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LOCATION AND CRITICAL GROUPS OF CRITICAL POINTS IN BANACH SPACES WITH AN APPLICATION TO NONLINEAR EIGENVALUE PROBLEMS

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Abstract. We develop critical-point theory in Banach spaces in order to find critical points inside or outside of given convex subsets of the space. In addition to localizing critical points we also obtain results on their critical groups. The abstract theory can be applied to p-Laplacian equations in order to prove the existence of multiple sign-changing solutions.

1. Introduction

In this paper we develop variational methods for functionals $J \in \mathcal{C}^1(X, \mathbb{R})$ defined on a Banach space X in order to obtain critical points of J in a prescribed subset $D \subset X$. These localized variational methods yield interesting additional information on the critical points which can also be used to obtain new multiplicity results. The main motivation for this work comes from the problem of finding sign-changing solutions of quasilinear boundary-value problems like

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Δ_p is the p-Laplace operator, i.e., $\Delta_p u = \mathrm{div}(|\nabla \mathrm{u}|^{\mathrm{p}-2}\nabla \mathrm{u})$, with $p \in (1,\infty)$. The p-Laplacian arises in several applications, for instance when one investigates non-Newtonian fluids or nonlinear elasticity problems.

Many papers deal with the existence and structure of positive solutions; see e.g. the recent papers [10, 16, 23–25] and the references therein for the

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quasilinear case $p \neq 2$. In recent years the existence and structure of sign-changing solutions has found considerable interest and different approaches have been developed; see [3, 4, 7, 10, 11, 15, 18, 19, 28–30, 33, 34, 38]. However, all papers deal with the semilinear case $p = 2$ only. The basic ingredient in these approaches is the order structure of the Hilbert space $X = H^1_0(\Omega)$ given by the cone $P = \{u \in X : u \geq 0 \text{ almost everywhere}\}$, the maximum and comparison principles for the Laplace operator, and the regularity theory for the Laplacian. Assuming $J \in \mathcal{C}^2(X)$ one can use the negative gradient flow of J which leaves the order cones P, $-P$ or certain order intervals $[\varphi, \psi]$ positive invariant as a consequence of the comparison principle. Earlier influential studies on order structures and variational methods are [1] and [27].

There are several difficulties when one wants to treat the case $p \neq 2$. First of all, the regularity theory for $p \neq 2$ is not as powerful as for $p = 2$. A technical problem is also that one cannot in general linearize (1.1) around a specific solution u. This is related to the fact that one does not have a Morse lemma available for $p \neq 2$. For $p < 1$ the energy functional J associated to (1.1) is only of class $\mathcal{C}^1$ so that the Morse lemma cannot be generalized. Moreover, since $X = W^{1,p}_0(\Omega)$ is only a Banach space there is no gradient vector field available so that one needs to construct pseudogradient vector fields. This has to be done in such a way that certain order cones or order intervals are positive invariant with respect to the associated flow. It is the goal of this paper to provide deformation lemmas and critical-point theorems that are applicable to (1.1) for every $p \in (1, \infty)$ and which yield, for example, the existence of solutions in prescribed sets of the form $D = D_1 \smallsetminus D_2$ where $D_2 \subset D_1$ are given convex sets. Being forced to construct pseudogradient vector fields by the lack of a Hilbert-space structure and the lack of $\mathcal{C}^2$ differentiability of J we can then also weaken the regularity hypotheses on the nonlinearity f which are usually required in the papers on sign-changing solutions. Thus our method can even be used to improve various earlier papers dealing with the case $p = 2$.

In order to keep the paper within reasonable bounds we publish the applications of our theory to (1.1) separately (see [5, 6]) because the case $p \neq 2$ is quite technical. We do include a few results on the nonlinear eigenvalue problem

$$\begin{cases} -\Delta u = \lambda f(x, u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

demonstrating how the localization of the solutions yields new multiplicity theorems. Setting $\varepsilon = \lambda^{-1/2}$ problem (1.2) turns into a boundary-value problem for the singularly perturbed equation

$$-\varepsilon^2 \Delta u = f(x,u) \qquad \text{in } \Omega. \tag{1.3}$$

This type of equation has also received much attention in the last years. The existence, multiplicity, and shape of positive solutions of (1.2) or (1.3) has been investigated for instance in the papers [9, 12, 20, 24–26, 31, 32, 35]. In these papers multiple positive solutions were obtained for λ large provided that the nonlinearity is in a certain sense oscillatory around 0. In [32] these solutions were proved to obey the natural ordering; see also [17, 21] for similar results. Using the localized variational methods developed here we can also prove that problem (1.2) has multiple sign-changing solutions for λ large under similar oscillatory conditions on f. The extension of these results to the corresponding eigenvalue problem for the p-Laplace operator will appear in [6]. It is an interesting project to investigate the shape of the singular limit of our nodal solutions of (1.2) or (1.3) as $\lambda \to \infty$ or $\varepsilon \to 0$. Even in the case $p = 2$ such results exist so far only for positive solutions; see the above-mentioned recent papers and the references therein.

The paper is organized as follows. In Section 2 we state some localized critical-point theorems in Banach spaces. In Section 3 we construct pseudogradient vector fields and prove deformation lemmas. These results are more general than needed in this paper, but they have already turned out to be useful and important in various applications. The localized critical-point theorems are proved in Section 4. The applications to the nonlinear eigenvalue problem (1.2) will be presented in Sections 5 and 6.

2. Localized critical-point theorems

Let X be a Banach space and $J : X \to \mathbb{R}$ a $\mathcal{C}^1$-functional. Our goal is to find critical points of J inside a given convex set D_1 and outside of other convex sets D_i. We write $K = \{u \in X : \ J'(u) = 0\}$ for the set of critical points of J. Given $a \in \mathbb{R}$ we set $K_a = \{u \in K : J(u) = a\}$ and $J^a = \{u \in X : J(u) \le a\}$. Let Y and Z be Banach spaces such that Y is densely embedded into X and Z is compactly embedded into Y (which we denote by $Z \subset\subset Y$). We suppose that $K \subset Y$. Let $A \in \mathcal{C}(X,X)$ be such that $A(Y) \subset Z$. We assume one of the following hypotheses relating A to J.

(J_1) There exist $1 < p \le 2$, $a_1 > 0$, and $a_2 > 0$ such that

$$\langle J'(u), u - A(u)\rangle_{X^*,X} \ge a_1 \|u - A(u)\|_X^2 (\|u\|_X + \|A(u)\|_X)^{p-2}$$

and

$$\|J'(u)\|_{X^*} \le a_2 \|u - A(u)\|_X^{p-1}$$

hold for every $u \in X$.

(J_2) There exist $p \geq 2$, $a_1 > 0$, and $a_2 > 0$ such that

$$\langle J'(u), u - A(u)\rangle_{X^*,X} \geq a_1 \|u - A(u)\|_X^p$$

and

$$\|J'(u)\|_{X^*} \leq a_2 \|u - A(u)\|_X (\|u\|_X + \|A(u)\|_X)^{p-2}$$

hold for every $u \in X$.

In the application to (1.1) we have in mind that X is the Sobolev space $W_0^{1,\,p}(\Omega)$, $Y := X \cap \mathcal{C}^1(\overline{\Omega})$, and $Z := X \cap \mathcal{C}^{1,\alpha}(\overline{\Omega})$ endowed with the $\mathcal{C}^{1,\alpha}$-norm, for some fixed $0 < \alpha < 1$. The functional J is defined as

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \lambda \int_\Omega F(x,u),$$

and the operator A by

$$A(u) := \lambda(-\Delta_p + \lambda m h_p(\cdot))^{-1}(f(x,u) + m h_p(u)),$$

where $F(x,t) = \int_0^t f(x,s)ds$, m is a suitable positive number and $h_p(t) = |t|^{p-2}t$. The hypotheses (J_1) and (J_2) distinguish between the cases $1 < p \leq 2$ and $p \geq 2$. For $p = 2$ they essentially coincide with the standard definition of A being a pseudogradient vector field for J (except for the Lipschitz continuity which A need not satisfy).

Given a subset $D \subset Y$ we write $\mathrm{clos_Y(D)}$ for the closure of D in Y, and $\widetilde{D} := \mathrm{clos_X(D)}$ for the closure of D in X. For $C \subset D$ the notation $\mathrm{int_D(C)}$ denotes the interior of C in D with the topology induced from Y. Finally, $\mathrm{conv(C)}$ denotes the convex hull of C. Given a pair of convex sets $D_2 \subset D_1 \subset Y$ we need the following hypotheses.

(P_1) $A(\widetilde{D_1}) \subset D_1 \cap Z$ is bounded in Z.

(P_2) $A(\widetilde{D_2}) \subset \mathrm{int_{D_1}(D_2) \cap Z}$.

(P_3) $\mathrm{int_{D_1}(D_2)}$ is convex;
if $C \subset \mathrm{int_{D_1}(D_2)}$ is compact, then $\mathrm{clos_Y(conv(C)) \subset int_{D_1}(D_2)}$;
if $a \in D_2$ and $b \in \mathrm{int_{D_1}(D_2)}$, then $ta + (1-t)b \in \mathrm{int_{D_1}(D_2)}$ for $0 \leq t < 1$.

Before stating the main result we first give a simple example showing that the properties listed in (P_3) are not consequences of the convexity of D_1 and D_2. Indeed, let H be a separable Hilbert space with orthonormal basis $\{e_n : n \in \mathbb{N}\}$. Let $H_1 = e_1^\perp$ be the closed subspace of H orthogonal to e_1 and let $P : H \to H_1$ be the orthogonal projection. Consider the sets

$$D_1 = \mathrm{clos(conv(\{e_1, -e_1\} \cup \{e_k/k : k \geq 2\}))}$$
$$D_2 = \{x \in D_1 \ : \ 2Px \in P(D_1)\}.$$

Then $D_2 \subset D_1$ are closed, convex subsets of H, but $\mathrm{int}_{D_1}\mathrm{D}_2$ is not convex, $\mathrm{clos(conv(C))} \not\subset \mathrm{int}_{D_1}\mathrm{D}_2$ for some compact $C \subset \mathrm{int}_{D_1}\mathrm{D}_2$ and $ta + (1-t)b \notin \mathrm{int}_{D_1}\mathrm{D}_2$ for some $a \in D_2$, $b \in \mathrm{int}_{D_1}(\mathrm{D}_2)$ and $0 \le t < 1$. In fact, we have $C := \{e_1, -e_1\} \subset \mathrm{int}_{D_1}(\mathrm{D}_2)$, $0 \in D_2$, but $te_1 \notin \mathrm{int}_{D_1}(\mathrm{D}_2)$ for $-\frac{1}{2} < t < \frac{1}{2}$.

For an isolated critical point $u \in K$ with $J(u) = c$ we define the critical groups of J at u by

$$C_q(J, u) = H_q(J^c \cap U,\ J^c \cap U \setminus \{u\}), \quad q \in \mathbb{Z},$$

where U is an open neighborhood of u in Y and H_* is singular homology theory (with coefficients in an arbitrary group G which we drop from the notation). The pair $(J^c \cap U,\ J^c \cap U \setminus \{u\})$ inherits the topology from Y. Its homology is independent of the choice of U due to the excision property of H_*. By a result of Palais [36] we may as well take an open neighborhood of u in X or Z and obtain isomorphic groups.

Now we can formulate our main results.

Theorem 2.1. *Assume either* (J_1) *or* (J_2). *Let* $D_2 \subset D_1 \subset Y$ *be closed, convex sets satisfying* (P_1)–(P_3). *If* J *satisfies the Palais-Smale condition and*

$$-\infty < \inf_{u \in D_1} J(u) < \inf_{u \in D_2} J(u), \tag{2.1}$$

then the following conclusions hold.

a) *There exists a critical point* u_0 *of* J *in* $D_1 \setminus D_2$ *such that*

$$J(u_0) = \inf_{u \in D_1} J(u).$$

b) *Fix* $u^* \in D_1$ *with* $J(u^*) < \inf_{u \in D_2} J(u)$, *e.g.* $u^* = u_0$, *and define* $\Gamma := \{h \in \mathcal{C}([0,1], D_1) : h(0) \in D_2,\ h(1) = u^*\}$ *and*

$$c = \inf_{h \in \Gamma} \max_{0 \le t \le 1} J(h(t)).$$

Then $c \ge \inf_{u \in D_2} J(u)$ *and there exists* $u_1 \in K \cap (D_1 \setminus D_2)$ *such that* $J(u_1) = c$.

c) *If* J *has only a finite number of critical points in* $D_1 \setminus D_2$ *and if they lie in* $\mathrm{int}_{\mathrm{Y}}(\mathrm{D}_1)$, *then*

$$C_q(J, u_0) \cong \delta_{q0} G, \quad q \in \mathbb{Z}$$

and

$$C_1(J, u_1) \neq 0.$$

Theorem 2.2. *Assume either* (J_1) *or* (J_2). *Let* $D_2, D_3 \subset D_1 \subset Y$ *be closed, convex sets and set* $D_4 := D_2 \cap D_3$. *Suppose the properties* (P_1)–(P_3) *hold for* $D_i \subset D_1$, $i = 2, 3, 4$. *If* J *satisfies the Palais-Smale condition and if there* *exist* $h_1, h_2 \in \mathcal{C}([0,1], D_1)$ *such that* $h_1(0) = h_2(0) \in D_2$, $h_1(1) = h_2(1) \in D_3$, $h_2([0,1]) \subset D_2 \cup D_3$, $h_1([0,1]) \cap h_2([0,1]) = \{h_1(0), h_1(1)\}$, *and*

$$-\infty < \inf_{u \in D_1} J(u) \leq \max_{0 \leq t \leq 1} J(h_1(t)) < \inf_{u \in D_4} J(u), \tag{2.2}$$

then the following conclusions hold.

a) *Setting* $\Gamma_1 := \{h \in \mathcal{C}([0,1], D_1) : h(0) \in D_2,\ h(1) \in D_3\}$ *and*

$$c_1 = \inf_{h \in \Gamma_1} \max_{0 \leq t \leq 1} J(h(t)),$$

there exists $w_1 \in (K \cap D_1) \setminus (D_2 \cup D_3)$ *with* $J(w_1) = c_1$.

b) *Setting* $\Gamma_2 := \{h \in \mathcal{C}([0,1], D_2 \cup D_3) : h(0) = h_1(0),\ h(1) = h_1(1)\}$ *and*

$$\gamma_2 = \inf_{h \in \Gamma_2} \max_{0 \leq t \leq 1} J(h(t)),$$

there exists $w_2 \in (K \cap D_1) \setminus (D_2 \cup D_3)$ *with* $c_2 := J(w_2) \geq \gamma_2$.

c) *If* J *has only a finite number of critical points in* $D_1 \setminus (D_2 \cup D_3)$ *and if they lie in* $\mathrm{int_Y(D_1)}$, *then*

$$C_1(J, w_1) \neq 0, \quad C_2(J, w_2) \neq 0.$$

The value γ_2 from Theorem 2.2b) is in general not the critical value $c_2 = J(w_2)$. In the proof of Theorem 2.2 we will give a homological description of c_2.

