

LOCAL AND GLOBAL WELL-POSEDNESS RESULTS FOR FLOWS OF INHOMOGENEOUS VISCOUS FLUIDS

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Abstract. This paper is devoted to the study of density-dependent, incompressible Navier-Stokes equations with periodic boundary conditions, or in the whole space. We aim at stating well-posedness in functional spaces as close as possible to the ones imposed by the scaling of the equations. Preliminary results have been obtained in [5] under the assumption that the density is close to a constant. Getting rid of this assumption (by allowing smoother data if necessary) is the main motivation of the present paper. Local well-posedness is stated for data (ρ_0, u_0) such that $(\rho_0 - \text{cste}) \in H^{\frac{N}{2}+\alpha}$ and $\inf \rho_0 > 0$, and $u_0 \in H^{\frac{N}{2}-1+\beta}$. The indices $\alpha, \beta > 0$ may be taken arbitrarily small. We further derive a blow-up criterion which entails global well-posedness in dimension $N = 2$ if there is no vacuum initially.

INTRODUCTION

In this paper, we are concerned with the following model of incompressible viscous fluids with variable density:

$$\begin{cases} \partial_t \rho + \operatorname{div} \rho u = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0, \quad (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases} \quad (0.1)$$

In the above equation, $\rho = \rho(t, x) \in \mathbb{R}^+$ stands for the density, and $u = u(t, x) \in \mathbb{R}^N$, for the velocity field. The term $\nabla \Pi$ (the gradient of the pressure) is the Lagrange multiplier associated to the constraint $\operatorname{div} u = 0$. The initial conditions (ρ_0, u_0) and the external force f are prescribed. Unless otherwise specified, we shall assume throughout this paper that the space variable x belongs to \mathbb{R}^N or to the torus \mathbb{T}^N .

The existence of global weak solutions for (0.1) under the assumptions that $\rho_0 \in L^\infty$ is nonnegative, that $\operatorname{div} u_0 = 0$, and that $\sqrt{\rho_0} u_0 \in L^2$ has

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been studied by different authors. It is based on the energy equality

$$\|\sqrt{\rho}u(t)\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau = \|\sqrt{\rho_0}u_0\|_{L^2}^2 + 2 \int_0^t (\rho f \cdot u)(\tau, x) dx d\tau. \quad (0.2)$$

In dimension $N = 2$ and under the additional assumption that $\nabla u_0 \in L^2$, smoother weak solutions may be built. Their existence stems from a quasi-conservation law involving the norm of ∇u in $L^\infty(0, T; L^2)$, and of $\partial_t u$, $\nabla \Pi$, and $\nabla^2 u$ in $L^2(0, T; L^2)$.

For both types of weak solutions however, the problem of uniqueness has not been solved. We refer to [1] and to [16] for an overview of results on weak solutions. Some recent improvements have been obtained by B. Desjardins in [9], [10], and [11].

The question of unique resolvability for (0.1) has been first addressed by O. Ladyzhenskaya and V. Solonnikov in the late seventies (see [14]). The authors consider system (0.1) in a bounded domain Ω with homogeneous Dirichlet boundary conditions for u . Under the assumption that $u_0 \in W^{2-\frac{2}{q}, q}$ ($q > N$) is divergence-free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\overline{\Omega})$ is bounded away from zero, the results are the following:

- Global well-posedness in dimension $N = 2$,
- Local well-posedness in dimension $N = 3$. If in addition u_0 is small in $W^{2-\frac{2}{q}, q}$, then global well-posedness holds true.

Similar qualitative results may be obtained in the framework of Sobolev spaces H^s (see [17]). The case of unbounded domains has been investigated by S. Itoh and A. Tani in [13]. In this framework, system (0.1) has been shown to be locally well-posed.

In the present paper, we aim at proving similar qualitative results in the whole space \mathbb{R}^N or in the torus \mathbb{T}^N under *weaker* regularity assumptions. The functional framework that we are going to use is inspired by works on *homogeneous* Navier-Stokes equations:

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla \Pi = 0, \\ \operatorname{div} v = 0, \quad v|_{t=0} = v_0. \end{cases} \quad (0.3)$$

Well-posedness for (0.3) has been studied by a number of authors. The standard proof is based on contractive-mapping arguments and on smoothing properties for the semigroup of the heat equation. That approach has been initiated by H. Fujita and T. Kato in [12] and is particularly efficient in functional spaces which have the same *scaling invariance* as (0.3), in the

sense that their norm is invariant for all $\ell > 0$ by the transformation

$$v_0(x) \mapsto \ell v_0(\ell x), \quad v(t, x) \mapsto \ell v(\ell^2 t, \ell x). \tag{0.4}$$

It leads to the following type of statement:

“Abstract well-posedness result”: *Let $E \subset \mathcal{S}'(\mathbb{R}^N)$ and $F \subset C(\mathbb{R}^+; E)$ be two functional spaces whose norm is invariant by (0.4). For $T > 0$, let F_T be the local version of F pertaining to functions defined on $[0, T]$. Under appropriate compatibility conditions on E and F , the following result holds: for any divergence-free data $v_0 \in E$, there exists $T > 0$ such that (NSI) has a unique local solution $v \in F_T$. If $\|v_0\|_E \ll \mu$, then that solution is global.*

The first example for $N = 3$ has been given by H. Fujita and T. Kato in [12]: $E = \dot{H}^{\frac{1}{2}}$ and

$$F = \left\{ u \in C(\mathbb{R}^+; \dot{H}^{\frac{1}{2}}) : t^{\frac{1}{4}} \nabla u \in C(\mathbb{R}^+; L^2) \quad \text{and} \quad t^{\frac{1}{4}} \nabla u \xrightarrow{t \rightarrow 0} 0 \right\}.$$

Since then, (0.3) has been shown to be well-posed in a number of functional spaces: Lebesgue spaces L^N , Besov spaces $\dot{B}_{p,r}^{\frac{N}{p}-1}$, etc.

Now, one can wonder whether Fujita and Kato’s approach is relevant for *inhomogeneous fluids*. If one believes so, scaling considerations should help us to find an adapted functional framework. As system (0.1) is invariant under the transformation

$$\begin{aligned} (\rho_0(x), u_0(x)) &\mapsto (\rho_0(\ell x), \ell u_0(\ell x)), \\ (\rho(t, x), u(t, x), \Pi(t, x)) &\mapsto (\rho(\ell^2 t, \ell x), \ell u(\ell^2 t, \ell x), \ell^2 \Pi(\ell^2 t, \ell x)), \end{aligned} \tag{0.5}$$

choosing initial data (ρ_0, u_0) such that $\nabla \rho_0$ and u_0 belong to an *adapted space* E for the “standard” Navier-Stokes equations should give satisfactory results. Hence, taking (ρ_0, u_0) in $W^{1,N} \times (L^N)^N$ should be a possible choice (see [6] and note the discrepancy with Ladyzhenskaya and Solonnikov’s assumptions).

In the present work, we rather consider Sobolev spaces H^s . Therefore, (0.5) suggests choosing initial data (ρ_0, u_0) such that $\nabla \rho_0$ and u_0 belong to $\dot{H}^{\frac{N}{2}-1}$. As system (0.1) degenerates if ρ vanishes or becomes unbounded, we further assume that $\rho_0^{\pm 1} \in L^\infty$. We also suppose that ρ tends to some positive constant (say 1) at infinity (or has average 1 in the periodic case). Denoting $1/\rho_0 = 1 + a_0$ and $1/\rho = 1 + a$, system (0.1) may be rewritten as

$$\begin{cases} \partial_t a + u \cdot \nabla a = 0, \\ \partial_t u + u \cdot \nabla u + (1 + a)(\nabla \Pi - \mu \Delta u) = f, \\ \operatorname{div} u = 0, \quad (a, u)|_{t=0} = (a_0, u_0). \end{cases} \tag{0.6}$$

For (at least) technical reasons however, we did not state well-posedness in the critical Sobolev spaces described above. Either the data have to be (slightly) more regular, or ρ_0 , close to a constant. In the latter case, the critical regularity exponent may be attained in *Besov spaces* $\dot{B}_{2,1}^s$ (an important property of these spaces is that $\dot{B}_{2,1}^{\frac{N}{2}} \hookrightarrow L^\infty$): in [5], we prove the following:

Theorem 0.1. *There exists a constant c depending only on N , and such that for all $u_0 \in \dot{B}_{2,1}^{\frac{N}{2}-1}$ with $\operatorname{div} u_0 = 0$, and $a_0 \in \dot{B}_{2,1}^{\frac{N}{2}}$ with $\|a_0\|_{\dot{B}_{2,1}^{\frac{N}{2}}} \leq c$, there is a $T \in (0, \infty]$ such that system (0.6) has a unique solution $(a, u, \nabla \Pi)$ with $a \in C_b([0, T]; \dot{B}_{2,1}^{\frac{N}{2}})$, $u \in C_b([0, T]; \dot{B}_{2,1}^{\frac{N}{2}-1}) \cap L^1(0, T; \dot{B}_{2,1}^{\frac{N}{2}+1})$, and $\nabla \Pi \in L^1(0, T; \dot{B}_{2,1}^{\frac{N}{2}-1})$. If in addition $\|u_0\|_{\dot{B}_{2,1}^{\frac{N}{2}-1}} \leq c\mu$, one can take $T = +\infty$.*

In the present paper, we consider more drastic perturbations of homogeneous fluids. We are going to show that if we allow for a bit of extra regularity, local well-posedness holds:

Theorem 0.2. *Let $\alpha > 0$ and $\beta \in (0, +\infty) \cap (\alpha - 1, \alpha + 1]$. Let $a_0 \in H^{\frac{N}{2}+\alpha}$ (with in addition $\nabla a_0 \in L^\infty$ if $\alpha = 1$) such that $\underline{b} \stackrel{\text{def}}{=} \inf_x (1 + a_0(x)) > 0$, $u_0 \in H^{\frac{N}{2}-1+\beta}$ with $\operatorname{div} u_0 = 0$, and $f \in \tilde{L}_{loc}^1(H^{\frac{N}{2}-1+\beta})$. There exists a $T > 0$ such that system (0.6) has a unique solution $(a, u, \nabla \Pi)$ in the space*

$$E_T^{\alpha, \beta} \stackrel{\text{def}}{=} \tilde{C}_T(H^{\frac{N}{2}+\alpha}) \times (\tilde{C}_T(H^{\frac{N}{2}+\beta-1}) \cap \tilde{L}_T^1(H^{\frac{N}{2}+\beta+1}))^N \times (\tilde{L}_T^1(H^{\frac{N}{2}+\beta-1}))^N$$

(with in addition $\nabla a \in L^\infty(0, T; L^\infty)$ if $\alpha = 1$). Moreover, $\underline{b} \leq 1 + a$, and the energy equality (0.2) is satisfied.

Above, $\tilde{C}_T(H^s)$ stands for a (large) subspace of $C([0, T]; H^s)$, and $\tilde{L}_T^1(H^\sigma)$ is slightly larger than $L^1(0, T; H^\sigma)$. The reader is referred to Definition 1.6 for more details.

Remark 0.3. The time T of local existence may be explicitly bounded below in terms of $\|a_0\|_{H^{\frac{N}{2}+\alpha}}$, $\|u_0\|_{H^{\frac{N}{2}+\beta-1}}$, $\|f\|_{\tilde{L}^1(H^{\frac{N}{2}+\beta-1})}$, μ , \underline{b} , α , β , and N (see inequality (4.13) below).

In addition, one can state the following blow-up criterion:

Theorem 0.4. *Let a_0 , u_0 , and f satisfy the assumptions of Theorem 0.2 and $(a, u, \nabla \Pi)$ solve (0.6) with data (a_0, u_0, f) . Assume that $(a, u, \nabla \Pi)$ is*

smooth on $[0, T^*)$ (i.e., belongs to $E_T^{\alpha, \beta}$ whenever $T < T^*$) and satisfies

$$\sup_{t \in [0, T^*)} \|a(t)\|_{H^{\frac{N}{2} + \alpha'}} < +\infty$$

for some $\alpha' > 0$, and

$$\int_0^{T^*} \|\nabla u(t)\|_{B_{\infty, \infty}^{\frac{2}{p}-2}}^p dt < +\infty \quad \text{for some } p \in (1, +\infty), \quad \text{or}$$

$$\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt < +\infty.$$

Then $(a, u, \nabla \Pi)$ may be continued beyond T^* into a smooth solution of (0.6).

Remark 0.5. Above $B_{\infty, \infty}^{\frac{2}{p}-2}$ stands for a Besov space (see Definition 1.1).

Let us just stress that $H^{\frac{N}{2} + \frac{2}{p} - 2}$ is continuously embedded in $B_{\infty, \infty}^{\frac{2}{p}-2}$.

Remark 0.6. Theorem 0.4 obviously implies that for data satisfying the assumptions of Theorem 0.2, the maximal existence time in $E_T^{\alpha', \beta'}$ with $0 < \alpha' \leq \alpha$ and $0 < \beta' \leq \beta$ is the same as in $E_T^{\alpha, \beta}$.

This blow-up criterion will enable us to prove global well-posedness for large data in dimension $N = 2$ if there is no vacuum initially:

Theorem 0.7. Let α and β be as in Theorem 0.2. Let $a_0 \in H^{1+\alpha}$ (with in addition $\nabla a_0 \in L^\infty$ if $\alpha = 1$) such that $\underline{b} \stackrel{\text{def}}{=} \inf_x (1 + a_0(x)) > 0$, $u_0 \in H^\beta$ with $\text{div } u_0 = 0$, and $f \in \tilde{L}_{loc}^1(H^\beta) \cap \tilde{L}_{loc}^2(\mathbb{R}^+; L^2)$. System (0.6) has a unique global solution $(a, u, \nabla \Pi)$ which belongs to $E_T^{\alpha, \beta}$ for all $T > 0$ (with in addition $\nabla a \in L_{loc}^\infty(\mathbb{R}^+; L^\infty)$ if $\alpha = 1$).

The paper is structured as follows. In Section 1, we present some basic tools in Fourier analysis: Littlewood-Paley decomposition and paraproduct, and define Sobolev, Besov, and $\tilde{L}_T^r(H^s)$ spaces. The next section is devoted to the proof of estimates for the linearized system (0.6). In Section 3, we state a stability result which entails uniqueness and continuity with respect to the data. Section 4 is devoted to the proof of local well-posedness and blow-up criterion in dimension N whereas Section 5 deals with global well-posedness in the two-dimensional case. Some technical estimates needed in the proof of the blow-up criterion are postponed to the appendix.

Notation: Summation convention on repeated indices will be used.

