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THE MILLER SCHEME IN SEMIGROUP THEORY

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Abstract. In this paper we apply a set-up introduced by R. K. Miller to transform a linear, inhomogeneous Cauchy problem for the generator of a semigroup on a Banach space into a homogeneous one for a matrix operator, which is the generator of a semigroup on a suitable product space. By using restriction theorems and extrapolation spaces we obtain new results for the inhomogeneous Cauchy problem for Hille-Yosida operators in Favard spaces.

1. Introduction

In [10], R. K. Miller proposed a method for reducing an inhomogeneous, linear Volterra integral equation in a Banach space to a homogeneous differential equation in the product of two Banach spaces containing the initial value and the forcing term of the integral equation. Since then this method has been successfully used by many authors, e.g., R. Grimmer, W. Desch, and W. Schappacher et al. (see references in [13], p. 338).

In this paper we use this scheme to show that suitable matrix operators are generators of strongly continuous semigroups on product spaces. These abstract results are then applied to the Cauchy problem

$$(P) \quad \begin{cases} u'(t) = \Lambda u(t) + f(t), \ t \geq 0, \\ u(0) = x, \end{cases}$$

where $\Lambda : D(\Lambda) \subset E \to E$ is the generator of a semigroup on the Banach space E. We will obtain a solution of (P) as the first component of the

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solution $\mathcal{U}$ of the problem

$$(Q) \quad \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), \ t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f}, \end{cases}$$

where $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ is the generator of a semigroup on the Banach space $Z = E_1 \oplus \mathcal{F}$. Here, E_1 is a suitably chosen Banach space with $E_1 \hookrightarrow E$ and $\mathcal{F}$ is a space of functions from $\mathbb{R}_+$ to E.

By using some interpolation spaces (e.g., the Favard spaces of Λ) and the associated extrapolation spaces we will extend the results of R. Phillips [12] and T. Kato [9] for equation (P) as well as those of G. Da Prato and E. Sinestrari ([3], [4]) in the case when Λ is only a Hille-Yosida operator. In the last section we will illustrate these results in a simple but significant example.

We will use the following notation. If E is a Banach space, then $\mathcal{L}(E)$ denotes the space of all bounded, linear operators on E. For a linear operator $\Lambda : D(\Lambda) \subset E \to E$, we endow its domain $D(\Lambda)$ with the graph norm and write $\rho(\Lambda)$ for its resolvent set. If I is an interval in $\mathbb{R}$, then $L^1(I;E)$ denotes the Banach space of the (classes of) functions $u : I \to E$ which are Bochner integrable. We will use also the following spaces:

$$C(I;E) = \{u : \ I \to E \text{ continuous}\},$$

$$BUC(I;E) = \{u : \ I \to E \text{ bounded and uniformly continuous}\},$$

$$C^1(I;E) = \{u : \ I \to E \text{ continuously differentiable}\},$$

$$BUC^1(I;E) = \left\{u : \ I \to E, \ u, \ u' \in BUC(E)\right\},$$

$$W^{1,1}(I;E) = \{u \in C(I;E) \cap L^1(I;E), \ \exists\, v \in L^1(I;E),$$

$$u(s_1) - u(s_2) = \int_{s_1}^{s_2} v(\sigma)d\sigma \text{ for } s_1, \ s_2 \in I\}.$$

When $I = \mathbb{R}_+ = [0, +\infty)$ we will simply write $C(\mathbb{R}_+;E) = C(E)$ and so on.

2. SEMIGROUPS AND FAVARD SPACES

We start from a Banach space $(E, \|\cdot\|)$ and the generator $\Lambda : \ D(\Lambda) \subset E \to E$ of a (strongly continuous) semigroup $(e^{\Lambda t})_{t \geq 0}$ on E. If

$$\|e^{\Lambda t}\|_{\mathcal{L}(E)} \leq M e^{\omega t}, \quad t \geq 0, \tag{2.1}$$

we write $\Lambda \in \mathcal{G}(M, \omega)$. We also set

$$\omega' := \max(\omega, 0). \tag{2.2}$$

We list some known definitions and results (see [2], [7], [8], and [11]).

Definition 2.1. *The Favard space associated with* Λ *is the Banach space*

$$D_\Lambda(1,\infty) := \Big\{x \in E;\ [x]_{1,\infty} := \sup_{0<t\le 1} \frac{\|e^{\Lambda t}x - x\|}{t} < \infty\Big\}$$

endowed with the norm $\|x\|_{D_\Lambda(1,\infty)} := \|x\| + [x]_{1,\infty}$.

This space is a limiting case of the real interpolation spaces $D_\Lambda(\theta,p)$ defined by J. L. Lions for $\theta \in (0,1)$ and $p \in [1,\infty]$ (see Section 3.4 of [2]). We have $D(\Lambda) \hookrightarrow D_\Lambda(1,\infty)$, and the two spaces coincide if, e.g., E is a reflexive space.

An equivalent definition (which can be used also for operators Λ satisfying only $\varrho(\Lambda) \supseteq (\omega,+\infty)$ for some $\omega \in \mathbb{R}$) is given by the following.

Theorem 2.2. *The Favard space is given by*

$$D_\Lambda(1,\infty) := \Big\{x \in E;\ [x]'_{1,\infty} := \sup_{\lambda>\omega} \|\lambda(\lambda-\Lambda)^{-1}x\|\Big\},$$

and

$$\|x\|'_{D_\Lambda(1,\infty)} := \|x\| + [x]'_{1,\infty}$$

is a norm equivalent to $\|x\|_{D_\Lambda(1,\infty)}$.

The following result will be useful in the sequel.

Theorem 2.3. *Let* F *be a Banach space and* $F \subseteq E$. *We have*

$$F \hookrightarrow D_\Lambda(1,\infty) \tag{2.3}$$

if and only if there is $\hat{c} > 0$ *such that*

$$\sup_{0<t\le 1} \Big\|\frac{1}{t}\int_0^t e^{\Lambda s}x\,ds\Big\|_{D(\Lambda)} \le \hat{c}\|x\|_F \quad \textit{for each } \ x \in F. \tag{2.4}$$

Proof. For $x \in E$ and $t > 0$ we have

$$\Big\|\frac{1}{t}\int_0^t e^{\Lambda s}x\ ds\Big\|_{D(\Lambda)} = \Big\|\frac{1}{t}\int_0^t e^{\Lambda s}x\ ds\Big\| + \Big\|\frac{e^{\Lambda t}x - x}{t}\Big\|. \tag{2.5}$$

Hence, if (2.3) holds and $x \in F$, we deduce $x \in D_\Lambda(1,\infty)$, and so

$$\Big\|\frac{1}{t}\int_0^t e^{\Lambda s}x\,ds\Big\|_{D(\Lambda)} \le Me^{\omega'}\|x\| + [x]_{1,\infty} \text{ for } t \in (0,1];$$

hence, (2.4) follows.

If (2.4) is true, then given $x \in F$ we have

$$\Big\|\frac{1}{t}\int_0^t e^{\Lambda s}x\,ds\Big\| \le \hat{c}\|x\|_F \text{ for } t \in (0,1].$$

By taking the limit as $t \to 0$ we conclude $\|x\| \le \hat{c}\|x\|_F$. By (2.4) and (2.5) we also have

$$\left\| \frac{e^{\Lambda t}x - x}{t} \right\| \le \hat{c}\|x\|_F \text{ for } t \in (0,1];$$

hence, $[x]_{1,\infty} \le 2\hat{c}\|x\|_F$. In conclusion, (2.3) holds. □

In this paper $(S(t))_{t\ge 0}$ will always denote the left-translation semigroup on some function space on $\mathbb{R}_+$; i.e., if $f:\ \mathbb{R}_+ \to E$, then $S(t)f$ is the function

$$(S(t)f)(s) = f(t+s), \quad s \ge 0. \tag{2.6}$$

The following theorem is well known (see e.g. [2] or [7], II.2.10).

Theorem 2.4. *The semigroup $(S(t))_{t\ge 0}$ is strongly continuous on the space $L^1(E)$ with generator $\frac{d}{ds}$ and domain $W^{1,1}(E)$, where $\frac{d}{ds}$ denotes the derivative almost everywhere of a function in $W^{1,1}(E)$. Moreover, $(S(t))_{t\ge 0}$ is a strongly continuous semigroup on the space $BUC(E)$ with generator $\frac{d}{ds}$ and domain $BUC^1(E)$.*

We will also use the following restriction theorem for semigroups.

