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GLOBAL SOLUTIONS OF NAVIER-STOKES EQUATIONS WITH LARGE L^2 NORMS IN A NEW FUNCTION SPACE

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Abstract. First we prove certain pointwise bounds for the fundamental solutions of the perturbed linearized Navier-Stokes equation (Theorem 1.1). Next, utilizing a new framework with very little L^p theory or Fourier analysis, we prove existence of global classical solutions for the full Navier-Stokes equation when the initial value has a small norm in a new function class of Kato type (Theorem 1.2). The smallness in this function class does not require smallness in L^2 norm. Furthermore we prove that a Leray-Hopf solution is regular if it lies in this class, which allows much more singular functions then before (Corollary 1). For instance this includes the well-known result in [25]. A further regularity condition (form boundedness) was given in Section 5. We also give a different proof about the L^2 decay of Leray-Hopf solutions and prove pointwise decay of solutions for the three-dimensional Navier-Stokes equations (Corollary 2, Theorem 1.2). Whether such a method exists was asked in a survey paper [2].

1. Introduction

There are two goals for the paper. The first is to establish certain pointwise bounds for the fundamental solutions of the perturbed linearized Navier-Stokes equation

$$\begin{cases} \Delta u(x,t) - b(x,t)\nabla u(x,t) - \nabla P(x,t) - \partial_t u(x,t) = 0, \\ \qquad\qquad\qquad\qquad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ div\ u = 0,\ n \geq 3, \qquad u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

Here Δ is the standard Laplacian, $u(x,t), u_0(x), b(x,t) \in \mathbf{R}^n$, $P(x,t) \in \mathbf{R}$ and $b\nabla u = \sum_{i=1}^n b_i \partial_{x_i} u$. This linear system is also known as the Oseen flow. The bounds we will prove were previously known only for the case of Stokes flow, i.e., when $b \equiv 0$ in (1.1). See [7], e.g.

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The second goal is to establish more general conditions which imply regularity of weak solutions to Navier-Stokes equations (1.2) and to prove existence of global classical solutions, when the initial value has a small norm in a certain function class of Kato type. Since smallness in this class does not require smallness in L^2 norm, we have thus proven the existence of global classical solutions for *some* initial values with arbitrarily large L^2 norms. Recall that the Cauchy problem for Navier-Stokes equations is

$$\begin{cases} \Delta u(x,t) - u\nabla u(x,t) - \nabla P(x,t) - \partial_t u(x,t) = 0, \\ \qquad\qquad\qquad\qquad (x,t) \in \mathbf{R}^n \times (0,\infty), \\ div\ u = 0,\ n \geq 3, \qquad u(x,0) = u_0(x). \end{cases} \tag{1.2}$$

Here and always u is a vector-valued function, which means a function whose range is a subset of $\mathbf{R}^n$.

Let us recall some of the recent advances in the problem of finding global (strong) solutions for (1.2). Due to the large number of pertinent papers, we may miss some of them. Kiselev and Ladyzhenskaya [12] proved that (1.2) for $n = 3$ has a global solution provided that $\|u_0\|_{W^{2,2}(\mathbf{R}^3)}$ is sufficiently small. See also the work of Kato and Ponce [14]. Fabes, Jones, and Riviére [7] showed that (1.2) has a global solution when $\|u_0\|_{L^{n+\epsilon}(\mathbf{R}^n)} + \|u_0\|_{L^{n-\epsilon}(\mathbf{R}^n)}$ is sufficiently small. Here ϵ is a small positive constant. Kato [11] proved that (1.2) has a global strong solution when u_0 is small in the $L^n(\mathbf{R}^n)$ sense. Later Giga and Miyakawa [9] and M. Taylor [28] proved the same result for small u_0 in a certain Morrey space. A similar result was obtained by Cannone [3] and Planchon [21] for initial data in certain Besov spaces. Related results can also be found in papers by Iftimie [10] and Lions and Masmoudi [18]. Most recently Koch and Tataru [15] proved global existence when u_0 is a small function in the so-called BMO^{-1} class ([15, page 24]). A function is in this class if its convolution with the heat kernel is in a type of Morrey space. The result in [15] recovers all the above-mentioned results on global existence of strong solutions. In the proofs, many authors have applied quite involved tools in harmonic analysis.

In this paper we find that the Navier-Stokes equation has certain global solutions in another natural function class of Kato type. In Remark 1.2 below, we will see that this class is different from the X class in [15] and hence different from all the spaces preceding [15]. This class has some interesting qualities, which makes it rather promising. The first is that these functions can be quite singular. For example, they do not have to be in any L^p_{loc} class for any $p > 1$. The singular set could be dimension $n-1$, etc. (see Remark 1.2 below). Yet Leray-Hopf solutions in this class must be smooth. The second is that this class gives rise to *pointwise* bounds for the global solutions in a natural manner. The third is that within this class, the proof of global existence is distinct and self-contained, using very little L^p theory or Fourier analysis, which have been vital components in all previous arguments. Directness of the proof indicates that it may be useful in future studies.

As another application we give a different proof about the L^2 decay of Leray-Hopf solutions of three-dimensional Navier-Stokes equations. (See Corollary 2 below.)

Roughly speaking, a function is in a Kato, or Schechter, or Stummel class if its convolution with certain kernel functions satisfies suitable boundedness or smallness assumptions. The kernel functions are usually chosen as the fundamental solutions of some elliptic or parabolic equations. It is well known that Kato-class functions are natural choices of function spaces in the regularity theory of solutions of elliptic and parabolic equations. This fact has been well documented in the paper [26]. In this paper we show that a suitable Kato class is also a natural function space in the study of Navier-Stokes equation.

Now let us introduce the main function class for Theorem 1.1, which will be called K_1. Then we will explain its many nice properties. For instance it is known that a Leray-Hopf solution (see Remark 1.1 for a definition) of the Navier-Stokes equations in $L^{p,q}$ ($\frac{n}{p} + \frac{2}{q} < 1$) space is regular ([25]). We will show that the function class defined below properly contains this $L^{p,q}$ space. Moreover Leray-Hopf solutions in this class are also regular (Corollary 1).

Definition 1.1. *A vector-valued function $b = b(x,t) \in L^1_{loc}(\mathbf{R}^{n+1})$ is in class K_1 if it satisfies the following condition:*

$$\lim_{t\to l} \sup_{x\in\mathbf{R}^n,\ l>0} \int_l^t \int_{\mathbf{R}^n} [K_1(x,t;y,s) + K_1(x,s;y,l)]|b(y,s)|\,dy\,ds = 0, \tag{1.3}$$

where

$$K_1(x,t;y,s) = \frac{1}{[|x-y| + \sqrt{t-s}]^{n+1}}, \quad t \ge s, x \ne y. \tag{1.4}$$

For convenience, we introduce the notation

$$B(b,l,t) \equiv \sup_{x\in\mathbf{R}^n} \int_l^t \int_{\mathbf{R}^n} [K_1(x,t;y,s) + K_1(x,s;y,l)]|b(y,s)|\,dy\,ds, \tag{1.5}$$

$$B(b,l,\infty) \equiv \sup_{x\in\mathbf{R}^n, t>l} \int_l^t \int_{\mathbf{R}^n} [K_1(x,t;y,s) + K_1(x,s;y,l)]|b(y,s)|\,dy\,ds. \tag{1.6}$$

These quantities will serve as replacements of the L^p and other norms used by other authors. The reader may wonder whether the appearance of two K_1s (K_1 and its conjugate) is necessary in the definition. At this time we do not know the answer. However, a similar class was defined by the author for the heat equation [31]. There K_1 is replaced by the gradient of the heat kernel. As pointed out in [8], using one kernel will result in a different class.

Remark 1.0. In case b is independent of time, an easy computation shows

$$\begin{aligned} B(b,0,\infty) &\equiv 2 \sup_{x \in \mathbf{R}^n} \int_0^\infty \int_{\mathbf{R}^n} K_1(x,s;y,0)|b(y)|\, dy\, ds \\ &= c_n \sup_{x \in \mathbf{R}^n} \int_{\mathbf{R}^n} \frac{|b(y)|}{|x-y|^{n-1}} dy. \end{aligned}$$

This last quantity was introduced in [22] and more explicitly in [4]. See also [31] for the time-dependent case. As in [31], it is also easy to see that for a time-independent function b, $b \in K_1$ if

$$\lim_{r \to \infty} \sup_x \int_{|x-y| \le r} \frac{|b(y)|}{|x-y|^{n-1}} dy = 0$$

and $B(b,0,\infty) < \infty$.

The main results of the paper are the next two theorems and corollaries.

Theorem 1.1. *Suppose b is in class K_1. Then* (1.1) *has a fundamental solution* (*matrix*) $E = E(x,t;y,s)$ *in the following sense.*

(*i*) *Let*

$$u(x,t) = \int_{\mathbf{R}^n} E(x,t;y,0) u_0(y) dy,$$

where $u_0 \in C_0^\infty(\mathbf{R}^n)$ is vector valued and divergence free. For any vector-valued $\phi \in C_0^\infty(\mathbf{R}^n \times (-\infty,\infty))$ with div $\phi = 0$, there holds

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} \langle u, \partial_t \phi + \Delta \phi \rangle dx\, dt - \int_0^\infty \int_{\mathbf{R}^n} \langle b \nabla u, \phi \rangle dx\, dt \\ &\quad = - \int_{\mathbf{R}^n} \langle u_0(x), \phi(x,0) \rangle dx. \end{aligned}$$

Furthermore,

$$\sum_{i=1}^n \partial_{x_i} E_{ij}(x,t;y,s) = 0 \quad \text{for all } j = 1, \dots, n.$$

(*ii*)
$$\lim_{t \to 0} \int_{\mathbf{R}^n} E(x,t;y,0)\phi(y) dy = \phi(x).$$

Here ϕ is a smooth, vector-valued function in $\mathbf{R}^n$ with div $\phi = 0$.

(*iii*) *There exists $\delta > 0$ depending only on b and n such that*

$$|E(x,t;y,s)| \le \frac{C_\delta}{(\ |x-y| + \sqrt{t-s}\ )^n}, \quad \textit{when} \quad 0 < t-s \le \delta.$$

Suppose in addition that

$$\lim_{T\to\infty} \sup_{x,t>T} \int_T^t \int_{\mathbf{R}^n} [K_1(x,t;y,s) + K_1(x,s;y,T)]|b(y,s)|\,dy\,ds \le \mu, \tag{1.7}$$

where μ is a small, positive constant depending only on n. Then there exists $T_0 > 0$ depending only on b and n such that

$$|E(x,t;y,s)| \le \begin{cases} \frac{C_\delta}{(\ |x-y|+\sqrt{t-s}\ )^n}, & \textit{when} \ \ 0 < t-s \le \delta \ \ \textit{or} \ \ T_0 \le s \le t, \\ \frac{C_{\delta,\epsilon}}{(\ |x-y|+\sqrt{t-s}\ )^{n-\epsilon}}, & \textit{otherwise.} \end{cases}$$

Here $\epsilon > 0$ is any sufficiently small number and $C_{\delta,\epsilon}$ depends only on b, ϵ, and n.

Next we turn to the Navier-Stokes equations (1.2).

Definition 1.2. *Following standard practice, we say that u is a* (*weak*) *solution of* (1.2) *if the following holds:*

For any vector-valued $\phi \in C_0^\infty(\mathbf{R}^n \times (-\infty,\infty))$ with div $\phi = 0$, u satisfies div $u = 0$ and

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} \langle u, \partial_t\phi + \Delta\phi\rangle dx\,dt - \int_0^\infty \int_{\mathbf{R}^n} \langle u\nabla u, \phi\rangle dx\,dt \\ &= -\int_{\mathbf{R}^n} \langle u_0(x), \phi(x,0)\rangle dx. \end{aligned} \tag{1.8}$$

We will use this definition for solutions of (1.2) throughout the paper, unless stated otherwise. Note that we need only that the above integrals make sense and that there are no a priori assumptions about which spaces u and ∇u lie in.

Remark 1.1. Solutions thus defined are more general than Leray-Hopf solutions, which in addition require also $\|u(\cdot,t)\|_{L^2(\mathbf{R}^n)} < \infty$ for all $t > 0$ and $\|\nabla u\|_{L^2(\mathbf{R}^n\times(0,\infty))} < \infty$.