Throughout the paper, C stands for a “harmless constant” whose precise meaning will be clear from the context. We shall sometimes alternatively

use the notation $A \lesssim B$ instead of $A \leq CB$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.

We denote $x \vee y = \min(x, y)$ and \mathbb{A}^N is either \mathbb{R}^N or \mathbb{T}^N . The notation \mathcal{P} stands for the L^2 projector on solenoidal vector fields, while \mathcal{Q} stands for the L^2 projector on gradient-type vector fields. Note that $\mathcal{P}u + \mathcal{Q}u = u$ whenever u has coefficients in L^2 .

For X a Banach space, we denote by $C([0, T]; X)$ the set of continuous functions on $[0, T)$ with values in X , and by $C_b([0, T]; X)$ the subset of bounded functions of $C([0, T]; X)$. For $p \in [1, +\infty]$, the notation $L^p(0, T; X)$ stands for the set of measurable functions on $[0, T)$ with values in X , such that $t \mapsto \|f(t)\|_X$ belongs to $L^p(0, T)$.

1. THE FUNCTIONAL TOOL BOX

The proof of most of the results presented in the paper requires a dyadic decomposition of Fourier variables, which is called the *Littlewood-Paley decomposition*. Let us briefly explain how it may be built in the case $x \in \mathbb{R}^N$. The case of periodic boundary conditions is very similar (see e.g. [4]).

Let (χ, φ) be a couple of C^∞ functions with

$$\text{Supp } \chi \subset \{|\xi| \leq \frac{4}{3}\}, \quad \text{Supp } \varphi \subset \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\} \text{ and } \chi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1.$$

Let $\varphi_q(\xi) = \varphi(2^{-q}\xi)$, $h_q = \mathcal{F}^{-1}\varphi_q$, and $\check{h} = \mathcal{F}^{-1}\chi$. The dyadic blocks are defined by

$$\begin{aligned} \Delta_q u &\stackrel{\text{def}}{=} 0 \quad \text{if } q \leq -1, \quad \Delta_{-1} u \stackrel{\text{def}}{=} \chi(D)u = \int_{\mathbb{R}^N} \check{h}(y)u(x-y) dy, \\ \Delta_q u &\stackrel{\text{def}}{=} \varphi(2^{-q}D)u = \int_{\mathbb{R}^N} h_q(y)u(x-y) dy \quad \text{if } q \geq 0. \end{aligned}$$

We shall also use the following low-frequency cut-off:

$$S_q u \stackrel{\text{def}}{=} \sum_{k \leq q-1} \Delta_k u = \chi(2^{-q}D)u.$$

One can easily prove that

$$\forall u \in \mathcal{S}'(\mathbb{R}^N), \quad u = \sum_{q \in \mathbb{Z}} \Delta_q u. \quad (1.1)$$

The Littlewood-Paley decomposition has nice properties of quasi-orthogonality:

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if } |k-q| \geq 2 \text{ and } \Delta_k(S_{q-1}u \Delta_q u) \equiv 0 \quad \text{if } |k-q| \geq 5. \quad (1.2)$$

A number of functional spaces may be characterized in terms of Littlewood-Paley decomposition. Let us give the definition of (nonhomogeneous) Besov spaces:

Definition 1.1. For $s \in \mathbb{R}$, $(p, r) \in [1, +\infty]^2$, and $u \in \mathcal{S}'(\mathbb{R}^N)$, we set

$$\|u\|_{B_{p,r}^s} \stackrel{\text{def}}{=} \left(\sum_{q \geq -1} 2^{rsq} \|\Delta_q u\|_{L^p}^r \right)^{\frac{1}{r}},$$

with the usual modification if $r = +\infty$.

We then define the Besov space $B_{p,r}^s = \{u \in \mathcal{S}' : \|u\|_{B_{p,r}^s} < +\infty\}$.

The definition of $B_{p,r}^s$ does not depend on the choice of the Littlewood-Paley. One can further remark that H^s coincide with $B_{2,2}^s$.

Let us now state some classical properties for these Besov spaces.

Proposition 1.2. *The following properties hold true:*

i) *Derivatives: we have*

$$\|\nabla u\|_{B_{p,r}^{s-1}} \lesssim \|u\|_{B_{p,r}^s}.$$

ii) *Sobolev embeddings: If $p_1 \leq p_2$ and $r_1 \leq r_2$, then $B_{p_1,r_1}^s \hookrightarrow B_{p_2,r_2}^{s-N(\frac{1}{p_1}-\frac{1}{p_2})}$. If $s_1 > s_2$ and $1 \leq p, r_1, r_2 \leq +\infty$, then $B_{p,r_1}^{s_1} \hookrightarrow B_{p,r_2}^{s_2}$.*

iii) *Algebraic properties: for $s > 0$, $B_{p,r}^s \cap L^\infty$ is an algebra. So is H^s if $s > N/2$.*

iv) *Real interpolation: $(B_{p,r}^{s_1}, B_{p,r}^{s_2})_{\theta,r'} = B_{p,r'}^{\theta s_1 + (1-\theta)s_2}$.*

Let us recall some classical estimates in Sobolev spaces for the product of two functions.

Proposition 1.3. *The following estimates hold true:*

$$\|uv\|_{H^s} \lesssim \|u\|_{L^\infty} \|v\|_{H^s} + \|v\|_{L^\infty} \|u\|_{H^s} \quad \text{if } s > 0, \quad (1.3)$$

$$\|u\|_{H^{s_1}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \quad \text{if } s_1 + s_2 > 0, \quad s_1 \leq \frac{N}{2} \quad \text{and} \quad s_2 > \frac{N}{2}, \quad (1.4)$$

$$\|uv\|_{H^{s_1+s_2-\frac{N}{2}}} \lesssim \|u\|_{H^{s_1}} \|v\|_{H^{s_2}} \quad \text{if } s_1 + s_2 > 0, \quad \text{and} \quad s_1, s_2 < \frac{N}{2}, \quad (1.5)$$

$$\|uv\|_{H^s} \lesssim \|u\|_{H^s} \|v\|_{L^\infty \cap H^{\frac{N}{2}}} \quad \text{if } |s| < \frac{N}{2}. \quad (1.6)$$

More accurate results may be obtained by means of (basic) paradifferential calculus, a tool which was introduced by J.-M. Bony in [2].

The paraproduct between f and g is defined by

$$T_f g \stackrel{\text{def}}{=} \sum_{q \in \mathbb{N}} S_{q-1} f \Delta_q g.$$

Denoting

$$R(f, g) \stackrel{\text{def}}{=} \sum_{q \geq -1} \Delta_q f \widetilde{\Delta}_q g$$

with

$$\widetilde{\Delta}_q g \stackrel{\text{def}}{=} (\Delta_{q-1} + \Delta_q + \Delta_{q+1})g, \quad \text{and} \quad T'_f g \stackrel{\text{def}}{=} T_f g + R(f, g),$$

we have the following so-called Bony's decomposition:

$$fg = T_f g + T_g f + R(f, g) = T'_f g + T_g f.$$

A bunch of continuity results for the paraproduct T and the remainder R are available. For instance, the following results hold true (see their proof in [18], Section 4.4):

Proposition 1.4. *For all $s \in \mathbb{R}$, $\sigma > 0$, and $1 \leq p, r \leq +\infty$, the paraproduct is a bilinear, continuous application from $B_{\infty, \infty}^{-\sigma} \times B_{p, r}^s$ to $B_{p, r}^{s-\sigma}$, and from $L^\infty \times B_{p, r}^s$ to $B_{p, r}^s$.*

The remainder is bilinear continuous from $B_{p, r}^{s_1} \times B_{p, \infty}^{s_2}$ to $B_{p, r}^{s_1 + s_2 - \frac{N}{p}}$ whenever $s_1 + s_2 > N \max(0, -1 + 2/p)$.

Remark 1.5. According to (1.2), the paraproduct may be rewritten as

$$T_u v = \sum_{q \geq 1} S_{q-1} u \Delta_q \left((1 - \chi)(D)v \right).$$

Thus, the low frequencies of v do not matter in the bilinear estimates for the paraproduct. Therefore, one has for instance for all $s \in \mathbb{R}$,

$$\|T_u v\|_{B_{p, r}^s} \lesssim \|u\|_{L^\infty} \|\nabla v\|_{B_{p, r}^{s-1}}.$$

The study of nonstationary PDEs requires spaces of the type $L_T^\rho(X) \stackrel{\text{def}}{=} L^\rho(0, T; X)$ for appropriate Banach spaces X . In our case, we expect X to be a Sobolev or a Besov space, so that it is natural to localize the equations through Littlewood-Paley decomposition. We then get estimates for each dyadic block and perform integration in time. But, in doing so, we obtain bounds in spaces which are not of type $L^\rho(0, T; H^s)$ (except if $\rho = 2$). That remark naturally leads to the following definition (introduced in [3]):

Definition 1.6. For $\rho \in [1, +\infty]$, $s \in \mathbb{R}$, and $T \in [0, +\infty]$, we set

$$\|u\|_{\tilde{L}_T^\rho(H^s)} \stackrel{\text{def}}{=} \left(\sum_{q \geq -1} 2^{2qs} \left(\int_0^T \|\Delta_q u(t)\|_{L^2}^\rho dt \right)^{\frac{2}{\rho}} \right)^{\frac{1}{2}}$$

and denote by $\tilde{L}_T^\rho(H^s)$ the subset of distributions $u \in \mathcal{S}'(0, T \times \mathbb{A}^N)$ with finite $\|u\|_{\tilde{L}_T^\rho(H^s)}$ norm. When $T = +\infty$, the index T is omitted and $\tilde{L}_{loc}^\rho(H^s) \stackrel{\text{def}}{=} \cap_{T>0} \tilde{L}_T^\rho(H^s)$. We further denote $\tilde{C}_T(H^s) \stackrel{\text{def}}{=} C([0, T]; H^s) \cap \tilde{L}_T^\infty(H^s)$ and $\tilde{L}_T^\rho(H^s \cap L^\infty) \stackrel{\text{def}}{=} \tilde{L}_T^\rho(H^s) \cap L_T^\rho(L^\infty)$.

Of course, one can also define the spaces $\tilde{L}_T^\rho(B_{p,r}^s)$ pertaining to the Besov space $B_{p,r}^s$.

Let us remark that by virtue of Minkowski's inequality, we have

$$\|u\|_{\tilde{L}_T^\rho(H^s)} \leq \|u\|_{L_T^\rho(H^s)} \quad \text{if } \rho \leq 2$$

and

$$\|u\|_{L_T^\rho(H^s)} \leq \|u\|_{\tilde{L}_T^\rho(H^s)} \quad \text{if } \rho \geq 2,$$

and one can easily prove that, whenever $\epsilon > 0$,

$$\|u\|_{\tilde{L}_T^\rho(H^s)} \lesssim \|u\|_{L_T^\rho(H^{s+\epsilon})} \quad \text{and} \quad \|u\|_{L_T^\rho(H^s)} \leq \|u\|_{\tilde{L}_T^\rho(H^{s+\epsilon})}. \quad (1.7)$$

We will often use the following interpolation inequality:

$$\|u\|_{\tilde{L}_T^\rho(H^s)} \leq \|u\|_{\tilde{L}_T^{\rho_1}(H^{s_1})}^\theta \|u\|_{\tilde{L}_T^{\rho_2}(H^{s_2})}^{1-\theta} \quad (1.8)$$

with $\frac{1}{\rho} = \frac{\theta}{\rho_1} + \frac{1-\theta}{\rho_2}$ and $s = \theta s_1 + (1-\theta)s_2$.

Remark 1.7. The product, the paraproduct, and the remainder are continuous in a number of spaces $\tilde{L}_T^\rho(B_{p,r}^s)$. The indices s , p , and r behave as in Propositions 1.3 and 1.4, and the indices pertaining to the time integrability behave as in Hölder's inequality. For example, inequality (1.3) becomes

$$\|uv\|_{\tilde{L}_T^\rho(H^s)} \lesssim \|u\|_{L_T^{\rho_1}(L^\infty)} \|v\|_{\tilde{L}_T^{\rho_2}(H^s)} + \|v\|_{L_T^{\rho_2}(L^\infty)} \|u\|_{\tilde{L}_T^{\rho_1}(H^s)}$$

whenever $s > 0$, $1 \leq \rho, \rho_1, \rho_2 \leq +\infty$, and $1/\rho = 1/\rho_1 + 1/\rho_2$.

2. THE LINEARIZED EQUATIONS

2.1. The transport equation. Estimates in Sobolev spaces for transport equations are standard. We need the following proposition, the proof of which may be found in [4, 8].

Proposition 2.1. *Let $s > -1 - N/2$. Let v be a solenoidal vector field. Suppose also that $a_0 \in H^s$, $g \in \tilde{L}_T^1(H^s)$, and that $a \in \tilde{L}_T^\infty(H^s) \cap C([0, T]; \mathcal{S}'(\mathbb{R}^N))$ solves*

$$\begin{cases} \partial_t a + v \cdot \nabla a = g, \\ a|_{t=0} = a_0. \end{cases} \tag{2.1}$$

Then $a \in \tilde{C}_T(H^s)$ and there exists a constant C depending only on s and N , and such that the following inequality holds on $[0, T]$:

$$\|a\|_{\tilde{L}_t^\infty(H^s)} \leq e^{CV(t)} (\|a_0\|_{H^s} + \|g\|_{\tilde{L}_t^1(H^s)}), \tag{2.2}$$

with
$$V(t) = \begin{cases} \int_0^t \|\nabla v(\tau)\|_{B_{2,\infty}^{\frac{N}{2}} \cap L^\infty} d\tau & \text{if } s < 1 + N/2, \\ \int_0^t \|\nabla v(\tau)\|_{H^{s-1}} d\tau & \text{if } s \geq 1 + N/2. \end{cases}$$

If $s = N/2 + 1$, in (2.2), it is understood that H^s has been replaced by $H^s \cap \text{Lip}$ and $\|\nabla v(\tau)\|_{H^{s-1}}$ with $\|\nabla v(\tau)\|_{H^{s-1} \cap L^\infty}$.

2.2. The momentum equation. This section is devoted to the proof of estimates for the following linear system:

$$\begin{cases} \partial_t u + v \cdot \nabla u + b(\nabla \Pi - \mu \Delta u) = f, \\ \text{div } u = 0, \quad u|_{t=0} = u_0, \end{cases} \tag{2.3}$$

where b, f, v , and u_0 are given. We assume that $\underline{b} \stackrel{\text{def}}{=} \inf_x b(x) > 0$ and that b tends to some positive constant (say 1 with no loss of generality) at infinity. Besides, $a \stackrel{\text{def}}{=} b - 1$ belongs to $\tilde{L}_T^\infty(H^{\frac{N}{2} + \alpha})$ for some $\alpha > 0$.