Theorem 2.5. *Let $\Lambda:\ D(\Lambda) \subset E \to E$ be the generator of the semigroup $(e^{\Lambda t})_{t\ge 0}$ on E, and let E_1 be a Banach space such that $E_1 \hookrightarrow E$. If the restrictions $e^{\Lambda t}|_{E_1}$ define a semigroup on E_1, then its generator Λ_1 is $\Lambda|_{D(\Lambda_1)}$, where $D(\Lambda_1) := \{x \in D(\Lambda) \cap E_1;\ \Lambda x \in E_1\}$. Conversely, if the operator Λ_1 generates a semigroup, then this semigroup coincides with $(e^{\Lambda t}|_{E_1})_{t\ge 0}$.*

Proof. For the first part see [7], II.2.3. For the second part, observe that for sufficiently large λ we have $(\lambda - \Lambda_1)^{-1} = (\lambda - \Lambda)^{-1}|_{E_1}$, and so

$$\int_0^{+\infty} e^{-\lambda t} e^{\Lambda_1 t} x\, dt = \int_0^{+\infty} e^{-\lambda t} e^{\Lambda t} x\, dt \text{ for each } x \in E_1.$$

By the uniqueness of the Laplace transform, this implies (see page 260 of [6]) that $e^{\Lambda_1 t}x = e^{\Lambda t}x$ for $t \ge 0$ and $x \in E_1$. □

3. The Miller scheme

We will describe the Miller scheme, appearing first in [10], to transform the inhomogeneous Cauchy problem,

$$(P) \qquad \begin{cases} u'(t) = \Lambda u(t) + f(t), & t \ge 0, \\ u(0) = x, \end{cases}$$

into a homogeneous Cauchy problem. We will also show that this construction is unique in a certain sense.

We suppose in (P) that (at least) $f \in L^1_{loc}(E)$ and $u \in W^{1,1}_{loc}(E)$ so that the equation (P) holds almost everywhere in $\mathbb{R}_+$. In this case it is known that the only possible solution is given by

$$u(t) = e^{\Lambda t}x + (e^{\Lambda} * f)(t), \quad t \geq 0,$$

where

$$(e^{\Lambda} * f)(t) = \int_0^t e^{\Lambda(t-s)} f(s)\, ds.$$

If we now want that the solution u of (P) should be given by the first component of the solution of a homogeneous Cauchy problem,

$$(Q) \qquad \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f}, \end{cases}$$

where $\mathcal{A}\colon\ D(\mathcal{A}) \subset Z \to Z$ is the generator of a suitable semigroup on the product space $Z := E \oplus L^1(E)$, then this semigroup can be defined in only one way and gives rise to the Miller scheme. This is stated more precisely in the following theorem.

Theorem 3.1. *Let $(G(t))_{t\geq 0}$ be a semigroup on $Z := E \oplus L^1(E)$ with generator $\mathcal{A}\colon\ D(\mathcal{A}) \subset Z \to Z$ such that if $\binom{x}{f} \in D(\mathcal{A})$, then*

$$u(t) := pr_1 G(t) \begin{pmatrix} x \\ f \end{pmatrix} \tag{3.1}$$

is a solution of (P). Then we have

$$G(t) = \begin{pmatrix} e^{\Lambda t} & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0, \tag{3.2}$$

where $R(t)$ is defined by

$$R(t)f := (e^{\Lambda} * f)(t) \tag{3.3}$$

for each $f \in L^1(E)$ and $S(t)$ is the translation given by (2.6). The same result holds by substituting $BUC(E)$ for $L^1(E)$.

Proof. We first prove the following assertion. If $f \in L^1(0,T;E)$ is such that

$$\int_0^t e^{\Lambda(t-s)} f(s)\, ds = 0, \quad t \in [0,T], \tag{3.4}$$

then $f(t) = 0$ for $t \in [0,T]$ almost everywhere. To this end fix $\tau \in (0,T]$ and integrate (3.4); i.e.,

$$0 = \int_0^\tau dt \int_0^t e^{\Lambda(t-s)} f(s)\, ds = \int_0^\tau ds \int_s^\tau e^{\Lambda(t-s)} f(s)\, dt.$$

By applying Λ we obtain

$$0 = \Lambda \int_0^\tau ds \int_s^\tau e^{\Lambda(t-s)} f(s)\, dt = \int_0^\tau (e^{\Lambda(\tau-s)} - I) f(s) ds$$
$$= \int_0^\tau e^{\Lambda(\tau-s)} f(s)\, ds - \int_0^\tau f(s) ds,$$

and so by (3.4)

$$\int_0^\tau f(s) ds = 0 \text{ for all } \tau \in (0, T];$$

hence, $f = 0$ almost everywhere in $[0, T]$.

Let us now prove the theorem. For any $t \geq 0$, the functions $\binom{x}{f} \to pr_1 G(t)\binom{x}{f}$ and $\binom{x}{f} \to e^{\Lambda(t)}x + R(t)f$ are continuous from Z to E. Since, by assumption, they coincide on the dense subspace $D(\mathcal{A})$, we have

$$pr_1 G(t)\begin{pmatrix} x \\ f \end{pmatrix} = e^{\Lambda t}x + R(t)f \quad \text{for each } \begin{pmatrix} x \\ f \end{pmatrix} \in Z.$$

Let us define operators $P(t)$ from Z into $L^1(E)$ by requiring

$$G(t)\begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} e^{\Lambda t}x + R(t)f \\ P(t)\binom{x}{f} \end{pmatrix} \quad \text{for each } \begin{pmatrix} x \\ f \end{pmatrix} \in Z. \tag{3.5}$$

Since $pr_1 G(\tau) G(t)\binom{x}{f} = pr_1 G(\tau + t)\binom{x}{f}$ for $t \geq 0$ and $\binom{x}{f} \in Z$, we deduce

$$e^{\Lambda\tau} R(t)f + R(\tau)P(t)\begin{pmatrix} x \\ f \end{pmatrix} = R(\tau + t)f. \tag{3.6}$$

Setting $f = 0$ we obtain $R(\tau)P(t)\binom{x}{0} = 0$. From the above assertion we deduce

$$P(t)\begin{pmatrix} x \\ 0 \end{pmatrix} = 0. \tag{3.7}$$

Setting $x = 0$, (3.6) becomes

$$e^{\Lambda\tau} R(t)f + R(\tau)P(t)\begin{pmatrix} 0 \\ f \end{pmatrix} = R(\tau + t)f;$$

i.e.,

$$\int_0^t e^{\Lambda(\tau+t-s)} f(s) ds + \int_0^\tau e^{\Lambda(\tau-s)} P(t)\begin{pmatrix} 0 \\ f \end{pmatrix}(s) ds = \int_0^{\tau+t} e^{\Lambda(\tau+t-s)} f(s) ds.$$

We deduce

$$\int_0^\tau e^{\Lambda(\tau-s)} P(t)\begin{pmatrix} 0 \\ f \end{pmatrix}(s) ds = \int_t^{\tau+t} e^{\Lambda(\tau+t-s)} f(s) ds = \int_0^\tau e^{\Lambda(\tau-s)} f(t+s) ds,$$

and so

$$R(\tau)[P(t)\binom{0}{f} - S(t)f] = 0.$$

Again by the above assertion we obtain

$$P(t)\binom{0}{f} = S(t)f.$$

From this and (3.7) we deduce that $P(t)\binom{x}{f} = S(t)f$, and from (3.5) we deduce (3.2).

The proof of the second case in the theorem is analogous. □

This theorem suggests that the study of the inhomogeneous problem (P) in E can be made through the homogeneous problem (Q) in Z, and that (P) is well-posed if $\mathcal{A}$ is the generator of a semigroup in Z.

In the next section we prove this fact when $Z = E \oplus L^1(E)$. Then we prove restriction theorems for the semigroup $(G(t))_{t\geq 0}$. Each result has important applications to problem (P), giving solutions with different types of regularity.

4. Homogenization in L^1

† In this section we show that the family $(G(t))_{t\geq 0}$ of matrix operators defined by (3.2) is a semigroup on $Z = E \oplus L^1(E)$, and we then characterize its generator. As an application we obtain easily a first generalization of the classical temporal regularity result by Phillips (see [12] or [7], Proposition VI.7.4–7.6).

Theorem 4.1. *The family of matrix operators*

$$G(t) = \begin{pmatrix} e^{\Lambda t} & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0, \tag{4.1}$$

is a semigroup on the space $Z = E \oplus L^1(E)$. Its generator $\mathcal{A}\colon\ D(\mathcal{A}) \subset Z \to Z$ is given by

$$\mathcal{A} = \begin{pmatrix} \Lambda & \delta \\ 0 & \frac{d}{ds} \end{pmatrix}, \tag{4.2}$$

$$D(\mathcal{A}) = D(\Lambda) \oplus W^{1,1}(E), \tag{4.3}$$

where for each $f \in W^{1,1}(E)$ we have set $\delta f := f(0)$ and $\frac{d}{ds}$ is the derivative almost everywhere.

†(Here "homogenization" means "reduction of an inhomogeneous problem to an equivalent homogeneous one.")