Theorem 1.2. *There exists a positive number η depending only on the dimension n such that, if $\sup_x \int_{\mathbf{R}^n} \frac{|u_0(y)|}{|x-y|^{n-1}} dy < \eta$ and div $u_0 = 0$, then the Navier-Stokes equations have a global solution u satisfying $B(u,0,\infty) < c\eta$. Moreover, there exists $C > 0$ such that*

$$|u(x,t)| \le C \int_{\mathbf{R}^n} \frac{|u_0(y)|}{(|x-y| + \sqrt{t})^n} dy.$$

If in addition $u_0 \in L^2(\mathbf{R}^n)$, then u is a classical solution when $t > 0$.

Remark 1.2. Let $b = b(x_1, x_2, x_3)$ be a function from $\mathbf{R^3}$ to $\mathbf{R^3}$. If b is compactly supported and $|b| \sim \frac{1}{|x_1| \ |\ln|x_1||^\delta}$ with $\delta > 2$, it is easy to check that b satisfies (1.3) and hence is in class K_1. The proof is given at the end of Section 2. The class of functions u satisfying $B(u, 0, \infty) < \infty$ is different from the function space X in [15], where the solutions of the Navier-Stokes equations in that paper reside. It is easy to find a function in X but not in that class. On the other hand, the above function b is in class K_1 and $B(b, 0, \infty) < \infty$ but it is not in X. Functions in X must be locally square integrable (see p. 24 in [15]). However b is not in L^p_{loc} for any $p > 1$. We should mention that the above function b is in the BMO^{-1} class, the space of initial values in [15].

The next two corollaries are direct consequences of a small part of Theorem 1.1 (part (iii)). Their proofs, depending on Lemma 3.1 below, are independent of the rest of Theorem 1.1 and Theorem 1.2.

Corollary 1. *Let $u \in K_1$ be a Leray-Hopf solution of the Navier-Stokes equations. Then u is classical when $t > 0$.*

By the example in Remark 1.2, we see that the function class K_1 permits solutions which apparently are much more singular then previously known. In case the spatial dimension is 3, a solution can have an apparent singularity of certain type that is not L^p_{loc} for any $p > 1$ and of dimension 1. One can also construct time-dependent functions in K_1 with quite singular behavior. Nonetheless the solution is regular. Notice also that there is no smallness assumption on the solution u as long as it is in K_1. By Proposition 2.1 below, Corollary 1 contains the well-known regularity result of [25]. The result here also differs from the borderline case in [27] since K_1 and the space $L^{p,q}$ with $n/p + 2/q = 1$ are different. By a direct computation, one can also show that it contains the Morrey-type class in [20]. In Section 5, we will propose a form-bounded condition on u, containing this borderline case when $n = 3$, which will imply the regularity of u. As explained there, this condition seems to be one of the widest possible to date.

Corollary 2. *Let u be any Leray-Hopf solution of the three-dimensional Navier-Stokes equations. Suppose in addition that $u_0 \in L^1(\mathbf{R}^n)$; then, for $n = 3$ and $C = C(u_0, n)$,*

$$\Big[\int_{\mathbf{R}^n} |u(x,t)|^2 dx \Big]^{1/2} \leq \frac{C}{t^{n/4}} \Big[\int_{\mathbf{R}^n} |u_0(x)| dx + \int_{\mathbf{R}^n} |u_0(x)|^2 dx \Big].$$

Remark 1.3. In the interesting papers [23, 24] and [13], the result in Corollary 2 (and more) was proved for some Leray-Hopf solutions or for all admissible solutions in the sense of [1] (see also [17]). This extra restriction was removed in [30]. Fourier analysis is the main tool in the proof of the decay. The corollary provides an alternative proof without using the Fourier transform. However, in the papers [24], [13], and [30] a similar decay property has been established for some or all Leray-Hopf solutions under the weaker assumption that $u_0 \in L^r(\mathbf{R}^n)$ when $1 < r < 2$. We are not able to do it for all solutions for $r \geq 5/4$. However the above method provides even pointwise decay of solutions (Theorem 1.2). Whether such a method exists was asked in a survey paper [2].

Remark 1.4. Since u_0 can be quite singular, at the first glance the solution u in Theorem 1.2 may be too singular to satisfy (1.8). However we will show that all terms in the definition are justified. The uniqueness question of the solution is also interesting. We will not address the problem in this paper. Some recent development can be found in [5].

Let us outline the proof of Theorem 1.2. The strategy is to use a fixed-point argument. The novelty is a number of new inequalities involving the kernel function K_1 and its relatives. We will use these inequalities to show that the Navier-Stokes equations are globally well posed in class under the norm $B(u, 0, \infty)$, provided the initial value u_0 satisfies that $B(u_0, 0, \infty)$ is smaller than a dimensional constant.

We will use $C, c, c_1, \dots$ to denote positive constants which may change from line to line. The rest of the paper is organized as follows. In Section 2 we present some elementary properties of the function class K_1. Theorems 1.1 and 1.2 will be proven in Sections 3 and 4 respectively. The proofs are independent, except for the use of Lemma 3.1. The corollaries will be proven at the end of Section 3.

2. Preliminaries

Proposition 2.1. *Suppose $b \in L^{p,q}(\mathbf{R}^n \times \mathbf{R})$ with $\frac{n}{p} + \frac{2}{q} < 1$. Then $b \in K_1$; i.e.,*

$$\lim_{h \to 0} B(b, t-h, t) = 0$$

uniformly for all t. In particular, if $b \in L^{n+\epsilon}(\mathbf{R}^n)$ for $\epsilon > 0$, then the above holds.

Proof. For completeness, we recall that

$$\|b\|_{L^{p,q}} = \Big[\int_{\mathbf{R}} \big(\int_{\mathbf{R}^n} |b(y,s)|^p dy \big)^{q/p} ds \Big]^{1/q}.$$

Using Hölder's inequality twice on (1.5) (taking $l = t - h$), we obtain

$$B(b, t-h, t) \le 2\Big[\int_{t-h}^{t} \big(\int_{\mathbf{R}^n} |b(y,s)|^p dy\big)^{q/p} ds\Big]^{1/q}$$
$$\times \sup_x \Big[\int_{t-h}^{t} \big(\int_{\mathbf{R}^n} \frac{1}{(|x-y| + \sqrt{t-s})^{p'(n+1)}} dy\big)^{q'/p'} ds\Big]^{1/q'}.$$

Here $p' = p/(p-1)$ and $q' = q/(q-1)$. Since

$$\int_{\mathbf{R}^n} \frac{dy}{(|x-y| + \sqrt{t-s})^{p'(n+1)}} = \frac{(t-s)^{n/2}}{(t-s)^{p'(n+1)/2}} \int_{\mathbf{R}^n} \frac{dy}{(1+|y|)^{p'(n+1)}}$$
$$= \frac{c}{(t-s)^{(p'(n+1)-n)/2}},$$

we see that

$$B(b, t-h, t) \le c\|b\|_{L^{p,q}} \Big(\int_{t-h}^{t} \frac{ds}{(t-s)^{(p'(n+1)-n)q'/(2p')}}\Big)^{1/q'}.$$

By the assumption that $\frac{n}{p} + \frac{2}{q} < 1$, one has

$$(1 - \frac{1}{p'})n + (1 - \frac{1}{q'})2 < 1.$$

Simple computation then leads to $\mu \equiv (p'(n+1) - n)q'/(2p') < 1$. This implies $B(b, t-h, t) \le c\|b\|_{L^{p,q}} h^{(1-\mu)/q'}$. □

Remark. One can also define the class K_1 for functions in various domains, as suggested by the referee.

Proposition 2.2. *Suppose, for* $\epsilon > 0$, $\|b\|_{L^{n+\epsilon,\infty}} + \|b\|_{L^{n-\epsilon,\infty}} < \infty$. *Then*

$$B(b, 0, \infty) \le C(\|b\|_{L^{n+\epsilon,\infty}} + \|b\|_{L^{n-\epsilon,\infty}}).$$

Proof. This is similar to that of Proposition 2.1, so we will be very brief. We write

$$\int_0^t \int_{\mathbf{R}^n} K_1(x,t;y,s)|b(y,s)|\, dy\, ds$$
$$= \int_{t-1}^{t} \int_{\mathbf{R}^n} K_1(x,t;y,s)|b(y,s)|\, dy\, ds + \int_0^{t-1} \int_{\mathbf{R}^n} K_1(x,t;y,s)|b(y,s)|\, dy\, ds.$$

As in the proof of Proposition 2.1, using the assumption that $\|b\|_{L^{n+\epsilon,\infty}} < \infty$ and applying Hölder's inequality, we see that the first integral on the right-hand side is finite. Using the assumption that $\|b\|_{L^{n-\epsilon,\infty}} < \infty$ and applying Hölder's inequality, we see that the second integral on the right-hand side is also finite. This finishes the proof. □

Here we give a proof that the function in Remark 1.2 is in class K_1.

Since b is independent of time and $b \in L^1_{loc}(\mathbf{R}^3)$, by a simple localization of the integral in (1.3), it is enough to prove that

$$\lim_{r\to 0}\sup_x \int_{B(x,r)} \frac{|b(y)|}{|x-y|^2} dy = 0. \tag{2.1}$$

Here $dy = dy_1 dy_2 dy_3$. Since $b = b(y)$ depends only on y_1, by direct computation, one sees that

$$\begin{aligned}
&\int_{B(x,r)} \frac{|b(y)|}{|x-y|^2} dy \\
&\le c\int_{x_1-r}^{x_1+r} |b(y_1)| \int_{x_2-r}^{x_2+r}\int_{x_3-r}^{x_3+r} \frac{1}{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2} dy_2 dy_3 dy_1 \\
&\le c\int_{x_1-r}^{x_1+r} \frac{|\ln|x_1-y_1||}{|y_1||\ln|y_1||^\delta} dy_1.
\end{aligned}$$

By the assumption that $\delta > 2$, it is easy to show that (2.1) holds.

3. Proof of Theorem 1.1. Bounds for fundamental solutions

The proof of Theorem 1.1 is divided into several parts. We begin with some lemmas. At the end of the section, the corollaries will be proven. The proof of the corollaries depends only on Lemma 3.1. So it can be read separately from the proof of Theorem 1.1.

Lemma 3.1. *The following inequalities hold for all $x, y, z \in \mathbf{R}^n$ and $t > \tau > 0$.*

$$K_0 * bK_1 \equiv \int_0^t \int_{\mathbf{R}^n} \frac{1}{(|x-z|+\sqrt{t-\tau})^n} \frac{|b(z,\tau)|}{(|z-y|+\sqrt{\tau})^{n+1}} dz\, d\tau \le C\frac{B(b,0,t)}{(|x-y|+\sqrt{t})^n}, \tag{3.1}$$

$$K_1 * bK_1 \equiv \int_0^t \int_{\mathbf{R}^n} \frac{1}{(|x-z|+\sqrt{t-\tau})^{n+1}} \frac{|b(z,\tau)|}{(|z-y|+\sqrt{\tau})^{n+1}} dz\, d\tau \le C\frac{B(b,0,t)}{(|x-y|+\sqrt{t})^{n+1}}. \tag{3.2}$$

Here and later $K_0(x,t;y,s) \equiv \frac{1}{(|x-y|+\sqrt{t-s})^n}$.