Theorem 2.3. *Suppose* J *is even,* A *is odd, and the hypotheses of Theorem* 2.2 *hold with* $D_1 = -D_1$, $D_2 = -D_3$, *and* $0 \in D_4$. *If there exists an odd map* $h_0 \in \mathcal{C}(S^{k-1}, D_1)$, S^{k-1} *the unit sphere in* $\mathbb{R}^k$, *such that*

$$-\infty < \inf J|_{D_1} \leq \max J \circ h_0 < \inf J|_{D_4}, \tag{2.3}$$

then J *has at least* $k-1$ *pairs of critical points in* $D_1 \setminus (D_2 \cup D_3)$ *with energy in* $[\inf J|_{D_1}, \max J \circ h_0]$ *and at least* $k-1$ *pairs of critical points in* $D_1 \setminus (D_2 \cup D_3)$ *with energy in* $[\inf J|_{D_4}, \max J \circ h]$, *where* $h \in \mathcal{C}(B^k, D_1)$ *is any odd extension of* h_0 *to the closed unit ball* $B^k \subset \mathbb{R}^k$.

In the proof of Theorem 2.3 we will give a minimax description of the critical values.

3. Deformation lemmas

The main technical difficulty in the proofs of the localized critical-point theorems is to construct appropriate deformations. This will be done in this section.

Throughout this section we fix Banach spaces $Z \subset\subset Y \subset X$, a functional $J \in \mathcal{C}^1(X)$ and $A \in \mathcal{C}(X, X)$ with $A(Y) \subset Z$ as in Section 2. In particular,

$K = \{u \in X : J'(u) = 0\} \subset Y$. We assume that (J_1) or (J_2) hold. Let $D_1, \dots, D_n \subset Y$ be closed (in Y) and convex. As in Section 2 we write $\widetilde{D_i} = \mathrm{clos_X(D_i)}$. Now we define

$$\mathcal{Q}^a := (J^a \cap D_1) \cup \bigcup_{i=2}^{n} D_i, \quad K_a^* := (K_a \cap D_1) \setminus \bigcup_{i=2}^{n} D_i, \quad \mathcal{Q}_*^a := \mathcal{Q}^a \setminus K_a^*.$$

Next we fix two real numbers $a < b$ and consider the set

$$\mathcal{M}_a^b = \mathcal{Q}_*^b \setminus \mathcal{Q}^a = J^b \cap D_1 \smallsetminus \Big(J^a \cup K_b \cup \bigcup_{i=2}^{n} D_i\Big).$$

Observe that these sets are contained in Y.

In addition to (J_1) and (J_2) we need a third assumption on J.

(J_3) J satisfies the Palais-Smale condition $(PS)_c$ for all $c \in [a, b]$, $K \cap \mathcal{M}_a^b = \emptyset$, and K_a^* is a finite set.

Now we can formulate our deformation results.

Proposition 3.1. *Suppose J satisfies (J_3) in addition to (J_1) or (J_2). Suppose moreover that (P_1)–(P_3) hold for $D_i \subset D_1$, $i = 2, \dots, n$. Then $\mathcal{Q}^a$ is a strong deformation retract of $\mathcal{Q}_*^b$; that is, there exists $\Phi \in \mathcal{C}([0,1] \times \mathcal{Q}_*^b, \mathcal{Q}_*^b)$ such that $\Phi(0, u) = u$ for $u \in \mathcal{Q}_*^b$, $\Phi(1, u) \in \mathcal{Q}^a$ for $u \in \mathcal{Q}_*^b$ and $\Phi(t, u) = u$ for $0 \le t \le 1$ and $u \in \mathcal{Q}^a$. In addition, if J is even, A is odd, and $\{D_1, \dots, D_n\} = \{-D_1, \dots, -D_n\}$, then Φ is odd in u.*

Proposition 3.2. *Suppose J satisfies the Palais-Smale condition in addition to (J_1) or (J_2). Suppose moreover that (P_1)–(P_3) hold for $D_i \subset D_1$, $i = 2, \dots, n$. Let $c \in \mathbb{R}$ and $\mathcal{O}$ be any neighborhood of K_c^* in X. Then there exists $\epsilon > 0$ and $\sigma^* \in \mathcal{C}\big((\mathcal{Q}^{c+\epsilon} \setminus \mathcal{O}) \cup \mathcal{Q}^{c-\epsilon}, \mathcal{Q}^{c-\epsilon}\big)$ such that $\sigma^*|_{\mathcal{Q}^{c-\epsilon}} = \mathrm{id}$ and $J(\sigma^*(u)) \le J(u)$ for $u \in \mathcal{Q}^{c+\epsilon} \setminus \mathcal{O}$. Moreover, σ^* is odd if J is even, A is odd, $\mathcal{O} = -\mathcal{O}$, and $\{D_1, \dots, D_n\} = \{-D_1, \dots, -D_n\}$.*

For the proofs of these propositions we first need to construct a special pseudogradient vector field.

Lemma 3.3. *If (J_1) holds, then there exists a locally Lipschitz-continuous map $B : X_0 = X \smallsetminus K \to Z$ such that*

(i) *$B(\widetilde{D}_i \cap X_0) \subset \mathrm{conv(A(\widetilde{D}_i))}$, the convex hull of $A(\widetilde{D}_i)$, $i = 1, \dots, n$;*
(ii) *for a_1 as in (J_1) and $u \in X_0$,*

$$\langle J'(u), u - B(u)\rangle_{X^*,X} \ge \frac{a_1}{2}\|u - A(u)\|_X^2(\|u\|_X + \|A(u)\|_X)^{p-2}, \tag{3.1}$$

$$\frac{1}{2}\|u - B(u)\|_X \le \|u - A(u)\|_X \le 2\|u - B(u)\|_X; \tag{3.2}$$

(iii) *if J is even, A is odd, and $\{D_1, \dots, D_n\} = \{-D_1, \dots, -D_n\}$, then B is odd.*

Proof. We consider the continuous maps $\Delta_1, \Delta_2 : X_0 \to (0, \infty)$ defined by

$$\Delta_1(u) = \frac{1}{2}\|u - A(u)\|_X \tag{3.3}$$

and

$$\Delta_2(u) = \frac{a_1}{2a_2}\|u - A(u)\|_X^{3-p}(\|u\|_X + \|A(u)\|_X)^{p-2}. \tag{3.4}$$

For $u \in X_0$ there exists $\gamma(u) > 0$ such that

$$\mathcal{N}(u) := \{v \in X : \|v - u\|_X < \gamma(u)\} \subset X_0$$

and

$$\|A(v) - A(w)\|_X < \min\{\Delta_1(v),\ \Delta_1(w),\ \Delta_2(v),\ \Delta_2(w)\}, \tag{3.5}$$

holds for all $v, w \in \mathcal{N}(u)$.

We now construct a locally finite open refinement $\{U_\lambda : \lambda \in \Lambda\}$ of the covering $\{\mathcal{N}(u) : u \in X_0\}$ of X_0 with the property that

$$\text{if } U_\lambda \cap \widetilde{D}_i \neq \emptyset \text{ for } i \in I \subset \{1, \dots, n\}, \text{ then } U_\lambda \cap \bigcap_{i \in I} \widetilde{D}_i \neq \emptyset. \tag{3.6}$$

Let $\{V_\mu : \mu \in M\}$ be a locally finite open refinement of $\{\mathcal{N}(u) : u \in X_0\}$. We call a subset $I \subset \{1, \dots, n\}$ maximal for V_μ if $V_\mu \cap \bigcap_{i \in I} \widetilde{D}_i \neq \emptyset$ but $V_\mu \cap \bigcap_{i \in J} \widetilde{D}_i = \emptyset$ for every $J \subset \{1, \dots, n\}$ with $I \subsetneq J$. Clearly, V_μ may have several maximal sets. If $\widetilde{D}_i \cap V_\mu \neq \emptyset$, then i is in some maximal set for V_μ; it may appear in several maximal sets. If $V_\mu \cap \widetilde{D}_i = \emptyset$ for every $i \in \{1, \dots, n\}$, then $I = \emptyset$ is the only maximal set for V_μ. For $\mu \in M$ let

$$\mathcal{J}_\mu := \{I \subset \{1, \dots, n\} : I \text{ is maximal for } V_\mu\}.$$

The covering

$$\{U_\lambda : \lambda \in \Lambda\} := \Big\{V_\mu \setminus \bigcup_{i \in \{1, \dots, n\} \setminus I} \widetilde{D}_i : \mu \in M,\ I \in \mathcal{J}_\mu\Big\}$$

is a locally finite open refinement of $\{\mathcal{N}(u) : u \in X_0\}$ and has the property (3.6). This follows easily from the construction and the fact that

$$\bigcup_{I \in \mathcal{J}_\mu} \Big(V_\mu \setminus \bigcup_{i \in \{1, \dots, n\} \setminus I} \widetilde{D}_i\Big) = V_\mu$$

for every $\mu \in M$. Let π_λ, $\lambda \in \Lambda$, be a locally finite partition of unity subordinated to $\{U_\lambda : \lambda \in \Lambda\}$ such that the $\pi_\lambda : X_0 \to \mathbb{R}$ are Lipschitz continuous. For each $\lambda \in \Lambda$ we set

$$I_\lambda := \{i \in \{1, \dots, n\} : U_\lambda \cap \widetilde{D}_i \neq \emptyset\}.$$

If $I_\lambda = \emptyset$ we choose $a_\lambda \in U_\lambda \cap Y$ arbitrarily; otherwise we choose $a_\lambda \in U_\lambda \cap \bigcap_{i\in I_\lambda} \widetilde{D}_i$ which is possible due to (3.6). Now we define

$$B : X_0 \to Z, \quad B(u) := \sum_{\lambda\in\Lambda} \pi_\lambda(u) A(a_\lambda).$$

If $u \in \widetilde{D}_i \cap X_0$, then $u \in \widetilde{D}_i \cap U_\lambda$ for any λ with $\pi_\lambda(u) \neq 0$. From the construction it follows that $a_\lambda \in U_\lambda \cap \widetilde{D}_i$ for any λ with $\pi_\lambda(u) \neq 0$. This implies $B(u) \in \mathrm{convA}(\widetilde{\mathrm{D}}_\mathrm{i})$, so Lemma 3.3(i) is proved. That $B : X_0 \to Z$ is locally Lipschitz continuous is a consequence of the Lipschitz continuity of the π_λ, the local finiteness of $\{U_\lambda : \lambda \in \Lambda\}$, and the facts that $A(Y) \subset Z$ and $A(\widetilde{D}_i) \subset Z$ for $i = 1, \dots, n$. In order to see (ii) we observe that

$$\|B(u) - A(u)\|_X \le \sum_{\lambda\in\Lambda} \pi_\lambda(u) \|A(a_\lambda) - A(u)\|_X, \quad u \in X_0. \tag{3.7}$$

If $\pi_\lambda(u) \neq 0$ for some $\lambda \in \Lambda$, then $u, a_\lambda \in U_\lambda \subset \mathcal{N}(u_\lambda)$ for some $u_\lambda \in X_0$, so by (3.5), (3.3), and (3.7)

$$\|B(u) - A(u)\|_X < \frac{1}{2} \|u - A(u)\|_X \quad \text{for } u \in X_0,$$

which implies (3.2). From (3.5), (3.4), and (3.7) it follows that

$$\|B(u) - A(u)\|_X < \frac{a_1}{2a_2} \|u - A(u)\|_X^{3-p} (\|u\|_X + \|A(u)\|_X)^{p-2} \quad \text{for } u \in X_0.$$

Combining this with (J_1) we obtain (3.1).

