Advances in Differential Equations Volume 9, Numbers 5-6, May/June 2004, Pages 563–586

ON AN EVOLUTION SYSTEM DESCRIBING SELF-GRAVITATING FERMI–DIRAC PARTICLES

Piotr Biler
Instytut Matematyczny, Uniwersytet Wrocławski
pl. Grunwaldzki 2/4, 50–384 Wrocław, Poland

Philippe Laurençot
Mathématiques pour l'Industrie et Physique, CNRS UMR 5640
Université Paul Sabatier – Toulouse 3, 118 route de Narbonne
F–31062 Toulouse cedex 4, France

Tadeusz Nadzieja
Instytut Matematyki, Uniwersytet Zielonogórski
ul. Szafrana 4a, 65–516 Zielona Góra, Poland

(Submitted by: Herbert Amann)

Abstract. The global-in-time existence of solutions for a system describing the interaction of gravitationally attracting particles that obey the Fermi–Dirac statistics is proved. Stationary solutions of that system are also studied.

1. Introduction and derivation of the system

Our aim in this paper is to study a nonlinear, nonlocal, parabolic system with nonlinear diffusion describing the evolution of a cloud of self-gravitating particles that obey the Fermi–Dirac statistics. This model has been introduced in [14] on the basis of considerations of kinetic equations, and has been studied in [13, 10].

Unlike the models of interacting particles where the particles are subject to linear Brownian diffusion (see, e.g., [7, 4, 5, 6, 23, 24]), the assumption that the density $0 \le f = f(x, v, t)$ of particles at the point $(x, t) \in \Omega \times \mathbb{R}^+$, $\Omega \subset \mathbb{R}^d$, moving at the velocity $v \in \mathbb{R}^d$ is bounded by, say $\eta_0 > 0$, leads to mathematically completely different models. They involve nonlinear diffusion resembling that for fast-diffusing gases (at large densities).

The plan of this paper is following: after discussing the derivation of the system of partial differential equations in Section 1 and properties of

Accepted for publication: January 2004.
AMS Subject Classifications: 35Q, 35K60, 35B40, 82C21.

auxiliary functions in Section 2, we will study the evolution problem in Section 3, and describe steady states in Section 4. The main outcome of our study is that for the systems of Fermi–Dirac self-gravitating particles, the gravitational collapse does not occur in dimensions $d \le 3$, which contrasts markedly with the linear Brownian diffusion case [7, 9].

The main steps in the derivation of the model (1.7)–(1.8) below are as follows:

– Suppose that the local entropy of the system is given by

$$S = \frac{1}{\int_{\mathbb{R}^d} f\,dv} \int_{\mathbb{R}^d} \left(\frac{f}{\eta_0} \log \frac{f}{\eta_0} + \left(1 - \frac{f}{\eta_0}\right) \log \left(1 - \frac{f}{\eta_0}\right)\right) dv$$

with some fixed $\eta_0 > 0$.

– The evolution of the density f, described generally by a kinetic equation $f_t + v \cdot \nabla_x f - \nabla\phi \cdot \nabla_v f = -\nabla_v J$, subject to mass density and the energy density constraints, is governed by the maximum entropy production principle (MEPP), discussed in detail in [14]. This principle determines the dissipation flux $-\nabla_v J$, and then the equation (1.7) up to the diffusion coefficient after the following procedure:

– averaging f over the velocities v, and the passage to the limit of large friction (or large times), thus obtaining "hydrodynamic" equations in the (x,t) space.

Such a procedure leads to the following form of the distribution function,

$$f(x,v,t) = \eta_0 \frac{1}{1 + \lambda e^{\beta |v|^2/2}},$$

which is called the Fermi–Dirac distribution, with the fugacity $\lambda = \lambda(x,t)$ and the inverse temperature $\beta = 1/\vartheta$. Note that $0 \le f(x,v,t) \le \eta_0$, and for large $|v|$, f resembles a Maxwell–Boltzmann distribution. Thus, the spatiotemporal density is

$$n(x,t) = \int_{\mathbb{R}^d} f(x,v,t)\,dv,$$

and the pressure is

$$p(x,t) = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 f(x,v,t)\,dv.$$

In other words, we have $(r = |v|)$

$$\begin{aligned} n(x,t) &= \int_0^\infty \frac{\eta_0}{1 + \lambda e^{\beta r^2/2}} \sigma_d r^{d-1}\,dr \qquad (1.1) \\ &= \eta_0 2^{d/2-1} \sigma_d \beta^{-d/2} \int_0^\infty \frac{y^{d/2-1}\,dy}{1 + \lambda e^{y}} = \eta_0 2^{d/2-1} \sigma_d \vartheta^{d/2} I_{d/2-1}(\lambda), \end{aligned}$$

where σ_d is the area of the unit sphere in $\mathbb{R}^d$, and I_α denotes the Fermi integral of order $\alpha > -1$,

$$I_\alpha(\lambda) = \int_0^\infty \frac{y^\alpha \, dy}{1 + \lambda e^y}, \tag{1.2}$$

defined for all $\lambda > 0$. Similarly we get

$$\begin{aligned} p(x,t) &= \frac{1}{d}\int_0^\infty \eta_0 \frac{r^2}{1 + \lambda e^{\beta r^2/2}} \sigma_d r^{d-1} \, dr \\ &= \eta_0 2^{d/2} \omega_d \beta^{-d/2-1} \int_0^\infty \frac{y^{d/2} \, dy}{1 + \lambda e^y} = \eta_0 2^{d/2} \omega_d \vartheta^{d/2+1} I_{d/2}(\lambda), \end{aligned} \tag{1.3}$$

where $\omega_d = \sigma_d / d$ is the volume of the unit ball in $\mathbb{R}^d$.

Remark. The bound $0 \le f \le \eta_0$ at the starting point of the derivation of equations may lead to arbitrarily large values of the density n in the position space. This bound has nothing to do with the Pauli exclusion principle in the (x,t) space, which would bound *a priori* the density n, as is, e.g., in [23].

The total mass of the system is $M = \int_\Omega n(x,t) \, dx$. The no-flux condition (1.9) below will guarantee the conservation of mass during the evolution.

The particles generate the gravitational potential φ that satisfies the Poisson equation

$$\Delta\varphi = \sigma_d G n,$$

where G is the gravitation constant.

The total energy of the system is thus

$$E = \frac{1}{2}\int_\Omega \int_{\mathbb{R}^d} f(x,v,t)|v|^2 \, dv \, dx + \frac{1}{2}\int_\Omega n(x,t)\varphi(x,t) \, dx,$$

or

$$E = \frac{d}{2}\int_\Omega p(x,t) \, dx + \frac{1}{2}\int_\Omega n(x,t)\varphi(x,t) \, dx.$$

The MEPP implies the following form of the mean field equation for n and p:

$$n_t = \nabla \cdot \left(D \left(\nabla p + n \nabla \varphi \right) \right),$$

where the diffusion coefficient D may depend on n, p, ϑ, x, t, $\ldots$. A natural choice arising from the analysis in [14] is

$$D = -\frac{I_{d/2-1}(\lambda)}{\lambda I'_{d/2-1}(\lambda)}. \tag{1.4}$$

Note that the diffusion coefficient in [14, Section 5.2] is $\vartheta D/\xi$ with a positive parameter ξ, and $\xi \to \infty$ corresponds to the high friction limit in the evolution equation above. Similarly, we can get rid of constants σ_d and G

in the Poisson equation. Thus, the relations between the density n in (1.1) and the pressure p in (1.3) are now given implicitly by

$$n = \frac{\mu}{2}\vartheta^{d/2} I_{d/2-1}(\lambda), \tag{1.5}$$

$$p = \frac{\mu}{d}\vartheta^{d/2+1} I_{d/2}(\lambda), \tag{1.6}$$

with $\mu = \eta_0 2^{d/2} G\sigma_d^2$, and the Fermi integrals defined above in (1.2). After the normalization (1.5)–(1.6) (different from that in [14]) of the density, pressure, and the equation for the potential, we will consider the system

$$n_t = \nabla \cdot (D(\nabla p + n\nabla\varphi)), \tag{1.7}$$

$$\Delta\varphi = n, \tag{1.8}$$

with the natural no-flux boundary condition

$$(\nabla p + n\nabla\varphi) \cdot \overline{\nu} = 0 \tag{1.9}$$

($\overline{\nu}$ is the unit exterior normal vector to $\partial\Omega$), the Dirichlet boundary condition

$$\varphi_{|\partial\Omega} = 0, \tag{1.10}$$

and an initial condition

$$n(x,0) = n_0(x). \tag{1.11}$$

Note that another condition for φ, the "free" (physically acceptable) condition

$$\varphi = E_d * n \tag{1.12}$$

with E_d being the fundamental solution of the Laplacian in $\mathbb{R}^d$, can be considered. In the case of radially symmetric solutions (1.10) is equivalent to (1.12) by adding a constant to the potential φ; cf. also the discussion of this issue in [7, 8, 3]. We will study in this paper mainly the Dirichlet condition for φ, but some comments on the problem with the free condition will be given.

