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# INSTANTANEOUS SHRINKING OF THE SUPPORT IN DEGENERATE PARABOLIC EQUATIONS WITH STRONG ABSORPTION

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**Abstract.** We investigate the phenomenon of instantaneous shrinking of the support of nonnegative solutions to the Cauchy problem in $\mathbb{R}^n$ for

$$u_t = f(u)\Delta u - g(u), \quad \text{where } f(0) = 0.$$

Among other results, it is shown by means of comparison and integral techniques that under some structural assumptions on $f$ and $g$, a necessary *and* sufficient condition on the growth of $g$ near zero for instantaneous shrinking to occur is $\int_0^1 \frac{ds}{g(s)} < \infty$.

## Introduction

For nonnegative bounded solutions $u$ of the diffusion problem

$$\begin{aligned} u_t &= f(u)\Delta u - g(u) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u|_{t=0} &= u_0, \end{aligned} \tag{0.1}$$

it is easy to guess—by means of "naive" comparison with flat solutions depending on $t$ only—that if the absorption term $g \geq 0$ is strong enough such that $\int_0 \frac{ds}{g(s)} < \infty$, then $u$ must vanish after some finite extinction time $T_{ext}$.

If one is in particular interested in the evolution of the "wetted" region $\{u(t) > 0\}$, one will therefore be led to the following question: Suppose that the mass is initially spread all over $\mathbb{R}^n$; that is, $\{u_0 > 0\} = \mathbb{R}^n$. Is the same true up to $t = T_{ext}$ or at least for small times, or do we find $\{u(t) > 0\} \neq \mathbb{R}^n$ for arbitrarily small positive $t$? Clearly, if $u_0 \equiv const.$, then the solution does not depend on $x$ and thus the first alternative holds. But for $u_0(x) \to 0$ merely as $|x| \to \infty$ it is shown in [7]—for the case of the heat equation ($f \equiv 1$) with an absorption term $g$ satisfying slightly sharper

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conditions than mentioned above—that the support *shrinks instantaneously* to a bounded set in the sense that $\{u(t) > 0\}$ is bounded for all positive $t$.

Evidently, the occurrence of the latter phenomenon essentially relies on the presence of diffusion; thus, one may ask whether such a shrinking process may be inhibited or postponed if the influence of diffusion is weakened near points where $u$ is small which is e.g. the case when $f$ is continuous with $f(0) = 0$. Focusing on this *degenerate diffusion* case, the present paper aims at finding sufficient conditions for instantaneous shrinking phenomena either to occur or to be definitely absent.

In order to formulate our main results, we briefly specify the class of problems to be considered by stating the following hypotheses, which will be assumed throughout this work.

(H1) $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is nonnegative with $\nabla u_0 \in L^2(\mathbb{R}^n)$.

(H2) $f \in C^0([0,\infty)) \cap W^{1,\infty}_{loc}((0,\infty))$ is positive in $(0,\infty)$ with $f(0) = 0$ and there exist $s_0 > 0$ and $\nu \in (0,1]$ such that

$$\frac{sf'(s)}{f(s)} \geq \nu \quad \forall\, s \in (0, s_0).$$

(H3) $g \in C^0([0,\infty)) \cap W^{1,\infty}_{loc}((0,\infty))$ is positive on $(0,\infty)$ with $g(0) = 0$.

We remark that we require the "$\liminf\limits_{s \searrow 0}$" estimate in (H2) mainly in view of the existence theory (cf. Section 1). In Section 2, we will inter alia derive the following:

a) If $g$ is nondecreasing with $\int_0^1 \frac{ds}{\sqrt{sg(s)}} < \infty$ and $u_0(x) \to 0$ as $|x| \to \infty$, then $\{u(t) > 0\}$ is bounded for all $t > 0$ (Theorem 2.2).

b) If $\int_0^1 \frac{ds}{g(s)} < \infty$ and $u_0$ decays fast enough in space, then $\{u(t) > 0\}$ has finite measure for almost every $t > 0$, provided that $(f/g)'$ is bounded below near zero (Theorem 2.4).

c) If, conversely, $\int_0^1 \frac{ds}{g(s)} = \infty$ and both $f/g$ and $(f/g)'$ are monotonic and bounded above near zero, then $\overline{\{u(t) > 0\}} \supset \{u_0 > 0\}$ for all $t > 0$ (Theorem 2.8).

Let us illustrate these results by a few examples. First, for the prototype equation

$$u_t = u^p \Delta u - u^q, \qquad p > 0, q > 0, \tag{0.2}$$

a) and c) assert that instantaneous shrinking occurs if and only if $q < 1$. This generalizes the results in [7] and [15] obtained for (0.2) with $p \in (0,1)$;

in this case, namely, (0.2) transforms into the porous-medium equation

$$v_t = \Delta v^m - v^r \chi_{\{v>0\}}$$

via the substitution $v(x,t) := au^{1-p}(bx,t)$ with $a := (1-p)^{-\frac{1-p}{1-q}}$ and $b := (1-p)^{\frac{p+1-q}{1-q}}$, where then $m = \frac{1}{1-p} \in (1,\infty)$ and $r := \frac{q-p}{1-p} \in (1-m,1)$. In order to demonstrate the sharpness of b) and c) in a large class of nonpathological cases, let us refine the scale of absorption intensity in (0.2) in the critical regime $q \nearrow 1$ and consider

$$u_t = u^p \Delta u - u \cdot \Big(\ln \frac{1}{u}\Big)_+^{\alpha}, \qquad p > 0, \alpha > 0. \tag{0.3}$$

Then b) states that

$$\alpha > 1 \tag{0.4}$$

is sufficient for instantaneous shrinking to occur, provided that $p \geq 1$, that is, *the degeneracy in* (0.1) *is strong enough*. Conversely, c) rules out shrinking for $\alpha \leq 1$ if $p > 2$. These results extend Theorems 3.5 and 3.8 in [15], which also show necessity and sufficiency of (0.4) for (0.3), but only in the *weakly degenerate* case $p \in (0,1)$.

Not surprisingly, (0.4) continues to be necessary and sufficient also for the very strongly degenerate equation

$$u_t = e^{-\frac{1}{u}} \Delta u - u \cdot \Big(\ln \frac{1}{u}\Big)_+^{\alpha}, \qquad \alpha > 0.$$

More generally, to the best of our knowledge the existing literature concentrates on problems that involve either regular diffusion (i.e., $f \equiv 1$; cf. [7]), singular diffusion (with $f(s) \to +\infty$ as $s \to 0$; see [5]), or weakly degenerate diffusion, whereas nothing seems to be known e.g. for $f(s) = s^p$ with $p \geq 1$ so far. Indeed, the previously known results which go furthest for degenerate equations seem to be those in [15], where instantaneous shrinking is shown to occur if

$$\int_0^1 \frac{ds}{g(s)} < \infty \qquad \text{and} \qquad \int_0^1 \frac{\sqrt{f(s)}}{\sqrt{sg(s)}} ds < \infty,$$

but only if both $g$ and $\frac{g}{f}$ are nondecreasing near zero. In particular, the latter condition excludes any $f$ satisfying $f(s) \leq cs$ for small $s$ (and some $c > 0$), because in this case the finiteness of $\int_0^1 \frac{ds}{g(s)}$ implies

$$\limsup_{s \to 0} \frac{g(s)}{f(s)} = \infty,$$

whence $\frac{g}{f}$ cannot be nondecreasing near zero. Therefore, for instance, the choice $f(s) = s^p$ is admissible there only for $p < 1$.