Finally, if J is even, A is odd, and $\{D_1, \dots, D_n\} = \{-D_1, \dots, -D_n\}$, then we may assume $\gamma(-u) = \gamma(u)$ for $u \in X_0$. We enlarge the covering $\{U_\lambda : \lambda \in \Lambda\}$ by adding the sets $-U_\lambda$, $\lambda \in \Lambda$. Then the vector field

$$B : X_0 \to Z, \quad B(u) = \sum_{\lambda\in\Lambda} \big(\pi_\lambda(u) - \pi_\lambda(-u)\big) A(a_\lambda)$$

is odd and satisfies (i) and (ii). □

Lemma 3.4. *If* (J_2) *holds, then the same conclusions as in Lemma* 3.3 *hold except that* (3.1) *is replaced by*

$$\langle J'(u), u - B(u)\rangle_{X^*,X} \ge \frac{a_1}{2} \|u - A(u)\|_X^p, \quad u \in X_0. \tag{3.8}$$

Proof. The proof is almost the same as the proof of Lemma 3.3, only (3.4) should be replaced with

$$\Delta_2(u) = \frac{a_1}{2a_2} \|u - A(u)\|_X^{p-1} (\|u\|_X + \|A(u)\|_X)^{2-p}. \qquad \square \tag{3.9}$$

Now we consider the semiflow φ^t on X_0 associated to the vector field $-\mathrm{id}+\mathrm{B}$:

$$\frac{\partial}{\partial t}\varphi^t(u) = -\varphi^t(u) + B(\varphi^t(u)), \quad \varphi^0(u) = u. \tag{3.10}$$

Let $[0,\tau(u))$ be the maximal interval of existence of $\varphi^t(u)$. Since $Y \hookrightarrow X$ and $B : X_0 \to Z \subset Y$ is locally Lipschitz continuous we see that the solution of (3.10) considered on $Y_0 := Y \setminus K$ is the same as $\varphi^t(u)$ if $u \in Y_0$. Observe that $\varphi^t(u)$ is odd in u if B is odd.

If $D_i \subset D_1$ satisfy (P_1) and (P_2), then $B(\widetilde{D}_i \cap X_0) \subset D_i$. It follows that $\widetilde{D}_i$ and D_i are invariant under φ^t for $i = 1,\dots,n$; see [33], for instance. Moreover, we have

Lemma 3.5. *If (P_1)–(P_3) hold for $D_i \subset D_1$, then $\varphi^t(u) \in \mathrm{int}_{D_1}(D_i)$ for $u \in D_i \setminus K$, $0 < t < \tau(u)$.*

Proof. We fix $u \in D_i \setminus K$. Then $\varphi^t(u) \in D_i \setminus K$ for $0 \le t < \tau(u)$. Since $A(\widetilde{D}_i) \subset \mathrm{int}_{D_1}(D_i)$ and $\mathrm{int}_{D_1}(D_i)$ is convex by (P_3), Lemma 3.3(i) implies $B(\widetilde{D}_i \setminus K) \subset \mathrm{int}_{D_1}(D_i)$. Therefore $B(\varphi^t(u)) \in \mathrm{int}_{D_1}(D_i)$ for $0 \le t < \tau(u)$. Next we see that for a given $t \in (0,\tau(u))$ the set $\{B(\varphi^s(u)) : 0 \le s \le t\} \subset \mathrm{int}_{D_1}(D_i)$ is compact. This together with (P_3) implies

$$\frac{1}{e^t - 1}\int_1^{e^t} B(\varphi^{\ln s}(u))ds \in \mathrm{int}_{D_1}(D_i).$$

Using (P_3) once more, we have

$$\varphi^t(u) = e^{-t}u + (1-e^{-t})\frac{1}{e^t-1}\int_1^{e^t} B(\varphi^{\ln s}(u))ds \in \mathrm{int}_{D_1}(D_i). \qquad \square$$

Lemma 3.6. *Suppose (P_1) holds for D_1 and let $G \subset D_1 \setminus K$ be bounded in X. Then there exists $c > 0$ such that*

$$\|\varphi^t(u)\|_X + \|A(\varphi^t(u))\|_X \le c$$

for all $u \in G$ and $0 \le t < \tau(u)$.

Proof. Note that

$$\varphi^t(u) = e^{-t}u + e^{-t}\int_0^t e^s B(\varphi^s(u))ds \tag{3.11}$$

for $0 \le t < \tau(u)$ and that $\varphi^t(u) \in D_1$ for all $u \in G$ and all $0 \le t < \tau(u)$. Since $B(\widetilde{D}_1 \cap X_0) \subset \mathrm{conv}(A(\widetilde{D}_1))$ and since $A(\widetilde{D}_1)$ is bounded in Z, the set

$$\left\{e^{-t}\int_0^t e^s B(\varphi^s(u))ds : u \in G, 0 \le t < \tau(u)\right\}$$

is bounded in Z, and therefore bounded in X. Consequently, $\{\varphi^t(u) : u \in G,\ 0 \le t < \tau(u)\}$ is bounded in X, which implies the lemma. □

For $u \in \mathcal{M}_a^b$ we define

$$\tau_1(u) := \sup\{\tau : 0 < \tau < \tau(u) \text{ and } \varphi^t(u) \in \mathcal{M}_a^b \text{ for } 0 \le t \le \tau\}.$$

Clearly, we have $0 < \tau_1(u) \le \tau(u)$.

Lemma 3.7. *Suppose J satisfies (J_3) in addition to (J_1) or (J_2). Suppose moreover that (P_1)–(P_3) hold for $D_i \subset D_1$, $i = 2, \dots, n$. Then for every $u \in \mathcal{M}_a^b$ the limit $\lim_{t\to\tau_1(u)^-} \varphi^t(u)$ exists in Y.*

Proof. We fix $u \in \mathcal{M}_a^b$. The result is obvious if $\tau_1(u) < \tau(u)$, so we only need to consider the case $\tau_1(u) = \tau(u)$.

The set $\left\{e^{-t}\int_0^t e^s B(\varphi^s(u))ds : 0 \le t < \tau(u)\right\}$ is bounded in Z and relatively compact in Y because $B(\widetilde{D}_1 \cap X_0) \subset \mathrm{conv}(\mathrm{A}(\widetilde{\mathrm{D}}_1))$ and $A(\widetilde{D}_1)$ is bounded in Z. Now (3.11) implies that the orbit $\mathcal{O}(u) := \{\varphi^t(u) : 0 \le t < \tau(u)\}$ is relatively compact in Y. Then a standard argument shows that

$$\omega(u) = \bigcap_{t\ge 0} \mathrm{clos_Y}\Big(\bigcup_{\mathrm{s\ge t}}\{\varphi^{\mathrm{s}}(\mathrm{u})\}\Big) \subset \mathrm{K}.$$

From this we deduce $\omega(u) \subset K \cap \mathrm{clos_Y}(\mathcal{M}_{\mathrm{a}}^{\mathrm{b}}) \cap \mathrm{J}^{\mathrm{a}} = \mathrm{K}_{\mathrm{a}}^*$. Here we used that $K \cap D_i \subset \mathrm{int_{D_1}(D_i)}$ for $i \ge 2$ by (P_2) and that $K \cap \mathcal{M}_a^b = \emptyset$. Now K_a^* is finite by (J_3) so $\varphi^t(u)$ must converge in Y towards a critical point of J in K_a^*. □

Setting $\hat{p} := \max\{2, p\}$ we also consider the flow ψ^t on X_0 defined by the equation

$$\frac{\partial}{\partial t}\psi^t(u) = \frac{-\psi(t) + B(\psi(t))}{\|\psi(t) - B(\psi(t))\|_X^{\hat{p}}}, \quad \psi^0(u) = u. \tag{3.12}$$

We let $\eta(u) \in (0, \infty]$ be the maximal time of existence of $\psi^t(u)$. As before we see that $\psi^t(u) \in Y_0$ provided $u \in Y_0$. Clearly $\psi^t(u)$ is just a reparametrization of $\varphi^t(u)$:

$$\psi^s(u) = \varphi^t(u) \quad \text{if} \quad s = \int_0^t \|\varphi^r(u) - B(\varphi^r(u))\|_X^{\hat{p}} dr.$$

For $u \in \mathcal{M}_a^b$ we define

$$\eta_1(u) = \int_0^{\tau_1(u)} \|\varphi^r(u) - B(\varphi^r(u))\|_X^{\hat{p}} dr.$$

Lemma 3.8. *Suppose J satisfies (J_3) in addition to (J_1) or (J_2). Suppose moreover that (P_1)–(P_3) hold for $D_i \subset D_1$, $i = 2, \dots, n$. If $G \subset \mathcal{M}_a^b$ is bounded in X, then $\eta_1(G)$ is bounded.*

Proof. Lemmas 3.3, 3.4, and 3.6 imply

$$\frac{d}{ds} J(\psi^s(u)) = -\frac{\langle J'(\psi^s(u)), \psi^s(u) - B(\psi^s(u))\rangle_{X^*,X}}{\|\psi^s(u) - B(\psi^s(u))\|_X^{\hat{p}}} \le -C_1$$

for $u \in G$ and $0 \le s < \eta(u)$. Here and in the sequel, C_i means a positive constant. On the other hand

$$-\int_0^{\eta_1(u)} \frac{d}{ds} J(\psi^s(u))\, ds = J(\psi^0(u)) - \lim_{t \to \eta_1(u)^-} J(\psi^t(u)) \le b - a$$

for $u \in G$. Therefore, $\eta_1(u) \le (b-a)/C_1$ for all $u \in G$. □

Lemma 3.9. *Suppose J satisfies (J_3) in addition to (J_1) or (J_2). Suppose moreover that (P_1)–(P_3) hold for $D_i \subset D_1$, $i = 2, \dots, n$. Then $\eta_1 : \mathcal{M}_a^b \to \mathbb{R}^+$ is continuous with respect to $\|\cdot\|_Y$.*

Proof. Let $u \in \mathcal{M}_a^b$ and set $\sigma(u) = \lim_{t \to \tau_1(u)^-} \varphi^t(u) = \lim_{s \to \eta_1(u)^-} \psi^s(u)$. We distinguish four cases in terms of the location of $\sigma(u)$.

(i) $\sigma(u) \in (J^b \setminus J^a) \cap \partial_{D_1} (\bigcup_{i=2}^n D_i)$,
(ii) $\sigma(u) \in (J^{-1}(a) \cap D_1) \setminus (\bigcup_{i=2}^n D_i \cup K)$,
(iii) $\sigma(u) \in J^{-1}(a) \cap \partial_{D_1} (\bigcup_{i=2}^n D_i)$,
(iv) $\sigma(u) \in K_a^*$.

In case (i), we have $\sigma(u) \notin K$, so $\eta_1(u) < \eta(u)$. By Lemma 3.5

$$\psi^s(u) \in \bigcup_{i=2}^n \mathrm{int}_{D_1}(D_i) \qquad \text{for } \eta_1(u) < s < \eta(u).$$

If $0 \le s < \eta_1(u)$, then $\psi^s(u) \notin \bigcup_{i=2}^n D_i$. Thus for any $\epsilon > 0$ small there exists $\delta > 0$ such that, if $v \in \mathcal{M}_a^b$ and $\|v - u\|_Y < \delta$, then $\psi^{\eta_1(u)+\epsilon}(v) \in \cup_{i=2}^n \mathrm{int}_{D_1}(D_i)$ and $\psi^{\eta_1(u)-\epsilon}(v) \notin \cup_{i=2}^n D_i$. Therefore, $\eta_1(u) - \epsilon \le \eta_1(v) \le \eta_1(u) + \epsilon$ if $v \in \mathcal{M}_a^b$ and $\|v - u\|_Y < \delta$; hence, η_1 is continuous at u.

Case (ii) is similar to case (i), and the result in case (iii) is a combination of those in cases (i) and (ii).

In case (iv) we have $\tau_1(u) = \tau(u)$ and $\eta_1(u) = \eta(u)$. For $s < \eta_1(u)$, we have $J(\psi^s(u)) > a$; therefore, if $v \in \mathcal{M}_a^b$ and $\|v - u\|_Y$ is small enough, then $J(\psi^s(v)) > a$ and $\eta_1(v) > s$. This proves the lower semicontinuity of η_1 at u. If η_1 were not continuous at u, then there would exist $\epsilon_0 > 0$ and a sequence

$(v_n)_n$ in $\mathcal{M}_a^b$ with $\|v_n - u\|_Y \to 0$ and such that $\eta_1(v_n) > \eta_1(u) + \epsilon_0$. In the case $1 < p \le 2$ Lemmas 3.3 and 3.6 imply

$$\begin{aligned} &J(\psi^{\eta_1(u)-\epsilon}(v_n)) - J(\psi^{\eta_1(u)+\epsilon_0}(v_n)) \\ &\qquad = \int_{\eta_1(u)-\epsilon}^{\eta_1(u)+\epsilon_0} \langle -J'(\psi^s(v_n)), \frac{d}{ds}\psi^s(v_n)\rangle_{X^*,X} ds \\ &\qquad \ge \frac{a_1}{8}\int_{\eta_1(u)-\epsilon}^{\eta_1(u)+\epsilon_0} \big(\|\psi^s(v_n)\|_X + \|A(\psi^s(v_n))\|_X\big)^{p-2} ds \ge C_2(\epsilon+\epsilon_0). \end{aligned}$$

Since $\eta_1(v_n) > \eta_1(u) + \epsilon_0$ we have

$$J(\psi^{\eta_1(u)-\epsilon}(v_n)) \ge a + C_2(\epsilon+\epsilon_0).$$

Now we let $v_n \to u$ and $\epsilon \to 0$ obtaining $a \ge a + C_2\epsilon_0$, a contradiction.

The case $p > 2$ can be dealt with analogously using Lemma 3.4 instead of 3.3. □

Now we are ready to prove the deformation results.

