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CONDITIONAL AND UNCONDITIONAL WELL-POSEDNESS FOR NONLINEAR EVOLUTION EQUATIONS

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Abstract. Attention is given to the question of well-posedness in Hadamard's classical sense for nonlinear evolution equations of the form

$$\frac{du}{dt} + Lu = N(u), \qquad u(0) = \phi. \tag{0.1}$$

In view are various classes of nonlinear wave equations, nonlinear Schrödinger equations, and the (generalized) KdV equations. Equations of type (0.1) are often well posed in a scale X_s, say, of Banach spaces, at least for s large enough. Here, increasing values of s correspond to more regularity; thus $X_r \subset X_s$ if $r > s$. For smaller values of s, some equations of the form in (0.1) are well-posed in a conditional sense that the uniqueness aspect depends upon the imposition of auxiliary conditions. In the latter context, it is natural to inquire whether or not the auxiliary conditions are essential to securing uniqueness.

It is shown here that for a conditionally well-posed Cauchy problem (0.1), the auxiliary specification is removable if a certain persistence of regularity holds. As a consequence, it will transpire that a conditionally well-posed problem (0.1) is (unconditionally) well-posed if the aforementioned persistence property holds.

These results are applied to study several recent conditional well-posedness results for the KdV equation, nonlinear Schrödinger equations, and nonlinear wave equations. It is demonstrated that the auxiliary conditions used to secure the uniqueness are all removable and the

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corresponding Cauchy problems are, in fact, unconditionally well-posed as long as their classical solutions exist globally. In addition, the well-posedness for an initial-boundary-value problem for the KdV equation posed in a quarter plane is also considered. An affirmative answer is provided for a uniqueness question left open in a recent paper of Colliander and Kenig [14].

1. Introduction

This paper is concerned with well-posedness of the Cauchy problem for several classes of nonlinear partial differential equations which include the Korteweg-de Vries equation, nonlinear Schrödinger equations, and nonlinear wave equations. The Cauchy problem for these equations may be written in the abstract form

$$\frac{du}{dt} + Lu = N(u), \qquad u(0) = \phi, \tag{1.1}$$

where L is a linear operator, N is a possibly time-dependent nonlinear operator and the initial datum ϕ belongs to a Banach space X_s with index $s \in R$. The scale of Banach spaces X_s has the property $X_{s_2} \subset X_{s_1}$ if $s_1 \leq s_2$, where the notation $X_{s_2} \subset X_{s_1}$ means not only that X_{s_2} is a subset of X_{s_1}, but also that the inclusion mapping is continuous and has dense range. The following definition of well-posedness in Hadamard's sense is standard.

Definition 1.1. *The Cauchy problem* (1.1) *is said to be well-posed in the space* X_s *if for any* $r > 0$ *there is a* $T = T(r) > 0$ *such that*

(a) *for any* $\phi \in X_s$ *with* $\|\phi\|_{X_s} \leq r$, (1.1) *admits a unique solution* u *in the space* $C([0,T]; X_s)$, *and*
(b) *the solution* u *depends continuously on its initial data* ϕ *in the sense that the mapping* $\phi \mapsto u$ *is continuous from* $\{\phi : \|\phi\|_{X_s} \leq r\}$ *to* $C([0,T]; X_s)$.

The well-posedness described by Definition 1.1 is *local* in character since the T in the definition depends on r. If T can be specified independently of r in Definition 1.1, then (1.1) is said to be globally well-posed in the space X_s.

The nonlinear evolution equations to be considered in this paper are often known to be well-posed in the space X_s in the strict sense of Definition 1.1 when s is sufficiently large. When s is not so large, it may arise that some auxiliary condition is needed to secure uniqueness. The Cauchy problem (1.1) in this case may be well-posed in the following weaker sense.

Definition 1.2. *The Cauchy problem* (1.1) *is said to be well-posed in the space* X_s *if for any* $r > 0$ *there is a* $T = T(r) > 0$ *such that*

(a) *for any* $\phi \in X_s$ *with* $\|\phi\|_{X_s} \le r$, (1.1) *admits exactly one solution* u *in the space* $C([0,T];X_s)$ *satisfying the auxiliary condition*

$$u \in \mathcal{Y}_s^T, \tag{1.2}$$

where $\mathcal{Y}_s^T$ *is an auxiliary metric space;*

(b) *the solution* u *depends continuously on its initial data* ϕ *just as in* (*b*) *of Definition* 1.1.

Because of the auxiliary condition (1.2), the well-posedness of (1.1) in the sense of Definition 1.2 is here termed *conditional well-posedness* following Kato's lead [18]. When contrast is helpful, the well-posedness of (1.1) in the sense of Definition 1.1 will be called *unconditional well-posedness*.

It is worth reviewing Kato's point of view concerning this matter. This is most transparently done in a concrete setting. In the example to follow, we will need recourse to $L_2(R^m)$, the usual Hilbert space of measurable functions defined on R^m whose square is Lebesgue integrable there. We also need the L_2-based Sobolev classes $H^j(R^m)$, $j = 1, 2, \ldots$, which is the subset of $L_2(R^m)$ consisting of those functions all of whose distributional derivatives up to order j also lie in $L_2(R^m)$. Consider the Cauchy problem

$$\begin{cases} u_t = i(\Delta u - F(u)), & t \ge 0,\ x \in R^m, \\ u(x,0) = \phi(x), & x \in R^m \end{cases} \tag{1.3}$$

for a nonlinear Schrödinger equation in dimension $m \ge 1$. Kato used this example in [18] to illustrate the difference between conditional and unconditional well-posedness. To be precise, assume the potential function F in (1.3) to be a C^1 mapping of the complex plane C to itself which vanishes at the origin and is such that for some $k \ge 1$,

$$DF(\xi) = O(|\xi|^{k-1}) \qquad \text{as } |\xi| \to \infty, \tag{1.4}$$

where DF is the differential of F viewed as a self-mapping of R^2. Then the following two theorems about (1.3) hold.

Theorem A. *In* (1.4), *assume that* $k < 1 + 4(m-2)$ (*no assumption on* k *if* $m = 1$). *Let* $r > 0$ *be given. There exists a* $T = T(r) > 0$ *such that*

(i) *for any* $\phi \in H^1 \equiv H^1(R^m)$ *with* $\|\phi\|_{H^1} \le r$, (1.3) *admits a unique solution* $u \in C([0,T];H^1)$, *and*

(ii) *the solution* u *has the additional properties*

$$\partial_x u \in L_\lambda([0,T];L_q(R^m)) \quad \textit{for} \quad \frac{1}{q} + \frac{2}{m\lambda} = 1,\ \frac{1}{2} - \frac{1}{m} < \frac{1}{q} < \frac{1}{2}. \tag{1.5}$$

Theorem B. *Suppose in hypothesis* (1.4) *that* $k < 1 + 4/m$. *Given* $r > 0$, *there exists a* $T = T(r) > 0$ *such that for any* $\phi \in L_2 \equiv L_2(R^m)$ *with* $\|\phi\|_{L_2} \leq r$, (1.3) *admits one and only one solution* u *with the properties*

(i) $u \in C([0, T]; L_2(R^m))$ *and*
(ii) $u \in L_\lambda([0, T]; L_q(R^m))$ *for* $\frac{1}{q} + \frac{2}{m\lambda} = 1$, $\frac{1}{2} - \frac{1}{m} < \frac{1}{q} < \frac{1}{2}$.

Remark. In both Theorem A and Theorem B, the solution u depends continuously on its initial value ϕ in the corresponding spaces.