Finally, returning to the particular problem (0.2) once again, we will in the case $q \in (0, \min\{1, p\})$ be able to relax the decay condition in b) to see that

d) if $u_0 \in L^r(\mathbb{R}^n)$ for some $r > 0$, then $\{u(t) > 0\}$ has finite measure for almost every $t > 0$ (Theorem 2.6).

## 1. Existence of weak solutions

First of all, since the loss of uniform parabolicity in (0.1) should drastically influence regularity of solutions near points where they are small (cf. e.g. [2]), we will expect solutions to exist in a generalized sense only rather than in the classical sense. For equations of type (0.1), various weak-solution concepts have been used (see e.g. [14], or [16]). For the "porous-medium case" $f(s) = s^p$, $p \in (0, 1)$, we also refer to [11] for a concept of continuous weak solutions in one space dimension.

**Definition 1.1.** *A function*

$$u \in L^\infty(\mathbb{R}^n \times (0,\infty)) \cap C^0([0,\infty); L^1_{loc}(\mathbb{R}^n)) \cap L^2_{loc}([0,\infty); W^{1,2}_{loc}(\mathbb{R}^n))$$

*with* $u_t \in L^2_{loc}(\mathbb{R}^n \times [0,\infty))$ *is called a weak solution of* (0.1) *if* $u|_{t=0} = u_0$ *and if for all* $T > 0$ *and any* $\varphi \in C_0^\infty(\mathbb{R}^n \times [0,T])$

$$\int_0^T \int_{\mathbb{R}^n} u_t \varphi + \int_0^T \int_{\mathbb{R}^n} \Big[ f'(u) |\nabla u|^2 \varphi + f(u) \nabla u \cdot \nabla \varphi \Big] + \int_0^T \int_{\mathbb{R}^n} g(u) \varphi = 0.$$

In the special case $f(s) = s$, weak solutions of this type have been proved to exist—but to be *not* unique—in [14] (cf. also [4]). In the present setting we proceed in quite a similar way, namely via approximation of (0.1) by suitable nondegenerate problems:

Fix a sequence of numbers $1 \geq \varepsilon_j \searrow 0$ and let, for $\varepsilon = \varepsilon_j$, $g_\varepsilon(s) := \chi(\frac{s}{\varepsilon}) g(s)$ with a nondecreasing $\chi \in C^\infty([0,\infty))$ that fulfills $\chi(\sigma) \equiv 0$ for $\sigma \leq 1$ and $\chi(\sigma) \equiv 1$ for $\sigma \geq 2$. Then $g_\varepsilon \in W^{1,\infty}_{loc}([0,\infty))$ is nondecreasing with $g_\varepsilon(s) = 0$ for $s \leq \varepsilon$, $g_\varepsilon(s) = g(s)$ for $s \geq 2\varepsilon$, and $g_\varepsilon \nearrow g$ as $\varepsilon = \varepsilon_j \searrow 0$. Furthermore, let $u_{0\varepsilon} \in C^3(\mathbb{R}^n)$ satisfy $u_0 + \varepsilon \leq u_{0,\varepsilon} \leq u_0 + 2\varepsilon$ in $\mathbb{R}^n$ and $u_{0\varepsilon} \searrow u_0$ as well as $\nabla u_{0\varepsilon} \to \nabla u_0$ in $L^2(\mathbb{R}^n)$ as $\varepsilon = \varepsilon_j \searrow 0$.

We claim that the approximating problem

$$\begin{aligned} u_{\varepsilon t} &= f(u_\varepsilon) \Delta u_\varepsilon - g_\varepsilon(u_\varepsilon) \quad \text{in } \mathbb{R}^n \times (0,\infty), \\ u_\varepsilon|_{t=0} &= u_{0\varepsilon}, \end{aligned} \tag{1.1}$$

has a classical solution $u_\varepsilon$. Indeed, consider in $B_R := B_R(0), R > 1$, the Dirichlet problems

$$\begin{aligned} u_{\varepsilon Rt} &= f(u_{\varepsilon R})\Delta u_{\varepsilon R} - g_\varepsilon(u_{\varepsilon R}) \quad \text{in } B_R \times (0,\infty), \\ u_{\varepsilon R}|_{\partial B_R} &= u_{0\varepsilon R}|_{\partial B_R}, \\ u_{\varepsilon R}|_{t=0} &= u_{0\varepsilon R}, \end{aligned} \tag{1.2}$$

where $\varepsilon \le u_{0\varepsilon R} \in C^3(\bar{B}_R)$ is such that $u_{0\varepsilon R} \to u_{0\varepsilon}$ in $C^3_{loc}(\mathbb{R}^n)$ and

$$\|u_{0\varepsilon R}\|_{L^\infty(\mathbb{R}^n)} \le c$$

as well as

$$\|\nabla u_{0\varepsilon R}\|_{L^2(\mathbb{R}^n)} \le c.$$

For general purpose it suffices to consider $u_{0\varepsilon R} := u_{0\varepsilon}|_{B_R}$, but later on (cf. the proof of Theorem 2.4) we shall employ a different approximation.

Applying standard comparison arguments to $u_{\varepsilon R}$, we see that $\varepsilon \le u_\varepsilon \le \|u_0\|_{L^\infty(\mathbb{R}^n)} + 1 =: M$, and since $g_\varepsilon \nearrow g$ as $\varepsilon \searrow 0$, the $u_\varepsilon$ decrease monotonically, $u_\varepsilon \searrow u$, to some nonnegative $u \in L^\infty(\mathbb{R}^n \times (0,\infty))$.

In order to utilize standard compactness methods, we prepare some $\varepsilon$-independent estimates which seem to be available only if (H1) and (H2) are fulfilled.

**Lemma 1.1.** *Suppose the conditions* (H1)–(H3) *are met and* $u_{0\varepsilon}$ *and* $u_{0\varepsilon R}$ *fulfill the assumptions made above. Then there is a constant* $c$ *independent of* $\varepsilon$ *and* $R$ *such that*

$$\|u_{\varepsilon t}\|_{L^2(\mathbb{R}^n \times (0,\infty))} \le c, \tag{1.3}$$

$$\|\nabla u_{\varepsilon R}\|_{L^\infty((0,\infty);L^2(B_R))} \le c, \tag{1.4}$$

$$\|\nabla u_\varepsilon\|_{L^\infty((0,\infty);L^2(\mathbb{R}^n))} \le c, \tag{1.5}$$

*and for all* $\gamma \in (1-\nu, 1)$*, any* $T > 0$ *and all compact* $K \subset \mathbb{R}^n$*, the estimate*

$$\int_0^T \int_K u_\varepsilon^{\gamma-1} |f'(u_\varepsilon)| \cdot |\nabla u_\varepsilon|^2 \le c(\gamma, T, K) \tag{1.6}$$

*holds for some* $c(\gamma, T, K) > 0$.

