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LOCAL ESTIMATES AND LARGE SOLUTIONS FOR SOME ELLIPTIC EQUATIONS WITH ABSORPTION

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Abstract. We prove local estimates for solutions of equation (1.1), which are applied to study existence and uniqueness of so-called large solutions, i.e., solutions which blow up uniformly at the boundary, as well as to the study of solutions in the whole space.

1. Introduction

In this paper we consider the nonlinear elliptic equation

$$-\Delta u + h(u) + g(u)|\nabla u|^2 = f\,, \quad \text{in } \Omega \subseteq \mathbf{R}^N, \tag{1.1}$$

where h and g are continuous functions, and we investigate under which assumptions it is possible to have local estimates for solutions (or subsolutions) of (1.1). Here by local estimates we mean that an interior local universal bound exists for solutions u of (1.1), so that u can be estimated on compact subsets inside Ω independently from its behavior at the boundary (or equivalently, without information at infinity if Ω should be unbounded). The importance of such local estimates is related to the possibility to construct solutions in a bounded domain with uniform blow-up at the boundary (so-called *large solutions*, or *explosive solutions*), as well as to solve equation (1.1) in unbounded domains without prescribing any behavior on the datum f at infinity.

In case $g \equiv 0$, this kind of question has been deeply investigated since the works [16] and [23], where J.B. Keller and R. Osserman proved that local estimates hold for positive solutions of the semilinear equation

$$-\Delta u + h(u) = f \tag{1.2}$$

if and only if $\int_a^{+\infty} (h(s)s)^{-\frac{1}{2}} ds < \infty$ (the so-called Keller-Osserman condition). Several other related features of equation (1.2) have been studied later, such as questions concerning large solutions (see e.g. [2, 3, 19, 20, 21, 25])

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or existence and uniqueness results for entire solutions ([6], [13]). Some generalizations to p-Laplace-type operators have also been proved ([4], [12], [18]).

The possibility of having local estimates in the presence of gradient-dependent, lower-order terms has also been the object of research; besides some results which are obtained still using the Keller-Osserman condition (see [11] and [14]), we mention the fundamental results obtained in [17] on large solutions of the model problem (connected to stochastic control problems)

$$-\Delta u + u + |\nabla u|^q = f\,, \qquad q > 1\,.$$

Existence and asymptotic behavior of large solutions of the equation

$$-\Delta u + h(u) = \pm|\nabla u|^q\,, \qquad q > 1\,,$$

were also proved in [1] mainly for the case $q \neq 2$.

We focus our attention here on the particular case of natural growth terms ($q = 2$) motivated by recent results on removable singularities and other local absorption phenomena for this type of equation (see [8] and [22]). This is in fact a special case among other possible growths of the first-order term, since there is always a semilinear (variational) structure behind equation (1.1) given through the change of unknown $w = \int_0^u \exp(-G(s))ds$, $G(s) = \int_0^s g(t)dt$. This allows a detailed study of the influence of the gradient-dependent term with respect to the zero-order term. In particular, two different cases appear, whether

$$\int_0^{+\infty} \exp(-G(s))\,ds < +\infty \tag{1.3}$$

or

$$\int_0^{+\infty} \exp(-G(s))\,ds = +\infty\,. \tag{1.4}$$

The case in which (1.4) holds true corresponds to a weak absorption of the gradient term and can be easily reduced to the known results proved for (1.2). In particular, under assumption (1.4), local estimates hold for positive subsolutions of (1.1) if and only if

$$\int_a^{+\infty} \Big(h(s)\,\exp(G(s))\,\int_0^s \exp(-G(t))dt\Big)^{-\frac{1}{2}} ds < +\infty \qquad \forall a > 0\,.$$

As a matter of fact, just some microscopic improvement with respect to the classical Keller-Osserman condition can be observed, and the results are directly deduced from the known theory after the change of unknown mentioned before.

Assumption (1.3) corresponds instead to a stronger influence of the first-order term, where main novelties appear with respect to (1.2); it is interesting to note that (1.3) is also the optimal assumption for having other local effects such as removable singularity properties (see [8] and [22]). Under (1.3), one new feature is that the study of (1.1) is related to properties of bounded positive solutions v of

$$-\Delta v = \rho(v), \tag{1.5}$$

where $\rho(s)$ is a positive function. The existence of universal interior upper bounds for (1.1) is then equivalent to existence of positive subsolutions for (1.5), therefore connected to $\lambda_{1,\Omega}$, the first eigenvalue of the Laplacian in Ω. If $f = 0$ and (1.3) holds, we prove then local estimates for positive subsolutions of (1.1) provided

$$\lambda_{1,\Omega} < \liminf_{s\to+\infty} \frac{h(s)\exp(-G(s))}{\int_s^{+\infty}\exp(-G(t))dt}\,.$$

A first interesting consequence is that local estimates may depend on the domain Ω, whereas for the semilinear equation (1.2) the Keller-Osserman condition is independent of the domain. In particular, estimates will hold under weaker assumptions if $\Omega = \mathbf{R}^N$, giving rise to weak maximum principles without a condition at infinity or to some Liouville-type results in $\mathbf{R}^N$ (see Section 2.3).

Our main application of these local estimates concerns existence and uniqueness of so-called large solutions, i.e., solutions of

$$\begin{cases} -\Delta u + h(u) + g(u)|\nabla u|^2 = f & \text{in } \Omega,\\ \displaystyle\lim_{\mathrm{dist}(x,\partial\Omega)\to 0} u(x) = +\infty\,. \end{cases} \tag{1.6}$$

Note that due to the relationship between (1.1) and (1.5), if $\exp(-G(s)) \in L^1(\mathbf{R}^+)$, the asymptotic behavior near the boundary of such explosive solutions can also be estimated using results on the corresponding semilinear equations which are possibly singular at zero (e.g., see [10]). Alternatively, one obtains sharp estimates by directly looking at the corresponding ODE's solutions. It turns out that a strongly absorbing gradient nonlinearity easily provides uniqueness of solutions; this is the content of our main result.

Theorem 1.1. *Let Ω be a bounded, smooth domain in $\mathbf{R}^N$, and let $f \in L^\infty(\Omega)$. Assume that h is increasing, that g is nondecreasing and such that*

$$\lim_{s\to+\infty} h(s) = +\infty\,, \qquad \lim_{s\to+\infty} g(s) = +\infty\,,$$

and that there exists $L > 0$ such that $h(s)s \geq 0$ and $g(s)s \geq 0$ for any s with $|s| > L$. Then there exists a unique solution u of (1.6). *Moreover, $u \in C^1(\Omega)$*

and satisfies

$$u(x) = F^{-1}(d_\Omega(x)) + o(1) \quad \text{as} \ \ d_\Omega(x) \to 0, \ \ \text{with}$$

$$F(t) = \int_t^{+\infty} \frac{e^{-G(s)}}{\left(\int_L^s h(\xi)e^{-2G(\xi)}d\xi\right)^{\frac{1}{2}}}ds,$$

where $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$.

Note that in particular this result applies to give existence and uniqueness of large solutions for the model example

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u + |u|^{\beta-1}u|\nabla u|^2 = f & \text{in } \Omega, \\ \lim\limits_{\operatorname{dist}(x,\partial\Omega)\to 0} u(x) = +\infty, \end{cases}$$

for any $\beta > 0$ and any $\alpha > 0$. With respect to the semilinear case (1.2), it should be noted that the presence of the gradient term not only allows existence of large solutions even for $\alpha \in (0, 1]$, but also notably simplifies the uniqueness result, for any value of $\alpha > 0$.

