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EXISTENCE OF THE NONTOPOLOGICAL CONDENSATE IN SELF-DUAL CHERN-SIMONS GAUGE THEORY

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Abstract. This note is concerned with the recent paper, "Non-topological N-vortex condensates for the self-dual Chern-Simons theory" by M. Nolasco. Making modifications to her arguments, we show that the existence of "nontopological" multivortex condensates actually follows with any prescribed vortex points.

1. Introduction

In recent years, charged vortex solutions in (2+1)-dimensional abelian Chern-Simons vortex theory have received much attention, because of their relation to many fields of physics such as high-critical-temperature super conductivity, some condensed matter systems, the charged anyon model, and so on ([5], [11]). Mathematical proof of the existence of stationary vortex solutions, called the *vortex condensate,* has been made by [1], [3], [4], [8], [9], and [10] on the periodic cell domain satisfying a suitable gauge-invariant periodicity, that is, the 't Hooft boundary condition.

Let Ω be the fundamental cell domain in $\mathbf{R}^2$ generated by linearly independent vectors $e_1 = (a, 0)$ and $e_2 = (0, b)$,

$$\Omega = \Big\{ x = (x_1, x_2) \in \mathbf{R}^2 : -\frac{a}{2} \le x_1 \le \frac{a}{2}, \quad -\frac{b}{2} \le x_2 \le \frac{b}{2} \Big\},$$

and let $p_1, \ldots, p_s \in \Omega \setminus \partial\Omega$ be s distinct vortex points with multiplicities $m_1, \ldots, m_s \in \mathbf{N}$. Then, after the reduction process of Taubes, existence of Chern-Simons N-vortex condensates is reduced to finding a solution $u = u_\kappa$

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to

$$\begin{cases} -\Delta u = \dfrac{4}{\kappa^2} e^u(1-e^u) - 4\pi \displaystyle\sum_{j=1}^{s} m_j \delta_{p_j} \quad \text{in } \Omega \\ u : \text{doubly periodic on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\kappa > 0$ is the Chern-Simons coupling parameter and $\sum_{j=1}^{s} m_j = N$. From the maximum principle follows that $e^u < 1$ on Ω, while integrating (1.1) over Ω implies

$$\int_\Omega e^u(1-e^u)dx = \pi N \kappa^2.$$

Hence, we obtain $e^{u_\kappa}(1-e^{u_\kappa}) \to 0$ almost everywhere as $\kappa \to 0$ passing through a subsequence. Actually, we are interested in the following cases:

- "topological" N-vortex condensates: $e^{u_\kappa} \to 1$ locally uniformly on $\Omega \setminus \{p_1, \ldots, p_s\}$ as $\kappa \to 0$.
- "non-topological" N-vortex condensates: $e^{u_\kappa} \to 0$ locally uniformly on $\Omega \setminus \{p_1, \ldots, p_s\}$ as $\kappa \to 0$.

The existence of "topological" N-vortex condensates was solved affirmatively in [10]. On the other hand, only partial results have been known concerning "nontopological" N-vortex condensates ([10], [9], [4], [3]).

In a recent paper, M. Nolasco asserted that "nontopological" N-vortex condensate exists for any given vortex points in Ω. Unfortunately, some estimates are not described in detail, and there seem to be several gaps in the argument. In this note, first, we shorten and clarify her arguments in use of the abstract analysis (Section 3). Second, we correct and provide the proof of many technical estimates needed in the main part (Sections 4 and 5). In connection, the main assertion of [7] below is established completely.

Theorem 1. *Given $p_j \in \Omega \setminus \partial\Omega$ and $m_j \in \mathbf{N}$ $(j = 1, \ldots, s)$, we have $\bar{\kappa} > 0$ such that if $\kappa \in (0, \bar{\kappa})$ there is a solution $u = u_\kappa$ to (1.1) satisfying the following:*

(1) $e^{u_\kappa} < 1$ *on* Ω.
(2) $e^{u_\kappa} \to 0$ *in* $C^q_{loc}(\Omega \setminus \{p_1, \ldots, p_s\}$ *for any* $q \geq 0$ *as* $\kappa \to 0$.
(3) $\frac{4}{\kappa^2} e^{u_\kappa}(1-e^{u_\kappa}) \rightharpoonup 4\pi \sum_{j=1}^{s} m_j \delta_{p_j}$ *in the sense of measures on* Ω *as* $\kappa \to 0$.

In spite of the above-mentioned technical improvements, we reproduce some parts of [7] for completeness.

2. Radially symmetric vortex

First, we extend the solution $u = u(x)$ to (1.1) for all of $x \in \mathbf{R^2}$ by the periodicity. That is, $\widetilde{u}(x) = u(x + n_1e_1 + n_2e_2)$, which satisfies that

$$\begin{cases} -\Delta\widetilde{u} = \dfrac{4}{\kappa^2}e^{\widetilde{u}}(1 - e^{\widetilde{u}}) - 4\pi \displaystyle\sum_{n\in\mathbf{Z^2}}\sum_{j=1}^{s} m_j\delta_{p_j^n} & \text{in } \mathbf{R^2} \\ \widetilde{u}(x + e_i) = \widetilde{u}(x) & \text{for } x \in \mathbf{R^2} \text{ and } i = 1, 2, \end{cases} \tag{2.1}$$

where $p_j^n = p_j + n_1e_1 + n_2e_2$ with $(n_1, n_2) \in \mathbf{Z}^2$ denotes the periodic lattice of vortex points for $j = 1, \dots, s$. Next, we introduce the scaling parameters δ, ε in $0 < \delta \ll \varepsilon$ as $\kappa = 2\varepsilon\delta$, and set that $\widehat{u}(x) = \widetilde{u}(\delta x)$ and $\widehat{p}_j^n = \frac{1}{\delta}p_j^n$ for $j = 1, \dots, s$. Then, it follows that

$$\begin{cases} -\Delta\widehat{u} = \dfrac{1}{\varepsilon^2}e^{\widehat{u}}(1 - e^{\widehat{u}}) - 4\pi \displaystyle\sum_{n\in\mathbf{Z^2}}\sum_{j=1}^{s} m_j\delta_{\widehat{p}_j^n} & \text{in } \mathbf{R^2} \\ \widehat{u}(x + \widehat{e}_i) = \widehat{u}(x) \qquad \text{for } x \in \mathbf{R^2} \text{ and } i = 1, 2 \end{cases} \tag{2.2}$$

for $\widehat{e}_i := \frac{1}{\delta}e_i$, because we have $a^2\delta(x) = \delta(\frac{x}{a})$ for any $a > 0$ and $x \in \mathbf{R^2}$. This is the equation that we solve by the implicit-function theorem, taking suitable approximate solutions. It is done by "gluing" radially symmetric vortex solutions of Chae-Imanuvilov [2], and thus, we recall the latter's work for the moment.

Namely, we consider the N-vortex condensate at the origin,

$$\begin{cases} -\Delta u = \dfrac{1}{\varepsilon^2}e^{u}(1 - e^{u}) - 4\pi N\delta_0 & \text{in } \mathbf{R^2} \\ u(x) \to -\infty & \text{as } |x| \to \infty, \end{cases} \tag{2.3}$$

where the case $N = 0$ is allowed. Chae-Imanuvilov [2] constructs a solution to (2.3) as a perturbation from $\log\rho_N$, where ρ_N is the radially symmetric function defined by

$$\rho_N(|x|) = \frac{8(N+1)^2|x|^{2N}}{(1 + |x|^{2N+2})^2}. \tag{2.4}$$

Actually, it is a solution to the Liouville equation

$$-\Delta\log\rho_N = \rho_N - 4\pi N\delta_0 \quad \text{in } \mathbf{R^2}. \tag{2.5}$$

Next, we introduce the auxiliary function $w = w_N(|x|) \in C^2(\mathbf{R^2})$ by

$$-\Delta w = \rho_N(x)w - \rho_N(x)^2 \quad \text{in } \mathbf{R^2}, \tag{2.6}$$

and make a change of variables in (2.3):

$$u(|x|) = \log(\varepsilon^2\rho_N(|x|)) + \varepsilon^2 w_N(|x|) + \varepsilon^2 v(|x|). \tag{2.7}$$

Then, this new unknown $v = v(|x|)$ has to satisfy

$$-\Delta v = \frac{1}{\varepsilon^2}\rho_N e^{\varepsilon^2(v+w_N)} - \rho_N^2 e^{2\varepsilon^2(v+w_N)} - \frac{1}{\varepsilon^2}\rho_N + \Delta w_N \tag{2.8}$$

in $\mathbf{R^2}$. Now, we take Hilbert spaces

$$\begin{aligned} X &= \Big\{u \in W^{2,2}_{loc}(\mathbf{R^2}) : \|u\|_X^2 = (u,u)_X < +\infty\Big\} \\ Y &= \Big\{u \in L^2(\mathbf{R^2}) : \|u\|_Y^2 = (u,u)_Y < +\infty\Big\} \end{aligned} \tag{2.9}$$

with the inner products $(\ ,\ )_X$, $(\ ,\ )_Y$ defined by

$$\begin{aligned} (u,v)_Y &= \int_{\mathbf{R}^2} \big(1+|x|^{2+\alpha}\big) uv dx \\ (u,v)_X &= (\Delta u, \Delta v)_Y + \int_{\mathbf{R}^2} uv\big(1+|x|^{2+\alpha}\big)^{-1} dx \end{aligned} \tag{2.10}$$

for $\alpha \in (0, \frac{1}{2})$. Further, X^r and Y^r denote the spaces of radially symmetric functions in X and Y, respectively. Then, we have the following (Lemmas 1.1 and 2.2 of [2]).

Lemma 1. *It holds that*

$$\big|v(x)\big| \le \big\|v\big\|_X \big(\log^+ \big|x\big| + 1\big) \tag{2.11}$$

for $v \in X$ *and* $x \in \mathbf{R}^2$.

Lemma 2. *We have* $C, \tilde{C} > 0$ *such that*

$$\begin{aligned} \big|w_N\big(\big|x\big|\big)\big| &\le C\big(\log^+ \big|x\big| + 1\big) \qquad \textit{for all } x \in \mathbf{R^2} \\ w_N\big(\big|x\big|\big) &= -\widetilde{C}\log^+ \big|x\big| + o\big(\log\big|x\big|\big) \qquad \textit{as} \quad \big|x\big| \to \infty. \end{aligned} \tag{2.12}$$

By using those lemmas, we can realize

$$P_N(v,\varepsilon) = \Delta v + \frac{1}{\varepsilon^2}\rho_N e^{\varepsilon^2(v+w_N)} - \rho_N^2 e^{2\varepsilon^2(v+w_N)} - \frac{1}{\varepsilon^2}\rho_N - \rho_N w_N + \rho_N^2 \tag{2.13}$$

as a smooth mapping from a bounded neighborhood of $(0,0)$ in $X^r \times \mathbf{R}$ into Y^r. Furthermore, $v \in X^r$ is a solution to (2.8) for $\varepsilon > 0$ if and only if $P_N(v,\varepsilon) = 0$, and we have $P_N(0,0) = \lim_{\varepsilon\downarrow 0} P_N(0,\varepsilon) = 0$ by the choice of w_N.