Proof of Proposition 3.1. We define $\Phi : [0,1] \times \mathcal{Q}_*^b \to \mathcal{Q}_*^b$ by

$$\Phi(t,u) = \begin{cases} u & \text{if } (t,u) \in [0,1] \times \mathcal{Q}^a, \\ \psi^{t\eta_1(u)}(u) & \text{if } (t,u) \in [0,1) \times \mathcal{M}_a^b, \\ \sigma(u) & \text{if } (t,u) \in \{1\} \times \mathcal{M}_a^b. \end{cases}$$

In order to complete the proof we just need to prove the continuity of Φ. We show that Φ is continuous at the point (t,u) in the case $(t,u) \in \{1\} \times \mathcal{M}_a^b$ and $\sigma(u) \in K_a^*$. The other cases are similar or even simpler.

For (t,u) as above we argue by contradiction. Suppose there exist sequences $[0,1) \ni t_n \to 1$, $\mathcal{M}_a^b \ni u_n \to u$ in Y, and $\epsilon_0 > 0$ such that

$$\|\Phi(t_n,u_n) - \sigma(u)\|_Y \ge \epsilon_0 \qquad \text{for all } n.$$

$\{\Phi(t_n,u_n)\}$ is relatively compact in Y because $\|u_n - u\|_Y \to 0$. Hence, there exists $\epsilon_1 > 0$ such that

$$\|\Phi(t_n,u_n) - \sigma(u)\|_X \ge \epsilon_1 \qquad \text{for all } n.$$

Since $\sigma(u)$ is an isolated critical point we may assume that there exists $\delta_0 > 0$ such that, if $v \in J^{-1}[a,b]$ and $\epsilon_1/2 \le \|v - \sigma(u)\|_X \le \epsilon_1$, then $\|J'(v)\|_X \ge \delta_0$. For $j \in \mathbb{N}$ we choose n_j with $t_{n_j} > 1 - j^{-1}$ and $\|\Phi(1-j^{-1}, u_{n_j}) - \Phi(1 - j^{-1}, u)\|_X < j^{-1}$. Then $\lim_{j\to\infty} \|\Phi(1-j^{-1}, u_{n_j}) - \sigma(u)\|_X = 0$. So for j large enough, we can choose t_j' and t_j'' with $1 - j^{-1} < t_j' < t_j'' \le t_{n_j}$ such that

$$\|\Phi(t_j', u_{n_j}) - \sigma(u)\|_X = \epsilon_1/2, \qquad \|\Phi(t_j'', u_{n_j}) - \sigma(u)\|_X = \epsilon_1,$$

and

$$\|J'(\Phi(t,u_{n_j}))\|_{X^*} \geq \delta_0 \quad \text{for } t_j' \leq t \leq t_j''.$$

Note that, as $j \to \infty$,

$$a \leq J(\Phi(t_j'',u_{n_j})) < J(\Phi(t_j',u_{n_j})) < J(\Phi(1-j^{-1},u_{n_j})) \to J(\sigma(u)) = a.$$

From the above facts and using Lemmas 3.3, 3.4, and 3.6, a standard argument leads to

$$\epsilon_1/2 \leq \|\Phi(t_j',u_{n_j}) - \Phi(t_j'',u_{n_j})\|_X \leq C_3\big(J(\Phi(t_j',u_{n_j})) - J(\Phi(t_j'',u_{n_j}))\big) \to 0,$$

a contradiction. □

Proof of Proposition 3.2. We first observe that K_c^* is compact in Y and therefore compact in X. In fact, $K_c^* = A(K_c^*)$ is bounded in Z and relatively compact in Y because $K_c^* \subset D_1$ and $A(\widetilde{D}_1)$ is bounded in Z. On the other hand, K_c^* is closed in Y because $K_c \cap D_1$ is closed in D_1 and $K_c^* = K_c \cap D_1 \setminus \bigcup_{i=2}^n \mathrm{int}_{D_1} D_i$ as implied by (P_2).

Now we choose $\delta > 0$ small enough so that

$$\mathcal{N}_\delta := \{u \in X : \mathrm{dist}_X(u, K_c^*) < \delta\} \subset \mathcal{O}$$

and

$$\mathrm{clos}_X(\mathcal{N}_\delta) \cap \Big(\bigcup_{i=2}^n \widetilde{D}_i\Big) = \emptyset.$$

For $\epsilon > 0$ small enough we have $\bigcup_{b\in[c-\epsilon,c+\epsilon]} K_b^* \subset \mathcal{N}_{\delta/4}$. Setting $G_\epsilon := J^{-1}([c-\epsilon,c+\epsilon]) \cap D_1 \cap (\mathcal{N}_\delta \setminus \mathcal{N}_{\delta/2})$, it follows that there exist $\delta_1 > 0$ and $\epsilon_1 > 0$ so that

$$\|J'(u)\|_{X^*} \geq \delta_1 \quad \text{for } u \in G_\epsilon,\ 0 < \epsilon \leq \epsilon_1.$$

Next we claim that there exists $\epsilon \in (0,\epsilon_1)$ such that for $u \in \mathcal{Q}^{c+\epsilon} \setminus (\mathcal{Q}^{c-\epsilon} \cup \mathcal{O})$ there exists a unique $\tau_2(u) < \tau(u)$ with $\varphi^{\tau_2(u)}(u) \in \mathcal{Q}^{c-\epsilon}$ and $\varphi^t(u) \notin \mathcal{Q}^{c-\epsilon} \cup \mathcal{N}_{\delta/2}$ for $t \in [0,\tau_2(u))$. This follows easily from the fact that $\|J'(u)\|$ is bounded away from 0 for $u \in \mathcal{N}_\delta \setminus \mathcal{N}_{\delta/2}$ by the Palais-Smale condition; hence,

$$\|\varphi^{t_1}(u) - \varphi^{t_2}(u)\|_X \leq C_4\big(J(\varphi^{t_1}(u)) - J(\varphi^{t_2}(u))\big)$$

provided $u \in X$ satisfies $\varphi^t(u) \in \mathcal{N}_\delta \setminus \mathcal{N}_{\delta/2}$ for $t \in [t_1,t_2]$. Here we again used Lemmas 3.3, 3.4, and 3.6. As in the proof of Lemma 3.9 one sees that $\tau_2(u)$ is continuous at $u \in \mathcal{Q}^{c+\epsilon} \setminus (\mathcal{Q}^{c-\epsilon} \cup \mathcal{O})$ with respect to $\|\cdot\|_Y$. Now we define $\sigma^* : \mathcal{Q}^{c+\epsilon} \setminus (\mathcal{Q}^{c-\epsilon} \cup \mathcal{O}) \to \mathcal{Q}^{c-\epsilon}$ by

$$\sigma^*(u) = \begin{cases} u & \text{if } u \in \mathcal{Q}^{c-\epsilon}, \\ \varphi^{\tau_2(u)}(u) & \text{if } u \in \mathcal{Q}^{c+\epsilon} \setminus (\mathcal{Q}^{c-\epsilon} \cup \mathcal{O}). \end{cases}$$

Then σ^* is continuous with respect to $\|\cdot\|_Y$. If J is even, A is odd, $\mathcal{O} = -\mathcal{O}$, and $\{D_1,\dots,D_n\} = \{-D_1,\dots,-D_n\}$, then it is easy to see that σ^* is odd. □

We end this section with an example showing that the maps τ_1 and τ_2 are discontinuous with respect to $\|\cdot\|_X$. This is why we have taken into account the topologies of Y and Z.

Example 3.10. Let $f \in C^1(\mathbb{R})$ be bounded and increasing with bounded f', and set $F(t) = \int_0^t f(s)ds$. The functional

$$J : H_0^1(\Omega) \to \mathbb{R}, \quad J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega F(u),$$

is of class $\mathcal{C}^2$ and satisfies $J'(u) = u - Au$ with $Au = (-\Delta)^{-1}f(u)$. Assume there exist $\varphi_1 \ll \varphi_2 \ll \psi_2 \ll \psi_1$ such that $\varphi_i \ll A\varphi_i$ and $A\psi_i \ll \psi_i$ for $i = 1,2$. Here, we use $u \ll v$ to denote $u, v \in C_0^1(\overline{\Omega})$ satisfying $u(x) < v(x)$ for $x \in \Omega$ and $\partial u/\partial\nu > \partial v/\partial\nu$ on $\partial\Omega$, where ν is the outer unit normal to $\partial\Omega$. In this case the spaces from the first paragraph of Section 2 are defined as $X = H_0^1(\Omega)$, $Y = C_0^1(\overline{\Omega})$, and $Z = C_0^{1,\alpha}(\overline{\Omega})$, with α a fixed number in $(0,1)$. Set $D_i = \{u \in Y : \varphi_i \le u \le \psi_i\}$ for $i = 1,2$ and consider the negative gradient flow φ^t for J, so φ^t solves (3.10) with $B = A$. For $u \in D_1 \setminus D_2$ we define

$$\tau^*(u) := \sup\{\tau : 0 < \tau < \tau(u) \text{ and } \varphi^t(u) \in D_1 \setminus D_2 \text{ for } 0 \le t \le \tau\}.$$

Claim: If $u \in D_1 \setminus D_2$ and $0 < \tau^*(u) < \tau(u)$, then τ^* is discontinuous at u with respect to $\|\cdot\|_X$.

Since $0 < \tau^*(u) < \tau(u)$ we have $\varphi^{\tau^*(u)}(u) \in \partial_{D_1}(D_2) = \partial_Y(D_2)$. It follows that $\varphi_2 \le \varphi^{\tau^*(u)}(u) \le \psi_2$ but not $\varphi_2 \ll \varphi^{\tau^*(u)}(u) \ll \psi_2$. Without loss of generality we assume $\varphi_2 \not\ll \varphi^{\tau^*(u)}(u)$. Then there exists $x_0 \in \Omega$ such that

$$\varphi^{\tau^*(u)}(u)|_{x_0} = \varphi_2(x_0), \tag{3.13}$$

or there exists $x_0 \in \partial\Omega$ such that

$$\frac{\partial}{\partial\nu}\varphi^{\tau^*(u)}(u)|_{x_0} = \frac{\partial}{\partial\nu}\varphi_2(x_0). \tag{3.14}$$

Choose $u_n \in D_1 \setminus D_2$, $n \in \mathbb{N}$, and $\delta > 0$ such that $\|u_n - u\|_X \to 0$, and

$$(u_n - u)|_{x_0} < -\delta \tag{3.15}$$

or

$$\frac{\partial}{\partial\nu}(u_n - u)|_{x_0} > \delta \tag{3.16}$$

hold for all n, respectively. We choose $\delta_1 > 0$ with $0 < \tau^*(u) - \delta_1 < \tau^*(u) + \delta_1 < \tau(u)$ and such that

$$\|\varphi^t(u) - \varphi^{\tau^*(u)}(u)\|_Y < \frac{\delta}{4}e^{-(\tau^*(u)+\delta_1)} \quad \text{if } |t - \tau^*(u)| < \delta_1. \tag{3.17}$$

We fix $p > N$ and want to show that

$$\|f(\varphi^t(u_n)) - f(\varphi^t(u))\|_p \overset{n\to\infty}{\longrightarrow} 0 \quad \text{uniformly in } t \in [0, \tau^*(u) + \delta_1]. \tag{3.18}$$

In order to see this we set $L := \|f'\|_\infty$ and fix $T > \max\{\|\varphi_1\|_\infty, \|\psi_1\|_\infty\}$. Let $\epsilon > 0$ be arbitrary and choose $\epsilon_1 < \epsilon/(2|\Omega|)^{1/p}L$. Since $\|u_n - u\|_X \to 0$ as $n \to \infty$ there exists n_0 such that

$$\|\varphi^t(u_n) - \varphi^t(u)\|_2 < \frac{\epsilon_1 \cdot \epsilon^{p/2}}{\sqrt{2}(2LT)^{p/2}} \quad \text{for } n \geq n_0 \text{ and } 0 \leq t \leq \tau^*(u) + \delta_1.$$

Denote $\Omega_{t,n} := \{x \in \Omega : |(\varphi^t(u_n)(x) - \varphi^t(u)(x))| \geq \epsilon_1\}$. Then we have

$$|\Omega_{t,n}| < \frac{\epsilon^p}{2(2LT)^p} \quad \text{for } n \geq n_0 \text{ and } 0 \leq t \leq \tau^*(u) + \delta_1.$$

This implies, for $n \geq n_0$ and $0 \leq t \leq \tau^*(u) + \delta_1$,

$$\begin{aligned} \|f(\varphi^t(u_n)) - f(\varphi^t(u))\|_p^p &= \Big(\int_{\Omega_{t,n}} + \int_{\Omega\setminus\Omega_{t,n}}\Big)|f(\varphi^t(u_n)) - f(\varphi^t(u))|^p \\ &\leq (2LT)^p|\Omega_{t,n}| + (L\epsilon_1)^p|\Omega| < \epsilon^p. \end{aligned}$$

This proves (3.18). Since $L^p(\Omega) \overset{(-\Delta)^{-1}}{\longrightarrow} W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \hookrightarrow C_0^1(\overline{\Omega}) = Y$, there exists $n_1 \geq n_0$ such that

$$e^{-t}\int_0^t e^s\|A(\varphi^s(u_n)) - A(\varphi^s(u))\|_Y ds < \frac{\delta}{4}e^{-(\tau^*(u)+\delta_1)} \tag{3.19}$$

for $n \geq n_1$ and $t \in [0, \tau^*(u) + \delta_1]$. Note that

$$\begin{aligned} \varphi^t(u_n) - \varphi^{\tau^*(u)}(u) &= \big(\varphi^t(u) - \varphi^{\tau^*(u)}(u)\big) + e^{-t}(u_n - u) \\ &\quad + e^{-t}\int_0^t e^s\big(A(\varphi^s(u_n)) - A(\varphi^s(u))\big)ds. \end{aligned}$$

If (3.13) holds, then (3.15), (3.17), and (3.19) imply

$$(\varphi^t(u_n) - \varphi^{\tau^*(u)}(u))|_{x_0} < -\frac{\delta}{2}e^{-(\tau^*(u)+\delta_1)}$$

for $t \in [\tau^*(u) - \delta_1, \tau^*(u) + \delta_1]$ and $n \geq n_1$. In case (3.14) we have

$$\frac{\partial}{\partial\nu}(\varphi^t(u_n) - \varphi^{\tau^*(u)}(u))|_{x_0} > \frac{\delta}{2}e^{-(\tau^*(u)+\delta_1)}$$

by (3.16)–(3.19). Therefore $\varphi^t(u_n) \notin D_2$ if $t \in [\tau^*(u) - \delta_1, \tau^*(u) + \delta_1]$ and $n \geq n_1$. Thus we have proved the claim that τ^* is discontinuous at u with respect to $\|\cdot\|_X$. As a consequence of the claim, τ_1 and τ_2 are discontinuous with respect to $\|\cdot\|_X$ in this example.