As pointed out by Kato, "in Theorem A, part (i) constitutes a self-contained theorem by itself. Part (ii) is simply a 'bonus,' which may or may not appear in the theorem. For this reason, we may say that (ii) is a *removable auxiliary condition,* and that (1.3) is *unconditionally* well-posed in H^1. In most such cases, auxiliary conditions (spaces) originate as tools for constructing the solution. Whether or not to retain such removable spaces in the theorem is largely a matter of taste. In Theorem B, the $L_r([0, T]; L_q)$ are auxiliary spaces. Unlike in Theorem A, part (ii) is an essential part of the theorem; without such a condition for at least a pair of (r, q), uniqueness might not hold or even make sense. In this case we say that (1.3) is *conditionally* well-posed in L_2 with the auxiliary space $L_r([0, T]; L_q)$." The following remarks are also quoted from Kato [18].

Remarks. (i) Practically, conditional well-posedness is not a definitive notion; it may turn out that an auxiliary condition, so far supposed to be necessary, is in fact removable.

(ii) In a conditionally well-posed case, it is possible that there is another solution with a different auxiliary space. If two auxiliary conditions lead to the same solution u, we say that they are consistent.

These remarks of Kato lead one to ponder the following questions after having demonstrated that a particular Cauchy problem (1.1) is conditionally well-posed.

Questions: (1) If there are two or more different auxiliary conditions (1.2), are they consistent?

(2) Is the auxiliary condition (1.2) removable?

It seems important to assure that all auxiliary conditions are consistent for a conditionally well-posed Cauchy problem (1.1). Otherwise, after proving that (1.1) is conditionally well-posed, one is still left wondering if there are other, inconsistent auxiliary conditions. It would often be more than a little inconvenient if such inconsistent auxiliary conditions exist, for then the uniqueness of solutions in the space $C([0, T]; X_s)$ would be broken. For many conditionally well-posed Cauchy problems (1.1) currently under study, it seems unlikely that there are two inconsistent auxiliary conditions. However, it would be satisfactory if one had theory delineating when such an event might occur.

Naturally, it would be very helpful to have a sufficient condition to guarantee the consistency of any pair of auxiliary conditions for a conditionally well-posed Cauchy problem (1.1). Such a sufficient condition will be provided in this paper. It will be shown that any two auxiliary conditions for a conditionally well-posed Cauchy problem are consistent if both of them possess the property of *persistence of regularity* (see Section 2 for a precise definition). This sufficient condition turns out to be very general and is satisfied by many conditionally well-posed Cauchy problems (1.1) studied in the recent literature.

The direct way to address point (2) is to establish uniqueness in the space $C([0,T];X_s)$ or in a weaker space. Indeed, this is the approach used by Kato in [18] for a class of nonlinear Schrödinger equation and by Zhou [39] for a class of nonlinear wave equations. However, for many conditionally well-posed Cauchy problems studied in the recent literature, it seems difficult to establish uniqueness directly in the space $C([0,T];X_s)$.

In this paper, we adopt an indirect approach to address Question 2. A concept of *mild solution* borrowed from semigroup theory will be introduced. A sequence of steps making use of mild solutions leads to the conclusion that a conditionally well-posed Cauchy problem having persistence of regularity is unconditionally well-posed, which is to say that the auxiliary condition is removable.

The paper is organized as follows. Section 2, concerned with the abstract Cauchy problem (1.1), develops the general theory just described. The remainder of the paper shows the efficacy of the general theory in concrete contexts. Section 3 is devoted to the Korteweg-de Vries equation. After a brief précis of the existing conditional and unconditional well-posedness results, the theory developed in Section 2 is brought to bear. It is concluded that the conditionally well-posed problems in the literature are in fact unconditionally well-posed. Section 4 deals in a similar vein with Schrödinger-type equations, whilst Section 5 countenances classes of nonlinear wave equations. In Section 6, we broaden the range of concrete problems by considering an initial-boundary-value problem for the Korteweg-de Vries equation. Our theory is applicable even in this considerably more complicated situation. The paper concludes with a brief retrospective.

2. Abstract evolution equations

In this section, consideration is given to the abstract Cauchy problem (1.1) described in Section 1. Throughout this section, it is assumed that

(i) s_1 and s_2 are two real numbers with $s_1 < s_2$ and the Banach space X_{s_2} is densely and continuously embedded into the Banach space X_{s_1};
(ii) the Cauchy problem (1.1) is *unconditionally* well-posed in the space X_{s_2};
(iii) the Cauchy problem (1.1) is *conditionally* well-posed in the space X_{s_1} with auxiliary spaces $\mathcal{Y}^T_{s_1}$.

We first address the issue of consistency of auxiliary conditions. For this, it is useful to have the following concept, mentioned already, of *persistence of regularity* for (1.1).

Definition 2.1 (Persistence of regularity)**.** *The Cauchy problem* (1.1), *conditionally well-posed in the space X_{s_1} with the auxiliary space $\mathcal{Y}^T_{s_1}$, is said to have* persistence of regularity *if any solution $u \in C([0,T];\, X_{s_1})$ of* (1.1) *with initial data $u(0) = \phi \in X_{s_1}$ also belongs to the space $C([0,T]; X_{s_2})$ if in fact $\phi \in X_{s_2}$.*

Remark. The thrust of this condition is not only that the solution u lies in $C([0,T^*]; X_{s_2})$ for some $T^* > 0$, but also that $T^* = T$. There is no reason in general why the existence time T^* in X_{s_2} should be as large as the existence time T for X_{s_1}, for example.

Theorem 2.2 (Consistency of auxiliary conditions)**.** *Suppose* (1.1) *is conditionally well-posed in the space X_{s_1} with two different auxiliary spaces $\mathcal{Y}^T_{s_1}$ and $\mathcal{W}^T_{s_1}$. If both of these conditionally well-posed problems have the property of persistence of regularity, then the two auxiliary conditions are consistent.*

Proof. For given $\phi \in X_{s_1}$, let $T = T(\|\phi\|_{X_{s_1}})$ be the minimum of the existence intervals T_1 and T_2 associated with $\mathcal{Y}^T_{s_1}$ and $\mathcal{W}^T_{s_1}$ and let u and v be the solutions of (1.1) associated with the auxiliary spaces $\mathcal{Y}^T_{s_1}$ and $\mathcal{W}^T_{s_1}$, respectively. We claim that $u = v$ in the space $C([0,T]; X_{s_1})$. Toward establishing this, let $\{\phi_n\}$ be a sequence in X_{s_2} which converges to ϕ in the space X_{s_1}. Let u_n be the solution of (1.1) with $u_n(0) = \phi_n$ associated with the auxiliary space $\mathcal{Y}^T_{s_1}$ and v_n be the solution of (1.1) with $v_n(0) = \phi_n$ associated with the auxiliary space $\mathcal{W}^T_{s_1}$ for $n = 1, 2, \ldots$. (Without loss of generality, we may take it that $\|\phi_n\|_{X_{s_1}} \le \|\phi\|_{X_{s_1}}$ for all n so that each u_n and v_n exist on the time interval $[0,T]$.) Then u_n converges to u and v_n

converges to v in the space $C([0,T];X_{s_1})$ because of continuous dependence. As a consequence of persistence of regularity, both u_n and v_n belong to the space $C([0,T];X_{s_2})$ and solve (1.1) with the same initial value ϕ_n. Since (1.1) is unconditionally well-posed in the space X_{s_2}, it follows that for $n = 1, 2, \ldots,$

$$u_n = v_n \quad \text{in the space } C([0,T];X_{s_2})\,,$$

which implies that

$$u = v \text{ in the space } C([0,T];X_{s_1}).$$

The proof is complete. □

Next, attention is turned to understanding when the auxiliary condition (1.2) is removable. The following definition of *mild* solution for (1.1) in the space $C([0,T];X_{s_1})$ is borrowed from semigroup theory (see, for example, [31]).

