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LOCAL AND GLOBAL SOLUTIONS FOR A NONLINEAR DIRAC SYSTEM

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Abstract. We construct local low-regularity solutions of the generalized Dirac-Klein-Gordon equations as well as global solutions with small initial data.

1. Introduction

In this paper we study local low-regularity solutions of the generalized Dirac-Klein-Gordon equations (Reed [19], Chadam [4]) in three space dimensions as well as global small-data low-regularity solutions. Global small-data solutions to this system have recently been studied in Machihara [14]. In the special case $2\gamma = \kappa + 1$ our result improves that of [14] by ϵ derivatives in ψ. The equations are

$$i\mathcal{D}\psi - M\psi = i\phi\gamma^0 \left(\bar{\psi}\psi\right)^{\gamma-1} \psi \tag{1.1a}$$

$$\Box\phi + m^2\phi = \left(\bar{\psi}\psi\right)^{\kappa} \tag{1.1b}$$

$$\psi(0,\cdot) = \psi_0 \tag{1.1c}$$

$$\phi(0,\cdot) = \phi_0 \ , \quad \partial_t\phi(0,\cdot) = \phi_1, \tag{1.1d}$$

where ψ is a 4-spinor field, ϕ is a real-valued scalar field, $\mathcal{D} = \gamma^\mu\partial_\mu$ ([1]) is the Dirac operator, and $\gamma^\mu, \mu = 0, \dots, 3$ are the 4×4 Dirac matrices,

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \ \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \ j = 0, 1, 2,$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

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[1] We use the summation notation with Greek indices summed from 0 to 3 and Latin indices from 1 to 3.

are the Pauli matrices. The Dirac matrices satisfy $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I_4$, where $g = diag(1, -1, -1, -1)$ is the Minkowski metric. As a consequence we have $\mathcal{D}^2 = \Box$, where $\Box = \partial_t^2 - \Delta$ is the wave operator. The quantity $\bar{\psi}\psi$ is defined by $\bar{\psi}\psi = \psi^\dagger\gamma^0\psi$, where $\psi^\dagger$ is the complex conjugate transpose of the 4-spinor ψ. If we write ψ as a column vector, $\psi = (\psi_1, \dots, \psi_4)^T$, then

$$\bar{\psi}\psi = |\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2 . \tag{1.2}$$

Finally, M and m are constants (masses), and γ and κ are positive integers. The classical Dirac-Klein-Gordon system corresponds to $\gamma = \kappa = 1$. We have to assume that γ and κ are integers because the quantity $\bar{\psi}\psi$ is not positive and the powers $\left(\bar{\psi}\psi\right)^{\gamma-1}$ and $\left(\bar{\psi}\psi\right)^{\kappa}$ would not otherwise be well defined ([2]). The assumption that γ and κ are integers implies that

$$f(\psi) = \gamma^0 \left(\bar{\psi}\psi\right)^{\gamma-1} \psi,\ G(\psi) = \left(\bar{\psi}\psi\right)^{\kappa} \tag{1.3}$$

are homogeneous polynomials in $\psi_1, \dots, \psi_4$ of degrees $2\gamma - 1$ and 2κ respectively. We also define

$$F(\psi, \phi) = i\phi\gamma^0 \left(\bar{\psi}\psi\right)^{\gamma-1} \psi. \tag{1.4}$$

We assume that

$$\psi_0 \in H^\mu\left(\mathbb{R}^3\right)\ ,\ \phi_0 \in H^\nu\left(\mathbb{R}^3\right)\ ,\ \phi_1 \in H^{\nu-1}\left(\mathbb{R}^3\right). \tag{1.5}$$

For guidance on how to choose μ and ν we turn to the scaling properties of the system. They suggest the following critical values (see [14]):

$$\mu_{cr} = \frac{3}{2} - \frac{3}{2(\gamma + \kappa - 1)}\ ,\ \nu_{cr} = \frac{7}{2} - \frac{3\kappa}{\gamma + \kappa - 1}. \tag{1.6}$$

One expects the Cauchy problem (1.1) to be ill posed if the data have fewer derivatives than these critical values and well posed if they have more. In Section 3 we shall work with $\mu = \mu_{cr} + \epsilon$ and $\nu = \nu_{cr}$, where ϵ is an arbitrarily small positive number. In Section 4 we discuss how ϵ can be removed if the assumption of small data is added.

The most natural space in which to construct solutions of equations of the form $\Box u = G(u, \partial u)$ is the energy space,

$$u \in C^0\left([0,T], H^1\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T], L^2\left(\mathbb{R}^3\right)\right),\ \text{with}\ \int_0^T \|\Box u\|_{L^2}\,dt < \infty. \tag{1.7}$$

However, for many equations the critical Sobolev exponent is strictly larger than 1, so a generalized energy space is used. It consists of all

$$u \in C^0\left([0,T], H^s\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T], H^{s-1}\left(\mathbb{R}^3\right)\right) \tag{1.8a}$$

[2]See however Remark 3.2 on p. 19.

such that

$$\int_0^T \|\Box u\|_{H^{s-1}}\,dt < \infty, \tag{1.8b}$$

where $s > 1$.

In the classical existence theory for nonlinear wave equations one starts by using a generalized energy estimate (see Lemma 2.8). This brings in the term $\int_0^T \|G(u, \partial u)\|_{H^{s-1}}\,dt$. If, for example, $G(u, \partial u) = u\partial u$ ([3]), then one needs to estimate $\int_0^T \|u\partial u\|_{H^{s-1}}\,dt$. This, in turn, brings in a term of the form $\int_0^T \|u\partial^s u\|_{L^2}\,dt$, and with $u(t, \cdot) \in H^s$, this can be estimated only by $\int_0^T \|u\|_{L^\infty} \|\partial^s u\|_{L^2}\,dt$. In the classical existence theory the Sobolev inequality is employed at this stage to deal with the L^∞ norm. This imposes the restriction $s > \frac{3}{2}$, which, in many cases, is rather severe. To deal with this problem Strichartz estimates were later used (see Lemmata 2.1 and 2.2). In effect, they replace the quantity (1.8b) by a mixed space-time norm $\|\Box u\|_{L_t^r L_x^p}$. The L^∞ norm of the solution never comes up, and existence theorems can be proved for much smaller values of s. Of course, property (1.8b) is usually lost.

Small values of s are interesting because of the possibility of proving global existence for large data via local existence for rough data([4]). When this is possible, abandoning the generalized energy space (1.8) is a small price to pay to gain global existence. Similar considerations apply to nonlinear Dirac equations where the charge replaces the energy (see [7] and [1]).

To prove global existence for a system like (1.1) one would have to use the conservation of charge,

$$\int |\psi(t, x)|^2\,dx = \int |\psi(0, x)|^2\,dx. \tag{1.9}$$

This however corresponds to $\mu = 0$ in (1.5) and falls in the regime where we don't expect the Cauchy problem to be well posed. Global existence for large data is one of the most important open problems for (1.1).

Since we seem to have no hope of proving global existence via local existence for rough data there is no reason for abandoning the natural spaces in which solutions to our system should be constructed. We shall therefore

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[3] Many interesting systems have nonlinearities of this form; see for example [11] for Maxwell-Klein-Gordon. Equation (1.1a) takes that form too if we apply $\mathcal{D}$ to both sides.

[4] See for example [11] and [12], where local existence with $s = 1$ is combined with conservation of energy to give global existence.

stick to the generalized energy space for ϕ,

$$\phi \in C^0([0,T], H^\nu(\mathbb{R}^3)) \cap C^1\left([0,T], H^{\nu-1}\left(\mathbb{R}^3\right)\right), \text{ with } \int_0^T \|\Box\phi\|_{H^{\nu-1}}\, dt < \infty \tag{1.10}$$

and the corresponding generalized charge space for ψ,

$$\psi \in C^0\left([0,T], H^\mu\left(\mathbb{R}^3\right)\right), \text{ with } \int_0^T \|\mathcal{D}\psi\|_{H^\mu}\, dt < \infty. \tag{1.11}$$

We shall start our proof by using the generalized energy estimate and the generalized charge estimate. As we discussed above, this will bring in L^∞ norms. We shall deal with them using a little-known Strichartz estimate (see Lemma (2.3)).

Notation. We denote by H^s_p the standard Sobolev spaces in $\mathbb{R}^3$. Mixed space-time norms are defined by

$$\|u\|_{L^r_t L^p_x} = \left(\int_0^T \left(\int_{\mathbb{R}^3} |u(t,x)|^p\, dx\right)^{\frac{r}{p}} dt\right)^{\frac{1}{r}}.$$

We shall frequently use obvious abbreviated notation, as for example $\int_0^T \|\phi\|_{L^\infty} \|\psi\|^{2\gamma-2}_{L^\infty}\, dt$, which stands for $\int_0^T \|\phi(t,x)\|_{L^\infty_x} \|\psi(t,x)\|^{2\gamma-2}_{L^\infty_x}\, dt$.