Our paper is planned as follows: the study of the strong absorption case in its different aspects, which is the core of this work, is contained in the next section. The third section deals with the weak absorption case; since this case can be reduced to the known semilinear problem we prove only the basic local estimates and make some remarks, including the possibility of having $g(s)s \leq 0$ (repulsive gradient terms). Finally, the last section suggests some extensions to more general nonlinearities as lower-order terms. We also mention that an extension of some of these results to p-Laplace-type operators is also possible; details will appear elsewhere. Moreover, in [24] some related results on uniqueness of solutions on $\mathbf{R}^N$ without a condition at infinity have also been proved.

2. The strong-absorption case

We assume, in the following, that $h(s)$ and $g(s)$ are continuous functions, and we set

$$G(s) = \int_0^s g(t)dt\,.$$

We deal here with the case that $\exp(-G(s)) \in L^1(\mathbf{R})$, which corresponds to a strong absorption of the gradient term in (1.1). Although not always necessary, we take Ω to be a C^2 subset in $\mathbf{R}^N$.

2.1. **Local estimates.** We denote by $\lambda_{1,\Omega}$ the first eigenvalue of the Laplacian (with Dirichlet boundary conditions) and by $\varphi_{1,\Omega}$ the corresponding positive first eigenfunction normalized so that $\max_{\overline{\Omega}} \varphi_{1,\Omega} = 1$; if $\Omega = B(0,1)$ is the unit ball, we will simply write Λ_1 and Φ_1 respectively. We also denote $d_\Omega(x) = \operatorname{dist}(x, \partial\Omega)$, and $\mathbf{R}^+ = (0, +\infty)$. Finally, it will be useful to introduce the following functions:

$$\theta(s) := \frac{h(s)\exp(-G(s))}{\int_s^{+\infty} \exp(-G(t))dt}\,, \qquad \psi(s) = \int_s^{+\infty} \exp(-G(t))dt\,. \tag{2.1}$$

Theorem 2.1. *Let $f \in L^\infty(\Omega)$. Assume*

$$\int_0^{+\infty} \exp(-G(s))ds < +\infty\,, \tag{2.2}$$

and that

$$\lambda_{1,\Omega} < \liminf_{s\to+\infty} \frac{(h(s) - \|f\|_{L^\infty(\Omega)})\exp(-G(s))}{\int_s^{+\infty} \exp(-G(t))dt}\,. \tag{2.3}$$

Then there exists a positive, decreasing function $F(s)$, with $\lim_{s\to 0^+} F(s) = +\infty$, such that any solution $u \in H^1(\Omega) \cap L^\infty(\Omega)$ of

$$\begin{cases} -\Delta u + h(u) + g(u)|\nabla u|^2 \le f & \text{in } \mathcal{D}'(\Omega), \\ u \ge 0 & \text{in } \Omega, \end{cases} \tag{2.4}$$

satisfies

$$u(x) \le F(d_\Omega(x)) \quad \text{for almost every } x \in \Omega.$$

In particular, if $f = 0$ and $\theta(s)$ is increasing, we have (almost everywhere in Ω)

$$u(x) \le \min\Big\{\theta^{-1}\Big(\frac{\Lambda_1}{d_\Omega(x)^2}\Big)\,,\, \psi^{-1}\Big(\varphi_{1,\Omega}(x)\,\psi(\theta^{-1}(\lambda_{1,\Omega}))\Big)\Big\}. \tag{2.5}$$

Proof. Let us set $v = \psi(u)$. Then v is a positive function which belongs to $H^1(\Omega) \cap L^\infty(\Omega)$ and satisfies, in a weak sense,

$$-\Delta v \ge (h(u) - f)\exp(-G(u)) \ge \eta(u)\psi(u)\,, \tag{2.6}$$

where

$$\eta(s) = \inf_{r\ge s} \frac{(h(r) - \|f\|_{L^\infty(\Omega)})\exp(-G(r))}{\int_r^{+\infty} \exp(-G(t))dt}\,.$$

Note that η is a continuous, nondecreasing function on $\mathbf{R}^+$, such that

$$\lim_{s\to+\infty} \eta(s) > \lambda_{1,\Omega}$$

by (2.3). Moreover, if η is not increasing (i.e., it has some flat zones), we can construct an increasing function $\tilde\eta(s)$ such that $\eta(s) \ge \tilde\eta(s)$ for any $s \in \mathbf{R}^+$

and $\lim_{s\to+\infty}\tilde\eta(s) = \lim_{s\to+\infty}\eta(s)$. Denoting $\rho(s) = s\tilde\eta(\psi^{-1}(s))$, we have that $\frac{\rho(s)}{s}$ is a decreasing function such that $\lim_{s\to 0^+}\frac{\rho(s)}{s} > \lambda_{1,\Omega}$, and from (2.6) v satisfies

$$-\Delta v \geq \rho(v) \qquad \text{in } \Omega. \tag{2.7}$$

Moreover, there exists T such that $\rho(T) \geq T\lambda_{1,\Omega}$. Setting $\varphi = T\,\varphi_{1,\Omega}$, we obtain, since $\varphi \leq T$, that

$$-\Delta\varphi = \lambda_1\, T\,\varphi_{1,\Omega} \leq \rho(T)\,\varphi_{1,\Omega} \leq \rho(\varphi) \qquad \text{in } \Omega.$$

Therefore, v and φ are positive supersolution and subsolution, respectively, of the problem

$$\begin{cases} -\Delta z = \rho(z) & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Due to the fact that $\frac{\rho(s)}{s}$ is decreasing, by a well-known comparison result (see for instance [9] or [7]) we deduce that $v(x) \geq T\varphi_{1,\Omega}(x)$ for almost every $x \in \Omega$. Since Ω is smooth, we conclude that there exists a positive constant C such that $v(x) \geq C\,d_\Omega(x)$. Recalling that $v = \psi(u)$, we can set $F(s) = \psi^{-1}(C\,s)$, and we obtain $u(x) \leq F(d_\Omega(x))$ for almost every $x \in \Omega$.

In particular, if $f = 0$ and $\theta(s)$ (defined in (2.1)) is increasing, we have $\frac{\rho(s)}{s} = \theta(\psi^{-1}(s))$ and $T = \psi(\theta^{-1}(\lambda_{1,\Omega}))$. The estimate found so far then becomes

$$u(x) \leq \psi^{-1}(\varphi_{1,\Omega}(x)\,\psi(\theta^{-1}(\lambda_{1,\Omega})))\,. \tag{2.9}$$

A possibly different estimate can be found as follows: let $x_0 \in \Omega$ be fixed, and consider the ball $B_{R_0}(x_0)$ centered at x_0 of radius $R_0 = d_\Omega(x_0)$. Applying the previous estimate to the domain $B_{R_0}(x_0)$, and using that (by radial symmetry) the first eigenfunction and eigenvector satisfy, respectively, $\varphi_1(x_0) = 1$ and $\lambda_1 = \frac{\Lambda_1}{R_0^2}$, we get $u(x_0) \leq \theta^{-1}(\frac{\Lambda_1}{R_0^2})$, so that we obtain

$$u(x) \leq \theta^{-1}(\frac{\Lambda_1}{d^2_\Omega(x)})\,, \tag{2.10}$$

for almost every $x \in \Omega$. Combining (2.9) and (2.10) yields (2.5). □

Remark 2.2. The two estimates contained in (2.5) correspond to a stronger or weaker influence of the gradient term with respect to the zero-order term. It is worth making precise that (2.10) has to be understood as significant provided $\frac{\Lambda_1}{d_\Omega(x)^2} < \sup\theta$; otherwise, the right-hand side should be meant as $+\infty$; in particular, it applies in case θ is unbounded. Note also that (2.9) depends on the regularity of the domain through its first eigenfunction; it is not essential to ask *a priori* that Ω is smooth. To compare with the results in