**Proof.** We multiply (1.2) by $\Delta u_{\varepsilon R}$ and integrate over $B_R \times (\tau, T), \tau > 0$, to obtain

$$\frac{1}{2}\int_{B_R} |\nabla u_{\varepsilon R}(t)|^2 + \int_\tau^T \int_{B_R} f(u_{\varepsilon R})|\Delta u_{\varepsilon R}|^2 + \int_\tau^T \int_{B_R} g'_\varepsilon(u_{\varepsilon R})|\nabla u_{\varepsilon R}|^2$$
$$= \frac{1}{2}\int_{B_R} |\nabla u_{\varepsilon R}(\tau)|^2.$$

Note that $\nabla u_{\varepsilon R}$ is continuous in $B_R \times [0,\infty)$ and bounded in $B_R \times (0,1)$ (see Theorem V.5.3 and Theorem V.6.2 in [13]). As furthermore $\nabla u_{0\varepsilon}$ is bounded in $L^2(\mathbb{R}^n)$, we may let $\tau \to 0$, so that the latter inequality results in

$$\|\nabla u_{\varepsilon R}\|_{L^\infty((0,\infty);L^2(B_R))} + \|\sqrt{f(u_{\varepsilon R})}\Delta u_{\varepsilon R}\|_{L^2(B_R\times(0,\infty))} \le c$$

for some $c$ independent of $\varepsilon$ and $R$. Thus, (1.4) has been proved and, moreover, (1.2) shows that also $\|u_{\varepsilon Rt}\|_{L^2(B_R\times(0,\infty))}$ is uniformly bounded, whence (1.3) and (1.5) follow from Fatou's lemma. Next, fix $\gamma \in (1-\nu,1)$ and let

$$\phi(s) := \begin{cases} s^{\gamma-1}, & s \in (0,s_0),\\ s_0^{\gamma-2} f(s_0)\cdot \frac{s}{f(s)}, & s \ge s_0, \end{cases}$$

and test (1.1) with $\phi(u_\varepsilon)\cdot\psi^2(x)$, where $\psi$ is in $C_0^\infty(\mathbb{R}^n)$ with $\psi \equiv 1$ in $K$. Writing $\Phi(s) := \int_0^s \phi(\sigma)d\sigma$, for $T>0$ we have

$$I_1+I_2+I_3+I_4 := \int_{\mathbb{R}^n}\Phi(u_\varepsilon(T))\cdot\psi^2 + \int_0^T\int_{\mathbb{R}^n}(f\phi)'(u_\varepsilon)|\nabla u_\varepsilon|^2\psi^2 \qquad (1.7)$$
$$+\,2\int_0^T\int_{\mathbb{R}^n}\psi\cdot(f\phi)(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\psi + \int_0^T\int_{\mathbb{R}^n}(g_\varepsilon\phi)(u_\varepsilon)\cdot\psi^2 = \int_{\mathbb{R}^n}\Phi(u_{0\varepsilon})\cdot\psi^2.$$

As $\gamma > 0$ and $u_{0\varepsilon}$ is bounded, the term on the right is bounded, while $I_1 + I_4 \ge 0$. Moreover, from

$$(f\phi)'(s) = \begin{cases} s^{\gamma-1}f'(s) - (1-\gamma)s^{\gamma-2}f(s) \ge (1-\frac{1-\gamma}{\nu})s^{\gamma-1}f'(s) \ge cs^{\gamma-1}f'(s) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } s<s_0,\\ s_0^{\gamma-2}f(s_0) \ge \tilde c \ge cs^{\gamma-1}|f'(s)| \qquad\qquad \text{for } s\in[s_0,M], \end{cases}$$

we see that $\frac{(f\phi)^2}{(f\phi)'}(u_\varepsilon) \le c$ and thus

$$\begin{aligned}|I_3| &= \Big|2\int_0^T\int_{\mathbb{R}^n}\psi\cdot(f\phi)(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla\psi\Big|\\ &\le \frac12\int_0^T\int_{\mathbb{R}^n}(f\phi)'(u_\varepsilon)|\nabla u_\varepsilon|^2\psi^2 + c\int_0^T\int_{\mathbb{R}^n}\frac{(f\phi)^2}{(f\phi)'}(u_\varepsilon)|\nabla\psi|^2 \le \frac12 I_2 + c.\end{aligned}$$

Finally,

$$I_2 \ge c\int_0^T\int_K u_\varepsilon^{\gamma-1}|f'(u_\varepsilon)|\cdot|\nabla u_\varepsilon|^2,$$

whereby (1.6) is proved. □

**Remark.** In the case of the porous-medium equation, that is, for $p \in (0,1)$ in $u_t = u^p\Delta u - u^q\chi_{\{u>0\}}$, Lemma 1.1 (and accordingly Theorem 1.2 below) can be modified so as to cover actually any $q \in (p-1,1)$. This allows us to choose all subcritical $r \in (-m,1)$ in $v_t = \Delta v^m - v^r \chi_{\{v>0\}}$—for criticality of $r=-m$ in one space dimension, see [11] and [17]. The modification mainly consists of testing (1.2) with $\frac{u_{\varepsilon Rt}}{u^p_{\varepsilon R}}$ (rather than $\Delta u_{\varepsilon R}$) and finally yields (1.3)–(1.6) under the additional decay assumption $u_0 \in L^{1+q-p}(\mathbb{R}^n)$.

We may now pass to the limit $\varepsilon \to 0$ to obtain

**Theorem 1.2.** *The function* $u = \lim\limits_{\varepsilon\to 0} u_\varepsilon$ *is a weak solution of* (0.1).

**Proof.** Fix $T>0$. Then Lemma 1.1 ensures that

$$u_{\varepsilon t} \rightharpoonup u_t \quad \text{in } L^2(\mathbb{R}^n \times (0,T)), \tag{1.8}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2(\mathbb{R}^n \times (0,T)), \tag{1.9}$$

and

$$u_\varepsilon \to u \quad \text{in } C^0([0,T]; L^1_{loc}(\mathbb{R}^n)). \tag{1.10}$$

Thus, $u|_{t=0} = u_0$ and in the weak formulation of (1.1), that is,

$$\int_0^T \int_{\mathbb{R}^n} u_{\varepsilon t}\varphi + \int_0^T \int_{\mathbb{R}^n} f'(u_\varepsilon)|\nabla u_\varepsilon|^2 \varphi + \int_0^T \int_{\mathbb{R}^n} f(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi$$
$$+ \int_0^T \int_{\mathbb{R}^n} g_\varepsilon(u_\varepsilon)\varphi = 0 \qquad \forall\, \varphi \in C_0^\infty(\mathbb{R}^n \times [0,T]),$$

we may go to the limit $\varepsilon \searrow 0$ in the first, third, and fourth term on the left. What remains to show is that for such $\varphi$,

$$\int_0^T \int_{\mathbb{R}^n} f'(u_\varepsilon)|\nabla u_\varepsilon|^2 \varphi \to \int_0^T \int_{\mathbb{R}^n} f'(u)|\nabla u|^2 \varphi. \tag{1.11}$$