2. Linear estimates

In this section we collect the Strichartz and other estimates we need for our proof. Estimates (2.4)–(2.5) and (2.6)–(2.7) are the standard Strichartz estimates for the wave and the Dirac equation respectively. The estimates of Lemma 2.3 are less well known but very useful. They are related to the following question. Let u solve $\Box u = G$, $u(0,\cdot) = u_0$, $\partial_t u(t,\cdot) = u_1$, with data in the energy class; i.e., $u_0 \in H^1$, $u_1 \in L^2$, $G \in L^1_t L^2_x$. Is it true that

$$\left(\int_0^T \|u(t,x)\|^2_{L^\infty_x}\, dt\right)^{\frac{1}{2}} \le C\Big[\|u_0\|_{H^1} + \|u_1\|_{L^2} + \int_0^T \|G(t,\cdot)\|_{L^2}\, dt\Big]? \tag{2.1}$$

This estimate would be very useful for many purposes. Unfortunately, it turns out that the energy can control $\|u\|_{L^2_t L^\infty_x}$ only in the spherically symmetric case (see the counterexamples in [11, 15]). Two questions arise naturally from these counterexamples. First of all, what norms can the energy actually control? And second, if we want to control $\|u\|_{L^2_t L^\infty_x}$, how much more regularity do we need? The answer to the first question is given in estimate (2.5). For $p \in (6,\infty)$ that estimate gives (set $\beta = 0$)

$$\left(\int_0^T \|u(t,\cdot)\|^{\frac{2p}{p-6}}_{L^p_x}\, dt\right)^{\frac{p-6}{2p}} \le C\Big[\|u_0\|_{H^1} + \|u_1\|_{L^2} + \int_0^T \|G(t,\cdot)\|_{L^2}\, dt\Big]. \tag{2.2}$$

This means that the energy actually controls the mixed norms $\|u\|_{L_t^r L_x^p}$, where $r = \frac{2p}{p-6}$, with arbitrarily large p, and that it fails to do so only at the endpoint $p = \infty$. The answer to the second question is given by estimate (2.9). According to that estimate, if the data have ϵ more derivatives than the energy class, then certain integrals of the L^∞ norm in space can be controlled. Indeed, setting $\alpha = 1 + \epsilon$ in (2.9) we find

$$\Big(\int_0^T \|\phi(t,\cdot)\|_{L^\infty}^p \, dt\Big)^{\frac{1}{p}} \leq CT^\theta \Big[\|\phi_0\|_{H^{1+\epsilon}} + \|\phi_1\|_{H^\epsilon} + \int_0^T \|F(t,\cdot)\|_{H^\epsilon} \, dt\Big], \tag{2.3}$$

where ϵ is an arbitrarily small positive number, $p \in (2, \frac{2}{1-2\epsilon})$, and θ is a positive number depending only on p and ϵ.

Lemma 2.1. (Strichartz estimates for the wave equation) *Let u solve*

$$\Box u = G, \ u(0,\cdot) = u_0, \ \partial_t u(0,\cdot) = u_1.$$

(1) *If $\beta \in [0,\infty)$, $0 \leq \alpha < 1$, $p = \frac{2}{1-\alpha}$, and $q = \frac{2p}{p-2}$, then*

$$\Big(\int_0^T \|u(t,\cdot)\|_{H_p^\beta}^q \, dt\Big)^{\frac{1}{q}} \leq C\Big[\|u_0\|_{H^{\alpha+\beta}} + \|u_1\|_{H^{\alpha+\beta-1}} + \int_0^T \|G(t,\cdot)\|_{H^{\alpha+\beta-1}} \, dt\Big], \tag{2.4}$$

where C is independent of T.

(2) *If $\beta \in [0,\infty)$ and $6 < p < \infty$, then*

$$\Big(\int_0^T \|u(t,\cdot)\|_{H_p^\beta}^{\frac{2p}{p-6}} \, dt\Big)^{\frac{p-6}{2p}} \leq C\Big[\|u_0\|_{H^{1+\beta}} + \|u_1\|_{H^\beta} + \int_0^T \|G(t,\cdot)\|_{H^\beta} \, dt\Big], \tag{2.5}$$

where C is independent of T.

Proof. Estimate (2.4) in the special case $\beta = 0$ is the standard Strichartz estimate; see for example [13]. The general case follows by applying this special case to $(I - \Delta)^\beta u$.

Estimate (2.5) with $\beta = 0$ is due to Pecher [17]. Also see [13] and the discussion on p. 101 of [20]. The general case follows by applying this special case to $(I - \Delta)^\beta u$. □

Lemma 2.2. (Strichartz estimates for the Dirac equation) *Let the 4-spinor ψ solve*

$$\mathcal{D}\psi = F, \ \psi(0,\cdot) = \psi_0.$$

(1) *If* $\beta \in [0, \infty)$, $0 \le \alpha < 1$, $p = \frac{2}{1-\alpha}$, *and* $q = \frac{2p}{p-2}$, *then*

$$\Big(\int_0^T \|\psi(t,\cdot)\|^q_{H^\beta_p} \, dt \Big)^{\frac{1}{q}} \le C \Big[\|\psi_0\|_{H^{\alpha+\beta}} + \int_0^T \|F(t,\cdot)\|_{H^{\alpha+\beta-1}} \, dt \Big], \qquad (2.6)$$

where C *is independent of* T.

(2) *If* $\beta \in [0, \infty)$ *and* $6 < p < \infty$, *then*

$$\Big(\int_0^T \|\psi(t,\cdot)\|^{\frac{2p}{p-6}}_{H^\beta_p} \, dt \Big)^{\frac{p-6}{2p}} \le C \Big[\|\psi_0\|_{H^{1+\beta}} + \int_0^T \|F(t,\cdot)\|_{H^\beta} \, dt \Big], \qquad (2.7)$$

where C *is independent of* T.

Proof. In the special case $\mathcal{D}\psi = 0$ and $\psi(0,\cdot) = \psi_0$, use $\mathcal{D}^2 = \Box$ to conclude that $\Box\psi = 0$ with initial data $\psi(0,\cdot) = \psi_0$ and $\partial_t \psi(0,\cdot) = -\gamma^0\gamma^j\partial_j\psi_0$. The result then follows from Lemma 2.1. The general case follows from this special one and Duhamel's principle. □

Lemma 2.3. *Let* ψ *solve*

$$\mathcal{D}\psi = F, \quad \psi(0,\cdot) = \psi_0,$$

and let ϕ *solve*

$$\Box\phi = G, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1.$$

Let $\alpha \in (1, \frac{3}{2}]$ *and* $p \in (2, \infty)$, *with* $\frac{3}{2} - \alpha < \frac{1}{p}$. *Then there are* $\delta > 0$ *and* $C > 0$ *depending only on* α *and* p, *such that for all* $T > 0$,

$$\Big(\int_0^T \|\psi(t,\cdot)\|^p_{L^\infty} \, dt \Big)^{\frac{1}{p}} \le CT^\theta \Big[\|\psi_0\|_{H^\alpha} + \int_0^T \|F(t,\cdot)\|_{H^\alpha} \, dt \Big] \qquad (2.8)$$

$$\Big(\int_0^T \|\phi(t,\cdot)\|^p_{L^\infty} \, dt \Big)^{\frac{1}{p}} \le CT^\theta \Big[\|\phi_0\|_{H^\alpha} + \|\phi_1\|_{H^{\alpha-1}} + \int_0^T \|F(t,\cdot)\|_{H^{\alpha-1}} \, dt \Big]. \qquad (2.9)$$

Proof. See [7]. □

We shall also make frequent use of the following "fractional Leibniz rule" to estimate various products.

Lemma 2.4. (Fractional Leibniz Rule) *Let* $\mu \ge 0$, $1 \le l, p, s < \infty$, *and* $1 \le q, r \le \infty$, *with* $\frac{1}{l} = \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{1}{s}$. *Let* $f \in H^\mu_p \cap L^r$ *and* $g \in H^\mu_q \cap L^s$. *Then* $fg \in H^\mu_l$ *and*

$$\|fg\|_{H^\mu_l} \le C \left[\|f\|_{H^\mu_p} \|g\|_{L^q} + \|f\|_{L^r} \|g\|_{H^\mu_s} \right]. \qquad (2.10)$$

Proof. See Lemma 4.2 in [9] and the Appendix of [8]. □

As an immediate consequence we have the following

Lemma 2.5. *Let $\mu \geq 0$, $1 \leq p \leq \infty$ and $1 \leq r, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Let P be a homogeneous polynomial of degree d. Then, for any $\psi \in H_q^\mu \cap L^{(d-1)p}$ we have $P(\psi) \in H_r^\mu$ and*

$$\|P(\psi)\|_{H_r^\mu} \leq C \, \|\psi\|_{L^{(d-1)p}}^{d-1} \, \|\psi\|_{H_q^\mu} \, . \tag{2.11}$$

With $p = \infty$ this becomes

$$\|P(\psi)\|_{H_q^\mu} \leq C \, \|\psi\|_{L^\infty}^{d-1} \, \|\psi\|_{H_q^\mu} \, . \tag{2.12}$$

Proof. The proof is by induction on d. For a more general version see Proposition 25 in [5], and also [6]. □

We shall also use the following estimates for differences. The proofs are similar to those of the estimates of Lemma 2.5 and are again based on the fractional Leibniz rule of Lemma 2.4.