Proof. It can be seen directly that $G(t) \in \mathcal{L}(Z)$ and the family $G(t)$, $t \geq 0$, satisfies the semigroup property. Its strong continuity is a consequence of the fact that

$$\lim_{t \to 0} \|R(t)f\|_{L^1(E)} = 0 \tag{4.4}$$

for each $f \in L^1(E)$. Hence $(G(t))_{t \geq 0}$ is a semigroup. If $\hat{\mathcal{A}} : D(\hat{\mathcal{A}}) \subset Z \to Z$ is its generator, $\binom{x}{f} \in D(\hat{\mathcal{A}})$, and $\hat{\mathcal{A}}\binom{x}{f} = \binom{y}{g}$, we have

$$\lim_{t \to 0} \| \frac{e^{\Lambda t}x - x}{t} - y + \frac{R(t)f}{t} \| = 0, \tag{4.5}$$

$$\lim_{t \to 0} \| \frac{S(t)f - f}{t} - g \|_{L^1(E)} = 0. \tag{4.6}$$

From this and Theorem 2.4 we deduce that $f \in W^{1,1}(E)$ and $g = f'$ almost everywhere in $\mathbb{R}_+$.

From the continuity of f at 0 we obtain

$$\lim_{t \to 0} \| \frac{1}{t} R(t)f - f(0) \| = 0,$$

and so from (4.5)

$$\lim_{t \to 0} \frac{e^{\Lambda t}x - x}{t} = y - f(0);$$

hence, $x \in D(\Lambda)$ and $\Lambda x = y - f(0)$.

This shows that $\hat{\mathcal{A}}$ is a restriction of $\mathcal{A}$ as defined by (4.2)–(4.3). However, if $\binom{x}{f} \in D(\Lambda) \oplus W^{1,1}(E)$, we see that (4.5)–(4.6) hold with $y = \Lambda x$ and $g = f'$ almost everywhere in $\mathbb{R}_+$. Hence $D(\hat{\mathcal{A}}) = D(\Lambda) \oplus W^{1,1}(E)$, and so $\mathcal{A} = \hat{\mathcal{A}}$. $\square$

As a corollary we obtain a generalization of the Phillips temporal-regularity result for problem (P).

Theorem 4.2. *If*

$$f \in W^{1,1}(E) \quad \text{and} \quad x \in D(\Lambda), \tag{4.7}$$

then the function

$$u(t) = e^{\Lambda t}x + (e^{\Lambda} * f)(t), \quad t \geq 0, \tag{4.8}$$

satisfies

$$u \in C^1(E) \cap C(D(\Lambda)) \tag{4.9}$$

and is the unique solution of problem

$$(P) \quad \begin{cases} u'(t) = \Lambda u(t) + f(t), \ t \geq 0, \\ u(0) = x. \end{cases}$$

Setting

$$x_1 := \Lambda x + f(0), \tag{4.10}$$

we have

$$u'(t) = e^{\Lambda t} x_1 + (e^{\Lambda} * f')(t), \quad t \geq 0, \tag{4.11}$$

and the estimates

$$\|u(t)\| \leq M e^{\omega t} \left(\|x\| + \int_0^t e^{-\omega s} \|f(s)\| ds \right), \tag{4.12}$$

$$\|u'(t)\| \leq M e^{\omega t} \left(\|x_1\| + \int_0^t e^{-\omega s} \|f'(s)\| ds \right). \tag{4.13}$$

Proof. In the space $Z = E \oplus L^1(E)$ and for the operator $\mathcal{A}$ defined above we consider the problem

$$(Q) \quad \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), \ t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f}. \end{cases}$$

Now, setting $\mathcal{U}(t) = \binom{u(t)}{v(t)}$, this can be written as

$$(\hat{Q}) \quad \begin{cases} u'(t) = \Lambda u(t) + \delta(v(t)), \quad t \geq 0, \\ v'(t) = \frac{d}{ds}(v(t)), \quad t \geq 0, \\ u(0) = x, \\ v(0) = f. \end{cases}$$

From Theorem 4.1 we know that $\mathcal{A}$ generates the semigroup $(G(t))_{t\geq 0}$ and $\binom{x}{f} \in D(\mathcal{A})$. Hence (Q) has a solution $\mathcal{U} \in C^1(Z)$ given by $\mathcal{U}(t) = G(t)\binom{x}{f}$; i.e,

$$\begin{cases} u(t) = e^{\Lambda t} x + (e^{\Lambda} * f)(t), \\ v(t) = S(t) f. \end{cases} \tag{4.14}$$

For $t, s \geq 0$ we have $v(t)(s) = (S(t)f)(s) = f(t+s)$, and so $\delta(v(t)) = f(t)$. Substituting this in the first equation of $(\hat{Q})$, we deduce that u given by (4.14) is a solution of (P) and $u \in C^1(E)$. From (P) we also deduce that $u \in C(D(\Lambda))$.

By using (Q) we obtain

$$G'(t)\begin{pmatrix} x \\ f \end{pmatrix} = \mathcal{A}G(t)\begin{pmatrix} x \\ f \end{pmatrix} = G(t)\mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} = G(t)\begin{pmatrix} x_1 \\ f' \end{pmatrix},$$

which implies (4.11). Estimates (4.12)–(4.13) are a consequence of (4.8) and (4.11).

To prove the uniqueness for problem (P) let us suppose that $u \in C^1(E) \cap C(D(\Lambda))$ is a solution of (P) with $f = 0$ and $x = 0$. Setting $\mathcal{U}(t) = \binom{u(t)}{0}$, we see that $\mathcal{U} \in C^1(Z) \cap C(D(\mathcal{A}))$ and $\mathcal{U}$ is a solution of (Q) with $x = 0$ and $f = 0$. Hence $u(t) \equiv 0$, and the conclusion follows. □

5. Restriction theorems in L^1

The result of the previous section shows that a generation theorem for the semigroup $(G(t))_{t\geq 0}$ yields a regularity theorem for problem (P). Since we already know that $(G(t))_{t\geq 0}$ is a semigroup in $Z = E \oplus L^1(E)$, we shall first characterize (using Theorem 2.5) some subspaces of Z where $(G(t))_{t\geq 0}$ still induces a semigroup.

Theorem 5.1. *Let E_1 and $\mathcal{F}$ be Banach spaces such that*

$$E_1 \hookrightarrow E \quad \text{and} \quad \mathcal{F} \hookrightarrow L^1(E). \tag{5.1}$$

Setting

$$Z_* := E_1 \oplus \mathcal{F} \quad \text{and} \quad G_*(t) := G(t)|_{Z_*}, \tag{5.2}$$

we have that $(G_*(t))_{t\geq 0}$ is a semigroup in Z_* if and only if the following properties hold.

(i) *$(e^{\Lambda t}|_{E_1})_{t\geq 0}$ is a semigroup on E_1.*
(ii) *$(S(t)|_{\mathcal{F}})_{t\geq 0}$ is a semigroup on $\mathcal{F}$.*
(iii) *If $f \in \mathcal{F}$, then $R(t)f \in E_1$ for $t \geq 0$ and $R(t) \in \mathcal{L}(\mathcal{F}, E_1)$.*
(iv) *There exists $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|R(t)f\|_{E_1} \leq \gamma(t)\|f\|_{\mathcal{F}} \text{ for } f \in \mathcal{F} \text{ and } t \geq 0.$$

(v) *Given $f \in \mathcal{F}$ we have*

$$\lim_{t\downarrow 0} \|R(t)f\|_{E_1} = 0.$$

If $(G_*(t))_{t\geq 0}$ is a semigroup in Z_*, then its generator $\mathcal{A}_*$ is the restriction of $\mathcal{A}$ with domain

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus W^{1,1}(E);\ x,\ x_1 \in E_1 \ \text{and}\ f,\ f' \in \mathcal{F} \right\}, \tag{5.3}$$

where $x_1 := \Lambda x + f(0)$.

Proof. If $(G_*(t))_{t\geq 0}$ is a semigroup in Z_*, consider the subspaces $\{\binom{x}{0}; x \in E_1\}$ and $\{\binom{0}{f}; f \in \mathcal{F}\}$ to see that properties (i)–(v) hold with $\gamma(t) := \|G_*(t)\|_{\mathcal{L}(Z_*)}$.

Conversely, if these are true, it can be checked that $(G(t)|_{Z_*})_{t\geq 0}$ is a semigroup in Z_*. So by virtue of Theorem 2.5 its generator $\mathcal{A}_*$ is the restriction of $\mathcal{A}$ with domain

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{A}) \cap Z_*;\ \mathcal{A}\begin{pmatrix} x \\ f \end{pmatrix} \in Z_* \right\},$$

which coincides with (5.3) □

As in Theorem 4.2 we can deduce a temporal regularity result for problem (P).