For this purpose, we first note that since $\liminf_{s\to 0} \frac{sf'(s)}{f(s)} > 0$ and $f(s) > 0$ for $s>0$, there is $\eta > 0$ such that

$$f_0(s) := \begin{cases} f(s), & s \le \eta, \\ f(\eta), & s > \eta, \end{cases}$$

defines a nondecreasing function $f_0 \le f$ on $[0,M]$. Writing

$$F_0(s) := \int_0^s f_0(\sigma)d\sigma, \qquad s \ge 0,$$

we multiply (1.1) by $\frac{f_0}{f}(u_\varepsilon) \cdot (F_0(u_\varepsilon) - F_0(u)) \cdot \psi$, $0 \le \psi \in C_0^\infty(\mathbb{R}^n)$, to see that

$$0 = \int_0^T \int_{\mathbb{R}^n} u_{\varepsilon t} \cdot \frac{f_0}{f}(u_\varepsilon)(F_0(u_\varepsilon) - F_0(u))\psi$$

$$+\int_0^T\int_{\mathbb{R}^n} f_0'(u_\varepsilon)|\nabla u_\varepsilon|^2(F_0(u_\varepsilon)-F_0(u))\psi$$
$$+\int_0^T\int_{\mathbb{R}^n} f_0(u_\varepsilon)\nabla u_\varepsilon\cdot\nabla(F_0(u_\varepsilon)-F_0(u))\psi$$
$$+\int_0^T\int_{\mathbb{R}^n} f_0(u_\varepsilon)(F_0(u_\varepsilon)-F_0(u))\nabla u_\varepsilon\cdot\nabla\psi$$
$$+\int_0^T\int_{\mathbb{R}^n} g_\varepsilon(u_\varepsilon)\frac{f_0}{f}(u_\varepsilon)(F_0(u_\varepsilon)-F_0(u))\psi =: I_1+\cdots+I_5. \tag{1.12}$$

Due to the dominated convergence theorem, $F_0(u_\varepsilon)\to F_0(u)$ in $L^2_{loc}(\mathbb{R}^n\times[0,T])$, so that (1.3) and (1.9) show that $I_1+I_4+I_5\to 0$ as $\varepsilon\to 0$, while $I_2\ge 0$ since $u_\varepsilon\ge u$ and $F_0$ is increasing. But

$$I_2=\int_0^T\int_{\mathbb{R}^n}\Big|\nabla(F_0(u_\varepsilon)-F_0(u))\Big|^2\psi+\int_0^T\int_{\mathbb{R}^n}\psi\cdot f_0(u)f_0(u_\varepsilon)\nabla u\cdot\nabla u_\varepsilon$$
$$-\int_0^T\int_{\mathbb{R}^n} f_0^2(u)|\nabla u|^2\psi,$$

where once more due to the fact that $u_\varepsilon\ge u$ and (1.9) the last two terms cancel each other in the limit $\varepsilon\to 0$. Therefore, (1.12) implies

$$\liminf_{\varepsilon\to 0}\int_0^T\int_{\mathbb{R}^n}\Big|\nabla(F_0(u_\varepsilon)-F_0(u))\Big|^2\psi\le 0$$

and thus, since $\psi$ was arbitrary, that

$$\nabla F_0(u_\varepsilon)\to\nabla F_0(u)\quad \text{in } L^2_{loc}(\mathbb{R}^n\times[0,T]). \tag{1.13}$$

Now for $0\le\varphi\in C_0^\infty(\mathbb{R}^n\times[0,T])$ and any $\delta>0$, we have

$$\Big|\int_0^T\int_{\mathbb{R}^n} f'(u_\varepsilon)|\nabla u_\varepsilon|^2\varphi-\int_0^T\int_{\mathbb{R}^n} f'(u)|\nabla u|^2\varphi\Big|$$
$$\le\int_0^T\int_{\mathbb{R}^n}\chi_{\{u_\varepsilon\ge\delta\}}\Big|f'(u_\varepsilon)|\nabla u_\varepsilon|^2-f'(u)|\nabla u|^2\Big|\varphi$$
$$+\int_0^T\int_{\mathbb{R}^n}\chi_{\{u_\varepsilon<\delta\}}|f'(u_\varepsilon)|\cdot|\nabla u_\varepsilon|^2\varphi+\int_0^T\int_{\mathbb{R}^n}|f'(u)|\cdot|\nabla u|^2\varphi$$
$$=: J_1+J_2+J_3, \tag{1.14}$$

where we may use (1.6) in estimating

$$J_2\le\delta^{1-\gamma}\int_0^T\int_{\mathbb{R}^n}\chi_{\{u_\varepsilon<\delta\}}u_\varepsilon^{\gamma-1}|f'(u_\varepsilon)|\cdot|\nabla u_\varepsilon|^2\le c\delta^{1-\gamma}, \tag{1.15}$$

while $\{u_\varepsilon < \delta\} \subset \{u < \delta\}$ implies that similarly

$$J_3 \le c\delta^{1-\gamma}. \tag{1.16}$$

Finally,

$$\begin{aligned} J_1 &= \int_0^T \int_{\mathbb{R}^n} \chi_{\{u_\varepsilon \ge \delta\}} \left| \frac{f'}{f_0^2}(u_\varepsilon)|\nabla F_0(u_\varepsilon)|^2 - \frac{f'}{f_0^2}(u)|\nabla F_0(u)|^2 \right| \cdot \varphi \\ &\le \int_0^T \int_{\mathbb{R}^n} \chi_{\{u_\varepsilon \ge \delta\}} \frac{f'}{f_0^2}(u_\varepsilon) \Big| |\nabla F_0(u_\varepsilon)|^2 - |\nabla F_0(u)|^2 \Big| \cdot \varphi \\ &+ \int_0^T \int_{\mathbb{R}^n} \chi_{\{u_\varepsilon \ge \delta\}} \left| \frac{f'}{f_0^2}(u_\varepsilon) - \frac{f'}{f_0^2}(u) \right| \cdot |\nabla F_0(u)|^2 \varphi =: J_{11} + J_{12}. \end{aligned}$$

For fixed $\delta > 0$, we find $\left|\frac{f'}{f_0^2}(s)\right| \le c(\delta, M)$ in $(\delta, M)$, so that (1.13) guarantees that $J_{11} \to 0$ as $\varepsilon \to 0$. Furthermore, the integrand in $J_{12}$ tends to zero for almost every $(x,t) \in \mathbb{R}^n \times (0,T)$ and is uniformly bounded by $2c(\delta, M)|\nabla F_0(u)|^2 \varphi \in L^1(\mathbb{R}^n \times (0,T))$, whence $J_{12} \to 0$ as $\varepsilon \to 0$ by the dominated convergence theorem, and the proof of (1.11) follows upon letting $\varepsilon$ and then $\delta$ tend to zero in (1.14). □

In the sequel, by "$u$" we will exclusively mean the weak solution constructed above.

In the case of sufficiently strong absorption, we indeed have finite-time extinction for our solution:

**Corollary 1.3.** *Suppose*

$$\int_0^1 \frac{ds}{g(s)} < \infty.$$

*Then there exists $T_{ext} > 0$ such that*

$$u(t) \equiv 0 \quad \forall\, t \ge T_{ext}.$$

**Proof.** For $\varepsilon > 0$, let $v_\varepsilon(t)$ solve $v_\varepsilon' = -g_\varepsilon(v_\varepsilon)$ on $(0,\infty)$, $v_\varepsilon(0) = M = \|u_0\|_{L^\infty(\mathbb{R}^n)} + 1$. Then, using the comparison principle in $B_R$ and letting $R \to \infty$, we obtain $u_\varepsilon(x,t) \le v_\varepsilon(t)$ for all $t$; in particular, $u \le u_\varepsilon \le 2\varepsilon$ for all $t \ge \int_{2\varepsilon}^M \frac{ds}{g(s)}$ and thus for all $t \ge \int_0^M \frac{ds}{g(s)}$. □

## 2. Instantaneous shrinking phenomena

Our first result on shrinking may be regarded as a simple extension of Theorem 2.2 in [7] to the present situation. The proof is based on comparison and yields as a by-product a quantitative estimate for the radius of the smallest ball which contains $\{u(t) > 0\}$ (cf. (2.5) and Corollary 2.3). For the sake of completeness, we present the whole proof in its modified version.