The model we consider is a simplified version of (5.18) in [14], where the full system takes into account the conservation of the energy, the angular and linear momenta for the cloud of particles in rotational and linear motion. Our analysis thus concerns the *isothermal* model, where the temperature ϑ is put constant as it was in [7]. The analysis in the case of time-dependent but space homogeneous temperature $\vartheta = \vartheta(t)$ is more complicated; cf. [12, 21, 16, 9] for the linear diffusion case. However, if $\vartheta =$ const we cannot take into account the relation $E =$ const valid in the *microcanonical ensemble*, but not verified in the *canonical ensemble*.

Note that other models can take into account also spatial nonhomogeneities of the temperature; cf. [24] for the particles in an exterior potential, and [6] for self-interacting particles. Those systems are much more difficult to analyze rigorously, even for the linear (i.e., Brownian) diffusion studied in the references above.

Notation. In the sequel $|\,.\,|_p$ will denote the $L^p(\Omega)$ norms, $\|\,.\,\|_{H^k}$ will be used for the Sobolev space $H^k(\Omega)$ norm, and $\|\,.\,\|_{\mathcal{C}^\varepsilon}$ – for the Hölder space $\mathcal{C}^\varepsilon$ norm. The letter C will denote inessential constants which may vary from line to line.

2. Properties of Fermi integrals and auxiliary functions

Concerning the properties of the Fermi integrals let us notice that $I_0(\lambda) = \log(1+1/\lambda)$, because

$$\frac{d}{d\lambda} I_0(\lambda) = -\frac{1}{\lambda(1+\lambda)}. \tag{2.1}$$

For $\alpha > -1$ and $|1/\lambda| < 1$ the function I_α has the asymptotic expansion in the powers of $1/\lambda$

$$\begin{aligned} I_\alpha(\lambda) &= \frac{1}{\lambda}\int_0^\infty y^\alpha e^{-y}\big(1+\frac{1}{\lambda}e^{-y}\big)^{-1} dy = \frac{1}{\lambda}\int_0^\infty y^\alpha e^{-y}\Big(\sum_{k=0}^\infty (-1)^k\Big(\frac{1}{\lambda}\Big)^k e^{-ky}\Big)\,dy \\ &= \Gamma(\alpha+1)\sum_{k=0}^\infty \frac{(-1)^k}{(k+1)^{\alpha+1}}\Big(\frac{1}{\lambda}\Big)^{k+1}. \end{aligned} \tag{2.2}$$

For $\alpha > -1$ we have

$$\frac{d}{d\lambda} I_\alpha(\lambda) = -\int_0^\infty \frac{y^\alpha e^y\,dy}{(1+\lambda e^y)^2} = \int_0^\infty \frac{y^\alpha}{\lambda}\,\frac{d}{dy}\Big(\frac{1}{1+\lambda e^y}\Big)dy = -\frac{\alpha}{\lambda} I_{\alpha-1}(\lambda), \tag{2.3}$$

the second line being valid for $\alpha > 0$ only. Moreover, the formula

$$\frac{d^2}{d\lambda^2} I_\alpha(\lambda) = 2\int_0^\infty \frac{y^\alpha e^{2y}\,dy}{(1+\lambda e^y)^3} \tag{2.4}$$

for the second derivative holds. Therefore, I_α is a decreasing convex function of λ.

It is important that the Fermi integrals (1.2) have the asymptotics

$$I_\alpha(\lambda) \sim \frac{1}{\alpha+1}\,(-\log\lambda)^{\alpha+1} \quad \text{as} \quad \lambda \searrow 0, \tag{2.5}$$

and

$$I_\alpha(\lambda) \sim \frac{\Gamma(\alpha+1)}{\lambda} \quad \text{as} \quad \lambda \nearrow \infty. \tag{2.6}$$

In fact, for $\alpha > 0$ these relations are consequences of

$$\begin{aligned} I_\alpha(\lambda) &\geq \int_0^{-\log\lambda} \frac{y^\alpha\,dy}{1+\lambda e^y} \\ &\geq \int_0^{-\log\lambda} y^\alpha\left(1-\lambda e^y\right)\,dy = \frac{(-\log\lambda)^{\alpha+1}}{\alpha+1} - (-\log\lambda)^\alpha \end{aligned}$$

for $0 < \lambda \leq 1$, and

$$\begin{aligned} I_\alpha(\lambda) &\leq \int_0^{-\log\lambda} y^\alpha\,dy + \int_{-\log\lambda}^\infty \frac{y^\alpha\,dy}{1+\lambda e^y} \\ &\leq \frac{(-\log\lambda)^{\alpha+1}}{\alpha+1} + \int_0^\infty (v-\log\lambda)^\alpha\,e^{-v}\,dv \\ &\leq \frac{(-\log\lambda)^{\alpha+1}}{\alpha+1} + (-\log\lambda)^\alpha \int_0^\infty \left(1+\frac{v}{\log 2}\right)^\alpha e^{-v}\,dv, \end{aligned}$$

for $0 < \lambda \leq 1/2$. The asymptotics for large λ follow from the inequalities

$$\frac{\Gamma(\alpha+1)}{\lambda+1} \leq I_\alpha(\lambda) \leq \frac{\Gamma(\alpha+1)}{\lambda}$$

valid for $\lambda \geq 1$.

From the physical point of view it is worth noting that in the *classical limit* $\lambda \nearrow \infty$ (i.e., the small density $n \to 0$) the relation

$$\frac{p}{n} = \frac{2}{d}\vartheta\frac{I_{d/2}(\lambda)}{I_{d/2-1}(\lambda)} = \vartheta\Big(1+\frac{1}{2^{d/2+1}\lambda} - \cdots\Big) \sim \vartheta \tag{2.7}$$

holds.

The completely degenerate case (*white dwarf* in astrophysics), when f is close to η_0, or $\lambda \searrow 0$ (i.e., $n \to \infty$), by (2.5) corresponds to the relation

$$\frac{p}{n^{1+2/d}} \sim \frac{2}{d+2}\left(\frac{d}{\mu}\right)^{2/d} = \text{const}, \tag{2.8}$$

i.e., to a polytropic equation of state of a gas; cf. [15].

In order to eliminate the variable λ we introduce auxiliary functions D, V, F, and H defined by

$$D(z) = -\frac{z}{I_{d/2-1}^{-1}\left(2z(\mu\vartheta^{d/2})^{-1}\right)\Big(I'_{d/2-1}\circ I_{d/2-1}^{-1}\Big)\left(2z(\mu\vartheta^{d/2})^{-1}\right)}, \tag{2.9}$$

$$V(z) = zD(z), \tag{2.10}$$

$$F'(z) = (D(z))^2, \quad F(0) = 0, \tag{2.11}$$

$$h(z) = \int_1^z \frac{F'(y)}{V(y)}\,dy, \quad H(z) = \int_1^z h(y)\,dy. \tag{2.12}$$

We gather some properties of these functions in the next lemma.

Lemma 2.1. *The function V belongs to $\mathcal{C}^2([0,\infty))$, is nonnegative, and can be extended to an element (still denoted by V) of $\mathcal{C}^2([-\delta,\infty))$ for some $\delta > 0$. Similarly, the function F belongs to $\mathcal{C}^3([0,\infty))$, is nonnegative, increasing, and convex. It can be extended to a convex increasing function in $\mathcal{C}^3([-\delta,\infty))$. The function H is nonnegative and convex on $[0,\infty)$.*

Proof. First note that $V(0) = F(0) = 0$, and $V'(0) = F'(0) = 1$. Next, using the expansion (2.2) for $\alpha = d/2-1$, observe that $I_{d/2-1}$ is analytic at $z = \infty$, with $I_{d/2-1}(z) \sim \Gamma(\alpha+1)/z$, so that $I^{-1}_{d/2-1}$ is analytic at 0. Then, the function D is a quotient of two compositions of analytic functions, and assumes a nonzero finite value at $z = 0$. Finally, V and F are analytic at $z = 0$, so that the analytic extensions of V and F satisfy the conclusion of Lemma 2.1.

Since $h(z) \sim \log z$ as $z \to 0$, and h is increasing, the function H is well defined and convex. □

Recalling (1.5) and (1.6), we see that

$$p(z) = \frac{\mu}{d}\vartheta^{d/2+1}\left(I_{d/2} \circ I^{-1}_{d/2-1}\right)\left(2z(\mu\vartheta^{d/2})^{-1}\right) \tag{2.13}$$

and

$$p'(z) = \vartheta D(z). \tag{2.14}$$

It is of interest to note that the asymptotic relations

$$V(z) \sim K_1 z^{1+2/d}, \quad V'(z) \sim K_1 \frac{d+2}{d} z^{2/d}, \tag{2.15}$$

$$F'(z) \sim K_2 z^{4/d}, \tag{2.16}$$

$$p(z) \sim \frac{d}{d+2} K_3 z^{1+2/d}, \quad p'(z) \sim K_3 z^{2/d}, \tag{2.17}$$

$$h(z) \sim K_4 z^{2/d}, \quad H(z) \sim \frac{d}{d+2} K_4 z^{1+2/d} \tag{2.18}$$

hold as $z \to \infty$ for some positive constants K_1, K_2, K_3, and K_4, which follows from the asymptotic properties (2.5) of the Fermi integrals.