[1] and [17], in the model case $g(s) = 1$, we have $\psi(s) = e^{-s}$ and $\theta(s) = h(s)$, so that

$$u(x) \leq \min\left\{h^{-1}\Big(\frac{\Lambda_1}{d_\Omega(x)^2}\Big)\,,\,\Big|\log\Big(\varphi_{1,\Omega}(x)\,\exp(-h^{-1}(\lambda_{1,\Omega}))\Big)\Big|\right\},$$

and we deduce in particular (for smooth domains) that

$$\text{if } h(t) = o(e^{2t}) \text{ as } t \to +\infty, \text{ then } u(x) \leq \log(\tfrac{1}{d_\Omega(x)}) + c,$$
$$\text{if } e^{2t} = o(h(t)) \text{ as } t \to +\infty, \text{ then } u(x) \leq h^{-1}(\tfrac{\Lambda_1}{d_\Omega(x)^2}).$$

Remark 2.3. In the previous result both h and g do not need to be everywhere positive. Surely, assumption (2.3) implies that $h(s)$ has to be positive for large s, while, as far as g is concerned, splitting it as $g(s) = g^+(s) - g^-(s)$, (2.2) implies that $\exp(-G^+(s)) \in L^1(\mathbf{R}^+)$, where $G^+(s) = \int_0^s g^+(t)dt$. A possible choice of changing sign $g(s)$ is allowed if, for example, $g^-(s) \in L^1(\mathbf{R}^+)$ and $\exp(-G^+(s)) \in L^1(\mathbf{R}^+)$.

Finally, let us mention that, in the previous result, one can also allow $h(s)$ to be bounded if $\|f\|_{L^\infty(\Omega)}$ is small enough; however, this amounts roughly speaking to taking g unbounded.

The following easy example shows the optimality of the previous result.

Example 2.4. Let u be a positive solution of

$$-\Delta u + h(u) + |\nabla u|^2 = 0\,, \tag{2.11}$$

where h is increasing. Since $v = \exp(-u)$ solves

$$-\Delta v = h(\log(\frac{1}{v}))v \quad \text{in } \Omega, \tag{2.12}$$

if $\lim_{s\to+\infty} h(s) = \lambda \leq \lambda_{1,\Omega}$ and if $v = 0$ on $\partial\Omega$ it follows that $v \equiv 0$ in the whole of Ω. This amounts to saying that u has no bound from above inside Ω.

In the general case, if u is a solution of (1.1) with $f = 0$, then the function

$$z = \log(\int_0^{+\infty} \exp(-G(s))ds) - \log(\int_u^{+\infty} \exp(-G(s))ds)$$

solves

$$-\Delta z + \tilde{h}(z) + |Dz|^2 = 0\,,$$

where $\tilde{h}(z) = \frac{h(u)\exp(-G(u))}{\int_u^{+\infty}\exp(-G(t))dt}$. A local bound for u exists if and only if a local bound for z exists. Since assumption (2.3) is equivalent to $\liminf_{s\to+\infty} \tilde{h}(s) > \lambda_{1,\Omega}$, we have just shown the optimality of this condition for (2.11). Similarly, in order to have local estimates for solutions of (1.1) with **any** $L^\infty(\Omega)$ right-hand side, one needs to assume that h goes to infinity as s tends to infinity. Note that in general if $h \equiv 0$ such local estimates cannot hold unless one has some further information at least on a subset of the boundary.

We easily deduce from the previous result an estimate for solutions of

$$-\Delta u + h(u) + g(u)|\nabla u|^2 = f \qquad \text{in } \Omega. \tag{2.13}$$

Corollary 2.5. *Let $\Omega \subseteq \mathbf{R}^N$, and let $f \in L^\infty(\Omega)$. Assume that* $\exp(-G(s)) \in L^1(\mathbf{R})$ *and that*

$$\lim_{s\to\pm\infty} h(s) = \pm\infty \tag{2.14}$$

$$\liminf_{s\to\pm\infty} \frac{h(s)\exp(-G(s))}{\int_s^{\pm\infty}\exp(-G(t))dt} > \lambda_{1,\Omega}\,. \tag{2.15}$$

Then for any compact subset $K \subset \Omega$ there exists a constant C_K, depending on K and $\|f\|_{L^\infty(\Omega)}$, such that any solution $u \in H^1(\Omega)\cap L^\infty(\Omega)$ of (2.13) *satisfies*

$$\|u\|_{L^\infty(K)} \le C_K\,.$$

Proof. Let u be a solution of (2.13). By Kato's inequality, it follows that $u^+ \in H^1(\Omega)\cap L^\infty(\Omega)$ and satisfies (in a weak sense)

$$-\Delta u^+ + \chi_{u>0}(h(u) + g(u)|\nabla u|^2) \le \|f\|_{L^\infty(\Omega)}\,.$$

Without loss of generality (up to adding a constant to f), we can assume that $h(0) = 0$. Then u^+ satisfies

$$-\Delta u^+ + h(u^+) + g(u^+)|\nabla u^+|^2 \le \|f\|_{L^\infty(\Omega)}\,.$$

Thanks to (2.14) we have

$$\liminf_{s\to+\infty} \frac{(h(s) - \|f\|_{L^\infty(\Omega)})\exp(-G(s))}{\int_s^{+\infty}\exp(-G(t))dt} = \liminf_{s\to+\infty} \frac{h(s)\exp(-G(s))}{\int_s^{+\infty}\exp(-G(t))dt}\,,$$

and then (2.15) allows us to apply Theorem 2.1 to u^+ (if Ω should be unbounded, it is enough to take a smooth, bounded subdomain including K). Similarly, u^- satisfies

$$-\Delta u^- + \tilde h(u^-) + \tilde g(u^-)|\nabla u^-|^2 \le \|f\|_{L^\infty(\Omega)}\,,$$

where $\tilde h(s) = -h(-s)$ and $\tilde g(s) = -g(-s)$; one can easily check that (2.14) and (2.15) (as far as conditions at $-\infty$ are concerned) imply that $\tilde h$ and $\tilde g$ satisfy the hypotheses of Theorem 2.1, which then also apply to u^-. We conclude that a universal upper bound (depending only on K and $\|f\|_{L^\infty(\Omega)}$) holds in K for all solutions of (2.13). □

2.2. **Large solutions.** We deal here with solutions of

$$\begin{cases} -\Delta u + h(u) + g(u)|\nabla u|^2 = f & \text{in } \Omega, \\ \lim\limits_{d_\Omega(x)\to 0} u(x) = +\infty\,. \end{cases} \tag{2.16}$$

By a solution of (2.16) we mean a function $u \in H^1_{loc}(\Omega)\cap C(\Omega)$ satisfying the equation inside the domain in a weak sense (i.e., with compactly supported test functions) and the uniform blow-up condition at the boundary.

We assume as before that

$$\exp(-G(s)) \in L^1(\mathbf{R}^+)\,, \tag{2.17}$$

and we use the same notation as in (2.1). Our main result is the following (recall that we are asking Ω to be smooth).