Recall that if a nonnegative, continuous function $h$ is nondecreasing, the finiteness of $\int_0^1 \frac{ds}{\sqrt{sh(s)}}$ implies that of $\int_0^1 \frac{ds}{h(s)}$ (cf. Lemma 2.1 in [7]).

**Lemma 2.1.** *Suppose that there exist $s_1 > 0$ and a nondecreasing $0 \le g_0 \in C^0([0,s_1]) \cap W^{1,\infty}_{loc}((0,s_1])$ such that $g_0 \le g$ and $\int_0^{s_1} \frac{ds}{\sqrt{sg_0(s)}} < \infty$. If $u_0$ satisfies $u_0(x) \to 0$ as $|x| \to \infty$, then $\{u(t) > 0\}$ is bounded for all $t > 0$.*

A trivial consequence of this lemma will be

**Theorem 2.2.** *Assume that*

$$g \text{ is nondecreasing} \tag{2.1}$$

*with*

$$\int_0^1 \frac{ds}{\sqrt{sg(s)}} < \infty \tag{2.2}$$

*and that $u_0$ satisfies, in addition to* (H1),

$$u_0(x) \to 0 \quad \text{as } |x| \to \infty.$$

*Then*

$$\{u(t) > 0\} \text{ is bounded for all } t > 0. \tag{2.3}$$

**Proof of the Lemma.** First, we redefine $g$ in such a way that it is constant on $(M, \infty)$, where again $M = \|u_0\|_{L^\infty(\mathbb{R}^n)} + 1$. Since for sufficiently small $\alpha > 0$,

$$\bar{g}_0(s) := \begin{cases} \alpha g(s), & s \le s_1, \\ \alpha g(s_1), & s > s_1, \end{cases}$$

is in $C^0([0,\infty)) \cap C^{1-}_{loc}((0,\infty))$ and nondecreasing with $\bar{g}_0 \le g$ and $\int_0^1 \frac{ds}{\sqrt{s\bar{g}_0(s)}} < \infty$, we may assume without loss of generality that $g_0$ itself has all these properties. Using a cutoff function $\chi$ as in the construction of $g_\varepsilon$, we set $g_{0\varepsilon}(s) := \chi(\frac{s}{\varepsilon}) \cdot g_0(s)$. (We could also define $g_{0\varepsilon} := \frac{g}{g_0} g_\varepsilon$, of course.)

Fix $t_0 > 0$ and $x_0 = (x_{01}, \ldots, x_{0n}) \in \mathbb{R}^n$ with $u(x_0, t_0) > 0$. Let $\varepsilon \in (0,1)$ be small such that

$$2(n+1)\varepsilon < u(x_0, t_0), \tag{2.4}$$

and define $\varphi_\varepsilon : [0,\infty) \to \mathbb{R}$ and $\varrho_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_\varepsilon'(t) = \frac{g_{0\varepsilon}(\varphi_\varepsilon(t))}{n+1}, \quad \varphi_\varepsilon(0) = 2\varepsilon,$$

and

$$\varrho_\varepsilon''(\xi) = \frac{g_{0\varepsilon}(\varrho_\varepsilon(\xi))}{(n+1)C_0}, \quad \varrho_\varepsilon(0) = 2\varepsilon,\ \varrho_\varepsilon'(0) = 0,$$

respectively, where $C_0 := \sup_{\varepsilon\in(0,1)} \|f(u_\varepsilon)\|_{L^\infty(\mathbb{R}^n\times[0,\infty))}$. As $g_{0\varepsilon}(s) = g_0(s)$ for $s \geq 2\varepsilon$, it follows that $\varphi_\varepsilon \geq \varphi$ and $\varrho_\varepsilon \geq \varrho$, with $\varphi$ and $\varrho$ defined by

$$\int_0^{\varphi(t)} \frac{ds}{g_0(s)} = \frac{t}{n+1}$$

and

$$\int_0^{\varrho(\pm\xi)} \frac{1}{\sqrt{\int_0^s g_0(\sigma)d\sigma}} ds = \sqrt{\frac{2}{(n+1)C_0}}\ |\xi|,$$

where the appearing integrals exist due to (2.2), the inequalities

$$\frac{s}{2} g_0(\frac{s}{2}) \leq \int_0^s g_0(\sigma)d\sigma \leq s g_0(s)$$

and the above remark. In view of (2.3) and the fact that $\varrho(\xi) \to \infty$ as $|\xi| \to \infty$, the expressions

$$R_1(\tau) := \sup\{|x| : u_0(x) \geq \varphi(\tau)\}$$

and

$$R_2(m) := \sqrt{n}\ \sup\{r > 0 : \varrho(r) \leq m\}$$

are both finite for any positive $\tau$ and $m$. We claim that $x_0 \in \bar{B}_{R_1(t_0)+R_2(M)}(0)$; that is,

$$\{u(t_0) > 0\} \subset \bar{B}_{R_1(t_0)+R_2(M)}(0). \tag{2.5}$$

Indeed, abbreviate $R := R_2(M)$ and let

$$w_\varepsilon(x,t) := \varphi_\varepsilon(t_0 - t) + \sum_{i=1}^n \varrho_\varepsilon(x_i - x_{0i}), \qquad (x,t) \in \bar{B}_R(x_0) \times [0, t_0].$$

Then

$$w_{\varepsilon t} - f(u_\varepsilon)\Delta w_\varepsilon + g_{0\varepsilon}(w_\varepsilon) = -\varphi_\varepsilon'(t_0 - t) - f(u_\varepsilon) \cdot \sum_{i=1}^n \varrho_\varepsilon''(x_i - x_{0i})$$
$$+ g_{0\varepsilon}\Big(\varphi_\varepsilon(t_0 - t) + \sum_{i=1}^n \varrho_\varepsilon(x_i - x_{0i})\Big)$$

$$\geq -\varphi_\varepsilon'(t_0-t) - C_0 \cdot \sum_{i=1}^n \varrho_\varepsilon''(x_i - x_{0i})$$
$$+ \frac{1}{n+1} g_{0\varepsilon}(\varphi_\varepsilon(t_0-t)) + \sum_{i=1}^n \frac{g_{0\varepsilon}(\varrho_\varepsilon(x_i-x_{0i}))}{n+1} = 0$$

due to the monotonicity of $g_{0\varepsilon}$. For $x \in \partial B_R(x_0)$, we have

$$w_\varepsilon \geq \varrho_\varepsilon(\max_i |x_i - x_{0i}|) \geq \varrho\Big(\frac{1}{\sqrt{n}}|x-x_0|\Big) = \varrho\Big(\frac{R}{\sqrt{n}}\Big) \geq M \geq u_\varepsilon,$$

so that if also $w_\varepsilon$ were greater than $u_\varepsilon$ in $B_R(x_0) \times \{0\}$, then the comparison principle would tell us that $w_\varepsilon \geq u_\varepsilon$ in $\bar{B}_R(x_0) \times [0, t_0]$, contradicting $w_\varepsilon(x_0, t_0) = 2(n+1)\varepsilon < u_\varepsilon(x_0, t_0)$ which we have asserted by (2.4). Thus, there must be some $\bar{x} \in \bar{B}_R(x_0)$ such that $w_\varepsilon(\bar{x}, 0) \leq u_{0\varepsilon}(\bar{x})$, that is,

$$u_{0\varepsilon}(\bar{x}) \geq \varphi_\varepsilon(t_0) + \sum_{i=1}^n \varrho_\varepsilon(\bar{x}_i - x_{0i}).$$