Proof. Since

$$|x-z| + \sqrt{t-\tau} + |y-z| + \sqrt{\tau} \ge |x-y| + \sqrt{t},$$

we have either

$$|x-z|+\sqrt{t-\tau} \geq \frac{1}{2}(|x-y|+\sqrt{t}), \tag{3.3}$$

or

$$|z-y|+\sqrt{\tau} \geq \frac{1}{2}(|x-y|+\sqrt{t}). \tag{3.4}$$

Suppose (3.3) holds; then

$$K_0 * bK_1 \leq \frac{2^n}{(|x-y|+\sqrt{t})^n} \int_0^t \int_{\mathbf{R^n}} \frac{|b(z,\tau)|}{(|z-y|+\sqrt{\tau})^{n+1}} dz\, d\tau.$$

That is,

$$K_0 * bK_1 \leq \frac{2^n B(b,0,t)}{(|x-y|+\sqrt{t})^n}. \tag{3.5}$$

Suppose (3.4) holds but (3.3) fails; then

$$|z-y|+\sqrt{\tau} \geq \frac{1}{2}(|x-y|+\sqrt{t}) \geq |x-z|+\sqrt{t-\tau}.$$

Therefore,

$$\begin{aligned}&\frac{1}{(|x-z|+\sqrt{t-\tau})^n\ (|z-y|+\sqrt{\tau})^{n+1}}\\ &\qquad \leq \frac{1}{(|x-z|+\sqrt{t-\tau})^{n+1}(|z-y|+\sqrt{\tau})^n}.\end{aligned}$$

This shows

$$\begin{aligned}&\frac{1}{(|x-z|+\sqrt{t-\tau})^n\ (|z-y|+\sqrt{\tau})^{n+1}}\\ &\qquad \leq \frac{2^n}{(|x-z|+\sqrt{t-\tau})^{n+1}(|x-y|+\sqrt{t})^n}.\end{aligned}$$

Substituting this into (3.1), we obtain

$$K_0 * bK_1 \leq \frac{2^n}{(|x-y|+\sqrt{t})^n} \int_0^t \int_{\mathbf{R^n}} \frac{|b(z,\tau)|}{(|x-z|+\sqrt{t-\tau})^{n+1}} dz\, d\tau.$$

That is,

$$K_0 * bK_1 \leq \frac{2^n B(b,0,t)}{(|x-y|+\sqrt{t})^n}.$$

Clearly, the only remaining case to consider is when both (3.3) and (3.4) hold. However, this case is already covered by (3.5). Thus (3.1) is proven. Similarly (3.2) is proven. $\square$

Remark 3.1. By the same argument, one can prove that, for $f = f(x,t) \geq 0$,

$$K_0 * fK_0(x,t;y,s) \equiv \int_s^t \int_{\mathbf{R}^n} K_0(x,t;z,\tau) f(z,\tau) K_0(z,\tau;y,s) dz\, d\tau$$
$$\leq c \sup_{x,t} \int_s^t \int_{\mathbf{R}^n} [K_0(x,t;z,\tau) + K_0(x,\tau;z,s)] |f(z,\tau)| dz\, d\tau\ K_0(x,t;y,s)$$

for all x and $t > 0$.

Lemma 3.2. *Suppose* $\lim_{h\to 0} \sup_t B(b, t-h, t) = 0$. *Define formally a function*

$$E(x,t;y,s) = E_0(x,t;y,s) - \int_s^t \int_{\mathbf{R}^n} E_0(x,t;z,\tau) \sum_{i=1}^n b_i(z,\tau) \partial_{z_i} E(z,\tau;y,s) dz\, d\tau, \qquad (3.6)$$

where E_0 *is the fundamental solution of the Stokes flow, i.e.,* (1.1) *with* $b \equiv 0$. *Then there exists* $\delta > 0$ *and* $C = C(\delta)$ *such that*

$$|E(x,t;y,s)| \leq \frac{C}{(|x-y| + \sqrt{t-s})^n},$$

$$|\nabla_x E(x,t;y,s)| + |\nabla_y E(x,t;y,s)| \leq \frac{C}{(|x-y| + \sqrt{t-s})^{n+1}}$$

when $0 < t - s \leq \delta$.

Furthermore, the above two inequalities hold for all $t > s > 0$ *provided* $B(b, 0, \infty)$ *is smaller than a suitable positive constant depending only on* n.

Proof. We can write (3.6) succinctly as

$$E(x,t;y,s) = E_0(x,t;y,s) - E_0 * (b\nabla E)(x,t;y,s).$$

Expanding this formally, we obtain

$$E(x,t;y,s) = E_0(x,t;y,s) + \sum_{j=1}^{\infty} (-1)^j E_0 * (b\nabla E_0)^{*j}(x,t;y,s). \qquad (3.7)$$

Here $*$ means convolution and $b\nabla E_0 \equiv \sum_{i=1}^n b_i(z,\tau)\partial_{z_i} E_0(z,\tau;y,s)$.

It is well known that (see [7] e.g.),

$$|E_0(x,t;y,s)| \leq \frac{c}{(|x-y| + \sqrt{t-s})^n},$$

$$|\nabla_x E_0(x,t;y,s)| \leq \frac{C}{(|x-y| + \sqrt{t-s})^{n+1}}.$$

Using these bounds on E_0 and ∇E_0 together with Lemma 3.1, we deduce

$$|E_0 * b\nabla E_0(x,t;y,s)| \le \frac{c_0 B(b,s,t)}{(|x-y|+\sqrt{t-s})^n}.$$

Here $c_0(>0)$ depends only on n. By induction, it is easy to see that

$$|E_0 * (b\nabla E_0)^{*j}(x,t;y,s)| \le \frac{[c_0 B(b,s,t)]^j}{(|x-y|+\sqrt{t-s})^n}.$$

It follows from (3.7) that

$$|E(x,t;y,s)| \le \frac{c}{(|x-y|+\sqrt{t-s})^n} \sum_{j=0}^{\infty} [c_0 B(b,s,t)]^j.$$

Using the condition on b, we can choose δ sufficiently small so that $c_0 B(b,s,t) < 1$. Then (3.7) is uniformly convergent. Moreover, there exists $C > 0$ such that

$$|E(x,t;y,s)| \le \frac{C}{(|x-y|+\sqrt{t-s})^n}$$

when $0 < t-s < \delta$. This proves the first inequality in the lemma.

The inequality on $|\nabla_x E|$ can be derived similarly. The only change is to use the inequality

$$|\nabla E_0 * b\nabla E_0(x,t;y,s)| \le \frac{c_0 B(b,s,t)}{(|x-y|+\sqrt{t-s})^{n+1}},$$

which is part of Lemma 3.1.

In order to prove the estimate on $|\nabla_y E|$, we use the assumption that $div\, b = 0$. This implies, after integration by parts and a standard limiting argument,

$$\begin{aligned} E(x,t;y,s) &= E_0(x,t;y,s) \\ &\quad + \int_s^t \int_{\mathbf{R}^n} \sum_{i=1}^n \partial_{z_i} E_0(x,t;z,\tau) b_i(z,\tau) E(z,\tau;y,s) n \, dz \, d\tau. \end{aligned}$$

We remark that the integration by parts is rigorous by the just-proven bounds on $|\nabla_x E|$. This shows

$$\begin{aligned} \nabla_y E(x,t;y,s) &= \nabla_y E_0(x,t;y,s) \\ &\quad + \int_s^t \int_{\mathbf{R}^n} \sum_{i=1}^n \partial_{z_i} E_0(x,t;z,\tau) b_i(z,\tau) \nabla_y E(z,\tau;y,s) dz \, d\tau. \end{aligned}$$

From here the bound on $\nabla_y E$ can be obtained just like that for $\nabla_x E$.

If $B(b, 0, \infty)$ is sufficiently small, then all the above arguments hold for all $t > s > 0$. This proves the last statement in the lemma. □

Lemma 3.3. *Suppose*

$$\lim_{T\to\infty} \sup_t B(b, T, t) = \lim_{T\to\infty} \sup_{x,t>T} \int_T^t \int_{\mathbf{R^n}} K_1(x, t; y, s)|b(y, s)|\, dy\, ds$$

is sufficiently small. Then there exists $T_0 > 0$ and $C = C(T_0)$ such that

$$|E(x, t; y, s)| \le \frac{C}{(|x - y| + \sqrt{t - s})^n},$$

$$|\nabla_x E(x, t; y, s)| + |\nabla_y E(x, t; y, s)| \le \frac{C}{(|x - y| + \sqrt{t - s})^{n+1}}$$

when $t > s > T_0$. Here E is defined in (3.6).

Proof. The proof is almost identical to that of Lemma 3.2. The only difference is that all integration takes place in the region $\mathbf{R}^n \times (T_0, \infty)$. □

Now we are ready to give a

Proof of Theorem 1.1. Let us assume that b is a smooth function with compact support. This does not reduce any generality since all constants below depend on b only in terms of the quantity in Definition 1.1. The smoothness or the size of the support of b is irrelevant. In the sequel we can apply the limiting argument at the end of Section 4 when b is not smooth. We mention that the limiting argument, designed for the full Navier-Stokes equations, is more than enough to cover the linear case.

We are going to prove that the function $E = E(x, t; y, s)$ defined in (3.6) and extended by using a reproducing formula is the right choice for the fundamental solution of (1.1).

The order of the proof is (iii), then (ii), then (i).

Proof of (iii). By Lemmas 3.2 and 3.3, all we need to consider is the case $t - s \ge \delta$ and $s \le T_0$. Here δ and T_0 are the control constants in the above lemmas. For clarity we divide the case into two separate parts.

Part 1. We assume that $0 \le s < t \le 2T_0$ and $t - s \ge \delta$, $s \le T_0$. Since δ is fixed, we can use the semigroup property of E to write

$$\begin{aligned} E(x, t; y, s) = \int\int\cdots\int E(x, t; z_1, t - \delta)E(z_1, t - \delta; z_2, t - 2\delta) \\ \cdots E(z_k, t - k\delta; y, s)dz_1dz_2\cdots dz_k. \end{aligned} \tag{3.8}$$

Here all integration takes place in $\mathbf{R}^n$ and k is an integer such that $0 < t - k\delta - s \le \delta$. We are going to show that the integrals in the above reproducing formula are actually absolutely convergent. Here is how. Considering the integral

$$J \equiv \int_{\mathbf{R}^n} \frac{1}{(|x-z|+\sqrt{\delta})^n} \frac{1}{(|z-y|+\sqrt{\delta})^n} dz,$$

clearly,

$$J \leq \int_{|x-z|\geq|x-y|/2} \cdots dz + \int_{|z-y|\geq|x-y|/2} \cdots dz \equiv J_1 + J_2.$$

When $|x-z| \geq |x-y|/2$, we have

$$|z-y| \leq |x-y| + |x-z| \leq 3|x-z|.$$

Hence $|x-z| \geq |z-y|/3$. This shows

$$J_1 \leq \frac{c}{(|x-y|+\sqrt{\delta})^{n-\epsilon}} \int_{\mathbf{R}^n} \frac{dz}{(|z-y|+\sqrt{\delta})^{n+\epsilon}}.$$

Here $\epsilon(>0)$ is an arbitrary small constant. This shows, by direct computation,

$$J_1 \leq \frac{c\delta^{-\epsilon/2}}{(|x-y|+\sqrt{\delta})^{n-\epsilon}}.$$

Similarly,

$$J_2 \leq \frac{c\delta^{-\epsilon/2}}{(|x-y|+\sqrt{\delta})^{n-\epsilon}}.$$

These imply that

$$J \leq \frac{c\delta^{-\epsilon/2}}{(|x-y|+\sqrt{\delta})^{n-\epsilon}}. \tag{3.9}$$

Applying (3.9) to (3.8) $k-1$ times, we obtain

$$\begin{aligned} &|E(x,t;y,s)| \leq c_k \delta^{-(k-1)\epsilon/2} \\ &\times \int_{\mathbf{R}^n} \frac{1}{(|x-z_k|+\sqrt{\delta})^{n-(k-1)\epsilon}} \frac{dz_k}{(|z_k-y|+\sqrt{t-k\delta-s})^n}. \end{aligned}$$

Without loss of generality we can reduce δ suitably so that $\delta/2 \leq t-k\delta-s \leq \delta$. So applying the same technique as in the proof of (3.9), we know that

$$|E(x,t;y,s)| \leq c_k \delta^{-k\epsilon/2} \frac{1}{(|x-y|+\sqrt{\delta})^{n-k\epsilon}}.$$

Since k is finite, we can choose ϵ sufficiently small so that $n-\epsilon k$ is as close to n as possible. Note also that $t-s \leq 2T_0$. So, in case $0 \leq s < t \leq 2T_0$

and $s \le T_0$, one has

$$|E(x,t;y,s)| \le c_k \delta^{-k\epsilon/2} \frac{1}{(|x-y|+\sqrt{t-s})^{n-\epsilon}}. \tag{3.10}$$

Here we have renamed $k\epsilon$ as ϵ. This finishes Part 1.

Part 2. We assume that $t \ge 2T_0$, $s \le T_0$, and $t-s \ge \delta$.