Moreover, from (2.14), (2.11), $\vartheta F''(z) = 2D(z)p''(z)$, and the inequality

$$p''(z) > 0 \quad \text{for} \quad z \geq 0, \tag{2.19}$$

it follows that the functions p and F are increasing and strictly convex on $[0,\infty)$. To prove the property (2.19) let us show that, more generally,

Lemma 2.2. *For $\alpha > \beta$, $I_\alpha \circ I_\beta^{-1}$ is an increasing convex function, while for $\alpha < \beta$ this is an increasing concave function.*

The desired property (2.19) for p as a function of z will follow immediately for $\alpha = d/2$ and $\beta = d/2 - 1$.

Proof. Setting $g = I_\alpha \circ I_\beta^{-1}$ we have $g' = I'_\alpha \circ I_\beta^{-1}/(I'_\beta \circ I_\beta^{-1}) > 0$. Next we consider $G = g' \circ I_\beta = I'_\alpha / I'_\beta$ and

$$G'(z) = \left(I''_\alpha(z) I'_\beta(z) - I'_\alpha(z) I''_\beta(z)\right) \left(I'_\beta(z)\right)^{-2} = -2J(z) \left(I'_\beta(z)\right)^{-2}.$$

Here we have

$$\begin{aligned} J(\lambda) &= \int_0^\infty \int_0^\infty \left(\frac{y^\alpha e^{2y}}{(1+\lambda e^y)^3} \frac{v^\beta e^v}{(1+\lambda e^v)^2} - \frac{y^\alpha e^y}{(1+\lambda e^y)^2} \frac{v^\beta e^{2v}}{(1+\lambda e^v)^3} \right) dv\, dy \\ &= \int_0^\infty \int_0^\infty \frac{y^\alpha v^\beta e^{y+v}}{(1+\lambda e^y)^3 (1+\lambda e^v)^3} \left(e^y - e^v\right) dv\, dy \end{aligned}$$

because the derivatives of the Fermi integrals are calculated from the formulae (2.3) and (2.4). After symmetrization the quantity J becomes

$$J(\lambda) = \frac{1}{2} \int_0^\infty \int_0^\infty \frac{e^{y+v} y^\alpha v^\alpha}{(1+\lambda e^y)^3 (1+\lambda e^v)^3} \left(v^{\beta-\alpha} - y^{\beta-\alpha}\right) \left(e^y - e^v\right) dv\, dy.$$

Now it is clear that for $\alpha > \beta$ we get $J(\lambda) > 0$; i.e., $\operatorname{sign}(\alpha - \beta) g'' > 0$. □

3. The evolution problem

From now on we suppose that $d \le 3$, the temperature is a fixed positive constant $\vartheta > 0$, and the diffusion coefficient in (1.7) is given by (1.4).

Since we have

$$\nabla p = p'(n) \nabla n = \vartheta D(n) \nabla n, \tag{3.1}$$

by (2.14), the initial–boundary-value problem (1.7)–(1.8) reads

$$n_t = \nabla \cdot \left(\vartheta\ F'(n)\ \nabla n + V(n)\ \nabla \varphi\right) \quad \text{in} \quad \Omega \times (0,\infty)\,, \tag{3.2}$$

$$\Delta \varphi = n \quad \text{in} \quad \Omega \times (0,\infty)\,, \tag{3.3}$$

$$\left(\vartheta F'(n)\ \nabla n + V(n)\ \nabla \varphi\right) \cdot \bar\nu = \varphi = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty)\,, \tag{3.4}$$

$$n(0) = n_0 \quad \text{in} \quad \Omega\,. \tag{3.5}$$

Theorem 3.1. *If $d \le 3$ and $n_0 \in L^2(\Omega)$, then there exists a global-in-time solution $(n, \varphi) \in \mathcal{C}\left([0,T]; w - L^2(\Omega)\right) \times L^\infty\left(0,T; H^2(\Omega)\right)$ of the system (3.2)–(3.5) such that $p(n) \in L^2(0,T; H^1(\Omega))$, $F(n) \in L^2(0,T; W^{1,6/5}(\Omega))$, and*

$$\int_\Omega (n(x,t) - n_0(x))\, \chi\, dx \quad + \quad \int_\Omega \nabla \chi \cdot (\vartheta\ \nabla F(n) + V(n) \nabla \varphi)\ dx = 0,$$

$$\Delta\varphi \quad = \quad n, \quad \varphi = 0 \;\; \text{on } \partial\Omega,$$

for all $T > 0$ *and each test function* $\chi \in W^{1,6}(\Omega)$.

In addition, for all $t \geq 0$, *we have the following estimates:*

$$|n(t)|_{1+2/d} + \|\varphi(t)\|_{H^1} \leq K_0\,, \tag{3.6}$$

$$\int_0^\infty |V(n)|_1^{-1}\;\; |\vartheta\;\nabla F(n) + V(n)\;\nabla\varphi|_1^2\;\; ds \leq K_0\,, \tag{3.7}$$

$$|n(t)|_2 + \int_t^{t+1} \|p(n)\|_{H^1}^2\;\; ds \leq K_0\,, \tag{3.8}$$

where K_0 *depends only on* n_0, Ω, ϑ, *and* d.

In order to study the well-posedness of (3.2)–(3.5), we consider the following regularized initial–boundary-value problem with

$$n_0 \in \mathcal{C}^\infty(\bar\Omega) \quad \text{such that} \quad n_0 \geq 0\,, \qquad M = |n_0|_1, \quad \text{and} \quad \varphi_0 = 0\,. \tag{3.9}$$

For $k \geq 1$, we introduce the parabolic system

$$n_t = \nabla\cdot\big(\vartheta\; F'(n)\;\nabla n + V(n)\;\nabla\varphi\big) \qquad \text{in} \qquad \Omega\times(0,\infty)\,, \tag{3.10}$$

$$\varphi_t - k\;\Delta\varphi = -k\;n \qquad \text{in} \qquad \Omega\times(0,\infty)\,, \tag{3.11}$$

$$\big(\vartheta\; F'(n)\;\nabla n + V(n)\;\nabla\varphi\big)\cdot\bar\nu = \varphi = 0 \qquad \text{on} \qquad \partial\Omega\times(0,\infty)\,, \tag{3.12}$$

$$(n(0),\varphi(0)) = (n_0,\varphi_0) \qquad \text{in} \qquad \Omega\,. \tag{3.13}$$

3.1. **Well-posedness of (3.10)–(3.13).**

Theorem 3.2. *The initial–boundary-value problem,* (3.10)–(3.13), *has a unique global-in-time classical solution* $(n,\varphi) \in \mathcal{C}(\bar\Omega\times[0,\infty);\mathbb{R}^2)\cap\mathcal{C}^{2,1}(\bar\Omega\times(0,\infty);\mathbb{R}^2)$ *such that*

$$n(t) \geq 0 \quad \textit{and} \quad \int_\Omega n(x,t)\;dx = M \quad \textit{for} \quad t \geq 0,$$

and

$$|n(t)|_{1+2/d} + \|\varphi(t)\|_{H^1} + \frac{1}{k}\;\int_0^\infty |\varphi_t|_2^2\;ds \leq C_0\,, \tag{3.14}$$

$$\int_0^\infty |V(n)|_1^{-1}\;\; |\vartheta\;\nabla F(n) + V(n)\;\nabla\varphi|_1^2\;\; ds \leq C_0 \tag{3.15}$$

for all $t \geq 0$, *where the constant* C_0 *depends only on* M, $|H(n_0)|_1$, Ω, ϑ, *and* $d \leq 3$. *Furthermore, for any* $T > 0$, *there is a constant* $C_1(T)$ *depending only on* $|n_0|_2$, $|H(n_0)|_1$, ϑ, d, Ω, *and* T *such that*

$$|n(t)|_2 + \int_0^T \|p(n)\|_{H^1}^2\;ds \leq C_1(T) \quad \textit{for } 0 \leq t \leq T. \tag{3.16}$$

We first use the theory developed by Amann [1] to prove the local well-posedness of (3.10)–(3.13).

Proposition 3.3. *The initial–boundary-value problem* (3.10)–(3.13) *has a unique maximal classical solution* $(n,\varphi) \in \mathcal{C}(\bar\Omega \times [0,T_m);\mathbb{R}^2) \cap \mathcal{C}^{2,1}(\bar\Omega \times (0,T_m);\mathbb{R}^2)$ *for some* $T_m \in (0,\infty]$. *In addition,* $n(t) \geq 0$ *for* $t \in [0,T_m)$.