Before stating our results let us introduce the following notation:

$$\mathcal{A}_T \stackrel{\text{def}}{=} \begin{cases} 1 + \underline{b}^{-1} \|\nabla b\|_{\tilde{L}_T^\infty(H^{\frac{N}{2} + \alpha - 1})} & \text{if } \alpha \neq 1, \\ 1 + \underline{b}^{-1} \|\nabla b\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}}) \cap L_T^\infty(L^\infty)} & \text{if } \alpha = 1. \end{cases} \tag{2.4}$$

The following result has been proved in [7]:

Proposition 2.2. *Let $\mu \geq 0$, $s \in (0, \alpha + N/2]$, and $\alpha > 0$. Assume that $\underline{b} \stackrel{\text{def}}{=} \inf_x b_0 > 0$ and that $a \stackrel{\text{def}}{=} b - 1 \in \tilde{L}_T^\infty(H^{\frac{N}{2} + \alpha})$ (and also $\nabla a \in L^\infty(0, T; L^\infty)$ if $\alpha = 1$). Let u_0 be a divergence-free vector field with coefficients in H^s , and f a time-dependent vector field with coefficients in $\tilde{L}_T^1(H^s)$.*

Let v be a time-dependent, divergence-free vector field such that $\nabla v \in L^1(0, T; B_{2,\infty}^{\frac{N}{2}} \cap L^\infty)$ if $s < N/2 + 1$ and $\nabla v \in L^1(0, T; H^s)$ if $s > N/2 + 1$, and assume in addition that $s \neq 1 + N/2$ if $\nabla v \not\equiv 0$. Let $u \in \tilde{L}_T^\infty(H^s)$ be a solution of (2.3) on $[0, T] \times \mathbb{A}^N$ for some $\nabla \Pi \in \tilde{L}_T^1(H^s)$.

Denote $\underline{\mu} \stackrel{\text{def}}{=} \underline{b}\mu$ and $\kappa \stackrel{\text{def}}{=} s/\alpha'$ with $\alpha' > 0$ such that

$$\alpha' \leq \min\left(1, \alpha, \frac{s}{2}\right) \quad \text{if} \quad \left[s < \frac{N}{2} + \alpha \quad \text{or} \quad \left(s = \frac{N}{2} + \alpha \quad \text{and} \quad \alpha > 1\right)\right],$$

$$\alpha' \in (0, \alpha) \cap \left(0, \frac{s}{2}\right] \quad \text{if} \quad \left[s = \frac{N}{2} + \alpha \quad \text{and} \quad \alpha \leq 1\right].$$

There exists $C = C(s, N, \alpha, \alpha')$ such that

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2})} \\ & \leq Ce^{C\mathcal{A}_T^\kappa V(T)} \left(\|u_0\|_{H^s} + \mathcal{A}_T^\kappa \left(\|f\|_{\tilde{L}_T^1(H^s)} + \underline{\mu}\mathcal{A}_T \|u\|_{\tilde{L}_T^1(H^{s+2-\alpha'})} \right) \right), \\ & \underline{b}\|\nabla\Pi\|_{\tilde{L}_T^1(H^s)} \\ & \lesssim \mathcal{A}_T^\kappa \left(\|\mathcal{Q}f\|_{\tilde{L}_T^1(H^s)} + \underline{\mu}(\mathcal{A}_T - 1)\|u\|_{\tilde{L}_T^1(H^{s+2})} + \int_0^T V'(t)\|u(t)\|_{H^s} dt \right), \end{aligned}$$

with

$$V(T) \stackrel{\text{def}}{=} \int_0^T \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2}} \cap L^\infty} dt \quad \text{if} \quad s < \frac{N}{2} + 1$$

and

$$V(T) \stackrel{\text{def}}{=} \int_0^T \|\nabla v\|_{H^{s-2}} dt \quad \text{if} \quad s > \frac{N}{2} + 1.$$

If $v = u$, the above estimates hold with $V(T) = \int_0^T \|\nabla u\|_{L^\infty} dt$ (even if $s = 1 + \frac{N}{2}$).

The following proposition will enable us to derive a blow-up criterion which entails global existence in dimension $N = 2$ if $\mu > 0$.

Proposition 2.3. *Let $\alpha > 0$, $p \in (1, +\infty)$, p' be the conjugate exponent of p , and $s \in (0, N/2 + 2/p - 1) \cap (0, N/2 + \alpha]$. Let α' and κ be defined as in Proposition 2.2. There exists $C = C(s, N, \alpha, \alpha', p)$ such that*

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2})} \\ & \leq Ce^{C\mathcal{A}_T^{\kappa p} \frac{V(T)}{\underline{\mu}^{p-1}}} \left(\|u_0\|_{H^s} + \mathcal{A}_T^\kappa \|f\|_{\tilde{L}_T^1(H^s)} + \underline{\mu}\mathcal{A}_T^{1+\kappa} \|u\|_{\tilde{L}_T^1(H^{s-\alpha'+2})} \right) \end{aligned}$$

with $V(T) \stackrel{\text{def}}{=} \int_0^T \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2} - \frac{2}{p}}}^p dt$. In the case $v = u$ under the weaker condition

$s \in (0, \alpha + N/2]$, the estimate above holds with $V(T) = \int_0^T \|\nabla u\|_{B_{\infty,\infty}^{-\frac{2}{p'}}}^p dt$.

Proof. To simplify the presentation, we assume that $\alpha \neq 1$. The case $\alpha = 1$ may be handled by changing $\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}$ into $\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha}) \cap L_T^\infty(\text{Lip})}$.

Let $R_q^j \stackrel{\text{def}}{=} \Delta_q(b\Delta u^j) - \text{div}(b\Delta_q \nabla u^j)$. Applying Δ_q to (2.3), we get $\partial_t \Delta_q u - \mu \text{div}(b\Delta_q \nabla u) + v \cdot \nabla \Delta_q u + \Delta_q \nabla \Pi = \Delta_q f - \Delta_q(a \nabla \Pi) + [v, \Delta_q] \cdot \nabla u + \mu R_q$. Using integration by parts and Bony's decomposition, we can write

$$|(\Delta_q(a \nabla \Pi) | \Delta_q u)| \leq |(\Delta_q T_{\nabla a} \Pi | \Delta_q u)| + |(\Delta_q T'_{\nabla \Pi} a | \Delta_q u)|. \quad (2.5)$$

Therefore, taking the L^2 scalar product with $\Delta_q u$ yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Delta_q u\|_{L^2}^2 + \underline{\mu} \|\nabla \Delta_q u\|_{L^2}^2 &\leq \|\Delta_q u\|_{L^2} \left(\mu \|R_q\|_{L^2} \right. \\ &\quad \left. + \|[v, \Delta_q] \cdot \nabla u\|_{L^2} + \|\Delta_q T_{\nabla a} \Pi\|_{L^2} + \|\Delta_q T'_{\nabla \Pi} a\|_{L^2} + \|\Delta_q \mathcal{P}f\|_{L^2} \right). \end{aligned}$$

According to Bernstein inequality, there exists a $\kappa > 0$ such that for all $q \geq 0$, we have $\|\Delta_q \nabla u\|_{L^2} \geq \kappa 2^q \|\Delta_q u\|_{L^2}$. Hence, elementary computations yield

$$\begin{aligned} \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu} \|u\|_{\tilde{L}_T^1(H^{s+2})} &\lesssim \|u_0\|_{H^s} + \underline{\mu} \|\Delta_{-1} u\|_{L_T^1(L^2)} \\ &\quad + \|T_{\nabla a} \Pi\|_{\tilde{L}_T^1(H^s)} + \|T'_{\nabla \Pi} a\|_{\tilde{L}_T^1(H^s)} + \|\mathcal{P}f\|_{\tilde{L}_T^1(H^s)} \\ &\quad + \left(\sum_{q \geq -1} 2^{2qs} \|R_q\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}} + \left(\sum_{q \geq -1} 2^{2qs} \|[v, \Delta_q] \cdot \nabla u\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}}. \end{aligned} \quad (2.6)$$

In the appendix of [7], the following estimate is proved:

$$\begin{aligned} &\left(\sum_{q \geq -1} 2^{2qs} \|\Delta_q(a \Delta u) - \text{div}(a \Delta_q \nabla u)\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}} \\ &\lesssim \|\nabla a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha-1})} \|\nabla u\|_{\tilde{L}_T^1(H^{s+1-\alpha'})} \end{aligned} \quad (2.7)$$

provided α' satisfies the conditions of Proposition 2.2.

As $0 < s < N/2 + 2/p - 1$, Lemma A.1 enables us to bound the last term in (2.6):

$$\begin{aligned} &\left(\sum_q 2^{2qs} \|[v^j, \Delta_q] \partial_j u\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}} \\ &\lesssim K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|\nabla u(t)\|_{H^{s-1}} dt. \end{aligned}$$

Plugging the above inequality and (2.7) in (2.6), we get

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2})} \lesssim \|u_0\|_{H^s} + \|f\|_{\tilde{L}_T^1(H^s)} + \underline{\mu}\|\Delta_{-1}u\|_{L_T^1(L^2)} \quad (2.8) \\ & + \|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})} \|\nabla\Pi\|_{\tilde{L}_T^1(H^{s-\alpha'})} + \mu\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})} \|\nabla u\|_{\tilde{L}_T^1(H^{s+1-\alpha'})} \\ & + K\|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|u\|_{H^s} d\tau. \end{aligned}$$

On the other hand, $\nabla\Pi$ solves the following elliptic equation:

$$\operatorname{div}(b\nabla\Pi) = \operatorname{div} F, \quad (2.9)$$

with $F = f + \mu a \Delta u - v \cdot \nabla u$.

In [7], the following result is stated:

Proposition 2.4. *Let $m \in [1, +\infty]$, $\epsilon > 0$, $\alpha > 0$, and $s \in \mathbb{R}$ be such that $1 \vee \alpha \leq |\sigma| \leq \alpha + N/2$. Let \mathcal{A}_T be defined in (2.4). The following estimate holds true:*

$$\underline{b}\|\nabla\Pi\|_{\tilde{L}_T^m(H^\sigma)} \lesssim \mathcal{A}_T^{\frac{|\sigma|+\epsilon}{1\vee\alpha}} \|\mathcal{Q}F\|_{\tilde{L}_T^m(H^\sigma)}. \quad (2.10)$$

If $1 \leq m \leq 2$ and $\sigma \geq \alpha \vee 1$, or $2 \leq m \leq \infty$ and $\sigma \leq -(\alpha \vee 1)$, inequality (2.10) holds with $\epsilon = 0$.

Therefore, we are led to estimate $\mathcal{Q}F$ in $\tilde{L}_T^1(H^{s-\alpha'})$. This may be done by making use of Bony's decomposition and by taking advantage of $\operatorname{div} u = \operatorname{div} v = 0$, which implies $\operatorname{div}(v \cdot \nabla u) = \operatorname{div}(u \cdot \nabla v)$. One ends up with the following formula:

$$\mathcal{Q}F = \mathcal{Q}f + \mathcal{Q}(\mu a \Delta u - T_{\nabla u} v - T'_{\nabla v} u). \quad (2.11)$$

Now, apply estimate (A.10) (with $s - \alpha'$ instead of s) for bounding the third term in the definition of $\mathcal{Q}F$ (this is allowed as $s - \alpha' < N/2 + 2/p - 1$), and Lemma A.2 for the last term. One eventually gets

$$\begin{aligned} & \|\mathcal{Q}F\|_{\tilde{L}_T^1(H^{s-\alpha'})} \lesssim \|\mathcal{Q}f\|_{\tilde{L}_T^1(H^{s-\alpha'})} + \mu\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})} \|\Delta u\|_{\tilde{L}_T^1(H^{s-\alpha'})} \\ & + L\|u\|_{\tilde{L}_T^1(H^{s+2-\alpha'})} + L^{-\frac{p}{p'}} \int_0^T \|u\|_{H^{s-\alpha'}} \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p dt. \end{aligned}$$

Plug this latter inequality in (2.10), then in (2.8), and choose $K = \eta \underline{\mu}$ and $L = \eta \underline{\mu} \mathcal{A}_T^{-\kappa}$ with η suitably small. Gronwall's lemma completes the proof.

If in addition $u = v$, one can get better estimates for the commutator $[u, \Delta_q] \cdot \nabla u$ and the last two terms in $\mathcal{Q}F$. Indeed, the last two inequalities

in Lemma A.1 and the last one in Lemma A.2 enable us to get

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2})} \\ & \lesssim \|f\|_{\tilde{L}_T^1(H^s)} + \underline{\mu}\|\Delta_{-1}u\|_{L_T^1(L^2)} + \|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}\|\nabla\Pi\|_{\tilde{L}_T^1(H^{s-\alpha})} \\ & \quad + \mu\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}\|\nabla u\|_{\tilde{L}_T^1(H^{s+1-\alpha})} \\ & \quad + K\|u\|_{\tilde{L}_T^1(H^{s+2})} + \|u_0\|_{H^s} + K^{-\frac{p}{p'}}\int_0^T\|\nabla u\|_{B_{\infty,\infty}^{-\frac{2}{p'}}}^p\|u\|_{H^s}d\tau, \\ & \|\mathcal{Q}F\|_{\tilde{L}_T^1(H^{s-\alpha'})} \lesssim \|\mathcal{Q}f\|_{\tilde{L}_T^1(H^{s-\alpha'})} + \mu\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}\|\Delta u\|_{\tilde{L}_T^1(H^{s-\alpha'})} \\ & \quad + L\|u\|_{\tilde{L}_T^1(H^{s+2-\alpha'})} + L^{-\frac{p}{p'}}\int_0^T\|u\|_{H^{s-\alpha'}}\|\nabla u\|_{B_{\infty,\infty}^{-\frac{2}{p'}}}^p dt, \end{aligned}$$

and the same arguments as before yield the result. □

2.3. Low regularity estimates for the momentum. Let us briefly study the case when $-N/2 < s \leq 0$, a result which turns out to be of importance for proving uniqueness in the case $N = 2, 3$. We have

Proposition 2.5. *Let $\alpha, \epsilon \in (0, 1)$ and $s \in (-N/2, 0]$. Denote*

$$\mathcal{A}_T \stackrel{\text{def}}{=} 1 + \underline{b}^{-1}\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}.$$

There exists $C = C(s, N, \alpha, \epsilon)$ such that

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2})} \leq Ce \\ & \quad \times \left(\|u_0\|_{H^s} + \mathcal{A}_T^{\frac{2(\alpha\vee 1)+\epsilon-s}{\alpha\vee 1}}\|f\|_{\tilde{L}_T^1(H^s)} + \underline{\mu}\mathcal{A}_T^{\frac{\epsilon+3(\alpha\vee 1)-s}{\alpha\vee 1}}\|u\|_{\tilde{L}_T^1(H^{s+2-\alpha\vee 1})} \right). \end{aligned}$$