Lemma 2.6. *Let $\mu \geq 0$, $1 \leq p, a, b \leq \infty$, and $1 \leq q, c < \infty$, with $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{2}$. Let P be a homogeneous polynomial of degree d. Then,*

$$\begin{aligned} \|P(\psi_1) - P(\psi_2)\|_{H^\mu} &\leq C \, \|\psi_1 - \psi_2\|_{H_q^\mu} \left[\|\psi_1\|_{L^{p(d-1)}}^{d-1} + \|\psi_2\|_{L^{p(d-1)}}^{d-1} \right] \\ &+ \|\psi_1 - \psi_2\|_{L^a} \left[\|\psi_1\|_{L^{b(d-2)}}^{d-2} + \|\psi_2\|_{L^{b(d-2)}}^{d-2} \right] \, \left[\|\psi_1\|_{H_c^\mu} + \|\psi_2\|_{H_c^\mu} \right] . \end{aligned} \tag{2.13}$$

In particular ($q = c = 2$ and $p = a = b = \infty$),

$$\begin{aligned} \|P(\psi_1) - P(\psi_2)\|_{H^\mu} &\leq C \, \|\psi_1 - \psi_2\|_{H^\mu} \left[\|\psi_1\|_{L^\infty}^{d-1} + \|\psi_2\|_{L^\infty}^{d-1} \right] \\ &+ \|\psi_1 - \psi_2\|_{L^\infty} \left[\|\psi_1\|_{L^\infty}^{d-2} + \|\psi_2\|_{L^\infty}^{d-2} \right] \, \left[\|\psi_1\|_{H^\mu} + \|\psi_2\|_{H^\mu} \right] . \end{aligned} \tag{2.14}$$

Finally, we state the generalized charge, energy, and Sobolev estimates we shall need. The proofs are well known.

Lemma 2.7. (Generalized Charge Estimate) *Let $\mu \in \mathbb{R}$. If ψ solves*

$$\mathcal{D}\psi = F, \quad \psi(0, \cdot) = \psi_0,$$

then for all t,

$$\|\psi(t, \cdot)\|_{H^\mu} \leq C \Big[\, \|\psi_0\|_{H^\mu} + \int_0^t \|F(\tau, \cdot)\|_{H^\mu} \, d\tau \Big] \tag{2.15}$$

with C independent of t.

Lemma 2.8. (Generalized Energy Estimate) *Let $\nu \in \mathbb{R}$. If ϕ solves*

$$\Box\phi = G, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1,$$

then for all t,

$$\begin{aligned} &\|\phi(t,\cdot)\|_{H^\nu} + \|\partial_t\phi(t,\cdot)\|_{H^{\nu-1}} \\ &\qquad \leq C\ (1+|t|)\Big[\ \|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}} + \int_0^t \|G(\tau,\cdot)\|_{H^{\nu-1}}\,d\tau\Big] \end{aligned} \tag{2.16}$$

with C independent of t.

Lemma 2.9. (Sobolev Embeddings) *Let $\mu \in (0,\infty)$ and $1 \leq p \leq q < \infty$, and suppose $\frac{3}{q} = -\mu + \frac{3}{p}$. Then $H_p^\mu \hookrightarrow L^q$.*

3. Local low-regularity solutions for large data

Theorem 3.1. *Let γ and κ be positive integers* ([5]) *such that*

$$\gamma \geq 2\ , \quad \kappa \geq 2,\ 2\kappa+1 < 4\gamma,\ 2\gamma \leq \kappa+2. \tag{3.1}$$

Let $\mu = \mu_{cr} + \epsilon$ and $\nu = \nu_{cr}$, where

$$0 < \epsilon < \frac{4\gamma - 2\kappa - 1}{2(\gamma+\kappa-1)}. \tag{3.2}$$

Fix initial data

$$\psi_0 \in H^\mu\left(\mathbb{R}^3\right),\ \phi_0 \in H^\nu\left(\mathbb{R}^3\right),\ \phi_1 \in H^{\nu-1}\left(\mathbb{R}^3\right). \tag{3.3}$$

Then there exists $T > 0$, depending only on γ, κ, ϵ, and the quantity $D_0 = \|\psi_0\|_{H^\mu} + \|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}}$, such that the Cauchy problem for the generalized Dirac-Klein-Gordon equations has a unique solution (ψ,ϕ) with

$$\phi \in C^0\left([0,T], H^\nu\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,T], H^{\nu-1}\left(\mathbb{R}^3\right)\right) \tag{3.4}$$

$$\int_0^T \|\Box\phi\|_{H^{\nu-1}}\,dt < \infty \tag{3.5}$$

$$\psi \in C^0\left([0,T], H^\mu\left(\mathbb{R}^3\right)\right) \tag{3.6}$$

$$\int_0^T \|\mathcal{D}\psi\|_{H^\mu}\,dt < \infty. \tag{3.7}$$

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[5]Because of the assumption that γ and κ are both integers, the only possibilities allowed by (3.1) are $4\gamma = 2\kappa+2,\ 2\kappa+4$; i.e., $2\gamma = \kappa+1\ ,\ \kappa+2$. See however Remark 3.2 on page 19.

Proof. The mass terms $M\psi$ and $m^2\phi$ can be moved to the right-hand side of the equations where they can easily be treated as lower-order terms. So, for the sake of simplicity, we take $M = m = 0$. We note for future use that our assumptions imply

$$\epsilon \le 1/2,\ 1 < \mu < \frac{3}{2},\quad 1 < \nu \le \frac{3}{2},\quad \mu < \nu < \mu + 1. \tag{3.8}$$

Fix a number $T \in (0,1)$, to be determined later. Some of the constants denoted by C in this proof depend on T. Since $T < 1$ and these constants are bounded for bounded T we may replace them all by absolute constants.

Define

$$D_0 = \|\psi_0\|_{H^\mu} + \|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}}\,. \tag{3.9}$$

Define X to be the space of all pairs (ψ, ϕ) such that

$$\psi \in C^0\left([0,T] \to H^\mu\right),\ \phi \in C^0\left([0,T] \to H^\nu\right) \cap C^1\left([0,T] \to H^{\nu-1}\right) \tag{3.10a}$$

$$\psi(0,\cdot) = \psi_0, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1 \tag{3.10b}$$

$$\int_0^T \|\mathcal{D}\psi(t,\cdot)\|_{H^\mu}\, dt \le D_0,\ \int_0^T \|\Box\phi(t,\cdot)\|_{H^{\nu-1}}\, dt \le D_0. \tag{3.10c}$$

We equip X with the metric ([6])

$$\begin{aligned} d\left((\psi_1,\phi_1),(\psi_2,\phi_2)\right) = \sup_{t\in[0,T]} [\ &\|\psi_1(t,\cdot) - \psi_2(t,\cdot)\|_{H^\mu} \\ &+ \|\phi_1(t,\cdot) - \phi_2(t,\cdot)\|_{H^\nu} + \|\partial_t\phi_1(t,\cdot) - \partial_t\phi_2(t,\cdot)\|_{H^{\nu-1}}] \\ + \int_0^T \|\mathcal{D}\psi_1(t,\cdot) - \mathcal{D}\psi_2(t,\cdot)\|_{H^\mu}\, dt &+ \int_0^T \|\Box\phi_1(t,\cdot) - \Box\phi_2(t,\cdot)\|_{H^\mu}\, dt. \end{aligned} \tag{3.11}$$

Equipped with d, X is a complete metric space.

The solution will be constructed as a fixed point of the following map:

$$X \to X, \qquad (\psi,\phi) \mapsto (\Psi,\Phi), \tag{3.12}$$

where, given $(\psi,\phi) \in X$, the pair (Ψ,Φ) is the unique solution of

$$\mathcal{D}\Psi = \phi\gamma^0\left(\bar\psi\psi\right)^{\gamma-1}\psi = \phi f(\psi) \tag{3.13a}$$

$$\Box\Phi = \left(\bar\psi\psi\right)^\kappa = g(\psi) \tag{3.13b}$$

$$\Psi(0,\cdot) = \psi_0 \tag{3.13c}$$

$$\Phi(0,\cdot) = \phi_0, \quad \partial_t\Phi(0,\cdot) = \phi_1. \tag{3.13d}$$

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[6]Although the last line in (3.11) dominates the first two via the generalized charge and energy estimates, it turns out that it is more convenient to define our metric in this way.

We claim that, if T is sufficiently small, then (3.12) does map X into X, and it is a contraction.