Theorem 5.2. *Let E_1 and $\mathcal{F}$ satisfy the assumptions of Theorem* 5.1. *If*

$$f \in W^{1,1}(E) \ \ \textit{and}\ x \in D(\Lambda) \tag{5.4}$$

are such that

$$f,\ f' \in \mathcal{F} \ \ \textit{and}\ \ x,\ x_1 \in E_1,\ \ \textit{for}\ \ x_1 := \Lambda x + f(0), \tag{5.5}$$

then

$$u(t) = e^{\Lambda t}x + (e^{\Lambda} * f)(t), \quad t \geq 0, \tag{5.6}$$

is the unique solution

$$u \in C^1(E_1) \cap C(D(\Lambda)) \tag{5.7}$$

of problem (P). Moreover, we have

$$u'(t) = e^{\Lambda t}x_1 + (e^{\Lambda} * f')(t), \quad t \geq 0. \tag{5.8}$$

In addition, if $\|e^{\Lambda t}\|_{\mathcal{L}(E_1)} \leq M_1 e^{\omega_1 t}$, $t \geq 0$, then

$$\|u(t)\|_{E_1} \leq M_1 e^{\omega_1 t}\|x\|_{E_1} + \gamma(t)\|f\|_{\mathcal{F}}, \tag{5.9}$$

$$\|u'(t)\|_{E_1} \leq M_1 e^{\omega_1 t}\|x_1\|_{E_1} + \gamma(t)\|f'\|_{\mathcal{F}} \quad \textit{for}\ \ t \geq 0. \tag{5.10}$$

Proof. We can proceed as in the proof of Theorem 4.2 with $G(t)$ and Z replaced by $G_*(t)$ and Z_*. □

We now apply Theorem 5.1 to the case in which $\mathcal{F} = L^1(F)$ with F a suitable Banach space such that $F \hookrightarrow E$. With this choice the conditions (i)–(v) in Theorem 5.1 can be reduced.

Proposition 5.3. *Let E_1 and F be Banach spaces satisfying the following properties.*

(i) *$E_1 \hookrightarrow E$ and $F \hookrightarrow E$.*
(ii) *If $f \in L^1(F)$, then $R(t)f \in E_1$ for all $t \geq 0$.*
(iii) *There exists $\gamma :\ \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that*

$$\|R(t)f\|_{E_1} \leq \gamma(t)\|f\|_{L^1(F)} \ \ \textit{for}\ \ f \in L^1(F) \ \ \textit{and}\ \ t \geq 0. \tag{5.11}$$

Then for each $f \in L^1(F)$ *and* $t > 0$ *we have*

$$\|R(t)f\|_{E_1} \leq \gamma(t)\|f\|_{L^1(0,\,t;F)}, \tag{5.12}$$

and for each $T > 0$

$$\|R(t)f\|_{E_1} \leq \gamma(T)\|f\|_{L^1(0,\,t;F)} \ \textit{for all}\, t \in [0,T], \tag{5.13}$$

and so

$$\lim_{t\downarrow 0}\|R(t)f\|_{E_1} = 0. \tag{5.14}$$

Proof. Let (i)–(iii) hold. Given $f \in L^1(F)$ and $t > 0$, set

$$f_0(s) := \begin{cases} f(s), & 0 \leq s \leq t, \\ 0, & s > t. \end{cases}$$

Since $f_0 \in L^1(F)$, we deduce from (5.11) that

$$\|R(t)f\|_{E_1} = \|R(t)f_0\|_{E_1} \leq \gamma(t)\|f_0\|_{L^1(F)} = \gamma(t)\|f\|_{L^1(0,\,t;F)},$$

i.e., estimate (5.12). Fix now $\delta > 0$ and set

$$f_\delta(s) := \begin{cases} 0, & 0 \leq s \leq \delta, \\ f(s-\delta), & s > \delta. \end{cases}$$

Since $f_\delta \in L^1(F)$, we obtain from (5.12)

$$\begin{aligned} \| \int_0^t e^{\Lambda(t-s)} f(s)\, ds\|_{E_1} &= \| \int_\delta^{t+\delta} e^{\Lambda(t+\delta-s)} f(s-\delta)\, ds\|_{E_1} \\ &= \| \int_0^{t+\delta} e^{\Lambda(t+\delta-s)} f_\delta(s)\, ds\|_{E_1} \\ &\leq \gamma(t+\delta)\|f_\delta\|_{L^1(0,\,t+\delta;F)} = \gamma(t+\delta)\|f\|_{L^1(0,\,t;F)}. \end{aligned}$$

If $0 < t < T$, we obtain (5.13) by setting $\delta := T - t$. From this we deduce (5.14). □

We can now apply the Restriction Theorem 5.1 to the case $\mathcal{F} := L^1(F)$ taking into account the preceding proposition.

Theorem 5.4. *Let* E_1 *and* F *be Banach spaces such that*

$$E_1 \hookrightarrow E \quad \textit{and} \quad F \hookrightarrow E. \tag{5.15}$$

Setting

$$Z_* := E_1 \oplus L^1(F) \quad \textit{and} \quad G_*(t) := G(t)|_{Z_*},$$

we have that $(G_*(t))_{t\geq 0}$ *is a semigroup on* Z_* *if and only if the following properties hold.*

(i) $(e^{\Lambda t}|_{E_1})_{t\geq 0}$ *is a semigroup on* E_1.
(ii) *If* $f \in L^1(F)$, *then* $R(t)f \in E_1$ *for* $t \geq 0$.
(iii) *There exists* $\gamma : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *such that*

$$\|R(t)f\|_{E_1} \leq \gamma(t)\|f\|_{L^1(F)} \quad \text{for } f \in L^1(F) \text{ and } t \geq 0.$$

If $(G_*(t))_{t\geq 0}$ *is a semigroup on* Z_*, *then its generator* $\mathcal{A}_*$ *is the restriction of* $\mathcal{A}$ *to the domain*

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus W^{1,1}(F);\ x,\ x_1 \in E_1 \right\}, \tag{5.16}$$

where $x_1 := \Lambda x + f(0)$.

Proof. The assumptions of Theorem 5.1 are satisfied with $\mathcal{F} = L^1(E)$ because (ii) and (iii) imply (5.14) by virtue of Proposition 5.3. From (5.3) we deduce that

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus W^{1,1}(E);\ x,\ x_1 \in E_1;\ f,\ f' \in L^1(F) \right\}, \tag{5.17}$$

which coincides with (5.16). □

The temporal regularity result for problem (P) which can be deduced from Theorem 5.2 is the following.

Theorem 5.5. *Let* E_1 *and* F *satisfy* (5.15) *and* (i)–(ii) *of Theorem* 5.4. *Given* $f \in W^{1,1}(F)$ *and* $x \in D(\Lambda) \cap E_1$ *such that* $x_1 := \Lambda x + f(0) \in E_1$, *then* u, *given by* (5.6), *is the unique solution of problem* (P) *such that* $u \in C^1(E_1) \cap C(D(\Lambda))$. *Moreover,* (5.8)–(5.10) *hold with* $\mathcal{F} = L^1(F)$.

6. Favard spaces and regularity

We will see now that if we choose in the Restriction Theorem 5.1 the space $E_1 := D(\Lambda)$ and $\mathcal{F} = L^1(F)$, then all the conditions (i)–(v) can be summarized by the imbedding of F into the Favard space of Λ as in (2.3). To this end we will need the following result by W. Desch-W. Schappacher (see [5]).

Theorem 6.1. *If* $f \in L^1(D_\Lambda(1,\infty))$, *then*

$$e^{\Lambda} * f \in C(D(\Lambda)) \tag{6.1}$$

and

$$\|(e^{\Lambda} * f)(t)\|_{D(\Lambda)} \leq M e^{\omega t} \int_0^t e^{-\omega s} \|f(s)\|_{D_\Lambda(1,\infty)}\, ds \ \text{ for } t \geq 0. \tag{6.2}$$

Proof. Given $f \in L^1(D_\Lambda(1,\infty))$ there exist $f_n \in W^{1,1}(D_\Lambda(1,\infty))$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} \|f_n - f\|_{L^1(D_\Lambda(1,\infty))} = 0. \tag{6.3}$$

Hence, for each $t > 0$ we obtain

$$\lim_{n\to\infty} \|(e^\Lambda * f_n)(t) - (e^\Lambda * f)(t)\| = 0. \tag{6.4}$$

If $g \in W^{1,1}(D_\Lambda(1,\infty))$, we deduce from Theorem 4.2 that

$$e^\Lambda * g \in C(D(\Lambda)).$$

For $0 < h \leq 1$ and $0 \leq t$ we have the estimate

$$\begin{aligned} \| \int_0^t e^{\Lambda(t-s)} g(s)\, ds\| + \| \frac{e^{\Lambda h} - I}{h} \int_0^t e^{\Lambda(t-s)} g(s)\, ds\| \\ \leq M e^{\omega t} \int_0^t e^{-\omega s} \Big(\|g(s)\| + \sup_{0<h\leq 1} \| \frac{e^{\Lambda h} - I}{h} g(s)\| \Big)\, ds. \end{aligned}$$

