij} is small enough. We have the following estimates:

Lemma 5.1.
$$|P\delta|^2 = S_4 - 16w_3\frac{H(a,a)}{\lambda^2} + O\Big(\frac{\log(\lambda d)}{(\lambda d)^4}\Big),$$
where w_3 is the volume of the sphere S^3 and $S_4 = 16w_3/3$.

Lemma 5.2.
$$\int_\Omega KP\delta^4 = K(a)S_4 + \frac{4}{3}w_3\frac{\Delta K(a)}{\lambda^2} - 64w_3K(a)\frac{H(a,a)}{\lambda^2} + O\Big(\frac{1}{(\lambda d)^3}\Big).$$

Lemma 5.3. *For $i \neq j$*
$$\big(P\delta_i, P\delta_j\big) = 16w_3\Big(\varepsilon_{ij} - \frac{H(a_i,a_j)}{\lambda_i\lambda_j}\Big) + O\Big(\varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^4}\Big).$$

Lemma 5.4. *For $i \neq j$*
$$\int_\Omega KP\delta_i^3P\delta_j = K(a_i)\big(P\delta_i, P\delta_j\big) + O\Big(\varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \frac{\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/2}}{\lambda_i} + \sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^4}\Big).$$

Lemma 5.5.
$$\big(P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}\big) = 16w_3\frac{H(a,a)}{\lambda^2} + O\Big(\frac{\log(\lambda d)}{(\lambda d)^4}\Big).$$

Lemma 5.6.
$$\int KP\delta^3\lambda\frac{\partial P\delta}{\partial\lambda} = 2K(a)\big(P\delta, \lambda\frac{\partial P\delta}{\partial\lambda}\big) - \frac{2}{3}w_3\frac{\Delta K(a)}{\lambda^2} + O\Big(\frac{1}{(\lambda d)^3}\Big).$$

Lemma 5.7. *For $i \neq j$*
$$\big(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) = 16w_3\Big(\lambda_i\frac{\partial\varepsilon_{ij}}{\partial\lambda_i} + \frac{H(a_i,a_j)}{\lambda_i\lambda_j}\Big) + O\Big(\sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^4} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1})\Big).$$

Lemma 5.8. *For $i \neq j$*
$$\int KP\delta_j^3\lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}$$
$$= K(a_j)\big(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) + O\Big(\sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^4} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \frac{\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/2}}{\lambda_j}\Big).$$

Lemma 5.9. *For $i \neq j$*
$$\int KP\delta_j\lambda_i\frac{\partial(P\delta_i^3)}{\partial\lambda_i}$$
$$= K(a_i)\big(P\delta_j, \lambda_i\frac{\partial P\delta_i}{\partial\lambda_i}\big) + O\Big(\sum_{k=i,j}\frac{\log(\lambda_k d_k)}{(\lambda_k d_k)^4} + \frac{\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{\frac{1}{2}}}{\lambda_i} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1})\Big).$$

Lemma 5.10.

$$\big(P\delta, \frac{1}{\lambda}\frac{\partial P\delta}{\partial a}\big) = -\frac{8w_3}{\lambda^3}\frac{\partial H}{\partial a}(a,a) + O\big(\frac{1}{(\lambda d)^4}\big).$$

Lemma 5.11.

$$\int KP\delta^3\frac{1}{\lambda}\frac{\partial P\delta}{\partial a} = -K(a)\frac{16w_3}{\lambda^3}\frac{\partial H}{\partial a}(a,a) + c_4\frac{\nabla K(a)}{\lambda}(1+o(1)) + O\big(\frac{\log(\lambda d)}{(\lambda d)^4} + \frac{1}{\lambda^2}\big),$$

where c_4 is a positive constant.

Lemma 5.12. *For $i \neq j$*

$$\begin{aligned}&\big(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\big)\\ &= -\frac{16w_3}{\lambda_i^2\lambda_j}\frac{\partial H}{\partial a_i}(a_i,a_j) + \frac{16w_3}{\lambda_i}\frac{\partial \varepsilon_{ij}}{\partial a_i} + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^4} + \varepsilon_{ij}^{5/2}\lambda_j|a_i - a_j|\Big).\end{aligned}$$

Lemma 5.13. *For $i \neq j$*

$$\begin{aligned}&\int KP\delta_j^3\frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\\ &= K(a_j)\big(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\big) + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^4} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \frac{\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/2}}{\lambda_j}\Big).\end{aligned}$$

Lemma 5.14. *For $i \neq j$*

$$\begin{aligned}&\int KP\delta_j\frac{1}{\lambda_i}\frac{\partial (P\delta_i^3)}{\partial a_i}\\ &= K(a_i)\big(P\delta_j, \frac{1}{\lambda_i}\frac{\partial P\delta_i}{\partial a_i}\big) + O\Big(\sum_{k=i,j}\frac{1}{(\lambda_k d_k)^4} + \varepsilon_{ij}^2\log(\varepsilon_{ij}^{-1}) + \frac{\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/2}}{\lambda_i}\Big).\end{aligned}$$

Lemma 5.15.

$$\int_\Omega K(\sum_{i=1}^p \alpha_i P\delta_i)^3 v = O\Big(|v|_{H_0^1}\Big(\sum_{i=1}^p\frac{|\nabla K(a_i)|}{\lambda_i} + \frac{1}{(\lambda_i d_i)^2} + \sum_{i\neq j}\varepsilon_{ij}(\log\varepsilon_{ij}^{-1})^{1/2}\Big)\Big).$$