Fix $(\psi, \phi) \in X$. We first show that $(\Psi, \Phi) \in X$. By the generalized charge estimate of Lemma 2.7 and the generalized energy estimate of Lemma 2.8,

$$\sup_{t\in[0,T]} \Big[\|\Psi(t,\cdot)\|_{H^\mu} + \|\Phi(t,\cdot)\|_{H^\nu} + \|\partial_t\Phi(t,\cdot)\|_{H^{\nu-1}} \Big] \leq C\Big[D_0 + \int_0^T \|\mathcal{D}\Psi(t,\cdot)\|_{H^\mu}\,dt + \int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}}\,dt\Big]. \quad (3.14)$$

We estimate the first of the two integrals in the right-hand side of (3.14) as follows. Define p, q, and l by

$$\frac{1}{p} = \frac{1-(\nu-\mu)}{2},\ \ \frac{1}{q} = \frac{\nu-\mu}{2},\ \ \frac{1}{l} = \frac{\nu-\mu}{2(2\gamma-1)} + \frac{\mu-1}{3}. \quad (3.15)$$

Then

$$p, q \in (2,\infty),\ \ l \in (2,\infty),\quad \frac{1}{p} + \frac{1}{q} = \frac{1}{2}. \quad (3.16)$$

The first and the last of these are obvious, and the middle one follows from the fact that $\epsilon < 1$ and

$$\frac{1}{2} - \frac{1}{l} = \frac{4\gamma-5}{6(2\gamma-1)}\Big[1-\epsilon + \frac{6\gamma+12\kappa-9}{2(\gamma+\kappa-1)(4\gamma-5)}\Big]. \quad (3.17)$$

By the fractional Leibniz rule of Lemma 2.4

$$\int_0^T \|\mathcal{D}\Psi\|_{H^\mu}\,dt = \int_0^T \|\phi f(\psi)\|_{H^\mu}\,dt \quad (3.18)$$

$$\leq \int_0^T \|\phi\|_{H^\mu_p}\|f(\psi)\|_{L^q}\,dt + \int_0^T \|\phi\|_{L^\infty}\|f(\psi)\|_{H^\mu}\,dt =: I_1 + I_2. \quad (3.19)$$

We first estimate I_1. Observe that

$$\frac{3}{q(2\gamma-1)} = -(\mu-1) + \frac{3}{l}\ ,\quad 1 \leq l \leq q(2\gamma-1) < \infty. \quad (3.20)$$

Therefore, recalling that $f(\psi)$ is a homogeneous polynomial in ψ of degree $2\gamma-1$ and using the Sobolev embedding $H^{\mu-1}_l \hookrightarrow L^{q(2\gamma-1)}$, we obtain

$$\|f(\psi(t,\cdot))\|_{L^q} \leq C\,\|\psi(t,\cdot)\|^{2\gamma-1}_{L^{q(2\gamma-1)}} \leq C\,\|\psi(t,\cdot)\|^{2\gamma-1}_{H^{\mu-1}_l}. \quad (3.21)$$

Using this we get

$$I_1 \leq C\int_0^T \|\phi\|_{H^\mu_p}\|\psi\|^{2\gamma-1}_{H^{\mu-1}_l}\,dt. \quad (3.22)$$

If $l > 6$ we continue as follows. By Hölder's inequality,

$$I_1 \leq CT^{2(\gamma-1)\epsilon}\Big(\int_0^T \|\phi\|_{H_p^\mu}^q \, dt\Big)^{\frac{1}{q}}\Big(\int_0^T \|\psi\|_{H_l^{\mu-1}}^{\frac{2l}{l-6}} \, dt\Big)^{\frac{(l-6)(2\gamma-1)}{2l}}, \tag{3.23}$$

where we have used the fact that

$$2(\gamma-1)\epsilon + \frac{1}{q} + \frac{(l-6)(2\gamma-1)}{2l} = 1. \tag{3.24}$$

By the Strichartz estimates (2.4) with $\beta = \mu$ and $\alpha = 2(\frac{1}{2} - \frac{1}{p}) = \nu - \mu$, we have

$$\Big(\int_0^T \|\phi\|_{H_p^\mu}^q \, dt\Big)^{\frac{1}{q}} \leq C\Big[\, \|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}} + \int_0^T \|\Box\phi\|_{H^{\nu-1}} \, dt\Big] \leq CD_0. \tag{3.25}$$

By the Strichartz estimates (2.7),

$$\Big(\int_0^T \|\psi\|_{H_l^{\mu-1}}^{\frac{2l}{l-6}} \, dt\Big)^{\frac{(l-6)(2\gamma-1)}{2l}} \leq C\Big[\, \|\psi_0\|_{H^\mu} + \int_0^T \|\mathcal{D}\psi\|_{H^\mu} \, dt\Big]^{2\gamma-1} \leq CD_0^{2\gamma-1}. \tag{3.26}$$

Using (3.25) and (3.26) in (3.23) we obtain

$$I_1 \leq CT^{2(\gamma-1)\epsilon} D_0^{2\gamma}. \tag{3.27}$$

This completes the estimate of I_1 in the case $l > 6$. If $2 < l \leq 6$ we can simply use the Sobolev embedding $H^\mu \hookrightarrow H_l^{\mu-1}$ to continue (3.22) as follows:

$$\begin{aligned} I_1 &\leq C\int_0^T \|\phi\|_{H_p^\mu} \|\psi\|_{H^\mu}^{2\gamma-1} \, dt \leq C \sup_{t\in[0,T]} \|\psi(t,\cdot)\|_{H^\mu}^{2\gamma-1} \int_0^T \|\phi\|_{H_p^\mu} \, dt \\ &\leq C \sup_{t\in[0,T]} \|\psi(t,\cdot)\|_{H^\mu}^{2\gamma-1} \, T^{1-\frac{1}{q}} \Big(\int_0^T \|\phi\|_{H_p^\mu}^q \, dt\Big)^{\frac{1}{q}}. \end{aligned} \tag{3.28}$$

For the ϕ-term we use (3.25) exactly as before. For the ψ-term we use the generalized charge estimate of Lemma 2.7 to get

$$\sup_{t\in[0,T]} \|\psi(t,\cdot)\|_{H^\mu} \leq C\Big[\, \|\psi_0\|_{H^\mu} + \int_0^T \|\mathcal{D}\psi\|_{H^\mu} \, dt\Big] \leq CD_0. \tag{3.29}$$

Therefore,

$$I_1 \leq CT^{1-\frac{1}{q}} D_0^{2\gamma}. \tag{3.30}$$

Next, we estimate I_2. Using first Lemma 2.5 and then (3.29) we obtain

$$\|f(\psi(t,\cdot))\|_{H^\mu} \leq C\,\|\psi(t,\cdot)\|_{L^\infty}^{2\gamma-2} \|\psi(t,\cdot)\|_{H^\mu} \leq C\,\|\psi(t,\cdot)\|_{L^\infty}^{2\gamma-2} D_0; \tag{3.31}$$

therefore,

$$I_2 \leq CD_0 \int_0^T \|\phi\|_{L^\infty} \|\psi\|_{L^\infty}^{2\gamma-2} \, dt. \tag{3.32}$$

Observe that, since $1 < \mu < \frac{3}{2}$ and $1 < \nu \leq \frac{3}{2}$, we have $0 < \frac{3}{2} - \mu < \frac{1}{2}$ and $0 \leq \frac{3}{2} - \nu < \frac{1}{2}$. Also

$$2(\gamma-1)\big(\frac{3}{2} - \mu\big) + \big(\frac{3}{2} - \nu\big) = 1 - 2(\gamma-1)\epsilon < 1.$$

Therefore, we can find $A, B \in (2, \infty)$ such that

$$\frac{3}{2} - \mu < \frac{1}{A} < \frac{1}{2}, \quad \frac{3}{2} - \nu < \frac{1}{B} < \frac{1}{2}, \quad 2(\gamma-1)\frac{1}{A} + \frac{1}{B} < 1. \tag{3.33}$$

Then

$$\int_0^T \|\phi\|_{L^\infty} \|\psi\|_{L^\infty}^{2\gamma-2} \, dt \leq T^{\delta_1} \Big(\int_0^T \|\phi\|_{L^\infty}^B \, dt \Big)^{\frac{1}{B}} \Big(\int_0^T \|\psi\|_{L^\infty}^A \, dt \Big)^{\frac{2\gamma-2}{A}}, \tag{3.34}$$

where $\delta_1 = 1 - \big(\frac{2\gamma-2}{A} + \frac{1}{B}\big)$. By (2.9) there exists $\delta_2 > 0$ such that

$$\Big(\int_0^T \|\phi\|_{L^\infty}^B \, dt \Big)^{\frac{1}{B}} \leq CT^{\delta_2} \Big[\|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}} + \int_0^T \|\Box\phi\|_{H^{\nu-1}} \, dt \Big] \leq CT^{\delta_2} D_0, \tag{3.35}$$

and, similarly, by (2.8), there exists $\delta_3 > 0$ such that

$$\Big(\int_0^T \|\psi\|_{L^\infty}^A \, dt \Big)^{\frac{1}{A}} \leq CT^{\delta_3} \Big[\|\psi_0\|_{H^\mu} + \int_0^T \|\mathcal{D}\psi\|_{H^\mu} \, dt \Big] \leq CT^{\delta_3} D_0; \tag{3.36}$$

therefore,

$$\int_0^T \|\phi\|_{L^\infty} \|\psi\|_{L^\infty}^{2\gamma-2} \, dt \leq CT^\delta D_0^{2\gamma-1}, \tag{3.37}$$

where $\delta = \delta_1 + \delta_2 + 2\delta_3(\gamma - 1)$. Using (3.37) in (3.32) we conclude that

$$I_2 \leq CT^\delta D_0^{2\gamma}. \tag{3.38}$$

From (3.19), (3.27), (3.30), and (3.38) we conclude that

$$\int_0^T \|\mathcal{D}\Psi(t,\cdot)\|_{H^\mu} \, dt \leq CT^{\sigma_1} D_0^{2\gamma} = CT^{\sigma_1} D_0^{2\gamma-1} D_0, \tag{3.39}$$

where $\sigma_1 = \min\{1 - \frac{1}{q}, 2(\gamma-1)\epsilon, \delta\}$. Therefore, if T is small enough so that

$$CT^{\sigma_1} D_0^{2\gamma-1} \leq 1, \tag{3.40}$$

then

$$\int_0^T \|\mathcal{D}\Psi(t,\cdot)\|_{H^\mu} \, dt \leq D_0 \tag{3.41}$$

as required.

