Advances in Differential Equations Volume 9, Numbers 5-6, May/June 2004, Pages 699–720

ON A VARIATIONAL CHARACTERIZATION OF A PART OF THE FUČÍK SPECTRUM AND A SUPERLINEAR EQUATION FOR THE NEUMANN P-LAPLACIAN IN DIMENSION ONE

Eugenio Massa
Dip. di Mat, Università degli Studi, Via Saldini 50, 20133 Milano, Italy

(Submitted by: Jean Mawhin)

Abstract. In the first part of this paper a variational characterization of parts of the Fučík spectrum for the p-Laplacian in an interval is given. The proof uses a linking theorem on suitably constructed sets in $W^{1,p}(0,1)$. In the second part, a superlinear equation with Neumann boundary conditions on an interval is considered, where the nonlinearity intersects all but the first eigenvalues. It is proved that under certain conditions this equation is solvable for arbitrary forcing terms. The proof uses a comparison of the minimax levels of the functional associated to this equation with suitable minimax values related to the Fučík spectrum.

1. Introduction

The main theme of this paper is the following superlinear equation with the p-Laplacian operator:

$$\begin{cases} -[\psi(u')]' = \lambda\psi(u) + g(x,u) + h(x) & \text{in} \quad (0,1) \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.1}$$

where $\psi(s) = \begin{cases} |s|^{p-2}s & s \neq 0 \\ 0 & s = 0 \end{cases}$, $p > 1$,

$$g \in \mathcal{C}^0([0,1]\times\mathbb{R}), \qquad \lim_{s\to-\infty}\frac{g(x,s)}{\psi(s)} = 0, \quad \lim_{s\to+\infty}\frac{g(x,s)}{\psi(s)} = +\infty \tag{H1}$$

uniformly with respect to $x \in [0,1]$ and $h \in L^q([0,1])$, with $1/p + 1/q = 1$.

In order to study problem (1.1) we will consider also the following Fučík problem with Neumann boundary conditions in dimension 1:

$$\begin{cases} -[\psi(u')]' = \lambda^+\psi(u^+) - \lambda^-\psi(u^-) & \text{in } (0,1) \\ u'(0) = u'(1) = 0 \end{cases}, \tag{1.2}$$

Accepted for publication: February 2004.
AMS Subject Classifications: 34L30, 49J35.

where $u^+(x) = \max\{0, u(x)\}$ and $u^-(x) = \max\{0, -u(x)\}$.

The notion of Fučík spectrum was introduced in [10] and [5] for the linear operator (that is, the case $p = 2$), and it was extended to the p-Laplacian by many authors; it is defined as the set $\Sigma \subset \mathbb{R}^2$ of the points (λ^+, λ^-) for which there exists a nontrivial solution of problem (1.2).

To know the Fučík spectrum is important in many applications, for example in the study of problems with nonlinearities which have the same order of growth as $\psi(s)$ at both $+\infty$ and $-\infty$, but with different multiplicative coefficients: if the coefficients correspond to a point (λ^+, λ^-) which is not in the Fučík spectrum, then it is possible to guarantee a priori estimates for the solutions and the PS condition for the associated functional.

If one has also a variational characterization of this spectrum, then other interesting results can be obtained; cf. [3] for the p-Laplacian and [6, 8, 4, 13] for analogous results with the Laplacian operator. However, [3] deals only with the first nontrivial curve of the Fučík spectrum.

In the one-dimensional case the Fučík spectrum for the p-Laplacian may be exactly calculated (see Section 2.1.1), and it is composed of a sequence of disjoint curves (we will call them Σ_k, $k = 1, 2, \dots$). Taking advantage of this fact, we will derive a variational characterization of points lying on one of the curves Σ_k with $k \geq 3$; in particular, we will prove

Theorem 1.1. *Let $\alpha^+ \geq \alpha^-$ and $(\alpha^+, \alpha^-) \in \Sigma_2$; then we can find and characterize variationally an intersection of the half-line $\{(\alpha^+ + t, \alpha^- + rt),\ t > 0\}$ with the Fučík spectrum, for each value of $r \in (0, 1]$.*

The cases $\alpha^+ \leq \alpha^-$ and $r \in [1, +\infty)$ can be done in a similar way.

Then, exploiting the variational characterization of these points on the Fučík spectrum, we will prove existence results for problem (1.1) when λ lies between the asymptotes of the second and the third curve of the Fučík spectrum. The proof uses the variational characterization to make a comparison of these minimax levels with those of the functional associated to problem (1.1), in order to prove the existence of a linking structure for this last functional.

Some hypotheses on the growth at infinity of the nonlinearity g will be needed in order to obtain the PS condition for the functional associated to problem (1.1): defining $G(x, s) = \int_0^s g(x, \xi)d\xi$, we ask

$$\exists \theta \in (0, \frac{1}{p}), s_0 > 0 \ \text{ s.t. } \ 0 < G(x, s) \leq \theta s g(x, s) \quad \forall s > s_0\,; \tag{H2}$$

$$\exists s_1 > 0,\ C_0 > 0 \ \text{ s.t. } \ G(x, s) \leq \frac{1}{p} s g(x, s) + C_0 \quad \forall s < -s_1\,. \tag{H3}$$

Moreover, for certain "resonant" values of the parameter λ, we will need the nonresonance condition

$$\exists \rho_0 > 0,\ \ M_0 \in \mathbb{R}\ \text{ s.t. }\ G(x,s) + h(x)s \le M_0\ \text{ a.e. }\ x \in [0,1],\ \forall s < -\rho_0\,. \tag{HR}$$

The exact statement of the results is this: let $\{\lambda^- = \lambda_k^*\}$ be the asymptote of the curve Σ_k of the Fučík spectrum for problem (1.2); then we have the following:

Theorem 1.2. *Under hypotheses* (H1), (H2), *and* (H3), *if* $p \ge 2$ *and* $\lambda \in (\lambda_2^*, \lambda_3^*)$, *then there exists a solution of problem* (1.1) *for all* $h \in L^q(0,1)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.3. *Under hypotheses* (H1), (H2), (H3), *and* (HR), *with* $p \ge 2$, $h \in L^q(0,1)$ *where* $\frac{1}{p} + \frac{1}{q} = 1$, *if* $\lambda = \lambda_k^*$ *for* $i = 2$ *or* $i = 3$, *then there exists a solution of problem* (1.1).

Remark 1.4. The hypotheses (H1) to (H3) are satisfied for example by the function $g(x,s) = e^s$; in this case, in order to satisfy also (HR) we will also need the condition $h(x) \ge 0$ almost everywhere.

Another example of a nonlinearity satisfying also (HR) and where there is some more freedom on h, is when g behaves at $-\infty$ as $|s|^\delta$ with $\delta \in (0, p-1)$, so that h may be chosen arbitrarily in $L^\infty(0,1)$.

Theorem 1.2 extends the result obtained in [21], where the existence is proved for $\lambda \in (0, \lambda_2^*)$. The result in [21] was derived for the Laplacian operator (a similar result was obtained in [7]) and then extended in a straightforward way to the p-Laplacian case.

In [13] we extended the result for the Laplacian to higher values of the parameter λ by making use of a variational characterization of the Fučík spectrum of the Laplacian; however, this variational characterization fails for $p \ne 2$ since it relies on the Hilbert-space structure of $W^{1,2}$.

Theorem 1.3 deals with some kind of resonance (as will be clear from the proofs below); the analog for $p = 2$ was obtained in [13] for any λ_k^* while, under different hypotheses, in [7] and [17] the case λ_1^* and λ_2^* respectively were considered.

For what concerns the variational characterization of the Fučík spectrum for the p-Laplacian we cite [3], where the second curve in any spatial dimension is characterized.

Another interesting variational characterization of the Fučík spectrum for the p-Laplacian is given in [15], where some pieces of the spectrum near the diagonal are characterized. Other characterizations, for the linear case $p = 2$, may be found in [8, 6, 20, 13].

1.1. **Description of the paper.** If we consider the linear case $p = 2$, let Ω be a bounded domain in $\mathbb{R}^N$, and denote by $0 \leq \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots$ the eigenvalues of $-\Delta$ in $W^{1,2}(\Omega)$ and with $(\phi_k, \quad k = 1, 2, \dots)$ the corresponding eigenfunctions; then given a point $a \in (\lambda_k, \lambda_{k+1})$ and the functional $J_a : W^{1,2}(\Omega) \to \mathbb{R}$,

$$J_a(u) = \int_\Omega |\nabla u|^2 - a \int_\Omega u^2 \ , \tag{1.3}$$

we have a natural splitting $W^{1,2}(\Omega) = V \oplus Z$, where $V = span\{\phi_1, \dots, \phi_k\}$.