To find continuous $\varepsilon \mapsto v_\varepsilon \in X^r$ in a neighborhood of $(0,0)$ satisfying $P_N(v_\varepsilon, \varepsilon) = 0$, we take the bounded linear operator

$$L_N^r = D_v P_N(0,0) = \Delta + \rho_N : X^r \to Y^r.$$

It is proven in [2] that L^r_N is onto and $\mathrm{Ker}(L^r_N)$ is equal to $\mathrm{span}\{\phi_N\}$ for $\phi_N(|x|) = (1-|x|^{2N+2})/(1+|x|^{2N+2})$. Therefore, taking

$$H^r_N = \{u \in X^r : (u, \phi_N)_X = 0\} = X^r/\mathrm{Ker}L^r_N,$$

we can apply the standard implicit-function theorem and obtain $v^*_{\varepsilon,N} \in H^r_N$ in a neighborhood of the origin, satisfying $P_N(v^*_{\varepsilon,N}, \varepsilon) = 0$ for $0 < \varepsilon \ll 1$. We can check that $v^*_{\varepsilon,N}$ is a smooth function, and by (2.11) it holds that

$$|v^*_{\varepsilon,N}(|x|)| \leq C(\varepsilon)(\log^+|x| + 1) \tag{2.14}$$

with $C(\varepsilon) = \|v^*_{\varepsilon,N}\|_X \to 0$ as $\varepsilon \downarrow 0$. Thus,

$$u^*_{\varepsilon,N}(|x|) = \log\left(\varepsilon^2\rho_N(|x|)\right) + \varepsilon^2 w_N(|x|) + \varepsilon^2 v^*_{\varepsilon,N}(|x|) \tag{2.15}$$

is a solution to (2.3) satisfying

$$e^{u^*_{\varepsilon,N}} = O(|x|^{-2N-4-\beta(\varepsilon)}) \tag{2.16}$$

as $|x| \to \infty$ for some $\beta(\varepsilon) > 0$ in $\lim_{\varepsilon\downarrow 0}\beta(\varepsilon) = 0$, and hence is a "nontopological" solution.

3. Linearization

The process of gluing requires fine analysis to the linearized operator around the Chae-Imanuvilov solution, namely, $A_{\varepsilon,N} = D_v P_N(v^*_{\varepsilon,N}, \varepsilon) : X \to Y$. Although the proof of this part is not described in detail in [7], we can justify the statement by using the perturbation theory for Fredholm operators. Actually, this operator is given as

$$A_{\varepsilon,N} = \Delta + \rho_N e^{\varepsilon^2(w_N + v^*_{\varepsilon,N})} - 2\varepsilon^2\rho_N^2 e^{2\varepsilon^2(w_N+v^*_{\varepsilon,N})}. \tag{3.1}$$

Because $e^{\varepsilon^2(w_N+v^*_{\varepsilon,N})} = 1 + \varepsilon^2 w_N + o(\varepsilon^2)$ holds by (2.12) and (2.14), we have

$$A_{\varepsilon,N} = L_N + \varepsilon^2 B_N + o(\varepsilon^2) \tag{3.2}$$

in the operator norm, where

$$L_N = \Delta + \rho_N \tag{3.3}$$

and

$$B_N = \rho_N w_N - 2\rho_N^2. \tag{3.4}$$

Now, we recall the following (Lemma 2.4, Proposition 2.2, and Lemma 2.5 of [2]).

Lemma 3. *Ker* $L_N =$ *span* $\left\{\phi_N, \phi_N^+, \phi_N^-\right\}$ *for*

$$\phi_N(x) = \frac{1-|x|^{2N+2}}{1+|x|^{2N+2}}, \qquad \phi_N^+(x) = \frac{|x|^{N+1}\cos(N+1)\theta}{1+|x|^{2N+2}}$$
$$\phi_N^-(x) = \frac{|x|^{N+1}\sin(N+1)\theta}{1+|x|^{2N+2}}. \tag{3.5}$$

Lemma 4.

$$Im\ L_N = \left\{f \in Y : \int_{\mathbf{R}^2} f\phi_N^\pm dx = 0\right\}.$$

Lemma 5.

$$I_N^\pm \equiv \left(B_N\phi_N^\pm, \phi_N^\pm\right)_{L^2(\mathbf{R^2})} \begin{cases} <0 & (N=1,2,\dots) \\ =0 & (N=0) \end{cases}$$

We also make use of $\left(B_N\phi_N^+, \phi_N^-\right)_{L^2(\mathbf{R^2})} = \left(B_N\phi_N, \phi_N^\pm\right)_{L^2(\mathbf{R^2})} = 0$ valid for $N \in \mathbf{N}\cup\{0\}$. Furthermore, we have $\phi_N^\pm \in H_N$ for

$$H_N = \{u \in X : (u, \phi_N)_X = 0\}, \tag{3.6}$$

and the orthogonal decomposition $X = H_N \oplus H_N^\perp$ with $H_N^\perp = \text{span}\{\phi_N\}$.

The following lemma assures the injectivity of $A_{\varepsilon,N}\big|_{H_N} : H_N \to Y$, and we provide the detailed proof for completeness. Actually, it simplifies the original and justifies Lemma 4.2 of [7]. Let us note that (4.18) of [7] does not hold for the case of $N = 0$.

Lemma 6. *If* $N \geq 1$*, we have* $\varepsilon_0 > 0$ *and* $C > 0$ *such that*

$$\left\|A_{\varepsilon,N}v\right\|_Y \geq C\varepsilon^2\left\|v\right\|_X$$

for any $\varepsilon \in (0, \varepsilon_0)$ *and* $v \in H_N$*.*

Proof. If this is not the case, we have $\varepsilon_n \downarrow 0$ and $v_n \in H_N$ with $\left\|v_n\right\|_X = 1$ and $\varepsilon_n^{-2}\left\|A_{\varepsilon,N}v_n\right\|_Y \to 0$. Then, we can extract a subsequence, denoted by the same symbol, satisfying $v_n \rightharpoonup \bar{v}$ weakly in X for some $\bar{v} \in H_N$. This implies $A_{\varepsilon_n,N}v_n \rightharpoonup L_N\bar{v}$ weakly in Y, and hence $L_N\bar{v} = 0$ follows from $\left\|A_{\varepsilon_n,N}v_n\right\| = o\left(\varepsilon_n^2\right)$. Namely, we have $\bar{v} \in \text{Ker}L_N \cap H_N$, and hence

$$\bar{v} = C_+\phi_N^+ + C_-\phi_N^-$$

holds with some $C_+, C_- \in \mathbf{R}$ by Lemma 3. Now, we claim that $v_n \to \bar{v}$ strongly in X, and therefore, that $\bar{v} \not\equiv 0$ or equivalently, $C_+^2 + C_-^2 \neq 0$.

In fact, by (3.2) and our assumption, we have $\left\|L_N(v_n - \bar{v})\right\|_Y = o(1)$, which leads to

$$\left\|\Delta(v_n - \bar{v})\right\|_Y \leq \left\|\rho_N(v_n - \bar{v})\right\|_Y + o(1)$$

as $n \to \infty$. On the other hand, Rellich-Kondrachov's theorem and the growth estimate (2.11) guarantee the compactness of $K_1 = \rho_N : X \to Y$, and therefore, $\Delta v_n \to \Delta \bar{v}$ strongly in Y. On the other hand, $K_2 = (1 + \left|x\right|^{2+\alpha})^{-1/2} : X \to L^2(\mathbf{R^2})$ is also compact, and then we conclude that $v_n \to \bar{v}$ strongly in X as is desired.

Now, we show that this is a contradiction in the case of $N \neq 0$. In fact, for $\bar{w} = \frac{\bar{v}}{1+\left|x\right|^{2+\alpha}} \in Y$ it holds that

$$\frac{1}{\varepsilon_n^2}\left|\left(A_{\varepsilon_n,N} v_n, \bar{w}\right)_Y\right| \leq \frac{1}{\varepsilon_n^2}\left\|A_{\varepsilon_n,N} v_n\right\|_Y \left\|\bar{w}\right\|_Y \to 0,$$

while (3.2) implies

$$\frac{1}{\varepsilon_n^2}\left(A_{\varepsilon_n,N} v_n, \bar{w}\right)_Y = \frac{1}{\varepsilon_n^2}\left(L_N v_n, \bar{w}\right)_Y + \left(B_N v_n, \bar{w}\right)_Y + o(1).$$

We also have from Lemma 4 that

$$\frac{1}{\varepsilon_n^2}\left(L_N v_n, \bar{w}\right)_Y = \frac{1}{\varepsilon_n^2}\left(L_N v_n, C_+\phi_N^+ + C_-\phi_N^-\right)_{L^2(\mathbf{R^2})} = 0,$$

and hence it follows that $\left(B_N v_n, \bar{w}\right)_Y = o(1)$. Therefore, because $v_n \to \bar{v}$ strongly in X, we have

$$\left(B_N \bar{v}, \bar{w}\right)_Y = \left(B_N \bar{v}, \bar{v}\right)_{L^2(\mathbf{R^2})} = 0.$$

However, it holds that

$$\begin{aligned}\left(B_N \bar{v}, \bar{v}\right)_{L^2(\mathbf{R^2})} &= C_+^2\left(B_N \phi_N^+, \phi_N^+\right)_{L^2(\mathbf{R}^2)} + C_-^2\left(B_N \phi_N^-, \phi_N^-\right)_{L^2(\mathbf{R}^2)} \\ &+2C_+C_-(B_N\phi_{+,N}, \phi_{-,N})_{L^2(\mathbf{R}^2)} = C_+^2 I_N^+ + C_-^2 I_N^- < 0\end{aligned}$$

by $N \neq 0$ and Lemma 5, and this contradiction proves the lemma in consideration. □

It is asserted in [7] that $\mathrm{Im}(A_{\varepsilon,N}\big|_{H_N})$ is closed in Y and $A_{\varepsilon,N}|_{H_N}$ is surjective if $\varepsilon > 0$ is sufficiently small (Lemmas 4.3 and 4.4), and therefore, $A_{\varepsilon,N}\big|_{H_N} : H_N \to X$ has a bounded inverse (Lemma 4.5). But for example, (4.29) of [7] does not follow directly from the proof there, because H_N is not contained in $L^2(\mathbf{R}^2)$ densely. Actually, H_N is the Hilbert space associated with L_N, and we can justify this part by using perturbation theory of Fredholm operators in the case of $N \geq 1$. (If $N = 0$, then a radially symmetric solution on the whole space actually exists, but its linearized operator is not invertible. See [2].) Let us recall that a bounded linear operator $T : E \to F$, between Banach spaces E and F, is *Fredholm* if $\mathrm{Ker}(T)$ is of finite dimension in E, and $\mathrm{Im}(T)$ is closed and has a finite codimension in F, and that its *index* is defined by $\mathrm{Index}(T) = \dim \mathrm{Ker}(T) - \mathrm{codim}\, \mathrm{Im}(T)$. Then, we can make use of the following abstract theorem of Gohberg and Krein ([6]):

Theorem 2. *Let E and F be Banach spaces, and assume that the bounded linear operator $T : E \to F$ is Fredholm. Then, there is $\gamma > 0$ such that if $B : E \to F$ is a bounded linear operator with $\|B\| < \gamma$, then $T + B$ is also Fredholm, and it holds that*

$$\textit{dim Ker } (T+B) \le \textit{dim Ker } (T), \quad \textit{codim Im } (T+B) \le \textit{codimIm } (T)$$
$$\textit{Index } (T+B) = \textit{Index } (T).$$

Actually, $B_{\varepsilon.N} \equiv A_{\varepsilon,N} - L_N : X \to Y$ is a bounded linear operator satisfying $\lim_{\varepsilon \downarrow 0} \|B_{\varepsilon,N}\| = 0$ by (3.2). On the other hand, we have for $L_N|_{H_N} : H_N \to Y$ the following.