Next we estimate the second integral on the right-hand side of (3.14). Using the fractional Leibniz rule (2.11) we have

$$\int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}}\,dt = \int_0^T \|g(\psi)\|_{H^{\nu-1}}\,dt \le C\int_0^T \|\psi\|^{2\kappa-1}_{L^{(2\kappa-1)p}}\|\psi\|_{H^{\nu-1}_q}\,dt, \tag{3.42}$$

where p and q are as in (3.15). Define r by

$$\frac{1}{r} = \frac{1}{(2\kappa-1)p} + \frac{\mu-1}{3} = \frac{1-(\nu-\mu)}{2(2\kappa-1)} + \frac{\mu-1}{3}. \tag{3.43}$$

By Sobolev,

$$\|\psi(t,\cdot)\|_{L^{(2\kappa-1)p}} \le C\,\|\psi(t,\cdot)\|_{H^{\mu-1}_r}; \tag{3.44}$$

therefore,

$$\int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}}\,dt \le C\int_0^T \|\psi\|^{2\kappa-1}_{H^{\mu-1}_r}\|\psi\|_{H^{\nu-1}_q}\,dt. \tag{3.45}$$

Notice that $r\in(2,\infty)$. Indeed,

$$\frac{1}{2} - \frac{1}{r} = \frac{2(2\kappa-1)\left(\frac{3}{2}-\mu\right) + (4\kappa-5) + 3(\nu-\mu)}{6(2\kappa-1)} > 0. \tag{3.46}$$

If $6<r<\infty$ we continue (3.45) as follows:

$$\int_0^T \|\Box\Phi\|_{H^{\nu-1}}\,dt \le CT^{2\kappa\epsilon}\Big(\int_0^T \|\psi\|^{\frac{2r}{r-6}}_{H^{\mu-1}_r}\,dt\Big)^{\frac{(r-6)(2\kappa-1)}{2r}}\Big(\int_0^T \|\psi\|^p_{H^{\nu-1}_q}\,dt\Big)^{\frac{1}{p}}, \tag{3.47}$$

where we have used the fact that

$$2\kappa\epsilon + \frac{(r-6)(2\kappa-1)}{2r} + \frac{1}{p} = 1.$$

By (2.7),

$$\Big(\int_0^T \|\psi\|^{\frac{2r}{r-6}}_{H^{\mu-1}_r}\,dt\Big)^{\frac{r-6}{2r}} \le C\Big[\|\psi_0\|_{H^\mu} + \int_0^T \|\mathcal{D}\psi\|_{H^\mu}\,dt\Big] \le CD_0. \tag{3.48}$$

By (2.6) (with $\beta=\nu-1$ and $\alpha = 2(\frac{1}{2}-\frac{1}{q}) = 1-\nu+\mu$),

$$\Big(\int_0^T \|\psi\|^p_{H^{\nu-1}_q}\,dt\Big)^{\frac{1}{p}} \le C\Big[\|\psi_0\|_{H^\mu} + \int_0^T \|\mathcal{D}\psi\|_{H^\mu}\,dt\Big] \le CD_0. \tag{3.49}$$

Therefore,

$$\int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}}\,dt \le CT^{2\kappa\epsilon}D_0^{2\kappa}. \tag{3.50}$$

This completes the case $r > 6$. If $2 < r \leq 6$ we have an easier task since we can then use a simple Sobolev embedding theorem and estimate

$$\|\psi(t,\cdot)\|_{H_r^{\mu-1}} \leq C \|\psi(t,\cdot)\|_{H^\mu}.$$

Then we can continue (3.45) as follows:

$$\int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}} dt \leq C \sup_{t\in[0,T]} \|\psi(t,\cdot)\|_{H^\mu}^{2\kappa-1} \int_0^T \|\psi\|_{H_q^{\nu-1}} dt$$
$$\leq C \sup_{t\in[0,T]} \|\psi(t,\cdot)\|_{H^\mu}^{2\kappa-1} T^{1-\frac{1}{p}} \Big(\int_0^T \|\psi\|_{H_q^{\nu-1}}^p dt \Big)^{\frac{1}{p}} \leq CT^{1-\frac{1}{p}} D_0^{2\kappa}, \quad (3.51)$$

where we have used (3.49) in the last step. We have shown that

$$\int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}} dt \leq CT^{\sigma_2} D_0^{2\kappa} = CT^{\sigma_2} D_0^{2\kappa-1} D_0, \quad (3.52)$$

where $\sigma_2 = \min\{1-\frac{1}{p}, 2\kappa\epsilon\}$. Therefore, if T is small enough so that

$$CT^{\sigma_2} D_0^{2\kappa-1} \leq 1, \quad (3.53)$$

then

$$\int_0^T \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}} dt \leq D_0 \quad (3.54)$$

as required.

We have shown that (3.12) does map X into X. We now show that, if T is sufficiently small, it is a contraction. Let $(\psi_1,\phi_1),(\psi_2,\phi_2) \in X$ and let (Ψ_1,Φ_1) and (Ψ_2,Φ_2) be their images under the map (3.12). Then

$$d\left((\Psi_1,\Phi_1),(\Psi_2,\Phi_2)\right)$$
$$\leq C\Big[\int_0^T \|\mathcal{D}\Psi_1 - \mathcal{D}\Psi_2\|_{H^\mu} dt + \int_0^T \|\Box\Phi_1 - \Box\Phi_2\|_{H^{\nu-1}} dt\Big]. \quad (3.55)$$

We have

$$\mathcal{D}\Psi_1 - \mathcal{D}\Psi_2 = (\phi_1-\phi_2) f(\psi_1) + \phi_2 \left(f(\psi_1) - f(\psi_2)\right). \quad (3.56)$$

For the first term we have

$$\int_0^T \|(\phi_1-\phi_2) f(\psi_1)\|_{H^\mu} dt \leq \int_0^T \|\phi_1-\phi_2\|_{H_p^\mu} \|f(\psi_1)\|_{L^q} dt$$
$$+ \int_0^T \|\phi_1-\phi_2\|_{L^\infty} \|f(\psi_1)\|_{H^\mu} dt =: A_1 + A_2, \quad (3.57)$$

where p and q are as in (3.15). The estimates for A_1 and A_2 are exactly the same as the estimates for I_1 and I_2 respectively, with $\phi_1 - \phi_2$ in the role of

ϕ (note that $\phi_1 - \phi_2$ has zero initial data) and ψ_1 in the role of ψ. We get, for $i = 1, 2$,

$$A_i \leq CT^{\theta_1} D_0^{2\gamma-1} \int_0^T \|\Box\phi_1 - \Box\phi_2\|_{H^{\nu-1}}\, dt \leq CT^{\theta_1} D_0^{2\gamma-1} d\left((\psi_1, \phi_1), (\psi_2, \phi_2)\right) \tag{3.58}$$

for some positive number θ_1. We are done with the first term in the right-hand side of (3.56). For the second term we have

$$\int_0^T \|\phi_2 \left(f(\psi_2) - f(\psi_1)\right)\|_{H^\mu}\, dt \leq \int_0^T \|\phi_2\|_{H_p^\mu} \|f(\psi_2) - f(\psi_1)\|_{L^q}\, dt$$
$$+ \int_0^T \|\phi_2\|_{L^\infty} \|f(\psi_2) - f(\psi_1)\|_{H^\mu}\, dt =: B_1 + B_2 \tag{3.59}$$

with the same p and q as above. We first estimate B_1. Recall that $f(\psi)$ is a homogeneous polynomial in ψ of degree $2\gamma - 1$. Therefore,

$$\begin{aligned} \|f(\psi_2) - f(\psi_1)\|_{L^q} &\leq C \left\| |\psi_1 - \psi_2| \left(|\psi_1|^{2\gamma-2} + |\psi_2|^{2\gamma-2} \right) \right\|_{L^q} \\ &\leq C \|\psi_1 - \psi_2\|_{L^{q(2\gamma-1)}} \left[\|\psi_1\|_{L^{q(2\gamma-1)}}^{2\gamma-2} + \|\psi_2\|_{L^{q(2\gamma-1)}}^{2\gamma-2} \right] \\ &\leq C \|\psi_1 - \psi_2\|_{H_l^{\mu-1}} \left[\|\psi_1\|_{H_l^{\mu-1}}^{2\gamma-2} + \|\psi_2\|_{H_l^{\mu-1}}^{2\gamma-2} \right], \end{aligned}$$

where we have used $H_l^{\mu-1} \hookrightarrow L^{q(2\gamma-1)}$ as in (3.21). Therefore,

$$B_1 \leq C \int_0^T \|\phi_2\|_{H_p^\mu} \|\psi_1 - \psi_2\|_{H_l^{\mu-1}} \|\psi_1\|_{H_l^{\mu-1}}^{2\gamma-2}\, dt$$
$$+ \int_0^T \|\phi_2\|_{H_p^\mu} \|\psi_1 - \psi_2\|_{H_l^{\mu-1}} \|\psi_2\|_{H_l^{\mu-1}}^{2\gamma-2}\, dt := B_{11} + B_{12}. \tag{3.60}$$