For related deformation theorems with respect to $\|\cdot\|_X$ we refer the reader to [13].

4. Proofs of the critical-point theorems

The proof of Theorem 2.1 is similar to the proof of Theorem 2.2 and somewhat easier. We therefore leave this proof to the reader and proceed to the

Proof of Theorem 2.2. First we observe that $c_1 < \inf_{u \in D_4} J(u)$. For $b \in \mathbb{R}$ we set $\mathcal{R}^b = (J^b \cap D_1) \cup (D_2 \cup D_3 \setminus D_4)$ considered as a subspace of Y. Given $0 < \epsilon < \inf_{u \in D_4} J(u) - c_1$ it follows from the definition of c_1 that there is a path $h : [0,1] \to \mathcal{R}^{c_1+\epsilon}$ joining $h(0) \in D_2 \setminus D_4$ with $h(1) \in D_3 \setminus D_4$. It also follows that $h(0)$ and $h(1)$ lie in different path components of $\mathcal{R}^{c_1-\epsilon}$. Now the long exact homology sequence of the pair $(\mathcal{R}^{c_1+\epsilon}, \mathcal{R}^{c_1-\epsilon})$ implies that $H_1(\mathcal{R}^{c_1+\epsilon}, \mathcal{R}^{c_1-\epsilon}) \neq 0$. Recalling the notation $\mathcal{Q}^a = (J^a \cap D_1) \cup D_2 \cup D_3$ from Section 3, the excision property of homology groups yields $H_1(\mathcal{Q}^{c_1+\epsilon}, \mathcal{Q}^{c_1-\epsilon}) \neq 0$. By Theorem 3.1 there exists a sequence of critical points $v_k \in \mathcal{M}_{c_1-1/k}^{c_1+1/k} \cap K$, $k \in \mathbb{N}$. Here $\mathcal{M}_a^b = J^b \cap D_1 \setminus \big(J^a \cup K_b \cup D_2 \cup D_3\big)$ as in Section 3. The Palais-Smale condition implies (possibly after passing to a subsequence) $\lim_{n\to\infty} \|v_n - w_1\|_X = 0$ for some $w_1 \in \widetilde{D}_1 \cap K_{c_1} \subset D_1$. If $w_1 \in D_2 \cup D_3$, then we would have $w_1 = Aw_1 \in (\mathrm{int}_{D_1}(D_2)) \cup (\mathrm{int}_{D_1}(D_3))$. Since $v_n = Av_n \in Z$ the sequence $(v_n)_n$ is relatively compact in Y; hence, $\lim_{n\to\infty} \|v_n - w_1\|_Y = 0$. Then $v_n \in (\mathrm{int}_{D_1}(D_2)) \cup (\mathrm{int}_{D_1}(D_3))$, which contradicts the fact that $v_n \in \mathcal{M}_{c_1-1/n}^{c_1+1/n}$. Therefore $w_1 \in (K_{c_1} \cap D_1) \setminus (D_2 \cup D_3) = K_{c_1}^*$.

Now suppose $(K \cap D_1) \setminus (D_2 \cup D_3)$ is a finite set. Theorem 3.1 and the excision property of homology theory yield for $\epsilon > 0$ small enough

$$0 \neq H_1(\mathcal{Q}^{c_1+\epsilon}, \mathcal{Q}^{c_1-\epsilon}) \cong H_1(\mathcal{Q}^{c_1}, \mathcal{Q}_*^{c_1}) \cong H_1(\mathcal{S}^{c_1}, \mathcal{S}^{c_1} \setminus K_{c_1}),$$

where $\mathcal{S}^{c_1} = (J^{c_1} \cap D_1) \setminus (D_2 \cup D_3)$. Now suppose that

$$\mathcal{S}^{c_1} \cap K = \{u_1^*, u_2^*, \ldots, u_m^*\} \subset \mathrm{int}_Y(D_1).$$

Then we have for $r > 0$ small enough

$$
\begin{aligned}
&H_1(\mathcal{S}^{c_1}, \mathcal{S}^{c_1} \setminus K_{c_1}) \\
&\cong H_1\Big(\mathcal{S}^{c_1} \cap \Big(\bigcup_{i=1}^m B_Y(u_i^*, r)\Big), (\mathcal{S}^{c_1} \setminus K_{c_1}) \cap \Big(\bigcup_{i=1}^m B_Y(u_i^*, r)\Big)\Big) \\
&\cong \bigoplus_{i=1}^m H_1\big(J^{c_1} \cap B_Y(u_i^*, r), (J^{c_1} \cap B_Y(u_i^*, r)) \setminus \{u_i^*\}\big) \cong \bigoplus_{i=1}^m C_1(J, u_i^*),
\end{aligned}
$$

where $B_Y(u_i^*, r) = \{u \in Y : \|u - u_i^*\|_Y < r\}$. Thus there exists $w_1 \in K_{c_1}^*$ with $C_1(J, w_1) \neq 0$.

Next we prove the result concerning w_2. First we observe that $h_2 \in \Gamma_2$. Setting $C_i := h_i([0,1])$, $i = 1, 2$, we write

$$i^\gamma : C_1 \cup C_2 \hookrightarrow \mathcal{Q}^\gamma = (J^\gamma \cap D_1) \cup D_2 \cup D_3$$

for the inclusion and $i_*^\gamma : H_1(C_1 \cup C_2) \to H_1(\mathcal{Q}^\gamma)$ for the homomorphism induced on the first homology level. Now we can define the critical value

$$c_2 := \sup\{\gamma \geq \max J \circ h_1 : i_*^\gamma \text{ is a monomorphism}\}.$$

Since the inclusion

$$C_1 \cup C_2 \hookrightarrow P := \{(1-s)h_1(t) + sh_2(t) : 0 \leq s, t \leq 1\}$$

is nullhomotopic, we have $i_*^\gamma = 0$ for $\gamma \geq \max J(P)$, and therefore $c_2 \leq \max J(P)$. In order to see $c_2 \geq \gamma_2$ we observe that $\gamma_2 > \max J \circ h_1$ by (2.2). Let α be any number satisfying $\max J \circ h_1 < \alpha < \gamma_2$ and define $A_1 = J^\alpha \cap D_1$ and $A_2 = D_2 \cup D_3$ so that $\mathcal{Q}^\alpha = A_1 \cup A_2$. We consider the following diagram of Mayer-Vietoris sequences:

$$
\begin{array}{ccccccc}
H_1(C_1) \oplus H_1(C_2) & \to & H_1(C_1 \cup C_2) & \xrightarrow{\partial} & \widetilde{H}_0(C_1 \cap C_2) & \to & \widetilde{H}_0(C_1) \oplus \widetilde{H}_0(C_2) \\
 & & \downarrow i_*^\alpha & & \downarrow j_* & & \\
H_1(A_1) \oplus H_1(A_2) & \to & H_1(A_1 \cup A_2) & \to & \widetilde{H}_0(A_1 \cap A_2) & \to & \widetilde{H}_0(A_1) \oplus \widetilde{H}_0(A_2),
\end{array}
$$

where j_* is induced by inclusion. We may assume without loss of generality that the paths h_1 and h_2 contain no loops so that $H_1(C_1) \oplus H_1(C_2) \cong \widetilde{H}_0(C_1) \oplus \widetilde{H}_0(C_2) = 0$. Then the exactness of the first row implies that ∂ is an isomorphism. Since $\alpha < \gamma_2$ there is no path in $J^\alpha \cap (D_2 \cup D_3) = A_1 \cap A_2$ joining $h_1(0)$ and $h_1(1)$. Thus $C_1 \cap C_2$ consists of two points which are in different path components of $A_1 \cap A_2$. Hence j_* is a monomorphism which implies that i_*^α is also a monomorphism. This proves $c_2 \geq \alpha$ and thus $c_2 \geq \gamma_2$. In order to see that c_2 is in fact a critical value we fix any $\epsilon > 0$ with $\epsilon < c_2 - \max J \circ h_1$ and write $i : \mathcal{Q}^{c_2 - \epsilon} \hookrightarrow \mathcal{Q}^{c_2+\epsilon}$ for the inclusion. Since $i^{c_2+\epsilon} = i \circ i^{c_2-\epsilon}$, we see that i_* is not a monomorphism, and therefore

$H_2(\mathcal{Q}^{c_2+\epsilon}, \mathcal{Q}^{c_2-\epsilon}) \neq 0$. Since this holds for every $\epsilon > 0$ small, a similar argument as in the proof of part a) yields a critical point $w_2 \in K^*_{c_2}$.

If J has only a finite number of critical points in $D_1 \setminus (D_2 \cup D_3)$, then we can choose as before $w_2 \in K^*_{c_2}$ with $C_2(J, w_2) \neq 0$. □

In order to prove Theorem 2.3 we recall the notion of genus. Given a space A with continuous involution $A \ni a \mapsto \overline{a} \in A$, the genus of A is defined by

$$\mathrm{gen(A)} := \inf\{\mathrm{k} \geq 0 : \exists \mathrm{f} \in \mathcal{C}(\mathrm{A}, \mathbb{R}^\mathrm{k} \setminus \{0\}), \mathrm{f \text{ is odd}}\}.$$

Here we call a map $f : A \to B$ between spaces with involution odd if $f(\overline{a}) = \overline{f(a)}$ for every $a \in A$. The involution on S^{k-1} is the antipodal map, as always in this paper where all spaces are subsets of vector spaces. As usual we set $S^{-1} = \emptyset$ and $\inf \emptyset = \infty$. The Borsuk-Ulam theorem implies $\mathrm{gen(S^{k-1})} = \mathrm{k}$. We make use of the following properties of the genus (all spaces have an involution, subspaces are invariant).

Monotonicity. If there exists an odd continuous map $f : A \to B$, then $\mathrm{gen(A)} \leq \mathrm{gen(B)}$.

Subadditivity. If A is a metric space, $A_1, A_2 \subset A$ are locally closed, and $A = A_1 \cup A_2$, then $\mathrm{gen(A)} \leq \mathrm{gen(A_1)} + \mathrm{gen(A_2)}$.

Continuity. If A is a metric space and $B \subset A$ is closed, then there exists an open neighborhood U of B in A with $\mathrm{gen(U)} = \mathrm{gen(B)}$.

A set $B \subset A$ is locally closed if $B = O \cap C$ is the intersection of an open subset O and a closed subset C of A. The proofs are easy and essentially standard except, maybe, for the subadditivity, which we did not find in the literature. This property is first proved if A_1 and A_2 are both open. The general version then follows easily by applying the continuity property. The more general version yields in particular that $\mathrm{gen(A)} \leq \mathrm{gen(B)} + \mathrm{gen(A \setminus B)}$ if A is a metric space and $B \subset A$ is closed or open.

Let $\varphi^t(u)$ be the solution of (3.10) for $u \in Y_0 = Y \setminus K$ $(0 \leq t < \tau(u))$. For two positive invariant sets $F \subset M \subset Y_0$ we define

$$C_M(F) := \{u \in M : u \in K \cap F \text{ or } \varphi^t(u) \in F \text{ for some } 0 < t < \tau(u)\}. \quad (4.1)$$

We write $\partial_M C_M(F)$ to denote the relative boundary of $C_M(F)$ in M; see [33] for properties of $\partial_M C_M(F)$.

Lemma 4.1. *Suppose the hypotheses of Theorem* 2.3 *hold. Then we have*

a) $G \setminus (D_2 \cup D_3) \neq \emptyset$ *for any* $G = -G \subset D_1 \setminus D_4$ *with* $\mathrm{gen(G)} \geq 2$.

b) *If there exists an odd map* $h_0 \in \mathcal{C}(S^{k-1}, D_1)$ *with*

$$\max_{x \in S^{k-1}} J(h_0(x)) < \inf_{u \in D_4} J(u), \quad (4.2)$$

then

$$\big(G \setminus (D_2 \cup D_3)\big) \cap \partial_{D_1}(C_{D_1}(D_4)) \neq \emptyset$$

for any $G = h(B^k \setminus B)$ *where* $h \in \mathcal{C}(B^k, D_1)$ *is odd,* $h(x) = h_0(x)$ *if* $x \in S^{k-1}$*,* $B = -B \subset B^k$ *is locally closed, and* $\mathrm{gen(B)} \le \mathrm{k-2}$.