Taking ∂B_V to be the boundary of the unit ball in the L^2 norm in V, one knows that there exists $\mu > 0$ such that

$$J_a(u) \leq -\mu < 0 \qquad \text{for all } u \in \partial B_V \ , \tag{1.4}$$

$$J_a(u) \geq \mu \|u\|^2_{W^{1,2}} \geq 0 \qquad \text{for all } u \in Z \ , \tag{1.5}$$

and that the two sets link (for a definition of the concept of linking see for example [18]). The existence of this structure allows us to characterize the eigenvalue λ_{k+1} as

$$\lambda_{k+1} = a + \inf_{\gamma \in \Gamma} \sup_{u \in \gamma(B^k)} J_a(u) \ , \tag{1.6}$$

where the family Γ is defined as

$$\Gamma = \{\gamma \in \mathcal{C}^0(B^k; \partial B_{L^2}) : \gamma|_{\partial B^k} \text{ is a homeomorphism onto } \partial B_V\} \,, \tag{1.7}$$

B_{L^2} denoting the unit ball in the L^2 norm in $W^{1,2}$ and $B^k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : \sum_{i=1}^k x_i^2 \leq 1\}$.

In [13] we derived a deformation of the above structure to obtain a characterization of the Fučík spectrum.

In the case $p \neq 2$ we have no more the Hilbert structure for the space $W^{1,p}$. But, for $k = 1, 2$, we will build suitable sets to play the same role as ∂B_V for the functional

$$J_\alpha(u) = \int_\Omega |\nabla u|^p - \alpha^+ \int_\Omega (u^+)^p - \alpha^- \int_\Omega (u^-)^p \ , \tag{1.8}$$

for a suitable $(\alpha^+, \alpha^-) \in \mathbb{R}^2$, in order to characterize a point in the Fučík spectrum on the second curve and then another one above it.

In particular (see Section 3), we will first reformulate in a somewhat different way the variational characterization of the second curve of the Fučík spectrum of the p-Laplacian obtained in [3] (in this part we can still work in any spatial dimension with both Neumann or Dirichlet boundary conditions); then, using this last characterization and restricting to the one-dimensional Neumann problem, we will obtain (in Section 3.3) the variational characterization claimed in Theorem 1.1.

Finally, having recovered a variational characterization as in [13], we may apply it to the "ψ-superlinear" problem (1.1) when λ is between the asymptotes of Σ_2 and Σ_3 or coincides with one of them (resonant case): in Section 4, a comparison of the obtained minimax levels with those of the functional associated to problem (1.1) will allow us to prove the existence of a linking structure for this last functional, and then to prove Theorems 1.2 and 1.3.

In Section 5 the complete proof of the PS condition for the functional associated to problem (1.1) is reported. We remark that this proof is the only point at which we use the hypothesis $p \geq 2$ which appears in Theorems 1.2 and 1.3.

2. The p-Laplacian operator

2.1. The eigenvalue and Fučík problems. The "natural" eigenvalue problem for the p-Laplacian operator is

$$\begin{cases} -\nabla \cdot [\psi(\nabla u)] = \lambda \psi(u) & \text{in } \Omega \\ Bu = 0 & \text{in } \partial\Omega, \end{cases} \tag{2.1}$$

where $Bu = 0$ represents Neumann or Dirichlet boundary conditions. Actually the two sides of the equation have the same degree of homogeneity, and so if $\bar{u}$ is a nontrivial solution then so is $t\bar{u}$ for each $t \in \mathbb{R}$. In this sense we will call in the following "ψ-linear" the rate of growth of ψ and "ψ-superlinear" (respectively "ψ-sublinear") the higher (respectively lower) rates of growth.

Much less is known about this eigenvalue problem than in the case $p = 2$. For the Dirichlet problem (but the same proofs may be adapted to the Neumann case) it is known (see [1] and [11]) that there exists a first eigenvalue λ_1, that it is simple and isolated, and that the related eigenfunction ϕ_1 does not change sign; moreover, this first eigenvalue may be characterized as

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^p : u \in W; \quad \|u\|_{L^p} = 1 \right\}. \tag{2.2}$$

(Here and in the following we denote by W the space $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, depending on the boundary conditions under consideration.) Then there exists a diverging sequence of eigenvalues which may be characterized variationally (see [15]), but it is not clear in general whether there exist other eigenvalues or not.

In an analogous way the natural formulation of the Fučík problem is

$$\begin{cases} -\nabla \cdot [\psi(\nabla u)] = \lambda^+ \psi(u^+) - \lambda^- \psi(u^-) & \text{in } \Omega \\ Bu = 0 & \text{in } \partial\Omega \end{cases}. \tag{2.3}$$

2.1.1. *The one-dimensional Neumann case.* The one-dimensional Neumann case is studied in [9] and [19], where it is shown that both the usual and the Fučík spectrum have the same qualitative shape as in the linear case: this is due to the possibility of using as in the linear case the uniqueness of the solution of the initial-value problem.

In particular, the eigenvalues are all simple and form a discrete and diverging sequence $0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$, where the corresponding eigenfunctions (which will be denoted by ϕ_k, $k = 1, 2, 3, \ldots$ and chosen such that $\|\phi_k\|_{L^p} = 1$ and $\phi_1 = const > 0$) change sign $k-1$ times; the Fučík spectrum is composed of monotone-decreasing curves arising from the diagonal points (λ_k, λ_k) with $k = 2, 3, \ldots$ (we will call each of these curves Σ_k), and by the two lines $\{\lambda^+ = \lambda_1\}$ and $\{\lambda^- = \lambda_1\}$ (which we will call Σ_1): here too the corresponding nontrivial solutions change sign $k-1$ times and may be divided into the two positive homogeneous families of solutions which are positive or negative at 0. In particular, the nontrivial solutions corresponding to a point in the curve Σ_2 are composed of a positive half-bump followed by a negative one and vice versa.

Another property which we will use is that each curve Σ_k with $k \geq 2$ lies completely in the quadrant $\lambda^\pm > \lambda_k^*$ and admits the asymptotes $\{\lambda^\pm = \lambda_k^*\}$, the values λ_k^* being distinct (and increasing in k).

Remark 2.1. The values λ_k and λ_k^* may be explicitly written, following [19] and [12],

$$\lambda_k = ((k-1)\pi_p)^p \quad \text{and} \quad \lambda_k^* = ((k-1)\frac{\pi_p}{2})^p \,, \tag{2.4}$$

where $\pi_p = \frac{2}{p}\frac{\sqrt[p]{p-1}}{\sin(\pi/p)}\pi$.

2.2. **Some useful lemmas.** We give the following lemmas, which will be used repeatedly throughout the proofs; their proof is just an application of Hölder's inequality. From now on we will denote by $q = \frac{p}{p-1}$ the dual exponent of p.

Lemma 2.2. *$u \in L^p(\Omega)$ implies $\psi(u) \in L^q(\Omega)$ and $\|\psi(u)\|_{L^q} = \|u\|_{L^p}^{p-1}$.*

Corollary 2.3. *For $u, v \in L^p(\Omega)$, we have $\psi(u)v \in L^1$ and we may estimate*

$$\Big| \int_\Omega \psi(u)v \Big| \leq \|u\|_{L^p}^{p-1} \|v\|_{L^p} \,. \tag{2.5}$$

Moreover,

Lemma 2.4. *$u_n \to u$ in $L^p(\Omega)$ implies $\int_\Omega \psi(u_n)v \to \int_\Omega \psi(u)v$ for all $v \in L^p$.*

Proof. Since $u_n \to u$ in L^p, up to a subsequence we have convergence almost everywhere, and we may find a function $k \in L^p$ such that $|u_n| \leq k$ almost everywhere, so that $|\psi(u_n)v| \leq |k|^{p-1}|v|$, which is an L^1 function by the previous lemma, and so the dominated-convergence theorem gives $\int_\Omega \psi(u_n)v \to \int_\Omega \psi(u)v$. This procedure may be applied to any subsequence, and then the result is true also without passing to a subsequence. □

In the course of the following sections we will use several times the fact that the operator $T : W \to W^*$ defined as $\langle Tu, v\rangle = \int_\Omega \psi(\nabla u)\nabla v$ satisfies the following property S^+:

Definition 2.5. *The operator $T : E \to E^*$ has the property S^+ if*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty}\langle Tu_n - Tu, u_n - u\rangle \leq 0 \text{ implies } u_n \to u.$$

Remark that the condition $\limsup_{n\to+\infty}\langle Tu_n - Tu, u_n - u\rangle \leq 0$ may be replaced by $\limsup_{n\to+\infty}\langle Tu_n, u_n - u\rangle \leq 0$, since by weak convergence $\lim_{n\to+\infty}\langle Tu, u_n - u\rangle = 0$.