In particular, $u_0(\bar{x}) \geq u_{0,\varepsilon}(\bar{x}) - 2\varepsilon \geq \varphi_\varepsilon(t_0) \geq \varphi(t_0)$, so that $|x| \leq R_1(t_0)$, and $\varrho(\bar{x}_i - x_{0i}) \leq \varrho_\varepsilon(\bar{x}_i - x_{0i}) \leq u_{0\varepsilon}(\bar{x}) \leq M$ for all $i = 1, \ldots, n$, whence $|\bar{x} - x_0| \leq R_2(M)$. Thus, $|x_0| \leq R_1(t_0) + R_2(M)$, which proves (2.5). □

The following quantitative estimate clearly continues to hold in the nondegenerate case $f \equiv 1$ (i.e., $p = 0$), which is however not considered here. Similar results for the fast diffusion case $f(s) = s^{-r}$, $r > 0$, can be found in [5].

**Corollary 2.3.** *Consider*

$$u_t = u^p \Delta u - u^q, \quad p > 0,\ q \in (0,1).$$

*If*

$$u_0(x) \leq c_1(1+|x|)^{-\alpha} \quad \textit{for some } \alpha > 0,$$

*then there is* $C = C(p, q, c_1) > 0$ *such that*

$$\{u(t) > 0\} \subset \Big\{x \in \mathbb{R}^n : |x| < C\big(1 + t^{-\frac{1}{(1-q)\alpha}}\big)\Big\} \quad \forall\, t > 0. \tag{2.6}$$

**Proof.** The claim follows easily from (2.5) if we note that in the present case $\varphi(t) = \big(\frac{t}{n+1}\big)^{\frac{1}{1-q}}$ and $\varrho(\xi) = c_2|\xi|^{\frac{2}{1-q}}$ with $c_2^{1-q} = \frac{(1-q)^2}{2(n+1)(q+1)(c_1+2)^p}$. Thus, $R_1(t) \leq \big[c_1(n+1)^{\frac{1}{1-q}}\big]^{\frac{1}{\alpha}} \cdot t^{-\frac{1}{(1-q)\alpha}}$ and $R_2(M) \leq \big(\frac{c_1+2}{c_2}\big)^{\frac{1-q}{2}}$, whereby (2.6) follows. □

The gap appearing between the extinction condition $\int_0^1 \frac{ds}{g(s)} < \infty$ (see Theorem 2.8 for its necessity for extinction) and (2.2) seems unnatural; for

$f \equiv 1$ and in one space dimension, however, it has been shown in [7] that if (2.2) is violated, then there exist positive initial data vanishing at infinity which do not lead to an instantaneous shrinking. This was generalized in [15], where a similar result was proved for $f(s) = s^p$ with $p \in (0,1)$ (and more general $f$, as mentioned in the introduction). Despite the outcome of Theorem 2.8 below, we have to leave open the question of whether or not (2.2) continues to be sharp in *this* sense if diffusion is strongly degenerate. Moreover, we do not know whether the assertion in the above lemma remains true if $g$ does not possess a monotonic minorant with the required properties. In order to obtain shrinking results also for $g$ belonging to a wider class, we both impose a somewhat more restrictive decay condition on $u_0$ and content ourselves with a weaker definition of "instantaneous shrinking" implicitly contained in

**Theorem 2.4.** *Suppose*

$$\int_0^1 \frac{ds}{g(s)} < \infty \tag{2.7}$$

*and*

$$\liminf_{s \searrow 0} \Big(\frac{f}{g}\Big)'(s) > -\infty. \tag{2.8}$$

*If* $u_0$ *satisfies* (H1) *and*

$$\int_{\mathbb{R}^n} \int_0^{u_0(x)} \frac{ds}{g(s)} dx < \infty, \tag{2.9}$$

*then*

$$\{u(t) > 0\} \textit{ has finite measure for a.e. } t > 0. \tag{2.10}$$

**Proof.** Let $\Phi(s) := \int_0^s \frac{d\sigma}{g(\sigma)}$, $s \geq 0$. Since $\Phi(0) = 0$ and $\Phi'(s) \to +\infty$ as $s \searrow 0$, we have $\Phi(s) \geq s^2$ for small $s$, and hence (2.9) implies $u_0 \in L^2(\mathbb{R}^n)$. Thus, we may construct the $u_{0\varepsilon R}$ in (1.2) in such a way that $u_{0\varepsilon R} \leq u_{0\varepsilon}$ and $u_{0\varepsilon R}|_{\partial B_R} = \varepsilon$; for example, let $u_{0\varepsilon R} := (u_{0\varepsilon} - \varepsilon)\psi_R + \varepsilon$, where $0 \leq \psi_R$ is in $C_0^\infty(B_R)$ with $\psi_R \equiv 1$ in $B_{R-1}$ and $|\nabla \psi_R| \leq 2$.

Once again fix a nondecreasing $\chi \in C^\infty([0,\infty))$ with $\chi_{(2,\infty)} \leq \chi \leq \chi_{(1,\infty)}$, and let $\chi_\delta(s) := \chi(\frac{s}{\delta})$ for $\delta > 0$. Multiplying (1.2) by $\frac{\chi_\delta}{g}(u_\varepsilon)$, $0 < \varepsilon < \frac{\delta}{2}$, we infer from an integration by parts over $B_R \times (0,T)$, $T > 0$, that

$$\int_{B_R} \Phi_\delta(u_{\varepsilon R}(T)) + \int_0^T \int_{B_R} \frac{g_\varepsilon}{g}(u_{\varepsilon R}) \cdot \chi_\delta(u_{\varepsilon R})$$

$$= \int_{B_R} \Phi_\delta(u_{0\varepsilon R}) - \int_0^T \int_{B_R} \Big(\frac{f}{g}\Big)'(u_{\varepsilon R}) \cdot \chi_\delta(u_{\varepsilon R}) |\nabla u_{\varepsilon R}|^2$$
$$- \int_0^T \int_{B_R} \frac{f}{g}(u_{\varepsilon R}) \cdot \chi'_\delta(u_{\varepsilon R}) |\nabla u_{\varepsilon R}|^2,$$

where we have set $\Phi_\delta(s) := \int_0^s \frac{\chi_\delta(\sigma)}{g(\sigma)} d\sigma$. Now the last term on the right is nonpositive, while the second is bounded above by

$$cT \cdot \sup_{t\in(0,T)} \int_{\mathbb{R}^n} |\nabla u_{\varepsilon R}|^2 \le cT,$$

according to (2.8) and (1.4). Since furthermore $\Phi_\delta(u_{0\varepsilon R}) \le \Phi_\delta(u_{0\varepsilon})$, passing to the limit $R \to \infty$ gives

$$\int_0^T \int_{\mathbb{R}^n} \frac{g_\varepsilon}{g}(u_\varepsilon) \cdot \chi_\delta(u_\varepsilon) \le \int_{\mathbb{R}^n} \Phi_\delta(u_\varepsilon) + cT,$$

where we note that $\Phi_\delta(u_{0\varepsilon})$ is in $L^1(\mathbb{R}^n)$ for all $\varepsilon < \frac{\delta}{2}$, because $u_{0\varepsilon} \le u_0 + 2\varepsilon$ in $\mathbb{R}^n$ implies $\{u_{0\varepsilon} > \delta\} \subset \{u_0 > \delta - 2\varepsilon\}$ and the latter set has finite measure since otherwise $\Phi(u_0)$ would not be integrable, contradicting (2.9). Letting $\varepsilon \to 0$ and applying twice the monotone convergence theorem, we see that

$$\int_0^T \int_{\mathbb{R}^n} \chi_\delta(u) \le \int_{\mathbb{R}^n} \Phi_\delta(u_0) + cT \le \int_{\mathbb{R}^n} \Phi(u_0) + cT$$

and thus

$$\int_0^T \Big|\{u(t) > 0\}\Big| dt < \infty,$$

so that $|\{u(t) > 0\}| < \infty$ for almost every $t \in (0,T)$ and hence for almost every $t > 0$. □