Proof. In the appendix of [7], the following commutator estimate is proved:

$$\left(\sum_{q \geq -1} 2^{2qs} \|[v, \Delta_q] \cdot \nabla u\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}} \lesssim \int_0^T \|\nabla v(t)\|_{B_{2,\infty}^{\frac{N}{2}} \cap L^\infty} \|u(t)\|_{H^s} dt.$$

Hence, from (2.6) and (2.7), we have

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^s)} + \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2})} \lesssim \underline{\mu}\|u\|_{\tilde{L}_T^1(H^{s+2-\alpha\vee 1})} \\ & \quad + \mu\|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}\|\nabla u\|_{\tilde{L}_T^1(H^{s+1-\alpha\vee 1})} + \|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}\|\nabla\Pi\|_{\tilde{L}_T^1(H^{s-\alpha\vee 1})} \\ & \quad + \|f\|_{\tilde{L}_T^1(H^s)} + \|u_0\|_{H^s} + \int_0^T \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2}} \cap L^\infty} \|u\|_{H^s} d\tau. \end{aligned}$$

Define F according to (2.11). Applying Proposition 2.4 gives

$$\underline{b} \|\nabla \Pi\|_{\tilde{L}_T^1(H^{s-\alpha\nu 1})} \lesssim \mathcal{A}_T^{\frac{\epsilon+\alpha\nu 1-s}{\alpha\nu 1}} \|\mathcal{Q}F\|_{\tilde{L}_T^1(H^{s-\alpha\nu 1})}. \tag{2.12}$$

Since $s + N/2 > 0$, we have

$$\begin{aligned} \|\mathcal{Q}F\|_{\tilde{L}_T^1(H^{s-\alpha\nu 1})} &\lesssim \|\mathcal{Q}f\|_{\tilde{L}_T^1(H^{s-\alpha\nu 1})} \\ &+ \mu \|a\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})} \|\Delta u\|_{\tilde{L}_T^1(H^{s-\alpha\nu 1})} + \int_0^T \|u\|_{H^s} \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2}} \cap L^\infty} dt. \end{aligned} \tag{2.13}$$

Straightforward computations complete the proof of Proposition 2.5. \square

3. STABILITY

Uniqueness is a straightforward consequence of the following proposition.

Proposition 3.1. *Let $T > 0$ and $\alpha \in (0, 1/2]$. Let $a_0^i \in H^{\frac{N}{2}+\alpha}$ and $u_0^i \in H^{\frac{N}{2}-1+\alpha}$ with $\operatorname{div} u_0^i = 0$, $f^i \in \tilde{L}_T^1(H^{\frac{N}{2}+\alpha-1})$, and $g^i \in \tilde{L}_T^1(H^{\frac{N}{2}+\alpha})$. Assume that $\underline{b} \stackrel{\text{def}}{=} 1 + \inf_x a_0^2(x) > 0$. Let $(a^i, u^i, \nabla \Pi^i)_{i=1,2} \in (E_T^{\alpha,\alpha})^2$ satisfy*

$$\begin{cases} \partial_t a^i + u^i \cdot \nabla a^i = g^i, \\ \partial_t u^i + u^i \cdot \nabla u^i + (1 + a^i)(\nabla \Pi^i - \mu \Delta u^i) = f^i, \\ \operatorname{div} u^i = 0, \quad (a^i, u^i)|_{t=0} = (a_0^i, u_0^i). \end{cases} \tag{3.1}$$

Let $K \geq 1$ be such that, for $i = 1, 2$,

$$\|a^i\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})} + \|u^i\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha-1})} + \|u^i\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\alpha+1})} + \|\nabla \Pi^i\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\alpha-1})} \leq K.$$

Let

$$(\delta a, \delta u, \nabla \delta \Pi) \stackrel{\text{def}}{=} (a^2 - a^1, u^2 - u^1, \nabla \Pi^2 - \nabla \Pi^1),$$

$$(\delta a_0, \delta u_0) \stackrel{\text{def}}{=} (a_0^2 - a_0^1, u_0^2 - u_0^1), \quad (\delta f, \delta g) \stackrel{\text{def}}{=} (f^2 - f^1, g^2 - g^1)$$

$$X(t) \stackrel{\text{def}}{=} \|\delta a\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}-1+\alpha})} + \|\delta u\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\alpha-2})} + \underline{\mu} \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha})} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(H^{\frac{N}{2}-2})}.$$

There exists a constant C_K depending only on $K, \underline{b}, \alpha, N, \mu$, and T , and such that for all $t \in [0, T]$ holds

$$X(t) \leq C_K (\|\delta a_0\|_{H^{\frac{N}{2}+\alpha-1}} + \|\delta u_0\|_{H^{\frac{N}{2}+\alpha-2}} + \|\delta f\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} + \|\delta g\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-1})}).$$

Proof. Remark that

$$\begin{cases} \partial_t \delta a + u^2 \cdot \nabla \delta a = \delta g - \delta u \cdot \nabla a^1, \\ \partial_t \delta u + u^2 \cdot \nabla \delta u + (1 + a^2)(\nabla \delta \Pi - \mu \Delta \delta u) \\ \quad = \delta f - \delta u \cdot \nabla u^1 - \delta a (\nabla \Pi^1 - \mu \Delta u^1), \\ \operatorname{div} \delta u = 0. \end{cases} \tag{3.2}$$

If $N \geq 4$, apply Proposition 2.2 with $s = N/2 - 2 + \alpha$ and $\alpha' = \alpha/2$. If $N = 2, 3$, apply Proposition 2.5 with $s = N/2 - 2 + \alpha$, and (2.12), (2.13). We get for $t \in [0, T]$

$$\begin{aligned} \|\delta u\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\alpha-2})} + \underline{\mu} \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha})} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(H^{\frac{N}{2}-2})} &\lesssim \mathcal{A}_T^\kappa e^{C\mathcal{A}_T^\kappa \int_0^T \|\nabla u^2\|_{H^{\frac{N}{2}} \cap L^\infty} d\tau} \\ &\times \left(\|\delta u_0\|_{H^{\frac{N}{2}-2+\alpha}} + \|\delta f\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} + \|\delta u \cdot \nabla u^1\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} \right. \\ &\quad \left. + \|\delta a(\nabla \Pi^1 - \mu \Delta u^1)\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} + \underline{\mu} \mathcal{A}_T \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\frac{\alpha}{2}})} \right), \end{aligned}$$

with a power $\kappa \geq 2$ depending only on N and α , and

$$\mathcal{A}_T = 1 + \frac{\|a^2\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})}}{\underline{b}}.$$

On the other hand, according to Proposition 2.1,

$$\begin{aligned} \|\delta a\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\alpha-1})} &\leq C e^{C \int_0^T \|\nabla u^2(\tau)\|_{H^{\frac{N}{2}} \cap L^\infty} d\tau} \\ &\times \left(\|\delta a_0\|_{H^{\frac{N}{2}+\alpha-1}} + \|\delta g\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-1})} + \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-1})} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} X(t) &\leq CK^{\kappa+1} e^{CK^{\kappa+1}} \left(\|\delta a_0\|_{H^{\frac{N}{2}+\alpha-1}} + \|\delta g\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-1})} + \|\delta u_0\|_{H^{\frac{N}{2}-2+\alpha}} \right. \\ &\quad + \|\delta f\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} + \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-1})} + \|\delta u \cdot \nabla u^1\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} \\ &\quad \left. + \|\delta a(\nabla \Pi^1 - \mu \Delta u^1)\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} + \underline{\mu} \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\frac{\alpha}{2}})} \right). \end{aligned} \tag{3.3}$$

Using Remark 1.7, Proposition 1.3, and Minkowski's inequality, we gather that

$$\begin{aligned} \|\delta u \cdot \nabla a^1\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-1})} &\lesssim \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}} \cap L^\infty)} \|\nabla a^1\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}-1+\alpha})}, \\ \|\delta u \cdot \nabla u^1\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} &\lesssim \int_0^t \|\nabla u^1(\tau)\|_{L^\infty \cap H^{\frac{N}{2}}} \|\delta u(\tau)\|_{H^{\frac{N}{2}-2+\alpha}} d\tau, \\ \|\delta a(\nabla \Pi^1 - \mu \Delta u^1)\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha-2})} &\lesssim \int_0^t \|\delta a(\tau)\|_{H^{\frac{N}{2}-1+\alpha}} \|(\nabla \Pi^1 - \mu \Delta u^1)(\tau)\|_{H^{\frac{N}{2}-1}} d\tau. \end{aligned}$$

On the other hand, combining (1.8), (1.7), and Young's inequality yields

$$\begin{aligned} \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}} \cap L^\infty)} &\lesssim \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\frac{\alpha}{2}})}, \\ &\lesssim \underline{\mu}^{-1} (\underline{\mu} t)^{\frac{\alpha}{4}} \left(\|\delta u\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}-2+\alpha})} + \underline{\mu} \|\delta u\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\alpha})} \right). \end{aligned}$$

Coming back to (3.3), one concludes that

$$X(t) \leq C_K \left(X_0 + (\underline{\mu}t)^{\frac{\alpha}{4}} X(t) + \int_0^t \left(\|\nabla u^1\|_{L^\infty \cap H^{\frac{N}{2}}} + \|\nabla \Pi^1\|_{H^{\frac{N}{2}-1}} \right) X(\tau) d\tau \right)$$

with $X_0 \stackrel{\text{def}}{=} \|\delta a_0\|_{H^{\frac{N}{2}+\alpha-1}} + \|\delta g\|_{\tilde{L}^1_T(H^{\frac{N}{2}+\alpha-1})} + \|\delta u_0\|_{H^{\frac{N}{2}-2+\alpha}} + \|\delta f\|_{\tilde{L}^1_T(H^{\frac{N}{2}+\alpha-2})}$.

Hence, denoting $\eta \stackrel{\text{def}}{=} \min \left(T, \frac{1}{\underline{\mu}} \left(\frac{1}{2C_K} \right)^{\frac{4}{\alpha}} \right)$, we get for $t \in [0, \eta]$,

$$X(t) \leq 2C_K \left(X_0 + \int_0^t \left(\|\nabla u^1(\tau)\|_{L^\infty \cap H^{\frac{N}{2}}} + \|\nabla \Pi^1(\tau)\|_{H^{\frac{N}{2}-1}} \right) X(\tau) d\tau \right).$$

Gronwall's lemma then implies that

$$X(t) \leq 2C_K X_0 e^{2C_K \int_0^t \left(\|\nabla u^1(\tau)\|_{L^\infty \cap H^{\frac{N}{2}}} + \|\nabla \Pi^1(\tau)\|_{H^{\frac{N}{2}-1}} \right) d\tau} \quad \text{for } t \in [0, \eta].$$

Now, starting from time $k\eta$ instead of 0 and assuming that $k\eta < T$, similar computations lead to

$$X(t) \leq 2C_K \left(X_0 + X(k\eta) + \int_{k\eta}^t \left(\|\nabla u^1\|_{L^\infty \cap H^{\frac{N}{2}}} + \|\nabla \Pi^1\|_{H^{\frac{N}{2}-1}} \right) X(\tau) d\tau \right)$$

for $t \in [k\eta, \min(T, (k+1)\eta)]$. Hence

$$X(t) \leq 2C_K (X_0 + X(k\eta)) e^{\int_{k\eta}^t \left(\|\nabla u^1(\tau)\|_{L^\infty \cap H^{\frac{N}{2}}} + \|\nabla \Pi^1(\tau)\|_{H^{\frac{N}{2}-1}} \right) d\tau},$$

which eventually leads to the desired estimate (up to a change of C_K). \square

4. LOCAL EXISTENCE

For the sake of conciseness, we assume throughout that $\alpha \neq 1$. The proof below may be carried out to the case $\alpha = 1$ by replacing $\|\cdot\|_{\tilde{L}^\infty_T(H^{\frac{N}{2}+\alpha})}$ with $\|\cdot\|_{\tilde{L}^\infty_T(H^{\frac{N}{2}+\alpha}) \cap L^\infty_T(\text{Lip})}$.

First step: Smooth solutions for approximate data. Let $(\phi^n)_{n \in \mathbb{N}} \in \mathcal{S}(\mathbb{R}^N)^{\mathbb{N}}$ be a sequence of nonnegative mollifiers. Denote $a_0^n \stackrel{\text{def}}{=} \phi^n \star a_0$, $u_0^n \stackrel{\text{def}}{=} \phi^n \star u_0$, and $f^n \stackrel{\text{def}}{=} \phi^n \star f$, where \star stands for convolution in the space variables. Obviously a_0^n and u_0^n belong to H^∞ , whereas $f^n \in L^1_{loc}([0, +\infty); H^\infty)$.