For the proof of this property see for example [16].

3. Variational characterization of parts of the Fučík spectrum of the p-Laplacian

In this section we will obtain the claimed variational characterization of the Fučík spectrum.

3.1. Some preliminary lemmas.

Consider, for a given point $(\alpha^+, \alpha^-) \in \mathbb{R}^2$ and $r \in (0, 1]$, the functional

$$J_\alpha(u) = \int_\Omega |\nabla u|^p - \alpha^+ \int_\Omega (u^+)^p - \alpha^- \int_\Omega (u^-)^p \tag{3.1}$$

and the manifold

$$Q_r = \{u \in W \text{ s.t. } V_r(u) = \int_\Omega (u^+)^p + r(u^-)^p = 1\}\ . \tag{3.2}$$

Remark 3.1. Note that the functional (respectively the manifold) are of class $\mathcal{C}^2$ for $p > 2$, $\mathcal{C}^1$ but not $\mathcal{C}^{1,1}$ for $p \in (1, 2)$, while for $p = 2$ they are $\mathcal{C}^{1,1}$, but not $\mathcal{C}^2$ unless $\alpha^+ = \alpha^-$ (respectively $r = 1$).

Definition 3.2. *For the derivative of the functional J_α restricted to Q_r we will consider the norm $\|J'_\alpha(u)\|_* = \inf_{t\in\mathbb{R}} \|J'_\alpha(u) - tV'_r(u)\|_{W^*}$.*

Lemma 3.3. *For $u \in Q_r$ we have that $1 \leq \int_\Omega |u|^p \leq 1/r$.*

Proof.

$$1 = \int_\Omega (u^+)^p + r(u^-)^p \le \int_\Omega (u^+)^p + (u^-)^p = \int_\Omega |u|^p$$
$$\le \Big(\int_\Omega (u^+)^p + r(u^-)^p\Big)/r = 1/r. \qquad \square$$

In the following we will also need some sort of PS condition: for $p < 2$ we need a stronger property; actually (see [2]) if Q_r is just of class $\mathcal{C}^1$, in order to use a deformation lemma we need to prove the existence of a converging subsequence for any PS sequence $\{u_n\}$ where $u_n \in Q_r^{\delta_n}$, δ_n being any sequence such that $\delta_n \to 0$ and $Q_r^{\delta_n} = \{u \in W \text{ s.t. } V_r(u) = 1 + \delta_n\}$.

Lemma 3.4. *The functional J_α constrained to Q_r satisfies the PS condition.*

Proof. We take two sequences $\delta_n \to 0$ and $\varepsilon_n \to 0^+$, a sequence $\{u_n\} \subseteq Q_r^{\delta_n}$, and a sequence $\{\beta_n\} \subseteq \mathbb{R}$, such that

$$\Big| \int_\Omega |\nabla u_n|^p - \alpha^+ \int_\Omega |u_n^+|^p - \alpha^- \int_\Omega |u_n^-|^p \Big| \le C \tag{3.3}$$

$$\Big| \int_\Omega \psi(\nabla u_n)\nabla v - \alpha^+ \int_\Omega \psi(u_n^+)v + \alpha^+ \int_\Omega \psi(u_n^-)v$$
$$+ \beta_n\Big(\int_\Omega \psi(u_n^+)v - r\psi(u_n^-)v\Big)\Big| \le \varepsilon_n \|v\|_W, \quad \forall v \in W\ . \tag{3.4}$$

Since $\{u_n\} \subseteq Q_r^{\delta_n}$, it is bounded in L^p, and then by equation (3.3) it is also bounded in W. Then, up to a subsequence, u_n converges weakly in W and strongly in L^p to some u.

The L^p convergence implies that $u \in Q_r$. Taking $v = u_n$ we get that

$$\Big(\int_\Omega |\nabla u_n|^p - \alpha^+ \int_\Omega |u_n^+|^p - \alpha^- \int_\Omega |u_n^-|^p\Big) + (1 + \delta_n)\beta_n \to 0. \tag{3.5}$$

Finally, with $v = u_n - u$, we have

$$\int_\Omega \psi(\nabla u_n)\nabla(u_n - u) - \alpha^+ \int_\Omega \psi(u_n^+)(u_n - u) + \alpha^- \int_\Omega \psi(u_n^-)(u_n - u)$$
$$-\Big(\int_\Omega |\nabla u_n|^p - \alpha^+ \int_\Omega |u_n^+|^p - \alpha^- \int_\Omega |u_n^-|^p\Big)\Big(\int_\Omega (\psi(u_n^+) - r\psi(u_n^-))(u_n - u)\Big)$$
$$\to 0, \tag{3.6}$$

where (estimating with equation (2.5)) all terms except the first go to zero, and then we conclude that $u_n \to u$ strongly in W by the property S^+ of the p-Laplacian. $\square$

Finally, it will be crucial in the following that

Proposition 3.5. *The critical points, at some level c, of J_α constrained to Q_r are nontrivial solutions of the Fučík problem* (2.3) *with coefficients* $(\lambda^+, \lambda^-) = (\alpha^+ + c, \alpha^- + rc)$*; that is, the criticality of c implies that* $(\alpha^+ + c, \alpha^- + rc) \in \Sigma$.

Proof. The criticality of u implies that there exists the Lagrange multiplier $\beta \in \mathbb{R}$ such that, $\forall v \in W$,

$$\int_\Omega \psi(\nabla u)\nabla v - \alpha^+ \int_\Omega \psi(u^+)v + \alpha^- \int_\Omega \psi(u^-)v + \beta\Big(\int_\Omega \psi(u^+)v - r\psi(u^-)v\Big) = 0. \tag{3.7}$$

Testing against u we get $\beta = -c$, and so u solves

$$\begin{aligned} -\Delta_p u &= \alpha^+\psi(u^+) - \alpha^-\psi(u^-) + c\psi(u^+) - cr\psi(u^-) \\ &= (\alpha^+ + c)\psi(u^+) - (\alpha^- + rc)\psi(u^-) \end{aligned} \tag{3.8}$$

in Ω, with the considered boundary conditions.

Finally, u is not trivial since it is in Q_r. □

3.2. **First nontrivial curve.** First we will reformulate in a slightly different way the variational characterization of the second curve of the Fučík spectrum of the p-Laplacian, made in [3].

In this part we can still work in any spatial dimension with both Neumann or Dirichlet boundary conditions. Consider, for a given $r \in (0, 1]$,

$$d_{\lambda_1,r} = \inf_{\delta \in \Gamma_{\lambda_1,r}} \sup_{u \in \delta([0,1])} J_{\lambda_1}(u)\ , \tag{3.9}$$

where

$$J_{\lambda_1}(u) = \int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p\ , \tag{3.10}$$

$$\Gamma_{\lambda_1,r} = \{\delta \in \mathcal{C}^0([0,1]; Q_r) : \delta(0) = \phi_1\ ,\ \delta(1) = -\frac{\phi_1}{\sqrt[p]{r}}\}\,. \tag{3.11}$$

In the next two lemmas we prove the existence of a linking structure for the functional (3.10).

Lemma 3.6. $\sup_{u\in\delta(\{0;1\})} J_{\lambda_1}(u) \le 0$*,* $\forall \delta \in \Gamma_{\lambda_1,r}$.

Proof. One needs only to note that $J_{\lambda_1}(\phi_1) = J_{\lambda_1}(-\frac{\phi_1}{\sqrt[p]{r}}) = 0$. □

Lemma 3.7. $+\infty > d_{\lambda_1,r} = \inf_{\delta\in\Gamma_{\lambda_1,r}} \sup_{u\in\delta([0,1])} J_{\lambda_1}(u) > 0$.

Proof. It is less than $+\infty$ since each $\delta([0,1])$ is a compact set.