- $\mathrm{Ker}(L_N\big|_{H_N}) = \mathrm{Ker}\big(L_N\big) \cap H_N = \mathrm{span}\, \{\phi_N^+, \phi_N^-\}$, and hence
$$\dim \mathrm{Ker}\ (L_N\big|_{H_N}) = 2$$
by Lemma 3.
- We have
$$L_N(X) = L_N(H_N \oplus H_N^\perp) = L_N\big(H_N \oplus \mathrm{span}\, \{\phi_N\}\big) = L_N(H_N) \oplus \{0\},$$
and hence it holds that
$$\mathrm{Im}(L_N\big|_{H_N}) = \mathrm{Im}(L_N) = \{f \in Y : \big(f, \phi_N^\pm\big)_{L^2(\mathbf{R^2})} = 0\}$$
by Lemma 4. In particular, $\mathrm{Im}(L_N\big|_{H_N})$ is closed in Y and
$$\mathrm{codim}\ \mathrm{Im}(L_N\big|_{H_N}) = 2.$$

Those facts guarantee that $L_N\big|_{H_N} : H_N \to Y$ is a Fredholm operator of index 0, and we can apply Theorem 2 for $E = H_N$, $F = Y$, $T = L_N\big|_{H_N}$, and $B = B_{\varepsilon,N}\big|_{H_N}$. Then, we conclude that

$$A_{\varepsilon,N}|_{H_N} = L_N\big|_{H_N} + B_{\varepsilon,N}\big|_{H_N}$$

is also a Fredholm operator of index 0 for $0 < \varepsilon \ll 1$. Because Lemma 6 guarantees the injectivity of $A_{\varepsilon,N}\big|_{H_N} : H_N \to Y$ for $N \ge 1$ and $0 < \varepsilon \ll 1$, we get that $A_{\varepsilon,N}\big|_{H_N}$ is also surjective then. Now, we conclude the following.

Lemma 7. *If $N \ge 1$, we have $\varepsilon_0 > 0$ such that $A_{\varepsilon,N}|_{H_N} : H_N \to Y$ is invertible for $\varepsilon \in (0, \varepsilon_0)$. Furthermore, there is $C > 0$ independent of ε such that*

$$\big\|\big(A_{\varepsilon,N}\big|_{H_N}\big)^{-1} u\big\|_X \le \frac{C}{\varepsilon^2} \|u\|_Y$$

for any $\varepsilon \in (0, \varepsilon_0)$ *and* $u \in Y$.

4. Gluing

Nolasco [7] constructed the approximate solution to (2.2) by "gluing" radially symmetric, single-vortex, entire solutions to (2.5), by using a partition of unity subordinate. More precisely, setting a false vortex point $p_0 = 0$ with multiplicity $m_0 = 0$, it studied the invertibility of the linearized operator. However, some modifications seem to be needed in this process. For example, $(\widehat{A}^n_{\varepsilon,0})^{-1}$ indicated in (6.36) of [7] does not exist.

To this end, we suppose that the multiplicity of p_1 is equal to

$$m_1 = \min_{j=1,\ldots,s} \{m_j\}$$

without loss of generality, and letting $r = \frac{1}{2} \min_{i\neq j} \left\{ \left|p_i - p_j\right|, \operatorname{dist}\left(p_j, \partial\Omega\right) \right\}$, we put that

$$B_j = \left\{ x \in \Omega : \left| x - p_j \right| < r \right\}, \quad j = 2, \ldots, s$$
$$B_1 = \left\{ x \in \mathbf{R^2} : \operatorname{dist}\big(x, \Omega \setminus \bigcup_{j=2,\ldots,s} B_j\big) < \frac{r}{2} \right\}.$$

Here, we take the agreement that $B_j = \emptyset$ $(j \geq 2)$ if $s = 1$. Let us note that $B_i \cap B_j = \phi$ holds for $i, j \in \{2, \ldots, s\}$, $i \neq j$, and $B_1 \cap B_j \neq \phi$ for $j = 2, \ldots, s$. By the definition, each B_j contains exactly one vortex point p_j with the multiplicity m_j. This setting of ours is different from that of [7], where B_0 prepared there contains no vortex points at all. In fact, it seems that the original proof of [7] does not work because of this B_0.

Given $n = (n_1, n_2) \in \mathbf{Z^2}$ and $j = 1, \ldots, s$, we set $B^n_j = B_j + n_1 e_1 + n_2 e_2$. Then, the collection $\{B^n_j\}_{n\in\mathbf{Z^2}, j=1,\ldots,s}$ forms a locally finite, periodic, open covering of $\mathbf{R^2}$, and therefore, we can take an associated partition of unity $\{\varphi^n_j\}_{n\in\mathbf{Z^2}, j=1,\ldots,s}$ such that $\varphi^n_j \in C^\infty_c(B^n_j)$, $0 \leq \varphi^n_j \leq 1$, and

$$\sum_{n\in\mathbf{Z^2}} \sum_{j=1}^{s} \varphi^n_j(x) = 1 \qquad \big(x \in \mathbf{R^2}\big).$$

Now, we recall that the scaling parameters $0 < \delta \ll \varepsilon$ are taken so that $\kappa = 2\varepsilon\delta$. Then, letting

$$\widehat{B}^n_j = \frac{1}{\delta} B^n_j, \qquad \widehat{\varphi}^n_j(x) = \varphi^n_j(\delta x)$$

for $n \in \mathbf{Z^2}$ and $j = 1, \ldots, s$, we have supp $\widehat{\varphi}_j^n \subset \widehat{B}_j^n$, $0 \le \widehat{\varphi}_j^n \le 1$, and

$$\sum_{n \in \mathbf{Z^2}} \sum_{j=1}^{s} \widehat{\varphi}_j^n(x) = 1 \qquad \bigl(x \in \mathbf{R^2}\bigr). \tag{4.1}$$

That is, $\{\widehat{\varphi}_j^n\}_{n,j}$ is a partition of unity subordinate to the locally finite periodic covering $\{\widehat{B}_j^n\}_{n,j}$ of $\mathbf{R^2}$. We also have

$$\sup_{x \in \widehat{B}_j^n} \bigl|\nabla \widehat{\varphi}_j^n\bigr| \le C\delta, \qquad \sup_{x \in \widehat{B}_j^n} \bigl|\Delta \widehat{\varphi}_j^n\bigr| \le C\delta^2 \tag{4.2}$$

and

$$\begin{array}{ll} \widehat{B}_i^n \cap \widehat{B}_j^n = \phi, & i, j \in \{2, \ldots, s\},\ i \ne j \ \ (\text{if } s \ge 2) \\ \widehat{B}_1^n \cap \widehat{B}_j^n \ne \phi, & j = 2, \ldots, s \ \ (\text{if } s \ge 2) \\ \widehat{B}_i^n \cap \widehat{B}_j^k = \phi, & k \ne n \text{ and } k \notin \langle n \rangle, i, j = 1, \ldots, s \\ \widehat{B}_1^n \cap \widehat{B}_1^k \ne \phi, & k \in \langle n \rangle, \end{array} \tag{4.3}$$

where $\langle n \rangle = \bigl\{k \in \mathbf{Z^2} : \bigl|n - k\bigr| = 1\bigr\}$ denotes the nearest neighborhood of $n \in \mathbf{Z^2}$. We put

$$\widehat{\rho}_j^n(x) = \rho_{m_j}\bigl(\bigl|x - \widehat{p}_j^n\bigr|\bigr) = \frac{8\bigl(m_j + 1\bigr)^2 |x - \widehat{p}_j^n|^{2m_j}}{(1 + |x - \widehat{p}_j^n|^{2m_j+2})^2} \tag{4.4}$$

and $\widehat{w}_j^n(x) = w_{m_j}(|x - \widehat{p}_j^n|)$, where ρ_N and w_N stand for the functions defined by (2.4) and (2.6) for $N \in \mathbf{N} \cup \{0\}$, respectively. Further, we set that

$$\widehat{\phi_j^n}(x) = \phi_{m_j}\bigl(\bigl|x - \widehat{p}_j^n\bigr|\bigr) = \frac{1 - |x - \widehat{p}_j^n|^{2m_j+2}}{1 + |x - \widehat{p}_j^n|^{2m_j+2}}. \tag{4.5}$$

We introduce Hilbert spaces

$$\begin{aligned} \widehat{X}_j^n &= \Bigl\{u \in W_{loc}^{2,2}(\mathbf{R^2}) : \bigl\|u\bigr\|_{\widehat{X}_j^n}^2 = (u, u)_{\widehat{X}_j^n} < +\infty\Bigr\} \\ \widehat{Y}_j^n &= \Bigl\{u \in L^2(\mathbf{R^2}) : \bigl\|u\bigr\|_{\widehat{Y}_j^n}^2 = (u, u)_{\widehat{Y}_j^n} < +\infty\Bigr\} \\ \widehat{H}_j^n &= \Bigl\{u \in \widehat{X}_j^n : (u, \widehat{\phi_j^n})_{\widehat{X}_j^n} = 0\Bigr\} \end{aligned} \tag{4.6}$$

with the inner products $(\ ,\ )_{\widehat{X}_j^n}$ and $(\ ,\ )_{\widehat{Y}_j^n}$ defined by

$$\begin{aligned} (u, v)_{\widehat{Y}_j^n} &= \int_{\mathbf{R}^2} \bigl(1 + \bigl(\delta\bigl|x - \widehat{p}_j^n\bigr|\bigr)^{2+\alpha}\bigr) uv\delta^2 dx \\ (u, v)_{\widehat{X}_j^n} &= \bigl(\Delta u, \Delta v\bigr)_{\widehat{Y}_j^n} + \int_{\mathbf{R}^2} \frac{uv\delta^2}{1 + (\delta|x - \widehat{p}_j^n|)^{2+\alpha}} dx. \end{aligned} \tag{4.7}$$

We also put that

$$\widehat{H}_\delta = \Big\{ u \in W^{2,2}_{loc}(\mathbf{R}^2) : \widehat{\varphi}^n_j u \in \widehat{H}^n_j \text{ for any } n, j \text{ and } \|u\|_{\widehat{H}_\delta} < +\infty \Big\}$$
$$\widehat{Y}_\delta = \Big\{ u \in L^2_{loc}(\mathbf{R}^2) : \widehat{\varphi}^n_j u \in \widehat{Y}^n_j \text{ for any } n, j \text{ and } \|u\|_{\widehat{Y}_\delta} < +\infty \Big\}, \quad (4.8)$$

where

$$\|u\|_{\widehat{H}_\delta} = \sup_{n,j} \|\widehat{\varphi}^n_j u\|_{\widehat{X}^n_j}, \qquad \|u\|_{\widehat{Y}_\delta} = \sup_{n,j} \|\widehat{\varphi}^n_j u\|_{\widehat{Y}^n_j}.$$