Theorem 2.6. *Let $f \in L^\infty(\Omega)$. Assume that h is increasing, that g is nondecreasing and such that*

$$\lim_{s\to+\infty} h(s) = +\infty\,, \qquad \lim_{s\to+\infty} g(s) = +\infty\,, \tag{2.18}$$

and that there exists $L > 0$ such that $h(s)s \geq 0$ and $g(s)s \geq 0$ for any s with $|s| > L$. Then there exists a unique solution u of (2.16). *Moreover, $u \in C^1(\Omega)$ and satisfies*

$$u(x) = F^{-1}(d_\Omega(x)) + o(1) \quad \text{as } \; d_\Omega(x) \to 0, \text{ where} \tag{2.19}$$

$$F(t) = \int_t^{+\infty} \frac{e^{-G(s)}}{\left(\int_L^s h(\xi)e^{-2G(\xi)}d\xi\right)^{\frac{1}{2}}}ds.$$

Proof. *Existence:* For any $n \in \mathbf{N}$, consider the sequence of smooth solutions u_n of

$$\begin{cases} -\Delta u_n + h(u_n) + g(u_n)|\nabla u_n|^2 = f & \text{in } \Omega, \\ u_n = n & \text{on } \partial\Omega. \end{cases} \tag{2.20}$$

Thanks to the monotonicity of h and g, we can apply the classical maximum principle to deduce that $\{u_n\}$ is increasing. Since both $h(s)$ and $g(s)$ are negative for $s < -L$, standard arguments (multiplying the equation by $(u_n^- - k)^+$) imply that u_n^- is bounded in $L^\infty(\Omega)$. Thanks to assumption (2.18), we also have

$$\liminf_{s\to+\infty} \frac{(h(s) - \|f\|_{L^\infty(\Omega)})\exp(-G(s))}{\int_s^{+\infty} \exp -G(t)dt} \geq \lim_{s\to+\infty} \frac{\exp(-G(s))}{\int_s^{+\infty} \exp -G(t)dt} = +\infty\,. \tag{2.21}$$

We can therefore apply Theorem 2.1 to u_n^+ and obtain that the sequence u_n^+ is bounded on every compact subset of Ω. Thus u_n is bounded in $L^\infty_{loc}(\Omega)$ and, by monotonicity, converges almost everywhere to a function $u \in L^\infty_{loc}(\Omega)$. Actually, further regularity results easily imply that $u \in C^1(\Omega)$ and solves the equation inside Ω. Since $u(x) \geq u_n(x)$ for every $x \in \Omega$ and any $n \in \mathbf{N}$, clearly u satisfies the uniform boundary condition in (2.16).

Uniqueness: We prove that *any* large solution u must satisfy (2.19), which implies that $u_1 - u_2$ tends to zero as $d_\Omega(x) \to 0$ for any couple of large solutions u_1, u_2; then applying the standard maximum principle we deduce the uniqueness result.

Let then u be any large solution; there exists $\delta_0 > 0$ such that the distance function $d_\Omega(x)$ is C^2 in $\Omega_0 = \{x : d_\Omega(x) < \delta_0\}$, and $|f(x)| \leq \frac{1}{2}h(u)$ in Ω_0. Thus u satisfies

$$\begin{aligned} &-\Delta u + \frac{1}{2}h(u) + g(u)|\nabla u|^2 \leq 0 \qquad \text{and} \\ &-\Delta u + \frac{3}{2}h(u) + g(u)|\nabla u|^2 \geq 0 \qquad \text{in } \Omega_0. \end{aligned}$$

Let $L_0 = \sup_{\Omega_0} |\Delta d_\Omega|$, and let ψ_ε be the solution of the ODE

$$\psi_\varepsilon'' = \frac{1}{2}h(\psi_\varepsilon) + g(\psi_\varepsilon)|\psi_\varepsilon'|^2 - L_0|\psi_\varepsilon|', \quad \psi_\varepsilon(0) = \sup_{\{d_\Omega(x)=\delta_0\}} u, \quad \psi_\varepsilon(\delta_0 - \varepsilon) = +\infty.$$

Then the function $w_\varepsilon(x) = \psi_\varepsilon(\delta_0 - d_\Omega(x))$ satisfies

$$\begin{aligned} -\Delta w_\varepsilon + \frac{1}{2}h(w_\varepsilon) + g(w_\varepsilon)|\nabla w_\varepsilon|^2 &= -\psi_\varepsilon'' + \frac{1}{2}h(\psi_\varepsilon) + g(\psi_\varepsilon)|\psi_\varepsilon'|^2 + \psi_\varepsilon' \Delta d_\Omega \\ &\geq -\psi_\varepsilon'' + \frac{1}{2}h(\psi_\varepsilon) + g(\psi_\varepsilon)|\psi_\varepsilon'|^2 - L_0|\psi_\varepsilon|' \end{aligned}$$

so that

$$-\Delta w_\varepsilon + \frac{1}{2}h(w_\varepsilon) + g(w_\varepsilon)|\nabla w_\varepsilon|^2 \geq 0 \qquad \text{in } \{x : \varepsilon < d_\Omega(x) < \delta_0\}.$$

Comparing u and w_ε, which are sub- and supersolutions in the set $\{x : \varepsilon < d_\Omega(x) < \delta_0\}$ and are ordered on the boundary, we conclude that $u \leq w_\varepsilon$. Letting ε tend to zero, clearly w_ε converges to $w = \psi(\delta_0 - d_\Omega(x))$, where ψ solves

$$\psi'' = \frac{1}{2}h(\psi) + g(\psi)|\psi'|^2 - L_0|\psi'|, \quad \psi(0) = \sup_{\{d_\Omega(x)=\delta_0\}} u\,, \quad \psi(\delta_0) = +\infty\,. \tag{2.22}$$

Therefore, we have $u \leq w$ in Ω_0. An estimate on w follows by studying the solution of (2.22); first note that there exists $\sigma_0 > 0$ such that ψ is increasing in (σ_0, δ_0), so that using also that $h(\psi) > 0$, we have

$$\psi'' \geq g(\psi)|\psi'|^2 - L_0\psi' \longrightarrow (e^{L_0 s}\psi')' \geq e^{L_0 s}\, g(\psi)|\psi'|^2,$$

and then, for every s in (σ_0, δ_0),

$$e^{L_0 s}\psi' \geq \psi'(\sigma_0)e^{-G(\psi(\sigma_0))}\, e^{G(\psi)} \longrightarrow L_0\psi' \leq L_0 \frac{e^{L_0\delta_0 + G(\psi(\sigma_0))}}{\psi'(\sigma_0)} |\psi'|^2 e^{-G(\psi)}.$$

There exists then a constant C_0 (depending on δ_0 and $\sup_{\{d_\Omega(x)=\delta_0\}} u$) such that ψ satisfies

$$\psi'' \geq \frac{1}{2}h(\psi) + |\psi'|^2(g(\psi) - C_0 e^{-G(\psi)})\,.$$

Integrating and setting $M = \int_0^\infty e^{-G(s)}ds$, we get

$$|\psi'|^2 e^{-2G(\psi)}\, e^{2C_0 M} \geq \int_L^\psi h(s)e^{-2G(s)}\, e^{2C_0 \int_0^s e^{-G}d\xi} ds \geq \int_L^\psi h(s)e^{-2G(s)}ds,$$

and then

$$\frac{\psi' e^{-G(\psi)}}{\left(\int_L^\psi h(s)e^{-2G(s)}ds\right)^{\frac{1}{2}}} \geq e^{-C_0 M}\,.$$

Thus we have, for every s in (σ_0, δ_0),

$$\psi(s) \leq F^{-1}(e^{-C_0 M}(\delta_0 - s))\,, \qquad F(t) = \int_t^\infty \frac{e^{-G(s)}}{\left(\int_L^s h(\xi)e^{-2G(\xi)}d\xi\right)^{\frac{1}{2}}}\, ds\,.$$

Recalling that $u(x) \leq \psi(\delta_0 - d_\Omega(x))$ we obtain

$$u(x) \leq F^{-1}(e^{-C_0 M}(d_\Omega(x))\,, \qquad F(t) = \int_t^\infty \frac{e^{-G(s)}}{\left(\int_L^s h(\xi)e^{-2G(\xi)}d\xi\right)^{\frac{1}{2}}}\, ds\,,$$

where C_0 is determined through the unique solution ψ of (2.22), and depends on δ_0 and $\sup_{\{d_\Omega(x)=\delta_0\}} u$.