We estimate B_{1i} $(i = 1, 2)$. Define l as in (3.15). If $6 < l < \infty$,

$$B_{1i} \leq CT^{2(\gamma-1)\epsilon} \Big(\int_0^T \|\phi_2\|_{H_p^\mu}^q\, dt \Big)^{\frac{1}{q}} \Big(\int_0^T \|\psi_1 - \psi_2\|_{H_l^{\mu-1}}^{\frac{2l}{l-6}}\, dt \Big)^{\frac{l-6}{2l}} \cdot$$
$$\cdot \Big(\int_0^T \|\psi_i\|_{H_l^{\mu-1}}^{\frac{2l}{l-6}}\, dt \Big)^{\frac{(l-6)(2\gamma-2)}{2l}}. \tag{3.61}$$

Working as in (3.25) and (3.26) we have

$$\Big(\int_0^T \|\phi_2\|_{H_p^\mu}^q\, dt \Big)^{\frac{1}{q}} \leq CD_0\ ,\ \Big(\int_0^T \|\psi_i\|_{H_l^{\mu-1}}^{\frac{2l}{l-6}}\, dt \Big)^{\frac{(l-6)(2\gamma-2)}{2l}} \leq CD_0^{2\gamma-2}.$$

On the other hand, by (2.7),

$$\Big(\int_0^T \|\psi_1-\psi_2\|_{H_l^{\mu-1}}^{\frac{2l}{l-6}}\,dt\Big)^{\frac{l-6}{2l}} \le C\int_0^T \|\mathcal{D}\psi_1-\mathcal{D}\psi_2\|_{H^\mu}\,dt \le Cd\left((\psi_1,\phi_1),(\psi_2,\phi_2)\right).$$

Therefore,

$$B_{1i}\le CT^{2(\gamma-1)\epsilon}D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \tag{3.62}$$

If, on the other hand, $2<l\le 6$, we use $H^\mu\hookrightarrow H_l^{\mu-1}$ to get

$$\begin{aligned} B_{1i}&\le C\int_0^T \|\phi_2\|_{H_p^\mu}\|\psi_1-\psi_2\|_{H^\mu}\|\psi_1\|_{H^\mu}^{2\gamma-2}\,dt\\ &\le C\sup_{t\in[0,T]}\|\psi_1(t,\cdot)-\psi_2(t,\cdot)\|_{H^\mu}\sup_{t\in[0,T]}\|\psi_i(t,\cdot)\|_{H^\mu}^{2\gamma-2}T^{1-\frac{1}{q}}\Big(\int_0^T\|\phi_2\|_{H_p^\mu}^q\,dt\Big)^{\frac{1}{q}}\\ &\le CT^{1-\frac{1}{q}}D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \end{aligned}$$

Therefore,

$$B_1\le CT^{\theta_2}D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right) \tag{3.63}$$

for some positive number θ_2.

Next we estimate B_2. By (2.14),

$$\begin{aligned} B_2&\le\int_0^T \|\phi_2\|_{L^\infty}\|\psi_1-\psi_2\|_{H^\mu}\Big[\|\psi_1\|_{L^\infty}^{2\gamma-2}+\|\psi_2\|_{L^\infty}^{2\gamma-2}\Big]\,dt\\ &\quad+\int_0^T \|\phi_2\|_{L^\infty}\|\psi_1-\psi_2\|_{L^\infty}\Big[\|\psi_1\|_{L^\infty}^{2\gamma-3}+\|\psi_2\|_{L^\infty}^{2\gamma-3}\Big]\ \left[\|\psi_1\|_{H^\mu}+\|\psi_2\|_{H^\mu}\right]\\ &=\sum_{i=1}^2\int_0^T \|\phi_2\|_{L^\infty}\|\psi_1-\psi_2\|_{H^\mu}\|\psi_i\|_{L^\infty}^{2\gamma-2}\,dt\\ &\quad+\sum_{i,j=1}^2\int_0^T \|\phi_2\|_{L^\infty}\|\psi_1-\psi_2\|_{L^\infty}\|\psi_i\|_{L^\infty}^{2\gamma-3}\|\psi_j\|_{H^\mu}\,dt\\ &=:\sum_{i=1}^2 B_{2i}+\sum_{i,j=1}^2 B_{2ij}. \end{aligned} \tag{3.64}$$

For B_{2i}, using an analog of (3.37), we have

$$\begin{aligned} B_{2i}&\le\sup_{t\in[0,T]}\|\psi_1(t,\cdot)-\psi_2(t,\cdot)\|_{H^\mu}\int_0^T\|\phi_2\|_{L^\infty}\|\psi_i\|_{L^\infty}^{2\gamma-2}\,dt\\ &\le CT^\delta D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \end{aligned} \tag{3.65}$$

To estimate B_{2ij}, let A and B be as in (3.33). Then

$$\begin{aligned} B_{2ij} &\le \sup_{t\in[0,T]} \|\psi_j(t,\cdot)\|_{H^\mu} \int_0^T \|\phi_2\|_{L^\infty} \|\psi_1-\psi_2\|_{L^\infty} \|\psi_i\|_{L^\infty}^{2\gamma-3}\,dt \\ &\le CD_0T^{\delta_1}\Big(\int_0^T \|\phi_2\|_{L^\infty}^B\,dt\Big)^{\frac{1}{B}}\Big(\int_0^T \|\psi_1\|_{L^\infty}^A\,dt\Big)^{\frac{2\gamma-3}{A}}\cdot \\ &\qquad\cdot\Big(\int_0^T \|\psi_1-\psi_2\|_{L^\infty}^A\,dt\Big)^{\frac{1}{A}}. \end{aligned} \tag{3.66}$$

Exactly as in (3.35) and (3.36), there exist positive numbers δ_2 and δ_3 such that

$$\Big(\int_0^T \|\phi_2\|_{L^\infty}^B\,dt\Big)^{\frac{1}{B}} \le CT^{\delta_2}D_0, \quad \Big(\int_0^T \|\psi_1\|_{L^\infty}^A\,dt\Big)^{\frac{2\gamma-3}{A}} \le CT^{\delta_3}D_0^{2\gamma-3}.$$

By the Strichartz estimate (2.7),

$$\Big(\int_0^T\|\psi_1-\psi_2\|_{L^\infty}^A\,dt\Big)^{\frac{1}{A}} \le C\int_0^T\|\mathcal{D}\psi_1-\mathcal{D}\psi_2\|_{H^\mu}\,dt \le Cd\left((\psi_1,\phi_1),(\psi_2,\phi_2)\right).$$

Therefore,

$$B_{2ij} \le CT^{\delta_1}D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \tag{3.67}$$

Using (3.65) and (3.67) in (3.64) we conclude that

$$B_2 \le CT^{\delta_1}D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \tag{3.68}$$

Putting everything together, we have

$$\int_0^T \|\mathcal{D}\Psi_1-\mathcal{D}\Psi_2\|_{H^\mu}\,dt \le CT^\theta D_0^{2\gamma-1}d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right) \tag{3.69}$$

for some positive exponent θ. This concludes the estimate of the first term on the right-hand side of (3.55).