Therefore, according to [7], there exists a $T^n \in (0, +\infty]$ such that system (0.6) with data (a_0^n, u_0^n) and external force f^n has a unique maximal solution $(a^n, u^n, \nabla \Pi^n)$ in

$$C([0, T^n]; H^\infty) \times (C([0, T^n]; H^\infty))^N \times (L^1_{loc}(0, T^n; H^\infty))^N.$$

Second step: Energy equality. Introducing $\rho^n \stackrel{\text{def}}{=} (1+a^n)^{-1}$ and $\rho_0^n \stackrel{\text{def}}{=} (1+a_0^n)^{-1}$ leads to the following system:

$$\begin{cases} \partial_t \rho^n + u^n \cdot \nabla \rho^n = 0, \\ \rho^n (\partial_t u^n + u^n \cdot \nabla u^n) - \mu \Delta u^n + \nabla \Pi^n = \rho^n f^n, \\ \operatorname{div} u^n = 0, \quad (\rho^n, u^n)|_{t=0} = (\rho_0^n, u_0^n). \end{cases}$$

Taking the L^2 scalar product of the momentum equation with u^n , doing integration by parts, then integrating in time yields

$$\begin{aligned} & \left\| \left(\sqrt{\rho^n} u^n \right) (t) \right\|_{L^2}^2 + 2\mu \int_0^t \|\nabla u^n(\tau)\|_{L^2}^2 d\tau \\ &= \left\| \sqrt{\rho_0^n} u_0^n \right\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^N} (\rho^n u^n \cdot f^n)(\tau, x) dx d\tau. \end{aligned} \quad (4.1)$$

Third step: Uniform bounds on $E_T^{\alpha, \beta}$ for some fixed T . Remark that

$$\begin{cases} \|a_0^n\|_{H^{\frac{N}{2}+\alpha}} \leq \|a_0\|_{H^{\frac{N}{2}+\alpha}}, \quad \|u_0^n\|_{H^{\frac{N}{2}+\beta-1}} \leq \|u_0\|_{H^{\frac{N}{2}+\beta-1}}, \\ \|f^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\beta-1})} \leq \|f\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\beta-1})}. \end{cases} \quad (4.2)$$

Hence, Proposition 2.1 yields

$$\forall t \in [0, T_n), \quad \|a^n\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha})} \leq e^{CV^n(t)} \|a_0\|_{H^{\frac{N}{2}+\alpha}}, \quad (4.3)$$

with

$$V^n(t) = \begin{cases} \int_0^t \|\nabla u^n(\tau)\|_{H^{\frac{N}{2}} \cap L^\infty} d\tau & \text{if } \alpha < 1, \\ \int_0^t \|\nabla u^n(\tau)\|_{H^{\frac{N}{2}+\alpha-1}} d\tau & \text{if } \alpha > 1. \end{cases}$$

By the maximum principle for transport equations, we have for $t \in [0, T_n)$,

$$\inf_x a^n(t, x) = \inf_x a_0^n(x) \geq \inf_x a_0(x).$$

Therefore,

$$\forall n \in \mathbb{N}, \forall t \in [0, T_n), \quad 1 + \inf_x a^n(t, x) \geq \underline{b}, \quad (4.4)$$

and, as a_0^n tends to a_0 in L^∞ , we have for large enough n ,

$$1 + \inf_x a^n(t, x) \leq 2\underline{b}. \quad (4.5)$$

Let $\mathcal{A}_t^n \stackrel{\text{def}}{=} 1 + \underline{b}^{-1} \|a^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\alpha})}$. Apply Proposition 2.2 with $s = N/2 - 1 + \beta$ (here comes the condition $\beta \leq 1 + \alpha$). Taking advantage of (4.2), (4.4), and (4.5), we get

$$U^n(t) \stackrel{\text{def}}{=} \|u^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}-1+\beta})} + \underline{\mu} \|u^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+1+\beta})} \leq C(\mathcal{A}_t^n)^\kappa e^{C(\mathcal{A}_t^n)^\kappa} \int_0^t \|\nabla u^n\|_{L^\infty} d\tau$$

$$\times \left(\|u_0\|_{H^{\frac{N}{2}-1+\beta}} + \|f\|_{\tilde{L}_t^1(H^{\frac{N}{2}-1+\beta})} + \underline{\mu} \mathcal{A}_t^n \|u^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+1+\beta-\alpha'})} \right), \quad (4.6)$$

for some positive exponent κ , and

$$\alpha' \stackrel{\text{def}}{=} \begin{cases} \min(1, \alpha, \frac{N}{4} + \frac{\beta-1}{2}) & \text{if } \beta < 1 + \alpha \text{ or } \alpha > 1, \\ \alpha' \in (0, \alpha) \cap (0, \frac{N}{4} + \frac{\beta-1}{2}] & \text{if } \beta = 1 + \alpha \text{ and } \alpha \leq 1. \end{cases}$$

By interpolation, we have

$$\|u^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+1+\beta-\alpha'})} \leq t^{\frac{\alpha'}{2}} \|u^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+1+\beta})}^{1-\frac{\alpha'}{2}} \|u^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}-1+\beta})}^{\frac{\alpha'}{2}} \leq (\underline{\mu}t)^{\frac{\alpha'}{2}} U^n(t). \quad (4.7)$$

Let us now bound $V^n(t)$. If $0 < \beta < 2$, standard embeddings and real interpolation yield

$$\begin{aligned} \int_0^t \|\nabla u^n(\tau)\|_{L^\infty \cap H^{\frac{N}{2}}} d\tau &\lesssim \int_0^t \|\nabla u^n(\tau)\|_{B_{2,1}^{\frac{N}{2}}} d\tau \\ &\lesssim \|\nabla u^n\|_{\tilde{L}_t^1(B_{2,\infty}^{\frac{N}{2}+\beta-2})}^{\frac{\beta}{2}} \|\nabla u^n\|_{\tilde{L}_t^1(B_{2,\infty}^{\frac{N}{2}+\beta})}^{1-\frac{\beta}{2}} \\ &\lesssim t^{\frac{\beta}{2}} \|\nabla u^n\|_{L_t^\infty(B_{2,\infty}^{\frac{N}{2}+\beta-2})}^{\frac{\beta}{2}} \|\nabla u^n\|_{\tilde{L}_t^1(B_{2,\infty}^{\frac{N}{2}+\beta})}^{1-\frac{\beta}{2}} \lesssim \underline{\mu}^{-1} (\underline{\mu}t)^{\frac{\beta}{2}} U^n(t). \end{aligned}$$

In the case $\alpha > 1$ and $\beta \in (\alpha - 1, \alpha + 1]$, similar arguments lead to

$$\begin{aligned} \int_0^t \|\nabla u^n(\tau)\|_{H^{\frac{N}{2}+\alpha-1}} d\tau &\lesssim t^{\frac{\beta+1-\alpha}{2}} \|\nabla u^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\beta-2})}^{\frac{\beta+1-\alpha}{2}} \|\nabla u^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\beta})}^{\frac{\alpha+1-\beta}{2}} \\ &\lesssim \underline{\mu}^{-1} (\underline{\mu}t)^{\frac{\beta+1-\alpha}{2}} U^n(t), \end{aligned}$$

whence, denoting $\gamma = \beta + \min(1 - \alpha, 0)$,

$$V^n(t) \lesssim \underline{\mu}^{-1} (\underline{\mu}t)^{\frac{\gamma}{2}} U^n(t). \quad (4.8)$$

Plugging (4.8) and (4.7) in (4.3) and (4.6), we conclude that

$$\|a^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\alpha})} \leq \|a_0\|_{H^{\frac{N}{2}+\alpha}} e^{\frac{C}{\underline{\mu}} (\underline{\mu}t)^{\frac{\gamma}{2}} U^n(t)}, \quad (4.9)$$

$$\begin{aligned} U^n(t) &\leq C(\mathcal{A}_t^n)^{\kappa+1} e^{\frac{C}{\underline{\mu}} (\mathcal{A}_t^n)^\kappa (\underline{\mu}t)^{\frac{\gamma}{2}} U^n(t)} \\ &\quad \times \left(\|u_0\|_{H^{\frac{N}{2}+\beta-1}} + \|f\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\beta-1})} + (\underline{\mu}t)^{\frac{\alpha'}{2}} U^n(t) \right). \end{aligned} \quad (4.10)$$

From (4.9), we gather

$$\mathcal{A}_t^n \leq \mathcal{A}_0 e^{\frac{C}{\underline{\mu}} (\underline{\mu}t)^{\frac{\gamma}{2}} U^n(t)} \quad \text{with} \quad \mathcal{A}_0 \stackrel{\text{def}}{=} 1 + \underline{b}^{-1} \|a_0\|_{H^{\frac{N}{2}+\alpha}}. \quad (4.11)$$

Let $t < T_n$ be so small as to satisfy

$$CU^n(t)2^\kappa \mathcal{A}_0^\kappa(\underline{\mu}t)^{\frac{\gamma}{2}} \leq \underline{\mu} \log 2 \quad \text{and} \quad C2^{\kappa+2} \mathcal{A}_0^{\kappa+1}(\underline{\mu}t)^{\frac{\alpha'}{2}} \leq 1/2. \quad (\mathcal{H})$$

From (4.9), (4.10), and (4.11), we easily infer that

$$\begin{cases} \|a^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\alpha})} \leq 2\|a_0\|_{H^{\frac{N}{2}+\alpha}}, \\ U^n(t) \leq C2^{\kappa+3} \mathcal{A}_0^{\kappa+1}(\|u_0\|_{H^{\frac{N}{2}+\beta-1}} + \|f\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\beta-1})}). \end{cases} \quad (4.12)$$

Fix a large T_0 , denote $U_0 \stackrel{\text{def}}{=} \|u_0\|_{H^{\frac{N}{2}+\beta-1}} + \|f\|_{\tilde{L}_{T_0}^1(H^{\frac{N}{2}+\beta-1})}$, and choose a $T \leq T_0$ such that

$$C^2 2^{2\kappa+3} \mathcal{A}_0^{2\kappa+1}(\underline{\mu}T)^{\frac{\gamma}{2}} U_0 \leq \underline{\mu} \log 2 \quad \text{and} \quad C2^{\kappa+2} \mathcal{A}_0^{\kappa+1}(\underline{\mu}T)^{\frac{\alpha'}{2}} \leq 1/2. \quad (4.13)$$

Then a standard bootstrap argument shows that (\mathcal{H}) and (4.12) are satisfied whenever $t \leq T$ and $t < T_n$. A continuation argument enables us to conclude that $T_n > T$ and that (4.12) holds on $[0, T]$. As Proposition 2.2 also yields uniform bounds in $\tilde{L}_T^1(H^{\frac{N}{2}+\beta-1})$ for the pressure, the sequence $(a^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$ is thus uniformly bounded in $E_T^{\alpha, \beta}$.

Fourth step: Convergence of the sequence in small norm. Let $\gamma \in (0, \min(\alpha, \beta, 1/4))$. We are going to show that $(a^n, u^n, \nabla \Pi^n)$ is a Cauchy sequence in $E_T^{\gamma-1, \gamma-1}$. For $n \in \mathbb{N}$ and $p \in \mathbb{N}$, denote

$$\delta a_p^n \stackrel{\text{def}}{=} a^{n+p} - a^n, \quad \delta u_p^n \stackrel{\text{def}}{=} u^{n+p} - u^n, \quad \delta f_p^n \stackrel{\text{def}}{=} f^{n+p} - f^n, \quad \nabla \delta \Pi_p^n \stackrel{\text{def}}{=} \nabla \Pi^{n+p} - \nabla \Pi^n,$$

$$K \stackrel{\text{def}}{=} \sup_{n \in \mathbb{N}} (\|a^n\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\gamma})} + \|u^n\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\gamma-1})} + \underline{\mu} \|u^n\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma+1})} + \|\nabla \Pi^n\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-1})}),$$

$$\begin{aligned} \delta X_p^n(t) \stackrel{\text{def}}{=} & \|\delta a_p^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\gamma-1})} + \|\delta u_p^n\|_{\tilde{L}_t^\infty(H^{\frac{N}{2}+\gamma-2})} \\ & + \underline{\mu} \|\delta u_p^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}+\gamma})} + \|\nabla \delta \Pi_p^n\|_{\tilde{L}_t^1(H^{\frac{N}{2}-2})}. \end{aligned}$$

Apply Proposition 3.1 with $\alpha = \gamma$ to $(a^n, u^n, \nabla \Pi^n)$ and $(a^{n+p}, u^{n+p}, \nabla \Pi^{n+p})$. We get for some constant $C_{K,T}$ depending only on K, T, \underline{b}, N , and γ ,

$$\delta X_p^n(T) \leq C_{K,T} (\|\delta a_p^n(0)\|_{H^{\frac{N}{2}+\gamma-1}} + \|\delta u_p^n(0)\|_{H^{\frac{N}{2}+\gamma-2}} + \|\delta f_p^n\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-2})}). \quad (4.14)$$

Let $\epsilon > 0$. There exists an $n_0 \in \mathbb{N}$ for which for all $p \in \mathbb{N}$ and $n \geq n_0$,

$$\|\delta a_p^n(0)\|_{H^{\frac{N}{2}+\gamma-1}} + \|\delta u_p^n(0)\|_{H^{\frac{N}{2}+\gamma-2}} + \|\delta f_p^n\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-2})} \leq \epsilon. \quad (4.15)$$

Hence for $n \geq n_0$,

$$\delta X_p^n(T) \leq C_{K,T} \epsilon \tag{4.16}$$

so that $(a^n, u^n, \nabla \Pi^n)_{n \in \mathbb{N}}$ tends to some limit $(a, u, \nabla \Pi)$ in $E_T^{\gamma-1, \gamma-1}$.

Fifth step: Checking that the limit belongs to $E_T^{\alpha, \beta}$ and satisfies (0.6). Taking advantage of the uniform bounds of step 3, we gather that $(a, u, \nabla \Pi)$ belongs to

$$\tilde{L}_T^\infty(H^{\frac{N}{2}+\alpha}) \times (\tilde{L}_T^\infty(H^{\frac{N}{2}+\beta-1}) \cap \tilde{L}_T^1(H^{\frac{N}{2}+\beta+1}))^N \times (\tilde{L}_T^1(H^{\frac{N}{2}+\beta-1}))^N.$$

Besides, arguing by interpolation, we infer that convergence holds in every intermediate space $E_T^{\alpha', \beta'}$ with $\alpha' < \alpha$ and $\beta' < \beta$, which suffices to pass to the limit in (0.6). Passing to the limit in (4.1) shows that $(\rho = 1/(1+a), u, \nabla \Pi)$ satisfies the energy equality (0.2).

We still have to show that $a \in C([0, T]; H^{\frac{N}{2}+\alpha})$ and $u \in C([0, T]; H^{\frac{N}{2}+1+\beta})$. Proposition 2.1 yields the result for a . On the other hand, as $\partial_t u - \mu \Delta u \in \tilde{L}_T^1(H^{\frac{N}{2}+\beta-1})$, Proposition 2.1 in [3] yields the continuity result for u . \square

Sixth step: Uniqueness. This is a mere consequence of Proposition 3.1

Last step: Blow-up criterion. This subsection is devoted to the proof of Theorem 0.4. Let us treat only the case $1 < p < +\infty$; the case $p = 1$ (which relies on Proposition 2.2) is left to the reader.

According to (1.7), one can assume with no loss of generality that $a \in \tilde{L}_{T^*}^\infty(H^{\frac{N}{2}+\alpha'})$ for some $\alpha' > 0$.

$$\text{Let } \gamma = \min(\beta, 1 + \alpha') \text{ and } K \stackrel{\text{def}}{=} \|a\|_{\tilde{L}_{T^*}^\infty(H^{\frac{N}{2}+\alpha'})} + \|\nabla u\|_{L_{T^*}^1(B_{\infty, \infty}^{\frac{2}{p}-2})}.$$

Applying Proposition 2.3 with $s = \frac{N}{2} - 1 + \gamma$ yields

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\gamma-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma+1})} + \|\nabla \Pi\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-1})} \\ & \leq C_K (\|u_0\|_{H^{\frac{N}{2}+\gamma-1}} + \|f\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(H^{\frac{N}{2}+1+\gamma-\epsilon})}), \end{aligned}$$

for some $\epsilon > 0$ and C_K depending only on K .