Definition 2.3 (Mild solution)**.** *For given $\phi \in X_{s_1}$, a function $u \in C([0,T];X_{s_1})$ is said to be a mild solution of* (1.1) *if there exists a sequence $\{\phi_n\}$ in the space X_{s_2} such that ϕ_n converges to ϕ as $n \to \infty$ in the space X_{s_1} and the solutions u_n of* (1.1) *with initial value ϕ_n lie in $C([0,T];X_{s_2})$ and converge to u in the space $C([0,T];X_{s_1})$.*

The proposition below provides a sufficient condition to ensure the uniqueness of mild solutions of (1.1) in the space $C([0,T];X_{s_1})$.

Proposition 2.4 (Uniqueness of mild solutions)**.** *Let $\{\phi_n\}$ be a sequence in X_{s_2} and suppose that $T > 0$ is such that the corresponding solution sequence $\{u_n\}$ of* (1.1) *lies in the space $C([0,T];X_{s_2})$. In this circumstance, if $\{\phi_n\}$ being a Cauchy sequence in X_{s_1} implies that $\{u_n\}$ is a Cauchy sequence in $C([0,T];X_{s_1})$, then for any $\phi \in X_{s_1}$,* (1.1) *admits at most one associated mild solution of* (1.1) *in the space $C([0,T];X_{s_1})$.*

Proof. Suppose that (1.1) admits two mild solutions u and v in the space $C([0,T];X_{s_1})$ for a given initial value $\phi \in X_{s_1}$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences in $C([0,T];X_{s_2}))$ such that both u_n and v_n solve the equation in (1.1) for $n = 1, 2, \ldots$, with $u_n \to u$ and $v_n \to v$, as $n \to \infty$, in the space $C([0,T];X_{s_1})$. For $m = 1, 2, \ldots,$ define

$$\psi_m = \begin{cases} u_n(0) & \text{if } m = 2n, \\ v_n(0) & \text{if } m = 2n-1 \end{cases}$$

for some integer n, and

$$w_m = \begin{cases} u_n & \text{if } m = 2n, \\ v_n & \text{if } m = 2n-1 \end{cases}$$

for some integer n. Then $w_m \in C([0,T];X_{s_2})$ solves (1.1) with the initial value $\psi_n \in X_{s_2}$. But $\{\psi_m\}$ is a Cauchy sequence in X_{s_1} since both $\{u_n(0)\}$ and $\{v_n(0)\}$ converge to ϕ in the space X_{s_1}. Then, by the assumption of the proposition, $\{w_n\}$ is a Cauchy sequence in the space $C([0,T];X_{s_1})$, and consequently, the two limit points u and v of the sequence $\{w_n\}$ in the space $C([0,T];X_{s_1})$ must coincide; that is, $u \equiv v$. The proof is complete. □

The next proposition connects solutions u of (1.1) given by conditional well-posedness in the space X_{s_1} to mild solutions.

Proposition 2.5. *Suppose that the conditionally well-posed Cauchy problem* (1.1) *with auxiliary space $\mathcal{Y}_{s_1}^T$ has the property of persistence of regularity. Then all its X_{s_1}-solutions are mild solutions.*

Proof. For a given $\phi \in X_{s_1}$, let $T = T(\|\phi\|_{X_{s_1}}) > 0$ and let $u \in C([0,T];X_{s_1})$ be the corresponding solution of (1.1) satisfying the auxiliary condition $u \in \mathcal{Y}_{s_1}^T$. Since X_{s_2} is densely and continuously embedded into X_{s_1}, there is a sequence $\{\phi_n\}$ in X_{s_2} such that ϕ_n converges to ϕ in X_{s_1}. For each n, let $u_n \in C([0,T];X_{s_1})$ be the solution of (1.1) satisfying the auxiliary condition (1.2) with the initial value ϕ_n. (Again, without loss of generality, it has been supposed that $\|\phi_n\|_{X_{s_1}} \leq \|\phi\|_{X_{s_1}}$ so that the existence time for each u_n is at least T.) Because of continuous dependence, u_n converges to u in the space $C([0,T];X_{s_1})$. On the other hand, the sequence $\{u_n\}$ also belongs to the space $C([0,T];X_{s_2})$ because of persistence of regularity. The solution u is therefore a mild solution by Definition 2.3. The proof is complete. □

These simple considerations lead to the following two interesting conclusions.

Theorem 2.6 (Removable auxiliary condition)**.** *The auxiliary condition* (1.2) *is removable if the corresponding conditionally well-posed Cauchy problem* (1.1) *has the property of persistence of regularity.*

Proof. If the corresponding conditionally well-posed Cauchy problem (1.1) has the property of persistence of regularity, then, by Proposition 2.5, its solutions are mild solutions whose uniqueness is guaranteed by Proposition 2.4 because of continuous dependence. The auxiliary condition (1.2) is therefore removable. □

Remark 2.7. If, in addition, we assume that $T(\|\phi\|_{X_{s_2}}) \geq T(\|\phi\|_{X_{s_1}})$ for any $\phi \in X_{s_2}$, then auxiliary condition (1.2) is removable only if the corresponding conditionally well-posed Cauchy problem (1.1) has the property of persistence of regularity.

Indeed, in this case, if a conditionally well-posed Cauchy problem (1.1) does not have the property of persistence of regularity, then there exists a $\psi \in X_{s_2}$ and $T > 0$ such that (1.1) with initial value ψ admits two solutions u, $v \in C([0,T]; X_{s_1})$ with $v \in C([0,T]; X_{s_2})$ and $u \in \mathcal{Y}_{s_1}^T$ which are different. The auxiliary condition (1.2) is therefore necessary for uniqueness.

Theorem 2.8 (Unconditional well-posedness). *If the Cauchy problem* (1.1) *is conditionally well-posed in* X_{s_1} *with auxiliary space* $\mathcal{Y}_s^T$ *and it has the property of persistence of regularity, then* (1.1) *is unconditionally well-posed in* X_{s_1} *in the sense that for any* $r > 0$ *there is a* $T = T(r) > 0$ *such that for any* $\phi \in X_{s_1}$ *with* $\|\phi\|_{X_{s_1}} \leq r$, (1.1) *admits a unique mild solution* $u \in C([0,T]; X_{s_1})$, *and that solution depends continuously on its initial data in the corresponding spaces.*

Proof. The uniqueness of mild solution follows from Proposition 2.4. The existence and the continuous dependence follow from the conditional well-posedness of (1.1) in the space X_{s_1} and Proposition 2.5. □

3. The Korteweg-de Vries equation

In the section, attention is given to the Cauchy problem for the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \qquad u(x,0) = \phi(x), \qquad x,\, t \in R \tag{3.1}$$

with $\phi \in H^s(R)$. The study of the problem (3.1) in the classical Sobolev spaces H^s began with Temam [36] and Sjöberg [34] and was followed by many others. The first well-posedness results in the sense of Definition 1.1 were given independently by Kato [15] and Bona and Smith [1]. Kato showed (3.1) to be globally well-posed in $H^s(R)$ for $s \geq 3$ whereas Bona and Smith [1] and Bona and Scott [2] showed it to be globally well-posed in $H^s(R)$ for $s \geq 2$. Kato [16, 17] later improved his original result, achieving the best unconditional well-posedness result to date, *viz.*

Theorem 3.1 (Kato). *The Cauchy problem* (3.1) *is unconditionally well-posed in the space* $H^s(R)$ *for any* $s > 3/2$.

There are two types of conditional well-posedness results for the Cauchy problem (3.1). One of them, due to Kenig, Ponce, and Vega [20, 22], is summarized as follows.