Furthermore, $T_m = \infty$ *if there are* $\varepsilon > 0$ *and a locally bounded function* $\omega : [0,\infty) \to [0,\infty)$ *such that, for every* $T > 0$,

$$\|n(t)\|_{\mathcal{C}^\varepsilon} + \|\varphi(t)\|_{\mathcal{C}^\varepsilon} \leq \omega(T) \quad \text{for} \quad t \in [0,T_m) \cap [0,T]\,. \tag{3.17}$$

Proof. We set $D_0 = (-\delta,\infty) \times \mathbb{R}$, $u = (n,\varphi)$ with $u_0 = (n_0,\varphi_0)$, and define $a \in \mathcal{C}^2(D_0;\mathcal{M}_2(\mathbb{R}))$ and $f \in \mathcal{C}^2(D_0;\mathbb{R}^2)$ by

$$a(u) = \begin{pmatrix} \vartheta\ F'(n) & V(n) \\ 0 & k \end{pmatrix}, \qquad f(u) = \begin{pmatrix} 0 \\ -k\ n \end{pmatrix}.$$

For $v \in D_0$, we introduce the operators

$$\begin{aligned} \mathcal{A}(v)u &= -\sum_{i=1}^d \sum_{j=1}^d \partial_i \left(a_{ij}(v)\ \partial_j u\right), \\ \mathcal{B}(v)u &= b\ \sum_{i=1}^d \sum_{j=1}^d \bar\nu_i \left(a_{ij}(v)\ \partial_j u\right) + (I_2 - b)\ u\,, \end{aligned}$$

where $a_{ij}(v) = a(v)\ \delta_{ij}$, $1 \leq i,j \leq d$, and

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

An abstract formulation of (3.10)–(3.13) reads

$$u_t + \mathcal{A}(u)u = f(u)\,, \tag{3.18}$$

$$\mathcal{B}(u)u = 0\,, \tag{3.19}$$

$$u(0) = u_0\,. \tag{3.20}$$

Thanks to Lemma 2.1, the eigenvalues of the matrix $a(v)$ are positive for each $v \in D_0$, and the boundary-value operator $(\mathcal{A},\mathcal{B})$ is of separated divergence form in the sense of [1, Section 4.3 (e)]. It then follows from [1, Section 4] that the boundary-value operator $(\mathcal{A},\mathcal{B})$ is normally elliptic in the sense of [1]. We may then apply [1, Theorem 14.4 and Theorem 14.6] to conclude that (3.18)–(3.20) has a unique maximal classical solution

$$u = (n,\varphi) \in \mathcal{C}(\bar\Omega \times [0,T_m);D_0) \cap \mathcal{C}^{2,1}(\bar\Omega \times (0,T_m);D_0)$$

for some $T_m \in (0,\infty]$. Also, since $n_0 \geq 0$, $V(0) = 0$, and the first component of $f(u)$ is equal to zero, the comparison principle (see, e.g., [1, Theorem 15.1] or [17, Corollary I.2.1]) implies that $n(t) \geq 0$ for $t \in [0, T_m)$. Furthermore, since f does not depend on ∇u and $n \geq 0$, Theorem 15.3 in [1] ensures that $T_m = \infty$ if there are $\varepsilon > 0$ and a locally bounded function $\omega : [0,\infty) \to [0,\infty)$ such that (3.17) holds true for every $T > 0$. $\square$

We now proceed to show that (3.17) is satisfied. We define the functional $\mathcal{W}$ by

$$\mathcal{W} = \int_\Omega \left(\vartheta\ H(n) + n\ \varphi + \frac{1}{2}\ |\nabla\varphi|^2 \right)\ dx$$

which will play the role of a Lyapunov functional for the regularized problem.

Lemma 3.4. *We have*

$$|n(t)|_1 = M \quad \text{for} \quad t \in [0, T_m)\,, \tag{3.21}$$

and there is a constant C_1 depending only on M, ϑ, and $|H(n_0)|_1$ such that

$$|n(t)|_{1+2/d} + \|\varphi(t)\|_{H^1} + \frac{1}{k}\ \int_0^t \int_\Omega |\varphi_t|^2\ dx\, ds \leq C_1\,, \tag{3.22}$$

$$\int_0^t |V(n)|_1^{-1}\ \ |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|_1^2\ \ ds \leq C_1\,, \tag{3.23}$$

for $t \in [0, T_m)$. In addition, $t \longmapsto \mathcal{W}(t)$ is a nonincreasing function on $[0, T_m)$.

Proof. It first readily follows from (3.10) and (3.12) by integration over $\Omega \times (0,t)$ that

$$\int_\Omega n(x,t)\ dx = \int_\Omega n_0(x)\ dx$$

for $t \in [0, T_m)$, whence (3.21) since n is nonnegative.

We next consider $\delta \in (0,1)$ and put

$$h_\delta(z) = h(\max\{z, \delta\})\,, \qquad H_\delta(z) = \int_1^z h_\delta(y)\ dy$$

for $z \geq 0$. Using the monotonicity of h, we realize that

$$\sup_{z\geq 0} |H_\delta(z) - H(z)| \leq \delta\ h(\delta) - H(\delta) \underset{\delta\to 0}{\longrightarrow} 0\,, \tag{3.24}$$

since $h(\delta) \sim \log\delta$ as $\delta \to 0$ and $H(0) = 0$.

We infer from (3.10), (3.11), and (3.12) that

$$\begin{aligned}\int_\Omega (\vartheta h_\delta(n) + \varphi) n_t\ dx &= -\int_\Omega (\vartheta \nabla h_\delta(n) + \nabla\varphi)\cdot(\vartheta\nabla F(n) + V(n)\nabla\varphi)\, dx \\ &= -\vartheta\ \int_{\{n\geq\delta\}} \nabla h(n)\cdot(\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ \ dx\end{aligned}$$

$$
\begin{aligned}
&- \int_{\{n\geq\delta\}} \nabla\varphi \cdot (\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ dx \\
&- \int_{\{n<\delta\}} \nabla\varphi \cdot (\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ dx \\
&= - \int_{\{n\geq\delta\}} V(n)\ |\nabla(\vartheta\ h(n) + \varphi)|^2\ dx \\
&- \int_{\{n<\delta\}} \nabla\varphi \cdot (\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ dx\,,
\end{aligned}
$$

where we have used the identity

$$
\vartheta\ F'(n)\ \nabla n + V(n)\ \nabla\varphi = V(n)\ \nabla(\vartheta\ h(n) + \varphi)
$$

to obtain the last equality. Since

$$
\int_\Omega h_\delta(n)\ n_t\ dx = \frac{d}{dt}\int_\Omega H_\delta(n)\ dx\,,
$$

and

$$
\begin{aligned}
\int_\Omega \varphi\ n_t\ dx &= \frac{d}{dt}\int_\Omega n\ \varphi\ dx - \int_\Omega n\ \varphi_t\ dx \\
&= \frac{d}{dt}\int_\Omega \left(n\ \varphi + \frac{1}{2}\ |\nabla\varphi|^2\right)\ dx + \frac{1}{k}\ \int_\Omega |\varphi_t|^2\ dx
\end{aligned}
$$

by (3.11), we end up with

$$
\begin{aligned}
\frac{d\mathcal{W}_\delta}{dt} &+ \frac{1}{k}\ |\varphi_t|_2^2 + \int_{\{n\geq\delta\}} V(n)\ |\nabla(\vartheta\ h(n) + \varphi)|^2\ dx \\
&= - \int_{\{n<\delta\}} \nabla\varphi \cdot (\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ dx\,,
\end{aligned}
$$

where

$$
\mathcal{W}_\delta = \int_\Omega \left(\vartheta\ H_\delta(n) + n\varphi + \frac{1}{2}\ |\nabla\varphi|^2\right) dx\,.
$$

Next it follows from the Hölder inequality that

$$
\begin{aligned}
&\int_{\{n\geq\delta\}} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ dx \\
\leq\ & |V(n)|_1^{1/2}\ \left(\int_{\{n\geq\delta\}} \frac{|\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|^2}{V(n)}\right)^{1/2},
\end{aligned}
$$

whence

$$
\left(\int_{\{n\geq\delta\}} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ dx\right)^2\ |V(n)|_1^{-1}
$$

$$\leq \int_{\{n\geq\delta\}} V(n)\ |\nabla\,(\vartheta\ h(n)+\varphi)|^2\ dx\,.$$

Consequently,

$$\begin{aligned}\frac{d\mathcal{W}_\delta}{dt} \quad &+ \quad \frac{1}{k}\ |\varphi_t|_2^2 + \Big(\int_{\{n\geq\delta\}} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ \ dx\Big)^2\ |V(n)|_1^{-1} \\ &\leq \quad -\int_{\{n<\delta\}} \nabla\varphi\cdot(\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ \ dx\,.\end{aligned}$$

Consider now $t_1\in[0,T_m)$ and $t_2\in(t_1,T_m)$. Integrating the above inequality over (t_1,t_2) yields

$$\begin{aligned}&\mathcal{W}_\delta(t_2) + \frac{1}{k}\ \int_{t_1}^{t_2} |\varphi_t|_2^2\ ds \\ + \quad &\int_{t_1}^{t_2} \Big(\int_{\{n\geq\delta\}} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ \ dx\Big)^2\ |V(n)|_1^{-1}\ ds \\ \leq \quad &\mathcal{W}_\delta(t_1) - \int_{t_1}^{t_2}\int_{\{n<\delta\}} \nabla\varphi\cdot(\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ \ dx\,ds\,. \end{aligned} \tag{3.25}$$