Similarly, we have that $u(x) \geq \varphi(\delta_0 - d_\Omega(x))$, where $\varphi(s)$ solves

$$\varphi'' = \frac{3}{2}h(\varphi) + g(\varphi)|\varphi'|^2 + L_0|\varphi'|, \qquad \varphi(0) = 0\,, \quad \varphi(\delta_0) = +\infty\,. \tag{2.23}$$

Indeed, the comparison with φ can be obtained considering first the solutions φ_n of

$$\varphi_n'' = \frac{3}{2}h(\varphi_n) + g(\varphi_n)|\varphi_n'|^2 + L_0|\varphi_n'|, \qquad \varphi_n(0) = 0\,, \quad \varphi_n(\delta_0) = n\,,$$

and then letting n tend to infinity. As before we get the estimate for φ (which can be supposed to be increasing):

$$\varphi' \geq \varphi'(0)\, e^{G(\varphi)}\,,$$

which yields

$$L_0\varphi' \leq L_0 \frac{1}{\varphi'(0)} |\varphi'|^2 e^{-G(\varphi)}.$$

Therefore, φ satisfies, for a constant $C_1 = \frac{L_0}{\varphi'(0)}$,

$$\varphi'' \leq \frac{3}{2} h(\varphi) + |\varphi'|^2 (g(\varphi) + C_1 e^{-G(\varphi)})$$

so that, integrating and setting $M = \int_0^\infty e^{-G(s)} ds$, we get

$$\begin{aligned} |\varphi'|^2 e^{-2G(\varphi)}\, e^{-2C_1 M} &\leq \varphi'(0)^2 + 3 \int_0^\varphi h(s) e^{-2G(s)}\, e^{-2C_1 \int_0^s e^{-G} d\xi} ds \\ &\leq \varphi'(0)^2 + 3 \int_0^\varphi h(s) e^{-2G(s)} ds. \end{aligned}$$

Since there exists a constant D_1 such that

$$\varphi'(0)^2 + 3 \int_0^\varphi h(s) e^{-2G(s)} ds \leq D_1 \int_L^{\varphi(t)} h(s) e^{-2G(s)} ds$$

for t in $(\frac{\delta_0}{2}, \delta_0)$ we conclude that for a positive constant Λ one has

$$\varphi(s) \geq F^{-1}(\Lambda(\delta_0 - s))\,, \qquad F(t) = \int_t^\infty \frac{e^{-G(s)}}{\left(\int_L^s h(\xi) e^{-2G(\xi)} d\xi\right)^{\frac{1}{2}}}\, ds\,,$$

and then, for $d_\Omega(x) < \frac{\delta_0}{2}$,

$$u(x) \geq F^{-1}(\Lambda(d_\Omega(x))\,, \qquad F(t) = \int_t^\infty \frac{e^{-G(s)}}{\left(\int_L^s h(\xi) e^{-2G(\xi)} d\xi\right)^{\frac{1}{2}}}\, ds\,,$$

where Λ is determined by δ_0 through the solution φ of (2.23). In conclusion, for any large solution u there exist a neighborhood of the boundary $\{x : d_\Omega(x) < \delta_1\}$ and two constants $0 < \lambda < \Lambda$ such that

$$F^{-1}(\Lambda d_\Omega(x)) \leq u(x) \leq F^{-1}(\lambda d_\Omega(x)) \qquad \forall x : \ d_\Omega(x) < \delta_1\,. \tag{2.24}$$

On the other hand we have, for any $\lambda < 1$,

$$0 \leq F^{-1}(\lambda s) - F^{-1}(s) \leq (1 - \lambda) \frac{s}{|F'(F^{-1}(\lambda s))|}.$$

By l'Hôpital's rule, using that $\lim_{s\to+\infty} g(s) = +\infty$, we obtain

$$\lim_{s\to+\infty} \frac{F(s)}{|F'(s)|} = 0;$$

hence,

$$(1-\lambda)\frac{s}{|F'(F^{-1}(\lambda s))|} = \frac{1-\lambda}{\lambda}\,\frac{F(F^{-1}(\lambda s))}{|F'(F^{-1}(\lambda s))|} \quad \overset{s\to 0}{\longrightarrow} 0\,.$$

We deduce that for any constant $\mu > 0$

$$|F^{-1}(\mu s) - F^{-1}(s)| \quad \overset{s\to 0}{\longrightarrow} 0$$

so that from (2.24) we obtain (2.19) and then uniqueness. □

Remark 2.7. The assumption that $\lim_{s\to+\infty} g(s) = +\infty$ can be relaxed in some cases; assume for example that $f \geq 0$; then there is uniqueness of positive large solutions if

$$g \text{ is nondecreasing, and } h(\lambda s) \geq \lambda h(s) \text{ for any } \lambda \geq 1, \text{ any } s \in \mathbf{R}^+.$$

Indeed, the same proof as above shows that if g is nondecreasing and bounded then $u_1 - u_2$ is bounded near the boundary, so that $\frac{u_1(x)}{u_2(x)} \to 1$ as $x \to \partial\Omega$. In particular, for any $\lambda > 1$, we have

$$\begin{aligned} &-\Delta(\lambda u_1) + h(\lambda u_1) + g(\lambda u_1)|\nabla(\lambda u_1)|^2 \\ &\qquad \geq -\Delta(\lambda u_1) + \lambda h(u_1) + \lambda g(u_1)|\nabla u_1|^2 = \lambda f \geq f\,, \end{aligned}$$

so that λu_1 is a supersolution and $\lambda\, u_1 > u_2$ at $\partial\Omega$ (in fact $(\lambda u_1 - u_2)^- \in H^1_0(\Omega)$). Thus one concludes that $\lambda u_1 \geq u_2$ for any $\lambda > 1$; then, letting $\lambda \to 1$ and changing the roles of u_1 and u_2, it follows that $u_1 = u_2$. This is actually the classical uniqueness argument which has also been used in many other papers for the semilinear case as well as in [17] with gradient terms, and which allows us to handle the case of linear or superlinear powers $h(s)$ for instance. With respect to these results, Theorem 2.6 points out that as soon as g is unbounded, uniqueness can be easily proved in much more generality; for example, we proved uniqueness for the problem

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u + |u|^{\beta-1}u|\nabla u|^2 = f & \text{in } \Omega, \\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u(x) = +\infty\,, \end{cases}$$

for any values of α, $\beta > 0$.

Remark 2.8. Besides uniqueness, Theorem 2.6 provides the asymptotic behavior of the solution at the boundary, through estimate (2.19). In the particular case $g(s) = 1$, similarly we get

$$\frac{u(x)}{F^{-1}(d_\Omega(x))} \to 1 \quad \text{as } d_\Omega(x) \to 0, \text{ where } F(t) = \int_t^{+\infty} \frac{e^{-G(s)}}{\left(\int_L^s h(\xi)e^{-2G(\xi)}d\xi\right)^{\frac{1}{2}}}ds.$$

If $h(s)\exp(-2s) \in L^1(\mathbf{R}^+)$ we recover (see also [17] and [1]) the estimate

$$\lim_{d_\Omega(x)\to 0} \frac{u(x)}{|\log(d_\Omega(x))|} = 1\,.$$

Note that this condition on h is sharp for having precisely this asymptotic at the boundary.

The assumptions on h and g can be slightly relaxed in the special case that $f = 0$. For instance, existence can be proved even if h is bounded, and uniqueness may hold even if g is decreasing.