Theorem 3.2 (Kenig, Ponce, and Vega)**.** *The Cauchy problem* (3.1) *is conditionally well-posed in the space* $H^s(R)$ *for* $s > 3/4$ *with the auxiliary condition that the solution* u *satisfies*

$$\Big(\int_{-T}^{T} \sup_{x\in R} |u(x,t)|^4 dt\Big)^{1/4} + \sup_{x\in R}\Big(\int_{-T}^{T} \big|D_x^{1+s}u(x,t)\big|^2\, dt\Big)^{1/2} < \infty \tag{3.2}$$

and

$$\Big(\int_{-\infty}^{\infty} \sup_{t\in(0,T)} |u(x,t)|^2 dx\Big)^{1/2} < \infty. \tag{3.3}$$

To describe a second type of conditional well-posedness, we need to recall the definition of the Bourgain spaces. For s, $b \in R$, the Bourgain space $X_{s,b}$ is $X_{s,b} = \{u \in \mathcal{S}'(R^2) : \|u\|_{X_{s,b}} < \infty\}$ where

$$\|u\|_{X_{s,b}} \equiv \Big(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+|\tau-\xi^3|)^{2b}(1+|\xi|)^{2s}|\hat{u}(\xi,\tau)|^2 d\tau\, d\xi\Big)^{1/2},$$

$\mathcal{S}'(R^2)$ is the space of tempered distributions, and $\hat{u}$ is the Fourier transform of u with respect to both its variables. For given $T > 0$,

$$\|u\|_{X^T_{s,b}} := \inf\ \{\|v\|_{X_{s,b}} : v = u \text{ on } R\times[0,T]\}$$

is the usual quotient norm. The following theorem, also due to Kenig, Ponce, and Vega [23], is the best conditional well-posedness result for (3.1) available so far in the literature.

Theorem 3.3 (Kenig, Ponce, and Vega)**.** [1] *For any* $s > -3/4$ *there exists* $b \in (\frac{1}{2}, 1)$ *such that the Cauchy problem* (3.1) *is conditionally well-posed in the space* $H^s(R)$ *with the auxiliary condition that*

$$\textit{the solution lies in } X^T_{s,b}. \tag{3.4}$$

Remark 3.4 The b in Theorem 3.3 depends only on s and can be chosen in such a way that b, as a function of s, is nondecreasing.

In light of the discussion in Section 2, one immediately wonders whether or not the very different auxiliary conditions (3.4) and (3.2)–(3.3) are consistent. And, of course, one would like to know if they are removable.

To address these issues, we first clarify the meaning of *solution* in the above theorems. When $s > 7/2$, the solution u given by Theorem 3.1 has the property that u, u_t, u_x, $u_{xxx} \in C(R\times[0,T])$ and the equation in (3.1)

[1] The theorem was first established by Bourgain [9] for $s = 0$ with $b = 1/2$, then improved by Kenig, Ponce, and Vega to $s > -5/8$ with some $b > 1/2$ in [22] prior to the publication of [23].

is satisfied pointwise. Such a solution is usually called *classical*. When $3 \leq s \leq 7/2$, the solution has the property that u_t, u_{xxx}, and uu_x all belong to the space $C([0,T];L_2(R))$. The equation (3.1) holds in this latter space and such solutions are usually referred to as *strong solutions*. When $s < 3$ the corresponding solution of (3.1) is a *weak* or *generalized solution* as defined now.

Definition 3.5 (generalized solution). *For given $T > 0$ and $s \in R$, a function $u \in C([0,T];H^s(R))$ is said to be a generalized solution of* (3.1) *if*

(a) *$u = \phi$ at $t = 0$;*
(b) *u^2 is a well-defined distribution;*
(c) *u solves the equation in* (3.1) *in the sense of distributions.*

Note that when $s \geq 0$, the condition (b) in the above definition is not needed as it is automatically satisfied (cf. [39]). Obviously, a strong or a classical solution of (3.1) is a generalized solution.

In an interesting paper, Zhou [39] established the uniqueness of generalized solutions of the Cauchy problem (3.1) in case $s \geq 0$. His uniqueness result provides an affirmative answer to both the questions raised above; the auxiliary conditions (3.2)–(3.3) and (3.4) are consistent and removable when $s \geq 0$. However, his approach seems unlikely to extend to negative values of s.

To address these questions for negative values of s, the concept of mild solution introduced in Section 2 is utilized. The theory is used where $X_s = H^s(R)$, $s_1 = s$, and $s_2 = \max\{s, 4\}$. Thus, as in Section 2, for some $T > 0$ and s given and fixed, a function $u \in C([0,T], H^s(R))$ is said to be a mild solution of (3.1) on the time interval $[0,T]$, if

(i) $u(x,0) = \phi(x)$ in $H^s(R)$;
(ii) there exists a sequence $\phi_n \in H^{s_2}(R)$ such that the corresponding strong solution u_n of (3.1) (with ϕ being replaced by ϕ_n) satisfies

$$\lim_{n\to\infty} \|u_n(\cdot,t) - u(\cdot,t)\|_{C([0,T];H^s(R))} = 0.$$

A mild solution is a strong solution when $s \geq 4$ and is a generalized solution when $s \geq 0$. But, when $s < 0$, it is not clear from the definition that a mild solution is a generalized solution since u^2 may not be a well-defined distribution. On the other hand, a generalized solution might not be a mild solution either. Nevertheless, all generalized solutions given by Theorem 3.2 and Theorem 3.3 are mild solutions. To see this is true, we show first that the conditional well-posedness results presented in Theorem 3.1–3.3 all have the property of persistence of regularity.

Proposition 3.6 (Persistence of regularity)**.** *Suppose $\phi \in H^4(R)$.*

(i) *If $3/2 < s < 4$ and $u \in C([0,T]; H^s(R))$ is the solution of* (3.1) *given by Theorem* 3.1 *with $u(x,0) = \phi(x)$, then $u \in C([0,T]; H^4(R))$.*

(ii) *If $3/4 < s \le 3/2$ and $u \in C([0,T]; H^s(R))$ is the solution of* (3.1) *given by Theorem* 3.2 *with $u(x,0) = \phi(x)$, then $u \in C([0,T]; H^4(R))$.*

(iii) *If $-3/4 < s \le 3/2$ and $u \in C([0,T]; H^s(R))$ is the solution of* (3.1) *given by Theorem* 3.3 *with $u(x,0) = \phi(x)$, then $u \in C([0,T]; H^4(R))$.*

Proof. Note at the outset that the strong solution corresponding to auxiliary data in $H^4(R)$ is not only unique, but globally defined (see e.g. [1] and [2]). Thus, the length of the time interval $[0,T]$ is not in question here. Assertion (i) is obviously true because of the unconditional well-posedness of (3.1) in the space $H^s(R)$ with $s > 3/2$. To see that (ii) is true, let v be the unique solution of (3.1) with the initial value ϕ given by Theorem 3.1. Then, clearly, we must have that for any $T > 0$, $v \in C([0,T]; H^4(R)) \cap C^1([0,T]; H^1(R))$. It follows that v also satisfies the auxiliary conditions (3.2) and (3.3). By the uniqueness result in Theorem 3.2, $u \equiv v$ in $C([0,T]; H^s(R))$ and therefore $u \in C([0,T]; H^4(R))$. Finally, to show that (iii) is true, let v be the unique solution of (3.1) with the initial value ϕ given by Theorem 3.1 and $w \in C([0,T]; H^4(R)) \cap X^T_{4,b}$ be the solution of (3.1) with the initial value ϕ given by Theorem 3.3 with $s = 4$. Because of the uniqueness result in Theorem 3.1, $v(x,t) \equiv w(x,t)$ for all $(x,t) \in R \times [0,T]$. On the other hand, $X^T_{4,b} \subset X^T_{s,b}$ for any $s \in (-3/4, 3/2]$. Thus w also satisfies the auxiliary condition (3.4) when $s \in (-3/4, 3/2]$. By the uniqueness portion of Theorem 3.3, $w = u$ in the space $C([0,T]; H^s(R))$. Consequently, $u = v$ in the space $C([0,T]; H^s(R))$ and $u \in C([0,T]; H^4(R))$. The proof is complete. □

Proposition 3.7. *Let $s > -3/4$ be given. Then, for any $\phi \in H^s(R)$, the corresponding solution u of* (3.1) *determined by one of Theorem* 3.1, *Theorem* 3.2, *or Theorem* 3.3 *is a mild solution.*

Proof. This follows directly from Proposition 2.5 and the last result since the conditionally well-posed Cauchy problem (3.1) determined by all those theorems has the property of persistence of regularity. □

The following uniqueness result holds for mild solutions of (3.1).