Clearly, (2.9) is indeed a condition on the decay of $u_0$:

**Corollary 2.5.** *There exists a one-parameter family $(w_\eta)_{\eta>0}$ of positive functions $w_\eta \in C^1(\mathbb{R}^n)$ such that*

i) $\lim_{\eta\to 0} w_\eta(x) = \infty$ *uniformly on compact subsets of* $\mathbb{R}^n$*, and*
ii) *if $u_0$ satisfies* (H1) *and $u_0 \le w_\eta$ in $\mathbb{R}^n$ for some $\eta > 0$, then $|\{u(t) > 0\}| < \infty$ for almost every $t > 0$.*

**Proof.** Redefining $g(s)$ to be constant for large $s$ if necessary, we may assume that

$$\Phi(s) := \int_0^s \frac{\sigma}{g(\sigma)} \to \infty \quad \text{as } s \to \infty,$$

whence the same is true for its inverse $\Phi^{-1}$. Thus, for example,

$$w_\eta(x) := \Phi^{-1}\left(\frac{1}{\eta(1+|x|)^{n+\eta}}\right), \quad \eta > 0,$$

fulfills $w_\eta(x) \to \infty$ uniformly on balls, but

$$\int_{\mathbb{R}^n} \Phi(w_\eta) = \frac{c}{\eta}\int_0^\infty \frac{r^{n-1}}{(1+r)^{n+\eta}} dr < \infty$$

for all $\eta > 0$. □

The basic idea in the proof of Theorem 2.4 can be exploited more effectively if only power-type terms enter our equation. In fact, we then arrive at the same conclusion as in Theorem 2.4 if merely some power of $u_0$ is integrable—a condition that clearly is neither better nor worse than the one required in Theorem 2.2.

**Theorem 2.6.** *Consider the equation*

$$u_t = u^p \Delta u - u^q, \qquad \text{with } p > 0,\ q \in (0, \min\{1, p\}).$$

*If $u_0$ satisfies* (H1) *and $u_0 \in L^r(\mathbb{R}^n)$ for some $r \in [1, \infty)$, then*

$$|\{u(t) > 0\}| < \infty \quad \text{for a.e. } t > 0.$$

**Proof.** Let $\tau, T > 0$ be given with $\tau < T$. We are done if we show that $|\{u(t) > 0\}| < \infty$ for almost every $t \in (\tau, T)$. Define $\psi_R : \mathbb{R}^n \to [0, 1]$ by

$$\psi_R(x) := \begin{cases} 1, & |x| \le R - 1, \\ R - |x|, & R - 1 < |x| \le R, \\ 0, & |x| > R, \end{cases}$$

and test (1.2) with $u_\varepsilon^{\gamma-1} \cdot \psi_R^2(x)$, $\gamma > \max\{0, 1-p\}$, to obtain for $0 \le t_0 < T$

$$\begin{aligned} &\frac{1}{\gamma}\int_{B_R} u_\varepsilon^\gamma(T) \cdot \psi_R^2 + (p+\gamma-1)\int_{t_0}^T \int_{B_R} u_\varepsilon^{p+\gamma-2}|\nabla u_\varepsilon|^2 \psi_R^2 \\ &\quad + \int_{t_0}^T \int_{B_R} u_\varepsilon^{\gamma-1} g_\varepsilon(u_\varepsilon) \cdot \psi_R^2 \\ &= \frac{1}{\gamma}\int_{B_R} u_\varepsilon^\gamma(t_0)\psi_R^2 - 2\int_{t_0}^T \int_{B_R} \psi_R u_\varepsilon^{p+\gamma-1} \nabla u_\varepsilon \cdot \nabla \psi_R \\ &\le \frac{1}{\gamma}\int_{B_R} u_\varepsilon^\gamma(t_0) \cdot \psi_R^2 + \eta \int_{t_0}^T \int_{B_R} u_\varepsilon^{2(p+\gamma-1)} \psi_R^2 |\nabla \psi_R|^2 \\ &\quad + c(\eta) T \sup_{t\in(0,T)} \int_{\mathbb{R}^n} |\nabla u_\varepsilon(t)|^2 \end{aligned}$$

for any $\eta > 0$, where we have used Young's inequality. Letting $\varepsilon$ tend to zero and observing $|\nabla\psi_R| \le 1$ as well as (1.5), we infer from this

$$\begin{aligned}
&\frac{1}{\gamma}\int_{B_R} u^\gamma(T)\cdot\psi_R^2 + \int_{t_0}^T\int_{B_R}\chi_{\{u>0\}}u^{q+\gamma-1}\psi_R^2 \\
&\le \frac{1}{\gamma}\int_{B_R} u^\gamma(t_0)\cdot\psi_R^2 + \eta\int_{t_0}^T\int_{B_R} u^{2(p+\gamma-1)}\psi_R^2 + c(\eta)T \\
&\le \frac{1}{\gamma}\int_{B_R} u^\gamma(t_0)\cdot\psi_R^2 + \eta\|u\|^{2p+\gamma-q-1}_{L^\infty(\mathbb{R}^n\times(0,T))}\int_{t_0}^T\int_{B_R}\chi_{\{u>0\}}u^{q+\gamma-1}\psi_R^2 + c(\eta)T,
\end{aligned}$$

since $p+\gamma-1>0$ and $2p+\gamma-q-1>p-q>0$, and hence, choosing $\eta$ small enough and letting $R\to\infty$,

$$\frac{1}{\gamma}\int_{\mathbb{R}^n} u^\gamma(T) + \frac{1}{2}\int_{t_0}^T\int_{\mathbb{R}^n}\chi_{\{u>0\}}u^{q+\gamma-1} \le \frac{1}{\gamma}\int_{\mathbb{R}^n} u^\gamma(t_0) + cT. \tag{2.11}$$

We claim that for all $s > \max\{0, 1-p\}$,

$$\int_{\mathbb{R}^n} u^s(T) + \int_\tau^T\int_{\mathbb{R}^n}\chi_{\{u>0\}}u^{q+s-1} \le c(s,\tau,T). \tag{2.12}$$

To prove this, let us fix $k_0\in\mathbb{N}$ such that $s+k_0(1-q)\ge r$ and assert that for all $k\in\{0,\dots,k_0\}$

$$\int_{\mathbb{R}^n} u^{s+(k_0-k)(1-q)}(T) + \int_{(1-2^{-k})\tau}^T\int_{\mathbb{R}^n}\chi_{\{u>0\}}u^{s+(k_0-k-1)(1-q)} \le c(k,\tau,T). \tag{2.13}$$

Indeed, for $k=0$ this derives from (2.11) upon the choices $\gamma := s+k_0(1-q)$ and $t_0 := 0$, and the observation that the term on the right of (2.11) is finite since $u_0\in L^r(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)\subset L^\gamma(\mathbb{R}^n)$.