Letting $h \downarrow 0$ on the left-hand side and using the fact that $e^\Lambda * g \in C(D(\Lambda))$ we obtain

$$\| \int_0^t e^{\Lambda(t-s)} g(s)\, ds\|_{D(\Lambda)} \leq M e^{\omega t} \int_0^t e^{-\omega s} \|g(s)\|_{D_\Lambda(1,\infty)}\, ds. \tag{6.5}$$

Replacing g by $f_n - f_m$ we deduce for $T > 0$ fixed

$$\sup_{0\leq t\leq T} \|e^\Lambda * (f_n - f_m)(t)\|_{D(\Lambda)} \leq M e^{\omega' T} \|f_n - f_m\|_{L^1(0,T;D_\Lambda(1,\infty))}.$$

By virtue of (6.3) we can conclude that $\{e^\Lambda * f\}$ is a Cauchy sequence in $C(0,T;D(\Lambda))$, and so (by using also (6.4)) we have

$$e^\Lambda * f \in C(0,T;D(\Lambda)) \tag{6.6}$$

and

$$\lim_{n\to\infty} \|e^\Lambda * f_n - e^\Lambda * f\|_{C(0,T;D(\Lambda))} = 0. \tag{6.7}$$

Taking $g = f_n$ in (6.5) and letting $n \to \infty$ we obtain from (6.7) and (6.3) that

$$\| \int_0^t e^{\Lambda(t-s)} f(s)\, ds\|_{D(\Lambda)} \leq M e^{\omega t} \int_0^t e^{-\omega s} \|f(s)\|_{D_\Lambda(1,\infty)}\, ds.$$

From this and (6.6) the conclusion follows. □

We now give a very simple form of the conditions (*ii*)–(*iii*) of Theorem 5.4 when $E_1 = D(\Lambda)$.

Proposition 6.2. *Let F be a Banach space such that $F \hookrightarrow E$. We have*

$$F \hookrightarrow D_\Lambda(1,\infty) \tag{6.8}$$

if and only if the following properties hold.

(i) *If $f \in L^1(F)$, then $R(t)f \in D(\Lambda)$ for all $t \geq 0$.*

(ii) *There exists $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|R(t)f\|_{D(\Lambda)} \leq \gamma(t)\|f\|_{L^1(F)} \quad \textit{for all } f \in L^1(F) \ \textit{ and } t \geq 0.$$

Proof. Let (i)–(ii) hold. We can use Proposition 5.3 with $E_1 = D(\Lambda)$ and deduce from (5.13) with $T = 1$ that

$$\|(e^\Lambda * f)(t)\|_{D(\Lambda)} \leq \gamma(1)\|f\|_{L^1(0,t,F)} \quad \text{for } f \in L^1(F) \ \text{ and } t \in (0,1]. \tag{6.9}$$

Take $x \in F$ and set

$$f(s) := \begin{cases} x, & 0 \leq s \leq 1, \\ 0, & s > 1. \end{cases}$$

We have $f \in L^1(F)$, and so (6.9) yields

$$\begin{aligned} \|e^{\Lambda t}x - x\| = \|\Lambda \int_0^t e^{\Lambda s}x \, ds\| &\leq \|(e^\Lambda * f)(t)\|_{D(\Lambda)} \\ &\leq \gamma(1)\|f\|_{L^1(0,t;F)} = \gamma(1)t\|x\|_F. \end{aligned} \tag{6.10}$$

Since there exists $c^* > 0$ such that for each $x \in F$ we have

$$\|x\| \leq c^*\|x\|_F,$$

we deduce from (6.10) that $x \in D_\Lambda(1,\infty)$ and

$$\|x\|_{D_\Lambda(1,\infty)} \leq (c^* + \gamma(1))\|x\|_F,$$

and so (6.8) is true.

Conversely, if (6.8) is valid, then $L^1(F) \hookrightarrow L^1(D_\Lambda(1,\infty))$, and so Theorem 6.1 implies that (i)–(ii) hold with $\gamma(t) = Me^{\omega' t}$. □

With this result the restriction theorem can be simplified in the case $E_1 = D(\Lambda)$ and $\mathcal{F} = L^1(F)$.

Theorem 6.3. *Let F be a Banach space such that $F \hookrightarrow E$ and set*

$$Z_* := D(\Lambda) \oplus L^1(F) \quad \textit{and} \quad G_*(t) := G(t)|_{Z_*}.$$

Then $(G_*(t))_{t\geq 0}$ is a semigroup in Z_* if and only if $F \hookrightarrow D_\Lambda(1,\infty)$. In this case its generator $\mathcal{A}_*$ is the restriction of $\mathcal{A}$ to

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus W^{1,1}(F);\ \Lambda x + f(0) \in D(\Lambda) \right\}. \tag{6.11}$$

Proof. It is a consequence of Theorem 5.4 with $E_1 = D(\Lambda)$ and Proposition 6.2. □

The application of Theorem 5.5 to the case $E_1 = D(\Lambda)$ and $F \hookrightarrow D_\Lambda(1,\infty)$ yields a new regularity result (which will be applied also in other situations in the next sections).

Theorem 6.4. *Let F be a Banach space such that*

$$F \hookrightarrow D_\Lambda(1,\infty). \tag{6.12}$$

Given

$$f \in W^{1,1}(F) \quad \text{and} \quad x \in D(\Lambda), \tag{6.13}$$

such that

$$x_1 = \Lambda x + f(0) \in D(\Lambda), \tag{6.14}$$

then

$$u(t) = e^{\Lambda t}x + (e^\Lambda * f)(t), \quad t \geq 0, \tag{6.15}$$

is the unique solution

$$u \in C^1(D(\Lambda)) \tag{6.16}$$

of (P). Moreover,

$$u'(t) = e^{\Lambda t}x_1 + (e^\Lambda * f')(t), \tag{6.17}$$

$$\|u(t)\|_{D(\Lambda)} \leq Me^{\omega t}\Big(\|x\|_{D(\Lambda)} + \int_0^t e^{-\omega s}\|f(s)\|_{D_\Lambda(1,\infty)}\, ds\Big), \tag{6.18}$$

$$\|u'(t)\|_{D(\Lambda)} \leq Me^{\omega t}\Big(\|x_1\|_{D(\Lambda)} + \int_0^t e^{-\omega s}\|f'(s)\|_{D_\Lambda(1,\infty)}\, ds\Big) \tag{6.19}$$

for $t \geq 0$. If in addition

$$D(\Lambda) \hookrightarrow F, \tag{6.20}$$

then we also have

$$\Lambda u \in C(F). \tag{6.21}$$

Proof. The first part is a consequence of Theorem 5.5, taking into account that the assumptions of Theorem 5.4, with $E_1 = D(\Lambda)$ and F satisfying (6.12), are satisfied because of Proposition 6.2.

The estimates (6.18)–(6.19) can be deduced from (6.15) and (6.17) because

$$\|e^{\Lambda t}\|_{\mathcal{L}(D(\Lambda))} \leq Me^{\omega t}, \quad t \geq 0,$$

and by using (6.2). If $D(\Lambda) \hookrightarrow F$, we deduce from (P) that $\Lambda u = u' - f \in C(D(\Lambda)) + W^{1,1}(F) \subset C(F)$, hence (6.21) holds. □

Remark 6.5. In the case $F \hookrightarrow D(\Lambda)$ the theorem is not interesting since then a solution $u \in C^1(D(\Lambda))$ of (P) can be obtained (by applying the classical Phillips theorem to the part of Λ in $D(\Lambda)$) under the weaker assumption $f \in W^{1,1}(D(\Lambda))$.

In Theorem 6.4 we deduced the existence of a solution of (P) from the fact that the restriction of $(G(t))_{t\geq 0}$ to $D(\Lambda) \oplus L^1(F)$ is a semigroup. The converse also holds, as is shown in the next theorem.

Theorem 6.6. *Let the following property hold for a Banach space $F \hookrightarrow E$. Given $f \in W^{1,1}(F)$ with $f(0) = 0$, then there exists a unique solution $u \in C^1(D(\Lambda))$ of the problem*

$$(P_0) \quad \begin{cases} u'(t) = \Lambda u(t) + f(t), \; t \geq 0, \\ u(0) = 0. \end{cases}$$

Then

$$F \hookrightarrow D_\Lambda(1, \infty). \tag{6.22}$$

Proof. By virtue of Theorem 6.3 and the last part of Theorem 2.5, the conclusion follows if we prove that the restriction of $\mathcal{A}$ to

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus W^{1,1}(F); \; \Lambda x + f(0) \in D(\Lambda) \right\}$$

is the generator of a semigroup in $Z_* = D(\Lambda) \oplus L^1(F)$.