Proposition 3.5 implies that the only critical points at level 0 on Q_r are $z_1 = \phi_1$ and $z_2 = -\frac{\phi_1}{\sqrt[p]{r}}$: let d be the distance between them.

Since $J_{\lambda_1}(u) \geq 0$ in Q_r by the variational characterization of λ_1, we have $d_{\lambda_1,r} \geq 0$.

Now suppose for the sake of contradiction that $d_{\lambda_1,r} = 0$; then for any sequence of positive reals $\varepsilon_n \to 0$ there would exist a sequence $\{\delta_n\} \subseteq \Gamma_{\lambda_1,r}$ such that

$$\sup_{u \in \delta_n([0,1])} J_{\lambda_1}(u) < \varepsilon_n \,, \tag{3.12}$$

and then also a sequence $\{u_n\} \subseteq Q_r$ such that

(1a) $u_n \in \delta_n([0,1])$, and then $J_{\lambda_1}(u_n) < \varepsilon_n$,
(2a) $\|u_n - z_i\|_W > d/4$ for $i = 1, 2$.

Since $\inf_{u \in Q_r} J_{\lambda_1}(u) = 0$ we are under the conditions to apply the Ekeland variational principle to each u_n, obtaining a sequence $\{w_n\} \subseteq Q_r$ such that

(1b) $0 \leq J_{\lambda_1}(w_n) \leq J_{\lambda_1}(u_n) < \varepsilon_n$,
(2b) $\|u_n - w_n\|_W \leq \sqrt{\varepsilon_n}$,
(3b) $\|J'_{\lambda_1}(w_n)\|_* \leq \sqrt{\varepsilon_n}$.

But then w_n would be a PS sequence for J_α constrained to Q_r, and so it would have a subsequence converging to one of the critical points at level 0 (z_1 or z_2), which is impossible considering properties (2a) and (2b).

We conclude then that $d_{\lambda_1,r} > 0$. □

Combining the previous two lemmas, the PS condition in Lemma 3.4, and Proposition 3.5, we can assert by a classical linking theorem (see for example [18]) that

Theorem 3.8. *The level $d_{\lambda_1,r}$ is critical for $J_{\lambda_1}(u)$ constrained to Q_r. That is, the point $(\lambda_1 + d_{\lambda_1,r}, \lambda_1 + r d_{\lambda_1,r}) \in \Sigma$.*

Moreover, we will see in Remark 3.10 that, in the one-dimensional Neumann case, this is actually the first intersection with the Fučík spectrum of the half-line $\{(\lambda_1 + t, \lambda_1 + rt),\, t > 0\}$.

As announced before, this is nothing but a different formulation of the variational characterization in [3]; however, it is in a useful form to be used in the following.

3.3. **Third (or higher) curve for the Neumann problem in one dimension.** Now we consider the one-dimensional Neumann case: we want to make one more step in the characterization of the Fučík spectrum.

The idea we are going to apply is to "build" a suitable set homeomorphic to ∂B^2 to be used as ∂B_V in equation (1.7) in order to recover (partially) a variational characterization as in [13].

3.3.1. *Construction of the set L_{α,r_1}.* We fix a point (α^+,α^-) on the curve Σ_2 with $\alpha^+ \geq \alpha^-$.

We define $r_1 = \frac{\alpha^- - \lambda_1}{\alpha^+ - \lambda_1} = \frac{\alpha^-}{\alpha^+}$; we call u_α one of the two solutions in Q_{r_1} of the Fučík problem (1.2) with coefficients (α^+,α^-), and $\overline{u}_\alpha$ the other one (namely $\overline{u}_\alpha(x) = u_\alpha(1-x)$). Then we consider the functional

$$J_\alpha(u) = \int_0^1 |u'|^p - \alpha^+ \int_0^1 (u^+)^p - \alpha^- \int_0^1 (u^-)^p \ . \tag{3.13}$$

Observe that for $u \in Q_{r_1}$ we have

$$J_\alpha(u) = J_{\lambda_1}(u) - (\alpha^+ - \lambda_1), \tag{3.14}$$

and so

$$\inf_{\delta \in \Gamma_{\lambda_1,r_1}} \sup_{u \in \delta([0,1])} J_\alpha(u) = d_{\lambda_1,r_1} - (\alpha^+ - \lambda_1) \geq 0\,; \tag{3.15}$$

actually it is not lower than zero since we chose $(\alpha^+,\alpha^-) \in \Sigma_2$, and so the point $(\lambda_1 + d_{\lambda_1,r}, \lambda_1 + r d_{\lambda_1,r})$ found in Theorem 3.8 has to be (α^+,α^-) itself or a point on a higher curve, implying $d_{\lambda_1,r_1} \geq \alpha^+ - \lambda_1$.

Reasoning in the same way we have also, by Lemma 3.6,

$$\sup_{u \in \delta(\{0;1\})} J_\alpha(u) = -(\alpha^+ - \lambda_1) < 0\,. \tag{3.16}$$

Now we look for a special $\delta \in \Gamma_{\lambda_1,r_1}$ such that $J_\alpha(u)|_{\delta([0,1])} \leq 0$: we will build the image of this δ as follows: take the path l on Q_{r_1} :

$$\overbrace{\phi_1 \tilde{u}_\alpha^+} \cup \overbrace{\tilde{u}_\alpha^+ u_\alpha} \cup \overbrace{u_\alpha (-\tilde{u}_\alpha^-)} \cup \overbrace{(-\tilde{u}_\alpha^-) \frac{-\phi_1}{\sqrt[p]{r_1}}},$$

where $\tilde{u}_\alpha^+ = \frac{u_\alpha^+}{\|u_\alpha^+\|_{L^p}}$, $\tilde{u}_\alpha^- = \frac{u_\alpha^-}{\sqrt[p]{r_1}\|u_\alpha^-\|_{L^p}}$, and the four arcs are taken projecting onto Q_{r_1} the segment that joins the two edges (note that these segments never pass through zero).

In the following lemma we verify that this is indeed what we were looking for:

Lemma 3.9. $\sup_{u \in l}(J_\alpha(u)) = 0$.

Proof. Let us start by observing that the Fučík equation in variational form,

$$\int_0^1 \psi(u_\alpha')v' = \alpha^+ \int_0^1 \psi(u_\alpha^+)v - \alpha^- \int_0^1 \psi(u_\alpha^-)v,$$

with test functions u_α^+ and u_α^-, gives

$$\int_0^1 |(u_\alpha^\pm)'|^p = \alpha^\pm \int_0^1 (u_\alpha^\pm)^p\,, \tag{3.17}$$

that is, $J_\alpha(u_\alpha^\pm)=0$; moreover, the homogeneity of J_α allows us to ignore the projection on Q_{r_1} in the proof.

Now we look at the four arcs:

- $\overbrace{\phi_1\tilde{u}_\alpha^+}$: let $v=t\phi_1+(1-t)u_\alpha^+$ so that $v'=(1-t)(u_\alpha^+)'$: v is everywhere nonnegative and then (since $[t\phi_1+(1-t)u_\alpha^+]\geq(1-t)u_\alpha^+$ everywhere)

$$\begin{aligned} J_\alpha(v) &= (1-t)^p\int_0^1|(u_\alpha^+)'|^p-\alpha^+\int_0^1[t\phi_1+(1-t)u_\alpha^+]^p \\ &\leq (1-t)^p\alpha^+\int_0^1(u_\alpha^+)^p-(1-t)^p\alpha^+\int_0^1(u_\alpha^+)^p=0\,. \end{aligned}$$

- $\overbrace{(-\tilde{u}_\alpha^-)(-\phi_1/\sqrt[p]{r_1})}$: in the same way: let $v=t(-\phi_1)+(1-t)(-u_\alpha^-)$ so that $v'=(1-t)(-u_\alpha^-)'$: v is everywhere nonpositive and then (since $[t\phi_1+(1-t)u_\alpha^-]\geq(1-t)u_\alpha^-$ everywhere) $J_\alpha(v)\leq 0$.
- $\overbrace{\tilde{u}_\alpha^+u_\alpha}$: here $v=tu_\alpha^++(1-t)u_\alpha=u_\alpha^++(1-t)(-u_\alpha^-)$: obviously u_α^+ and u_α^- are nonzero on disjoint sets; then

$$J_\alpha(v)=J_\alpha(u_\alpha^+)+(1-t)^pJ_\alpha(u_\alpha^-)=0\,.$$

- $\overbrace{-\tilde{u}_\alpha^-u_\alpha}$: here $v=t(-u_\alpha^-)+(1-t)u_\alpha=(-u_\alpha^-)+(1-t)(u_\alpha^+)$, and as before $J_\alpha(v)=J_\alpha(u_\alpha^-)+(1-t)^pJ_\alpha(u_\alpha^+)=0$. □

Now we consider the linear isometry $Y:W\to W:u(x)\mapsto u(1-x)$, and we observe that the functionals J_α and V_r are invariant under this transformation; that is, $J_\alpha(u)=J_\alpha(Yu)$ and $V_r(u)=V_r(Yu)$ for any $u\in W$.