Lemma 5.16. [5] [19] *For $\theta = \theta_{(a,\lambda)} = \delta_{(a,\lambda)} - P\delta_{(a,\lambda)}$ and H the regular part of the Green's function, we have the following estimates:*

$$\begin{array}{lcllcl}\theta(x) &=& \frac{\sqrt{8}}{\lambda}H(a,x) + O\big(\frac{1}{\lambda^3 d^4}\big), & |\theta|_{L^4} &=& O\big(\frac{1}{\lambda d}\big)\\ \frac{\partial\theta}{\partial a}(x) &=& \frac{\sqrt{8}}{\lambda}\frac{\partial H}{\partial a}(a,x) + O\big(\frac{1}{\lambda^3 d^5}\big), & |\frac{1}{\lambda}\frac{\partial\theta}{\partial a}|_{L^4} &=& O\big(\frac{1}{(\lambda d)^2}\big)\\ \lambda\frac{\partial\theta}{\partial\lambda}(x) &=& -\frac{\sqrt{8}}{\lambda}H(a,x) + O\big(\frac{1}{\lambda^3 d^4}\big), & |\lambda\frac{\partial\theta}{\partial\lambda}|_{L^4} &=& O\Big(\frac{1}{\lambda d}\Big)\end{array}$$

where $d = d(a, \partial\Omega)$.

Lemma 5.17. [5] [19] *For each $a \in \Omega$, near the boundary of Ω, let $n_a = n$ the outward normal to $\partial\Omega$ at a,*

$$\begin{array}{rclrcl} H(a,a) & = & (2d)^{-2} + o(d^{-2}), & H(x,a) & \leq & c\ \max(d_x, d_a)^{-2} \\ \frac{\partial H}{\partial n}(a,a) & = & \frac{2}{4d^3} + o\big(\frac{1}{d^3}\big), & |\frac{\partial H}{\partial x}|(x,a) & \leq & \frac{c}{d_x} H(x,a). \end{array}$$

Lemma 5.18. [9] [13] *Let $(x_1, x_2) \in \Omega^2$ such that $d_1 \leq d_2$ and $c_2 d_2 \leq |x_1 - x_2|$, where c_2 is a fixed constant. If d_1 is small enough, then $(\partial G/\partial n_1)(x_1, x_2) \leq 0$.*

References

[1] M. Ahmedou and K. El Mehdi, *Computation of the difference of topology at infinity for Yamabe-type problem on annuli-domains,* Duke Math. J., 94 (1998), 215–229, 231–255.

[2] M. Ahmedou and K. El Mehdi, *On an elliptic problem with critical nonlinearity in expanding annuli,* J. Funct. Anal., 163 (1999), 29–62.

[3] T. Aubin, "Some Nonlinear Problem in Defferential Geometry," Springer Verlag, New York, 1997.

[4] T. Aubin and A. Bahri, *Méthodes de topologie algebrique pour le problème de la courbure scalaire prescrite,* J. Math. Pures and Appl., 76 (1997), 525–549.

[5] A. Bahri, *Critical points at infinity in some variational problems,* Pitman Res. Notes Math. Ser., 182, Longman Sci. Tech. Harlow (1989).

[6] A. Bahri, *An invariant for Yamabe-type flows with applications to scalar-curvature problems in high dimension,* Duke Math. J., 281 (1996), 323–466.

[7] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical Sobolev exponent: The effect of the topology on the domain,* Comm. Pure Appl. Math., 41 (1988), 253–294.

[8] A. Bahri and J.M. Coron, *The scalar curvature problem on the standard three-dimensional sphere,* J. Func. Anal., 95 (1991), 106–172.

[9] A. Bahri, Y.Y. Li, and O. Rey, *On a variational problem with lack of compactness: The topological effect of the critical points at infinity,* Calculus of Variation and Partial Diff. Equa., 3 (1995), 67–94.

[10] A. Bahri and P. Rabinowitz, *Periodics orbits of Hamiltonian system on the standard three body type,* Ann. Ins. H. Poincaré Anal. non linéaire., 8 (1991), 561–649.

[11] M. Ben Ayed, Y. Chen, H. Chtioui, and M. Hammami, *On the prescribed scalar curvature problem on 4-manifolds,* Duke Math. J., 84 (1996), 633–667.

[12] M. Ben Ayed, H. Chtioui, and M. Hammami, *The scalar-curvature problem on higher dimensional spheres,* Duke Math. J., 93 (1998), 379–424.

[13] M. Ben Ayed, H. Chtioui, and M. Hammami, *A Morse lemma at infinity for Yamabe type problems on domains,* Ann. I. H. Poincaré, AN., 20 (2003), 543–577

[14] H. Brezis and J.M. Coron, *Convergence of solutions of H-systems or how to blow bubbles,* Arch. Rational Mech. Anal., 89 (1985), 21–56.

[15] E. Hebey, *La méthode de isometrie-concentration dans le cas d'un problème non linéaire sur les variétés compactes à bord,* Bull. Sci. Math., 116 (1992), 35–51.

[16] E. Hebey, *Asymptotics for some quasi-linear elliptic equations,* Diff. Int. Equat., 9 (1996), 89–106.

[17] E. Hebey, *Introduction d'analyse non linéaire sur les variétés,* Diderot Editeur, Paris, 1997.

[18] P.L. Lions, *The concentration compactness principle in the calculus of variations. The limit case,* Rev. Mat. Iberoamericana, 1 (1985), 165–201; 45–121.

[19] O. Rey, *The role of the Green's function in a nonlinear elliptic equation involving the critical Sobolev exponent,* J. Funct. Anal., 89 (1990), 1–52.

[20] M. Struwe, *A global compactness result for elliptic boundary value problem involving limiting nonlinearities*, Math. Z., 187 (1984), 511–517.