Proposition 3.8 (Uniqueness of mild solutions)**.** *For given $s > -3/4$ and $\phi \in H^s(R)$,* (3.1) *admits at most one mild solution.*

Proof. When $s > 3/2$, the uniqueness of mild solution can be established by the usual energy method. In case $s \le 3/2$, let $\{\phi_n\}_{n=1}^\infty$ be a sequence in the space $H^4(R)$ which is Cauchy in the space $H^s(R)$. Let $u_n \in C([0,T]; H^4(R))$ be the strong solution of (3.1) with $u_n(x,0) = \phi_n(x)$ for $n = 1, 2, \ldots$. By Proposition 2.4, it suffices to show that $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the space $C([0,T]; H^s(R))$. To this end, let $v_n \in C([0,T]; H^s(R))$ be the solution of (3.1) with $v_n(0) = \phi_n$ determined by Theorem 3.3, for $n = 1, 2, \ldots$. Because of continuous dependence, $\{v_n\}_{n=1}^{\infty}$ is a Cauchy sequence in the space $C([0,T]; H^s(R))$. On the other hand, by the persistence of regularity and the uniqueness of strong solutions, it follows that

$$v_n = u_n \,, \qquad \text{for } n = 1, 2, \ldots .$$

The sequence $\{u_n\}_{n=1}^{\infty}$ is therefore also a Cauchy sequence in the space $C([0,T]; H^s(R))$. The proof is complete. □

In consequence of these ruminations, Theorem 3.1, Theorem 3.2, and Theorem 3.3 can be recast as the following unconditional well-posedness result.

Theorem 3.9 (Unconditional well-posedness)**.** *Let $s > -3/4$ and $r > 0$ be given. There exists a constant $T = T(r) > 0$ such that for any $\phi \in H^s(R)$ with $\|\phi\|_{H^s(R)} \le r$,* (3.1) *admits a unique mild solution $u \in C([0,T]; H^s(R))$. The solution u depends continuously on its initial data ϕ in the corresponding spaces. In addition, the unique mild solution $u \in C([0,T]; H^s(R))$ has the following smoothing properties:*

$$\begin{cases} u \text{ satisfies the conditions (3.2) and (3.3)} & \text{if } s > 3/4, \\ u \text{ satisfies the condition (3.4)} & \text{if } s > -3/4. \end{cases}$$

Notice that the auxiliary conditions (3.2), (3.3), and (3.4) are not needed for the well-posedness and, in Kato's terminology, are simply a "bonus" coming from the well-posedness result.

4. Nonlinear Schrödinger equations

Studied here is the Cauchy problem for the nonlinear Schrödinger equation (1.3). It is shown that the auxiliary condition in Theorem B is not essential for uniqueness if (1.3) is unconditionally globally well-posed in the space $H^1(R^m)$. By Theorem 2.6, one need show only that the conditionally well-posed Cauchy problem (1.3) in the space $L_2(R^m)$ has the crucial property of persistence of regularity.

Proposition 4.1. *Assume that* (1.3) *is globally well-posed in $H^1(R^m)$. For $\phi \in H^1(R^m)$, let $T = T(\|\phi\|_{L_2(R^m)}) > 0$ and $u \in C([0,T]; L_2(R^m))$ be the* *solution of* (1.3) *guaranteed by Theorem B. Then, it is also the case that* $u \in C([0,T]; H^1(R^m))$.

Remark. Proposition 4.1 simply asserts persistence of regularity between $L_2(R^m)$ and $H^1(R^m)$.

Proof. Since (1.3) is globally well-posed in $H^1(R^m)$, for any $T > 0$, we may let $v \in C([0,T]; H^1(R^m))$ be the solution starting at $v(\cdot,0) = \phi(\cdot)$ whose existence is implied by our assumption. It follows from the Sobolev imbedding theorem that $v \in C([0,T]; L_q(R^m))$ for any q with

$$\frac{1}{2} - \frac{1}{m} < \frac{1}{q} < \frac{1}{2}\,,$$

and hence that $v \in L_\lambda([0,T]; L_q(R^m))$ for any $\lambda > 0$, and in particular for λ such that

$$\frac{1}{q} + \frac{2}{m\lambda} = 1\,.$$

Thus, the function v is a generalized solution of (1.3), lying in the space $C([0,T]; L_2(R^m))$, and satisfies the auxiliary condition, enunciated in Theorem B. In consequence of the conditional well posedness, it must be that $u = v$, whence $u \in C([0,T]; H^1(R^m))$ as announced. □

Mild solutions will again play a role in our analysis. The spaces X_s are $H^s(R^m)$, and $s_1 = 0$ and $s_2 = 1$ in this case. Thus, for given $T > 0$ and $\phi \in L_2(R^m)$, a function $u \in C([0,T]; L_2(R^m))$ is a mild solution of (1.3) with $u(0) = \phi$ if there exists a sequence $\{u_n\}_{n=1}^\infty$ in the space $C([0,T]; H^1(R^m))$ which solves (1.3) for $n = 1, 2, \ldots$, and

$$\lim_{n\to\infty} \|u_n(\cdot,t) - u(\cdot,t)\|_{C([0,T];L_2(R^m))} = 0.$$

In the light of Proposition 4.1, the uniqueness and existence of mild solutions is inferred from Proposition 2.4 and Proposition 2.5. Theorem B may therefore be restated as an unconditional well-posedness result.

Theorem 4.2 (Unconditional well-posedness). *Assume* $k < 1+4/m$ *in* (1.4) *and, in addition, that* (1.3) *is globally well-posed in the space* $H^1(R^m)$. *Then for any* $r > 0$, *there exists a* $T = T(r) > 0$ *such that for* $\phi \in L_2(R^m)$ *with* $\|\phi\|_{L_2} \le r$, (1.3) *admits a unique mild solution* $u \in C([0,T]; L_2(R^m))$, *which depends continuously on its initial value* ϕ *in the corresponding spaces. In addition, the solution* u *has the properties*

$$u \in L_\lambda([0,T); L_q) \quad \textit{for any } q, \lambda \textit{ such that } \ \frac{1}{q} + \frac{2}{m\lambda} = 1,\ \frac{1}{2} - \frac{1}{m} < \frac{1}{q} < \frac{1}{2}.$$

Remarks. (i) The assumption that (1.3) is globally well-posed in the space $H^1(R^m)$ in the above theorem is used to ensure that (1.3) possesses the property of persistence of regularity, which is central to the rest of the argument.

(ii) There are many works in the literature addressing the global well-posedness of (1.3) (cf. [11, 12, 13, 32]). Here is one of the relevant sufficient conditions, taken from [13].

Assume that $F(u) = Vu + f(\cdot, u(\cdot)) + (W * |u|^2)u$, *where* V, f, *and* W *are restricted as follows:*

(a) $V : R^m \to R$ *is real-valued potential with* $V \in L_p(R^m) \cap L_\infty(R^m)$ *for some* $p \geq 1,\ p > m/2$.

(b) $f : R^m \times [0, \infty) \to R$ *is measurable in* x *and continuous in* u, *and* $f(x, 0) = 0$, *almost everywhere on* R^m. *If* $m \geq 2$, *it is also presumed that there exist constants* C *and* $\alpha \in [0, \frac{4}{m-2})$ $(\alpha \in [0, \infty)$ *if* $m = 2)$ *such that*

$$|f(x,u) - f(x,v)| \leq C(1 + |v|^\alpha + |v|^\alpha)|v - u|,$$

for almost all $x \in R^m$ *and all* u, $v \in R$.

If $m = 1$, *assume instead that for every* M, *there exists* $L(M)$ *such that* $|f(x,v) - f(x,u)| \leq L(M)|v - u|$, *for almost all* $x \in R^m$ *and all* u *and* v *in* R *such that* $|u| + |v| \leq M$.