Moreover, let $Fix(Y)\subseteq W$ be the set of the fixed points of Y: we observe that $l\cap Fix(Y)=\{\phi_1;-\phi_1/\sqrt[p]{r_1}\}$.

These observations allow us to define

$$L_{\alpha,r_1}=l\cup Yl\,, \tag{3.18}$$

such that $L_{\alpha,r_1}\subseteq Q_{r_1}$ and is homeomorphic to ∂B^2.

Remark 3.10. At this point it is clear that (in the one-dimensional Neumann case) the level $d_{\lambda_1,r}$ defined in (3.9) corresponds to the first intersection with the Fučík spectrum of the halfline $\{(\lambda_1+t,\lambda_1+rt),\,t>0\}$: it cannot be lower (if it were it would give a new solution of the Fučík problem that we know does not exist), and we were able to give an example of a $\delta\in\Gamma_{\lambda_1,r}$ where $\sup J_\alpha(u)=0$, that is, $\sup J_{\lambda_1}(u)=\alpha^+-\lambda_1$, and then $d_{\lambda_1,r}=\alpha^+-\lambda_1$, where (α^+,α^-) was taken on the second curve.

3.3.2. *Linking structure.* Now we define the class

$$\Gamma_{\alpha,r_1} = \{\gamma \in \mathcal{C}^0(B^2; Q_{r_1}) : \gamma|_{\partial B^2} \text{ is a homeomorphism onto } L_{\alpha,r_1}\}. \quad (3.19)$$

We have that

Lemma 3.11. $\sup_{u\in\gamma(\partial B^2)} J_\alpha(u) = 0 \quad \forall\gamma \in \Gamma_{\alpha,r_1}$.

Proof. The results follows from the definition of L_{α,r_1} in (3.18), the invariance of J_α with respect to the map Y, and Lemma 3.9. □

Moreover,

Lemma 3.12. $+\infty > d_{\alpha,r_1} = \inf_{\gamma\in\Gamma_{\alpha,r_1}} \sup_{u\in\gamma(B^2)} J_\alpha(u) > 0$.

Proof. It is less than $+\infty$ since each $\gamma(B^2)$ is a compact set.

Proposition 3.5 implies that the only critical points at level 0 on Q_{r_1} are $z_1 = u_\alpha$ and $z_2 = \overline{u}_\alpha$: let $\hat{d}$ be such that $B_{\hat{d}}(u_\alpha)$ and $B_{\hat{d}}(\overline{u}_\alpha)$ are disjoint and contain neither ϕ_1 nor $-\frac{\phi_1}{\sqrt[p]{r_1}}$.

Lemma 3.11 implies that $d_{\alpha,r_1} \geq 0$, so suppose for the sake of contradiction that $d_{\alpha,r_1} = 0$: then for any sequence of positive reals $\varepsilon_n \to 0$ there would exist a sequence $\{\gamma_n\} \subseteq \Gamma_{\alpha,r_1}$ such that

$$\sup_{u\in\gamma_n(B^2)} J_\alpha(u) < \varepsilon_n\,, \quad (3.20)$$

and then also a sequence of paths $\{\delta_n\} \subseteq \Gamma_{\lambda_1,r_1}$ such that

(1a) $\delta_n([0,1]) \subseteq \gamma_n(B^2)$, and then $0 \leq \sup_{u\in\delta_n([0,1])} J_\alpha(u) < \varepsilon_n$ (see equation (3.15)),

(2a) $d(\delta_n([0,1]), z_i) > \hat{d}$ for $i = 1,2$.

Now we may apply to each δ_n the minimax principle derived from Ekeland's variational principle (see for example in [14]).

Actually (see equations (3.15) and (3.16) and Remark 3.10),

$$\inf_{\delta\in\Gamma_{\lambda_1,r_1}} \sup_{u\in\delta([0,1])} J_\alpha(u) = d_{\lambda_1,r_1} - (\alpha^+ - \lambda_1) = 0\,, \quad (3.21)$$

$$\sup_{u\in\delta(\{0;1\})} J_\alpha(u) = -(\alpha^+ - \lambda_1) < 0, \quad (3.22)$$

and the sequence δ_n above is minimizing for the value $\sup_{u\in\delta([0,1])} J_\alpha(u)$ with $\delta \in \Gamma_{\lambda_1,r_1}$.

So we obtain a sequence $\{w_n\} \subseteq Q_{r_1}$ such that

(1b) $-\varepsilon_n \leq J_\alpha(w_n) \leq \sup_{u\in\delta_n([0,1])} J_\alpha(u) < \varepsilon_n$,

(2b) $d(\delta_n([0,1]), w_n) \leq \sqrt{\varepsilon_n}$,

(3b) $\|J'_\alpha(w_n)\|_* \leq \sqrt{\varepsilon_n}$.

But then w_n would be a PS sequence for J_α constrained to Q_{r_1}, and so it would have a subsequence converging to one of the critical points at level 0 (z_1 or z_2), which is impossible considering properties (2a) and (2b).

We conclude then that $d_{\alpha,r_1} > 0$. □

3.3.3. *Characterization of a point above* Σ_2. Now, given an $r_2 \in (0,1]$ and $P_{r_1}^{r_2}$ being the radial projection from Q_{r_1} to Q_{r_2}, we define

$$\Gamma_{\alpha,r_2} = \{\gamma = P_{r_1}^{r_2} \circ \widetilde{\gamma} : \widetilde{\gamma} \in \Gamma_{\alpha,r_1}\}, \tag{3.23}$$

and we get from the previous two lemmas these corollaries:

Corollary 3.13. $\sup_{u\in\gamma(\partial B^2)} J_\alpha(u) \le 0 \quad \forall\gamma \in \Gamma_{\alpha,r_2}$.

Proof. The result of the projection is just multiplying by a positive scalar the point u, and then the effect on $J_\alpha(u)$ is multiplying by the p_{th} power of this scalar, which does not change the sign. □

Corollary 3.14. $+\infty > \inf_{\gamma\in\Gamma_{\alpha,r_2}} \sup_{u\in\gamma(B^2)} J_\alpha(u) > 0$.

Proof. As before the effect of the projection is just multiplying by a number that (on Q_{r_1}) is positive, bounded, and bounded away from zero, and then the result follows. □

From now on we can proceed as in [13]; that is, we define

$$d_{\alpha,r_2} = \inf_{\gamma\in\Gamma_{\alpha,r_2}} \sup_{u\in\gamma(B^2)} J_\alpha(u) > 0\,; \tag{3.24}$$

we deduce from Corollaries 3.13 and 3.14 and the PS condition in Lemma 3.4 that

Proposition 3.15. *The level* $d_{\alpha,r_2} > 0$ *is a critical value for* J_α *constrained to* Q_{r_2}*, the critical points associated to it are nontrivial solutions of the Fučík problem* (1.2)*, and then* $(\alpha^+ + d_{\alpha,r_2}, \alpha^- + r_2 d_{\alpha,r_2}) \in \Sigma$.

This proves Theorem 1.1.

Remark 3.16. Observe that we did not prove whether $(\alpha^+ + d_{\alpha,r_2}, \alpha^- + r_2 d_{\alpha,r_2})$ belongs to Σ_3 or to a higher curve; however, we know that it belongs to a curve Σ_h with $h \ge 3$, since $d_{\alpha,r_2} > 0$ and $(\alpha^+, \alpha^-) \in \Sigma_2$.

4. The "ψ-superlinear" problem

Since we reproduced the variational characterization as in [13], we may apply it to the "ψ-superlinear" problem (1.1) when λ is between the asymptotes of Σ_2 and Σ_3 or (resonant case) coincides with one of them.

We sketch here the basic ideas of the proofs of Theorems 1.2 and 1.3, which indeed follow closely those in [13].