Proof. a) This is a simple consequence of the standard properties of the genus.

b) Consider $G = h(B^k \setminus B)$ as in the statement of the theorem, and set $\mathcal{O} := \{x \in B^k : h(x) \in C_{D_1}(D_4)\}$. By Lemma 3.5 the set $C_{D_1}(D_4)$ is an open, symmetric subset of D_1; hence, $\mathcal{O}$ is an open, symmetric subset of B^k. By (4.2) and the definition of $C_{D_1}(D_4)$ in (4.1) we have

$$\sup_{x\in S^{k-1}} J(h(x)) < \inf_{u\in D_4} J(u) = \inf_{u\in C_{D_1}(D_4)} J(u) \le \inf_{x\in\mathcal{O}} J(h(x)).$$

Therefore $\overline{\mathcal{O}} \subset B^k \setminus S^{k-1}$ and $\mathcal{O}$ is a symmetric, bounded, open neighborhood of 0 in $\mathbb{R}^k$, which implies $\mathrm{gen}(\partial\mathcal{O}) = \mathrm{k}$. Since $C_{D_1}(D_4)$ is open in D_1 and h is continuous, we have $h(\partial\mathcal{O}) \subset \partial_{D_1}(C_{D_1}(D_4))$, and therefore $h\big(\partial\mathcal{O} \setminus B\big) \subset G \cap \partial_{D_1}(C_{D_1}(D_4))$. This implies

$$\mathrm{gen(G \cap \partial_{D_1} C_{D_1}(D_4))} \ge \mathrm{gen}\big(\partial\mathcal{O} \setminus \mathrm{B}\big) \ge \mathrm{gen}(\partial\mathcal{O}) - \mathrm{gen(B)} \ge 2.$$

Now b) follows from a) applied to $G \cap \partial_{D_1}(C_{D_1}(D_4))$. □

Proof of Theorem 2.3. We first prove the existence of $k-1$ pairs of critical points in $D_1 \setminus (D_2 \cup D_3)$ with energy in $[\inf J|_{D_1}, \max J \circ h_0]$. For $j = 2, 3, \dots, k$ we define $\Gamma_j := \{G \setminus (D_2 \cup D_3) : G = -G \subset D_1 \setminus D_4,\ \mathrm{gen(G)} \ge \mathrm{j}\}$ and

$$c_j = \inf_{C\in\Gamma_j} \sup_{u\in C} J(u).$$

Throughout this proof subsets of Y are considered with the Y-topology. Observe that $h_0(S^{k-1}) \setminus (D_2 \cup D_3) \in \Gamma_k$ by (2.3). According to Lemma 4.1a) we have

$$-\infty < \inf_{u\in D_1} J(u) \le c_2 \le \dots \le c_k \le \max_{x\in S^{k-1}} J(h_0(x)) < \inf_{u\in D_4} J(u).$$

For the first statement of Theorem 2.3 it remains to prove that $\mathrm{gen(K_c^*)} \ge \mathrm{l+1}$ if $c = c_1 = \dots = c_{i+l}$ for some $2 \le i \le i+l \le k$. Since $K_c^* \subset Y$ is compact, we may choose an open, symmetric neighborhood $\mathcal{O}$ of K_c^* in X such that $\mathrm{gen}(\mathcal{O}) = \mathrm{gen(K_c^*)}$. By Theorem 3.2 there exist $\epsilon > 0$ and an odd continuous map $\sigma^* : (\mathcal{Q}^{c+\epsilon} \setminus \mathcal{O}) \cup \mathcal{Q}^{c-\epsilon} \to \mathcal{Q}^{c-\epsilon}$ with $\sigma^*|_{\mathcal{Q}^{c-\epsilon}} = \mathrm{id}$ and $J(\sigma^*(u)) \le J(u)$ for $u \in \mathcal{Q}^{c+\epsilon} \setminus \mathcal{O}$. We can assume that $c + \epsilon < \inf J(D_4)$ because $c < \inf J(D_4)$. Since $c = c_{i+l}$, there exists $C = G \setminus (D_2 \cup D_3) \in \Gamma_{i+l}$ with $\sup J(C) < c + \epsilon$. Clearly $G' := \sigma^*(G \setminus \mathcal{O})$ satisfies $G' = -G' \subset \mathcal{Q}^{c-\epsilon}$; hence, $\sup J(G' \setminus (D_2 \cup D_3)) \le c - \epsilon = c_i - \epsilon$.

Next we show that $G' \subset D_1 \setminus D_4$. Consider an arbitrary element $\sigma^*(u) \in G'$, $u \in G \setminus \mathcal{O}$. If $\sigma^*(u) \in J^{c+\epsilon}$, then $\sigma^*(u) \notin D_4$ because $c + \epsilon < \inf J(D_4)$. On the other hand, if $\sigma^*(u) \notin J^{c+\epsilon}$, then $u \in D_2 \cup D_3$, and therefore $\sigma^*(u) = u \in G \subset D_1 \setminus D_4$. In any case $\sigma^*(u) \notin D_4$.

As a consequence of this we obtain $C = G' \setminus (D_2 \cup D_3) \in \Gamma_j$ with $j = \mathrm{gen}(\mathrm{G}')$. Therefore $c_j \leq \sup J(G' \setminus (D_2 \cup D_3)) < c_i$, and thus $j = \mathrm{gen}(\mathrm{G}') \leq \mathrm{i} - 1$. The standard properties of the genus imply

$$\mathrm{gen}(\mathrm{G}') + \mathrm{gen}(\mathrm{K}_\mathrm{c}^*) \geq \mathrm{gen}(\mathrm{G} \setminus \mathcal{O}) + \mathrm{gen}(\mathcal{O}) \geq \mathrm{gen}(\mathrm{G}) \geq \mathrm{i} + \mathrm{l};$$

hence, $\mathrm{gen}(\mathrm{K}_\mathrm{c}^*) \geq \mathrm{l} + 1$.

Now we prove the existence of $k-1$ pairs of critical points in $D_1 \setminus (D_2 \cup D_3)$ with energy in $[\inf J|_{D_4}, \max J \circ h]$, where $h \in \mathcal{C}(B^k, D_1)$ is any odd extension of h_0 to the closed unit ball $B^k \subset \mathbb{R}^k$. Setting

$$G := \{h \in \mathcal{C}(B_k, D_1) : \text{ h is odd and } h(x) = h_0(x) \text{ for } x \in S_{k-1}\}$$

and, for $j = 2, \dots, k$,

$$\Delta_j := \{h(B^k \setminus B) \setminus (D_2 \cup D_3) : \ h \in G,\, B = -B \subset B^k \text{ open}, \\ \mathrm{gen}(\mathrm{B}) \leq \mathrm{k} - \mathrm{j}\}$$

we define the critical values by

$$d_j = \inf_{C \in \Delta_j} \max_{u \in C} J(u), \quad j = 2, \dots, k.$$

Clearly we have $d_2 \leq \cdots \leq d_k \leq \max J \circ h$ for any $h \in G$. Moreover, Lemma 4.1b) and the definition of d_k imply

$$d_2 \geq \inf_{u \in \partial_{D_1} C_{D_1}(D_4)} J(u) \geq \inf_{u \in C_{D_1}(D_4)} J(u) = \inf_{u \in D_4} J(u) > -\infty\,. \tag{4.3}$$

It remains to prove that $\mathrm{gen}(\mathrm{K}_\mathrm{c}^*) \geq \mathrm{l} + 1$ if $c = d_i = \cdots = d_{i+l}$ for some $2 \leq i \leq i + l \leq k$. As above we choose an open, symmetric neighborhood $\mathcal{O}$ of K_c^* in X such that $\mathrm{gen}(\mathcal{O}) = \mathrm{gen}(\mathrm{K}_\mathrm{c}^*)$. We also choose $\epsilon > 0$ with $c - \epsilon > \max J \circ h_0$, and an odd and continuous map $\sigma^* : (\mathcal{Q}^{c+\epsilon} \setminus \mathcal{O}) \cup \mathcal{Q}^{c-\epsilon} \to \mathcal{Q}^{c-\epsilon}$ with $\sigma^*|_{\mathcal{Q}^{c-\epsilon}} = \mathrm{id}$ according to Theorem 3.2. By the definition of $c = c_{i+l}$ there exists $h \in G$ and $B = -B \subset B^k$ open with $\mathrm{gen}(\mathrm{B}) \leq \mathrm{k} - \mathrm{i} - \mathrm{l}$ so that $\sup J(C) < c + \epsilon$ for $C := h(B^k \setminus B) \setminus (D_2 \cup D_3) \in \Delta_{i+l}$. Setting $B' := B \cup h^{-1}(\mathcal{O})$ we have

$$h(B^k \setminus B') = h(B^k \setminus B) \setminus \mathcal{O} \subset (\mathcal{Q}^{c+\epsilon} \setminus \mathcal{O}) \cup \mathcal{Q}^{c-\epsilon},$$

and thus $\sigma^* \circ h(B^k \setminus B') \subset \mathcal{Q}^{c-\epsilon}$. Observing that $h' := \sigma^* \circ h \in G$ we obtain

$$C' := h'(B^k \setminus B') \setminus (D_2 \cup D_3) \in \Delta_j \quad \text{with } j := k - \mathrm{gen}(\mathrm{B}').$$

The definition of d_j now implies $d_j \le \max J(C') \le c-\epsilon = d_i-\epsilon < d_i$; hence, $\mathrm{gen}(\mathrm{B}') = \mathrm{k}-\mathrm{j} \ge \mathrm{k}+1-\mathrm{i}$. The properties of the genus yield the desired result:

$$\begin{aligned}\mathrm{gen}(\mathrm{K}_\mathrm{c}^*) = \mathrm{gen}(\mathcal{O}) &\ge \mathrm{gen}\big(\mathrm{h}^{-1}(\mathcal{O})\big) \ge \mathrm{gen}(\mathrm{B}') - \mathrm{gen}(\mathrm{B}) \\ &\ge k+1-i-(k-i-l) = l+1\end{aligned}$$

5. AN APPLICATION

We now apply our results to the semilinear eigenvalue problem (1.2)

$$\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega \end{cases}$$

from the introduction. Here λ is a positive parameter and $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain. The nonlinearity $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ or a suitable modification of f will be subcritical so that solutions of (1.2) correspond to critical points of the energy functional $J_\lambda : X = H_0^1(\Omega) \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda \int_\Omega F(x,u).$$

Here $F(x,t) := \int_0^t f(x,s)ds$ as usual. We need various hypotheses on f.

(f_0) $f \in \mathcal{C}^0(\overline{\Omega} \times \mathbb{R})$ satisfies $f(x,0) = 0$.
(f_1) $\lim_{t\to 0} f(x,t)/t = 0$ uniformly in x.
(f_2) There exist $t_- < 0 < t_+$ such that $f(x,t_-) = f(x,t_+) = 0$.
(f_3) There exists $\delta > 0$ such that $F(x,t) > 0$ for $0 < |t| \le \delta$.
(f_4) $\lim_{|t|\to\infty} f(x,t)/t = 0$ uniformly in x.
(f_5) $\lim_{|t|\to\infty} F(x,t) = +\infty$ uniformly in x.

In the following we use the notation

$$[t_-, t_+] := \{u \in Y : t_- \le u(x) \le t_+ \text{ for } x \in \overline{\Omega}\}$$

for the order interval in $Y = X \cap \mathcal{C}^1(\overline{\Omega})$ and set $\alpha_\lambda := \inf J_\lambda|_{[t_-,t_+]}$.

Theorem 5.1. *Assume* (f_0)–(f_3). *Then there exists* $\Lambda > 0$ *such that problem* (1.2) *with* $\lambda \ge \Lambda$ *has at least six solutions* u_1, u_2, v_1, v_2, w_1, *and* w_2 *with the following properties.*

(i) $u_1, u_2, v_1, v_2, w_1, w_2 \in [t_-, t_+]$.
(ii) u_1 *and* u_2 *are positive,* v_1 *and* v_2 *are negative, and* w_1 *and* w_2 *are sign changing.*
(iii) $\max\{J_\lambda(u_1),\ J_\lambda(v_1)\} \le J_\lambda(w_1) < 0$,
$\min\{J_\lambda(u_2),\ J_\lambda(v_2)\} > 0$,
$\max\{J_\lambda(u_2),\ J_\lambda(v_2)\} \le J_\lambda(w_2)$.

(iv) *If f is of class $\mathcal{C}^1$ and if* (1.2) *has only a finite number of positive solutions inside $[t_-, t_+]$, then $C_q(J_\lambda, u_1) \cong \delta_{q0}G$ and $C_q(J_\lambda, u_2) \cong \delta_{q1}G$ for all $q \in \mathbb{Z}$.*

(v) *If f is of class $\mathcal{C}^1$ and if* (1.2) *has only a finite number of negative solutions inside $[t_-, t_+]$, then $C_q(J_\lambda, v_1) \cong \delta_{q0}G$ and $C_q(J_\lambda, v_2) \cong \delta_{q1}G$ for all $q \in \mathbb{Z}$.*

(vi) *If f is of class $\mathcal{C}^1$ and if* (1.2) *has only a finite number of sign-changing solutions inside $[t_-, t_+]$, then $C_q(J_\lambda, w_1) \cong \delta_{q1}G$ for all $q \in \mathbb{Z}$ and $C_2(J_\lambda, w_2) \neq 0$.*

As a counterpart of Theorem 5.1, we have

Theorem 5.2. *Assume (f_0), (f_2), (f_4), and (f_5). Then there exists $\Lambda > 0$ such that problem* (1.2) *with $\lambda \geq \Lambda$ has at least six solutions u_1, u_2, v_1, v_2, w_1, and w_2 with the following properties.*

(i) $\min_\Omega v_i$, $\min_\Omega w_i < t_- < t_+ < \max_\Omega u_i$, $\max_\Omega w_i$.