To this end it is sufficient to prove (see Theorem III.6.7 of [6]) that $\mathcal{A}_*$ has a nonempty resolvent and the problem

$$\begin{cases} \mathcal{U}'(t) = & \mathcal{A}_* \mathcal{U}(t), \quad t \geq 0 \\ \mathcal{U}(0) = & \binom{x}{f} \end{cases}$$

has a unique solution $\mathcal{U} \in C^1(Z_*)$ for each $\binom{x}{f} \in D(\mathcal{A}_*)$. The verification of this assertion is left to the reader. □

Because of this result and Theorem 6.3 we deduce that if F satisfies the assumptions of Theorem 6.6, then $F \hookrightarrow D_\Lambda(1, \infty)$. This will be also a consequence of the next theorem (see [5]).

Theorem 6.7. *Let F be a Banach space such that $F \hookrightarrow E$. There exists $\tau > 0$ such that for each $f \in L^1(0, \tau; F)$ we have*

$$R(\tau) f \in D(\Lambda) \tag{6.23}$$

if and only if

$$F \hookrightarrow D_\Lambda(1, \infty). \tag{6.24}$$

Proof. The operator $\Lambda R(\tau): \ L^1(0,\tau;F) \to E$ is closed, hence continuous. Let $c > 0$ be such that for each $f \in L^1(0,\tau;F)$ we have

$$\|\Lambda R(\tau)f\| \leq c\|f\|_{L^1(0,\tau;F)}.$$

Fix $t \in (0,\tau)$ and $x \in F$. Since the function

$$f(s) = \begin{cases} 0, & 0 \leq s \leq \tau - t, \\ x, & \tau - t \leq s \leq \tau, \end{cases}$$

belongs to $L^1(0,\tau;F)$, we have

$$\begin{aligned} \|e^{\Lambda t}x - x\| &= \|\Lambda \int_0^t e^{\Lambda(t-s)}x \, ds\| = \|\Lambda \int_{\tau-t}^{\tau} e^{\Lambda(\tau-s)}x \, ds\| \\ &= \|\Lambda \int_0^{\tau} e^{\Lambda(\tau-s)}f(s) \, ds\| \leq c\|f\|_{L^1(0,\tau;F)} = ct\|x\|_F. \end{aligned}$$

From this the conclusion follows. □

Theorem 6.8. *Let F be a Banach space such that $F \hookrightarrow E$, and let $\tau > 0$ be such that for each $f \in W^{1,1}([0,\tau];F)$ with $f(0) = 0$, problem (P_0) has a unique solution $u \in C^1([0,\tau];D(\Lambda))$; then $F \hookrightarrow D_\Lambda(1,\infty)$.*

Proof. Let $g \in L^1([0,\tau];F)$. Setting $f(t) := \int_0^t g(s)\,ds$, we have $f \in W^{1,1}([0,\tau];F)$ and $f'(t) = g(t)$ almost everywhere in $(0,\tau)$. From the assumption, problem (P_0) has a unique solution $u \in C^1([0,\tau];D(\Lambda))$. Therefore, we obtain

$$(e^\Lambda * g)(\tau) = (e^\Lambda * f')(\tau) = u'(\tau) \in D(\Lambda).$$

So the conclusion is a consequence of Theorem 6.7. □

7. Hille-Yosida operators and spatial regularity

In this section we recall the definition and the main properties of Hille-Yosida operators. They were introduced in [3] and [4] as a generalization of generators of semigroups as required by certain differential problems (see the examples given in [4]) when studied in nonreflexive Banach spaces. We refer to this paper for the proofs and other information.

Definition 7.1. *A linear operator $A\ : D(A) \subset X \to X$ on a Banach space X is called a Hille-Yosida operator if there exist $M \geq 0$ and $\omega \in \mathbb{R}$ such that if $\lambda > \omega$, then $\lambda \in \rho(A)$ and $\|(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leq M(\lambda - \omega)^{-n}$ for each $n \in \mathbb{N}$ and $\lambda > \omega$. In this case we write $A \in HY(M,\omega)$.*

A Hille-Yosida operator A is even the generator of a semigroup if and only if $\overline{D(A)} = X$. In any case we obtain a generator of a semigroup on $\overline{D(A)}$ in the following way.

Definition 7.2. *For $A \in HY(M,\omega)$, the linear operator $A_0 : D(A_0) \subset X_0 \to X_0$ defined as*

$$\begin{cases} X_0 := \overline{D(A)}, \\ D(A_0) := \left\{ x \in D(A),\ Ax \in \overline{D(A)} \right\}, \quad A_0 x = Ax, \end{cases}$$

is a called the generator of the semigroup associated to the Hille-Yosida operator A.

In fact we have $A_0 \in \mathcal{G}(M,\omega)$, and moreover (see Proposition 3.2 of [11])

$$D(A_0) \subset D(A) \subset D_{A_0}(1,\infty). \tag{7.1}$$

Using Theorem 2.2 with $\Lambda = A$, we obtain the Favard space $D_A(1,\infty)$ of a Hille-Yosida operator which coincides with $D_{A_0}(1,\infty)$.

We can apply the results of Section 6 to the operator A_0 to obtain temporal and spatial regularity for the inhomogeneous problem

$$(\hat{P}) \quad \begin{cases} u'(t) = Au(t) + f(t),\ t \geq 0, \\ u(0) = x \end{cases}$$

corresponding to the Hille-Yosida operator A.

The next theorem gives a generalization of the spatial regularity result of Da Prato-Sinestrari (see Theorem 8.3 of [4]), as it provides a classical solution of $(\hat{P})$ when A is a Hille-Yosida operator and f is not (necessarily) differentiable almost everywhere and its image does not belong (necessarily) to $D(A)$ almost everywhere.

Theorem 7.3. (Spatial regularity) *Let $A : D(A) \subset X \mapsto X$ be a Hille-Yosida operator of type $HY(M,\omega)$. Given*

$$f \in L^1(D_A(1,\infty)) \tag{7.2}$$

and

$$x \in D(A) \text{ such that } Ax \in \overline{D(A)}, \tag{7.3}$$

there is a unique solution

$$u \in C(D(A)) \cap W^{1,1}_{loc}(X) \tag{7.4}$$

of problem $(\hat{P})$ for $t \in \mathbb{R}_+$ almost everywhere given by

$$u(t) = e^{A_0 t}x + (e^{A_0} * f)(t). \tag{7.5}$$

Moreover, we have

$$\|u(t)\|_{D(A)} \le Me^{\omega t}\left(\|x\|_{D(A)} + \int_0^t e^{-\omega s}\|f(s)\|_{D_A(1,\infty)}\,ds\right). \tag{7.6}$$

If in addition

$$f \in L^1(D_A(1,\infty)) \cap C(\overline{D(A)}), \tag{7.7}$$

then we also have

$$u \in C^1(X), \tag{7.8}$$

and the equation in $(\hat{P})$ *holds for each* $t \ge 0$.

Proof. Given f and x as in (7.2)–(7.3) define u by (7.5). We can apply Theorem 6.1 to the case in which

$$\begin{cases} E = \overline{D(A)}, \\ \Lambda = A_0, \\ D(\Lambda) = D(A_0) = \left\{x \in D(A);\ Ax \in \overline{D(A)}\right\}, \end{cases} \tag{7.9}$$

to infer that

$$u \in C(D(A_0)). \tag{7.10}$$

For $t > 0$ we have

$$\int_0^t u(s)\,ds = \int_0^t e^{A_0 s}x\,ds + \int_0^t ds \int_0^s e^{A_0(s-\sigma)}f(\sigma)\,d\sigma$$

and

$$\int_0^t ds \int_0^s e^{A_0(s-\sigma)}f(\sigma)\,d\sigma = \int_0^t d\sigma \int_\sigma^t e^{A_0(s-\sigma)}f(\sigma)\,ds;$$

hence,

$$A_0\int_0^t u(s)\,ds = e^{A_0 t}x - x + \int_0^t e^{A_0(t-\sigma)}f(\sigma)\,d\sigma - \int_0^t f(\sigma)\,d\sigma,$$

and therefore

$$u(t) = x + \int_0^t A_0 u(s)\,ds + \int_0^t f(\sigma)\,d\sigma. \tag{7.11}$$

From (7.10) and the fact that $f \in L^1(D_{A_0}(1,\infty))$ we deduce $u \in W^{1,1}_{loc}(X)$ and

$$u'(t) = A_0 u(t) + f(t) \quad \text{for } t \ge 0 \text{ a.e.} \tag{7.12}$$

If in addition $f \in C(X)$, we deduce from (7.11) that (7.12) holds for all $t \ge 0$ and $u \in C^1(X)$. □

8. Extrapolation spaces and temporal regularity

To obtain a temporal regularity result for problem $(\hat{P})$ with a Hille-Yosida operator A we need some basic facts about the extrapolation spaces (we refer the reader to [7], Section II.5, and [11] for the proofs and more information). In the sequel, $A_0 : D(A_0) \subset X_0 \mapsto X_0$ will be the generator of a semigroup $(e^{A_0 t})_{t \geq 0}$ on a Banach space X_0. For our results it is not restrictive to suppose that $A_0^{-1} \in \mathcal{L}(X_0)$.