By the semigroup property again

$$E(x,t;y,s) = \int E(x,t;z,1.5T_0)\ E(z,1.5T_0;y,t-s)dz. \tag{3.11}$$

By Lemma 3.3 and Part 1 just above, we have

$$|E(x,t;z,1.5T_0)| \le \frac{c}{(|x-z|+\sqrt{t-1.5T_0})^n},$$

$$|E(z,1.5T_0;y,s)| \le \frac{c}{(|z-y|+\sqrt{1.5T_0-s})^{n-\epsilon}}.$$

Substituting the above into (3.11), we see that

$$E(x,t;y,s) = c\int \frac{1}{(|x-z|+\sqrt{t-1.5T_0})^n} \frac{1}{(|z-y|+\sqrt{1.5T_0-s})^{n-\epsilon}} dz. \tag{3.12}$$

As before we split (3.12) as follows

$$E(x,t;y,s) \le c\int_{|x-z|\ge|x-y|/2} \cdots dz + c\int_{|z-y|\ge|x-y|/2} \cdots dz \equiv E_1+E_2. \tag{3.13}$$

When $|x-z| \ge |x-y|/2$, we have $|x-z| \ge |z-y|/3$. Note also $t-1.5T_0 \ge (1.5T_0-s)/3$. Hence,

$$E_1 \le \frac{c}{(|x-y|+\sqrt{t-1.5T_0})^{n-2\epsilon}} \int_{\mathbf{R}^n} \frac{dz}{(|z-y|+\sqrt{1.5T_0-s})^{n+\epsilon}},$$

which implies

$$E_1 \le \frac{c}{(|x-y|+\sqrt{t-1.5T_0})^{n-2\epsilon}} \frac{1}{(T_0-s)^{\epsilon/2}}. \tag{3.14}$$

Finally, we estimate E_2. When $|z-y| \ge |x-y|/2$, we have $|z-y| \ge |x-z|/3$. Since $t \ge 2T_0$ and $s \le T_0$, it is clear that

$$(|x-z|+\sqrt{t-1.5T_0})(|z-y|+\sqrt{1.5T_0-s}) \ge \frac{1}{8}(|x-z|+\sqrt{t})(|z-y|+\sqrt{T_0}).$$

By elementary computation, using the current assumptions on x, y, z, t, and T_0,

$$(|x-z|+\sqrt{t})(|z-y|+\sqrt{T_0}) \ge c(|x-z|+\sqrt{T_0})(|z-y|+\sqrt{t})$$

for some $c \in (0,1)$. Combining the last two inequalities we know that

$$
\begin{aligned}
&|E_2(x,t;y,s)| \\
&\le c \int_{|z-y|\ge|x-y|/2} \frac{1}{(|x-z|+\sqrt{t-1.5T_0})^n} \frac{1}{(|z-y|+\sqrt{1.5T_0-s})^{n-\epsilon}} dz \\
&\le c \int_{|z-y|\ge|x-y|/2} \frac{1}{(|x-z|+\sqrt{t})^n (|z-y|+\sqrt{T_0})^{n-\epsilon}} dz \\
&= c \int_{|z-y|\ge|x-y|/2} \frac{1}{(|x-z|+\sqrt{t})^\epsilon} \frac{dz}{[(|x-z|+\sqrt{t})\ (|z-y|+\sqrt{T_0})]^{n-\epsilon}} \\
&\le c \int_{|z-y|\ge|x-y|/2} \frac{1}{(|x-z|+\sqrt{t})^\epsilon} \frac{dz}{[(|x-z|+\sqrt{T_0})\ (|z-y|+\sqrt{t})]^{n-\epsilon}} \\
&\le \frac{c}{(|x-y|+\sqrt{t})^{n-2\epsilon}} \int_{|z-y|\ge|x-y|/2} \frac{1}{(|x-z|+\sqrt{t})^\epsilon} \frac{dz}{(|x-z|+\sqrt{T_0})^n}.
\end{aligned}
$$

Therefore,

$$
|E_2(x,t;y,s)| \le \frac{c}{(|x-y|+\sqrt{t})^{n-2\epsilon}} \int_{\mathbf{R}^n} \frac{dz}{(|x-z|+\sqrt{T_0})^{n+\epsilon}},
$$

which shows

$$
|E_2(x,t;y,s)| \le \frac{cT_0^{-\epsilon/2}}{(|x-y|+\sqrt{t-s})^{n-2\epsilon}}. \tag{3.15}
$$

Combining (3.14) and (3.15), we have

$$
|E(x,t;y,s)| \le \frac{C}{(|x-y|+\sqrt{t-s})^{n-2\epsilon}} \tag{3.16}
$$

when $t \ge 2T_0$, $s \le T_0$, and $t-s \ge \delta$. This proves (iii).

Proof of (ii). Since, for small t,

$$
\begin{aligned}
&E(x,t;y,0) = \\
&E_0(x,t;y,0) - \int_0^t \int_{\mathbf{R}^n} E_0(x,t;z,\tau) \sum_{i=1}^n b_i(z,\tau) \partial_{z_i} E(z,\tau;y,0) dz\, d\tau.
\end{aligned}
$$

From part (iii) of the theorem and Lemma 3.2, there exists $c > 0$ such that

$$
|E(x,t;y,0) - E_0(x,t;y,0)| \le c \int_0^t \int_{\mathbf{R}^n} K_0(x,t;z,\tau)|b(z,\tau)|K_1(z,\tau;y,0) dz\, d\tau
$$

when t is sufficiently small. By Lemma 3.1,

$$
|E(x,t;y,0) - E_0(x,t;y,0)| \le cB(b,0,t)K_0(x,t;y,0).
$$

Let $\phi = \phi(x)$ be a divergence-free, smooth, vector-valued function. Using the above inequality, we know

$$\begin{aligned}&\Big| \int_{\mathbf{R}^n} E(x,t;y,0)\phi(y)dy - \int_{\mathbf{R}^n} E_0(x,t;y,0)\phi(y)dy \Big| \\ &\le cB(b,0,t) \int_{\mathbf{R}^n} K_0(x,t;y,0)|\phi(y)|dy.\end{aligned}$$

Since ϕ is divergence-free, the above shows

$$\begin{aligned}&\Big| \int_{\mathbf{R}^n} E(x,t;y,0)\phi(y)dy - \int_{\mathbf{R}^n} G(x,t;y,0)\phi(y)dy \Big| \\ &\le cB(b,0,t) \int_{\mathbf{R}^n} K_0(x,t;y,0)|\phi(y)|dy.\end{aligned}$$

Here G is the fundamental solution of the heat equation. Using our main assumption that $B(b,0,t) \to 0$ when $t \to 0$ and the property of G, we obtain

$$\lim_{t\to 0} \int_{\mathbf{R}^n} E(x,t;y,0)\phi(y)dy = \phi(x).$$

This proves part (ii).

Proof of (i). Let $u_0 = u_0(x)$ be an initial value such that $div\, u_0 = 0$ in the weak sense. We need to prove that

$$u(x,t) \equiv \int_{\mathbf{R}^n} E(x,t;y,0)u_0(y)dy \tag{3.17}$$

is a solution to (1.1). Using a partition of unity, we can assume that the support of the test function is sufficiently narrow in the time direction. Hence, by the semigroup property, it is enough to prove (i) when t is sufficiently small. From formula (3.6) for E its clear that

$$\begin{aligned}u(x,t) = &\int_{\mathbf{R}^n} E_0(x,t;y,0)u_0(y)dy \\ &- \int_0^t \int_{\mathbf{R}^n} E_0(x,t;z,\tau) \sum_{l=1}^n b_l(z,\tau)\partial_{z_l} u(z,\tau)dz\, d\tau.\end{aligned}$$

The proof, following an argument in [7], goes as follows.

For simplicity we write

$$f = f(z,\tau) = b(z,\tau)\nabla_z u(z,\tau) \equiv \sum_{l=1}^n b_l(z,\tau)\partial_{z_l} u(z,\tau).$$

By Lemma 3.2

$$|f(z,\tau)| \le c|b(z,\tau)| \int_{\mathbf{R}^n} \frac{1}{(|z-w|+\sqrt{\tau})^{n+1}} |u_0(w)| dw.$$

Using Young's inequality and the extra assumption that b is smooth and compactly supported, $f(\cdot,t)$ is in $L^p(\mathbf{R}^n)$ for some $p>1$ and almost all t.

Let $R=(R_iR_j)_{n\times n}$, where R_i is the Riesz transform. It is known that (see [7] e.g.)

$$\begin{aligned} F(x,t) &\equiv \int_0^t \int_{\mathbf{R}^n} E_0(x,t;z,\tau) \sum_{l=1}^n b_l(z,\tau)\partial_{z_l} u(z,\tau) dz\, d\tau \qquad (3.18) \\ &= \int_0^t \int_{\mathbf{R}^n} E_0(x,t;z,\tau) f(z,\tau) dz\, d\tau \\ &= \int_0^t \int_{\mathbf{R}^n} G(x,t;z,\tau) f(z,\tau) dz\, d\tau - \int_0^t \int_{\mathbf{R}^n} G(x,t;z,\tau)(Rf)(z,\tau) dz\, d\tau, \end{aligned}$$

where G is the fundamental solution of the heat equation. We mention that the last term in (3.18) is valid since, by the extra assumption that b is smooth and compactly supported, $f(\cdot,t)$ is in $L^p(\mathbf{R}^n)$ for some $p>1$ and almost all t. However, as shown below, this term will be integrated out. Therefore the argument will be independent of the extra assumption on b eventually. Hence,

$$(\Delta - \partial_t)F(x,t) = -f(x,t) + (Rf)(x,t). \qquad (3.19)$$

Let ϕ be any suitable vector-valued test function; i.e., $\phi \in C_0^\infty(\mathbf{R}^n \times \mathbf{R})$ and $div\, \phi = 0$. By (3.19) and elementary properties of the heat equation, we have, for any $T>0$,

$$\begin{aligned} &\int_0^T \int_{\mathbf{R}^n} \langle F, \partial_t \phi + \Delta\phi\rangle(x,t) dx\, dy \\ &= -\int_0^T \int_{\mathbf{R}^n} \langle f, \phi\rangle(x,t) dx\, dt + \int_0^T \int_{\mathbf{R}^n} \langle Rf, \phi\rangle(x,t) dx\, dt. \end{aligned} \qquad (3.20)$$

Here $\langle \cdot\, , \cdot\rangle$ means the inner product in $\mathbf{R}^n$. Since $div\, \phi = 0$, quoting [7], we know that

$$\int_0^T \int_{\mathbf{R}^n} \langle Rf, \phi\rangle(x,t) dx\, dt = \int_0^T \int_{\mathbf{R}^n} \langle f, R\phi\rangle(x,t) dx\, dt = 0.$$

From (3.20), the above shows, since $f = b\nabla u$,

$$\int_0^T \int_{\mathbf{R}^n} \langle F, \partial_t \phi + \Delta\phi\rangle(x,t) dx\, dy = -\int_0^T \int_{\mathbf{R}^n} \langle b\nabla u, \phi\rangle(x,t) dx\, dt. \qquad (3.21)$$

According to (3.17), (3.6), and (3.18),

$$u(x,t)=\int_{\mathbf{R}^n}E_0(x,t;y,0)u_0(y)dy-F(x,t)=\int_{\mathbf{R}^n}G(x,t;y,0)u_0(y)dy-F(x,t).$$

Using this, we can compute as follows:

$$\begin{aligned}&\int_0^T\int_{\mathbf{R}^n}\langle u,\partial_t\phi+\Delta\phi\rangle(x,t)dx\,dy\\&=\int_0^T\int_{\mathbf{R}^n}\Big\langle\int Gu_0,\partial_t\phi+\Delta\phi\Big\rangle(x,t)dx\,dy\\&-\int_0^T\int_{\mathbf{R}^n}\langle F,\partial_t\phi+\Delta\phi\rangle(x,t)dx\,dt\\&=-\int_{\mathbf{R}^n}\langle u_0(x),\phi(x,0)\rangle dx+\int_0^T\int_{\mathbf{R}^n}\langle b\nabla u,\phi\rangle(x,t)dx\,dt.\end{aligned}$$

Here we just used (3.21)and an obvious property for $\int Gu_0$. Therefore,

$$\begin{aligned}&\int_0^T\int_{\mathbf{R}^n}\langle u,\partial_t\phi+\Delta\phi\rangle(x,t)dx\,dy-\int_0^T\int_{\mathbf{R}^n}\langle b\nabla u,\phi\rangle(x,t)dx\,dt\\&=-\int_{\mathbf{R}^n}\langle u_0(x),\phi(x,0)\rangle dx.\end{aligned}$$

In addition, since $\sum_{i=1}^n\partial_{x_i}(E_0)_{ij}(x,t;y,s)=0$, we know from (3.6) that

$$\sum_{i=1}^n\partial_{x_i}E_{ij}(x,t;y,s)=0.$$

This shows that $div\,u=0$. Hence u is a solution of (1.1). This proves part (i). The proof of Theorem 1.1 is complete. $\square$

Next we prove the two corollaries. The proof is in fact independent of that of Theorem 1.1. It relies only on Lemma 3.1 in the section.