(ii) *u_1 and u_2 are positive, v_1 and v_2 are negative, and w_1 and w_2 are sign changing.*

(iii) *$J_\lambda(u_1)$, $J_\lambda(v_1)$, $J_\lambda(w_1) < \alpha_\lambda < J_\lambda(u_2)$, $J_\lambda(v_2)$, $J_\lambda(w_2)$, where $\alpha_\lambda = \inf J_\lambda|_{[t_-, t_+]}$.*

In addition, (iv)–(vi) *of Theorem* 5.1 *continue to hold, with the word "inside" being replaced by "outside."*

The third theorem is

Theorem 5.3. *Assume (f_0), (f_1), (f_4), and, in addition (f_3) or (f_5). Then the same conclusions as in Theorem* 5.1 *hold with the words "inside $[t_-, t_+]$" being removed.*

If f is odd in t, then we can say much more about the number of sign-changing solutions.

Theorem 5.4. *Suppose in addition to the assumptions from Theorem* 5.1 *that f is odd in t. Then for any $k \in \mathbb{N}$ there exists $\Lambda_k > 0$ such that problem* (1.2) *with $\lambda \geq \Lambda_k$ has at least k pairs of sign-changing solutions inside $[t_-, t_+]$ with negative energy and at least k pairs of sign-changing solutions inside $[t_-, t_+]$ with positive energy.*

Theorem 5.5. *Suppose in addition to the assumptions from Theorem* 5.2 *that f is odd in t. Then for any $k \in \mathbb{N}$ there exists $\Lambda_k > 0$ such that problem* (1.2) *with $\lambda \geq \Lambda_k$ has at least k pairs of sign-changing solutions with energy less than α_λ and at least k pairs of sign-changing solutions with energy larger than α_λ. These solutions u satisfy*

$$\min_\Omega u < t_-, \quad \max_\Omega u > t_+.$$

Theorem 5.6. *Suppose in addition to the assumptions from Theorem* 5.3 *that f is odd in t. Then the same conclusions as in Theorem* 5.4, *with the words "inside $[t_-, t_+]$" being removed, hold.*

Of course, if the conditions (f_0)–(f_5) are all satisfied one can combine the above theorems to have new conclusions.

Remark 5.7. a) If $\lambda > 0$ is small, then problem (1.2) may have no nontrivial solution even if the conditions (f_0)–(f_5) are all satisfied.

b) Theorems 5.1 to 5.6 are valid if $-\Delta$ is replaced with general second-order elliptic operators of divergence form. Extensions and related results for the p-Laplace operator can be found in [5, 6].

c) The existence of two positive solutions and two negative solutions are well known in Theorems 5.1 to 5.3 (cf. [9, 12, 26]). The existence of the sign-changing solutions as well as the information on the location and the critical groups are new.

d) In Theorems 5.4 to 5.6, the sign-changing solutions have different features in terms of energy, location, and Morse indices. The existence of the solutions could also be proved using the methods from [2, 14, 37]. These methods however do not yield information about nodal properties of the solutions. Theorem 5.4 improves a result from Li and Wang [29, Theorem 4.7].

6. Proofs of the theorems from Section 5

We shall prove Theorems 5.1 to 5.6 under only slightly stronger assumptions. Namely, we require

- (f_0') $f \in \mathcal{C}^0(\overline{\Omega} \times \mathbb{R})$ satisfies $f(x,0) = 0$ and $\limsup_{t\to 0} |f(x,t)/t| < \infty$,
- (f_2') There exist $t_- < 0 < t_+$ such that $f(x,t_-) = f(x,t_+) = 0$ and $\liminf_{t\to t_\mp \pm 0} f(x,t)/(t - t_\mp) > -\infty$,

instead of (f_0) and (f_2), respectively. If the additional inequalities in (f_0') or (f_2') do not hold we may use an approximation argument. We leave this approximation argument to the reader.

Let $\lambda_1 < \lambda_2 \le \cdots$ be the eigenvalues of $-\Delta$ on $H_0^1(\Omega)$, and let $(e_i)_1^\infty$ be an orthonormal basis of $X := H_0^1(\Omega)$ consisting of associated eigenfunctions with $e_1 > 0$. We write $Y = X \cap \mathcal{C}^1(\overline{\Omega})$ and $Z = X \cap \mathcal{C}^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. For $u, v \in Y$ and $u \le v$ we use $[u, v]$ to represent the order interval $\{w \in Y : u \le w \le v\}$.

Proof of Theorem 5.1. We may assume $f(x,t) = 0$ for all $t \notin [t_-, t_+]$. Then all solutions of (1.2) are inside $[t_-, t_+]$, and J_λ is of class $\mathcal{C}^1$ satisfying the Palais-Smale condition. As a consequence of (f_0') and (f_2') there exists a number $K > 0$ such that

$$Kt_- < f(x,t) + Kt < Kt_+ \qquad \text{for } t_- < t < t_+,\ x \in \Omega \tag{6.1}$$

and

$$tf(x,t) + Kt^2 > 0 \qquad \text{for } t \neq 0,\ x \in \Omega.$$

We will work with the norm

$$\|u\|_\lambda^2 = \int_\Omega |\nabla u|^2 + \lambda K \int_\Omega u^2$$

on X, which is equivalent to the usual one. With respect to this norm we have $\nabla J_\lambda(u) = u - A_\lambda(u)$ where $A_\lambda(u) = \lambda(-\Delta + \lambda K)^{-1}(f(x,u) + Ku)$. Let $\phi_1^\lambda \in Z$ and $\psi_1^\lambda \in Z$ satisfy

$$-\Delta\phi_1^\lambda + \lambda K \phi_1^\lambda = \lambda K t_- \quad \text{and} \quad -\Delta\psi_1^\lambda + \lambda K \psi_1^\lambda = \lambda K t_+.$$

The strong maximum principle implies $t_- < \phi_1^\lambda(x) < \psi_1^\lambda(x) < t_+$ for $x \in \Omega$, $\phi_1^\lambda \ll 0 \ll \psi_1^\lambda$, ϕ_1^λ is decreasing in λ, and ψ_1^λ is increasing in λ. Recall that the symbol $\ll$ was defined in Example 3.10. We choose $r > 0$ such that the path $h_1 : [0,\pi] \to Y$ defined by $h_1(t) = r(e_1 \cos \pi t + e_2 \sin \pi t)$ satisfies

$$\phi_1^1 \ll h_1(t) \ll \psi_1^1 \quad \text{and} \quad \|h_1(t)\|_\infty < \delta \qquad \text{for } 0 \le t \le 1,$$

where δ is the number from condition (f_3). This implies

$$\phi_1^\lambda \ll h_1(t) \ll \psi_1^\lambda \qquad \text{for } \lambda \ge 1 \text{ and } 0 \le t \le 1 \tag{6.2}$$

and

$$\gamma := \min_{0 \le t \le 1} \int_\Omega F(x, h_1(t))\, dx > 0.$$

Define $\Lambda := \max\{1,\ \gamma^{-1} r^2\}$ and fix a number $\lambda \ge \Lambda$. Then we have

$$\max_{0 \le t \le 1} J_\lambda(h_1(t)) = \frac{1}{2} r^2 - \lambda \min_{0 \le t \le 1} \int_\Omega F(x, h_1(t))\, dx \le -\frac{1}{2} r^2. \tag{6.3}$$

By (f_1) there exists $t_\lambda > 0$ such that

$$-\frac{\lambda_1}{2\lambda}|t| \le f(x,t) \le \frac{\lambda_1}{2\lambda}|t| \qquad \text{for } |t| \le t_\lambda. \tag{6.4}$$

Picking $\delta_1 > 0$ small enough and setting $\phi_2^\lambda := -\delta_1 e_1$ and $\psi_2^\lambda := \delta_1 e_1$ we have $\|\phi_2^\lambda\|_\infty, \|\psi_2^\lambda\|_\infty < t_\lambda$, and $\phi_1^\lambda \ll \phi_2^\lambda \ll \psi_2^\lambda \ll \psi_1^\lambda$. The strong maximum principle together with (6.1) and (6.4) yields

$$\phi_i^\lambda \ll A_\lambda(u) \ll \psi_i^\lambda \qquad \text{for } u \in X \text{ with } \phi_i^\lambda \le u \le \psi_i^\lambda,\ i = 1, 2.$$

As a consequence of (6.4) we also have

$$J_\lambda(u) \ge c_\lambda \|u\|_\lambda^2 \qquad \text{for } u \in B_X(0, r_\lambda) \cap [\phi_2^\lambda, \psi_2^\lambda] \tag{6.5}$$

for some positive constants c_λ and r_λ.

Step 1. The existence of two positive solutions.

We define $D_i^+ := [0,\, \psi_i^\lambda]$, $i = 1, 2$, so that $\mathrm{int}_{\mathrm{D}_1^+}\mathrm{D}_2^+ = \{\mathrm{u} \in \mathrm{Y} : 0 \le \mathrm{u} \ll \psi_2^\lambda\}$. From the above considerations it follows easily that the hypotheses of Theorem 2.1 are satisfied for J_λ, A_λ, D_1^+, and D_2^+. According to Theorem 2.1, J_λ has two critical points $u_1, u_2 \in D_1^+ \setminus D_2^+$. These are positive solutions of problem (1.2) such that

$$J_\lambda(u_1) = \inf_{u \in D_1^+} J_\lambda(u) < 0 \quad \text{and} \quad J_\lambda(u_2) = \inf_{h \in \Gamma^+} \max_{0 \le t \le 1} J_\lambda(h(t)) =: c^+,$$

where $\Gamma^+ = \{h \in \mathcal{C}([0,1], D_1^+) : h(0) = h_1(0),\, h(1) \in D_2^+\}$. For any $h \in \Gamma^+$ we have $\max_{0 \le t \le 1} J_\lambda(h(t)) \ge c_\lambda r_\lambda^2$ by (6.3) and (6.5), and therefore $J_\lambda(u_2) \ge c_\lambda r_\lambda^2 > 0$. We also deduce from Theorem 2.1 that

$$C_q(J_\lambda, u_1) \cong \delta_{q0} G \quad \text{for } q \in \mathbb{Z}$$

and $C_1(J_\lambda, u_2) \ne 0$. By [4, Proposition 3.3], if f is $\mathcal{C}^1$ and problem (1.2) has only a finite number of positive solutions inside $[t_-, t_+]$, then

$$C_q(J_\lambda, u_2) \cong \delta_{q1} G \qquad \text{for } q \in \mathbb{Z},$$

which proves (iv) in Theorem 5.1.

Step 2. The existence of two negative solutions.

Setting $D_i^- := [\phi_i^\lambda, 0]$, $i = 1, 2$, we proceed as in Step 1 and obtain two negative solutions $v_1, v_2 \in D_1^- \setminus D_2^-$ such that (v) in Theorem 5.1 holds. Moreover,

$$J_\lambda(v_1) = \inf_{u \in D_1^-} J_\lambda(u) < 0 \quad \text{and} \quad J_\lambda(v_2) = \inf_{h \in \Gamma^-} \max_{0 \le t \le 1} J_\lambda(h(t)) =: c^- > 0,$$

where $\Gamma^- = \{h \in \mathcal{C}([0,1], D_1^-) : h(0) \in D_2^-,\, h(1) = h_1(1)\}$.

Step 3. The existence of two sign-changing solutions.

Here we define $D_1 := [\phi_1^\lambda, \psi_1^\lambda]$, $D_2 := [\phi_2^\lambda, \psi_1^\lambda]$, $D_3 := [\phi_1^\lambda, \psi_2^\lambda]$, and $D_4 := [\phi_2^\lambda, \psi_2^\lambda]$. As a consequence of (6.2), (6.3), and (6.5), we see that $h_1(0) = re_1 \in D_2$, $h_1(1) = -re_1 \in D_3$, and

$$-\infty < \inf_{u \in D_1} J_\lambda(u) \le \max_{0 \le t \le 1} J_\lambda(h_1(t)) < \inf_{u \in D_4} J_\lambda(u) = 0.$$

According to Theorem 2.2 problem (1.2) has two sign-changing solutions $w_1, w_2 \in D_1 \setminus (D_2 \cup D_3)$ such that

$$J_\lambda(w_1) = \inf_{h \in \Gamma_1} \max_{0 \le t \le 1} J_\lambda(h(t)) =: c_1 < 0$$

and

$$J_\lambda(w_2) \ge \inf_{h \in \Gamma_2} \max_{0 \le t \le 1} J_\lambda(h(t)) =: c_2 > 0,$$

where

$$\Gamma_1 = \{h \in \mathcal{C}([0,1], D_1) : h(0) \in D_2,\, h(1) \in D_3\}$$

and

$$\Gamma_2 = \{h \in \mathcal{C}([0,1], D_2 \cup D_3) : h(0) = h_1(0),\, h(1) = h_1(1)\}.$$

Part (vi) from Theorem 5.1 holds as before.

Step 4. The energy bounds.