Definition 8.1. *The extrapolation space X_{-1} of A_0 is the completion of $Y_0 = (X_0, \|\cdot\|_{-1})$ where*

$$\|x\|_{-1} = \|A_0^{-1} x\| \tag{8.1}$$

for $x \in X_0$. The extrapolated semigroup of $(e^{A_0 t})_{t \geq 0}$ is defined as the unique continuous extension of $e^{A_0 t} : Y_0 \subset X_{-1} \mapsto X_{-1}$ to X_{-1}. Its generator will be indicated by

$$A_{-1} : D(A_{-1}) \subset X_{-1} \mapsto X_{-1} \tag{8.2}$$

and is called the operator extrapolated from A_0.

The following theorems are proved in Section 1 of [11] (see also [7], Theorem II.5.5).

Theorem 8.2. *Under the above assumptions the following properties hold.*

(i) $D(A_{-1}) = X_0$ *with equivalent norms.*
(ii) $\|e^{A_{-1} t}\|_{\mathcal{L}(X_{-1})} = \|e^{A_0 t}\|_{\mathcal{L}(X_0)}, \quad t \geq 0.$
(iii) $\|A_{-1} x\|_{-1} = \|x\|, \quad x \in X_0.$
(iv) A_{-1} *is an extension of* A_0.
(v) A_{-1} *is an isometry from* $D_{A_0}(1, \infty)$ *onto* $D_{A_{-1}}(1, \infty)$.

Theorem 8.3. *Let $A : D(A) \subset X \mapsto X$ be a Hille-Yosida operator such that $A^{-1} \in \mathcal{L}(X)$. If $A_0 : D(A_0) \subset X_0 \mapsto X_0$ is the generator of the semigroup associated to A (see Definition 7.2), we have the inclusions*

$$\begin{aligned} D(A_0) \subset D(A) &\hookrightarrow D_{A_0}(1, \infty) \hookrightarrow X_0 \\ &= D(A_{-1}) \hookrightarrow X \hookrightarrow D_{A_{-1}}(1, \infty) \hookrightarrow X_{-1}. \end{aligned} \tag{8.3}$$

Moreover, A_{-1} is an extension of A, called again the extrapolated operator from A.

The set-up of extrapolation spaces allows us to apply the regularity results of Section 6 to a generalization of problem (P) obtained by replacing Λ by a Hille-Yosida operator or its extrapolated operator. In the first case, we immediately obtain a temporal regularity result due to Da Prato and Sinestrari (see [3] and [4]).

Theorem 8.4. *Let $A: D(A) \subset X \mapsto X$ be a Hille-Yosida operator. Given*

$$f \in W^{1,1}(X) \quad \text{and } x \in D(A) \tag{8.4}$$

such that

$$Ax + f(0) \in \overline{D(A)}, \tag{8.5}$$

then there is a unique

$$u \in C^1(X) \cap C(D(A)) \tag{8.6}$$

that is the solution of

$$(P') \quad \begin{cases} u'(t) = Au(t) + f(t), & t \geq 0, \\ u(0) = x. \end{cases}$$

Proof. Let us associate to A the generator A_0 (see Definition 7.2). If A_{-1} is the extrapolated operator from A_0, the assumptions of Theorem 6.4 are satisfied if we set

$$\begin{cases} E := X_{-1}, \\ \Lambda := A_{-1}, \\ D(\Lambda) := X_0 = \overline{D(A)}, \\ F := X. \end{cases} \tag{8.7}$$

In this case (6.13)–(6.14) are equivalent to (8.4)–(8.5) because from (6.13) and (6.14) we have

$$\begin{cases} x \in \overline{D(A)}, \\ A_{-1}x + f(0) \in \overline{D(A)}. \end{cases} \tag{8.8}$$

Since $f(0) \in X$, we obtain $A_{-1}x \in X$. This implies that $x \in D(A)$, and so (8.4) and (8.5) yield (8.8). □

In the next theorem we state a new result which gives a solution with the same time regularity but less space regularity so that A must be replaced by its extension A_{-1} (see [11], Corollary 3.5).

Theorem 8.5. (Temporal regularity) *Let $A: D(A) \subset X \mapsto X$ be a Hille-Yosida operator in $HY(M;\omega)$ and take*

$$f \in W^{1,1}(D_{A_{-1}}(1,\infty)) \quad \text{and } x \in D_A(1,\infty) \tag{8.9}$$

such that

$$A_{-1}x + f(0) \in \overline{D(A)}. \tag{8.10}$$

There is a unique

$$u \in C^1(X) \cap C(D_A(1,\infty)) \tag{8.11}$$

that is a solution of

$$(P_{-1}) \quad \begin{cases} u'(t) = A_{-1}u(t) + f(t), & t \geq 0, \\ u(0) = x. \end{cases}$$

The function u is given by

$$u(t) = e^{A_{-1}t}x + (e^{A_{-1}} * f)(t), \tag{8.12}$$

and we have

$$\|u(t)\| \leq Me^{\omega t}(\|x\| + \int_0^t e^{-\omega s}\|f(s)\|_{D_{A_{-1}}(1,\infty)}\, ds), \tag{8.13}$$

$$\|u'(t)\| \leq Me^{\omega t}(\|x_1\| + \int_0^t e^{-\omega s}\|f'(s)\|_{D_{A_{-1}}(1,\infty)}\, ds), \tag{8.14}$$

where $x_1 := A_{-1}x + f(0)$.

Proof. We can use Theorem 6.4 by setting

$$\begin{cases} E := X_{-1}, \\ \Lambda := A_{-1}, \\ D(\Lambda) := X_0 = \overline{D(A)}, \\ F := D_{A_{-1}}(1,\infty). \end{cases} \tag{8.15}$$

In this case, (8.9)–(8.10) are equivalent to (8.4)–(8.5) because from (6.13) and (6.14) we have

$$\begin{cases} x \in \overline{D(A)}, \\ A_{-1}x + f(0) \in \overline{D(A)}. \end{cases} \tag{8.16}$$

Since $f(0) \in D_{A_{-1}}(1,\infty)$, we conclude $A_{-1}x \in D_{A_{-1}}(1,\infty)$, which implies $x \in D_{A_0}(1,\infty) = D_A(1,\infty)$. So (8.9) and (8.10) yield (8.16). □

9. Homogenization in BUC

By slight modifications of the proofs, we can obtain the results of Theorems 4.1, 4.2, 5.1, and 5.2 with $L^1(E)$ and $W^{1,1}(E)$ replaced by $BUC(E)$ and $BUC^1(E)$, respectively.

In the case $E_1 = D(\Lambda)$ and $\mathcal{F} = BUC(F)$, the characterization of F such that the restriction of $(G(t))_{t\geq 0}$ to $D(\Lambda) \oplus BUC(F)$ is a semigroup is less restrictive than (6.8). To prove this we need the following.

Lemma 9.1. *Let F be a Banach space such that $F \hookrightarrow E$ and*

$$(F) \quad R(t)f \in D(\Lambda) \quad \text{for } t \geq 0 \ \text{ and } f \in BUC(F).$$

Then there exists $\gamma: \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|R(t)f\|_{D(\Lambda)} \leq \gamma(t)\|f\|_{BUC(F)} \text{ for } t \geq 0 \text{ and } f \in BUC(F) \tag{9.1}$$

and

$$\lim_{t \downarrow 0} \|R(t)f\|_{D(\Lambda)} = 0. \tag{9.2}$$

Proof. For each $t > 0$, $\Lambda R(t): BUC(F) \to E$ is a closed operator, hence bounded, and (9.1) holds.