We consider the second integral in (3.55). Using (2.13) we have

$$\begin{aligned} &\int_0^T \|\Box\Phi_1-\Box\Phi_2\|_{H^{\nu-1}}\,dt = \int_0^T \|g(\psi_1)-g(\psi_2)\|_{H^{\nu-1}}\,dt \\ &\le C\int_0^T \|\psi_1-\psi_2\|_{H_q^{\nu-1}}\Big[\|\psi_1\|_{L^{(2\kappa-1)p}}^{2\kappa-1}+\|\psi_2\|_{L^{(2\kappa-1)p}}^{2\kappa-1}\Big]\,dt \\ &+C\int_0^T \|\psi_1-\psi_2\|_{L^{(2\kappa-1)p}}\Big[\|\psi_1\|_{L^{(2\kappa-1)p}}^{2\kappa-2}+\|\psi_2\|_{L^{(2\kappa-1)p}}^{2\kappa-2}\Big]\cdot \\ &\qquad\cdot\Big[\|\psi_1\|_{H_q^{\nu-1}}+\|\psi_2\|_{H_q^{\nu-1}}\Big]\,dt \\ &=: J_1+J_2. \end{aligned} \tag{3.70}$$

For J_1 we work exactly as in the proof of (3.52) to obtain

$$J_1 \leq CT^{\sigma_2} D_0^{2\kappa-1} d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \tag{3.71}$$

For J_2, use $H_r^{\mu-1} \hookrightarrow L^{(2\kappa-1)p}$ to get

$$\begin{aligned} J_2 &\leq C\int_0^T \|\psi_1-\psi_2\|_{H_r^{\mu-1}}\left[\|\psi_1\|_{H_r^{\mu-1}}^{2\kappa-2}+\|\psi_2\|_{H_r^{\mu-1}}^{2\kappa-2}\right]\cdot \\ &\qquad\qquad \cdot\left[\|\psi_1\|_{H_q^{\nu-1}}+\|\psi_2\|_{H_q^{\nu-1}}\right]dt \\ &\leq C\sum_{i,j=1}^{2}\int_0^T \|\psi_1-\psi_2\|_{H_r^{\mu-1}}\|\psi_i\|_{H_r^{\mu-1}}^{2\kappa-2}\|\psi_j\|_{H_q^{\nu-1}}\,dt =: \sum_{i,j=1}^{2} J_{2ij}. \end{aligned} \tag{3.72}$$

Then, if $6<r<\infty$,

$$\begin{aligned} J_{2ij} &\leq CT^{2\kappa\epsilon}\Big(\int_0^T \|\psi_1-\psi_2\|_{H_r^{\mu-1}}^{\frac{2r}{r-6}}\,dt\Big)^{\frac{r-6}{2r}}\Big(\int_0^T \|\psi_i\|_{H_r^{\mu-1}}^{\frac{2r}{r-6}}\,dt\Big)^{\frac{(r-6)(2\kappa-2)}{2r}}\cdot \\ &\qquad \cdot\Big(\int_0^T \|\psi_j\|_{H_q^{\nu-1}}^{p}\,dt\Big)^{\frac{1}{p}}. \end{aligned} \tag{3.73}$$

As in (3.48) and (3.49), we have

$$\Big(\int_0^T \|\psi_i\|_{H_r^{\mu-1}}^{\frac{2r}{r-6}}\,dt\Big)^{\frac{(r-6)(2\kappa-2)}{2r}} \leq CD_0^{2\kappa-2},\ \Big(\int_0^T \|\psi_j\|_{H_q^{\nu-1}}^{p}\,dt\Big)^{\frac{1}{p}} \leq CD_0.$$

On the other hand, (2.7) gives

$$\Big(\int_0^T\|\psi_1-\psi_2\|_{H_r^{\mu-1}}^{\frac{2r}{r-6}}\,dt\Big)^{\frac{r-6}{2r}} \leq C\int_0^T\|\mathcal{D}\psi_1-\mathcal{D}\psi_2\|_{H^\mu}\,dt \leq d((\psi_1,\phi_1),(\psi_2,\phi_2)).$$

Therefore,

$$J_{2ij} \leq CT^{2\kappa\epsilon} D_0^{2\kappa-1} d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \tag{3.74}$$

If $2<r\leq 6$, use $H^\mu \hookrightarrow H_r^{\mu-1}$ to get

$$\begin{aligned} J_{2ij} &\leq C\sup_{t\in[0,T]}\|\psi(t,\cdot)-\psi(t,\cdot)\|_{H^\mu}\cdot\sup_{t\in[0,T]}\|\psi_i(t,\cdot)\|_{H^\mu}^{2\kappa-2}\int_0^T\|\psi_j\|_{H_q^{\nu-1}}\,dt \\ &\leq CD_0^{2\kappa-2}T^{1-\frac{1}{p}}\Big(\int_0^T\|\psi_j\|_{H_q^{\nu-1}}\,dt\Big)^{\frac{1}{p}} d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right) \\ &\leq CD_0^{2\kappa-1}T^{1-\frac{1}{p}} d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right). \end{aligned} \tag{3.75}$$

Putting everything together we conclude that

$$\int_0^T \|\Box\Phi_1-\Box\Phi_2\|_{H^{\nu-1}}\,dt \leq CD_0^{2\kappa-1}T^\theta d\left((\psi_1,\phi_1),\ (\psi_2,\phi_2)\right) \tag{3.76}$$

for some positive number θ.

Using (3.76) and (3.69) in (3.55) we obtain

$$d\left(\left(\Psi_1,\Phi_1\right),\ \left(\Psi_2,\Phi_2\right)\right) \leq CT^\theta \left(D_0^{2\gamma-1} + D_0^{2\kappa-1}\right) d\left(\left(\psi_1,\phi_1\right),\ \left(\psi_2,\phi_2\right)\right). \tag{3.77}$$

Therefore, if T is small enough so that

$$CT^\theta \left(D_0^{2\gamma-1} + D_0^{2\kappa-1}\right) < 1 \tag{3.78}$$

the map (3.12) is a contraction. This completes the proof of existence. Uniqueness is proved along the same lines. □

We end this section with some remarks on the parameters γ, κ, μ, and ν.

Remark 3.2. We have assumed that γ and κ are integers so that the quantities $\left(\overline{\psi}\psi\right)^{\gamma-1}$ and $\left(\overline{\psi}\psi\right)^{\kappa}$ are well defined. However, we didn't use the special structure of $\overline{\psi}\psi$ (it is a "null form" in the sense of Klainerman; see [10, 2, 3]). All we needed was the fact that it was quadratic in ψ. Using our method, it is possible to study a system of the form

$$\mathcal{D}\psi = \phi f(\psi)\ ,\ \Box\phi = G(\psi),$$

where f and G are assumed to be smooth and satisfy certain growth estimates,

$$|f(\psi)| \leq C\,|\psi|^{2\gamma-1},\ |G(\psi)| \leq C\,|\psi|^{2\kappa}$$

with corresponding growth assumptions on their derivatives. The Leibniz rules would have to be more sophisticated [16, 18], and one needs to use Besov spaces as in [14], but the main ideas in the proof remain the same. The conditions $\gamma,\kappa \geq 2$ become $\gamma,\kappa > 1$.

Remark 3.3. The use of the generalized charge estimate requires us to deal with the term $\int_0^T \left\|\phi(\overline{\psi}\psi)^{\gamma-1}\psi\right\|_{H^\mu} dt$. Since $\phi(t,\cdot) \in H^\nu$ we need to have $\mu \leq \nu$. Similarly, the use of the generalized energy estimate brings in the term $\int_0^T \left\|(\overline{\psi}\psi)^\kappa\right\|_{H^{\nu-1}} dt$. Since $\psi(t,\cdot) \in H^\mu$ we need to have $\nu - 1 \leq \mu$. The assumptions (3.1) are designed to guarantee the stronger conditions $\mu < \nu < \mu + 1$. The endpoint cases $\mu = \nu$ and $\nu = \mu + 1$ can easily be dealt with. We have decided not to do so in this paper for the sake of simplicity. The upper bounds of $\frac{3}{2}$ for μ and ν in (3.8) are not essential, since, if μ or ν is larger than $\frac{3}{2}$, then the Sobolev inequality can be used.

4. Global small-data low-regularity solutions

In this section we work with $\epsilon = 0$, i.e., $\mu = \mu_{cr}$ and $\nu = \nu_{cr}$, but add the assumption that the quantity

$$D_0 = \|\psi_0\|_{H^\mu} + \|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}}$$

is sufficiently small and restrict ourselves to the case $2\gamma = \kappa + 1$. Our result improves that of [14] by ϵ derivatives in μ. We shall use the following endpoint version of Lemma 2.3.