Finally, we take

$$\widehat{v}^n_{\varepsilon,j}(x) = v^*_{\varepsilon,m_j}\big(|x - \widehat{p}^n_j|\big) \quad (4.9)$$

for $v^*_{\varepsilon,N} = v^*_{\varepsilon,N}\big(|x|\big)$ constructed in Section 2, which solves that $P_N(v,\varepsilon) = 0$ for $0 < \varepsilon \ll 1$. This means that $P^n_j(\widehat{v}^n_{\varepsilon,j}, \varepsilon) = 0$ for

$$P^n_j(v,\varepsilon) = \Delta v + \frac{1}{\varepsilon^2}\widehat{\rho}^n_j e^{\varepsilon^2(v+\widehat{w}^n_j)} - (\widehat{\rho}^n_j)^2 e^{2\varepsilon^2(v+\widehat{w}^n_j)} - \frac{1}{\varepsilon^2}\widehat{\rho}^n_j - \widehat{\rho}^n_j\widehat{w}^n_j + (\widehat{\rho}^n_j)^2, \quad (4.10)$$

and the linearized operator $\widehat{A}^n_{\varepsilon,j} = D_v P^n_j(\widehat{v}^n_{\varepsilon,j}, \varepsilon) : \widehat{H}^n_j \to \widehat{Y}^n_j$, defined by

$$\widehat{A}^n_{\varepsilon,j} = \Delta + \widehat{\rho}^n_j e^{\varepsilon^2\left(\widehat{w}^n_j + \widehat{v}^n_{\varepsilon,j}\right)} - 2\varepsilon^2(\widehat{\rho}^n_j)^2 e^{2\varepsilon^2\left(\widehat{w}^n_j + \widehat{v}^n_{\varepsilon,j}\right)}, \quad (4.11)$$

is subject to Lemma 7. Namely, we have the following lemma. (See Proposition 5.2 of [7].) It will become clear from this lemma that sum of radial vortex solutions with vortex points $\widehat{p}^n_j$ $(j = 1, 2, \ldots, s)$ is a good approximate solution to (2.2).

Lemma 8. *There exist $\varepsilon_0 > 0$ and $C > 0$ independent of $0 < \delta \ll 1$ such that $\widehat{A}^n_{\varepsilon,j}\big|_{\widehat{H}^n_j} : \widehat{H}^n_j \to \widehat{Y}^n_j$ is invertible and satisfies that*

$$\big\|\big(\widehat{A}^n_{\varepsilon,j}\big|_{\widehat{H}^n_j}\big)^{-1} u\big\|_{\widehat{X}^n_j} \le \frac{C}{\varepsilon^2}\|u\|_{\widehat{Y}^n_j}$$

for any $\varepsilon \in (0, \varepsilon_0)$ and $u \in \widehat{Y}^n_j$.

Now, we take $z = z(x)$ by

$$\widehat{u}(x) = \sum_{n\in\mathbf{Z}^2}\sum_{j=1}^{s} \widehat{\varphi}^n_j(x)\widehat{u}^n_j(x) + \varepsilon^2 z(x) \quad (4.12)$$

in (2.2), where

$$\widehat{u}^n_j(x) = \log\big(\varepsilon^2\widehat{\rho}^n_j(x)\big) + \varepsilon^2\widehat{w}^n_j(x) + \varepsilon^2\widehat{v}^n_{\varepsilon,j}(x) \quad (4.13)$$

is a solution to

$$-\Delta\widehat{u}^n_j = \frac{1}{\varepsilon^2} e^{\widehat{u}^n_j}(1 - e^{\widehat{u}^n_j}) - 4\pi m_j \delta_{\widehat{p}^n_j} \quad (4.14)$$

in $\mathbf{R^2}$. Then, (2.2) is reduced to finding $z = z(x)$ satisfying

$$\begin{aligned}\Delta z + \frac{1}{\varepsilon^2}\sum_{n,j}\big\{(\Delta\widehat{\varphi}_j^n)\widehat{u}_j^n + 2\nabla\widehat{\varphi}_j^n\cdot\nabla\widehat{u}_j^n\big\}\\ +\frac{1}{\varepsilon^4}\big(\sum_{n,j}\widehat{\varphi}_j^n e^{2\widehat{u}_j^n} - \exp(2\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n)\exp(2\varepsilon^2 z)\big)\\ -\frac{1}{\varepsilon^4}\big(\sum_{n,j}\widehat{\varphi}_j^n e^{\widehat{u}_j^n} - \exp(\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n)\exp(\varepsilon^2 z)\big) = 0 \qquad (4.15)\end{aligned}$$

in $\mathbf{R^2}$, because $\sum_{n,j}\widehat{\varphi}_j^n m_j\delta_{\widehat{p}_j^n} - \sum_{n,j} m_j\delta_{\widehat{p}_j^n} \equiv 0$ holds true. In the rest of this section, we shall represent (4.15) in a simple form.

First, we take that

$$C_j^n(x) = (\Delta\widehat{\varphi}_j^n)\widehat{u}_j^n + 2\nabla\widehat{\varphi}_j^n\cdot\nabla\widehat{u}_j^n \qquad (4.16)$$

for $n \in \mathbf{Z^2}$, $j = 1,\dots,s$. Next, we put that

$$\begin{aligned}R_1^n(x) &\equiv \exp\big\{\sum_{l=1}^{s}\widehat{\varphi}_l^n(\widehat{u}_l^n - \widehat{u}_1^n)\big\}\times\exp\big\{\sum_{k\in\langle n\rangle}\widehat{\varphi}_1^k(\widehat{u}_1^k - \widehat{u}_1^n)\big\}\\ &= \exp\big\{\sum_{l=1}^{s}\widehat{\varphi}_l^n\big(\log\frac{\widehat{\rho}_l^n}{\widehat{\rho}_1^n} + \varepsilon^2(\widehat{w}_l^n - \widehat{w}_1^n) + \varepsilon^2(\widehat{v}_{\varepsilon,l}^n - \widehat{v}_{\varepsilon,1}^n)\big)\big\}\\ &\quad\cdot\exp\big\{\sum_{k\in<n>}\widehat{\varphi}_1^k\big(\log\frac{\widehat{\rho}_1^k}{\widehat{\rho}_1^n} + \varepsilon^2(\widehat{w}_1^k - \widehat{w}_1^n) + \varepsilon^2(\widehat{v}_{\varepsilon,1}^k - \widehat{v}_{\varepsilon,1}^n)\big)\big\}\\ R_j^n(x) &\equiv \exp\big\{\widehat{\varphi}_1^n(\widehat{u}_1^n - \widehat{u}_j^n)\big\}\\ &= \exp\big\{\widehat{\varphi}_1^n\big(\log\frac{\widehat{\rho}_1^n}{\widehat{\rho}_j^n} + \varepsilon^2(\widehat{w}_1^n - \widehat{w}_j^n) + \varepsilon^2(\widehat{v}_{\varepsilon,1}^n - \widehat{v}_{\varepsilon,j}^n)\big)\big\}, \qquad (4.17)\end{aligned}$$

for $j = 2,\dots,s$ if $s \geq 2$. Then, we obtain

$$\begin{aligned}R_1^n(x) = 1 \qquad &\text{on} \quad \big(\bigcup_{l=2}^{s}\widehat{B}_l^n \cup \bigcup_{k\in\langle n\rangle}\widehat{B}_1^k\big)^c\\ R_j^n(x) = 1 \qquad &\text{on} \quad \big(\widehat{B}_1^n\big)^c, \quad j = 2,\dots,s.\end{aligned}$$

On the other hand, we have from (4.3) that

$$\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n = \begin{cases}\widehat{\varphi}_1^n\widehat{u}_1^n + \widehat{\varphi}_j^n\widehat{u}_j^n & \big(x\in\widehat{B}_j^n, j = 2,\dots,s\big)\\ \sum_{l=1}^{s}\widehat{\varphi}_l^n\widehat{u}_l^n + \sum_{k\in\langle n\rangle}\widehat{\varphi}_1^k\widehat{u}_1^k & \big(x\in\widehat{B}_1^n\big).\end{cases} \qquad (4.18)$$

Here, we confirm that

$$\exp\big(\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n\big)=R_j^n(x)e^{\widehat{u}_j^n(x)}=\varepsilon^2\widehat{\rho}_j^n e^{\varepsilon^2(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n)}R_j^n\quad\text{on}\quad\widehat{B}_j^n \tag{4.19}$$

holds by using (4.17) and (4.18), where $n\in\mathbf{Z^2}$, $j=1,\dots,s$. In fact, in the case of $x\in\widehat{B}_j^n$ for $j\neq 1$, we have $\widehat{\varphi}_1^n(x)+\widehat{\varphi}_j^n(x)=1$, and hence it follows that

$$\exp\big(\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n\big)=e^{\widehat{\varphi}_1^n\widehat{u}_1^n+\widehat{\varphi}_j^n\widehat{u}_j^n}=e^{\widehat{\varphi}_1^n\widehat{u}_1^n-\widehat{\varphi}_1^n\widehat{u}_j^n}\cdot e^{(\widehat{\varphi}_1^n+\widehat{\varphi}_j^n)\widehat{u}_j^n}=R_j^n(x)e^{\widehat{u}_j^n}.$$

For $x\in\widehat{B}_1^n$, we have $\sum_{l=1}^s\widehat{\varphi}_l^n(x)+\sum_{k\in\langle n\rangle}\widehat{\varphi}_1^k(x)=1$, and therefore,

$$\begin{aligned}\exp\big(\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n\big)&=\exp\big(\sum_{l=1}^s\widehat{\varphi}_l^n\widehat{u}_l^n+\sum_{k\in<n>}\widehat{\varphi}_1^k\widehat{u}_1^k\big)\\&=\exp\big(\sum_{l=1}^s\widehat{\varphi}_l^n(\widehat{u}_l^n-\widehat{u}_1^n)+\sum_{k\in<n>}\widehat{\varphi}_1^k(\widehat{u}_1^k-\widehat{u}_1^n)\big)\\&\cdot\exp\big(\big(\sum_{l=1}^s\widehat{\varphi}_l^n+\sum_{k\in<n>}\widehat{\varphi}_1^k\big)\widehat{u}_1^n\big)=R_1^n(x)e^{\widehat{u}_1^n}\end{aligned}$$

holds true. This proves (4.19).

By (4.19), we have

$$\exp\big(\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n\big)=\sum_{n,j}\widehat{\varphi}_j^nR_j^ne^{\widehat{u}_j^n},\qquad\exp\big(2\sum_{n,j}\widehat{\varphi}_j^n\widehat{u}_j^n\big)=\sum_{n,j}\widehat{\varphi}_j^n(R_j^n)^2e^{\widehat{u}_j^n}$$

on $\mathbf{R^2}$. Therefore, (4.15) is equivalent to

$$F_{\varepsilon,\delta}(z)=0, \tag{4.20}$$

where

$$\begin{aligned}F_{\varepsilon,\delta}(z)&=\Delta z+\frac{1}{\varepsilon^2}\sum_{n,j}C_j^n(x)\\&+\frac{1}{\varepsilon^2}\sum_{n,j}\widehat{\varphi}_j^n\widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\big(e^{\varepsilon^2 z}R_j^n(x)-1\big)\\&-\frac{1}{\varepsilon^2}\sum_{n,j}\widehat{\varphi}_j^n(\widehat{\rho}_j^n)^2e^{2\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\big(e^{2\varepsilon^2 z}R_j^n(x)^2-1\big).\end{aligned} \tag{4.21}$$

This equation is solved by Banach's fixed-point theorem, and in the next section we study the linearized operator.