Theorem 2.9. *Assume that there exists $L > 0$ such that $h(s)s \geq 0$ for any s with $|s| > L$ and that the function $\theta(s)$ (defined in (2.1)) is increasing. If*

$$\lambda_{1,\Omega} < \lim_{s\to+\infty} \theta(s), \tag{2.25}$$

then there exists a solution u of

$$\begin{cases} -\Delta u + h(u) + g(u)|\nabla u|^2 = 0 & \text{in } \Omega,\\ \lim\limits_{\mathrm{dist}(x,\partial\Omega)\to 0} u(x) = +\infty\,. \end{cases} \tag{2.26}$$

If moreover either $h(s)e^{-2G(s)} \in L^1(\mathbf{R}^+)$ or $h(s)e^{-G(s)}$ is nondecreasing, then the solution is unique.

Proof. *Existence:* The existence of solutions is obtained as in Theorem 2.6 by passing to the limit in the approximating problems (2.20) (with $f = 0$). We just need to observe that the assumption that $h(s)s \geq 0$ for any s with $|s| > L$ is enough to ensure a uniform bound for u_n^-, while the bound for u_n^+ follows from (2.25) and Theorem 2.1. Moreover, the functions $v_n = \psi(u_n)$ (see (2.1)) satisfy

$$\begin{cases} -\Delta v_n = \theta(\psi^{-1}(v_n))\, v_n & \text{in } \Omega,\\ v_n = \int_n^{+\infty} e^{-G(s)} ds & \text{on } \partial\Omega, \end{cases} \tag{2.27}$$

and since $\theta(\psi^{-1}(s))$ is decreasing, a comparison result for problem (2.27) implies that the sequence v_n is decreasing; hence, u_n is an increasing sequence. This allows us to conclude that u_n almost everywhere converges to a function u which blows up uniformly at the boundary, and solves the equation inside Ω.

Uniqueness: Since u is a solution of (2.26) if and only if $v = \psi(u)$ is a solution of

$$\begin{cases} -\Delta v = \beta(v) & \text{in } \Omega,\\ \lim\limits_{d_\Omega(x)\to 0} v(x) = 0\,, \end{cases} \tag{2.28}$$

with $\beta(v) = h(u)e^{-G(u)}$, it is enough to prove the uniqueness of v. In case $h(s)e^{-2G(s)}$ is integrable at infinity, as in Theorem 2.6 we can prove that any solution v satisfies, in a neighborhood Ω_0 of $\partial\Omega$,

$$\lambda\, d_\Omega(x) \le v(x) \le \Lambda\, d_\Omega(x)$$

for some constants $\lambda, \Lambda > 0$. Therefore, since $\frac{\beta(s)}{s}$ is decreasing,

$$\int_\Omega \beta(v)\, dx = \int_{\Omega\setminus\Omega_0} \beta(v) + \int_{\Omega_0} \frac{\beta(v)}{v}\, v\, dx \le c + \frac{\Lambda}{\lambda}\int_{\Omega_0} \beta(\lambda\, d_\Omega(x))\, dx\,.$$

Using that β is integrable in zero, and since Ω is smooth, the last integral is finite, so we conclude that $\beta(v) \in L^1(\Omega)$. This easily implies that $v \in H_0^1(\Omega)$ (for instance using $(v-\varepsilon)^+$ as test function and letting ε tend to zero), so that the known uniqueness argument for weak solutions of (2.28) can be applied (again using that $\frac{\beta(s)}{s} = \theta(\psi^{-1}(s))$ is decreasing) to deduce that v is unique.

If instead $h(s)e^{-G(s)}$ is nondecreasing, then β is nonincreasing, and uniqueness follows immediately by choosing $(v_1 - v_2 - \varepsilon)^+$ (which is compactly supported in Ω) as test function in (2.28) written for v_1 and v_2. ☐

Remark 2.10. If $h(s)$ and $g(s)$ are both increasing, it follows that the function $\theta(s)$ defined in (2.1) is increasing, as is easily checked. On the other hand, assuming $\theta(s)$ to be increasing allows us to deal with a larger class of examples, as for the equation

$$-\Delta u + u + \frac{2}{1+|u|}|\nabla u|^2 = 0\,.$$

Moreover, the previous result implies the existence of large solutions even if g and h are both bounded, provided Ω has a small first eigenvalue. This is a new feature with respect to the results of J.B. Keller and R. Osserman for semilinear equations, where large solutions either exist for all bounded domains or they do not exist in any domain.

2.3. **Solutions in $\mathbf{R}^N$.** The precise nature of the local estimates obtained before yields some Liouville properties for solutions in the whole space. In particular, we have that a weak maximum principle holds without requiring any condition at infinity.

Theorem 2.11. *Assume that* $\exp(-G(s)) \in L^1(\mathbf{R}^+)$ *and*

$$\begin{gathered} h(s) > 0 \text{ for every } s > 0,\ h(0) = 0, \\ \liminf_{s\to+\infty} \frac{h(s)\exp(-G(s))}{\int_s^{+\infty}\exp(-G(t))dt} > 0\,. \end{gathered} \tag{2.29}$$

Then, if $u \in H^1_{loc}(\mathbf{R}^N) \cap L^\infty_{loc}(\mathbf{R}^N)$ *is a weak solution of*

$$-\Delta u + h(u) + g(u)|\nabla u|^2 \le 0 \quad \text{in } \mathbf{R}^N,$$

we have $u \le 0$.

Proof. Using Kato's inequality, we have that u^+ also satisfies

$$-\Delta u^+ + h(u^+) + g(u^+)|\nabla u^+|^2 \le 0 \quad \text{in } \mathbf{R}^N. \tag{2.30}$$

Let $\theta(s)$ be defined as in (2.1). Since $h(s) > 0$ for $s > 0$ and $h(0) = 0$, the same is true for $\theta(s)$. As in the proof of Theorem 2.1, we can eventually reduce to considering the case that $\theta(s)$ is increasing. It is well known, by scaling arguments, that the first eigenvector in the ball $B_n(0)$ of radius n is given by $\lambda_{1,n} = \frac{\Lambda_1}{n^2}$; if n is large enough so that $\frac{\Lambda_1}{n^2} < \liminf_{s\to+\infty} \theta(s)$, and due to (2.30), we can then apply Theorem 2.1 to u^+ in $B_n(0)$. In particular (2.5) gives

$$u^+(0) \le \theta^{-1}\big(\frac{\Lambda_1}{n^2}\big)$$

so that we deduce, as n tends to infinity, that $u^+(0) = 0$. This argument can be applied, through translation, to any point $x \in \mathbf{R}^N$; therefore, we have $u \le 0$. □

An immediate corollary concerns the nonexistence of nontrivial solutions to the equation

$$-\Delta u + h(u) + g(u)|\nabla u|^2 = 0 \qquad \text{in } \mathbf{R}^N. \tag{2.31}$$

Corollary 2.12. *Assume that* $\exp(-G(s)) \in L^1(\mathbf{R})$, $h(s)s > 0$, *and that*

$$\liminf_{s\to\pm\infty} \frac{h(s)\exp(-G(s))}{\int_s^{\pm\infty} \exp(-G(t))dt} > 0\,.$$

Then $u = 0$ *is the only solution of* (2.31).

An important consequence of local estimates is the possibility to have solutions on $\mathbf{R}^N$ without assuming a condition at infinity on the data.