Interpolating between $L^1(0, T; L^2)$ and $\tilde{L}_T^1(H^{\frac{N}{2}+1+\gamma})$ and using Young's inequality gives (up to a change of C_K),

$$\begin{aligned} & \|u\|_{\tilde{L}_T^\infty(H^{\frac{N}{2}+\gamma-1})} + \underline{\mu} \|u\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma+1})} + \|\nabla \Pi\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-1})} \\ & \leq C_K (\|u_0\|_{H^{\frac{N}{2}+\gamma-1}} + \|f\|_{\tilde{L}_T^1(H^{\frac{N}{2}+\gamma-1})} + \underline{\mu} T \|u\|_{L_T^\infty(L^2)}). \end{aligned}$$

On the other hand, using Theorem 3.1, one can easily check that any solution in $E_T^{\alpha, \beta}$ satisfies the energy equality (0.2). Hence the last term is finite so that

u belongs to $\tilde{L}_{T^*}^\infty(H^{\frac{N}{2}-1+\gamma}) \cap \tilde{L}_{T^*}^1(H^{\frac{N}{2}+1+\gamma})$, and $\nabla \Pi \in \tilde{L}_{T^*}^1(H^{\frac{N}{2}-1+\gamma})$. Coming back to the transport equation, we now get $a \in \tilde{L}_T^\infty(H^{\frac{N}{2}+\min(\alpha, 2+\alpha')})$.

Then one can use again the momentum equation to get additional regularity for u . Within a finite number of steps, one concludes that $(a, u, \nabla \Pi)$ belongs to

$$\tilde{L}_{T^*}^\infty(H^{\frac{N}{2}+\alpha}) \times (\tilde{L}_{T^*}^\infty(H^{\frac{N}{2}-1+\beta}) \cap \tilde{L}_{T^*}^1(H^{\frac{N}{2}+1+\beta}))^N \times (\tilde{L}_{T^*}^1(H^{\frac{N}{2}-1+\beta}))^N.$$

Then a standard continuation argument achieves the proof of Theorem 0.4.

5. GLOBAL WELL-POSEDNESS IN DIMENSION $N = 2$

Obviously, the proof used hitherto cannot yield global well-posedness even for small data. The reason why is that, owing to a bad control of the low frequencies of u , a growth in time appears in the estimate of Proposition 2.2 when bounding the term $\|u\|_{\tilde{L}_T^1(H^{s+2-\alpha'})}$. In the case of *bounded domains*, Poincaré’s inequality supplies for free the missing control. This is crucial for proving global results in dimension $N = 2$ and in dimension $N = 3$ for small initial velocity (see [14] and [6] for more details). In the whole space however, no inequality of this type is expected. More surprisingly, the situation does not improve in the torus as we do not know how to control the average of u !

In order to recover the missing control on the low frequencies of u , we shall make use of a pseudo-conservation law involving the H^1 norm of the velocity. At first sight, this seems to force us to make quite strong an assumption on the velocity, namely that $u_0 \in H^1$. We shall see later on how it can be removed.

5.1. Proof in the case $\beta = 1$ and $\alpha \in (0, 2)$.

First step: Local well-posedness. Theorem 0.2 supplies a *local strong solution* $(a, u, \nabla \Pi)$ which belongs to $E_T^{\alpha,1}$ (see the statement of Theorem 0.2). We denote by $T^* \in (0, +\infty]$ the maximum time of existence of $(a, u, \nabla \Pi)$, i.e., the supremum of all T such that $(a, u, \nabla \Pi)$ belongs to $E_T^{\alpha,1}$.

Second step: Existence of a global weak solution. In dimension $N = 2$, the existence of a global weak solution $(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi})$ which satisfies the energy inequality (0.2), and $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{A}^N)$, $\nabla \tilde{u} \in L_{loc}^\infty(\mathbb{R}^+; L^2) \cap L_{loc}^2(\mathbb{R}^+; H^1)$, $\partial_t \tilde{u}, \nabla \tilde{\Pi} \in L_{loc}^2(\mathbb{R}^+; L^2)$ under the hypotheses that $u_0 \in H^1$ and $\rho_0 \in L^\infty$ is bounded away from vacuum, is well known (see e.g. [1] or [16]).

The proof stems from the conservation of the L^∞ norm for $\rho_0^{\pm 1}$, a priori estimates involving the $L^\infty(0, T; H^1)$ and $L^2(0, T; H^2)$ norms of u (see Proposition 5.1 below) and compactness arguments. It is actually quite

straightforward if no vacuum initially (which is precisely the case we are interested in).

As people do not usually consider the case of a nonzero external force f , we briefly state a priori estimates in the case where f belongs to $L^2_{loc}(\mathbb{R}^+; L^2)$.

Proposition 5.1. *Let $(\rho, u, \nabla\Pi)$ solve*

$$\begin{cases} \partial_t \rho + v \cdot \nabla \rho = 0, \\ \rho(\partial_t u + v \cdot \nabla u) - \mu \Delta u + \nabla \Pi = \rho f, \\ \operatorname{div} u = 0, \end{cases}$$

for some divergence-free, time-dependent vector field v . There exists a universal constant C such that the following a priori estimate holds true:

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 + \int_0^t \left(\frac{\|\sqrt{\rho} \partial_t u\|_{L^2}^2}{2\mu} + \frac{\|\nabla \Pi\|_{L^2}^2}{\mu \|\rho_0\|_{L^\infty}} + \frac{\|\nabla^2 u\|_{L^2}^2}{\|\rho_0\|_{L^\infty}} \right) d\tau \\ \leq e^{\frac{C\|\rho_0\|_{L^\infty}^3}{\mu^3} \int_0^t \|v\|_{L^4}^4 d\tau} \left(\|\nabla u_0\|_{L^2}^2 + C \int_0^t \frac{\|\sqrt{\rho} f\|_{L^2}^2}{\mu} d\tau \right). \end{aligned}$$

Proof. Taking the L^2 scalar product of the momentum equation with $\partial_t u$ and using Hölder's and Young's inequalities yields

$$\|\sqrt{\rho} \partial_t u\|_{L^2}^2 + \mu \frac{d}{dt} \|\nabla u\|_{L^2}^2 \leq \|\sqrt{\rho} f\|_{L^2}^2 + \|\sqrt{\rho} v\|_{L^4}^2 \|\nabla u\|_{L^4}^2. \tag{5.1}$$

On the other hand, using maximal regularity for the Stokes equation

$$-\mu \Delta u + \nabla \Pi = \sqrt{\rho} (\sqrt{\rho} f + \sqrt{\rho} \partial_t u + \sqrt{\rho} v \cdot \nabla u), \quad \operatorname{div} u = 0,$$

gives

$$\begin{aligned} \mu \|\nabla^2 u\|_{L^2} + \|\nabla \Pi\|_{L^2} \\ \lesssim \sqrt{\|\rho\|_{L^\infty}} \left(\|\sqrt{\rho} f\|_{L^2} + \|\sqrt{\rho} \partial_t u\|_{L^2} + \|\sqrt{\rho} v\|_{L^4} \|\nabla u\|_{L^4} \right). \end{aligned}$$

Now applying Gagliardo-Nirenberg's inequality,

$$\|\nabla u\|_{L^4} \lesssim \sqrt{\|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2}}, \tag{5.2}$$

and Young's inequality yields

$$\begin{aligned} \mu \|\nabla^2 u\|_{L^2} + \|\nabla \Pi\|_{L^2} \\ \lesssim \|\rho\|_{L^\infty}^{\frac{1}{2}} \left(\|\sqrt{\rho} f\|_{L^2} + \|\sqrt{\rho} \partial_t u\|_{L^2} \right) + \frac{\|\rho\|_{L^\infty} \|\sqrt{\rho} v\|_{L^4}^2 \|\nabla u\|_{L^2}}{\mu}. \end{aligned} \tag{5.3}$$

Making use of (5.2) in (5.1), then plugging (5.3) in (5.1), eventually leads to

$$\|\sqrt{\rho}\partial_t u\|_{L^2}^2 + 2\mu \frac{d}{dt} \|\nabla u\|_{L^2}^2 \lesssim \|\sqrt{\rho}f\|_{L^2}^2 + \frac{\|\rho\|_{L^\infty}}{\mu^2} \|\sqrt{\rho}v\|_{L^4}^4 \|\nabla u\|_{L^2}^2. \quad (5.4)$$

Adding (5.4) and (5.3), using that $\|\rho(t)\|_{L^\infty} \leq \|\rho_0\|_{L^\infty}$, and performing a time integration yields

$$\begin{aligned} \|\nabla u(t)\|_{L^2}^2 + \int_0^t \left(\frac{\|\sqrt{\rho}\partial_t u\|_{L^2}^2}{2\mu} + \frac{\|\nabla \Pi\|_{L^2}^2}{\mu \|\rho_0\|_{L^\infty}} + \frac{\|\nabla^2 u\|_{L^2}^2}{\|\rho_0\|_{L^\infty}} \right) d\tau \\ \leq \|\nabla u_0\|_{L^2}^2 + C \int_0^t \frac{\|\sqrt{\rho}f\|_{L^2}^2}{\mu} d\tau + \frac{C \|\rho_0\|_{L^\infty}^3}{\mu^3} \int_0^t \|v\|_{L^4}^4 \|\nabla u\|_{L^2}^2 d\tau, \end{aligned}$$

and Gronwall’s lemma completes the proof. □

Of course the standard energy inequality (0.2) combined with Gagliardo-Nirenberg’s inequality yields bounds for the argument of the exponential.

Hence we are given the local, smooth, unique solution $(a, u, \nabla \Pi)$ and another solution $(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi})$ which is global and satisfies the properties described above.

Third step: Loosing a priori estimates for transport equations. At this point, there is no (known) way of proving uniqueness of $(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi})$. Only a part of the assumptions on the data have been used though. Indeed, we did not take advantage of $a_0 \in H^{1+\alpha}$. Does anything persist of the initial smoothness?

To answer this question, we observe that \tilde{a} is advected by the vector field \tilde{u} which belongs to $L^2_{loc}(\mathbb{R}^+; H^2)$. In view of Proposition 2.1, the $H^{1+\alpha}$ regularity of \tilde{a} would be preserved if in addition $\nabla \tilde{u}$ were in $L^1_{loc}(\mathbb{R}^+; L^\infty)$. But H^1 fails to be embedded in L^∞ in dimension $N = 2$! Nevertheless, the following statement holds true:

Proposition 5.2. *Assume that $N = 2$ and that $0 \leq s \leq 2$. Let $f_0 \in H^s$ and $v \in L^1(0, T; H^2)$. Then the transport equation*

$$\begin{cases} \partial_t f + v \cdot \nabla f = 0, \\ f|_{t=0} = f_0, \end{cases}$$

has a unique solution f which belongs to $C([0, T]; H^{s-\epsilon})$ for all $\epsilon > 0$. Moreover, there exists a constant $C = C(\epsilon, s)$ such that

$$\forall t \in [0, T], \|f(t)\|_{H^{s-\epsilon}} \leq C e^{C \left(\int_0^t \|\nabla v(t)\|_{H^1} dt \right)^2} \|f_0\|_{H^s}. \quad (5.5)$$

A similar proposition has been proved by B. Desjardins under a slightly stronger assumption: $v \in L^2(0, T; H^2)$ and in the case of a bounded domain or in \mathbb{T}^2 (see [9] and [11]). In [5], we investigate losing *a priori* estimates

for the transport equation and estimate (5.5) comes up as a particular case of Theorem 0.1 therein. As for existence, it may easily be obtained by smoothing out the data and using estimate (5.5).

Fourth step: Weak=strong uniqueness. According to the previous step,

$$\tilde{a} \in C(\mathbb{R}^+; H^{1+\frac{\alpha}{2}}), \quad \tilde{u} \in L_{loc}^\infty(\mathbb{R}^+; H^1) \cap L_{loc}^2(\mathbb{R}^+; H^2), \quad \nabla \tilde{\Pi} \in L_{loc}^2(\mathbb{R}^+; L^2).$$

We aim at proving that $(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi}) \equiv (a, u, \nabla \Pi)$ on $[0, T^*) \times \mathbb{A}^N$.

Let $\delta a \stackrel{\text{def}}{=} a - \tilde{a}$, $\delta u \stackrel{\text{def}}{=} u - \tilde{u}$, and $\nabla \delta \Pi \stackrel{\text{def}}{=} \nabla \Pi - \nabla \tilde{\Pi}$. We have

$$\begin{cases} \partial_t \delta a + u \cdot \nabla \delta a = -\delta u \cdot \nabla \tilde{a}, \\ \partial_t \delta u + u \cdot \nabla \delta u + (1+a)(\nabla \delta \Pi - \mu \Delta \delta u) = -\delta u \cdot \nabla \tilde{u} - \delta a(\nabla \tilde{\Pi} - \mu \Delta \tilde{u}), \\ \operatorname{div} \delta u = 0. \end{cases} \tag{5.6}$$

Hence, applying Proposition 2.1 with $s = \alpha/2$ and Proposition 2.5 with $s = \alpha/2 - 1$,

$$\begin{aligned} \|\delta a\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}})} &\leq C e^{C \int_0^t \|\nabla u(\tau)\|_{H^1 \cap L^\infty} d\tau} \|\delta u \cdot \nabla \tilde{a}\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}})}, \\ \|\delta u\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}-1})} + \|\delta u\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}+1})} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}-1})} &\leq C \mathcal{A}_t^\kappa e^{C \mathcal{A}_t^\kappa \int_0^t \|\nabla u(\tau)\|_{H^1 \cap L^\infty} d\tau} \\ &\quad \times \left(\|\delta u \cdot \nabla \tilde{u}\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}-1})} + \|\delta a(\nabla \tilde{\Pi} - \mu \Delta \tilde{u})\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}-1})} + \|\delta u\|_{\tilde{L}_T^1(H^1)} \right), \end{aligned}$$

with $\mathcal{A}_t \stackrel{\text{def}}{=} 1 + \underline{b}^{-1} \|a\|_{\tilde{L}_t^\infty(H^{1+\alpha})}$.

Let $X(t) \stackrel{\text{def}}{=} \|\delta a\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}})} + \|\delta u\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}-1})} + \|\delta u\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}+1})} + \|\nabla \delta \Pi\|_{\tilde{L}_t^1(H^{\frac{\alpha}{2}-1})}$.