5. Second linearization

We show the solvability of $F_{\varepsilon,\delta}(z) = 0$ by examining the method of [7] in detail. First, $F_{\varepsilon,\delta} : \widehat{H}_\delta \to \widehat{Y}_\delta$ is well defined by (2.12) and (2.14). More precisely, we have the following estimates (Lemmas 6.2, 6.3, and 6.4 of [7]).

Lemma 9. *We have*

$$\sup_{x\in\mathbf{R}^2} \widehat{\varphi}_j^n(x)\widehat{\rho}_j^n(x)\big|R_j^n(x) - 1\big| = O(\delta^{4-\beta(\varepsilon)}) \tag{5.1}$$

and

$$\big\|\sum_{n,j} C_j^n(x)\big\|_{\widehat{Y}_\delta} = O\big(\delta^2\big|\log\delta\big|\big) \tag{5.2}$$

as $\delta \downarrow 0$ *with* $\beta(\varepsilon) \to 0$, *where* $n \in \mathbf{Z}^2$ *and* $j = 1, \ldots, s$.

Proof. By (4.5) and

$$\operatorname{diam}(\widehat{B}_j^n) = O\big(\delta^{-1}\big), \tag{5.3}$$

we have $C_1, C_2 > 0$ such that

$$C_1\delta^{2m_j+4} \le \widehat{\rho}_j^n(x) \le C_2\delta^{2m_j+4},$$

where $n \in \mathbf{Z}^2, j = 1, \ldots, s$ and $x \in \widehat{B}_j^n$. Hence it holds that

$$\sup_{x\in\widehat{B}_j^n\cap\widehat{B}_1^n} \frac{\widehat{\rho}_j^n(x)}{\widehat{\rho}_1^n(x)} \le C\delta^{2m_j-2m_1} \le C \qquad \big(j = 2, \ldots, s\big) \tag{5.4}$$

$$\sup_{x\in\widehat{B}_1^k\cap\widehat{B}_1^n} \frac{\widehat{\rho}_1^k(x)}{\widehat{\rho}_1^n(x)} \le C \qquad \big(k \in \langle n\rangle\big).$$

In fact, the exponent of δ in (5.4) is nonnegative by our choice of m_1. Moreover by (2.12) and (2.14), we have

$$\sup_{x\in\widehat{B}_j^n\cap\widehat{B}_1^n} \big(\big|\widehat{w}_1^n(x) - \widehat{w}_j^n(x)\big| + \big|\widehat{v}_{\varepsilon,1}^n(x) - \widehat{v}_{\varepsilon,j}^n(x)\big|\big) \le C\big|\log\delta\big|$$

for $j = 2, \ldots, s$ and

$$\sup_{x\in\widehat{B}_1^k\cap\widehat{B}_1^n} \big(\big|\widehat{w}_1^n(x) - \widehat{w}_1^k(x)\big| + \big|\widehat{v}_{\varepsilon,1}^n(x) - \widehat{v}_{\varepsilon,1}^k(x)\big|\big) \le C\big|\log\delta\big|$$

for $k \in \langle n\rangle$. Therefore by (5.4) and (4.17), we have

$$\big|R_j^n(x)\big| \le \exp\big(C + C\varepsilon^2\big|\log\delta\big|\big) = C\delta^{-C\varepsilon^2} \tag{5.5}$$

for all $x \in \mathbf{R}^2$. This implies that

$$\sup_{x\in\mathbf{R}^2} \widehat{\varphi}_j^n(x)\widehat{\rho}_j^n(x)\big|R_j^n(x) - 1\big| \le \sup_{x\in\widehat{B}_j^n} \widehat{\rho}_j^n(x)\big(R_j^n(x) + 1\big) \le C\delta^{2m_j+4-\beta(\varepsilon)}$$

for $0 < \delta \ll 1$, where $\beta(\varepsilon) = C\varepsilon^2$. Thus, we obtain (5.1).

Now, we note the following: If f_ℓ^k ($k \in \mathbf{Z^2}$, $\ell \in \{1, \dots, s\}$) are functions satisfying $\mathrm{supp}(f_\ell^k) \subset \widehat{B}_l^k$, then it holds that

$$\Big\| \sum_{k,l} f_\ell^k \Big\|_{\widehat{Y}_\delta} \le C \sup_{k,\ell} \big\| f_\ell^k \big\|_{\widehat{Y}_l^k}, \tag{5.6}$$

where C is a constant independent of k and ℓ. In fact, we have

$$\Big\| \sum_{k,l} f_\ell^k \Big\|_{\widehat{Y}_\delta} = \sup_{n,j} \Big\| \widehat{\varphi}_j^n \sum_{k,l} f_\ell^k \Big\|_{\widehat{Y}_j^n} \le C \sup_{n,j} \sup_{\substack{k,\ell \\ \widehat{B}_j^n \cap \widehat{B}_l^k \neq \phi}} \big\| \widehat{\varphi}_j^n f_\ell^k \big\|_{\widehat{Y}_j^n},$$

and it holds that

$$\begin{aligned}
\big\| \widehat{\varphi}_j^n f_\ell^k \big\|_{\widehat{Y}_j^n} &= \int_{\mathbf{R}^2} \big(1 + \big(\delta \big| x - \widehat{p}_j^n \big|\big)^{2+\alpha}\big) \big| \widehat{\varphi}_j^n(x) f_\ell^k(x) \big|^2 \delta^2 dx \\
&= \int_{\widehat{B}_j^n \cap \widehat{B}_l^k} \frac{1 + (\delta |x - \widehat{p}_j^n|)^{2+\alpha}}{1 + (\delta |x - \widehat{p}_l^k|)^{2+\alpha}} \big(1 + \big(\delta \big| x - \widehat{p}_l^k \big|\big)^{2+\alpha}\big) \big| \widehat{\varphi}_j^n f_l^k \big|^2 \delta^2 dx \\
&\le C \int_{\widehat{B}_l^k} \big(1 + \big(\delta \big| x - \widehat{p}_l^k \big|\big)^{2+\alpha}\big) \big| f_l^k(x) \big|^2 \delta^2 dx
\end{aligned}$$

because of

$$\frac{1 + \big(\delta \big| x - \widehat{p}_j^n \big|\big)^{2+\alpha}}{1 + \big(\delta \big| x - \widehat{p}_l^k \big|\big)^{2+\alpha}} = O(1) \qquad \big(x \in \widehat{B}_j^n \cap \widehat{B}_l^k\big)$$

and $\widehat{\varphi}_j^n(x) \le 1$. This shows (5.6).

From this, we obtain that

$$\begin{aligned}
\Big\| \sum_{n,j} C_j^n \Big\|_{\widehat{Y}_\delta} &\le C \sup_{n,j} \big\| C_j^n \big\|_{\widehat{Y}_j^n} \\
&\le C \sup_{x \in \widehat{B}_j^n} \big(\big| \Delta \widehat{\varphi}_j^n(x) \big| \big| \widehat{u}_j^n(x) \big| + 2 \big| \nabla \widehat{\varphi}_j^n(x) \big| \big| \nabla \widehat{u}_j^n(x) \big| \big).
\end{aligned} \tag{5.7}$$

Similarly to before, we have from (2.11), (2.12), and (2.14) that

$$\big| \widehat{u}_j^n(x) \big| \le C \big| \log \big| x - \widehat{p}_j^n \big| \big|, \qquad \big| \nabla \widehat{u}_j^n(x) \big| \le \frac{C}{\big| x - \widehat{p}_j^n \big|} \tag{5.8}$$

as $|x| \to \infty$, where $C > 0$ is a constant independent of n and j. We also have (4.2), and hence (5.2) follows as

$$\Big\| \sum_{n,j} C_j^n(x) \Big\|_{\widehat{Y}_\delta} \le C \big(\delta^2 \log \frac{1}{\delta} + \delta^2 \big).$$

The proof is complete. □

Now, we see that $F_{\varepsilon,\delta}$ is a smooth map from the unit ball of $\widehat{H}_\delta$ to $\widehat{Y}_\delta$ if $0<\varepsilon\ll 1$. Furthermore, by (2.12), (2.14), and (5.3), we have

$$\sup_{x\in\widehat{B}_j^n} e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\le C\delta^{-C\varepsilon^2}, \tag{5.9}$$

and hence it follows that

$$\begin{aligned}&\left\|\widehat{\varphi}_j^n\widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\left[R_j^n-1\right]\right\|_{\widehat{Y}_j^n} \\ &\le \sup_{x\in\widehat{B}_j^n}\widehat{\varphi}_j^n(x)\widehat{\rho}_j^n(x)e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\left|R_j^n(x)-1\right|\le C\delta^{2m_j+4-C\varepsilon^2}=o\left(\delta^2\right)\end{aligned} \tag{5.10}$$

as $\delta\downarrow 0$. Similarly, we have

$$\left\|\widehat{\varphi}_j^n(\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\left[(R_j^n)^2-1\right]\right\|_{\widehat{Y}_j^n}\le o\left(\delta^2\right).$$

Therefore, Lemma 9 guarantees the following.

Lemma 10. *For $0<\delta\ll\varepsilon\ll 1$, it holds that*

$$\left\|F_{\varepsilon,\delta}(0)\right\|_{\widehat{Y}_\delta}\le\frac{C}{\varepsilon^2}\delta^2\left|\log\delta\right| \quad \text{as } \delta\downarrow 0.$$

Now, we turn to $A_{\varepsilon,\delta}\equiv DF_{\varepsilon,\delta}(0):\widehat{H}_\delta\to\widehat{Y}_\delta$. It is realized as

$$\begin{aligned}A_{\varepsilon,\delta}=\Delta&+\sum_{n,j}\widehat{\varphi}_j^n\widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}R_j^n(x)\\ &-2\varepsilon^2\sum_{n,j}\widehat{\varphi}_j^n(\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}R_j^n(x)^2.\end{aligned} \tag{5.11}$$

The next lemma is the same as Lemma 6.4 of [7]. There, it is shown that this operator is approximated locally by $\widehat{A}_{\varepsilon,j}^n$ of (4.11):

$$\widehat{A}_{\varepsilon,j}^n=\Delta+\widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}-2\varepsilon^2(\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}.$$

Lemma 11. *We have $C>0$ and $\beta(\varepsilon)\to 0$ as $\varepsilon\downarrow 0$ such that*

$$\left\|\left(A_{\varepsilon,\delta}-\widehat{A}_{\varepsilon,j}^n\right)(\widehat{\varphi}_j^n h)\right\|_{\widehat{Y}_j^n}\le C\delta^{4-\beta(\varepsilon)}\left\|\widehat{\varphi}_j^n h\right\|_{\widehat{X}_j^n}$$

for any n and j and $h\in\widehat{H}_\delta$ in the case of $0<\delta\ll\varepsilon\ll 1$.