On the one hand, it readily follows from (3.24) that

$$\lim_{\delta\to 0}\mathcal{W}_\delta(t_i) = \mathcal{W}(t_i) \quad \text{for} \quad i=1,2\,.$$

On the other hand, since $V(0)=0$, $\mathbf{1}_{\{n\geq\delta\}}\to\mathbf{1}_{\{n>0\}}$, and $\mathbf{1}_{\{n<\delta\}}\to\mathbf{1}_{\{n=0\}}$ as $\delta\to 0$, the regularity of (n,φ) and the Lebesgue dominated convergence theorem ensure that

$$\begin{aligned}&\lim_{\delta\to 0}\int_{t_1}^{t_2}\int_{\{n<\delta\}} \nabla\varphi\cdot(\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ \ dx\,ds \\ = \quad &\int_{t_1}^{t_2}\int_{\{n=0\}} \nabla\varphi\cdot(\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi)\ \ dx\,ds = 0\,,\end{aligned}$$

and

$$\begin{aligned}&\lim_{\delta\to 0}\int_{t_1}^{t_2} \Big(\int_{\{n\geq\delta\}} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ \ dx\Big)^2\ |V(n)|_1^{-1}\ ds \\ = \quad &\int_{t_1}^{t_2} \Big(\int_{\{n>0\}} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ \ dx\Big)^2\ |V(n)|_1^{-1}\ ds \\ = \quad &\int_{t_1}^{t_2} \Big(\int_{\Omega} |\vartheta\ \nabla F(n) + V(n)\ \nabla\varphi|\ \ dx\Big)^2\ |V(n)|_1^{-1}\ ds\,.\end{aligned}$$

We may then let $\delta \to 0$ in (3.25) and conclude that

$$\mathcal{W}(t_2) + \frac{1}{k}\int_{t_1}^{t_2} |\varphi_t|_2^2 ds + \int_{t_1}^{t_2} |\vartheta\nabla F(n) + V(n)\nabla\varphi|_1^2 \, |V(n)|_1^{-1} \; ds \leq \mathcal{W}(t_1). \tag{3.26}$$

The monotonicity of $\mathcal{W}$ and the bounds (3.22) and (3.23) are then straightforward consequences of (3.26) by the Poincaré inequality and Lemma 3.5 below. □

Lemma 3.5. *There are positive constants $C_{\mathcal{W}}$ and $D_{\mathcal{W}}$ depending on $d \leq 3$, Ω, ϑ, and M such that*

$$\mathcal{W} \geq C_{\mathcal{W}}(|n|_{1+2/d}^{1+2/d} + |\nabla\varphi|_2^2) - D_{\mathcal{W}}. \tag{3.27}$$

Proof. Let $\varepsilon \in (0,1)$. It follows from the continuous imbedding of $H^1(\Omega)$ in $L^6(\Omega)$ and the Hölder inequality that

$$\Big|\int_\Omega n\ \varphi\ dx\Big| \leq |n|_{6/5}\ |\varphi|_6 \leq C\ |n|_1^{(10-d)/12}\ |n|_{1+2/d}^{(d+2)/12}\ |\nabla\varphi|_2 \, .$$

We next infer from (3.21) and the Young inequality that

$$\Big|\int_\Omega n\ \varphi\ dx\Big| \leq \frac{1}{4}\ |\nabla\varphi|_2^2 + C\ |n|_{1+2/d}^{(d+2)/6} \leq \frac{1}{4}\ |\nabla\varphi|_2^2 + \varepsilon\ |n|_{1+2/d}^{1+2/d} + C(\varepsilon) \, .$$

Now, by (2.12) and (2.18), there is a constant $C > 0$ such that $h'(z) \geq C\ z^{2/d-1}$ for $z \geq 0$, from which we deduce that

$$H(z) \geq C_H\ z^{1+2/d} - C'_H \, , \qquad z \geq 0 \, , \tag{3.28}$$

for some $C_H > 0$ and $C'_H > 0$. It then follows from (3.28) and the previous inequality with $\varepsilon = \vartheta C_H/2$ that

$$\mathcal{W} \geq \vartheta\ C_H\ |n|_{1+2/d}^{1+2/d} - C - \frac{1}{4}\ |\nabla\varphi|_2^2 - \vartheta\frac{C_H}{2}\ |n|_{1+2/d}^{1+2/d} - C + \frac{1}{2}\ |\nabla\varphi|_2^2 \, ,$$

whence (3.27). □

Thanks to (3.22), we may improve the regularity of $\nabla\varphi$.

Lemma 3.6. *Let $q \in (1,\infty)$ and $T > 0$. There is a constant $C(q,T)$ depending only on q and T such that*

$$\int_0^t |\nabla\varphi(s)|_{15/4}^q\ ds \leq C(q,T) \quad \textit{for} \quad t \in [0,T_m) \cap [0,T] \, .$$

Proof. For $\tau \in [0,kT_m)$ and $x \in \Omega$, we put $\varphi_\star(x,\tau) = \varphi(x,\tau/k)$ and $n_\star(x,\tau) = n(x,\tau/k)$. Owing to (3.11) and (3.12), $\varphi_\star$ is a solution to

$$(\varphi_\star)_\tau - \Delta\varphi_\star = -n_\star \quad \text{in} \quad \Omega \times (0,kT_m) \, ,$$

with the homogeneous Dirichlet boundary conditions and $\varphi_\star(0) = 0$. We then infer from [18, Corollaire 1.1] that there is a constant $C(q)$ depending only on q such that

$$|\,(\varphi_\star)_\tau\,|_{L^q(0,kt;L^{1+2/d}(\Omega))} + |\Delta\varphi_\star|_{L^q(0,kt;L^{1+2/d}(\Omega))} \le C(q)\ |n_\star|_{L^q(0,kt;L^{1+2/d}(\Omega))}$$

for each $t \in (0, T_m)$. In terms of φ and n, the above estimate reads

$$\frac{1}{k}\ |\varphi_t|_{L^q(0,t;L^{1+2/d}(\Omega))} + |\Delta\varphi|_{L^q(0,t;L^{1+2/d}(\Omega))} \le C(q)\ |n|_{L^q(0,t;L^{1+2/d}(\Omega))}\,.$$

Now, if $t \in [0, T_m) \cap [0, T]$, it follows from (3.22) and the above inequality that

$$\int_0^t \|\varphi(s)\|^q_{W^{2,1+2/d}}\ ds \le C(q, T)\,.$$

Then we use the continuity of the imbedding of $W^{2,1+2/d}(\Omega)$ in $W^{1,15/4}(\Omega)$ to complete the proof of Lemma 3.6. □

Owing to Lemma 3.6, an L^2 estimate is available for n.

Lemma 3.7. *Let $T > 0$. There is a constant $C_2(T)$ depending only on $|n_0|_2$, $|H(n_0)|_1$, ϑ, and T such that*

$$|n(t)|_2 + \int_0^t \|p(n(s))\|^2_{H^1}\ ds \le C_2(T) \tag{3.29}$$

for $t \in [0, T_m) \cap [0, T]$.

Proof. Let $t \in [0, T_m) \cap [0, T]$. We multiply (3.10) by $2n$ and integrate over Ω to obtain

$$\frac{d}{dt}|n|_2^2 + 2\vartheta\ \int_\Omega F'(n)\ |\nabla n|^2\ dx = -2\ \int_\Omega V(n)\ \nabla n\cdot\nabla\varphi\ dx\,.$$

Since $V(n) = n\ \left(F'(n)\right)^{1/2}$, we have

$$2\Big|\int_\Omega V(n)\ \nabla n\cdot\nabla\varphi\ dx\Big| \le \vartheta\ \int_\Omega F'(n)\ |\nabla n|^2\ dx + \frac{1}{\vartheta}\ \int_\Omega n^2\ |\nabla\varphi|^2\ dx$$

by the Young inequality, whence

$$\frac{d}{dt}|n|_2^2 + \vartheta\ \int_\Omega F'(n)\ |\nabla n|^2\ dx \le \frac{1}{\vartheta}\ \int_\Omega n^2\ |\nabla\varphi|^2\ dx\,. \tag{3.30}$$

It follows from the Hölder inequality that

$$\int_\Omega n^2\ |\nabla\varphi|^2\ dx \le |n|^2_{30/7}\ |\nabla\varphi|^2_{15/4} \le \left|n^{1+2/d}\right|^{2d/(d+2)}_{30d/7(d+2)}\ |\nabla\varphi|^2_{15/4}\,.$$

Since $p(z) \sim C\ z$ as $z \to 0$, we infer from (2.17) that there is a positive constant C_p such that $p(z) \geq C_p\ z^{1+2/d}$ for all $z \geq 0$. Consequently,

$$\int_\Omega n^2\ |\nabla\varphi|^2\ dx \leq C\ |p(n)|_{30d/7(d+2)}^{2d/(d+2)}\ |\nabla\varphi|_{15/4}^2 \leq C\ \|p(n)\|_{H^1}^{2d/(d+2)}\ |\nabla\varphi|_{15/4}^2\,,$$

where the last inequality follows from the continuous imbedding of $H^1(\Omega)$ in $L^{30d/7(d+2)}(\Omega)$. We next use the Young inequality to conclude that

$$\int_\Omega n^2\ |\nabla\varphi|^2\ dx \leq \varepsilon\ \|p(n)\|_{H^1}^2 + C(\varepsilon)\ |\nabla\varphi|_{15/4}^{d+2} \tag{3.31}$$

for each $\varepsilon \in (0,1)$.