Lemma 4.1. *Let ψ solve*

$$\mathcal{D}\psi = F, \quad \psi(0,\cdot) = \psi_0,$$

and let ϕ solve

$$\Box\phi = G, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1.$$

Let $\alpha \in (1, \frac{3}{2})$ and $p \in (2, \infty)$ with $\frac{3}{2} - \alpha = \frac{1}{p}$. Then

$$\Big(\int_0^\infty \|\psi(t,\cdot)\|_{L^\infty}^p \, dt\Big)^{\frac{1}{p}} \le C\Big[\, \|\psi_0\|_{H^\alpha} + \int_0^\infty \|F(t,\cdot)\|_{H^\alpha} \, dt\Big] \tag{4.1}$$

$$\Big(\int_0^\infty \|\phi(t,\cdot)\|_{L^\infty}^p \, dt\Big)^{\frac{1}{p}} \le C\Big[\, \|\phi_0\|_{H^\alpha} + \|\phi_1\|_{H^{\alpha-1}} + \int_0^\infty \|F(t,\cdot)\|_{H^{\alpha-1}} \, dt\Big]. \tag{4.2}$$

Proof. See [7]. $\Box$

Theorem 4.2. *Let $\gamma, \kappa \ge 2$ be positive integers with $2\gamma = \kappa + 1$. Let*

$$\mu = \mu_{cr} = \frac{3}{2} - \frac{2}{2(\gamma+\kappa-1)}, \;\; \nu = \nu_{cr} = \frac{7}{2} - \frac{3\kappa}{\gamma+\kappa-1}.$$

Fix initial data,

$$\psi_0 \in H^\mu\left(\mathbb{R}^3\right), \phi_0 \in H^\nu\left(\mathbb{R}^3\right), \phi_1 \in H^{\nu-1}\left(\mathbb{R}^3\right).$$

Suppose that D_0 is sufficiently small. Then the Cauchy problem (1.1) *has a unique global solution* (ψ, ϕ) *with*

$$\phi \in C^0\left([0,\infty), H^\nu\left(\mathbb{R}^3\right)\right) \cap C^1\left([0,\infty), H^{\nu-1}\left(\mathbb{R}^3\right)\right),$$
$$\int_0^\infty \|\Box\phi\|_{H^{\nu-1}} \, dt < \infty, \quad \psi \in C^0\left([0,\infty), H^\mu\left(\mathbb{R}^3\right)\right), \quad \int_0^\infty \|\mathcal{D}\psi\|_{H^\mu} \, dt < \infty.$$

Proof. The proof is similar to that of Theorem 3.1, so we shall only sketch the main steps. We define X to be the space of all pairs (ψ, ϕ) such that

$$\psi \in C^0\left([0,\infty) \to H^\mu\right), \; \phi \in C^0\left([0,\infty) \to H^\nu\right) \cap C^1\left([0,\infty) \to H^{\nu-1}\right), \tag{4.3a}$$

$$\psi(0,\cdot) = \psi_0, \quad \phi(0,\cdot) = \phi_0, \quad \partial_t\phi(0,\cdot) = \phi_1, \tag{4.3b}$$

$$\int_0^\infty \|\mathcal{D}\psi(t,\cdot)\|_{H^\mu} \, dt \le D_0 \;, \; \int_0^\infty \|\Box\phi(t,\cdot)\|_{H^{\nu-1}} \, dt \le D_0. \tag{4.3c}$$

We consider the same map (3.12), and show that, if D_0 is sufficiently small, then it maps X into X and it is a contraction. As in (3.14),

$$\sup_{t\in[0,\infty)} [\|\Psi(t,\cdot)\|_{H^\mu} + \|\Phi(t,\cdot)\|_{H^\nu} + \|\partial_t\Phi(t,\cdot)\|_{H^{\nu-1}}] \leq C\Big[D_0 + \int_0^\infty \|\mathcal{D}\Psi(t,\cdot)\|_{H^\mu}\,dt + \int_0^\infty \|\Box\Phi(t,\cdot)\|_{H^{\nu-1}}\,dt\Big]. \quad (4.4)$$

We define p, q, and l as in (3.15). The first integral in (4.4) is then estimated exactly as in (3.19). Under the present assumptions, $l \in (6,\infty)$. Using the same Strichartz estimates as before, we can show

$$I_1 \leq CD_0^{2\gamma}. \quad (4.5)$$

For I_2 we have, as in (3.32),

$$I_2 \leq CD_0 \int_0^\infty \|\phi\|_{L^\infty}\,\|\psi\|_{L^\infty}^{2\gamma-2}\,dt. \quad (4.6)$$

We have $\mu \in \left(1,\frac{3}{2}\right)$, and since $2\gamma = \kappa+1$ we have $\nu = \frac{3}{2} - \frac{2}{3\kappa-1} \in \left(1,\frac{3}{2}\right)$ ([7]). Also $2(\gamma-1)\left(\frac{3}{2}-\mu\right) + \left(\frac{3}{2}-\nu\right) = 1$. Using the endpoint Strichartz estimates of Lemma 4.1 we have

$$\begin{aligned} I_2 &\leq CD_0\Big(\int_0^T \|\phi\|_{L^\infty}^{\frac{1}{\frac{3}{2}-\nu}}\,dt\Big)^{\frac{3}{2}-\nu}\Big(\int_0^T \|\psi\|_{L^\infty}^{\frac{1}{\frac{3}{2}-\mu}}\,dt\Big)^{(2\gamma-2)\left(\frac{3}{2}-\mu\right)} \\ &\leq CD_0\Big[\,\|\phi_0\|_{H^\nu} + \|\phi_1\|_{H^{\nu-1}} + \int_0^\infty \|\Box\phi\|_{H^{\nu-1}}\,dt\Big]\cdot \\ &\quad \cdot\Big[\,\|\psi_0\|_{H^\mu} + \int_0^\infty \|\mathcal{D}\psi\|_{H^\mu}\,dt\Big]^{2\gamma-2} \leq CD_0^{2\gamma}. \end{aligned} \quad (4.7)$$

Therefore,

$$\int_0^T \|\mathcal{D}\Psi\|_{H^\mu}\,dt \leq CD_0^{2\gamma}. \quad (4.8)$$

The second integral in (4.4) can be estimated exactly as in the proof of Theorem 3.1 ($r > 6$ under our assumptions, so there is only one case to consider). We obtain

$$\int_0^T \|\Box\Phi\|_{H^{\nu-1}}\,dt \leq CD_0^{2\kappa}. \quad (4.9)$$

Therefore, if D_0 is sufficiently small, the map (3.12) maps X into X. Working similarly we obtain the contraction property of this map. □

[7] $2\gamma = \kappa + 2$ would give $\nu = \nu_{cr} = \frac{3}{2}$ and would require a different approach.

REFERENCES

[1] N. Bournaveas, *A new proof of global existence for the Dirac Klein-Gordon equations in one space dimension,* J. Funct. Anal., 173 (2000), 203–213.

[2] N. Bournaveas, *Low regularity solutions of the Dirac Klein-Gordon equations in two space dimensions,* Comm. Partial Differential Equations, 26 (2001), 1345–1366.

[3] N. Bournaveas, *Local existence of energy class solutions for the Dirac-Klein-Gordon equations,* Comm. Partial Differential Equations, 24 (1999), 1167–1193.

[4] J.M. Chadam, *Asymptotic behavior of equations arising in quantum field theory,* Applicable Anal., 3 (1973/74), 377–402.

[5] M. Christ, *Lectures on singular integral operators,* "CBMS Regional Conference Series in Mathematics," 77, American Mathematical Society, Providence, RI, 1990.

[6] M. Christ and M.I. Weinstein, *Dispersion of small amplitude solutions of the generalized Korteweg-de Vries equation,* J. Funct. Anal., 100 (1991), 87–109.

[7] M. Escobedo and L. Vega, *A semilinear Dirac equation in $H^s(R^3)$ for $s > 1$,* SIAM J. Math. Anal., 28 (1997), 338–362.

[8] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations,* Comm. Pure Appl. Math., 41 (1988), 891–907.

[9] C. Kenig, G. Ponce, and L. Vega, *The initial value problem for a class of nonlinear dispersive equations,* "Functional-Analytic Methods for Partial Differential Equations (Tokyo, 1989)," 141–156, Lecture Notes in Math., 1450, Springer, Berlin, 1990.

[10] S. Klainerman, *On the mathematical theory of classical fields and general relativity,* "General Relativity and Gravitation 1992 (Cordoba)," 101–117, Inst. Phys., Bristol, 1993.

[11] S. Klainerman and M. Machedon, *On the Maxwell-Klein-Gordon equation with finite energy,* Duke Math. J., 74 (1994), 19–44.

[12] S. Klainerman and M. Machedon, *Finite energy solutions of the Yang-Mills equations in $\mathbb{R}^{3+1}$,* Ann. of Math., 142 (1995), 39–119.

[13] H. Lindblad and C. Sogge, *On existence and scattering with minimal regularity for semilinear wave equations,* J. Funct. Anal., 130 (1995), 357–426.

[14] S. Machihara, *Small data global solutions for Dirac-Klein-Gordon equation,* Differential Integral Equations, 15 (2002), 1511–1517.

[15] S. Montgomery-Smith, *Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations,* Duke Math. J., 91 (1998), 393–408.

[16] M. Nakamura and T. Ozawa, *Low energy scattering for nonlinear Schrödinger equations in fractional order Sobolev spaces,* Rev. Math. Phys., 9 (1997), 397–410.

[17] H. Pecher, *Nonlinear small data scattering for the wave and Klein-Gordon equations,* Math. Z., 185 (1984), 261–270.

[18] G. Ponce, *On the global well-posedness of the Benjamin-Ono equation,* Differential Integral Equations, 4 (1991), 527–542.

[19] M. Reed, "Abstract Nonlinear Wave Equations," Lecture Notes in Mathematics, Vol. 507, Springer-Verlag, Berlin-New York, 1976.

[20] C. Sogge, "Lectures on Nonlinear Wave Equations," Monographs in Analysis, II, International Press, Boston, MA, 1995.