Proof. In fact, we have

$$\left\|\left(A_{\varepsilon,\delta}-\widehat{A}_{\varepsilon,j}^n\right)(\widehat{\varphi}_j^n h)\right\|_{\widehat{Y}_j^n}\le\left\|\widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\left(\widehat{\varphi}_j^n R_j^n(x)-1\right)(\widehat{\varphi}_j^n h)\right\|_{\widehat{Y}_j^n}$$

$$+2\varepsilon^2\big\|(\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\big(\widehat{\varphi}_j^n R_j^n(x)^2-1\big)(\widehat{\varphi}_j^n h)\big\|_{\widehat{Y}_j^n}$$
$$+\Big\|\sum_{(k,l)\neq(n,j)}\widehat{\varphi}_l^k\widehat{\rho}_l^k e^{\varepsilon^2\left(\widehat{w}_l^k+\widehat{v}_{\varepsilon,l}^k\right)}R_l^k(x)(\widehat{\varphi}_j^n h)\Big\|_{\widehat{Y}_j^n}$$
$$+2\varepsilon^2\Big\|\sum_{(k,l)\neq(n,j)}\widehat{\varphi}_l^k(\widehat{\rho}_l^k)^2 e^{2\varepsilon^2\left(\widehat{w}_l^k+\widehat{v}_{\varepsilon,l}^k\right)}R_l^k(x)^2(\widehat{\varphi}_j^n h)\Big\|_{\widehat{Y}_j^n}. \qquad (5.12)$$

We apply (5.9) and (5.1) for the first term of the right-hand side, and get that

$$\big\|\widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\big(\widehat{\varphi}_j^n R_j^n(x)-1\big)(\widehat{\varphi}_j^n h)\big\|_{\widehat{Y}_j^n}$$
$$\leq \sup_{x\in\widehat{B}_j^n}\Big(\widehat{\rho}_j^n(x)e^{\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\big|R_j^n(x)-1\big|\Big)\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{Y}_j^n}\leq C\delta^{4-\beta(\varepsilon)}\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{X}_j^n}.$$

Similarly, we have

$$\big\|(\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n+\widehat{v}_{\varepsilon,j}^n\right)}\big(\widehat{\varphi}_j^n R_j^n(x)^2-1\big)(\widehat{\varphi}_j^n h)\big\|_{\widehat{Y}_j^n}\leq o\big(\delta^{4-\beta(\varepsilon)}\big)\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{X}_j^n}.$$

For the third and the fourth terms, we apply (5.6) and (5.9). Then, we get that

$$\Big\|\sum_{(k,l)\neq(n,j)}\widehat{\varphi}_l^k\widehat{\rho}_l^k e^{\varepsilon^2\left(\widehat{w}_l^k+\widehat{v}_{\varepsilon,l}^k\right)}R_l^k(x)(\widehat{\varphi}_j^n h)\Big\|_{\widehat{Y}_j^n}$$
$$\leq \sup_{(k,l)\neq(n,j)}\ \sup_{x\in\widehat{B}_j^n\cap\widehat{B}_l^k}\widehat{\rho}_l^k(x)e^{\varepsilon^2(\widehat{w}_l^k+\widehat{v}_{\varepsilon,l}^k)}R_l^k(x)\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{Y}_j^n}\leq C\delta^{4-\beta(\varepsilon)}\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{X}_j^n},$$

and

$$\Big\|\sum_{(k,l)\neq(n,j)}\widehat{\varphi}_l^k(\widehat{\rho}_l^k)^2 e^{2\varepsilon^2\left(\widehat{w}_l^k+\widehat{v}_{\varepsilon,l}^k\right)}R_l^k(x)^2(\widehat{\varphi}_j^n h)\Big\|_{\widehat{Y}_j^n}\leq o\big(\delta^{4-\beta(\varepsilon)}\big)\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{X}_j^n}.$$

From these estimates, we obtain the lemma. □

Now, we show the main result of section. (See Proposition 6.5 of [7].)

Theorem 3. *Given* $0<\varepsilon\ll 1$*, we have* $\delta=\delta_\varepsilon>0$ *such that* $A_{\varepsilon,\delta}=DF_{\varepsilon,\delta}(0):\widehat{H}_\delta\to\widehat{Y}_\delta$ *is invertible for any* $\delta\in(0,\delta_\varepsilon)$*. Moreover, it holds that* $\|A_{\varepsilon,\delta}^{-1}\|\leq\frac{C}{\varepsilon^2}$ *with a constant* $C>0$ *independent of* $\delta\in(0,\delta_\varepsilon)$ *and* $0<\varepsilon\ll 1$*.*

Proof. We follow the key idea of the original proof [7], and take the smooth function

$$\widehat{g}_j^n(x)=\frac{\widehat{\varphi}_j^n(x)}{\Big\{\sum_{k,l}\widehat{\varphi}_l^k(x)^2\Big\}^{1/2}}, \qquad (5.13)$$

where $n \in \mathbf{Z^2}$ and $j = 1, \dots, s$. Now, we introduce the auxiliary linear operator $S_{\varepsilon,\delta} : \widehat{H}_\delta \to \widehat{Y}_\delta$ by

$$S_{\varepsilon,\delta} h = \sum_{n\in\mathbf{Z^2}} \sum_{j=1,\dots,s} \widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} (\widehat{g}_j^n h), \tag{5.14}$$

where $h \in \widehat{Y}_\delta$. In fact, we have from (5.6) and Lemma 8 that

$$\left\|S_{\varepsilon,\delta} h\right\|_{\widehat{H}_\delta} \le C \sup_{n,j} \left\|\widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \widehat{g}_j^n\right\| \le \frac{C}{\varepsilon^2} \left\|h\right\|_{\widehat{Y}_\delta}, \tag{5.15}$$

with a constant $C > 0$ independent of δ in $0 < \delta \ll \varepsilon$.

We shall show that both $A_{\varepsilon,\delta} S_{\varepsilon,\delta} : \widehat{Y}_\delta \to \widehat{Y}_\delta$ and $S_{\varepsilon,\delta} A_{\varepsilon,\delta} : \widehat{H}_\delta \to \widehat{H}_\delta$ are invertible. In fact, we have

$$S_{\varepsilon,\delta} A_{\varepsilon,\delta} = I_{\widehat{H}_\delta} + \sum_{n,j} \widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \big(A_{\varepsilon,\delta} - \widehat{A}_{\varepsilon,j}^n\big) \widehat{g}_j^n - \sum_{n,j} \widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \left[A_{\varepsilon,\delta}, \widehat{g}_j^n\right], \tag{5.16}$$

with the commutator $[A_{\varepsilon,\delta}, \widehat{g}_j^n]$ given by

$$\left[A_{\varepsilon,\delta}, \widehat{g}_j^n\right] h = A_{\varepsilon,\delta}(\widehat{g}_j^n h) - \widehat{g}_j^n A_{\varepsilon,\delta} h = \Delta(\widehat{g}_j^n h) - \widehat{g}_j^n \Delta h = \left[\Delta, \widehat{g}_j^n\right] h \tag{5.17}$$

for $h \in \widehat{H}_\delta$. Then, we can prove that the error term $S_{\varepsilon,\delta} A_{\varepsilon,\delta} - I_{\widehat{H}_\delta}$ in (5.16) is small in the operator norm for $0 < \delta \ll \varepsilon$. Because the original proof of [7] is not described in detail at this stage, here we examine it in full length.

First, by using (5.6) and Lemmas 8 and 11, we have

$$\begin{aligned}
&\Big\| \sum_{n,j} \widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \big(A_{\varepsilon,\delta} - \widehat{A}_{\varepsilon,j}^n\big)(\widehat{g}_j^n h)\Big\|_{\widehat{H}_\delta} \\
&\le C \sup_{n,j} \left\|\widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \big(A_{\varepsilon,\delta} - \widehat{A}_{\varepsilon,j}^n\big)(\widehat{g}_j^n h)\right\|_{\widehat{X}_j^n} \\
&\le \frac{C}{\varepsilon^2} \sup_{n,j} \left\|\big(A_{\varepsilon,\delta} - \widehat{A}_{\varepsilon,j}^n\big)(\widehat{\varphi}_j^n h)\right\|_{\widehat{Y}_j^n} \le \frac{C}{\varepsilon^2} \delta^{4-\beta(\varepsilon)} \sup_{n,j} \left\|\widehat{\varphi}_j^n h\right\|_{\widehat{X}_j^n}
\end{aligned} \tag{5.18}$$

for any $h \in \widehat{H}_\delta$. Similarly, we have

$$\begin{aligned}
\Big\| \sum_{n,j} \widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \left[A_{\varepsilon,\delta}, \widehat{g}_j^n\right] h\Big\|_{\widehat{H}_\delta} &\le C \sup_{n,j} \left\|\widehat{g}_j^n (\widehat{A}_{\varepsilon,j}^n)^{-1} \left[A_{\varepsilon,\delta}, \widehat{g}_j^n\right] h\right\|_{\widehat{X}_j^n} \\
&\le \frac{C}{\varepsilon^2} \sup_{n,j} \left\| \left[\Delta, \widehat{g}_j^n\right] h\right\|_{\widehat{Y}_j^n}.
\end{aligned} \tag{5.19}$$

Now, we shall show that

$$\left\| \left[\Delta, \widehat{g}_j^n\right] h\right\|_{\widehat{Y}_j^n} = O\big(\delta\big) \sup_{n,j} \left\|\widehat{\varphi}_j^n h\right\|_{\widehat{X}_j^n} \tag{5.20}$$

holds as $\delta \downarrow 0$. Actually, this part requires several (rather delicate) modifications of the original paper. First, we note

$$\sup_{x\in\widehat{B}_j^n}\left|\Delta\widehat{g}_j^n(x)\right| \le C\delta^2, \qquad \sup_{x\in\widehat{B}_j^n}\left|\nabla\widehat{g}_j^n(x)\right| \le C\delta,$$

and $1+\left(\delta\left|x-\widehat{p}_j^n\right|\right)^{2+\alpha} = O(1)$ for $x \in \widehat{B}_j^n$, which implies that

$$\begin{aligned}
\left\| \left[\Delta, \widehat{g}_j^n\right] h\right\|_{\widehat{Y}_j^n}^2 &= \int_{\widehat{B}_j^n} \left(1+\left(\delta\left|x-\widehat{p}_j^n\right|\right)^{2+\alpha}\right)\left(2\nabla\widehat{g}_j^n\cdot\nabla h + (\Delta\widehat{g}_j^n)h\right)^2\delta^2 dx \\
&\le C\int_{\widehat{B}_j^n}\left(\left|\Delta\widehat{g}_j^n\right|^2 h^2 + \left|\nabla\widehat{g}_j^n\right|^2\left|\nabla h\right|^2\right)\delta^2 dx \\
&\le C\delta^4\int_{\widehat{B}_j^n} h^2\delta^2 dx + C\delta^2\int_{\widehat{B}_j^n}\left|\nabla h\right|^2\delta^2 dx. \qquad (5.21)
\end{aligned}$$

Here, we show the following.