Since $p'(n) = \vartheta\ (F'(n))^{1/2}$, it follows from (3.30), (3.31) (with $\varepsilon = 1/(2\vartheta^2)$), and the Poincaré inequality that

$$\frac{d}{dt}|n|_2^2 + \frac{1}{2\vartheta^2}\ \int_\Omega |\nabla p(n)|^2\ dx \leq C\ \left(|p(n)|_{L^1}^2 + |\nabla\varphi|_{15/4}^{d+2}\right)\,.$$

Now, $p(n) \leq C\ (1 + n^{1+2/d})$ by (2.17) and (3.22) yields

$$\frac{d}{dt}|n|_2^2 + \frac{1}{2\vartheta^2}\ |\nabla p(n)|_2^2\ \leq C\ \left(1 + |\nabla\varphi|_{15/4}^{d+2}\right)\,. \tag{3.32}$$

Now, (3.29) readily follows from (3.32) after integration with respect to time, thanks to (3.22), Lemma 3.6 (with $q = d + 2$), and the Poincaré inequality. □

Remark. Observe that, up to now, the estimates obtained on n and φ do not depend on $k \geq 1$.

We are now in a position to complete the proof of Theorem 3.2.

Proof of Theorem 3.2. We consider only $d = 3$, the cases $d = 1$ and $d = 2$ being handled in a similar way. Let $T > 0$ and $t \in [0, T_m) \cap [0, T]$. We claim that there is a positive constant $C_4(T)$ depending only on n_0 and T such that

$$|V(n)|_{L^{14/5}(\Omega\times(0,t))} + |\nabla V(n)|_{L^2(\Omega\times(0,t))} \leq C_4(T)\,. \tag{3.33}$$

Indeed, we infer from (2.15), (2.17), and Lemma 2.1 that $V(z) \leq C\ (1+z^{5/3})$ and $V'(z) \leq Cp'(z)$ for $z \geq 0$. Consequently, it follows from (3.29) that

$$\sup_{s\in[0,t]} |V(n(s))|_{6/5} + \int_0^t \|V(n(s))\|_{H^1}^2\ ds \leq C(T).$$

We next use the continuity of the imbedding of $H^1(\Omega)$ in $L^6(\Omega)$ and an interpolation argument to deduce (3.33).

We now employ a bootstrap argument to show that (3.17) holds true. It follows from (3.29), (2.17), and the continuity of the imbedding of $H^1(\Omega)$ in $L^6(\Omega)$ that

$$|n|_{L^{10/3}(0,t;L^{10}(\Omega))} \le |p(n)|_{L^2(0,t;L^6(\Omega))}^{3/5} \le C(T)\,,$$

which, together with (3.29), leads to

$$\int_0^t |n(s)|_{14/3}^{14/3}\, ds \le \int_0^t |n(s)|_{10}^{10/3}\ |n(s)|_2^{4/3}\, ds \le C(T)\,.$$

Therefore,

$$|n|_{L^{14/3}(\Omega\times(0,t))} \le C(T)\,,$$

and we infer from (3.11) and [17, Theorem IV.9.1 and Lemma II.3.3] that

$$|\nabla\varphi|_{L^{70}(\Omega\times(0,t))} + |\Delta\varphi|_{L^{14/3}(\Omega\times(0,t))} \le C(k,T)\,.$$

This estimate and (3.33) ensure that

$$|\nabla V(n)\cdot\nabla\varphi|_{L^{35/18}(\Omega\times(0,t))} + |V(n)\ \Delta\varphi|_{L^2(\Omega\times(0,t))} \le C(k,T)\,.$$

Since

$$n_t - \vartheta\Delta F(n) = \nabla V(n)\cdot\nabla\varphi + V(n)\ \Delta\varphi\,,$$

we use once more [17, Theorem IV.9.1] to obtain that

$$\|n\|_{W^{2,1}_{38/15}(\Omega\times(0,t))} \le C(k,T)\,,$$

which, in turn, implies that

$$\|n\|_{\mathcal{C}^\varepsilon([0,t])} \le C(k,T)$$

for $\varepsilon\in(0,1/38)$ by [17, Lemma II.3.3]. A similar estimate is then available for φ, from which we conclude that (3.17) holds true and complete the proof of Theorem 3.2. □

3.2. **Proof of Theorem 3.1.** We consider $n_0\in L^2(\Omega)$ such $n_0\ge 0$ almost everywhere in Ω and put $M=|n_0|_1$. Let $(n_{0,k})_{k\ge1}$ be a sequence of nonnegative functions in $\mathcal{C}^\infty(\bar\Omega)$ such that

$$|n_{0,k}|_1 = M \quad\text{and}\quad \lim_{k\to\infty}|n_{0,k}-n_0|_2 = 0\,. \tag{3.34}$$

For $k\ge1$, we denote by (n_k,φ_k) the unique classical solution to (3.10)–(3.13) with initial datum $(n_{0,k},0)$ given by Theorem 3.2. Fix $T>0$. Owing to (3.14), (3.16), and (3.34), there is a constant $C_5(T)$ such that

$$|n_k(t)|_2 \quad+\quad \|\varphi_k(t)\|_{H^1} + \frac{1}{k}\ \int_0^T |(\varphi_k)_t|_2^2\, ds$$

$$+ \quad \int_0^T \|p(n_k(s))\|_{H^1}^2 \, ds \le C_5(T) \tag{3.35}$$

for $t \in [0,T]$. Observe that (3.35), (2.17), (2.15), (2.16), and Lemma 2.1 imply that

$$|\nabla F(n_k)|_{6/5} \le C \, |n_k^{2/3} \, \nabla p(n_k)|_{6/5} \le C \, |\nabla p(n_k)|_2 \,,$$
$$|V(n_k) \, \nabla \varphi|_{3/2} \le C \, |p(n_k) \, \nabla \varphi_k|_{3/2} \le C \, |p(n_k)|_6 \,,$$

whence, thanks to the continuity of the imbedding of $H^1(\Omega)$ in $L^6(\Omega)$,

$$\int_0^T \left(|\nabla F(n_k)|_{6/5}^2 + |V(n_k) \, \nabla \varphi_k|_{3/2}^2 \right) \, ds \le C(T) \,. \tag{3.36}$$

We then deduce from (3.10) and (3.36) that

$$|(n_k)_t|_{L^2(0,T;W^{1,6/5}(\Omega)')} \le C(T) \,. \tag{3.37}$$

Consequently, owing to (3.37) and Lemma 2.1, the sequence (n_k) is bounded in $L^2(0,T;H^1(\Omega))$ and in $H^1(0,T;W^{1,6/5}(\Omega)')$.

Owing to the compactness of the imbedding of $H^1(\Omega)$ in $L^2(\Omega)$ and to the continuity of the imbedding of $L^2(\Omega)$ in $W^{1,6/5}(\Omega)'$, we infer from [20, Corollary 4] that (n_k) is relatively compact in $L^2(\Omega \times (0,T))$. Therefore, there are $n \in L^2(\Omega \times (0,T))$ and a subsequence of (n_k) (not relabeled) such that

$$n_k \longrightarrow n \quad \text{in} \quad L^2(\Omega \times (0,T)) \cap \mathcal{C}([0,T];W^{1,6/5}(\Omega)') \quad \text{and a.e. in} \quad \Omega \times (0,T) \,. \tag{3.38}$$

Let $\varphi \in L^\infty(0,T;H^2(\Omega))$ be the solution to

$$\Delta \varphi = n \quad \text{in} \quad \Omega \times (0,T) \,, \qquad \varphi = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \,. \tag{3.39}$$

It follows from (3.11) and (3.39) that $\varphi_k - \varphi$ is a solution of the Poisson equation

$$-\Delta(\varphi_k - \varphi) = n - n_k - \frac{1}{k} \, (\varphi_k)_t$$

with the homogeneous Dirichlet boundary conditions, and the right-hand side of the above equation converges to zero in $L^2(\Omega \times (0,T))$ as $k \to \infty$ by (3.35) and (3.38). Therefore,

$$\varphi_k \longrightarrow \varphi \quad \text{in} \quad L^2(0,T;H^2(\Omega)) \,. \tag{3.40}$$

Combining (3.35) with the convergence results (3.38) and (3.40) finally allow us to conclude that $(\nabla F(n_k))$ and $(V(n_k) \, \nabla \varphi_k)$ converge weakly to $\nabla F(n)$ and $V(n) \, \nabla \varphi$ in $L^{6/5}(\Omega \times (0,T))$ and $L^{3/2}(\Omega \times (0,T))$, respectively. It is now straightforward to pass to the limit as $k \to \infty$ in (3.10)–(3.13) and conclude that (n, φ) is a weak solution to (3.2)–(3.4) in the sense of Theorem 3.1.