The aim is to prove the existence of a nonconstrained critical point of the functional

$$F(u) = \frac{1}{p}\int_0^1 |u'|^p - \frac{\lambda}{p}\int_0^1 |u|^p - \int_0^1 G(x,u) - \int_0^1 hu \,, \tag{4.1}$$

in particular to show that the minimax level

$$f = \inf_{\gamma^* \in \Gamma^R_{\alpha,\bar r}} \sup_{u\in\gamma^*(B^2)} F(u)\,, \tag{4.2}$$

where

$$\Gamma^R_{\alpha,\bar r} = \{\gamma^* \in \mathcal{C}^0(B^2;W) : \gamma^*|_{\partial B^2} \text{ is a homeomorphism onto } RL_{\alpha,\bar r}\}\,, \tag{4.3}$$

is critical for a suitable $R > 0$ and arbitrary $\bar r \in (0,1]$, and then corresponds to a solution of problem (1.1).

To do this one estimates the functional F in terms of the functional J_α of the previous section, deriving from Corollary 3.13 and Proposition 3.15 the following two lemmas, which provide a linking structure for the functional F:

Lemma 4.1. *$\forall C \in \mathbb{R}$ we can find $R > 0$ such that*

$$\sup_{u\in\gamma^*(\partial B^2)} F(u) < C \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar r}\,. \tag{4.4}$$

Lemma 4.2. *There exists $\tilde C(\lambda,h,g)$ such that*

$$\sup_{u\in\gamma^*(B^2)} F(u) \geq -\tilde C(\lambda,h,g) \quad \forall \gamma^* \in \Gamma^R_{\alpha,\bar r}\,. \tag{4.5}$$

In Section 5 we will prove (see Lemma 5.1) that under hypotheses (H2) and (H3) the functional F satisfies the PS condition for $p \geq 2$ and any $\lambda > 0$, and then by using Lemma 4.1 with $C < -\tilde C(\lambda,h,g)$ and Lemma 4.2, we are under the conditions to apply a linking theorem that proves the criticality of the level f defined in (4.2), and then proves also Theorems 1.2 and 1.3.

4.1. **Sketch of the proofs.** Recall that in the one-dimensional Neumann case the asymptote of each Σ_k with $k = 2,3$ is at $\lambda^- = \lambda_k^*$ and that Σ_k lies entirely in the half-plane $\{\lambda^- > \lambda_k^*\}$. This structure of Σ implies that, having fixed $\lambda \in (\lambda_2^*, \lambda_3^*]$, it is always possible to find

- a point $(\alpha^+,\alpha^-) \in \Sigma_2$ and such that $\alpha^- < \lambda$,
- a $\delta > 0$ such that $\alpha^- < \lambda - \delta$ and (if $\lambda < \lambda_3^*$) also $\lambda + \delta < \lambda_3^*$.

Since $h \in L^q$ and using hypothesis (H1) we may estimate, with any $M \in \mathbb{R}$ and suitable constants $C_1(\delta,h)$, $C_2(\delta,g)$, and $C_3(M,g)$,

$$\Big|\int_0^1 G(x,-u^-) + hu\Big| \quad\leq\quad \frac{\delta}{p}\|u\|^p_{L^p} + C_1(\delta,h) + C_2(\delta,g)\,, \tag{4.6}$$

$$\int_0^1 G(x,u^+) \quad \geq \quad \frac{M}{p}\|u^+\|^p_{L^p} - C_3(M,g)\,. \tag{4.7}$$

Proof of Lemma 4.1. Using the above estimates one obtains (see [13] for the details) that for $u \in L_{\alpha,\bar r}$ and $\rho > 0$,

$$\frac{F(\rho u)}{\rho^p} \leq \frac{1}{p}J_\alpha(u) - \frac{\lambda-\delta-\alpha^-}{p}\int_0^1 |u|^p + \frac{C_1(\delta,h)+C_2(\delta,g)+C_3(\alpha^+-\alpha^-,g)}{\rho^p}, \tag{4.8}$$

from which, recalling that $J_\alpha(u) \leq 0$ by Corollary 3.13 and $\int_0^1 |u|^p \geq 1$ on $L_{\alpha,\bar r}$, one gets the claim since $\lambda - \delta - \alpha^- > 0$ by construction. □

Proof of Lemma 4.2. (See again [13] for the details.) One first fixes a $\gamma^* \in \Gamma^R_{\alpha,\bar r}$: since $\gamma^*(B^2)$ is a compact set in a space of continuous functions, this allows us to estimate the superlinear side of G as

$$G(x,s) \leq 1 + \frac{\mu_{\gamma^*}}{p}s^p \text{ for all } s \in [0, \max\{|u(x)| : x \in [0,1], u \in \gamma^*(B^2)\}] \tag{4.9}$$

for a suitable $\mu_{\gamma^*} > 0$ depending on γ^*.

Then (in the hypotheses of Theorem 1.2) defining $r_{\gamma^*} = \frac{\lambda+\delta-\alpha^-}{\lambda+\delta+\mu_{\gamma^*}-\alpha^+}$ and estimating in terms of J_α and $V_{r_{\gamma^*}}(u) = \int_0^1 (u^+)^p + r_{\gamma^*}\int_0^1 (u^-)^p$ one gets

$$\sup_{u\in\gamma^*(B^2)} F(u) \geq -C_1(\delta,h) - C_2(\delta,g) - 1 + \frac{1}{p}\sup_{u\in\gamma^*(B^2)}\Big[\Big(r_{\gamma^*}\frac{J_\alpha(u)}{V_{r_{\gamma^*}}(u)} - (\lambda+\delta-\alpha^-)\Big)\Big(\frac{V_{r_{\gamma^*}}(u)}{r_{\gamma^*}}\Big)\Big] \tag{4.10}$$

(the case in which $0 \in \gamma^*(B^2)$ so that $V_{r_{\gamma^*}}(u)$ becomes null may be treated easily). But $\sup_{u\in\gamma^*(B^2)} r_{\gamma^*}\frac{J_\alpha(u)}{V_{r_{\gamma^*}}(u)}$ is equivalent to $r_{\gamma^*}\sup_{u\in\gamma(B^2)} J_\alpha(u)$ for some $\gamma \in \Gamma_{\alpha,r_{\gamma^*}}$ (compare equation (3.19) and (4.3)); then using Proposition 3.15 and Remark 3.16 we obtain

$$\sup_{u\in\gamma^*(B^2)} r_{\gamma^*}\frac{J_\alpha(u)}{V_{r_{\gamma^*}}} \geq r_{\gamma^*} d_{\alpha,r_{\gamma^*}} > \lambda_3^* - \alpha^- > \lambda + \delta - \alpha^-\,. \tag{4.11}$$

This implies that the sup in the right-hand side of (4.10) is nonnegative, independently from the γ^* chosen, and hence the lemma is proved.

In the hypotheses of Theorem 1.3, one has a kind of resonance which creates difficulties for the last estimate above: actually we have no more

$\lambda_3^* > \lambda + \delta$, and then for Lemma 4.2 we need to impose also the hypothesis (HR) in order to estimate without need of the δ:

$$\int G(x, u^-) + hu \quad \leq \quad C_4(h, g) + \frac{1}{p}\int_0^1 (u^+)^p \,; \tag{4.12}$$

then equation (4.10) reads

$$\begin{aligned} &\sup_{u\in\gamma^*(B^2)} F(u) \qquad (4.13)\\ &\geq -C_4(h,g) - 1 + \frac{1}{p}\sup_{u\in\gamma^*(B^2)} \Big[\Big(r_{\gamma^*}\frac{J_\alpha(u)}{V_{r_{\gamma^*}}(u)} - (\lambda - \alpha^-)\Big)\Big(\frac{V_{r_{\gamma^*}}(u)}{r_{\gamma^*}}\Big)\Big] \end{aligned}$$

with $r_{\gamma^*} = \frac{\lambda-\alpha^-}{\lambda+\mu_{\gamma^*}+1-\alpha^+}$, and as before we may conclude since now we just need $r_{\gamma^*} d_{\alpha, r_{\gamma^*}} > \lambda_3^* - \alpha^- = \lambda - \alpha^-$. □

In the case $\lambda = \lambda_2^*$ we may repeat the same argument choosing $(\alpha^+, \alpha^-) = (\lambda_1, \lambda_1)$, $[0,1]$ in place of B^2, and comparing with the variational characterization of the first nontrivial curve in Section 3.2.