Proof of Corollary 1. It is well known that a Leray-Hopf solution is classical at least in a finite time interval. Let T_0 be the time such that $\|u(\cdot,t)\|_\infty$ is finite when $t\in(0,T_0)$ but $\lim_{t\to T_0^-}\|u(\cdot,t)\|_\infty=\infty$.

Let E_0 be the fundamental solution of the Stokes flow. Given $t_0\in(0,T_0)$, by [7] (Theorem 2.1), we have, for $t>t_0$,

$$\begin{aligned}u(x,t)=&\int_{\mathbf{R}^n}E_0(x,t;y,t_0)u(y,t_0)dy\\&+\int_{t_0}^t\int_{\mathbf{R}^n}b(y,s)\nabla_yE_0(x,t;y,s)u(y,s)\,dy\,ds,\end{aligned}\tag{3.22}$$

where $b(y,s) = u(y,s)$. Here we remark that the class of solutions in Theorem 2.1 of [7] contains Leray-Hopf solutions. Hence (3.22) is valid for all Leray-Hopf solutions.

The second integral in the last equality is absolutely convergent when $t \in (t_0, T_0 - \epsilon)$ for $\epsilon > 0$. This is so because $u \in L^\infty(\mathbf{R}^n \times (t_0, T_0 - \epsilon))$ and $b = u \in K_1$. Iterating (3.22), we obtain, as in the proof of Lemma 3.2,

$$|u(x,t)| \le C \int_{\mathbf{R}^n} \sum_{k=0}^{\infty} (cK_1|b|)^{*k} * K_0(x,t;y,t_0)\ |u_0(y)|dy.$$

Here

$$(K_1|b|) * K_0(x,t;y,0) \equiv \int_{t_0}^{t} \int_{\mathbf{R}^n} |b(z,\tau)| K_1(x,t;z,\tau) K_0(z,\tau;y,t_0) dz\, d\tau.$$

By Lemma 3.1,

$$(K_1|b|) * K_0(x,t;y,t_0) \le c_1 B(b,t_0,t) K_0(x,t;y,t_0).$$

Using induction, there holds

$$|u(x,t)| \le C \int_{\mathbf{R}^n} \frac{|u(y,t_0)|}{(|x-y| + \sqrt{t-t_0})^n} dy \ \sum_{j=0}^{\infty} [c_0 B(b,t_0,t)]^j. \tag{3.23}$$

Here c_0 depends only on n. The above series is convergent if

$$c_0 B(b,t_0,t) < 1. \tag{3.24}$$

Since $b = u \in K_1$, there exists a $\delta > 0$ depending only on the rate of convergence of (1.3) such that (3.24) holds when $0 < t - t_0 < \delta$. In this case we have

$$|u(x,t)| \le C_1(\delta) \int_{\mathbf{R}^n} \frac{|u(y,t_0)|}{(|x-y| + \sqrt{t-t_0})^n} dy. \tag{3.25}$$

Next, we choose, for the above δ, $t_0 = T_0 - (\delta/2)$. Then for $t \in (t_0, T)$, (3.25) implies

$$|u(x,t)| \le C_1(\delta) \Big[\int_{\mathbf{R}^n} \frac{dy}{(|x-y| + \sqrt{t-t_0})^{2n}} \Big]^{1/2} \Big[\int_{\mathbf{R}^n} |u(y,t_0)|^2 dy \Big]^{1/2}.$$

Hence

$$|u(x,t)| \le \frac{C_1(\delta)}{(t-t_0)^{n/4}} \|u_0\|_{L^2}.$$

Letting $t \to T_0$, we see that

$$\|u(\cdot, T_0)\|_{L^\infty} \le \frac{C_1(\delta)}{(T_0 - t_0)^{n/4}} \|u_0\|_{L^2} < \infty.$$

This contradiction shows that u is a bounded and hence a classical solution when $t > 0$. □

Proof of Corollary 2. It suffices to prove the corollary when t is sufficiently large. Let u be a Leray-Hopf solution to (1.2) with $n = 3$. It is well known that u becomes a bounded, classical solution when t is sufficiently large. Moreover, the L^2 norm of $u(\cdot, t)$ tends to zero as $t \to \infty$ ([11]). Hence for large t, the norm $\|u(\cdot,t)\|_{L^{n+\epsilon}(\mathbf{R}^n)} + \|u(\cdot,t)\|_{L^{n-\epsilon}(\mathbf{R}^n)}$ is sufficiently small by interpolation between L^2 and L^∞ norms. By Propositions 2.1 and 2.2, this implies that $B(u, t_0, \infty)$ is small when t_0 is large. Hence, by the same arguments from (3.22) to (3.23), we have

$$|u(x,t)| \le C \int_{\mathbf{R}^n} \frac{|u(y,t_0)|}{(|x-y| + \sqrt{t-t_0})^n} dy \sum_{j=0}^{\infty} [c_0 B(b, t_0, t)]^j \tag{3.26}$$

for all $t > t_0$. Since $B(b, t_0, t) \le B(b, t_0, \infty)$ and the latter is sufficiently small, (3.26) shows

$$|u(x,t)| \le C \int_{\mathbf{R}^n} \frac{|u(y,t_0)|}{(|x-y| + \sqrt{t-t_0})^n} dy \tag{3.27}$$

for all $t > t_0$. Applying Young's inequality on (3.27), we obtain

$$\|u(\cdot,t)\|_{L^2} \le c \sup_x \Big[\int_{\mathbf{R}^n} \frac{dy}{(|x-y| + \sqrt{t-t_0})^{2n}} \Big]^{1/2} \int_{\mathbf{R}^n} |u(y,t_0)| dy.$$

Hence, for $t > t_0$,

$$\|u(\cdot,t)\|_{L^2} \le \frac{c}{(t-t_0)^{n/4}} \int_{\mathbf{R}^n} |u(y,t_0)| dy. \tag{3.28}$$

From (3.22), since $div\, u = 0$, one has, from [7],

$$u(x,t) = \int_{\mathbf{R}^n} G(x,t;y,0) u_0(y) dy + \int_0^t \int_{\mathbf{R}^n} u(y,s) \nabla_y E_0(x,t;y,s) u(y,s) dy\, ds,$$

where G is the heat kernel. Therefore,

$$\begin{aligned} \int_{\mathbf{R}^n} |u(x,t)| dx \le & \int_{\mathbf{R}^n} |u_0(y)| dy \\ & + \int_0^t \int_{\mathbf{R}^n} |u(y,s)|^2 \int_{\mathbf{R}^n} \frac{dy}{(|x-y| + \sqrt{t-s})^{(n+1)}} dx\, dy\, ds. \end{aligned}$$

This implies

$$\int_{\mathbf{R}^n} |u(x,t_0)| dx \le \int_{\mathbf{R}^n} |u_0(y)| dy + \int_{\mathbf{R}^n} |u_0(y)|^2 dy \int_0^{t_0} \frac{1}{\sqrt{t_0 - s}} ds.$$

Substituting this into (3.28), we obtain

$$\|u(\cdot,t)\|_{L^2} \le \frac{c}{(t-t_0)^{n/4}}\Big[\int_{\mathbf{R}^n}|u_0(y)|dy + \sqrt{t_0}\int_{\mathbf{R}^n}|u_0(y)|^2 dy\Big]. \quad \square$$

4. Proof of Theorem 1.2. Existence of global solutions

Throughout the section and for a function $u = u(x,t)$, we will use the global norm

$$\|u\|_K \equiv \sup_{x\in\mathbf{R}^n, t>0}\int_0^t\int_{\mathbf{R}^n}[K_1(x,t;y,s)+K_1(x,s;y,0)]|u(y,s)|dy\,ds. \quad (4.1)$$

If it is finite. Here, as before,

$$K_1(x,t;y,s) = \frac{1}{(|x-y|+\sqrt{t-s})^{n+1}}.$$

We will also use the related kernel function

$$K_0(x,t;y,s) = \frac{1}{(|x-y|+\sqrt{t-s})^{n}}.$$

If u is independent of time, then by a simple computation, we see that

$$\|u\|_K = c\sup_{x\in\mathbf{R}^n}\int_{\mathbf{R}^n}\frac{|u(y)|}{|x-y|^{n-1}}dy. \quad (4.2)$$

We also use the following convention in the use of convolutions $*$, $\#$, and $\bullet$.

If f is a function depending both on x and t, then, for $i = 1, 2$,

$$K_i * f(x,t) \equiv \int_0^t\int_{\mathbf{R}^n} K_i(x,t;y,s)f(y,s)\,dy\,ds;$$

$$K_i \# f(x,t) \equiv \int_0^t\int_{\mathbf{R}^n} K_i(x,s;y,0)f(y,s)\,dy\,ds.$$

If f is an initial value depending only on x, then

$$K_i \bullet f(x,t) \equiv \int_{\mathbf{R}^n} K_i(x,t;y,0)f(y)dy.$$

The proof of Theorem 1.2 is divided into three steps. The only prerequisite in the previous section is Lemma 3.1.

Step 1. Four basic inequalities. In this step we prove the following four inequalities.

$$\int_0^t\int_{\mathbf{R}^n}K_1(x,t;y,s)\int_{\mathbf{R}^n}K_0(y,s;z,0)|u_0(z)|dz\,dy\,ds$$
$$\le c\int_{\mathbf{R}^n}\frac{|u_0(z)|}{|x-z|^{n-1}}dz \le c\|u_0\|_K. \quad (4.3)$$

$$\int_0^t \int_{\mathbf{R}^n} K_0(x,t;y,s) \int_{\mathbf{R}^n} K_1(y,s;z,0)|u_0(z)|dz\,dy\,ds$$
$$\leq c\int_{\mathbf{R}^n} \frac{|u_0(z)|}{|x-z|^{n-1}}dz \leq c\|u_0\|_K. \tag{4.4}$$

$$\int_0^t \int_{\mathbf{R}^n} K_1(x,s;y,0) \int_{\mathbf{R}^n} K_0(y,s;z,0)|u_0(z)|dz\,dy\,ds \leq c\|u_0\|_K. \tag{4.3'}$$

$$\int_0^t \int_{\mathbf{R}^n} K_0(x,s;y,0) \int_{\mathbf{R}^n} K_1(y,s;z,0)|u_0(z)|dz\,dy\,ds \leq c\|u_0\|_K. \tag{4.4'}$$

Sometimes, for simplicity, we also write (4.3)–(4.4') as

$$K_1 * K_0 \bullet |u_0|(x,t) \leq c\|u_0\|_K, \qquad K_0 * K_1 \bullet |u_0|(x,t) \leq c\|u_0\|_K$$
$$K_1 \# K_0 \bullet |u_0|(x,t) \leq c\|u_0\|_K, \qquad K_0 \# K_1 \bullet |u_0|(x,t) \leq c\|u_0\|_K$$

respectively.