If (2.13) has been proved for some $k<k_0$, it follows in particular that the space-time integral $\int_{(1-2^{-k})\tau}^{(1-2^{-k-1})\tau}\int_{\mathbb{R}^n} u^{s+(k_0-k-1)(1-q)}$ is bounded, so that there exists some $t_\star\in((1-2^{-k})\tau,(1-2^{-k-1})\tau)$ such that

$$\int_{\mathbb{R}^n} u^{s+(k_0-k-1)(1-q)}(t_\star) \le \frac{2^{k+1}}{\tau}\cdot c.$$

Now (2.13) for $k+1$ results from (2.11) if we set $\gamma := s+(k_0-k-1)(1-q)$ and $t_0 := t_\star$.

Next, as $q<\min\{1,p\}$, we may let $s := 1-q$ in (2.12) to achieve $\int_\tau^T |\{u(t)>0\}|dt < \infty$, whence $|\{u(t)>0\}|<\infty$ for almost every $t\in(\tau,T)$. □

Incidentally, from (2.12) we also gain that $u(t)$ decays in space—in a weakened sense—uniformly with respect to $t \in (\tau, T)$ (cf. also [7]):

**Corollary 2.7.** *Under the hypotheses of Theorem* 2.6, *we have*

$$u \in L^\infty_{loc}((0,\infty); L^s(\mathbb{R}^n)) \qquad \textit{for all } s > \max\{0, 1-p\}.$$

Let us finally show that (2.7) cannot be essentially weakened any further without losing the instantaneous shrinking phenomenon.

**Theorem 2.8.** *i) If*

$$\limsup_{s \searrow 0} \frac{g(s)}{s} < \infty,$$

*then for all $t > 0$, $u(t) > 0$ almost everywhere in $\{u_0 > 0\}$.*

*ii) If*

$$\int_0^1 \frac{ds}{g(s)} = \infty \tag{2.14}$$

*and*

$$\limsup_{s \searrow 0} \Big(\frac{f}{g}\Big)'(s) < \infty \tag{2.15}$$

*as well as*

$$\textit{either} \quad \limsup_{s \searrow 0} \frac{f}{g}(s) < \infty \tag{2.16}$$

$$\textit{or} \quad \limsup_{s \searrow 0} \frac{s^{2-\gamma} f(s)}{g^2(s)} < \infty \quad \textit{for some } \gamma \in (1-\nu, 1), \tag{2.17}$$

*then for all $t > 0$,*

$$u(t) > 0 \quad \textit{a.e. in } \{u_0 > 0\}. \tag{2.18}$$

**Remark.** In view of i), part ii) will be of interest only if $\limsup_{s \searrow 0} \frac{g(s)}{s} = \infty$. In this case, if $g'$ does not oscillate too heavily near $s = 0$ in the sense that $\liminf_{s \searrow 0} g'(s) > 0$, then (2.17) is implied by (2.16).

**Proof.** i) According to the assumption on $g$, there exist positive numbers $c_1$ and $s_1$ such that $g(s) \le c_1 s$ for all $s \in (0, s_1)$. Fix $x_0 \in \{u_0 > 0\}$ and a ball $B$ around $x_0$ such that $u_0 > 0$ in $\bar{B}$. Let $y_0 > 0$ be so small that $y_0 \le \inf_B u_0$ and $y_0 \le s_1$. Denoting by $\Theta$ the principal eigenfunction of $-\Delta$ in $B$ with $\max \Theta = 1$ and by $\lambda_1$ the corresponding eigenvalue, we can easily check that $v(x,t) := e^{-\alpha t} y_0 \cdot \Theta(x)$ is a subsolution of (1.1) in $B \times (0, \infty)$ if we choose $\alpha \ge c_1 + \lambda_1 \max_{[0,y_0]} f$. As $v \le u_\varepsilon$ on $\partial B$ and at $t = 0$, we conclude that $u_\varepsilon \geq v$ for all $\varepsilon$ and thus $u \geq v$ in $B \times (0,\infty)$, whence in particular $u(x_0,t) \geq ce^{-\alpha t}$.

ii) Let $\Phi(s) := \int_1^s \frac{d\sigma}{g(\sigma)}$, $s > 0$, and multiply (1.1) by $\frac{\psi(x)}{g(u_\varepsilon)}$, $0 \leq \psi \in C_0^\infty(\{u_0 > 0\})$, to see that

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(u_\varepsilon(t)) \cdot \psi &= \int_{\mathbb{R}^n} \Phi(u_{0\varepsilon}) \cdot \psi - \int_0^t \int_{\mathbb{R}^n} \frac{g_\varepsilon}{g}(u_\varepsilon) \cdot \psi \\ &\quad - \int_0^t \int_{\mathbb{R}^n} \Big(\frac{f}{g}\Big)'(u_\varepsilon)|\nabla u_\varepsilon|^2 \psi - \int_0^t \int_{\mathbb{R}^n} \frac{f}{g}(u_\varepsilon)\nabla u_\varepsilon \cdot \nabla\psi \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{2.19}$$

where

$$I_1 \geq \int_{\mathbb{R}^n} \Phi(u_0) \cdot \psi \geq -c$$

since $u_0 \geq c > 0$ in $\operatorname{supp}\psi$. Clearly, $g_\varepsilon \leq g$ implies $I_2 \geq -c$, while in accordance with (2.15) and (1.5),

$$I_3 \geq -c \int_0^t \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 \geq -c.$$

Similarly, if (2.16) holds, then

$$|I_4|^2 \leq c \int_0^t \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2 \leq c;$$

however, if (2.17) is satisfied, then

$$\begin{aligned}|I_4| &\leq \|\nabla\psi\|_{L^\infty(\mathbb{R}^n)} \Big(\int_0^t \int_{\operatorname{supp}\psi} u_\varepsilon^{\gamma-2} f(u_\varepsilon)|\nabla u_\varepsilon|^2\Big)^{\frac{1}{2}} \\ &\quad \times \Big(\int_0^t \int_{\operatorname{supp}\psi} \frac{u_\varepsilon^{2-\gamma} f(u_\varepsilon)}{g^2(u_\varepsilon)}\Big)^{\frac{1}{2}} \leq c\end{aligned}$$

due to (1.6) and the fact that

$$s^{\gamma-2} f(s) \leq c(|f'(s)| + 1) \quad \text{on} \quad \big(0, \sup_{\varepsilon\in(0,1)} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^n\times(0,\infty))}\big).$$

Consequently, applying the monotone convergence theorem to (2.19) yields

$$\int_{\mathbb{R}^n} \Phi(u(t)) \cdot \psi > -\infty,$$

from which we conclude in view of (2.14) that $u(t) > 0$ almost everywhere in $\{\psi > 0\}$ and thus, since $\psi$ was arbitrary, that $u(t) > 0$ almost everywhere in $\{u_0 > 0\}$. $\square$

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