In particular the critical level will be defined (for R large enough) by

$$f = \inf_{\delta^*\in\Gamma^R_{\lambda_1,\bar r}} \sup_{u\in\delta^*([0,1])} F(u) \,, \tag{4.14}$$

where

$$\Gamma_{\lambda_1,\bar r} = \{\delta^* \in \mathcal{C}^0([0,1]; W) \ \text{ s.t. } \ \delta^*(0) = R\phi_1 \,, \ \delta^*(1) = -R\frac{\phi_1}{\sqrt[p]{\bar r}}\} \,. \tag{4.15}$$

5. Proof of the PS condition

In this section we will prove the PS condition for functional (4.1) with $p \geq 2$. This proof is adapted from that in [8] for the periodic problem on an interval, with the Laplacian operator.

The exact statement of the result is

Lemma 5.1. *For $p \geq 2$, under hypotheses* (H1), (H2), *and* (H3) *with $h \in L^q(0,1)$, the functional* (4.1) *satisfies the PS condition for any $\lambda > 0$.*

We observe that, as in the case $p = 2$, when $\lambda \geq \lambda_2^*$ one needs also hypothesis (H3), which was not needed in [21].

First note that from hypothesis (H1) one can always make the following estimates: for any $\varepsilon > 0$, $\bar s \in \mathbb{R}$, and $M \in \mathbb{R}$, there exist C_M, $C_\varepsilon \in \mathbb{R}$ (of course depending also on $\bar s$) such that

$$g(x,s) \geq M\psi(s) - C_M \quad \text{for} \quad s > \bar s \,, \tag{5.1}$$

$$|g(x,s)| \leq \varepsilon\psi(-s) + C_\varepsilon \quad \text{for} \quad s \leq \bar s \,. \tag{5.2}$$

Let now $\{u_n\} \subseteq W^{1,p}(0,1)$ be a PS sequence; i.e., there exist $T > 0$ and $\varepsilon_n \to 0^+$ such that

$$|F(u_n)| = \Big|\frac{1}{p}\int_0^1 |u_n'|^p - \frac{\lambda}{p}\int_0^1 |u_n|^p - \int_0^1 G(x,u_n) - \int_0^1 hu_n\Big| \le T \ , \quad (5.3)$$

$$\begin{aligned}\big|\langle F'(u_n), v\rangle\big| = \Big|\int_0^1 \psi(u_n')v' - \lambda\int_0^1 \psi(u_n)v - \int_0^1 g(x,u_n)v - \int_0^1 hv\Big| \\ \le \varepsilon_n\|v\|_{W^{1,p}}, \quad \forall v \in W^{1,p} \ . \quad (5.4)\end{aligned}$$

(1) Suppose u_n is not bounded; then we can assume $\|u_n\|_{W^{1,p}} \ge 1$, $\|u_n\|_{W^{1,p}} \to +\infty$, and define $z_n = \frac{u_n}{\|u_n\|_{W^{1,p}}}$, so that z_n is a bounded sequence in $W^{1,p}$ and we can select a subsequence such that $z_n \to z_0$ weakly in $W^{1,p}$ and strongly in $L^p(0,1)$ and $\mathcal{C}^0[0,1]$.

(2) Claim: $z_0 \le 0$.

Proof of the claim. Considering $|\frac{\langle F'(u_n), z_0^+\rangle}{\|u_n\|^{p-1}_{W^{1,p}}}|$, we get

$$\begin{aligned}&\int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|^{p-1}_{W^{1,p}}} \quad (5.5)\\ &\le \Big|\int_0^1 \psi(z_n')(z_0^+)'\Big| + \lambda\Big|\int_0^1 \psi(z_n)z_0^+\Big| + \Big|\int_0^1 \frac{hz_0^+}{\|u_n\|^{p-1}_{W^{1,p}}}\Big| + \frac{\varepsilon_n\|z_0^+\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}} .\end{aligned}$$

Now for any $\bar{x}$ such that $z_0^+(\bar{x}) > 0$, we have that $u_n(\bar{x}) > 0$ for n big enough, and then we can use the estimate (5.1) to obtain

$$\frac{g(\bar{x},u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} \ge M\psi(z_n(\bar{x})) - \frac{C_M}{\|u_n\|^{p-1}_{W^{1,p}}} ; \quad (5.6)$$

by first taking the lim inf and then exploiting the arbitrariness of M we get

$$\lim_{n\to+\infty} \frac{g(\bar{x},u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} = +\infty \ . \quad (5.7)$$

Joining equations (5.1) and (5.2) with $\bar{s} = 0$ and dividing by $\|u_n\|^{p-1}_{W^{1,p}}$ we get

$$\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} \ge -\varepsilon\psi(|z_n|) - \frac{\max\{C_M;\, C_\varepsilon\}}{\|u_n\|^{p-1}_{W^{1,p}}} ; \quad (5.8)$$

since z_n is uniformly bounded by its $\mathcal{C}^0$ convergence and $\|u_n\|_{W^{1,p}} \ge 1$, this implies that the functions $\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}}$ are bounded below uniformly so that we can use Fatou's lemma and get from (5.5) and supposing $z_0^+ \not\equiv 0$

$$+\infty = \int_0^1 \lim_{n\to+\infty} \frac{g(x,u_n)z_0^+}{\|u_n\|_{W^{1,p}}^{p-1}} \le \liminf_{n\to+\infty} \int_0^1 \frac{g(x,u_n)z_0^+}{\|u_n\|_{W^{1,p}}^{p-1}} \tag{5.9}$$
$$\le \liminf_{n\to+\infty} \Big(\Big| \int_0^1 \psi(z_n')(z_0^+)' \Big| + \lambda \Big| \int_0^1 \psi(z_n) z_0^+ \Big| + \Big| \int_0^1 \frac{h z_0^+}{\|u_n\|_{W^{1,p}}^{p-1}} \Big| + \frac{\varepsilon_n \|z_0^+\|_{W^{1,p}}}{\|u_n\|_{W^{1,p}}^{p-1}} \Big).$$

The right-hand side can be estimated since the first two terms are bounded by $(1+\lambda)\|z_n\|_{W^{1,p}}^{p-1}\|z_0^+\|_{W^{1,p}} \le 1+\lambda$ and the last two clearly go to zero; then equation (5.9) gives rise to a contradiction unless $z_0 \le 0$. □

(3) Claim: Using hypotheses (H2) and (H3) we obtain a constant A such that

$$\int_{u_n>s_0} u_n g(x,u_n) \le A\|u_n\|_{W^{1,p}}\,, \tag{5.10}$$

at least for n big enough. For $p \ge 2$ this implies

$$\int_{u_n>s_0} u_n g(x,u_n) \le A\|u_n\|_{W^{1,p}}^{p-1}\,. \tag{5.11}$$

Proof of the claim. Considering first $|pF(u_n) - \langle F'(u_n), u_n\rangle|$, we get

$$\int_{u_n>s_0} g(x,u_n)u_n - pG(x,u_n) \le \int_{u_n\le s_0} pG(x,u_n) - g(x,u_n)u_n \tag{5.12}$$
$$+ (p-1)\Big|\int_0^1 hu_n\Big| + pT + \varepsilon_n\|u_n\|_{W^{1,p}}.$$

Then we estimate (using hypothesis (H3) in (5.14) and hypothesis (H2) in (5.16)):

$$\int_{-s_1\le u_n\le s_0} pG(x,u_n) - g(x,u_n)u_n \le \sup_{\substack{x\in[0,1],\\ s\in[-s_1,s_0]}} \Big(pG(x,s) - g(x,s)s\Big), \tag{5.13}$$

$$\int_{u_n\le -s_1} pG(x,u_n) - g(x,u_n)u_n \le pC_0, \tag{5.14}$$

$$\Big|\int_0^1 hu_n\Big| \le \|h\|_{L^q}\|u_n\|_{L^p} \le \|h\|_{L^q}\|u_n\|_{W^{1,p}}\,, \tag{5.15}$$

$$(1-p\theta)\int_{u_n>s_0} g(x,u_n)u_n \le \int_{u_n>s_0} g(x,u_n)u_n - pG(x,u_n)\,. \tag{5.16}$$

Since $(1-p\theta)>0$, joining all estimates from (5.12) to (5.16), we get

$$\int_{u_n>s_0} g(x,u_n)u_n \le \frac{A}{2}\|u_n\|_{W^{1,p}} + \frac{A}{2} \le A\|u_n\|_{W^{1,p}} \tag{5.17}$$

for some constant A. Since we are supposing $\|u_n\|_{W^{1,p}} \ge 1$, this implies (5.11) for $p \ge 2$. □