Here is a proof of (4.3). Denote the left-hand side of (4.3) by $I = I(x,t)$. It is clear that

$$I(x,t) = \int_{\mathbf{R}^n} \int_0^t J\ ds\ |u_0(z)|dz, \tag{4.5}$$

where

$$J \equiv \int_{\mathbf{R}^n} \frac{1}{(|x-y|+\sqrt{t-s})^{n+1}(|y-z|+\sqrt{s})^n}dy.$$

Obviously

$$J \leq \int_{|x-y|\geq|x-z|/2} \cdots dy + \int_{|y-z|\geq|x-z|/2} \cdots dy \equiv J_1 + J_2. \tag{4.6}$$

When $|x-y| \geq |x-z|/2$, we have $|x-y| \geq |y-z|/3$. Hence

$$J_1 \leq \frac{c}{(|x-z|+\sqrt{t-s})^n} \int_{|x-y|\geq|x-z|/2} \frac{1}{(|x-y|+\sqrt{t-s})(|y-z|+\sqrt{s})^n}dy$$
$$\leq \frac{c}{(|x-z|+\sqrt{t-s})^n} \int_{|x-y|\geq|x-z|/2} \frac{1}{(|y-z|+\sqrt{t-s})(|y-z|+\sqrt{s})^n}dy.$$

This implies

$$J_1 \leq \begin{cases} \frac{c}{(|x-z|+\sqrt{t-s})^n} \int_{|x-y|\geq|x-z|/2} \frac{1}{(|y-z|+\sqrt{s})^{n+1}}dy, & s \in (0,t/2) \\ \frac{c}{(|x-z|+\sqrt{t-s})^n} \int_{|x-y|\geq|x-z|/2} \frac{1}{(|y-z|+\sqrt{t-s})^{n+1}}dy, & s \in [t/2,t]. \end{cases}$$

Therefore,

$$J_1 \leq \begin{cases} \frac{c}{(|x-z|+\sqrt{t-s})^n\sqrt{s}}, & s \in (0,t/2) \\ \frac{c}{(|x-z|+\sqrt{t-s})^n\sqrt{t-s}}, & s \in [t/2,t]. \end{cases} \tag{4.7}$$

Integrating (4.7) from 0 to t, we obtain

$$\int_0^t J_1\, ds \le c\int_0^{t/2} \frac{ds}{(|x-z|+\sqrt{s})^n\sqrt{s}} + c\int_{t/2}^t \frac{ds}{(|x-z|+\sqrt{t-s})^n\sqrt{t-s}}.$$

By direct computation

$$\int_0^t J_1\, ds \le \frac{c}{|x-z|^{n-1}}. \tag{4.8}$$

Next we estimate $\int_0^t J_2\, ds$. When $|y-z| \ge |x-z|/2$, from (4.6), we have

$$J_2 \le \frac{c}{(|x-z|+\sqrt{s})^n}\int_{|y-z|\ge|x-z|/2} \frac{1}{(|y-z|+\sqrt{t-s})^{n+1}}dy.$$

Hence

$$J_2 \le \frac{c}{(|x-z|+\sqrt{s})^n\sqrt{t-s}}.$$

By a simple computation, we see that

$$\int_0^t J_2\, ds \le \int_0^t \frac{c}{(|x-z|+\sqrt{s})^n\sqrt{t-s}}ds \le \frac{c}{|x-z|^{n-1}}. \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.5), we deduce

$$I(x,t) \le c\int_{\mathbf{R}^n} \frac{|u_0(z)|}{|x-z|^{n-1}}dz \le c\|u_0\|_K.$$

This is (4.3). The proof for (4.4), (4.3') and (4.4') is similar.

Step 2. Solving an integral equation. The space of divergence-free, vector-valued functions equipped with the norm $\|\cdot\|_K$ (defined in (4.1)) is denoted by $\mathbf{S}$.

Given vector-valued functions $b = b(x,t)$ and $u_0 = u(x)$, let us consider the integral equation in $\mathbf{S}$

$$u(x,t) = \int_{\mathbf{R}^n} E_0(x,t;y,0)u_0(y)dy - \int_0^t\int_{\mathbf{R}^n} E_0(x,t;y,s)b(y,s)\nabla u(y,s)dy\, ds. \tag{4.10}$$

We will prove that (4.10) has a solution in $\mathbf{S}$ provided that $\|b\|_K < \eta$ and $u_0 \in K_1$. Here η is a sufficiently small number depending only on dimension.

It suffices to prove that the following series, obtained by iterating (4.10), is norm convergent in $\mathbf{S}$.

$$\begin{aligned} u(x,t) = &\int_{\mathbf{R}^n} E_0(x,t;y,0)u_0(y)dy \\ &- \int_0^t\int_{\mathbf{R}^n} E_0(x,t;y,s)b(y,s)\int_{\mathbf{R}^n} \nabla_y E_0(y,s;z,0)u_0(z)dz\, dy\, ds + \cdots \end{aligned}$$

By induction after exchanging the order of integration, we obtain

$$u(x,t) = \int E_0(x,t;z,0)u_0(z)dz$$
$$+ \int_{\mathbf{R}^n} \big[\sum_{j=1}^{\infty}(-1)^j E_0 * (b\nabla E_0)^{*j}\big](x,t;z,0)u_0(z)dz. \tag{4.11}$$

Here

$$[E_0 * b\nabla E_0](x,t;z,0) \equiv \int_0^t \int_{\mathbf{R}^n} E_0(x,t;y,s)b(y,s)\nabla_y E_0(y,s;z,0)dy\,ds.$$

By Lemma 3.1, we have

$$K_0 * |b|K_1 \le C_0 B(b,0,\infty)K_0.$$

This shows, by induction, that

$$\big|E_0 * (b\nabla E_0)^{*j}\big](x,t;z,0) \le [C_0 B(b,0,t)]^j K_0(x,t;z,0).$$

Substituting this into (4.11), we deduce

$$|u(x,t)| \le \sum_{j=0}^{\infty}[C_0 B(b,0,t)]^j \int_{\mathbf{R}^n} K_0(x,t;z,0)|u_0(z)|dz. \tag{4.12}$$

Note that $B(b,0,\infty) = \|b\|_K$. Hence, by (4.12),

$$(K_1 * |u|)(x,t) \le c\sum_{j=0}^{\infty}[C_0\|b\|_K]^j \int_0^t \int_{\mathbf{R}^n} K_1(x,t;y,s)$$
$$\times \int_{\mathbf{R}^n} K_0(y,s;z,0)|u_0(z)|dz\,dy\,ds. \tag{4.13}$$

This and (4.3) imply

$$(K_1 * |u|)(x,t) \le c\|u_0\|_K \sum_{j=0}^{\infty}[C_0\|b\|_K]^j.$$

By (4.12) and (4.3'),

$$(K_1\#|u|)(x,t) \le c\sum_{j=0}^{\infty}[C_0\|b\|_K]^j \int_0^t \int_{\mathbf{R}^n} K_1(x,s;y,0)$$
$$\times \int_{\mathbf{R}^n} K_0(y,s;z,0)|u_0(z)|dz\,dy\,ds$$
$$\le cK_1\#K_0 \bullet |u_0|(x,t)\ \sum_{j=0}^{\infty}[C_0\|b\|_K]^j \le c\|u_0\|_K \sum_{j=0}^{\infty}[C_0\|b\|_K]^j.$$

Hence, by (4.1),

$$\|u\|_K \leq c \sup(K_1 * |u| + K_1 \# |u|) \leq c\|u_0\|_K \sum_{j=0}^{\infty} [C_0 \|b\|_K]^j.$$

Therefore, (4.11) is norm convergent in $\mathbf{S}$ when $C_0\|b\|_K < 1$. In this case

$$\|u\|_K \leq \frac{c}{1 - C_0\|b\|_K} \|u_0\|_K.$$

Let $\eta = \frac{1}{2C_0}$ and $S_{K,\eta} \equiv \{u \in \mathbf{S} : \|u\|_K < \eta\}$. Given $b \in S_{K,\eta}$, by the above, u defined by (4.4) also belongs to $S_{K,\eta}$ provided that $\|u_0\|_K < \eta/(2c)$. The mapping T defined by $Tb = u$ maps $S_{K,\eta}$ to itself.

Next we prove that T is contraction if η is sufficiently small. Given $b_1, b_2 \in S_{K,\eta}$, let $Tb_1 = u_1$ and $Tb_2 = u_2$. Then it is easy to see that

$$u_1 - u_2 = \int_0^t \int E_0(b_1 - b_2)\nabla u_1 + \int_0^t \int E_0 b_2 \nabla(u_1 - u_2). \tag{4.14}$$

We denote

$$A = \int_0^t \int E_0(b_1 - b_2)\nabla u_1$$

and let E_1 be the fundamental solution of (1.1) with b replaced by b_1. Then

$$|\nabla u_1|(x,t) \leq \int_{\mathbf{R}^n} |\nabla_x E_1(x,t;z,0)||u_{0,1}(z)|dz,$$

where $u_{0,1}(z) = u_1(z,0)$. Using the gradient estimate on E_1 (Lemma 3.2), we have, since $\|b_1\|_k$ is sufficiently small,

$$|\nabla u_1|(x,t) \leq c \int_{\mathbf{R}^n} K_1(x,t;z,0)|u_{0,1}(z)|dz.$$

Hence,

$$|A(x,t)| \leq c \int_{\mathbf{R}^n} \int_0^t \int_{\mathbf{R}^n} K_0(x,t;y,s)|b_1 - b_2|(y,s) K_1(y,s;z,0)dy\, ds |u_{0,1}(z)|dz.$$

By Lemma 3.1,

$$|A(x,t)| \leq cB(b_1 - b_2, 0, t) \int_{\mathbf{R}^n} K_0(x,t;z,0)|u_{0,1}(z)|dz. \tag{4.15}$$

Similarly,

$$|\nabla_x A(x,t)| \leq cB(b_1 - b_2, 0, t) \int_{\mathbf{R}^n} K_1(x,t;z,0)|u_{0,1}(z)|dz. \tag{4.16}$$

Substituting (4.15) and (4.16) into (4.14) and using induction, we find

$$|u_1-u_2|(x,t) \le cB(b_1-b_2,0,t)\sum_{j=0}^{\infty}[c_0B(b_2,0,t)]^j \int_{\mathbf{R}^n} K_0(x,t;z,0)|u_{0,1}(z)|dz.$$

It follows that

$$(K * |u_1-u_2|)(x,t) \le c\|b_1-b_2\|_K \sum_{j=0}^{\infty}[c_0\|b_2\|_K]^j (K_1 * K_0 \bullet u_{0,1})(x,t),$$

$$(K\#|u_1-u_2|)(x,t) \le c\|b_1-b_2\|_K \sum_{j=0}^{\infty}[c_0\|b_2\|_K]^j (K_1\#K_0 \bullet u_{0,1})(x,t).$$

Using (4.3) and (4.3'), we have

$$\|u_1-u_2\|_K \le c\|b_1-b_2\|_K\|u_{0,1}\|_K \le c\|b_1-b_2\|_k\eta.$$

This implies that T is a contraction when η is sufficiently small.

Therefore T has a fixed point $S_{K,\eta}$, which satisfies

$$u(x,t) = \int_{\mathbf{R}^n} E_0(x,t;y,0)u_0(y)dy - \int_0^t \int_{\mathbf{R}^n} E_0(x,t;y,s)u(y,s)\nabla_y u(y,s)dy. \tag{4.17}$$

Step 3. Proving the solution of the integral equation is a solution of the Navier-Stokes equations. If we knew that the function $u(\cdot,t)\nabla u(\cdot,t)$ is in $L^p(\mathbf{R}^n)$ for some $p>1$, then by the argument of [7], reproduced in the proof of Theorem 1.1 (i) in Section 3, we would know that a solution to (4.17) is a solution to the Navier-Stokes equations. The L^p bound is needed since one needs to apply the Riesz transform on $u\nabla u$. However, we do not have this information at the moment. In fact we do not even know whether $u\nabla u$ is integrable for the moment. To overcome this difficulty, we carry out an approximation process.

In order to proceed, let us summarize the main properties obtained in the last step for the solution of (4.17).

(a) When $u_0 \in S_{K,\eta}$ and η is sufficiently small, the equation (4.17) has a solution in $S_{K,\eta}$.

(b) There exists $c = c(\eta) > 0$ such that

$$|u(x,t)| \le c\int_{\mathbf{R}^n} K_0(x,t;y,0)|u_0(y)|dy, \tag{4.18}$$

$$|\nabla u(x,t)| \le c\int_{\mathbf{R}^n} K_1(x,t;y,0)|u_0(y)|dy. \tag{4.19}$$

The first estimate is just (4.12), and the second can be obtained similarly by iterating the derivative of (4.10). Here we emphasize that all iterations are valid under the assumption of the smallness of the $\|\cdot\|_K$ norm of all relevant functions. The rest of the proof is divided into three parts.

Part 1. Stability. Let u_i $(\in S_{K,\eta})$, $i=1,2$, be the solutions to (4.17) with the initial values of $u_{0,i}$. We want to prove that the $\|u_1-u_2\|_K$ is dominated by $\|u_{0,1}-u_{0,2}\|_K$.