We may also pass to the limit in (3.14) and (3.15) and use classical lower-semicontinuity arguments to deduce that (3.6) and (3.7) hold true.

Next, since $\inf p' > 0$ by (2.17) and Lemma 2.1, it follows from (3.21), (3.32), and the Poincaré inequality that

$$\frac{d}{dt}|n_k|_2^2 + \gamma\ \left(|n_k|_2^2 + \|p(n_k)\|_{H^1}^2\right) \leq C\ \left(1 + |\nabla\varphi_k|_{15/4}^{d+2}\right)$$

for some positive constant γ. After integration with respect to time, we obtain

$$|n_k(t)|_2^2 \leq |n_{0,k}|_2^2\ e^{-\gamma t} + C\ \int_0^t \left(1 + |\nabla\varphi_k|_{15/4}^{d+2}\right)\ e^{\gamma(s-t)}\ ds \tag{3.41}$$

for $t \geq 0$, and

$$\begin{aligned}\int_t^{t+1} \|p(n_k(s))\|_{H^1}^2\ ds &\leq \int_{t-1}^t \int_\tau^{\tau+2} \|p(n_k(s))\|_{H^1}^2\ ds\, d\tau \\ &\leq C\ \left(1 + \int_{t-1}^t |n_k(\tau)|_2^2\ d\tau + \int_{t-1}^{t+2} |\nabla\varphi_k(s)|_{15/4}^{d+2}\ ds\right) \end{aligned}\tag{3.42}$$

for $t \geq 1$. Now, $|\nabla\varphi_k|_{15/4}$ is bounded in $L^q(0,t)$ for any $q \in (1,\infty)$ by Lemma 3.6, and we infer from (3.40) and the continuous imbedding of $H^2(\Omega)$ in $W^{1,15/4}(\Omega)$ that $(|\nabla\varphi_k - \nabla\varphi|_{15/4})$ converges to zero in $L^2(0,t)$. Consequently, $(|\nabla\varphi_k - \nabla\varphi|_{15/4})$ converges to zero in $L^{d+2}(0,t)$. We then may pass to the limit as $k \to \infty$ in (3.41) and (3.42) with the help of (3.38) and weak-convergence arguments for the left-hand sides and conclude that

$$|n(t)|_2^2 \leq |n_0|_2^2\ e^{-\gamma t} + C\ \int_0^t \left(1 + |\nabla\varphi|_{15/4}^{d+2}\right)\ e^{\gamma(s-t)}\ ds$$

for $t \geq 0$ and

$$\int_t^{t+1} \|p(n)\|_{H^1}^2\ ds \leq C\ \left(1 + \int_{t-1}^t |n(s)|_2^2\ ds + \int_{t-1}^{t+2} |\nabla\varphi(s)|_{15/4}^{d+2}\ ds\right)$$

for $t \geq 1$. Since φ is a solution to the Poisson equation (1.8), we have

$$|\nabla\varphi(s)|_{15/4} \leq C\ \|\varphi(s)\|_{W^{2,1+2/d}} \leq C\ |n(s)|_{1+2/d} \leq C$$

by (3.6). Inserting this estimate in the previous two inequalities yields first the boundedness of $|n|_2$ with respect to time, and then (3.8). □

Corollary 3.8. *Let* $n_0 \in L^2(\Omega)$ *be a nonnegative function, and denote by* (n,φ) *the corresponding solution of* (3.2)–(3.5) *given by Theorem* 3.1. *The trajectory* $\{(n(t),\varphi(t))\,, t \geq 0\}$ *is weakly relatively compact in the space* $L^2(\Omega) \times H^2(\Omega)$, *and the accumulation points are steady states, that is, solve* (4.1), (4.2), *and* (1.10).

Proof. The entropy dissipation formula (3.7) (which is a weaker form of (4.7)), together with the *a priori* estimates in Theorem 3.1 suffice to prove the relative compactness, and then to identify the possible accumulation points as steady states. □

4. The steady-state problem with fixed mass

The stationary solutions (N, Φ) of the problem (1.7)–(1.12) (or (1.7)–(1.10)) with arbitrary diffusion coefficient $\widetilde{D}$ and a constant ϑ, are characterized by the identity $\nabla(\vartheta \log \Lambda - \Phi) = 0$. Here, the relation between Λ and N is as was for λ and n in (1.5). This can be obtained formally from the equation (1.7) by taking the product with $\vartheta \log \Lambda - \Phi$ and integrating by parts, using of course the boundary conditions (1.9). Indeed, by (3.1) we get in such a way the identity $\int_\Omega N\widetilde{D}\, |\nabla(\vartheta \log \Lambda - \Phi)|^2 \, dx = 0$, and thus $\log \Lambda = \Phi/\vartheta - c$ where c is an appropriate integration constant.

Taking into account the relations (2.13), (1.5), and the Poisson equation (1.8), we arrive at the relation

$$\Delta \Phi = \frac{\mu}{2} \vartheta^{d/2} I_{d/2-1}(e^{\Phi/\vartheta - c}), \tag{4.1}$$

for the stationary potential Φ and the density N given by

$$N = \frac{\mu}{2} \vartheta^{d/2} I_{d/2-1}(e^{\Phi/\vartheta - c}).$$

The constant c is so that the mass constraint

$$\int_\Omega \Delta \Phi \, dx = \int_{\partial\Omega} \frac{\partial \Phi}{\partial \overline{\nu}} \, d\sigma = M \tag{4.2}$$

is satisfied. The equation (4.1) will be called the Poisson–Fermi–Dirac equation. It can be simplified a bit by introducing the new potential $\Psi = -\Phi/\vartheta$, leading to

$$\Delta \Psi + \frac{\mu}{2} \vartheta^{d/2-1} I_{d/2-1}\left(e^{-\Psi - c}\right) = 0 \tag{4.3}$$

with the constant c satisfying the mass constraint

$$\int_{\partial\Omega} \frac{\partial \Psi}{\partial \overline{\nu}} \, d\sigma = -\frac{M}{\vartheta}. \tag{4.4}$$

4.1. **Solutions satisfying the Dirichlet condition (1.10).** The problem (4.3)–(4.4) has the form

$$\Delta U + f(U + c) = 0, \qquad \int_{\partial\Omega} \frac{\partial U}{\partial \overline{\nu}} = -K \tag{4.5}$$

with a given constant K and the homogeneous Dirichlet condition (1.10). An analysis of the problem (4.5) can be found in [25, (2.2)–(2.3)] (see also [2]), where the following results have been proved using variational methods.

Assume that

(i) f is a continuous, nondecreasing function on $\mathbb{R}$ which is increasing whenever $f > 0$, and $\lim_{s\to\infty} f(s) = \infty$,

(ii) $\lim_{s\to\infty} f(s)/s^{p^*} = 0$, where $p^* = d/(d-2)$ if $d \geq 3$ or $p^* < \infty$ is arbitrary if $d = 2$.

Then the problem (4.5) with (1.10) has at least one solution for every $0 < K < \infty$.

Moreover, if $d \geq 3$ and

(iii) $\lim_{s\to\infty} f(s)/s^{p^*} = \kappa > 0$,

then the problem (4.5), (1.10) is solvable for each $0 < K < \widetilde{K}$, with some $\widetilde{K} < \infty$.

Remark. In fact one has $\kappa = 1$ in [25], but a simple scaling leads to the formulation (4.5).

In our case the nonlinearity in (4.5) is

$$f(s) = \frac{\mu}{2}\vartheta^{d/2-1} I_{d/2-1}\left(e^{-s}\right) \sim |s|^{d/2} \ll |s|^{d/(d-2)} \quad \text{if} \quad d \leq 3\,.$$

Thus, we obtain the following existence result:

Proposition 4.1. *For* $d = 1, 2, 3$*, given* $M > 0$ *there exists at least one solution* Φ *of the Poisson–Fermi–Dirac equation* (4.1) *satisfying the Dirichlet condition* (1.10) *and* (4.2).

For $d = 4$ *such a solution exists for all sufficiently small* $M > 0$.

The question of the existence of multiple solutions of the equation (4.1), and their stability as solutions of the evolution problem (1.7)–(1.11), is rather delicate. There are some numerical results in the case of radially symmetric solutions in the ball of $\mathbb{R}^3$ in [10] and [11].