Theorem 2.13. *Assume that* $\exp(-G(s)) \in L^1(\mathbf{R})$, *and that*

$$\lim_{s\to\pm\infty} h(s) = \pm\infty\,,$$

$$\liminf_{s\to\pm\infty} \frac{h(s)\exp(-G(s))}{\int_s^{\pm\infty} \exp(-G(t))dt} > 0\,.$$

Let $f \in L^\infty_{loc}(\mathbf{R}^N)$; *then there exists a solution* $u \in H^1_{loc}(\mathbf{R}^N) \cap L^\infty_{loc}(\mathbf{R}^N)$ *of*

$$-\Delta u + h(u) + g(u)|\nabla u|^2 = f \qquad \text{in } \mathbf{R}^N. \tag{2.32}$$

If moreover $h(s) > 0$ for every $s > 0$, $h(0) = 0$, and if $f \leq 0$, then we have $u \leq 0$ almost everywhere.

Proof. Let us fix $R_0 > 0$ such that

$$\frac{\Lambda_1}{R_0^2} < \min\left\{\liminf_{s\to+\infty} \frac{h(s)\exp(-G(s))}{\int_s^{+\infty}\exp(-G(t))dt}, \liminf_{s\to-\infty} \frac{h(s)\exp(-G(s))}{\int_s^{-\infty}\exp(-G(t))dt}\right\}. \tag{2.33}$$

For any $n > R_0$, we consider the Dirichlet problems

$$\begin{cases} -\Delta u_n + h(u_n) + g(u_n)|\nabla u_n|^2 = f & \text{in } B_n(0), \\ u_n = 0 & \text{on } \partial B_n(0), \end{cases} \tag{2.34}$$

which admit bounded, smooth solutions u_n. For any fixed $R > R_0$, we can apply Corollary 2.5 in $B_R(0)$ to u_n for any $n > R$ since the first eigenvalue in $B_R(0)$ is $\frac{\Lambda_1}{R^2} \leq \frac{\Lambda_1}{R_0^2}$ and since (2.33) holds true. We deduce that there exists a constant $C_R > 0$ (not depending on n) such that

$$\|u_n\|_{L^\infty(K)} \leq C_R \qquad \forall K \subset\subset B_R(0), \quad \forall n > R.$$

We can choose $K = B_{R_0}$, and successively $R = kR_0$, $k > 1$ so as to obtain estimates in $B_{kR_0}(0)$ for $\{u_n\}_{n>kR_0}$. By a diagonal argument, we construct a subsequence u_{n_k} of solutions of (2.34) which is bounded in $L^\infty_{loc}(\mathbf{R}^N)$; then, for any fixed, compact set K in $\mathbf{R}^N$, since $f \in L^\infty(K)$ and u_{n_k} is uniformly bounded on K, we can prove with known methods (see e.g. [5]) that u_{n_k} strongly converges in $H^1(K)$. The function u defined in each $B_{kR_0}(0)$ as the limit of u_{n_k} then belongs to $H^1_{loc}(\mathbf{R}^N) \cap L^\infty_{loc}(\mathbf{R}^N)$ and is a weak solution of (2.32). The last assertion of the theorem is just deduced from Theorem 2.11.

Remark 2.14. It is clear from the previous proof and from the general result of Theorem 2.1 that even if $h(s)$ is bounded we can still obtain solutions assuming the datum f sufficiently small ($\|f\|_{L^\infty(\mathbf{R}^N)} < \liminf_{s\to\pm\infty}|h(s)|$).

3. The weak absorption case

We assume here that the function $G(s) = \int_0^s g(t)\,dt$ satisfies

$$\int_0^{\pm\infty} \exp(-G(t))\,dt = \pm\infty, \tag{3.1}$$

which corresponds to a weaker growth at infinity of the gradient-dependent term. The following lemma shows that this case can be immediately reduced to the well-known semilinear case studied by J.B. Keller and R. Osserman.

Theorem 3.1. *Let Ω be a bounded subset of $\mathbf{R}^N$. Assume that $\exp(-G(s)) \notin L^1(\mathbf{R}^+)$, that $h(s)\exp(-G(s))$ is nondecreasing, and*

$$\int_a^{+\infty} \Big(h(s)\,\exp(G(s))\int_0^s \exp(-G(t))\Big)^{-\frac{1}{2}} dt < +\infty \qquad \forall a > 0\,. \tag{3.2}$$

Then there exists a positive, decreasing function $F(s)$, with $\lim_{s\to 0^+} F(s) = +\infty$, such that any solution $u \in H^1(\Omega)\cap L^\infty(\Omega)$ of

$$\begin{cases} -\Delta u + h(u) + g(u)|\nabla u|^2 \le 0 & \text{in } \mathcal{D}'(\Omega),\\ u \ge 0 & \text{in } \Omega, \end{cases} \tag{3.3}$$

satisfies $u(x) \le F(d_\Omega(x))$ for almost every $x \in \Omega$.

Proof. Defining $w = \varphi(u) = \int_0^u \exp(-G(t))\,dt$, we have that w satisfies the inequality $-\Delta w + \beta(w) \le 0$ in Ω, where $\beta(s) = h(\varphi^{-1}(s))\exp(-G(\varphi^{-1}(s)))$, which is nondecreasing. By the well-known results of J.B. Keller and R. Osserman, if

$$\int_a^{\infty} (\beta(s)s)^{-\frac{1}{2}}\;ds < \infty \qquad \forall a > 0\,, \tag{3.4}$$

then there exists a decreasing function $Q(r)$ such that $w(x) \le Q(d_\Omega(x))$ for any $x \in \Omega$. Since $\exp(-G(s)) \notin L^1(\mathbf{R}^+)$, we have that (3.4) is equivalent to (3.2). Under this latter assumption, we have then $u(x) = \varphi^{-1}(w) \le \varphi^{-1}(Q(d_\Omega(x)))$, and the claim follows setting $F(r) = \varphi^{-1}(Q(r))$, which is a decreasing function. □

Remark 3.2. Note that no assumption on the sign of g was made in the previous result, so that it applies to both absorbing and repulsive quadratic gradient terms. A comparison with the results of [16] and [23] is needed. The classical condition known for the semilinear case (i.e., $g = 0$) in order to have local estimates is

$$\int_a^{\infty} (h(s)s)^{-\frac{1}{2}}\;ds < \infty \qquad \forall a > 0\,. \tag{3.5}$$

Even if $g \ge 0$, there are still many examples where no improvement can be obtained with respect to condition (3.5), for instance if $g(s)$ behaves like $\frac{\lambda}{s}$ at infinity, with $\lambda < 1$. If however $g(s)$ behaves like $\frac{1}{s}$ at infinity, then local estimates hold for (3.3) provided

$$\int^{\infty} (h(s)s\log(|s|))^{-\frac{1}{2}}\;ds < \infty\,,$$

so that the first-order term allows condition (3.5) to be slightly relaxed.

If $g \le 0$, the gradient term is repulsive and (3.2) obviously implies (3.5). Even in this case, if $-\frac{\gamma}{s} \le g(s) \le 0$, with $\gamma > 0$, then the Keller-Osserman condition is still the optimal condition to have local estimates. Note also that, regardless of its sign, if $g \in L^1(\mathbf{R}^+)$ then assumption (3.2) reduces to (3.5); i.e., the condition for problem (1.1) to have local estimates is the same as if $g = 0$.

Under some monotonicity assumptions, the previous estimates allow us to deduce immediately the existence of large solutions.

Theorem 3.3. *Let* $f \in L^\infty(\Omega)$*. Assume that* $\exp(-G(s)) \notin L^1(\mathbf{R}^+)$*, that* (3.2) *holds true, and* $h(s)\exp(-G(s))$ *is increasing. Then there exists a solution* u *of* (2.16).