From the construction of u_i by (4.17) (replacing u_0 by $u_{0,i}$), we see that

$$\begin{aligned} &u_1(x,t)-u_2(x,t) \\ &\qquad \equiv J(x,t)+\int_0^t\int_{\mathbf{R}^n} E_0(x,t;y,s)u_2\nabla_y(u_1-u_2)(y,s)\,dy\,ds, \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} J(x,t)\equiv &\int_{\mathbf{R}^n} E_0(x,t;y,0)(u_{0,1}(y)-u_{0,2}(y))dy \\ &+\int_0^t\int_{\mathbf{R}^n} E_0(x,t;y,s)(u_1-u_2)\nabla_y u_1(y,s)dy\,ds. \end{aligned}$$

Applying (4.19), which obviously holds for u_1, we have

$$\begin{aligned} |J(x,t)|\le c&\int_{\mathbf{R}^n} K_0(x,t;y,0)|u_{0,1}(y)-u_{0,2}(y)|dy \\ &+c\int_0^t\int_{\mathbf{R}^n} K_0(x,t;y,s)|u_1-u_2|\int_{\mathbf{R}^n} K_1(y,s;z,0)u_{0,1}(y,s)dy\,ds. \end{aligned}$$

Using Lemma 3.1, we deduce

$$\begin{aligned} |J(x,t)|\le c&\int_{\mathbf{R}^n} K_0(x,t;y,0)|u_{0,1}(y)-u_{0,2}(y)|dy \\ &+cB(u_1-u_2,0,\infty)\int_{\mathbf{R}^n} K_0(x,t;y,0)|u_{0,1}(y)|dy. \end{aligned}$$

For simplicity we write the above as

$$|J(x,t)|\le cK_0\bullet|u_{0,1}-u_{0,2}|(x,t)+c\|u_1-u_2\|_K\ K_0\bullet|u_{0,1}|(x,t). \tag{4.21}$$

Similarly,

$$|\nabla J(x,t)|\le cK_1\bullet|u_{0,1}-u_{0,2}|(x,t)+c\|u_1-u_2\|_K\ K_1\bullet|u_{0,1}|(x,t). \tag{4.22}$$

Substituting (4.21) and (4.22) into (4.20) and iterating, we obtain

$$|u_1-u_2|(x,t)\le c\sum_{j=0}^{\infty} K_0*(|u_2|K_1)^{*j}\bullet|u_{0,1}-u_{0,2}|(x,t)$$

$$+ c\|u_1 - u_2\|_K \sum_{j=0}^{\infty} K_0 * (|u_2|K_1)^{*j} \bullet |u_{0,1}|(x,t)$$

$$\le c\sum_{j=1}^{\infty}[c_0 B(u_2,0,\infty)]^j \big(K_0 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K K_0 \bullet |u_{0,1}|(x,t)\big).$$

Here we just used Lemma 3.1 again.

Similarly, taking the gradient on both sides of (4.20) and integrating, there holds

$$|\nabla(u_1 - u_2)|(x,t) \le$$
$$c\sum_{j=1}^{\infty}[c_0 B(u_2,0,\infty)]^j \big(K_1 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_1 \bullet |u_{0,1}|(x,t)\big).$$

Since u_1 and u_2 are sufficiently small in the K norm, the above imply

$$|u_1 - u_2|(x,t) \le c\big(K_0 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_0 \bullet |u_{0,1}|(x,t)\big), \quad (4.23)$$

$$|\nabla(u_1 - u_2)|(x,t) \le c\big(K_1 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_1 \bullet |u_{0,1}|(x,t)\big). \quad (4.24)$$

These show that

$$K_1 * |u_1 - u_2|(x,t) \le$$
$$\big(K_1 * K_0 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_1 * K_0 \bullet |u_{0,1}|(x,t)\big), \quad (4.25)$$
$$|K_0 * \nabla(u_1 - u_2)|(x,t) \le$$
$$c\big(K_0 * K_1 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_0 * K_1 \bullet |u_{0,1}|(x,t)\big), \quad (4.26)$$
$$K_1 \# |u_1 - u_2|(x,t) \le$$
$$\big(K_1 \# K_0 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_1 \# K_0 \bullet |u_{0,1}|(x,t)\big), \quad (4.25')$$
$$|K_0 \# \nabla(u_1 - u_2)|(x,t) \le$$
$$c\big(K_0 \# K_1 \bullet |u_{0,1} - u_{0,2}|(x,t) + c\|u_1 - u_2\|_K \ K_0 \# K_1 \bullet |u_{0,1}|(x,t)\big). \quad (4.26')$$

Applying (4.3) and (4.4) to (4.25) and (4.26) respectively, we reach

$$K_1 * |u_1 - u_2|(x,t) \le c(\|u_{0,1} - u_{0,2}\|_K + c\|u_1 - u_2\|_K \|u_{0,1}\|_K), \quad (4.27)$$

$$|K_0 * \nabla(u_1 - u_2)|(x,t) \le c(\|u_{0,1} - u_{0,2}\|_K + c\|u_1 - u_2\|_K \|u_{0,1}\|_K). \quad (4.28)$$

Similarly, applying (4.3') and (4.4') to (4.25') and (4.26') respectively, we get

$$K_1 \# |u_1 - u_2|(x,t) \le c(\|u_{0,1} - u_{0,2}\|_K + c\|u_1 - u_2\|_K \|u_{0,1}\|_K), \quad (4.27')$$

$$|K_0 \# \nabla(u_1 - u_2)|(x,t) \le c(\|u_{0,1} - u_{0,2}\|_K + c\|u_1 - u_2\|_K \|u_{0,1}\|_K). \quad (4.28')$$

Since $\|u_{0,1}\|_K$ is small, (4.27) and (4.27') imply

$$\|u_1 - u_2\|_K = \sup(K_1 * |u_1 - u_2| + K_1\#|u_1 - u_2|)(x,t) \le c\|u_{0,1} - u_{0,2}\|_K. \tag{4.29}$$

Substituting (4.29) into (4.28), we see that

$$|K_0 * \nabla(u_1 - u_2)|(x,t) \le c\|u_{0,1} - u_{0,2}\|_K. \tag{4.30}$$

Part 2. Approximation. Given $u_0 \in S_{K,\eta}$, let $\{u_{0,k}\} \subset S_{K,\eta} \cap C_0^\infty(\mathbf{R}^n)$ be a sequence such that $|u_{0,k}(x)| \le |u_0(x)|$ and

$$\lim_{k\to\infty} \|u_{0,k} - u_0\|_K = 0. \tag{4.31}$$

Let u be a solution to the integral equation (4.17) and u_k be a solution to

$$\begin{aligned} u_k(x,t) = &\int_{\mathbf{R}^n} E_0(x,t;y,0)u_{0,k}(y)dy \\ &- \int_0^t \int_{\mathbf{R}^n} E_0(x,t;y,s)u_k(y,s)\nabla_y u_k(y,s)dy. \end{aligned} \tag{4.32}$$

According to (4.29) and (4.30) (replacing u_1 by u_k and u_2 by u),

$$\|u_k - u\|_K \le c\|u_{0,k} - u_0\|_K \tag{4.33}$$

$$|K_0 * \nabla(u_k - u)|(x,t) \le c\|u_{0,k} - u_0\|_K \tag{4.34}$$

$$|K_0\#\nabla(u_k - u)|(x,t) \le c\|u_{0,k} - u_0\|_K. \tag{4.34$'$}$$

Since $u_{0,k}$ is smooth and compactly supported, by (4.18) and (4.19), $u_k\nabla u_k \in L^p(\mathbf{R}^n \times [0,l))$ for some $p > 1$. Here $l > 0$. Hence by the argument in the proof of Theorem 1.1 (i), which is borrowed from [7], we know that u_k is a solution to the Navier-Stokes equations with initial value $u_{0,k}$, i.e., for any vector-valued $\phi \in C_0^\infty(\mathbf{R}^n \times (-\infty,\infty))$ with $div\, \phi = 0$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{R}^n} \langle u_k, \partial_t\phi + \Delta\phi\rangle dx\, dt - \int_0^\infty \int_{\mathbf{R}^n} \langle u_k\nabla u_k, \phi\rangle dx\, dt \\ &= - \int_{\mathbf{R}^n} \langle u_{0,k}(x), \phi(x,0)\rangle dx. \end{aligned} \tag{4.35}$$

Part 3. Taking the limit. We are going to show that each term in (4.35) converges as $k \to \infty$.

Since $K_1(x,t;y,s)$ is bounded away from 0 when t and x,y are finite, by (4.33), we see that

$$\|u_k - u\|_{L^1(\Omega)} \le C_\Omega\|u - u_k\|_K \to 0, \tag{4.36}$$

when $k \to \infty$. Here Ω is any bounded domain of $\mathbf{R}^n \times [0,\infty)$.

Next, notice that, by (4.19),

$$\begin{aligned}
&(K_1 * |u_k - u||\nabla u_k|)(x,t) = \int_0^t \int_{\mathbf{R}^n} K_1(x,t;y,s)|u_k - u||\nabla u_k|(y,s)dy\,ds \\
&\le c \int_0^t \int_{\mathbf{R}^n} K_1(x,t;y,s)|u_k - u|(y,s) \int_{\mathbf{R}^n} K_1(y,s;z,0)|u_{0,k}(z)|dz\,dy\,ds \\
&\le c \int_{\mathbf{R}^n} \int_0^t \int_{\mathbf{R}^n} K_1(x,t;y,s)|u_k - u|(y,s)K_1(y,s;z,0)\,dy\,ds|u_{0,k}(z)|dz \\
&\le c\|u_k - u\|_K \int_{\mathbf{R}^n} K_1(x,t;z,0)|u_0(z)|dz. \qquad (4.37)
\end{aligned}$$

Here we just used Lemma 3.1 and the fact that $|u_{0,k}(x)| \le |u_0(x)|$.

By (4.4), we know that for almost every (x,t),

$$\int_{\mathbf{R}^n} K_1(x,t;z,0)|u_0(z)|dz < \infty.$$

Let Ω be any bounded domain of $\mathbf{R}^n \times [0,\infty)$; we fix $(x,t) \in \Omega^c$ so that the above integral is finite. Since $K_1(x,t;y,s)$ is strictly positive when $(y,s) \in \Omega$, we have, by (4.37) and (4.33),

$$\lim_{k\to\infty} \int_\Omega |u_k - u||\nabla u_k|(y,s)\,dy\,ds \le C_\Omega \lim_{k\to\infty} (K_1 * |u_k - u||\nabla u_k|)(x,t) = 0. \qquad (4.38)$$

Similarly, by (4.18),

$$K_0 * |u||\nabla(u - u_k)| \le cK_0 * (|\nabla(u - u_k)|K_0 \bullet |u_0|).$$

Using (4.34), (4.34'), and Remark 3.1 after Lemma 3.1, one has

$$\begin{aligned}
&K_0 * |u||\nabla(u - u_k)|(x,t) \\
&\le c \sup(K_0 * |\nabla(u_k - u)| + K_0\#|\nabla(u_k - u)|) \int_{\mathbf{R}^n} K_0(x,t;z,0)|u_0(z)|dz. \\
&\le C\|u_k - u\|_K \int_{\mathbf{R}^n} K_0(x,t;z,0)|u_0(z)|dz.
\end{aligned}$$

By (4.4), the last integral in the above is finite for almost all (x,t). Therefore, as before

$$\lim_{k\to\infty} \int_\Omega |u||\nabla(u - u_k)|(y,s)\,dy\,ds = 0. \qquad (4.39)$$

Note that

$$
\begin{aligned}
&\int_{\Omega} |\langle u_k \nabla u_k, \phi\rangle - \langle u \nabla u, \phi\rangle| \; dx\, dt \\
&\le \int_0^\infty \int_\Omega |\langle (u_k - u)\nabla u_k, \phi\rangle| dx\, dt + \int_\Omega | < u \nabla (u_k - u), \phi\rangle | dx\, dt.
\end{aligned}
$$

Choosing an Ω that contains the support of ϕ we have, by (4.38), (4.39), and (4.35),

$$
\begin{aligned}
&\int_0^\infty \int_{\mathbf{R}^n} \langle u, \partial_t \phi + \Delta \phi\rangle dx\, dt - \int_0^\infty \int_{\mathbf{R}^n} \langle u \nabla u, \phi\rangle dx\, dt \\
&= - \int_{\mathbf{R}^n} \langle u_0(x), \phi(x,0)\rangle dx.
\end{aligned}
$$

This shows that u is a solution to the Navier-Stokes equations.

Finally, let us prove the last statement in Theorem 1.2. This is easy, now that we also assume $u_0 \in L^2(\mathbf{R}^n)$. By (4.18) and Hölder's inequality

$$
|u(x,t)|^2 \le c \int_{\mathbf{R}^n} \frac{1}{(|x-y| + \sqrt{t})^{2n}} dy \; \|u_0\|^2_{L^2} = \frac{c}{t^{n/2}} \|u_0\|^2_{L^2}.
$$

Therefore $u(x,t)$ is finite for all x and $t > 0$. Hence u is classical when $t > 0$. Note that no smallness of $\|u_0\|^2_{L^2}$ is required here. □

5. Improved sufficient condition for regularity

In this section we prove that a certain form-boundedness condition on the velocity is sufficient to imply regularity. Throughout the years, various conditions on u that imply regularity have been proposed. One of them is the Prodi-Serrin condition, which requires that $u \in L^{p,q}$ with $\frac{3}{p} + \frac{2}{q} \le 1$ for some $3 < p \le \infty$ and $q \ge 2$. See ([25, 27] e.g.) Recently the authors in [6] showed that the condition $p = 3$ and $q = \infty$ also implies regularity. In another development the author of [19] improved the Prodi-Serrin condition by a log factor, i.e., by requiring

$$
\int_0^T \frac{\|u(\cdot,t)\|^q_p}{1 + \log^+ \|u(\cdot,t)\|_p} dt < \infty,
$$

where $3/p + 2/q = 1$ and $3 < p < \infty$, $2 < q < \infty$.