Extend f *to* $R^m \times C$ *by setting* $f(x, z) = \frac{z}{|z|} f(x, |z|)$, *for all* $z \in C$, $z \neq 0$, *and almost all* $x \in R^m$. *Set*

$$F(x,z) = \int_0^{|z|} f(x,s)ds, \quad \textit{for all } z \in C \textit{ and almost all } x \in R^m,$$

and assume that $F(x, u) \leq A|u|^2(1 + |u|^\delta)$, *where* $0 \leq \delta < 4/m$.

(c) $W : R^m \to R$ *is an even, real-valued potential such that* $W \in L_q(R^m) \cap L_\infty(R^m)$, *for some* $q \geq 1,\ q > n/4$. *Assume also that* $W^+ \in L_\sigma(R^m) \cap L_\infty(R^m)$, *for some* $\sigma \geq 1$, $\sigma \geq m/2$ (*and* $\sigma > 1$ *if* $m = 2$).

The reader is referred to Bourgain's recent lecture notes [11] for more results on this aspect.

(iii) If (1.3) is not globally well-posed in $H^1(R^m)$, its solution may blow up in finite time. However, in this case there may exist a positive number $\beta > 0$ such that for given $\phi \in H^1(R^m)$ with $\|\phi\|_{L_2(R^m)} \leq \beta$, the corresponding solution u of (1.3) belongs to the space $C([0,T]; H^1(R^m))$. In this situation, Theorem 4.2 still holds for those initial data having L_2 norm less than β.

There is another more-general conditional well-posedness result that was provided by Kato [18] for the Cauchy problem (1.3). For $s \geq 0$, assume that $F(u)$ in (1.3) satisfies the following two assumptions.

F1 (smoothness) $F \in C^{[s]}(C, C)$, with $F(0) = 0$, where $[s]$ denotes the smallest positive integer $\geq s$, and

F2 (growth rate) if $s > m/2$, no further assumption is made, but if $s \leq m/2$ and if $F(\zeta)$ is a polynomial in ζ and $\bar{\zeta}$, then degree$(F) = k \leq \chi(s)$, where χ is an extended real-valued function given by

$$\chi(\sigma) = 1+4/(m-2\sigma), \qquad -\infty \leq \sigma \leq m/2, \qquad (\chi(-\infty) = 1,\ \chi(m/2) = \infty)$$

with inverse

$$\chi^{-1}(k) = m/2 - 2/(k-1), \quad 1 \leq k \leq \infty \quad (\chi^{-1}(1) = -\infty,\ \chi^{-1}(\infty) = m/2);$$

if $s \leq m/2$ and if F is not a polynomial, then

$$D^i F(\zeta) = O(|\zeta|^{k-i}), \qquad i = 0, 1, \ldots, [s], \qquad \text{as } |\zeta| \to \infty,$$

where k is such that $[s] \leq k \leq \chi(s)$.

Theorem 4.3 (Kato). *Assume that the potential function F satisfies conditions (F1) and (F2) as described above. The Cauchy problem (1.3) is conditionally well-posed in the space $H^s(R^m)$ for $s \geq 0$ with the auxiliary condition that*

$$\text{the solution lies in the space } \mathcal{Y}_s^T \tag{4.1}$$

where

$$\mathcal{Y}_s^T \equiv \bigcap_{q,\lambda} \left\{ L_\lambda([0,T]; L_q^s(R^m)) : \text{where } q, \lambda \text{ are such that } \frac{1}{q} + \frac{2}{m\lambda} = \frac{1}{2}, \lambda > \frac{1}{2} \right\}$$

and $L_q^s(R^m) = (1-\Delta)^{-s/2} L_q(R^m)$ is the standard Lebesgue space.

By directly showing uniqueness in a weaker function space (see [18] for details), Kato proved that the auxiliary condition (4.1) in Theorem 4.3 is removable if $s \geq m/2$, or if $s < m/2$ and

$$k < 1 + (4 \wedge (2s+2))/(m-2s) \quad (k \leq 2/(1-2s) \text{ if } m = 1). \tag{4.2}$$

Here $a \wedge b = \min\{a,\ b\}$. Whether or not the auxiliary space $\mathcal{Y}_s^T$ is removable when (4.2) is not satisfied is a question left open in [18]. By applying the theory developed in Section 2, an argument similar to that leading to Theorem 4.2 provides a partial affirmative answer for this open question, namely, for any $s \geq 0$, the auxiliary condition (4.1) is removable so long as (1.3) is globally well-posed in the space $H^{s'}(R^m)$ for some $s' > m/2$.

Theorem 4.3 can thus be restated as the following unconditional well-posedness result.

Theorem 4.4. *Assume that the potential function F satisfies conditions $(F1)$ and $(F2)$ and, in addition, that* (1.3) *is globally well-posed in the space $H^{s'}(R^m)$ for some $s' \geq m/2$. Then for any given $s \geq 0$ and $r > 0$, there exists a $T = T(r) > 0$ such that for any $\phi \in H^s(R^m)$ with $\|\phi\|_{H^s} \leq r$,* (1.3) *admits a unique mild solution $u \in C([0,T]; H^s(R^m))$ which continuously depends on its initial value ϕ in the corresponding spaces. Moreover, the solution u has the additional property that $u \in \mathcal{Y}_s^T$.*

5. Nonlinear wave equations

In this section, attention is turned to systems of nonlinear wave equations of the form

$$\Box\, \phi = F(\partial\phi) \tag{5.1}$$

in R^{1+3}, with initial values

$$\phi(x,0) = f_0(x), \qquad \partial_t\phi(x,0) = f_1(x)\,. \tag{5.2}$$

Here, $\Box$ denotes the standard four-dimensional d'Alembertian

$$-\partial_t^2 + \partial_1^2 + \partial_2^2 + \partial_3^2, \qquad \phi = (\phi^1, \phi^2, \ldots, \phi^n),$$

and the nonlinearity $F = (F^1, F^2, \ldots, F^n)$ has the form

$$F^i = \sum_{j,k} \Gamma^i_{j,k} B^i_{j,k}(\partial\phi^i, \partial\phi^k)$$

with $B^i_{j,k}(\partial\phi, \partial\omega) = Q(\phi,\omega)$ being any of the null forms

$$Q(\phi,\omega) = Q_{\alpha\beta}(\phi,\omega) = \partial_\alpha\partial_\beta\omega - \partial_\beta\phi\partial_\alpha\omega, \qquad 1 \leq \alpha < \beta \leq 3 \tag{5.3}$$

where the $\Gamma^i_{j,k}$ are constants. Equation (5.1) is a simple model for the Yang-Mills equations under Coulomb gauge conditions. There are several conditional well-posedness results for the Cauchy problem (5.1)–(5.2).