Given $f \in BUC(F)$ and $t > 0$ the function

$$\hat{f}(s) := \begin{cases} f(s), & 0 \leq s \leq t, \\ f(t), & s > t, \end{cases}$$

belongs to $BUC(F)$, and so from (9.1) we obtain

$$\|R(t)f\|_{D(\Lambda)} = \|R(t)\hat{f}\|_{D(\Lambda)} \leq \gamma(t)\|\hat{f}\|_{BUC(F)} = \gamma(t)\|f\|_{C(0,t;F)}. \tag{9.3}$$

If $f(0) = 0$ and $\delta > 0$, the function

$$f^*(s) := \begin{cases} 0, & 0 \leq s \leq \delta, \\ f(s-\delta), & s \geq \delta, \end{cases}$$

belongs to $BUC(F)$, and so from (9.1) we obtain

$$\begin{aligned} \|R(t)f\|_{D(\Lambda)} &= \Big\| \int_\delta^{t+\delta} e^{\Lambda(t+\delta-s)} f(s-\delta)\, ds \Big\|_{D(\Lambda)} \\ &= \Big\| \int_0^{t+\delta} e^{\Lambda(t+\delta)} f^*(s)\, ds \Big\|_{D(\Lambda)} \\ &\leq \gamma(t+\delta)\|f^*\|_{C(0,t;F)} = \gamma(t+\delta)\|f\|_{C(0,t;F)}. \end{aligned}$$

Fix $T > 0$ and set $\delta := T - t$ for $0 < t < T$. Then we obtain

$$\|R(t)f\|_{D(\Lambda)} \leq \gamma(t)\|f\|_{C(0,t;F)}.$$

Since $f(0) = 0$, we deduce (9.2). If $f(0) \neq 0$, we have

$$\|R(t)f(0)\|_{D(\Lambda)} = \| \int_0^t e^{\Lambda s} f(0) ds\| + \|e^{\Lambda t} f(0) - f(0)\|,$$

and so

$$\lim_{t \downarrow 0} \|R(t)f(0)\|_{D(\Lambda)} = 0.$$

Writing $f(t) := f(t) - f(0) + f(0)$, for each $f \in BUC(F)$ and $t \geq 0$ we obtain (9.2). □

As a consequence we can prove (with analogous proofs) a result which corresponds to Theorem 6.3 and 6.4.

Theorem 9.2. *Let F be a Banach space such that $F \hookrightarrow E$ and*

$$(F) \quad R(t)f \in D(\Lambda) \quad \text{for } t > 0 \text{ and } f \in BUC(F).$$

Then $G(t)|_{Z_*}$, where $Z_* := D(\Lambda) \oplus BUC(F)$ defines a semigroup, and its generator $\mathcal{A}_*$ is the restriction of $\mathcal{A}$ to

$$D(\mathcal{A}_*) = \left\{ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\Lambda) \oplus BUC^1(F);\ \Lambda x + f(0) \in D(\Lambda) \right\}. \tag{9.4}$$

Take

$$f \in BUC^1(F) \quad \text{and} \quad x \in D(\Lambda), \tag{9.5}$$

such that

$$x_1 := \Lambda x + f(0) \ \in D(\Lambda).$$

Then problem (P) has a unique solution $u \in C^1(D(\Lambda))$ given by

$$u(t) = e^{\Lambda t}x + (e^{\Lambda t} * f)(t), \quad t \geq 0.$$

Moreover, (6.17)–(6.19) *hold. If in addition*

$$D(\Lambda) \hookrightarrow F, \tag{9.6}$$

then we also have

$$\Lambda u \in C(F). \tag{9.7}$$

Theorem 9.3. *If*

$$F \hookrightarrow D_\Lambda(1, \infty), \tag{9.8}$$

then (F) holds while (in general) the converse is not true.

Proof. If $f \in BUC(F)$ and $t > 0$, $\hat{f} := f \cdot \chi_{[0,t]} \in L^1(F)$. Hence if (9.8) holds, we can use Proposition 6.2 to obtain

$$R(t)f = R(t)\hat{f} \in D(\Lambda);$$

i.e., (F) is verified.

Let Λ be the generator of a bounded, analytic semigroup. Then we have for the interpolation space

$$D_\Lambda(\theta, \infty) = \left\{ x \in E;\ \|x\|_\theta := \sup_{s>0} \|s^{1-\theta}\Lambda e^{\Lambda s}x\| < \infty \right\}$$

(see Section 3.5 of [2]) and for $0 < \theta < 1$ the inclusion $D_\Lambda(1, \infty) \hookrightarrow D_\Lambda(\theta, \infty)$. Now if $F \hookrightarrow D_\Lambda(\theta, \infty)$, we have for $f \in BUC(F)$ and $t > 0$

$$\sup_{s>0} \|s^{1-\theta}\Lambda e^{\Lambda s} f(t)\| \leq \|f(t)\|_{D_\Lambda(\theta,\infty)} \leq c\|f\|_{BUC(E)}$$

with $c > 0$ independent of t. Since $\|\Lambda e^{\Lambda(t-s)} f(s)\| \leq c\|f\|_{BUC(E)}(t-s)^{\theta-1}$ for $0 \leq s \leq t$, we deduce that $R(t)f \in D(\Lambda)$; i.e., (F) holds. □

10. An application

We wish to indicate the impact of the preceding abstract results on initial–boundary-value problems for partial differential equations of hyperbolic type by choosing the simplest nontrivial example (for more applications see [4]). Consider

$$\begin{cases} u_t(t,x) = -u_x(t,x) + f(t,x), \quad t \geq 0,\ 0 \leq x \leq 1, \\ u(t,0) = 0, \quad t \geq 0, \\ u(0,x) = u_0(x), \quad 0 \leq x \leq 1. \end{cases} \tag{10.1}$$

If we define

$$\begin{cases} X := C[0,1] \\ Au := -u' \\ D(A) := C_0^1[0,1] = \left\{u \in C^1[0,1];\ \ u(0) = 0\right\}, \end{cases} \tag{10.2}$$

then (10.1) can be written in abstract form as

$$\begin{cases} u'(t) = Au(t) + f(t), \quad t \geq 0, \\ u(0) = u_0. \end{cases} \tag{10.3}$$

In [11] the following results have been proved.

The operator $A \in HY(1,0)$ and $A_0 :\ D(A_0) \subset X_0 \mapsto X_0$, the generator associated to A, is given by

$$\begin{cases} X_0 = C_0[0,1] = \{u \in C[0,1];\ u(0) = 0\}, \\ A_0 u = -u', \\ D(A_0) = \left\{u \in C^1[0,1];\ u(0) = u'(0) = 0\right\}. \end{cases}$$

In addition,

$$D_A(1,\infty) = \{u \in C[0,1];\ u \text{ Lipschitz continuous;}\ \ u(0) = 0\} := Lip_0[0,1],$$

and if A_{-1} is the operator extrapolated from A_0, we have that

$$D_{A_{-1}}(1,\infty) \simeq L^\infty(0,1)$$

and $A_{-1}u = -u'$ almost everywhere when $u \in D_A(1,\infty)$.

The spatial regularity result of Theorem 7.3 gives the following for problem (10.1).

Theorem 10.1. *Let $f \in L^1(\mathbb{R}_+; Lip[0,1])$ and $u_0 \in C^1[0,1]$ be such that*

$$f(t,0) = u_0(0) = u_0'(0) = 0, \quad t \geq 0.$$

Then problem (10.1) has a solution u for $t \geq 0$ almost everywhere and $x \in [0,1]$ such that $u, u_x \in C(\mathbb{R}_+ \times [0,1])$, $u \in W^{1,1}_{loc}(\mathbb{R}_+; C[0,1])$, and for $(t,x) \in$

$\mathbb{R}_+ \times [0,1]$ *we have the estimate*

$$|u(t,x)| + |u_x(t,x)| \leq \|u_0\|_{C^1[0,1]} + \int_0^t \|f(s,\cdot)\|_{Lip[0,1]}\, ds.$$

If in addition $f \in C(\mathbb{R}_+ \times [0,1])$, *then we also have* $u_t \in C(\mathbb{R}_+ \times [0,1])$, *and* (10.1) *holds for all* $t \geq 0$ *and* $0 \leq x \leq 1$.

The temporal regularity result of Theorem 8.5 yields the following.

Theorem 10.2. *Let* $f \in W^{1,1}(\mathbb{R}_+; L^\infty(0,1))$ *and* $u_0 \in Lip[0,1]$ *be such that*

$$u_1 := u_0' + f(0,\cdot) \in C[0,1]$$

and

$$u_0(0) = u_1(0) = 0.$$

Then problem (10.1) *has a solution* u *for* $t \geq 0$ *and* $x \in [0,1]$ *almost everywhere such that*

$$u,\ u_t \in C(\mathbb{R}_+ \times [0,1]) \quad \text{and} \quad u \in C(\mathbb{R}_+; Lip[0,1]).$$

Moreover, for $(t,x) \in \mathbb{R}_+ \times [0,1]$ *the estimate*

$$|u(t,x)| \leq \|u_0\|_{C(0,1)} + \int_0^t \|f(s,\cdot)\|_{L^\infty(0,1)}\, ds,$$

and for $t \in \mathbb{R}_+$ *almost everywhere and* $x \in [0,1]$ *the estimate*

$$|u_t(t,x)| \leq \|u_1\|_{C[0,1]} + \int_0^t \|f_s(s,\cdot)\|_{L^\infty(0,1)}\, ds$$

holds.

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