(4) Claim: under hypothesis (H3),

$$\lim_{n\to+\infty}\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|^{p-1}_{W^{1,p}}} = 0\ . \tag{5.18}$$

Proof of the claim. Fix $\varepsilon>0$ and k such that $\frac{A}{k}\le\varepsilon$ and $k>s_0$. Estimate (5.2) shows that

$$\int_{u_n\le k}\frac{|g(x,u_n)|}{\|u_n\|_{W^{1,p}}} \le \int_0^1 \frac{\varepsilon|u_n|^{p-1}+C_\varepsilon}{\|u_n\|^{p-1}_{W^{1,p}}} \le \varepsilon C\frac{\|u_n\|^{p-1}_{L^p}}{\|u_n\|^{p-1}_{W^{1,p}}} + \frac{C_\varepsilon}{\|u_n\|^{p-1}_{W^{1,p}}}\ , \tag{5.19}$$

from which there exists $\bar n$ such that

$$\int_{u_n\le k}\frac{|g(x,u_n)|}{\|u_n\|^{p-1}_{W^{1,p}}} \le (C+1)\varepsilon \quad \text{for} \quad n>\bar n\ . \tag{5.20}$$

Since $k>s_0$ and using estimate (5.11), one has

$$\int_{u_n>k}\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}} \le \int_{u_n>k}\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}}\frac{u_n}{k} \le \int_{u_n>s_0}\frac{g(x,u_n)}{\|u_n\|^{p-1}_{W^{1,p}}}\frac{u_n}{k} \le \frac{A}{k} \le \varepsilon\ . \tag{5.21}$$

Then we conclude that for $n>\bar n$

$$\int_0^1 \frac{|g(x,u_n)|}{\|u_n\|^{p-1}_{W^{1,p}}} \le (2+C)\varepsilon\ ; \tag{5.22}$$

because of the arbitrariness of ε the claim is proved. □

(5) Claim: $z_n \to z_0$ strongly in $W^{1,p}$.

Proof of the claim. Considering $|\frac{\langle F'(u_n),(z_n-z_0)\rangle}{\|u_n\|^{p-1}_{W^{1,p}}}|$, we get

$$\begin{aligned}\Big|\int_0^1 \psi(z_n')(z_n'-z_0')\Big| &\le \lambda\int_0^1 |\psi(z_n)||z_n-z_0| + \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|^{p-1}_{W^{1,p}}}|z_n-z_0| \\ &+ \Big|\int_0^1 \frac{h(z_n-z_0)}{\|u_n\|^{p-1}_{W^{1,p}}}\Big| + \frac{\varepsilon_n\|z_n-z_0\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}}\ ;\end{aligned} \tag{5.23}$$

but now all the terms on the right go to zero (use equation (5.18) and the strong convergence of z_n in L^p and $\mathcal{C}^0$), and then we conclude that $z_n \to z_0$ strongly in $W^{1,p}$ by the property S^+ of the p-Laplacian. □

(6) Claim: under hypothesis (H3), $\lambda > 0$ implies $z_0 = 0$.

Proof of the claim. Consider $|\frac{\langle F'(u_n),v\rangle}{\|u_n\|^{p-1}_{W^{1,p}}}|$; for any $v \in W^{1,p}$ we get

$$\Big|\int_0^1 \psi(z_n')v' - \lambda \int_0^1 \psi(z_n)v\Big| \le \int_0^1 \frac{|g(x,u_n)|}{\|u_n\|^{p-1}_{W^{1,p}}}|v| + \Big|\int_0^1 \frac{hv}{\|u_n\|^{p-1}_{W^{1,p}}}\Big| + \frac{\varepsilon_n\|v\|_{W^{1,p}}}{\|u_n\|^{p-1}_{W^{1,p}}}, \tag{5.24}$$

but now the right-hand side goes to zero by equation (5.18), and so, taking the limit and using Lemma 2.4, we get

$$\int_0^1 \psi(z_0')v' - \lambda \int_0^1 \psi(z_0)v = 0 \quad \text{for any} \quad v \in W^{1,p}. \tag{5.25}$$

Finally, $v = 1$ gives, with $\lambda > 0$, that $\int_0^1 \psi(z_0) = 0$, but for a nonpositive function this implies $z_0 = 0$. □

(7) Claim: u_n is bounded.

Proof of the claim. The result follows since we get the contradiction $1 = \|z_n\|_{W^{1,p}} \to \|z_0\|_{W^{1,p}} = 0$. □

(8) The PS condition follows now with standard calculations from the boundedness of u_n.

Remark 5.2. The above proof may easily be adapted to the multidimensional Neumann problem under the hypothesis $p > N$, which guarantees the compact inclusion $W^{1,p}(\Omega) \subseteq \mathcal{C}^0(\bar{\Omega})$.

References

[1] A. Anane, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, C. R. Acad. Sci. Paris Sér. I Math., 305 (1987), 725–728.

[2] A. Bonnet, *A deformation lemma on a C^1 manifold*, Manuscripta Math., 81 (1993), 339–359.

[3] M. Cuesta, D. de Figueiredo, and J.-P. Gossez, *The beginning of the Fučik spectrum for the p-Laplacian*, J. Differential Equations, 159 (1999), 212–238.

[4] M. Cuesta and J.-P. Gossez, *A variational approach to nonresonance with respect to the Fučik spectrum*, Nonlinear Anal., 19 (1992), no. 5, 487–500.

[5] E.N. Dancer, *On the Dirichlet problem for weakly non-linear elliptic partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A, 76 (1976/77), 283–300.

[6] D.G. de Figueiredo and J.-P. Gossez, *On the first curve of the Fučik spectrum of an elliptic operator*, Differential Integral Equations, 7 (1994), 1285–1302.

[7] D.G. de Figueiredo and B. Ruf, *On a superlinear Sturm-Liouville equation and a related bouncing problem*, J. Reine Angew. Math., 421 (1991), 1–22.
[8] D.G. de Figueiredo and B. Ruf, *On the periodic Fučik spectrum and a superlinear Sturm-Liouville equation*, Proc. Roy. Soc. Edinburgh Sect. A, 123 (1993), 95–107.
[9] P. Drábek, "Solvability and Bifurcations of Nonlinear Equations," Pitman Research Notes in Mathematics Series, vol. 264, Longman Scientific & Technical, Harlow, 1992.
[10] S. Fučík, *Boundary value problems with jumping nonlinearities*, Časopis Pěst. Mat., 101 (1976), 69–87.
[11] P. Lindqvist, *On the equation* $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc., 109 (1990), 157–164.
[12] P. Lindqvist, *Some remarkable sine and cosine functions*, Ricerche Mat., 44 (1995), 269–290.
[13] E. Massa, *On a variational characterization of the Fučík spectrum of the Laplacian and a superlinear Sturm-Liouville equation*, accepted for publication in Proc. Roy. Soc. Edinburgh Sect. A.
[14] J. Mawhin and M. Willem, "Critical Point Theory and Hamiltonian Systems," Applied Mathematical Sciences, vol. 74, Springer-Verlag, New York, 1989.
[15] A.M. Micheletti and A. Pistoia, *On the Fučík spectrum for the p-Laplacian*, Differential Integral Equations, 14 (2001), 867–882.
[16] J. Nečas, "Introduction to the Theory of Nonlinear Elliptic Equations," Teubner-Texte zur Mathematik (Teubner Texts in Mathematics), vol. 52, BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1983, with German, French and Russian summaries.
[17] K. Perera, *Existence and multiplicity results for a Sturm-Liouville equation asymptotically linear at* $-\infty$ *and superlinear at* $+\infty$, Nonlinear Anal., TMA., 39 (2000), 669–684.
[18] P.H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," CBMS Regional Conference Series in Mathematics, vol. 65, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.
[19] W. Reichel and W. Walter, *Sturm-Liouville type problems for the p-Laplacian under asymptotic non-resonance conditions*, J. Differential Equations, 156 (1999), 50–70.
[20] M. Schechter, *Resonance problems with respect to the Fučík spectrum*, Trans. Amer. Math. Soc., 352 (2000), 4195–4205 (electronic).
[21] S. Villegas, *A Neumann problem with asymmetric nonlinearity and a related minimizing problem*, J. Differential Equations, 145 (1998), 145–155.