Theorem 5.1 (Klainerman and Machedon [25])**.** *The Cauchy problem* (5.1)–(5.2) *is conditionally well-posed in the space $H^s(R^n) \times H^{s-1}(R^n)$ for $s \geq 2$ with the auxiliary condition that the solution ϕ has the quantity*

$$\int_0^T \int_{R^3} \left(|Q(\phi,\phi)|^2 + |DQ(\phi,\phi)|^2\right) dx\, dt \tag{5.4}$$

finite for any of the null forms (5.3) *that appear in F and some $T > 0$.*

To describe other conditional well-posedness results for (5.1)–(5.2), we need to mention the spaces $H_{s,\delta}$ and $\mathcal{H}_{s,\delta}$. The space $H_{s,\delta}$ is the completion of the Schwartz space $S(R^4)$ with respect to the norm

$$N_{s,\delta}(u) = \left(\int_{-\infty}^{\infty}\int_{R^3} w_+^{2s}(\tau,\xi)w_-^{2\delta}(\tau,\xi)|\hat{u}(\tau,\xi)|^2 d\xi\, d\tau\right)^{1/2},$$

where $\hat{u}$ denotes the space-time Fourier transform of u and

$$w_\pm(\tau,\xi) = 1 + |(|\tau| \pm |\xi|)|\,.$$

The definition of the space $\mathcal{H}_{s,\delta}$ is a little bit more complicated. For $u \in S(R^4)$, define

$$M_\delta(u) = \sup_{|||b|||\le 1,\, N_{0.\delta}(v)\le 1} \int_{-\infty}^{\infty}\int_{R^3} \left| w_+^{1/2} w_-^{1/2} w_-^{2\delta-1}\hat{u}(\tau,\xi)\cdot \widehat{ub}(\tau,\xi)\right| d\tau\, d\xi,$$

where

$$|||b||| = \left\| \int_{-\infty}^{\infty} \left|\hat{b}(\tau,\cdot)\right| d\tau \right\|_{L_2(R^3)} = \|\hat{b}\|_{L_2(R^3;L_1(R))}.$$

The space $\mathcal{H}_{s,\delta}$ is the completion of the Schwartz space $S(R^4)$ with respect to the norm

$$\mathcal{N}_{s,\delta}(u) \equiv N_{s,\delta}(u) + M_\delta(u).$$

Theorem 5.2 (Y. Zhou [38])**.** *The Cauchy problem* (5.1)–(5.2) *is conditionally well-posed in the space* $H^s(R^n)\times H^{s-1}(R^n)$ *for* $s > 7/4$ *with the auxiliary condition*

$$\textit{the solution } \phi \textit{ belongs to the space } \ H_{s,s-1}. \tag{5.5}$$

Theorem 5.3 (Klainerman and Machedon [26, 27])**.** *The Cauchy problem* (5.1)–(5.2) *is conditionally well-posed in the space* $H^s(R^n)\times H^{s-1}(R^n)$ *for* $s > 3/2$ *with the auxiliary condition*

$$\textit{the solution } \phi \textit{ belongs to the space } \ \mathcal{H}_{s,s-1}. \tag{5.6}$$

In light of these results, the natural question to raise is whether or not the auxiliary conditions (5.4), (5.5), and (5.6) are removable.

When $s > 5/2$, uniqueness may be established for solutions in the space

$$C([0,T];H^s(R^3))\cap C^1([0,T];H^{s-1}(R^3))$$

by standard energy-type methods. In a recent paper, Zhou [40] demonstrated the uniqueness of solutions in the space

$$L_\infty([0,T];H^2(R^3))\cap W^1_\infty([0,T];H^1(R^3)).$$

Consequently, when $s \geq 2$, those auxiliary conditions are removable and the Cauchy problem (5.1)–(5.2) is unconditionally well-posed in the space $X_s = H^s(R^3) \times H^{s-1}(R^3)$.

To show the auxiliary conditions (5.5) and (5.6) are removable when $s < 2$, consider mild solutions of (5.1)–(5.2) which are defined as in Definition 2.3 with $X_{s_2} = H^2(R^3) \times H^1(R^3)$. A detailed study of the proofs offered in [38] and [27] reveals that both conditionally well-posed Cauchy problems (5.1)–(5.2) given by Theorem 5.2 and Theorem 5.3 have the property of persistence of regularity if they are globally well-posed in the space $H^{s'}(R^3) \times H^{s'-1}(R^3)$ for some $s' \geq 2$. Thus, by Theorem 2.6, the auxiliary conditions (5.5) and (5.6) are indeed removable when $s > 3/2$ and we have the following unconditional well-posedness result.

Theorem 5.4. *Assume that* (5.1)–(5.2) *is globally well-posed in the space* $H^{s'}(R^3) \times H^{s'-1}(R^3)$ *for some* $s' \geq 2$. *Let* $s > 3/2$ *and* $r > 0$ *be given. There exists a* $T = T(r) > 0$ *such that for any* $(f, g) \in H^s(R^3) \times H^{s-1}(R^3)$ *with*

$$\|(f, g)\|_{H^s(R^3) \times H^{s-1}(R^3)} \leq r,$$

the Cauchy problem (5.1)–(5.2) *admits a unique mild solution*

$$\phi \in C([0, T]; H^s(R^3)) \cap C^1([0, T]; H^{s-1}(R^3))$$

which depends continuously on its initial data. In addition the solution ϕ *has the property*

$$\begin{cases} \phi \text{ satisfies condition (5.4)} & \text{if } s \geq 2, \\ \phi \in \mathcal{H}_{s,s-1} & \text{if } s > 3/2. \end{cases}$$

6. Initial-boundary-value problems

The focus of our discussion thus far has been the pure Cauchy problem. It is our purpose in this final section to indicate the efficacy of the general theory for a broader range of problems. Here, attention is turned to an initial-boundary-value problem (IBVP) for the KdV-equation, namely the quarter-plane problem

$$\begin{cases} u_t + u_x + uu_x + u_{xxx} = 0, & x, \ t \in R^+, \\ u(x, 0) = \phi(x), \quad u(0, t) = h(t), & x, \ t \in R^+. \end{cases} \tag{6.1}$$

Definition 6.1 (well-posedness)**.** *Let* s, $s' \in R$ *be given. The IBVP* (6.1) *is* (*locally*) *well-posed in the space* $H^s(R^+) \times H^{s'}_{loc}(R^+)$ *if for any* $r > 0$

there exists a constant $T = T(r) > 0$ such that for given $\phi \in H^s(R^+)$ and $h \in H^{s'}_{loc}(R^+)$ satisfying certain compatibility conditions and

$$\|\phi\|_{H^s(R^+)} + \|h\|_{H^{s'}(0,T)} \leq r,$$

(6.1) *admits a unique solution $u = u(x,t)$ in the space $C([0,T]; H^s(R^+))$. Moreover, the solution depends continuously on its initial and boundary data (ϕ, h) in the corresponding spaces.*

Early well-posedness results for the IBVP (6.1) were presented by Bona and Winther in [3, 4].

Theorem 6.2 (Bona and Winther)**.** *The IBVP* (6.1) *is well-posed in the space $H^{3k+1}(R^+) \times H^{k+1}_{loc}(R^+)$ for $k = 1, 2, \ldots$.*

More recently, the present authors obtained the following well-posedness result for (6.1) (see [7]).

Theorem 6.3 (Bona, Sun and Zhang [7])**.** *The IBVP* (6.1) *is well-posed in the space $H^s(R^+) \times H^{(s+1)/3}_{loc}(R^+)$ for $s > 3/4$ with the auxiliary conditions*

$$\left(\sup_{0<x<+\infty} \int_0^T |\partial_x^{s+1} u(x,t)|^2 dt \right)^{\frac{1}{2}} \leq C \left(\|\phi\|_{H^s(R^+)} + \|h\|_{H^{(s+1)/3}(0,T)} \right), \tag{6.2}$$

$$\left(\int_0^T \sup_{0<x<+\infty} |\partial_x u(\cdot,t)|^4 dt \right)^{\frac{1}{4}} \leq C \left(\|\phi\|_{H^s(R)} + \|h\|_{H^{(s+1)/3}(0,T)} \right) \tag{6.3}$$

and

$$\left(\int_0^{+\infty} \sup_{0 \leq t \leq T} |u(x,t)|^2 dx \right)^{\frac{1}{2}} \leq C \left(\|\phi\|_{H^s(R)} + \|h\|_{H^{(s+1)/3}(0,T)} \right) \tag{6.4}$$

to insure uniqueness.

The following result for (6.1) was then established by Colliander and Kenig [14].