Remark 3.4. Uniqueness of large solutions can be sometimes deduced from previous results concerning semilinear equations. For example, consider the model case

$$\begin{cases} -\Delta u + u(\log(1+u))^q + \frac{|\nabla u|^2}{1+u} = 0 \quad \text{in } \Omega, \\ \lim\limits_{d_\Omega(x)\to 0} u(x) = +\infty\,. \end{cases} \tag{3.6}$$

If $q > 1$, the existence of a positive solution is given by Theorem 3.3. Since u is a solution of (3.6) if and only if $w = \log(1+u)$ is a solution of

$$\begin{cases} -\Delta w + (1-e^{-w})w^q = 0 \quad \text{in } \Omega, \\ \lim\limits_{d_\Omega(x)\to 0} w(x) = +\infty\,, \end{cases}$$

a direct application of Theorem III in [21] implies that w is unique; hence, (3.6) has a unique positive solution.

4. Generalizations

Since the local estimates have been proved for subsolutions of (1.1), they can be applied to equations having more general nonlinearities as for

$$-\Delta u + H(x,u,\nabla u) = f \qquad \text{in } \Omega, \tag{4.1}$$

where $H(x,s,\xi)$ is a Carathéodory function satisfying suitable structure conditions. Assume first that

$$\exists \quad L > 0 \ : \qquad H(x,s,\xi)s \geq 0\,, \quad \text{for any } s \in \mathbf{R}\text{: } |s| > L. \tag{4.2}$$

$$|H(x,s,\xi)| \leq b(|s|)(1+|\xi|^2) \quad \text{for any } s \in \mathbf{R},\ \xi \in \mathbf{R}^N \text{ and a.e. } x \in \Omega, \tag{4.3}$$

where $b \in C(\mathbf{R}^+, \mathbf{R}^+)$ is an increasing function. We look for existence of large solutions for problem (4.1); however, in a more general framework (without monotonicity assumptions on the lower-order term) we give a weaker meaning to the condition that u blows up at the boundary. We will say that u is a solution of

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = f \text{ in } \Omega, \\ u = +\infty \qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

if $u \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ is a distributional solution of the equation and if

$$\operatorname{ess}\lim_{s\to 0^+} \int_U u(r - s\nu(r))d\gamma(r) = +\infty \tag{4.5}$$

for every relatively open set $U \subset \partial\Omega$, where $\nu(r)$ denotes the outward unit normal at $r \in \partial\Omega$.

Theorem 4.1. *Let Ω be a C^2, bounded subset of $\mathbf{R}^N$, and let $f \in L^\infty(\Omega)$. Assume that $H(x, s, \xi)$ satisfies (4.2)–(4.3) and that there exist continuous functions $h(s)$ and $g(s)$ such that*

$$H(x, s, \xi) \geq h(s) + g(s)|\xi|^2 \quad \text{for any } s \in \mathbf{R}^+ \text{ and } \xi \in \mathbf{R}^N \text{ and a.e. } x \in \Omega, \tag{4.6}$$

with $\lim_{s\to+\infty} h(s) = +\infty$, and that one of the following assumptions holds:

$$(i) \int_0^{+\infty} \exp(-G(s))ds < +\infty, \text{ and } \liminf_{s\to+\infty} \frac{h(s)\exp(-G(s))}{\int_s^{+\infty}\exp(-G(t))dt} > \lambda_{1,\Omega}. \tag{4.7}$$

$$(ii) \int_0^{+\infty} \exp(-G(s))ds = +\infty\,, \ h(s)\exp(-G(s)) \text{ is increasing, and}$$

$$\int_1^{+\infty} \left(h(s)\exp(G(s)) \int_0^s \exp(-G(t)) \right)^{-\frac{1}{2}} < +\infty\,. \tag{4.8}$$

Then there exists a solution u of (4.4) in the sense specified above.

Proof. Let us consider a sequence of solutions of

$$\begin{cases} -\Delta u_n + H(x, u_n, \nabla u_n) = f \text{ in } \Omega, \\ u_n = n \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases} \tag{4.9}$$

whose existence is proved, for instance, in [5]. Thanks to (4.2) and since $u_n \geq 0$ on the boundary, one easily checks that u_n^- is bounded in $L^\infty(\Omega)$. Thanks to (4.6) and (4.7) or (4.8), we can use Theorem 2.1 or Theorem 3.1 to obtain an estimate on u_n^+ in $L^\infty_{loc}(\Omega)$, so that we conclude that u_n is bounded in $L^\infty_{loc}(\Omega)$. This easily implies that u_n is also bounded in $H^1_{loc}(\Omega)$. Following standard methods (as in [5] for example), one can prove the compactness of the sequence u_n in $H^1_{loc}(\Omega)$. Thanks to (4.3) we also deduce that $H(x, u_n, \nabla u_n)$ converges to $H(x, u, \nabla u)$ in $L^1_{loc}(\Omega)$, so that passing to the limit in (4.9) we obtain that u is a distributional solution of (4.4). Moreover, assumption (4.3) and the L^∞_{loc} bound on u_n imply (see [15]) that u_n is bounded in $C^{0,\alpha}_{loc}(\Omega)$ for some $\alpha \in (0,1)$, so that u_n converges to u uniformly on compact subsets of Ω. Thus u is a continuous function in Ω. Since, for any $k > 0$, $T_k(u) = \min(k, \max(u, -k))$ belongs to $H^1(\Omega)$ and $T_k(u) = k$ on $\partial\Omega$ in the sense of traces, we have that

$$\operatorname*{ess\,lim}_{s\to 0^+} \int_U u(r - s\nu(r))d\gamma(r) \geq \operatorname*{ess\,lim}_{s\to 0^+} \int_U T_k(u)(r - s\nu(r))d\gamma(r) = k$$

for every relatively open set $U \subset \partial\Omega$, so that u satisfies (4.5) since k is arbitrary. □

Remark 4.2. For such general nonlinear problems, we can alternatively define the condition that u blows up at the boundary by saying that $T_k(u) = k$ in the sense of traces of $H^1(\Omega)$, for any $k > 0$.

Similarly, one can deduce an existence result for the problem

$$-\Delta u + H(x, u, \nabla u) = f \quad \text{in } \mathbf{R}^N. \tag{4.10}$$

We omit the proof, which uses the same tools as above.

Theorem 4.3. *Let $f \in L^\infty_{loc}(\mathbf{R}^N)$, and let $H(x, s, \xi)$ satisfy* (4.2)–(4.3). *Assume that there exist continuous functions $h(s)$ and $g(s)$ such that*

$$H(x, s, \xi)s \geq h(s)s + g(s)s|\xi|^2, \quad \text{for any } s \in \mathbf{R},\ \xi \in \mathbf{R}^N \text{ and a.e. } x \in \mathbf{R}^N,$$

with $\lim_{s\to\pm\infty} h(s) = \pm\infty$, and that one of the following conditions hold:

$$(i) \quad \exp(-G(s)) \in L^1(\mathbf{R}) \quad \text{and} \quad \liminf_{s\to\pm\infty} \frac{h(s)\exp(-G(s))}{\int_s^{\pm\infty} \exp(-G(t))dt} > 0$$

$$(ii) \quad \int_0^{\pm\infty} \exp(-G(s)) = \pm\infty\,,\ \ h(s)\exp(-G(s)) \text{ is nondecreasing, and}$$

$$\left(h(s)\,\exp(G(s))\int_0^s \exp(-G(t))dt\right)^{-\frac{1}{2}} \in L^1((-\infty, 1) \cup (1, +\infty))\,.$$

Then there exists a weak solution u of (4.10), *in the sense that $u \in H^1_{loc}(\mathbf{R}^N) \cap L^\infty_{loc}(\mathbf{R}^N)$ and it is a distributional solution of the equation.*

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