It remains to prove the first and the third inequality of part (iii) from Theorem 5.1. First we observe that $D_2 \cap K \subset D_1^+$ and $D_3 \cap K \subset D_1^-$. This can be easily seen from (6.4). Thus we have

$$J_\lambda(u_1) = \inf_{u \in D_1^+} J_\lambda(u) = \inf_{u \in D_2} J_\lambda(u) \quad \text{and} \quad J_\lambda(v_1) = \inf_{u \in D_1^-} J_\lambda(u) = \inf_{u \in D_3} J_\lambda(u).$$

This together with the definition of c_1 implies

$$J_\lambda(w_1) \geq \max\{J_\lambda(u_1),\ J_\lambda(v_1)\},$$

and the first part of (iii) follows. For the third inequality of (iii) it suffices to prove $c_2 \geq \max\{c^+,\, c^-\}$, that is, to prove

$$\max_{0 \leq t \leq 1} J_\lambda(h(t)) \geq \max\{c^+,\, c^-\} \quad \text{for any } h \in \Gamma_2.$$

We fix $h \in \Gamma_2$ and consider a pseudogradient vector field $B : X_0 = X \setminus K \to Z$ as in Lemma 3.3. Let $\varphi^t(u)$ be the associated maximal semiflow defined for $u \in X_0$ and $0 \leq t < \tau(u)$. Now we define for $t \in [0,1]$ with $h(t) \in (D_2 \cup D_3) \setminus (D_1^+ \cup D_1^-)$

$$\tau_t := \sup\{\tau : 0 < \tau < \tau(h(t)) \text{ and } \varphi^s(h(t)) \notin D_1^+ \cup D_1^- \text{ for } 0 \leq s \leq \tau\}.$$

Since $\mathrm{clos}_Y((D_2 \cup D_3) \setminus (D_1^+ \cup D_1^-)) \cap K = \{0\}$ we have $\varphi^{\tau_t}(h(t)) \in \partial_{D_2}(D_1^+) \cup \partial_{D_3}(D_1^-)$ if $\tau_t < \tau(h(t))$, and $\varphi^s(h(t)) \in \mathrm{int}_Y(D_1^+) \cup \mathrm{int}_Y(D_1^-)$ if $\tau_t < s < \tau(h(t))$. Now an argument similar to that in the proof of Lemma 3.9 shows that $\varphi^{\tau_t}(h(t))$ is continuous in t if $\tau_t < \tau(h(t))$. Since 0 is a strict local minimizer of J_λ, we have $\lim_{s \to \tau_t^-} \phi^s(h(t)) = 0$ if $\tau_t = \tau(h(t))$, and $\lim_{s \to \tau_t^-} \phi^s(h(t))$ is continuous in t in this case also. Thus the path h^* defined by

$$h^*(t) = \begin{cases} h(t) & \text{if } h(t) \in D_1^+ \cup D_1^-, \\ \lim\limits_{s \to \tau_t^-} \phi^s(h(t)) & \text{if } h(t) \in (D_2 \cup D_3) \setminus (D_1^+ \cup D_1^-) \end{cases}$$

is continuous from $[0,1]$ to $D_1^+ \cup D_1^-$, and satisfies $h^*(0) = h_1(0)$, $h^*(1) = h_1(1)$, and $J_\lambda(h(t)) \geq J_\lambda(h^*(t))$ for $0 \leq t \leq 1$. Then the definitions of c^+ and c^- imply

$$\max_{0 \leq t \leq 1} J_\lambda(h(t)) \geq \max_{0 \leq t \leq 1} J_\lambda(h^*(t)) \geq \max\{c^+,\, c^-\}$$

for any $h \in \Gamma_2$, as required. □

Proof of Theorem 5.2. The proof of Theorem 5.2 is similar to that of Theorem 5.1, and we will indicate only the differences. First we pick two positive numbers δ_0 and γ_0 such that

$$\mathrm{meas}\{x \in \Omega : |e_1(x)\cos\pi t + e_2(x)\sin\pi t| \geq \delta_0\} \geq \gamma_0 \quad \text{for } 0 \leq t \leq 1. \tag{6.6}$$

By (f_5) there exist $T > T_1 > \max\{|t_-|,\, t_+\}$ such that $F(x,t) \geq 0$ for $|t| \geq T_1$ and $F(x,t) > 3K\gamma_0^{-1}|\Omega|$ for $|t| \geq T$, where $K := \max\{|F(x,t)| : x \in \overline{\Omega},\ |t| \leq T_1\}$. Setting $R := T\delta_0^{-1}$ and $h_1(t) := R(e_1 \cos\pi t + e_2 \sin\pi t)$, we have for $0 \leq t \leq 1$ and $u \in [t_-, t_+]$

$$\begin{aligned} J_\lambda(h_1(t)) - J_\lambda(u) &\leq \frac{1}{2}R^2 - \lambda\int_\Omega (F(x,h_1(t)) - F(x,u))\,dx \\ &\leq \frac{1}{2}R^2 - \lambda\int_{|h_1(t)|\geq T} F(x,h_1(t))\,dx + 2\lambda K|\Omega| \\ &\leq \frac{1}{2}R^2 - 3\lambda K\gamma_0^{-1}|\Omega|\mathrm{meas}\{|h_1| \geq T\} + 2\lambda K|\Omega|. \end{aligned}$$

This together with (6.6) implies for $0 \leq t \leq 1$ and $u \in [t_-, t_+]$

$$J_\lambda(h_1(t)) - J_\lambda(u) \leq \frac{1}{2}R^2 - \lambda K|\Omega| \leq -\frac{1}{2}R^2, \tag{6.7}$$

provided $\lambda \geq \Lambda := R^2(K|\Omega|)^{-1}$. Now we fix such a λ. By (f_4) there exists $T_\lambda > 0$ such that

$$|f(x,t)| < \frac{\lambda_1}{2\lambda}|t| \quad \text{for } |t| \geq T_\lambda.$$

Choose $C_\lambda > \max\{|\lambda f(x,t) - \lambda_1 t/2| : x \in \overline{\Omega},\ |t| \leq T_\lambda\}$ large enough so that the function $\psi_3^\lambda \in Z$ defined by

$$-\Delta\psi_3^\lambda - \frac{\lambda_1}{2}\psi_3^\lambda = C_\lambda$$

satisfies

$$-\psi_3^\lambda \ll \phi_1^\lambda \ll \psi_1^\lambda \ll \psi_3^\lambda \ \text{ and } \ -\psi_3^\lambda \leq h_1(t) \leq \psi_3^\lambda \quad \text{for } 0 \leq t \leq 1. \tag{6.8}$$

Here ϕ_1^λ and ψ_1^λ are as in the proof of Theorem 5.1. In addition, if $-\psi_3^\lambda \leq u \leq \psi_3^\lambda$, then $-\psi_3^\lambda \ll A_\lambda(u) \ll \psi_3^\lambda$. Combining (6.7) and (6.8) implies

$$-\infty < \inf_{u\in[-\psi_3^\lambda,\psi_3^\lambda]} J_\lambda(u) \leq \max_{0\leq t\leq 1} J_\lambda(h_1(t)) < \inf_{u\in[\phi_1^\lambda,\psi_1^\lambda]} J_\lambda(u).$$

Since $A_\lambda([t_-,t_+]) \subset [\phi_1^\lambda,\psi_1^\lambda]$, for every $u \in [t_-,t_+] \setminus [\phi_1^\lambda,\psi_1^\lambda]$ there exists $t > 0$ such that $\varphi^t(u) \in [\phi_1^\lambda,\psi_1^\lambda]$. Therefore,

$$\inf_{u\in[\phi_1^\lambda,\psi_1^\lambda]} J_\lambda(u) = \inf_{u\in[t_-,t_+]} J_\lambda(u) = \alpha_\lambda.$$

With the above facts, just as in the proof of Theorem 5.1, we obtain two positive solutions u_1 and u_2, two negative solutions v_1 and v_2, and two sign-changing solutions w_1 and w_2 satisfying

$$\min_\Omega v_i,\ \min_\Omega w_i < t_-, \quad \max_\Omega u_i,\ \max_\Omega w_i > t_+ \quad \text{for } i = 1, 2,$$

and

$$J_\lambda(u_1),\ J_\lambda(v_1),\ J_\lambda(w_1) < \alpha_\lambda \le J_\lambda(u_2),\ J_\lambda(v_2),\ J_\lambda(w_2), \tag{6.9}$$

and such that the conclusions (iv)–(vi) in Theorem 5.2 hold.

In order to complete the proof we need only to show that the second inequality in (6.9) is strict. Observe that each point in $K_{\alpha_\lambda} \cap [\phi_1^\lambda, \psi_1^\lambda]$ is a local minimizer of J_λ in Y and therefore a local minimizer of J_λ in X according to [8, 22]. Setting

$$\mathcal{N}(\delta) := \{u \in X : \mathrm{dist_X(u, K_{\alpha_\lambda}} \cap [\phi_1^\lambda, \psi_1^\lambda]) < \delta\},$$

the Palais-Smale condition implies the existence of $\delta_0 > 0$ and $b > 0$ such that

$$J_\lambda(u) \ge \alpha_\lambda + b \quad \text{for } u \in \big([\phi_1^\lambda, \psi_1^\lambda] \setminus \mathcal{N}(\delta_0)\big) \cup \partial_X \mathcal{N}(\delta_0) \tag{6.10}$$

and

$$h_1(t) \notin \mathcal{N}(\delta_0) \quad \text{for } 0 \le t \le 1. \tag{6.11}$$

The definitions of Γ^+, Γ^-, and Γ_2, together with (6.11), imply that each path $h \in \Gamma^+ \cup \Gamma^- \cup \Gamma_2$ intersects with $([\phi_1^\lambda, \psi_1^\lambda] \setminus \mathcal{N}(\delta_0)) \cup \partial_X \mathcal{N}(\delta_0)$. Thus by (6.10)

$$J_\lambda(u_2),\ J_\lambda(v_2),\ J_\lambda(w_2) \ge \inf_{h \in \Gamma^+ \cup \Gamma^- \cup \Gamma_2} \max_{0 \le t \le 1} J_\lambda(h(t)) \ge \alpha_\lambda + b > \alpha_\lambda$$

as required. □

In a similar way, we can prove Theorem 5.3.

Remark 6.1. If problem (1.2) has only six solutions outside of $[t_-, t_+]$ in the situation of Theorem 5.2, then we have $J_\lambda(w_i) \ge \max\{J_\lambda(u_i),\ J_\lambda(v_i)\}$ for $i = 1, 2$. But there is no such relation in the general case.

Proof of Theorem 5.4. Let D_i with $i = 1, \ldots, 4$ be as in Step 3 of the proof of Theorem 5.1. For $k \in \mathbb{N}$ and $r > 0$ we define $E^{k+1} := \mathrm{span}\{e_1, \ldots, e_{k+1}\}$ and $S_r^k := \{u \in E^{k+1} : \|u\|_X = r\}$. We fix $r > 0$ small so that

$$S_r^k \subset D_1 \quad \text{for } \lambda \ge 1 \tag{6.12}$$

and $\|u\|_\infty < \delta$ for $u \in S_r^k$. By (f_3) there exists $\Lambda_k > 0$ such that

$$\max_{u \in S_r^k} J_\lambda(u) < -1 \quad \text{for } \lambda \ge \Lambda_k. \tag{6.13}$$

By the proof of Theorem 5.1 we can assume that

$$J_\lambda(u) \geq c_\lambda \|u\|_\lambda^2 \quad \text{for } u \in B_X(0,r) \cup D_4. \tag{6.14}$$

Now (6.12)–(6.14) imply

$$-\infty < \inf_{u\in D_1} J_\lambda(u) \leq \max_{u\in S_r^k} J_\lambda(u) < -1 < 0 = \inf_{u\in D_4} J_\lambda(u).$$

By Theorem 2.3 and (4.3) in its proof, J_λ has at least k pairs of critical points in $D_1 \setminus (D_2 \cup D_3)$ with energy less than -1 and at least k pairs of critical points in $D_1 \setminus (D_2 \cup D_3)$ with energy not less than $\inf J_\lambda(\partial_{D_1}(C_{D_1}(D_4)))$. Here the set $C_{D_1}(D_4)$ is defined as in (4.1) using a semiflow $\varphi^t(u)$ as in the proof of Theorem 2.3.

It remains to show that $\inf J_\lambda(\partial_{D_1}(C_{D_1}(D_4))) > 0$. Indeed, there exists $r_* \in (0,r)$ with $\varphi^t(u) \to 0$ in Y as $t \to \tau(u)$, for every $u \in B_X(0,r_*) \cap Y$. Therefore,

$$\partial_{D_1} C_{D_1}(D_4) \cap B_X(0,r_*) = \emptyset. \tag{6.15}$$

For $u \in \partial_{D_1}(C_{D_1}(D_4))$ and $\epsilon > 0$, (6.15) implies the existence of $v \in C_{D_1}(D_4)$ such that $\|v\|_X > r_*$ and $J_\lambda(v) < J_\lambda(u) + \epsilon$. We choose $t > 0$ such that $\varphi^t(v) \in D_4$. If $\|\varphi^t(v)\|_X \geq r_*$, then (6.14) implies $J_\lambda(\varphi^t(v)) \geq c_\lambda r_*^2$. If $\|\varphi^t(v)\|_X < r_*$, then there must exist $t_1 \in (0,t)$ with $\|\varphi^{t_1}(v)\|_X = r_*$, so (6.14) implies $J_\lambda(\varphi^{t_1}(v)) \geq c_\lambda r_*^2$. In any case we have

$$J_\lambda(u) \geq J_\lambda(v) - \epsilon \geq c_\lambda r_*^2 - \epsilon,$$

which implies

$$\inf_{u\in\partial_{D_1} C_{D_1}(D_4)} J_\lambda(u) \geq c_\lambda r_*^2 > 0,$$

as required. □

The proofs of Theorems 5.5 and 5.6 are similar and left to the reader.

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