From the above estimates and Remark 1.7, we have on $[0, T^*)$

$$\begin{aligned} X(t) &\leq C e^{C \mathcal{A}_t^\kappa \int_0^t \|\nabla u(\tau)\|_{H^1 \cap L^\infty} d\tau} \left(\|\nabla \tilde{a}\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}})} \|\delta u\|_{\tilde{L}_t^1(H^1 \cap L^\infty)} \right. \\ &\quad \left. + \|\nabla \tilde{u}\|_{L_t^2(L^2)} \|\delta u\|_{L_t^2(H^{\frac{\alpha}{2}})} + \|\nabla \tilde{\Pi} - \mu \Delta \tilde{u}\|_{L_t^1(L^2)} \|\delta a\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}})} + \|\delta u\|_{\tilde{L}_t^1(H^1)} \right). \end{aligned}$$

Fix a $T < T^*$. Combining the above inequality and interpolation, we get

$$X(t) \leq C_K X(t) \left(t^{\frac{\alpha}{8}} (1 + \|\nabla \tilde{a}\|_{\tilde{L}_t^\infty(H^{\frac{\alpha}{2}})}) + t^{\frac{1}{2}} (\|\nabla \tilde{u}\|_{L_t^\infty(L^2)} + \|\nabla \tilde{\Pi} - \mu \Delta \tilde{u}\|_{L_t^2(L^2)}) \right)$$

for some constant C_K depending on the bounds of $(a, u, \nabla \Pi)$ in $E_T^{\alpha,1}$.

This entails $X \equiv 0$ on a some small interval $[0, \eta]$. A standard induction argument then yields $X \equiv 0$ on the whole interval $[0, T^*)$.

Fifth step: Continuation. Assume that $T^* < \infty$. In view of Propositions 5.1 and 5.2, a belongs to $L^\infty(0, T^*; H^{1+\frac{\alpha}{2}})$. On the other hand, as $(\rho, u, \nabla \Pi)$ satisfies the energy inequality (0.2), ∇u belongs to $L^2(0, T^*; L^2)$.

In dimension $N = 2$, we have $L^2 \hookrightarrow B_{\infty, \infty}^{-1}$; hence, the blow-up criterion given in Theorem 0.4 applies, and we conclude that $(\rho, u, \nabla \Pi)$ may be continued beyond T^* into a solution which belongs to $E_T^{\alpha, 1}$.

This contradicts the maximality of T^* . Hence $T^* = \infty$.

5.2. General case $\beta > 0$. Let us treat only the case $\beta \in (0, 1)$; the case $\beta \geq 1$ may be treated by combining the case $\beta = 1$ and Theorem 0.4.

Once again, Theorem 0.2 supplies a strong, unique solution $(a, u, \nabla \Pi)$ in $E_T^{\alpha, \beta}$ for some $T > 0$. In particular, u belongs to $L^2(0, T; H^{1+\beta})$, thus to $L^2(0, T; H^1)$. Hence there exists a $T_0 < T$ such that $u(T_0)$ belongs to H^1 . Now, one may go along the proof in the case $\beta = 1$ and get a unique, global, strong solution $(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi})$ (on the time interval $[T_0, +\infty)$) with data (a_{T_0}, u_{T_0}) . Uniqueness ensures that $(\tilde{a}, \tilde{u}, \nabla \tilde{\Pi}) \equiv (a, u, \nabla \Pi)$ on $[T_0, T]$. Therefore $(a, u, \nabla \Pi)$ may be continued globally. \square

Remark 5.3. One can object that the above argument does not give much information on the possible growth of the solution. Yet it works!

APPENDIX A

For the sake of completeness, we here give the proof of some technical estimates.

Lemma A.1. *Let $N \geq 2$, $1 < p < +\infty$, and $s \in (-N/2 - 1, 2/p + N/2 - 1)$. Denote $p' \stackrel{\text{def}}{=} (1 - 1/p)^{-1}$. Assume that $\text{div } v = 0$. There exists a constant C such that for all $K > 0$ the following estimate holds true:*

$$\left(\sum_q 2^{2qs} \|[v^j, \Delta_q] \partial_j u\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}} \leq C \left(K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{2, \infty}^{\frac{N}{2} - \frac{2}{p'}}}^p \|\nabla u(t)\|_{H^{s-1}} dt \right).$$

Besides, for all $s > \max(-1, -2/p')$ holds

$$\left(\sum_q 2^{2qs} \|[u^j, \Delta_q] \partial_j u\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}} \leq C \left(K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla u(t)\|_{B_{\infty, \infty}^{\frac{N}{2} - \frac{2}{p'}}}^p \|\nabla u(t)\|_{H^{s-1}} dt \right).$$

Proof. Let $\tilde{u} \stackrel{\text{def}}{=} u - \Delta_{-1}u$ and $\tilde{v} \stackrel{\text{def}}{=} v - \Delta_{-1}v$. We shall use throughout that for any $\sigma \in \mathbb{R}$ and $1 \leq p, r \leq +\infty$,

$$\|\tilde{u}\|_{B_{p,r}^\sigma} \lesssim \|\nabla u\|_{B_{p,r}^{\sigma-1}} \quad \text{and} \quad \|\tilde{v}\|_{B_{p,r}^\sigma} \lesssim \|\nabla v\|_{B_{p,r}^{\sigma-1}}. \quad (\text{A.1})$$

Split the commutator into six parts:

$$[v^j, \Delta_q] \partial_j u = \sum_{i=1}^6 R_q^i$$

with

$$\begin{aligned} R_q^1 &= [T_{\tilde{v}^j}, \Delta_q] \partial_j u = \sum_{|q'-q| \leq 4} [S_{q'-1} \tilde{v}^j, \Delta_q] \Delta_{q'} \partial_j u, \\ R_q^2 &= T'_{\Delta_q \partial_j u} \tilde{v}^j = \sum_{q' \geq q-2} S_{q'+2} \Delta_q \partial_j u \Delta_{q'} \tilde{v}^j, \\ R_q^3 &= -\Delta_q T_{\partial_j u} \tilde{v}^j = \sum_{|q'-q| \leq 4} \Delta_q (S_{q'-1} \partial_j u \Delta_{q'} \tilde{v}^j), \\ R_q^4 &= -\Delta_q \partial_j R(\tilde{u}, \tilde{v}^j) = \sum_{q' \geq q-3} \Delta_q \partial_j (\Delta_{q'} \tilde{u} \tilde{\Delta}_{q'} \tilde{v}^j), \\ R_q^5 &= [\Delta_{-1} v^j, \Delta_q] \partial_j u = \sum_{|q'-q| \leq 2} [\Delta_{-1} v^j, \Delta_q] \partial_j \Delta_{q'} u, \\ R_q^6 &= -\Delta_q R(\partial_j \Delta_{-1} u, \tilde{v}^j) = \sum_{q' \leq 1} \Delta_q (\partial_j \tilde{\Delta}_{q'} \Delta_{-1} u \Delta_{q'} \tilde{v}^j). \end{aligned}$$

Let $\mathcal{R}^i \stackrel{\text{def}}{=} \left(\sum_q 2^{2qs} \|R_q^i\|_{L_T^1(L^2)}^2 \right)^{\frac{1}{2}}$.

Bounds for \mathcal{R}^1 . The first-order Taylor's formula yields (see e.g. the proof of Lemma A.1 in [4])

$$\begin{aligned} \mathcal{R}^1 &\lesssim \sum_{|i| \leq 4} \left[\sum_q 2^{2q(s-1)} \left(\int_0^T \|\nabla S_{q+i-1} \tilde{v}^j\|_{L^\infty} \|\Delta_{q+i} \partial_j u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}}, \\ &\lesssim \sum_{|i| \leq 4} \left[\sum_q \left(\int_0^T 2^{-\frac{2(q+i)}{p'}} \|\nabla S_{q+i-1} \tilde{v}\|_{L^\infty} \left(2^{(q+i)(s-1)} \|\Delta_{q+i} \nabla u\|_{L^2} \right)^{\frac{1}{p}} \right. \right. \\ &\quad \left. \left. \times \left(2^{(q+i)(s+1)} \|\Delta_{q+i} \nabla u\|_{L^2} \right)^{\frac{1}{p'}} dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

Now, taking advantage of Young's inequality,

$$ab \leq \frac{K^{-\frac{p}{p'}}}{p} a^p + \frac{K}{p'} b^{p'}, \quad (\text{A.2})$$

we get

$$\begin{aligned} \mathcal{R}^1 &\lesssim K \sum_{i=-4}^4 \left[\sum_q \left(\int_0^T 2^{(q+i)(s+1)} \|\Delta_{q+i} \nabla u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &+ K^{-\frac{p}{p'}} \sum_{i=-4}^4 \left[\sum_q \left(\int_0^T \left(2^{-\frac{2(q+i)}{p'}} \|\nabla S_{q+i-1} \tilde{v}\|_{L^\infty} \right)^p 2^{(q+i)(s-1)} \|\Delta_{q+i} \nabla u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (\text{A.3})$$

Remark that since $p' < +\infty$, we have

$$2^{-\frac{2(q+i)}{p'}} \|\nabla S_{q+i-1} \tilde{v}\|_{L^\infty} \leq \sum_{q' \leq q+i-1} 2^{-\frac{2(q+i-q')}{p'}} 2^{-\frac{2q'}{p'}} \|\nabla \Delta_{q'} \tilde{v}\|_{L^\infty} \lesssim \|\nabla \tilde{v}\|_{B_{\infty, \infty}^{-\frac{2}{p'}}};$$

hence, applying (A.1) and Minkowski inequality for bounding the last term in (A.3) yields

$$\mathcal{R}^1 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{\infty, \infty}^{-\frac{2}{p'}}}^p \|\nabla u(t)\|_{H^{s-1}} dt. \quad (\text{A.4})$$

Bounds for \mathcal{R}^2 . We have

$$\mathcal{R}^2 \lesssim \left[\sum_q 2^{2qs} \left(\int_0^T \sum_{q' \geq q-2} \|S_{q'+2} \Delta_q \partial_j u \Delta_{q'} \tilde{v}^j\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}}, \quad (\text{A.5})$$

$$\begin{aligned} &\lesssim \left[\sum_q \left(\int_0^T \sum_{q' \geq q-2} 2^{q'(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_{q'} \tilde{v}\|_{L^2} 2^{q(s+1-\frac{N}{2}-\frac{2}{p})} \right. \right. \\ &\quad \left. \left. \times \|\Delta_q \nabla u\|_{L^\infty} 2^{(q-q')(\frac{N}{2}+1-\frac{2}{p'})} dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (\text{A.6})$$

Young's inequality (A.2) yields

$$\begin{aligned} 2^{q'(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_{q'} \tilde{v}\|_{L^2} 2^{q(s+1-\frac{N}{2}-\frac{2}{p})} \|\Delta_q \nabla u\|_{L^\infty} &\leq K 2^{q(s+1-\frac{N}{2})} \|\Delta_q \nabla u\|_{L^\infty} \\ &+ K^{-\frac{p'}{p}} \left(2^{q'(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_{q'} \tilde{v}\|_{L^2} \right)^p 2^{q(s-\frac{N}{2}-1)} \|\Delta_q \nabla u\|_{L^\infty}, \end{aligned}$$

whence

$$\begin{aligned} \mathcal{R}^2 &\lesssim K \left[\sum_q \left(\sum_{q' \geq q-2} 2^{(q-q')(\frac{N}{2}+1-\frac{2}{p'})} 2^{q(s+1-\frac{N}{2})} \|\Delta_q \nabla u\|_{L_T^1(L^\infty)} \right)^2 \right]^{\frac{1}{2}} \\ &+ K^{-\frac{p'}{p}} \left[\sum_q \left(\int_0^T \sum_{q' \geq q-2} 2^{(q-q')(\frac{N}{2}+1-\frac{2}{p'})} \left(2^{q'(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_{q'} \tilde{v}\|_{L^2} \right)^p \right. \right. \\ &\quad \left. \left. \times 2^{q(s-\frac{N}{2}-1)} \|\Delta_q \nabla u\|_{L^\infty} dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

According to Minkowski's inequality, the last term may be bounded by

$$K^{-\frac{p'}{p}} \int_0^T \|\tilde{v}\|_{B_{2,\infty}^{\frac{N}{2}+1-\frac{2}{p'}}}^p \left[\sum_q \left(\sum_{q' \geq q-2} 2^{(q-q')(\frac{N}{2}+1-\frac{2}{p'})} 2^{q(s-\frac{N}{2}-1)} \|\Delta_{q'} \nabla u\|_{L^\infty} \right)^2 \right]^{\frac{1}{2}} dt.$$

Now, as $1 - 2/p' + N/2 > 0$, convolution inequalities combined with (A.1) yield

$$\mathcal{R}^2 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(B_{\infty,2}^{s+1-\frac{N}{2}})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|\nabla u(t)\|_{B_{\infty,2}^{s-1-\frac{N}{2}}} dt. \quad (\text{A.7})$$

Let us derive additional estimates for \mathcal{R}^2 in the case $u = v$. Inequality (A.5) yields

$$\begin{aligned} \mathcal{R}^2 &\lesssim \left[\sum_q \left(\int_0^T \sum_{q' \geq q-2} 2^{q'(s+\frac{2}{p'})} \|\Delta_{q'} \tilde{u}\|_{L^2} 2^{-\frac{2q}{p'}} \|\Delta_q \nabla u\|_{L^\infty} 2^{(q-q')(s+\frac{2}{p'})} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \left[\sum_q \left(\int_0^T \sum_{q' \geq q-2} 2^{q'(s+2)} \|\Delta_{q'} \tilde{u}\|_{L_T^1(L^2)} 2^{(q-q')(s+\frac{2}{p'})} dt \right)^2 \right]^{\frac{1}{2}} \\ &\quad + \left[K^{-\frac{p'}{p}} \left(\int_0^T \sum_{q' \geq q-2} (2^{-\frac{2q}{p'}} \|\Delta_q \nabla u\|_{L^\infty})^p 2^{q'(s+2)} \|\Delta_{q'} \tilde{u}\|_{L_T^1(L^2)} 2^{(q-q')(s+\frac{2}{p'})} dt \right)^2 \right]^{\frac{1}{2}}, \end{aligned}$$

whence, if $s + 2/p' > 0$,

$$\mathcal{R}^2 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(B_{\infty,2}^{s+1-\frac{N}{2}})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla u(t)\|_{B_{\infty,\infty}^{\frac{2}{p'}}}^p \|\nabla u(t)\|_{B_{\infty,2}^{s-1-\frac{N}{2}}} dt. \quad (\text{A.8})$$

Bounds for \mathcal{R}^3 . We have

$$\begin{aligned} \mathcal{R}^3 &\lesssim \left[\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{qs} \|\Delta_{q'} \partial_j u\|_{L^\infty} \|\Delta_q \tilde{v}^j\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \left[\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{(q-q')(s+1-\frac{N}{2}-\frac{2}{p'})} (2^{q'(s+1-\frac{N}{2})} \|\Delta_{q'} \partial_j u\|_{L^\infty})^{\frac{1}{p'}} \right. \right. \\ &\quad \left. \left. \times (2^{q'(s-\frac{N}{2}-1)} \|\Delta_{q'} \partial_j u\|_{L^\infty})^{\frac{1}{p}} 2^{q(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_q \tilde{v}^j\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}}. \quad (\text{A.9}) \end{aligned}$$