Remark. For the existence of solutions of (4.3), (4.4) with Ψ satisfying either the free condition (1.12) or the Dirichlet condition (1.10) for small $M > 0$ and each $d \geq 3$, we refer the reader to [22]. These results are proved in the spirit of fixed-point theorems based on the compactness properties of the operator $N \mapsto \Psi$. Also, it is shown in [22] (by an application of the Pohozaev identity) that for $d \geq 5$, the equation (4.3) with the boundary condition (1.10) in a star-shaped domain has no solution for sufficiently large $M \gg 1$.

4.2. **Solutions satisfying the free condition (1.12).** An approach for the existence of solutions satisfying the free condition is possible using the (neg)entropy functional W defined by

$$W = \int_\Omega \left(\vartheta H(n) + \frac{1}{2} n\varphi \right) dx. \tag{4.6}$$

Note that for φ satisfying the Poisson equation (3.3) with the homogeneous Dirichlet condition, the functional W coincides with the entropy $\mathcal{W}$ considered in the preceding section. The entropy W satisfies, for all sufficiently regular solutions of the evolution problem constructed in Theorem 3.1, the relation

$$\frac{dW}{dt} = -\int_\Omega nD \left| \nabla \left(\vartheta h(n) + \varphi \right) \right|^2 \, dx \le 0\,. \tag{4.7}$$

Moreover, W is bounded from below by a result similar to Lemma 3.5.

Lemma 4.2. *If $d = 1, 2, 3$, then the entropy W controls from above the $L^{1+2/d}(\Omega)$ norm of n. More precisely, for each $0 < c_1 < \frac{d^{1+2/d}}{d+2} \mu^{-2/d}$ there exists a constant $c_2 = c_2(c_1, \Omega, M)$ such that*

$$W \ge c_1 |n|_{1+2/d}^{1+2/d} - c_2.$$

If $d = 4$ such an estimate is valid only for small mass $M = |n|_1$ since

$$W \ge \left(c_3 - C(\Omega) M^{1/2} \right) |n|_{3/2}^{3/2} - c_4,$$

holds for some $c_3 > 0$, $c_4 > 0$, and $C(\Omega) > 0$.

The proof follows from the idea used in Lemma 3.5, but now one has to use the Poisson equation (1.7) to estimate φ in terms of n. □

Remark. This result on the integrability $n \in L^{1+2/d}(\Omega)$ under the finiteness assumption on W was first obtained by R. Robert in [19] for the radially symmetric functions n, φ, and λ defined in the unit ball of $\mathbb{R}^3$ by a clever, simple argument on the level of kinetic equations.

Minimizers of the entropy W are steady states. Thus we have

Proposition 4.3. *For $d \le 3$ and given $M > 0$ there is a solution of the Poisson–Fermi–Dirac equation* (4.1) *satisfying the free condition* (1.12) *and* (4.2). *For $d = 4$ such a solution exists for all sufficiently small $M > 0$.*

Proof. By Lemma 4.2, we can take a minimizing sequence $n_k \in L^{1+2/d}(\Omega)$ for W such that

$$W(n_k) \to \inf\{W(n) : 0 \le n \in L^{1+2/d}(\Omega), \quad \int_\Omega n \, dx = M\} > -\infty.$$

Again by Lemma 4.2, a subsequence, still denoted by n_k, weakly converges to an element $n_\infty \in L^{1+2/d}(\Omega)$. Now we arrive at

$$\int_\Omega (n_k\varphi_k - n_\infty\varphi_\infty)\,dx = \int_\Omega n_k(\varphi_k - \varphi_\infty)\,dx + \int_\Omega (n_k - n_\infty)\varphi_\infty\,dx \to 0$$

since the sequence of the associated potentials (φ_k) converges in $L^{1+d/2}(\Omega)$, thanks to the compactness of the imbedding of $W^{2,1+2/d}(\Omega)$ into $L^{1+d/2}(\Omega)$ for $d \le 4$. The functional $\int_\Omega H(n)\,dx$ is convex by Lemma 2.1 and thus weakly lower semicontinuous. Therefore

$$\liminf_{k\to\infty} \int_\Omega H(n_k)\,dx \ge \int_\Omega H(n_\infty)\,dx$$

holds and the minimum of W is attained at n_∞. □

Acknowledgments. The preparation of this paper was partially supported by the KBN grants 2/P03A/011/19, 2/P03A/002/24, the POLONIUM project ÉGIDE–KBN 2003–05643SE, and by the EU network HYKE under the contract HPRN-CT-2002-00282. We acknowledge the hospitality of the Juliusz Schauder Center for Nonlinear Studies (Toruń), where we began some calculations related to this subject in November 2002. We are grateful to Pierre-Henri Chavanis and Carole Rosier for interesting and enlightening conversations. We thank Robert Stańczy and Gershon Wolansky for pertinent remarks.

References

[1] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, 9–126 in "Function Spaces, Differential Operators and Nonlinear Analysis," H. Triebel, H.J. Schmeisser, eds., Teubner-Texte Math., 133, Teubner, Stuttgart, 1993.

[2] H. Berestycki and H. Brezis, *On a free boundary problem arising in plasma physics*, Nonlinear Anal., 4 (1980), 415–436.

[3] P. Biler, J. Dolbeault, M.J. Esteban, P.A. Markowich, and T. Nadzieja, *Steady states for Streater's energy-transport models of self-gravitating particles*, 37–56 in "Transport in Transition Regimes," N. Ben Abdallah et al., eds., *IMA Volumes in Mathematics and Its Applications,* 135, Springer, New York, 2003.

[4] P. Biler, W. Hebisch, and T. Nadzieja, *The Debye system: existence and long time behaviour of solutions*, Nonlinear Anal., 23 (1994), 1189–1209.

[5] P. Biler, D. Hilhorst, and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, II*, Colloq. Math., 67 (1994), 297–308.

[6] P. Biler, A. Krzywicki, and T. Nadzieja, *Self-interaction of Brownian particles coupled with thermodynamic processes*, Rep. Math. Phys., 42 (1998), 359–372.

[7] P. Biler and T. Nadzieja, *Existence and nonexistence of solutions for a model of gravitational interaction of particles, I*, Colloq. Math., 66 (1994), 319–334.

[8] P. Biler and T. Nadzieja, *Structure of steady states for Streater's energy-transport models of gravitating particles*, Topol. Methods Nonlinear Anal., 19 (2002), 283–301.
[9] P. Biler and T. Nadzieja, *Global and exploding solutions in a model of self-gravitating systems*, Rep. Math. Phys., 52 (2003), 205–225.
[10] P.-H. Chavanis, *Phase transitions in self-gravitating systems: self-gravitating fermions and hard sphere models*, Phys. Rev. E, 65 (2002), 056123.
[11] P.-H. Chavanis, M. Ribot, C. Rosier, and C. Sire, *Collapses, explosions and hysteresis in self-gravitating systems: a thermodynamical approach*, in preparation.
[12] P.-H. Chavanis, C. Rosier, and C. Sire, *Thermodynamics of self-gravitating systems*, Phys. Rev. E, 66 (2002), 036105.
[13] P.-H. Chavanis and J. Sommeria, *Degenerate equilibrium states of collisionless stellar systems*, Mon. Not. R. Astron. Soc., 296 (1998), 569–578.
[14] P.-H. Chavanis, J. Sommeria, and R. Robert, *Statistical mechanics of two-dimensional vortices and collisionless stellar systems*, Astrophys. J., 471 (1996), 385–399.
[15] P.-H. Chavanis and C. Sire, *Anomalous diffusion and collapse of self-gravitating Langevin particles in D dimensions*, Phys. Rev. E 69 (2004), to appear.
[16] C.J. van Duijn, I.A. Guerra, and M.A. Peletier, *Global existence conditions for a non-local problem arising in statistical mechanics*, 1–21, to appear.
[17] O.A. Ladyženskaja, V.A. Solonnikov, and N.N. Ural'ceva, "Linear and Quasilinear Equations of Parabolic Type," Transl. Math. Monogr., 23, Amer. Math. Soc., Providence, RI, 1968.
[18] D. Lamberton, *Équations d'évolution linéaires associées à des semi-groupes de contraction dans les espaces L^p*, J. Funct. Anal., 72 (1987), 252–262.
[19] R. Robert, *On the gravitational collapse of stellar systems*, Classical Quantum Gravity, 15 (1998), 3827–3840.
[20] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Mat. Pura Appl., 146 (1987), 65–96.
[21] C. Sire and P.-H. Chavanis, *Thermodynamics and collapse of self-gravitating Brownian particles in D dimensions*, Phys. Rev. E, 66 (2002), 046133.
[22] R. Stańczy, *Steady states for a system describing Fermi–Dirac particles*, preprint.
[23] R.F. Streater, *A model of dense fluids*, in "Quantum Probability" (Gdańsk, 1997), 381–393, Banach Center Publications, 43, Polish Acad. Sci, Warsaw, 1998.
[24] R.F. Streater, *The Soret and Dufour effects in statistical dynamics*, Proc. R. Soc. London A, 456 (2000), 205–211.
[25] G. Wolansky, *Critical behaviour of semi-linear elliptic equations with sub-critical exponents*, Nonlinear Anal., 26 (1996), 971–995.