Theorem 6.4 (Colliander and Kenig)**.** *For any $\phi \in H^s(R^+)$ and $h \in H^{(s+1)/3}(R^+)$ with $0 \leq s \leq 1$ which satisfies the compatibility condition $\phi(0) = h(0)$ if $s > 1/2$, there exists a $T = T(\|\phi\|_{H^s(R^+)}, \|h\|_{H^{(s+1)/3}(R^+)}) > 0$ and a solution $u \in C([0,T]; H^s(R^+))$ of the IBVP* (6.1)*. The map $(\phi, h) \mapsto u$ is Lipschitz-continuous from $H^s(R^+) \times H^{(s+1)/3}(R^+)$ to $C([0,T]; H^s(R^+))$.*

This is not a well-posedness result in the sense of Definition 6.1 since uniqueness is not discussed. Actually, a well-posedness result is established for an integral equation

$$w = HS_1(\phi, h) + IHS_1(ww_x) \tag{6.5}$$

posed on the whole plane $R \times R$, where $HS_1(\phi, h)$ is an integral operator associated to the linear homogeneous problem

$$\begin{cases} v_t + v_x + v_{xxx} = 0, \quad x > 0,\ t \in (0,T)\,, \\ w(x,0) = \phi(x), \quad w(0,t) = h(t)\,, \quad x > 0,\ t \in (0,T) \end{cases}$$

and $IHS_1(f)$ is an integral operator associated with the linear inhomogeneous problem

$$\begin{cases} v_t + v_x + v_{xxx} = f, \quad x > 0,\ t \in (0,T)\,, \\ w(x,0) = 0, \quad w(0,t) = 0, \quad x > 0,\ t \in (0,T)\,. \end{cases}$$

The precise definition of the integral operators HS_1 and IHS_1 is given in [14]. The relation between (6.5) and the IBVP (6.1) is that a solution w of (6.5), when restricted to the domain $R^+ \times (0,T)$, solves (6.1). For the integral equation (6.5), Colliander and Kenig have the following well-posedness result.

Theorem 6.5 (Colliander and Kenig)**.** *Let* $0 \le s \le 1$ *be given. There exists a* $\delta > 0$ *such that, if* $(\phi, h) \in H^s(R^+) \times H^{(s+1)/3}(R^+)$ *satisfies*

$$\|(\phi, h)\|_{L_2(R^+)\times H^{1/3}(R^+)} \le \delta$$

and $\phi(0) = h(0)$ *when* $s > 1/2$*, then the integral equation* (6.5) *admits a unique solution* $u \in C(R; L_{2,x}(R))$ *satisfying the auxiliary condition*

$$\Lambda^{\alpha}_{s,b}(w) < \infty \tag{6.6}$$

for some $\alpha > 1/2$ *and* b *in the range* $0 < b < 1/2$*, where*

$$\Lambda^{\alpha}_{s,b}(w) \equiv \Big(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1+|\xi|)^{2s} (1+|\tau-\xi^3|)^{2b} |\hat{w}(\xi,\tau)|^2 d\xi\, d\tau$$
$$+ \int_{-\infty}^{\infty} \int_{-T}^{T} (1+|\tau|)^{2\alpha} |\hat{w}(\xi,\tau)|^2 d\xi\, d\tau \Big)^{\frac{1}{2}}.$$

The well-posedness results of (6.1) presented in Theorem 6.3 and Theorem 6.5 are conditional since auxiliary conditions are needed to ensure uniqueness. One naturally wonders whether or not the auxiliary conditions are essential for uniqueness.

The uniqueness issue in the context of Theorem 6.4 is serious. There are many ways to transform the IBVP (6.1) into an integral equations, each of which has a solution according to a suitable version of Theorem 6.4. The question is whether the solutions of those different integral equations are equal when restricted to the domain $R^+ \times (0,T)$ for some $T > 0$. For the linear problem this is established by Colliander and Kenig in [14], but the point is unresolved for the nonlinear problem.

The key to a resolution of these issues is to introduce the analogous mild solution for the IBVP (6.1) as described below and to demonstrate that the conditional well-posedness results presented in Theorem 6.3 and Theorem 6.5 have the property of persistence of regularity.

Definition 6.7 (Mild solution)**.** *Let $s < 3$ and $T > 0$ be given. For given $\phi \in H^s(R^+)$ and $h \in H^{(s+1)/3}_{loc}(R^+)$, a function $u \in C([0,T]; H^s(R^+))$ is said to be a mild solution of* (6.1) *on* $[0,T]$ *if there exists a sequence $\{u_n\}$ in the space $C([0,T]; H^3(R^+)) \cap C^1([0,T]; L_2(R^+))$ with*

$$\phi_n(x) = u_n(x,0), \qquad h_n(t) = u_n(0,t), \qquad n = 1, 2, \dots,$$

which is such that

(i) u_n *solves the equation in* (6.1) *in* $L_2(R^+)$ *for* $0 < t < T$;
(ii) $\lim_{n\to\infty} \sup_{0\le t\le T} \|u_n(\cdot,t) - u(\cdot,t)\|_{H^s(R^+)} = 0$;
(iii) $\lim_{n\to\infty} \|h_n - h\|_{H^{(s+1)/3}(0,T)} = 0$.

The following facts hold for mild solutions.

(a) The weak solutions given by Theorem 6.3 and Theorem 6.5 are all mild solutions.
(b) For given $\phi \in H^s(R^+)$ and $h \in H^{(s+1)/3}_{loc}(R^+)$ with $0 \le s \le 3$, there is at most one mild solution for the IBVP (6.1).

Consequently, the auxiliary conditions in Theorem 6.3 and Theorem 6.5 are not needed for uniqueness and can be removed. In particular, the two solutions given by Theorem 6.5 for two different integral equations are the same when restricted to the domain $R^+ \times (0,T)$. If the word *solution* in Definition 6.1 is understood as *mild solution*, then we arrive at the following unconditional well-posedness result for the IBVP (6.1).

Theorem 6.8 (Unconditional well-posedness)**.** *The IBVP* (6.1) *is unconditionally well-posed in the space $H^s(R^+) \times H^{(s+1)/3}_{loc}(R^+)$ for any $s \ge 0$. Its solution u has the additional smoothing properties*

$$\begin{cases} u \text{ satisfies conditions (6.2)–(6.4)} & \text{if } s > 3/4; \\ u \text{ satisfies condition (6.6)} & \text{if } 1 \ge s \ge 0. \end{cases}$$

The proof of this theorem is more technically involved than those in Section 3 for the KdV equation posed on the whole line R. The reader is referred to [7] for details.

7. Conclusion

The focus of our discussion has been the pure Cauchy problem

$$\frac{du}{dt} + Lu = N(u), \quad u(0) = \phi. \tag{7.1}$$

These are considered with initial data ϕ selected from a scale of Banach spaces X_s, $s \in R$, such as the classical Sobolev spaces $H^s(R^n)$. The issue of local-in-time well-posedness is considered, and it is assumed that (7.1) is indeed locally well-posed in X_s for s large enough, but that it is only conditionally well-posed for rough data corresponding to small values of s. This is the situation that obtains in a number of currently interesting situations. A general result is formulated to the effect that, in this circumstance, and for small values of s where only conditional well-posedness is known, (7.1) is well-posed in X_s if it possesses the property of persistence of regularity. As indicated in Remark 2.7, persistence of regularity is a necessary and sufficient condition in some circumstances. This is not a complicated result, but it has the salutary property of reducing the issue of whether or nor (7.1) is unconditionally well-posed to the question of persistence of regularity. The latter point is amenable to analysis by means other than classical techniques for proving uniqueness, which mostly fail in situations without sufficient regularity. The efficacy of our observation is then demonstrated by applying it to a number of concrete examples where the issue of unconditional well-posedness is open, including nonlinear wave equations, nonlinear Schrödinger-type equations, and the Korteweg-de Vries equation. It is intimated that the same consideration may apply to a considerable range of nonlinear evolution equations (for possible examples, see [8, 9, 10, 22, 24, 28, 33, 19, 29, 30]). It was also shown that these ideas have force when nonhomogeneous boundary conditions are in question, as for the Korteweg-de Vries equation.

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