Young's inequality yields

$$\begin{aligned} \mathcal{R}^3 &\lesssim K \left(\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{(q-q')(s+1-\frac{N}{2}-\frac{2}{p'})} 2^{q'(s+1-\frac{N}{2})} \|\Delta_{q'} \partial_j u\|_{L^\infty} dt \right)^2 \right)^{\frac{1}{2}} \\ &\quad + K^{-\frac{p}{p'}} \left(\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{(q-q')(s+1-\frac{N}{2}-\frac{2}{p'})} 2^{q'(s-\frac{N}{2}-1)} \|\Delta_{q'} \partial_j u\|_{L^\infty} \right. \right. \\ &\quad \left. \left. \times (2^{q(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_q \tilde{v}^j\|_{L^2})^p dt \right)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

As $s < N/2 + 2/p - 1$, convolution inequalities enable us to conclude that

$$\mathcal{R}^3 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(B_{\infty,2}^{s+1-\frac{N}{2}})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|\nabla u(t)\|_{B_{\infty,2}^{s-1-\frac{N}{2}}} dt. \quad (\text{A.10})$$

Let us now derive additional estimates for \mathcal{R}^3 in the case $u = v$. We have

$$\begin{aligned} \mathcal{R}^3 &\lesssim \left[\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{\frac{2}{p'}(q'-q)} 2^{-\frac{2q'}{p'}} \|\Delta_{q'} \partial_j u\|_{L^\infty} 2^{q(s+\frac{2}{p'})} \|\Delta_q \tilde{v}^j\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \left[\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{\frac{2}{p'}(q'-q)} 2^{q(s+2)} \|\Delta_q \tilde{u}^j\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\quad + K^{-\frac{p}{p'}} \left[\sum_q \left(\sum_{q' \leq q-2} \int_0^T 2^{\frac{2}{p'}(q'-q)} (2^{-\frac{2q'}{p'}} \|\Delta_{q'} \partial_j u\|_{L^\infty})^p 2^{qs} \|\Delta_q \tilde{u}^j\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}}, \end{aligned}$$

whence, since $2/p' > 0$,

$$\mathcal{R}^3 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(B_{\infty,2}^{s+1-\frac{N}{2}})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla u(t)\|_{B_{2,\infty}^{\frac{2}{p'}}}^p \|\nabla u(t)\|_{B_{\infty,2}^{s-1-\frac{N}{2}}} dt. \quad (\text{A.11})$$

Bounds for \mathcal{R}^4 . Combining Minkowski's and Young's inequalities yields

$$\begin{aligned} \mathcal{R}^4 &\leq \left[\sum_q \left(\sum_{q' \geq q-3} \int_0^T 2^{q(s+1+\frac{N}{2})} \|\Delta_{q'} \tilde{u}\|_{L^2} \|\tilde{\Delta}_{q'} \tilde{v}\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\leq \left[\sum_q \left(\sum_{q' \geq q-3} \int_0^T 2^{(q-q')(\frac{N}{2}+1+s)} \left(2^{q'(s+2)} \|\Delta_{q'} u\|_{L^2} \right)^{\frac{1}{p'}} \right. \right. \\ &\quad \left. \left. \times \left(2^{q's} \|\Delta_{q'} \tilde{u}\|_{L^2} \right)^{\frac{1}{p'}} 2^{q'(\frac{N}{2}+1-\frac{2}{p'})} \|\tilde{\Delta}_{q'} \tilde{v}\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \left[\sum_q \left(\sum_{q' \geq q-3} \int_0^T 2^{(q-q')(s+1+\frac{N}{2})} 2^{q'(s+2)} \|\Delta_{q'} u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\quad + K^{-\frac{p}{p'}} \left[\sum_q \left(\sum_{q' \geq q-3} \int_0^T 2^{(q-q')(s+1+\frac{N}{2})} 2^{q's} \|\Delta_{q'} u\|_{L^2} \right. \right. \\ &\quad \left. \left. \times \left(2^{q'(\frac{N}{2}+1-\frac{2}{p'})} \|\Delta_{q'} \tilde{v}\|_{L^2} \right)^p dt \right)^2 \right]^{\frac{1}{2}}. \end{aligned}$$

As $s+1+N/2 > 0$, convolution inequalities and (A.1) enable us to conclude that

$$\mathcal{R}^4 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|\nabla u(t)\|_{H^{s-1}} dt. \quad (\text{A.12})$$

If $u = v$ and $s > -1$, one can further write

$$\begin{aligned} \mathcal{R}^4 &\leq \left[\sum_q \left(\sum_{q' \geq q-3} \int_0^T 2^{q(s+1)} \|\Delta_{q'} \tilde{u}\|_{L^2} \|\tilde{\Delta}_{q'} \tilde{u}\|_{L^\infty} dt \right)^2 \right]^{\frac{1}{2}} \\ &\leq \left[\sum_q \left(\sum_{q' \geq q-3} \int_0^T 2^{(q-q')(s+1)} \left(2^{q'(s+2)} \|\Delta_{q'} u\|_{L^2} \right)^{\frac{1}{p'}} \right. \right. \\ &\quad \left. \left. \times \left(2^{q's} \|\Delta_{q'} \tilde{u}\|_{L^2} \right)^{\frac{1}{p}} 2^{q'(1-\frac{2}{p'})} \|\tilde{\Delta}_{q'} \tilde{u}\|_{L^\infty} dt \right)^2 \right]^{\frac{1}{2}}, \end{aligned}$$

whence, in view of Minkowski's and Young's inequalities,

$$\mathcal{R}^4 \lesssim K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\nabla u(t)\|_{B_{\infty,\infty}^{-\frac{2}{p'}}}^p \|\nabla u(t)\|_{H^{s-1}} dt. \quad (\text{A.13})$$

Bounds for \mathcal{R}^5 . According to the first-order Taylor's formula, we have

$$\|[\Delta_{-1} v^j, \Delta_q] \Delta_{q'} \partial_j u\|_{L^2} \lesssim 2^{-q} \|\nabla \Delta_{-1} v\|_{L^\infty} \|\nabla \Delta_{q'} u\|_{L^2}.$$

Now, straightforward computations yield

$$\begin{aligned} \mathcal{R}^5 &\leq \left[\sum_q \left(\sum_{|q'-q| \leq 2} \int_0^T 2^{q(s-1)} \|\Delta_{q'} \nabla u\|_{L^2} \|\Delta_{-1} \nabla v\|_{L^\infty} dt \right)^2 \right]^{\frac{1}{2}} \\ &\leq \left[\sum_q \left(\int_0^T \left(2^{q(s+1)} \|\nabla \Delta_q u\|_{L^2} \right)^{\frac{1}{p'}} \|\nabla \Delta_{-1} v\|_{L^\infty} \left(2^{q(s-1)} \|\Delta_q \nabla u\|_{L^2} \right)^{\frac{1}{p}} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \left(\sum_q \left(\int_0^T \|\Delta_{-1} \nabla v\|_{L^\infty}^p 2^{q(s-1)} \|\Delta_q \nabla u\|_{L^2} dt \right)^2 \right)^{\frac{1}{2}} \\ &\lesssim K \|\nabla u\|_{\tilde{L}_T^1(H^{s+1})} + K^{-\frac{p}{p'}} \int_0^t \|\Delta_{-1} \nabla v(t)\|_{L^\infty}^p \|\nabla u(t)\|_{H^{s-1}} dt. \quad (\text{A.14}) \end{aligned}$$

Bounds for \mathcal{R}^6 . Note that R_q^6 vanishes for $q > 3$. Therefore,

$$\begin{aligned} \mathcal{R}^6 &\lesssim \left[\sum_{q' \leq 1} \left(\int_0^T \|\Delta_{-1} \nabla u\|_{L^2} \|\Delta_{q'} \tilde{v}\|_{L^\infty} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \int_0^T \|\Delta_{-1} \nabla u\|_{L^2} dt + K^{-\frac{p}{p'}} \left[\sum_{q' \leq 1} \left(\int_0^T \|\Delta_{q'} \tilde{v}\|_{L^\infty}^p \|\Delta_{-1} \nabla u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \int_0^T \|\Delta_{-1} \nabla u\|_{L^2} dt + K^{-\frac{p}{p'}} \int_0^t \|\nabla v(t)\|_{B_{\infty,\infty}^{-\frac{2}{p'}}}^p \|\Delta_{-1} \nabla u(t)\|_{L^2} dt. \quad (\text{A.15}) \end{aligned}$$

Combining inequalities (A.4), (A.7), (A.10), (A.12), (A.14), and (A.15) with elementary embeddings yields the desired estimates. If in addition $u = v$, one can use

inequalities (A.4), (A.8), (A.11), (A.13), (A.14), and (A.15). The proof of lemma A.1 is complete. \square

Lemma A.2. *Let $1 < p < +\infty$ and $s > -\frac{N}{2} - 1$. Assume that $\operatorname{div} u = 0$. The following estimate holds true:*

$$\|T'_{\partial_j v^i} u^j\|_{\tilde{L}_T^1(H^s)} \lesssim K \|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\nabla v\|_{B_{2,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|u\|_{H^s} dt.$$

If in addition $s > -1$, one further has

$$\|T'_{\partial_j u^i} u^j\|_{\tilde{L}_T^1(H^s)} \lesssim K \|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\nabla u\|_{B_{\infty,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|u\|_{H^s} dt.$$

Proof. Let $\tilde{v} \stackrel{\text{def}}{=} v - \Delta_{-1}v$. The proof is based on the decomposition

$$T'_{\partial_j v^i} u^j = T_{\partial_j v^i} u^j + \partial_j R(\tilde{v}^i, u^j) + R(\partial_j \Delta_{-1} v^i, u^j). \quad (\text{A.16})$$

From Minkowski's and Young's inequalities, one infers

$$\begin{aligned} \|T_{\partial_j v^i} u^j\|_{\tilde{L}_T^1(H^s)} &\lesssim \left[\sum_q \left(\int_0^T 2^{qs} \sum_{|q'-q|\leq 4} \|S_{q'-1} \nabla v\|_{L^\infty} \|\Delta_{q'} u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \left[\sum_{\substack{q \\ |q'-q|\leq 4}} \left(\int_0^T 2^{-\frac{2q'}{p'}} \|S_{q'-1} \nabla v\|_{L^\infty} 2^{q'(s+\frac{2}{p'})} \|\Delta_{q'} u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \left[\sum_q \left(\int_0^T \|\nabla v\|_{B_{\infty,\infty}^{\frac{N}{2}-\frac{2}{p'}}} \left(2^{qs} \|\Delta_q u\|_{L^2} \right)^{\frac{1}{p}} \left(2^{q(s+2)} \|\Delta_q u\|_{L^2} \right)^{\frac{1}{p'}} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\nabla v\|_{B_{\infty,\infty}^{\frac{N}{2}-\frac{2}{p'}}}^p \|u\|_{H^s} dt. \end{aligned} \quad (\text{A.17})$$

For the second term, Bernstein's, Minkowski's, Young's, and convolution inequalities yield (here comes that $s > -N/2 - 1$)

$$\begin{aligned} \|\partial_j R(\tilde{v}^i, u^j)\|_{\tilde{L}_T^1(H^s)} &\leq \left[\sum_q 2^{2qs} \left(\int \sum_{q'\geq q-2} \|\partial_j \Delta_q(\tilde{\Delta}_{q'} \tilde{v}^i \Delta_{q'} u^j)\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \left[\sum_q \left(\int_0^T \sum_{q'\geq q-2} 2^{q(s+1+\frac{N}{2})} \|\tilde{\Delta}_{q'} \tilde{v}\|_{L^2} \|\Delta_{q'} u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \left[\sum_q \left(\int_0^T \sum_{q'\geq q-2} 2^{(q-q')(s+1+\frac{N}{2})} \left(2^{q'(s+2)} \|\Delta_{q'} u\|_{L^2} \right)^{\frac{1}{p'}} \right. \right. \\ &\quad \left. \left. \times 2^{q'(\frac{N}{2}-\frac{2}{p'}+1)} \|\tilde{\Delta}_{q'} \tilde{v}\|_{L^2} \left(2^{q's} \|\Delta_{q'} u\|_{L^2} \right)^{\frac{1}{p}} dt \right)^2 \right]^{\frac{1}{2}} \end{aligned}$$

$$\lesssim K \|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\tilde{v}\|_{B_{2,\infty}^{1-\frac{2}{p'}+\frac{N}{2}}}^p \|u\|_{H^s} dt. \tag{A.18}$$

For the last term in (A.16), one readily has

$$\begin{aligned} \|R(\partial_j \Delta_{-1} v^i, u^j)\|_{\tilde{L}_T^1(H^s)} &\lesssim \int_0^T \sum_{q \leq 3} \|\Delta_{-1} \nabla v\|_{L^\infty} \|\Delta_{q'} u\|_{L^2} dt \\ &\lesssim K \|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\nabla \Delta_{-1} v\|_{L^\infty}^p \|u\|_{H^s} dt. \end{aligned} \tag{A.19}$$

Plugging (A.17), (A.18), and (A.19) in (A.16) completes the proof in the general case.

If in addition $v = u$ and $s > -1$, one can alternatively make the following computations for bounding the second term:

$$\begin{aligned} \|\partial_j R(\tilde{u}^i, u^j)\|_{\tilde{L}_T^1(H^s)} &\lesssim \left[\sum_q \left(\int_0^T \sum_{q' \geq q-2} 2^{q(s+1)} \|\tilde{\Delta}_{q'} \tilde{u}\|_{L^\infty} \|\Delta_{q'} u\|_{L^2} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim \left[\sum_q \left(\int_0^T \sum_{q' \geq q-2} 2^{(q-q')(s+1)} \left(2^{q'(s+2)} \|\Delta_{q'} u\|_{L^2} \right)^{\frac{1}{p'}} \right. \right. \\ &\quad \left. \left. \times 2^{q'(1-\frac{2}{p'})} \|\tilde{\Delta}_{q'} \tilde{u}\|_{L^\infty} \left(2^{q's} \|\Delta_{q'} u\|_{L^2} \right)^{\frac{1}{p}} dt \right)^2 \right]^{\frac{1}{2}} \\ &\lesssim K \|u\|_{\tilde{L}_T^1(H^{s+2})} + K^{-\frac{p}{p'}} \int_0^T \|\tilde{u}\|_{B_{\infty,\infty}^{1-\frac{2}{p'}}}^p \|u\|_{H^s} dt, \end{aligned}$$

which completes the proof in this particular case.

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