Lemma 12. *It holds that*

$$\int_{\widehat{B}_j^n}\left(h^2+\left|\nabla h\right|^2\right)\delta^2 dx \le C\sup_{k,l}\left\|\widehat{\varphi}_l^k h\right\|_{\widehat{X}_l^k}^2 \qquad (5.22)$$

for any $h \in \widehat{H}_\delta$.

Proof. Given $(n,j) \in \mathbf{Z^2}\times\{1,\ldots,s\}$, we put that

$$J_j^n = \left\{(k,l)\in\mathbf{Z^2}\times\{1,\ldots,s\} : \widehat{B}_l^k\cap\widehat{B}_j^n \ne \phi\right\}. \qquad (5.23)$$

The cardinality of J_j^n satisfies that $\left|J\right| \equiv \sup_{n,j}\left|J_j^n\right| < +\infty$, because the periodic covering $\{\widehat{B}_j^n\}_{n,j}$ of $\mathbf{R}^2$ is locally finite. Therefore, Schwarz' inequality guarantees for any $x \in \widehat{B}_j^n$ that

$$1 \equiv \sum_{(k,l)\in J_j^n}\widehat{\varphi}_l^k(x) \le \Big\{\sum_{(k,l)\in J_j^n}\widehat{\varphi}_l^k(x)^2\Big\}^{\frac{1}{2}}\Big\{\sum_{(k,l)\in J_j^n}1^2\Big\}^{\frac{1}{2}} \le |J|\sum_{(k,l)\in J_j^n}\widehat{\varphi}_l^k(x)^2, \qquad (5.24)$$

and hence it holds that

$$\begin{aligned}
\int_{\widehat{B}_j^n} h^2\delta^2 dx &\le \left|J\right|\sum_{(k,l)\in J_j^n}\int_{\widehat{B}_j^n\cap\widehat{B}_l^k}(\widehat{\varphi}_l^k)^2 h^2\delta^2 dx \qquad (5.25)\\
&\le C\sup_{(k,l)\in J_j^n}\int_{\widehat{B}_l^k}\frac{(\widehat{\varphi}_l^k h)^2\delta^2}{\left(1+\left(\delta\left|x-\widehat{p}_l^k\right|\right)^{2+\alpha}\right)}dx \le C\sup_{k,l}\left\|\widehat{\varphi}_l^k h\right\|_{\widehat{X}_l^k}^2
\end{aligned}$$

because

$$C^{-1} \le 1 + \bigl(\delta\bigl|x - \widehat{p}_j^n\bigr|\bigr)^{2+\alpha} \le C$$

holds for all $x \in \widehat{B}_j^n$. Similarly, we have

$$\begin{aligned}\int_{\widehat{B}_j^n} \bigl|\nabla h\bigr|^2 \delta^2 dx &\le \bigl|J\bigr| \sum_{(k,l)\in J_j^n} \int_{\widehat{B}_j^n \cap \widehat{B}_l^k} (\widehat{\varphi}_l^k)^2 \bigl|\nabla h\bigr|^2 \delta^2 dx \\ &\le C \sum_{k,l} \int_{\widehat{B}_l^k} \bigl|\widehat{\varphi}_l^k \nabla h\bigr|^2 \delta^2 dx \\ &\le C \sup_{k,l} \int_{\widehat{B}_l^k} \bigl|\nabla(\widehat{\varphi}_l^k h)\bigr|^2 \delta^2 dx + C \sup_{k,l} \int_{\widehat{B}_l^k} \bigl|\nabla \widehat{\varphi}_l^k\bigr|^2 h^2 \delta^2 dx \\ &\le C \sup_{k,l} \int_{\widehat{B}_l^k} \bigl|\nabla(\widehat{\varphi}_l^k h)\bigr|^2 \delta^2 dx + C\delta^2 \sup_{k,l} \int_{\widehat{B}_l^k} h^2 \delta^2 dx \qquad (5.26)\end{aligned}$$

by $\bigl|\widehat{\varphi}_l^k \nabla h\bigr| \le \bigl|\nabla(\widehat{\varphi}_l^k h)\bigr| + \bigl|\nabla \widehat{\varphi}_l^k\bigr|\bigl|h\bigr|$. For the first term of the right-hand side, we note that $\widehat{\varphi}_l^k h \in C_c^1(\widehat{B}_l^k)$ holds by Sobolev's imbedding theorem, and hence it follows from Young's inequality that

$$\begin{aligned}&\int_{\widehat{B}_l^k} \bigl|\nabla(\widehat{\varphi}_l^k h)\bigr|^2 \delta^2 dx \le \int_{\widehat{B}_l^k} \bigl|(\widehat{\varphi}_l^k h)\Delta(\widehat{\varphi}_l^k h)\bigr| \delta^2 dx \\ &\le \frac{1}{2} \int_{\widehat{B}_l^k} \frac{(\widehat{\varphi}_l^k h)^2 \delta^2}{\bigl(1 + \bigl(\delta\bigl|x - \widehat{p}_l^k\bigr|\bigr)^{2+\alpha}\bigr)} dx \\ &\qquad + \frac{1}{2} \int_{\widehat{B}_l^k} \bigl|\Delta(\widehat{\varphi}_l^k h)\bigr|^2 \bigl(1 + \bigl(\delta\bigl|x - \widehat{p}_l^k\bigr|\bigr)^{2+\alpha}\bigr) \delta^2 dx \\ &\le \frac{1}{2} \bigl\|\widehat{\varphi}_l^k h\bigr\|_{\widehat{X}_l^k}^2 + \frac{1}{2} \bigl\|\Delta(\widehat{\varphi}_l^k h)\bigr\|_{\widehat{Y}_l^k}^2 \le \bigl\|\widehat{\varphi}_l^k h\bigr\|_{\widehat{X}_l^k}^2. \qquad (5.27)\end{aligned}$$

Therefore, combining (5.26), (5.27), and (5.25), we obtain

$$\int_{\widehat{B}_j^n} \bigl|\nabla h\bigr|^2 \delta^2 dx \le C\bigl(1 + \delta^2\bigr) \sup_{k,l} \bigl\|\widehat{\varphi}_l^k h\bigr\|_{\widehat{X}_l^k}^2,$$

and the proof is complete. □

By using (5.21) and Lemma 12, we obtain

$$\bigl\| \bigl[\Delta, \widehat{g}_j^n\bigr] h \bigr\|_{\widehat{Y}_j^n} \le C\bigl(\delta + \delta^2\bigr) \sup_{k,l} \bigl\|\widehat{\varphi}_l^k h\bigr\|_{\widehat{X}_l^k},$$

and thus (5.20) follows. Therefore, we have from (5.18) and (5.19) that

$$\bigl\|S_{\varepsilon,\delta} A_{\varepsilon,\delta} - I_{\widehat{H}_\delta}\bigr\| = \frac{C}{\varepsilon^2} O(\delta) \to 0$$

as $\delta \downarrow 0$. On the other hand, we have

$$A_{\varepsilon,\delta}S_{\varepsilon,\delta} = I_{\widehat{Y}_\delta} + \sum_{n,j} \widehat{g}_j^n \big(A_{\varepsilon,\delta} - \widehat{A}_{\varepsilon,j}^n\big)(\widehat{A}_{\varepsilon,j}^n)^{-1}\widehat{g}_j^n + \sum_{n,j} \big[A_{\varepsilon,\delta}, \widehat{g}_j^n\big]\,(\widehat{A}_{\varepsilon,j}^n)^{-1}\widehat{g}_j^n, \tag{5.28}$$

and it is shown that $\big\|A_{\varepsilon,\delta}S_{\varepsilon,\delta} - I_{\widehat{Y}_\delta}\big\| \to 0$ similarly.

Therefore, both $S_{\varepsilon,\delta}A_{\varepsilon,\delta} : \widehat{H}_\delta \to \widehat{H}_\delta$ and $A_{\varepsilon,\delta}S_{\varepsilon,\delta} : \widehat{Y}_\delta \to \widehat{Y}_\delta$ are invertible with the estimates $\big\|\big(A_{\varepsilon,\delta}S_{\varepsilon,\delta}\big)^{-1}\big\| \le 2$ and $\big\|\big(S_{\varepsilon,\delta}A_{\varepsilon,\delta}\big)^{-1}\big\| \le 2$ uniformly for $0 < \delta \ll \varepsilon \ll 1$. Thus, $(A_{\varepsilon,\delta})^{-1} : \widehat{Y}_\delta \to \widehat{H}_\delta$ exists as

$$\big(A_{\varepsilon,\delta}\big)^{-1} = S_{\varepsilon,\delta}\big(A_{\varepsilon,\delta}S_{\varepsilon,\delta}\big)^{-1} = \big(S_{\varepsilon,\delta}A_{\varepsilon,\delta}\big)^{-1}S_{\varepsilon,\delta}.$$

We also have by (5.15) that

$$\big\|\big(A_{\varepsilon,\delta}\big)^{-1}h\big\|_{\widehat{H}_\delta} = \big\|\big(S_{\varepsilon,\delta}A_{\varepsilon,\delta}\big)^{-1}S_{\varepsilon,\delta}h\big\|_{\widehat{H}_\delta} \le 2\big\|S_{\varepsilon,\delta}h\big\|_{\widehat{H}_\delta} \le \frac{C}{\varepsilon^2}\|h\|_{\widehat{Y}_\delta}$$

for $h \in \widehat{Y}_\delta$. The proof of Theorem 3 is complete. □

6. Completion of proof

We are in position to apply Banach's fixed-point theorem to the functional equation $F_{\varepsilon,\delta}(z) = 0$. Henceforth, we put that $B_r = \{z \in \widehat{H}_\delta : \|z\|_{\widehat{H}_\delta} \le r\}$ for $r > 0$, and introduce the nonlinear mapping $G_{\varepsilon,\delta} : B_1 \to \widehat{H}_\delta$ by

$$G_{\varepsilon,\delta}(z) = z - \big(A_{\varepsilon,\delta}\big)^{-1}F_{\varepsilon,\delta}(z). \tag{6.1}$$

Thus, we are seeking the fixed point of $G_{\varepsilon,\delta}$ in B_1. Actually, this is done by the following. (See Theorem 6.6 of [7].)