The form-boundedness condition, with its root in the perturbation theory of elliptic operators and mathematical physics, seems to be different from all the previous conditions. It seems to be one of the most general conditions under the available tools. This fact has been well documented in the theory of linear elliptic equations. See [26] e.g. Here, first of all it allows singularity of the form $c(t)/|x|$, which does not belong to any of the previous regularity classes. Here $c(t)$ is bounded. It also has the advantage of just requiring L^2 integrability of the velocity. Moreover, it contains the Prodi-Serrin condition except when p or q are infinite. It also contains suitable Morrey-Compamato-type spaces. However, we are not sure this condition contains the one in [19].

More precisely we have

Theorem 5.1. *Let u be a Leray-Hopf solution to the three-dimensional Navier-Stokes equation (1.2) in $\mathbf{R}^3 \times (0,\infty)$. Suppose for every $(x_0,t_0) \in \mathbf{R}^3 \times (0,\infty)$, there exists a cube $Q_r = B(x_0,r) \times [t_0 - r^2, t_0]$ such that u satisfies the form-bounded condition*

$$\int_{Q_r} |u|^2 \phi^2 \, dy \, ds \tag{5.0}$$
$$\leq \frac{1}{24}\Big(\int_{Q_r} |\nabla \phi|^2 \, dy \, ds + \sup_{s\in[t_0-r^2,t_0]} \int_{B(x_0,r)} \phi^2(y,s) dy \Big) + B(\|\phi\|_{L^2(Q_r)}).$$

Here ϕ is any smooth function vanishing on the parabolic side of Q_r and $B = B(t)$ is any given function which is bounded when t is bounded. Then u is a classical solution when $t > 0$.

The next corollary shows that the form-boundedness condition contains the famous Prodi-Serrin condition in the whole space, except for $p = \infty$ and $q = \infty$.

Corollary 3. *Suppose $u \in L^{p,q}(Q_r)$ with $3/p + 2/q = 1$ and neither p nor q is infinity. Then u satisfies (5.0) in $Q_{r'}$ where $r'(< r)$ is sufficiently small.*

Proof. Let ϕ be as in Theorem 5.1. Then, by Hölder's inequality

$$\int_{Q_{r'}} |u|^2 \phi^2 \, dy \, ds \leq \Big(\Big(\int_{Q_{r'}} |\phi|^{2a'} dy \Big)^{b'/a'} ds \Big)^{1/b'} \Big(\Big(\int_{Q_{r'}} |u|^{2a} dy \Big)^{b/a} ds \Big)^{1/b}.$$

Here $2a = p$, $2b = q$, and a' and b' are the conjugates of a and b respectively. Hence

$$\int_{Q_{r'}} |u|^2 \phi^2 \, dy \, ds \leq \|u\|^2_{L^{p,q}(Q_{r'})} \|\phi\|^2_{L^{2a',2b'}(Q_{r'})}.$$

Note that

$$\frac{3}{2a'} + \frac{2}{2b'} = \frac{3}{2}(1 - \frac{1}{a}) + 1 - \frac{1}{b} = \frac{5}{2} - (\frac{3}{2a} + \frac{2}{2b}) = \frac{5}{2} - (\frac{3}{p} + \frac{2}{q}) = \frac{3}{2}.$$

By [16, p. 152, Example 6.2], we have

$$\|\phi\|^2_{L^{2a',2b'}(Q_{r'})} \leq C\Big(\int_{Q_{r'}} |\nabla \phi|^2 \, dy \, ds + \sup_{s\in[t_0-(r')^2,t_0]} \int_{B(x_0,r')} \phi^2(y,s) dy \Big).$$

Choosing r' sufficiently small, we see that u satisfies (5.0). $\square$

Proof of Theorem 5.1. Let t_0 be the first moment of singularity formation. We will reach a contradiction. It is clear that we need only to prove that u is bounded in $Q_{r/8} = Q_{r/8}(x_0, t_0)$ for some $r > 0$. In fact the number 8 is not essential. Any number greater than 1 would work.

We will follow the idea in [27]. Consider the the equation for vorticity $w = \nabla \times u$. It is well known that, in the interior of Q_r, w is a classical solution to the parabolic system with singular coefficients

$$\Delta w - u\nabla w + w\nabla u - w_t = 0. \tag{5.1}$$

Let $\psi = \psi(y, s)$ be a standard cut-off function such that $\psi = 1$ in $Q_{r/2}$, $\psi = 0$ in Q^c_r, and such that $0 \le \psi \le 1$, $|\nabla\psi| \le C/r$, and $|\psi_t| \le C/r^2$. We can use $w\psi^2$ as a test function on (5.1) to obtain

$$\begin{aligned} &\int_{Q_r} |\nabla(w\psi)|^2 \, dy \, ds + \frac{1}{2}\int_{B(x_0,r)} |w\psi|^2(y, t_0) dy \\ &\qquad \le \frac{C}{r^2}\int_{Q_r} |w|^2 \, dy \, ds + |\int_{Q_r} u\nabla w \cdot w\psi^2 \, dy \, ds| + |\int_{Q_r} w\nabla u \cdot w\psi^2 \, dy \, ds| \\ &\qquad \equiv I_1 + I_2 + I_3. \end{aligned} \tag{5.2}$$

The term I_1 is already in good shape. Next, using integration by parts and the divergence-free condition on u, we have

$$I_2 = |\frac{1}{2}\int_{Q_r} u \cdot \nabla\psi\psi|w|^2 \, dy \, ds|.$$

Hence

$$I_2 \le \epsilon \int_{Q_r} |u|^2|w\psi|^2 \, dy \, ds + C_\epsilon \int_{Q_r} |\nabla\psi|^2|w|^2 \, dy \, ds. \tag{5.3}$$

Here $\epsilon > 0$ is arbitrary. Next

$$\begin{aligned} I_3 &= |\int_{Q_r} \Sigma_i w_i\psi\partial_i u \cdot w\psi \, dy \, ds| \\ &= |\int_{Q_r} \Sigma_i \partial_i(w_i\psi)u \cdot w\psi \, dy \, ds| + |\int_{Q_r} \Sigma_i w_i\psi u \cdot \partial_i(w\psi) \, dy \, ds| \\ &\le \frac{1}{2}\int_{Q_r} |\nabla(w\psi)|^2 \, dy \, ds + \frac{3}{2}\int_{Q_r} |u|^2|w\psi|^2 \, dy \, ds \\ &\quad + \frac{1}{4}\int_{Q_r} |\nabla(w\psi)|^2 \, dy \, ds + \int_{Q_r} |u|^2|w\psi|^2 \, dy \, ds. \end{aligned}$$

Substituting this and (5.3) into (5.2) and simplifying we obtain

$$\frac{1}{4}\int_{Q_r}|\nabla(w\psi)|^2\,dy\,ds+\frac{1}{2}\int_{B(x_0,r)}|w\psi|^2(y,t_0)dy \\ \le \frac{C+C_\epsilon}{r^2}\int_{Q_r}|w|^2\,dy\,ds+\frac{5+\epsilon}{2}\int_{Q_r}|u|^2|(w\psi)|^2\,dy\,ds.$$

After repeating the above in $Q_r\cap\{(y,s):s<t\}$ for all $t\in(t_0-r^2,t_0)$, one has

$$\frac{1}{4}\int_{Q_r}|\nabla(w\psi)|^2\,dy\,ds+\frac{1}{2}\sup_{t_0-r^2\le s\le t_0}\int_{B(x_0,r)}|w\psi|^2(y,s)dy \\ \le \frac{C}{r^2}\int_{Q_r}|w|^2\,dy\,ds+(5+\epsilon)\int_{Q_r}|u|^2|(w\psi)|^2\,dy\,ds.$$

By the form-boundedness assumption on u, we have

$$\int_{Q_r}|\nabla(w\psi)|^2\,dy\,ds+\sup_{t_0-r^2\le s\le t_0}\int_{B(x_0,r)}|w\psi|^2(y,s)dy \\ \le \frac{C_\epsilon}{r^2}\int_{Q_r}|w|^2\,dy\,ds+4(5+\epsilon)\frac{1}{24}\Big(\int_{Q_r}|\nabla(w\psi)|^2\,dy\,ds \\ +\sup_{t_0-r^2\le s\le t_0}\int_{B(x_0,r)}|w\psi|^2(y,s)dy\Big)+CB(\|w\|_{L^2(Q_r)}). \tag{5.4}$$

Hence, we can choose ϵ so small that

$$\int_{Q_r}|\nabla(w\psi)|^2\,dy\,ds+\sup_{t_0-r^2\le s\le t_0}\int_{B(x_0,r)}|w\psi|^2(y,s)dy \\ \le \frac{C_\epsilon}{r^2}\|w\|_{L^2(Q_r)}+CB(\|w\|_{L^2(Q_r)}). \tag{5.5}$$

Using standard results, we know that (5.5) implies that u is regular. Here is the proof.

From (5.5), it is clear that $\int_{Q_r}|curl\ (w\psi)|^2\,dy\,ds\le C$. Hence, since $\psi=1$ in $Q_{r/2}$,

$$\int_{Q_{r/2}}|\Delta u|^2\,dy\,ds\le C. \tag{5.6}$$

Let $\eta=\eta(y)$ be a cut-off function such that $\eta=1$ in $B(x_0,r/4)$ and $\eta=0$ in $B(x_0,r/2)^c$. Then for each $s\in[t_0-(r/4)^2,t_0]$, we have, in the weak sense,

$$\Delta(u\eta)=\eta\Delta u+2\nabla u\nabla\eta+u\Delta\eta\equiv f$$

in $Q_{r/2}$. By standard elliptic estimates, using the fact that $u\eta = 0$ on the boundary,

$$\|D^2 u(\cdot, s)\|_{L^2(B(x_0,r/4))} \le C\|f(\cdot, s)\|_{L^2(B(x_0,r/2))}.$$

This shows that

$$\|D^2 u\|_{L^2(Q_{r/4})} \le C\|\Delta u\|_{L^2(Q_{r/2})} + C\|u\|_{L^2(Q_{r/2})}.$$

By Sobolev imbedding,

$$\nabla u \in L^{6,2}(Q_{r/4}). \tag{5.7}$$

Next, from (1.22) on p. 316 of [29],

$$\|u(\cdot, s)\eta\|_{W^{1,2}} \le C\big(\|u(\cdot, s)\eta\|_{L^2} + \|div(u\eta)(\cdot, s)\|_{L^2} + \|curl(u(\cdot, s)\eta)\|_{L^2}\big).$$

Here all norms are over the ball $B(x_0, r/2)$. Therefore,

$$\|u\eta(\cdot, s)\|_{W^{1,2}} \le$$
$$C\Big(\|u(\cdot, s)\eta\|_{L^2} + \|u\nabla\eta(\cdot, s)\|_{L^2} + \|w\eta(\cdot, s)\|_{L^2} + \||u(\cdot, s)|\ |\nabla\eta|\|_{L^2}\Big).$$

It follows that

$$\|u(\cdot, s)\|_{W^{1,2}(B(x_0,r/4))} \le C.$$

From Sobolev imbedding we know that

$$u \in L^{6,\infty}(Q_{r/4}). \tag{5.8}$$

We treat u and ∇u as coefficients in equation (5.1). By (5.6) and (5.7), standard parabolic theory (see [16] e.g.) shows that w is bounded and Hölder continuous in $Q_{r/8}$. Here the bound depends only on the L^2 norm of w in Q_r and r. This is so because of the relation $3/6 + 2/\infty < 1$ for the norm of u and $3/6 + 2/2 < 2$ for the norm of ∇u. Now a standard bootstrapping argument shows that u is smooth.

Note that one can also use the Prodi-Serrin condition, which is implied by (5.8), to conclude that u is bounded and hence regular. $\square$

Remark. Currently we are not able to prove a local version of Theorem 5.1 due to a difficulty in obtaining an approximation argument for weak solutions under the form-boundedness condition. Under the Prodi-Serrin condition, this is done in [27].

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