Theorem 4. *Any $0 < \varepsilon \ll 1$ admits $\delta_\varepsilon > 0$ satisfying $\lim_{\varepsilon\downarrow 0}\delta_\varepsilon = 0$ such that if $\delta \in (0, \delta_\varepsilon)$ there is a unique fixed point $z_{\varepsilon,\delta}^*$ in B_{R_δ} of $G_{\varepsilon,\delta}$, where $R_\delta = \delta^\beta$ with $\beta \in (1, 2)$.*

Proof. We show that $G_{\varepsilon,\delta}$ is a contractive self-map on B_r for $r = R_\delta = O\big(\delta^\beta\big)$ chosen below. In fact, we have

$$DG_{\varepsilon,\delta}(z) = I - A_{\varepsilon,\delta}^{-1}DF_{\varepsilon,\delta}(z) = A_{\varepsilon,\delta}^{-1}\big(A_{\varepsilon,\delta} - DF_{\varepsilon,\delta}(z)\big),$$

and hence Theorem 3 guarantees that

$$\big\|DG_{\varepsilon,\delta}(z)\big\| \le \big\|A_{\varepsilon,\delta}^{-1}\big\|\big\|DF_{\varepsilon,\delta}(z) - A_{\varepsilon,\delta}\big\| \le \frac{C}{\varepsilon^2}\big\|DF_{\varepsilon,\delta}(z) - A_{\varepsilon,\delta}\big\| \tag{6.2}$$

for any $z \in B_1$. Here, we have

$$DF_{\varepsilon,\delta}(z) = \Delta + \sum_{n,j} \widehat{\varphi}_j^n \widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n + \widehat{v}_{\varepsilon,j}^n\right)} e^{\varepsilon^2 z} R_j^n(x)$$

$$-2\varepsilon^2 \sum_{n,j} \widehat{\varphi}_j^n (\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n + \widehat{v}_{\varepsilon,j}^n\right)} e^{2\varepsilon^2 z} R_j^n(x)^2, \tag{6.3}$$

and hence it follows that

$$\begin{aligned}
&\big\|\big(DF_{\varepsilon,\delta}(z) - A_{\varepsilon,\delta}\big)h\big\|_{\widehat{Y}_\delta} = \big\|\big(DF_{\varepsilon,\delta}(z) - DF_{\varepsilon,\delta}(0)\big)h\big\|_{\widehat{Y}_\delta} \\
&= \sup_{n,j} \big\|\widehat{\varphi}_j^n \big(DF_{\varepsilon,\delta}(z) - DF_{\varepsilon,\delta}(0)\big)h\big\|_{\widehat{Y}_j^n} \\
&\le C \sup_{n,j} \big\|\widehat{\varphi}_j^n \widehat{\rho}_j^n e^{\varepsilon^2\left(\widehat{w}_j^n + \widehat{v}_{\varepsilon,j}^n\right)} R_j^n(x)\big(e^{\varepsilon^2 z} - 1\big)\widehat{\varphi}_j^n h\big\|_{\widehat{Y}_j^n} \\
&+ C\varepsilon^2 \sup_{n,j} \big\|\widehat{\varphi}_j^n (\widehat{\rho}_j^n)^2 e^{2\varepsilon^2\left(\widehat{w}_j^n + \widehat{v}_{\varepsilon,j}^n\right)} R_j^n(x)^2 \big(e^{2\varepsilon^2 z} - 1\big)\widehat{\varphi}_j^n h\big\|_{\widehat{Y}_j^n}
\end{aligned} \tag{6.4}$$

for any $h \in \widehat{H}_\delta$. Because of the maximum principle to (4.14) we have $\widehat{u}_j^n \le 0$, and hence it follows from (4.19) that

$$\varepsilon^2 \widehat{\rho}_j^n e^{\varepsilon^2(\widehat{w}_j^n + \widehat{v}_{\varepsilon,j}^n)} R_j^n \le 1$$

in $\widehat{B}_j^n$. We also have

$$\big|e^{\varepsilon^2 z(x)} - 1\big| = \Big|\frac{d}{dt}\int_0^1 e^{\varepsilon^2 t z(x)} dt\Big| \le \varepsilon^2 \|z\|_\infty e^{\varepsilon^2 \|z\|_\infty} \tag{6.5}$$

for all $x \in \mathbf{R^2}$, and Sobolev's inequality implies that

$$\|u\|_\infty \le C\|u\|_{W^{2,2}} \approx \frac{C}{\delta}\|u\|_{\widehat{X}_j^n}$$

for $u \in \widehat{X}_j^n$ in supp $u \subset \widehat{B}_j^n$. From those relations we obtain

$$\begin{aligned}
\big\|\big(DF_{\varepsilon,\delta}(z) - A_{\varepsilon,\delta}\big)h\big\|_{\widehat{Y}_\delta} &\le C \sup_{n,j} \big\|\widehat{\varphi}_j^n z\big\|_\infty \big(e^{\varepsilon^2\|z\|_\infty} + e^{2\varepsilon^2\|z\|_\infty}\big)\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{X}_j^n} \\
&\le \frac{C}{\delta} \sup_{n,j} \big\|\widehat{\varphi}_j^n z\big\|_{\widehat{X}_j^n} \big(2 + 3\varepsilon^2\|z\|_\infty\big)\big\|\widehat{\varphi}_j^n h\big\|_{\widehat{X}_j^n} \\
&\le \frac{C}{\delta}\|z\|_{\widehat{H}_\delta}\big(1 + \frac{C}{\delta}\|z\|_{\widehat{H}_\delta}\big)\|h\|_{\widehat{H}_\delta}
\end{aligned} \tag{6.6}$$

by $e^x(1+e^x) = 2 + 3x + o(x)$ as $x \to 0$.

We conclude from (6.6) and (6.2) that

$$\big\|DG_{\varepsilon,\delta}(z)\big\| \le \frac{1}{2} \qquad \big(z \in B_{R_\delta}\big) \tag{6.7}$$

for $0 < \varepsilon \ll 1$ if

$$\frac{CR_\delta}{\delta}\big(1 + \frac{C}{\delta}R_\delta\big) \le \frac{1}{2}\varepsilon^2 \tag{6.8}$$

is the case. If we take $R_\delta = \delta^\beta$ for $\beta \in (1,2)$, this requirement (6.8) follows from $C\delta^{\beta-1}(1 + C\delta^{\beta-1}) \le \frac{\varepsilon^2}{2}$ or equivalently, $\delta^{\beta-1} \le \frac{\varepsilon^2}{4C}$. Therefore, setting

$$\delta_\varepsilon = \Big(\frac{\varepsilon^2}{4C}\Big)^{1/(\beta-1)} \tag{6.9}$$

we see that (6.7) holds for $\delta \in (0, \delta_\varepsilon)$ and $R_\delta = \delta^\beta$, which implies that $G_{\varepsilon,\delta}$ is a contraction on B_{R_δ}.

Next, we have for $z \in B_{R_\delta}$ that

$$\begin{aligned}\big\|G_{\varepsilon,\delta}(z)\big\|_{\widehat{H}_\delta} &\le \big\|G_{\varepsilon,\delta}(z) - G_{\varepsilon,\delta}(0)\big\|_{\widehat{H}_\delta} + \big\|G_{\varepsilon,\delta}(0)\big\|_{\widehat{H}_\delta}\\ &\le \frac{1}{2}\big\|z\big\|_{\widehat{H}_\delta} + \big\|A^{-1}_{\varepsilon,\delta}F_{\varepsilon,\delta}(0)\big\|_{\widehat{H}_\delta},\end{aligned}$$

where it follows from Theorem 3 and Lemma 10 that

$$\big\|A^{-1}_{\varepsilon,\delta}F_{\varepsilon,\delta}(0)\big\|_{\widehat{H}_\delta} \le \big\|A^{-1}_{\varepsilon,\delta}\big\|\big\|F_{\varepsilon,\delta}(0)\big\|_{\widehat{Y}_\delta} \le \frac{C}{\varepsilon^4}\delta^2\big|\log\delta\big|.$$

Therefore, if $\frac{C}{\varepsilon^4}\delta^2\big|\log\delta\big| \le \frac{1}{2}R_\delta$, then it holds that $G_{\varepsilon,\delta}\big(B_{R_\delta}\big) \subset B_{R_\delta}$. Because $R_\delta = \delta^\beta$, this is equivalent to $\delta^{2-\beta}\big|\log\delta\big| \le \frac{1}{2C}\varepsilon^4$, which means that $\delta \in (0,\delta_\varepsilon)$ for some $\delta_\varepsilon > 0$ with $\lim_{\varepsilon\downarrow 0}\delta_\varepsilon = 0$. The proof is complete. □

Theorem 4 assures that

$$\widehat{u}_{\varepsilon,\delta}(x) = \sum_{n\in\mathbf{Z^2}}\sum_{j=1}^{s}\widehat{\varphi}^n_j(x)\big(\log(\varepsilon^2\widehat{\rho}^n_j(x)) + \varepsilon^2\widehat{w}^n_j(x) + \varepsilon^2\widehat{v}^n_{\varepsilon,j}(x)\big) + \varepsilon^2 z^*_{\varepsilon,\delta}(x) \tag{6.10}$$

is a solution to (2.2). On the other hand, it is obvious that $z^*_{\varepsilon,\delta}(\cdot\ +\widehat{e}_k) \in B_{R_\delta}$ and $F_{\varepsilon,\delta}\big(z^*_{\varepsilon,\delta}\big(\cdot\ +\widehat{e}_k\big)\big) = F_{\varepsilon,\delta}(z^*_{\varepsilon,\delta}) = 0$. Hence from the uniqueness of the fixed point of $G_{\varepsilon,\delta}$ in B_{R_δ}, we have $z^*_{\varepsilon,\delta}(x+\widehat{e}_k) = z^*_{\varepsilon,\delta}(x)$ for any $x \in \mathbf{R^2}$ and $k = 1,2$. This implies $\widehat{u}_{\varepsilon,\delta}(x+\widehat{e}_k) = \widehat{u}_{\varepsilon,\delta}(x)$ for $k = 1,2$ and $\widehat{u}_{\varepsilon,\delta}$ is a doubly periodic solution to (2.2).

Back to (1.1), given $\kappa = 2\varepsilon\delta > 0$ with $0 < \varepsilon \ll 1$, we have $\delta_\varepsilon > 0$ such that if $\delta \in (0,\delta_\varepsilon)$ there is a solution $u_{\varepsilon,\delta}$ to (1.1). It has the form

$$u_{\varepsilon,\delta}(x) = \sum_{j=1}^{s}\phi_j(x)u^*_{\varepsilon,m_j}\big(\big|x - p_j\big|/\delta\big) + \varepsilon^2 z^*_{\varepsilon,\delta}\big(x/\delta\big) \tag{6.11}$$

for $x \in \Omega$, with $u^*_{\varepsilon,m_j}(|\cdot\ -p_j|/\delta)$ standing for the radially symmetric single-vortex solution with the vortex point p_j and multiplicity m_j. Once this expression is obtained, we can evaluate the asymptotic behavior of $|\phi_\kappa| = \exp(u_{\varepsilon,\delta}/2)$ exactly in the same way as in Proposition 7.1 of [7]. We just state it for completeness.

Theorem 5. *If $\kappa = 2\varepsilon\delta$ with $0 < \varepsilon \ll 1$ and $\delta \in (0, \delta_\varepsilon)$, then $\left|\phi_\kappa\right| = \exp\left(u_{\varepsilon,\delta}/2\right)$ defined for $u_{\varepsilon,\delta}$ given by (6.11) satisfies the following:*

(1) *$|\phi_\kappa| < 1$ on Ω and vanishes exactly at the vortex points p_j with multiplicity m_j $(j = 1, \ldots, s)$.*
(2) *$|\phi_\kappa| \to 0$ in $C^q_{loc}(\Omega \setminus \{p_1, \ldots, p_s\})$ for any $q \geq 0$ as $\kappa \to 0$.*
(3) *$\frac{4}{\kappa^2}|\phi_\kappa|^2(1 - |\phi_\kappa|^2) \rightharpoonup 4\pi \sum_{j=1}^{s} m_j \delta_{p_j}$ in the sense of measures on Ω as $\kappa \downarrow 0$.*

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Note added in proof: In the preliminary version, we showed Theorem 1 in the case of $s \geq 2$: "Non-topological condensate in self-dual gauge theory," Banach Center Publication, to appear.