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EXISTENCE AND MULTIPLICITY FOR PERTURBATIONS OF AN EQUATION INVOLVING A HARDY INEQUALITY AND THE CRITICAL SOBOLEV EXPONENT IN THE WHOLE OF $\mathbb{R}^N$

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Abstract. In order to obtain solutions to the problem

$$\begin{cases} -\Delta u = \dfrac{A+h(x)}{|x|^2}u + k(x)u^{2^*-1}, \ x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases}$$

h and k must be chosen taking into account not only the size of some norm but the shape. Moreover, if $h(x) \equiv 0$, to reach a multiplicity of solutions, some hypotheses about the local behavior of k close to the points of maximum are needed.

1. Introduction

In this paper we will consider the following class of problems:

$$\begin{cases} -\Delta u = \Big(\dfrac{A+h(x)}{|x|^2}\Big)u + k(x)u^{2^*-1}, \ x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $2^* = \frac{2N}{N-2}$, and h and k are continuous, bounded functions, for which we will state appropriate complementary hypotheses. Here $\mathcal{D}^{1,2}(\mathbb{R}^N)$

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denotes the closure space of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$||u||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} := \Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx \Big)^{1/2}.$$

By the Sobolev inequality we can see that $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the class of functions in $L^{2^*}(\mathbb{R}^N)$ the distributional gradient of which satisfies

$$\Big(\int_{\mathbb{R}^N} |\nabla u|^2 dx \Big)^{1/2} < \infty.$$

For $h \equiv 0$ and $k \equiv 1$, the problem is studied by S. Terracini in [18]. In [11] the existence of a positive solution is proved in the case $h = 0$ by using the perturbative method by Ambrosetti-Badiale in [2], even for a more general class of differential operators related to the Caffarelli-Kohn-Nirenberg inequalities that contains our operator. By the perturbative nature of the method, the solutions found in [11] are close to some radial solutions to the unperturbed problem. On the other hand, in [16] Smets obtains the existence of a positive solution for problem (1.1) with $h = 0$, k bounded, $k(0) = \lim_{|x|\to\infty} k(x)$, and dimension $N = 4$.

In this paper we study the existence of positive solutions in the case in which either $h \equiv 0$ and $k \not\equiv 1$ or $k \equiv 1$ and $h \not\equiv 0$ satisfying suitable assumptions. Our results hold in any dimension and are proved using the *concentration-compactness* arguments by P.L. Lions.

It is known that the general problem has an obstruction provided by a Pohozaev-type identity that shows us the particularity of this problem, that is, *the existence of a positive solution depends not only on the size of the functions h and k but also on their shape.* More precisely, assume that u is a variational solution to our equation with $h, k \in \mathcal{C}^1$. Multiplying the equation by $\langle x, \nabla u\rangle$ and with a convenient argument of approximation we get that necessarily

$$\frac{\lambda}{2} \int \langle \nabla h(x), x\rangle \frac{u^2}{|x|^2} dx + \frac{1}{2^*} \int \langle \nabla k(x), x\rangle |u|^{2^*} dx = 0.$$

This behavior makes the problem more interesting to analyze. The existence part of the paper is mainly based on the *concentration-compactness arguments* by P.L. Lions (see [13] and [14]) and involves some qualitative properties of the coefficients that avoids the Pohozaev-type obstruction. We also obtain multiplicity of positive solutions by using variational and topological arguments.

The organization of the paper is as follows. Section 2 is devoted to the study of nonexistence and existence for $k \equiv 1$ and h satisfying suitable conditions. As pointed out above, we mainly use the *concentration-compactness* principle by P.L. Lions. The main result in this part is Theorem 2.5. Section 3 deals with the existence and multiplicity results for the case in which $h \equiv 0$ and k satisfies some convenient conditions. In this part of the paper we will use techniques that previously had been introduced to study related problems by Tarantello in [17] and refined by Cao-Chabrowsky in [6] (see also the references therein).We use this approach in the case that the function k achieves its maximum at a finite number of points. The main result in Section 3 is Theorem 3.13. In Section 4 we study a more general class of functions k; i.e., we treat the case in which k can reach its maximum at infinitely many points, but having only accumulation points at finite distance from the origin. To analyze this case we use the *Lusternik-Schnirelman category*. This point of view is inspired by the study of multiplicity of positive solutions to subcritical problems done by R. Musina in [15]. After several technical lemmas the main result contained in Section 4 is Theorem 4.5.

In a forthcoming paper we will discuss the case of critical equations related to the so-called Caffarelli-Kohn-Nirenberg inequalities.

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2. Perturbation in the linear term

We will study perturbations of a class of elliptic equations in $\mathbb{R}^N$ related to a Hardy inequality interacting with a nonlinear term involving the critical Sobolev exponent. Precisely we will consider the following problem:

$$\begin{cases} -\Delta u = \dfrac{A + h(x)}{|x|^2} u + u^{2^*-1}, \ x \in \mathbb{R}^N, \\ \quad u > 0 \text{ in } \mathbb{R}^N, \ \text{and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{2.1}$$

where $N \geq 3$ and $2^* = \frac{2N}{N-2}$. Hypotheses on h will be given below. To be precise we recall the Hardy inequality.

Lemma 2.1. (Hardy inequality) *Assume that* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$*; then* $\frac{u}{|x|} \in L^2(\mathbb{R}^N)$ *and*

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \le C_N \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

where $C_N = \left(\frac{2}{N-2}\right)^2$ *is optimal and not attained.*

Hereafter we will call $\Lambda_N := C_N^{-1} = \frac{(N-2)^2}{4}$. See for instance [9] for a proof.

The case $h = 0$ of (2.1) has been studied by S. Terracini in [18]; she shows, in particular, that

(1) if $A \ge \Lambda_N$, then problem (2.1) has no positive solution in $\mathcal{D}'(\mathbb{R}^N)$;
(2) if $A \in (0, \Lambda_N)$, then problem (2.1) has the one-dimensional $\mathcal{C}^2$ manifold of positive solutions

$$Z_A = \left\{ w_\mu : w_\mu(x) = \mu^{-\frac{N-2}{2}} w^{(A)}\left(\frac{x}{\mu}\right),\ \mu > 0 \right\}, \tag{2.2}$$

where

$$w^{(A)}(x) = \frac{\left(N(N-2)\nu_A^2\right)^{\frac{N-2}{4}}}{\left(|x|^{1-\nu_A}(1 + |x|^{2\nu_A})\right)^{\frac{N-2}{2}}}, \text{ and } \nu_A = \left(1 - \frac{A}{\Lambda_N}\right)^{\frac{1}{2}}. \tag{2.3}$$

Moreover, if we set

$$Q_A(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - A \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx,$$

then we obtain that

$$\overline{S} \equiv \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{Q_A(u)}{\|u\|_{2^*}^2} = \frac{Q_A(w_\mu)}{\|w_\mu\|_{2^*}^2} = S\left(1 - \frac{A}{\Lambda_N}\right)^{\frac{N-1}{N}}, \tag{2.4}$$

where S is the best constant in the Sobolev inequality. Notice that $\overline{S}$ is attained exactly in the family w_μ defined in (2.2).

2.1. **Nonexistence results.** We begin by proving some nonexistence results that show the fact that in this kind of problem both the size and the shape of the perturbation are important. Define

$$Q(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} \left(\frac{A + h(x)}{|x|^2}\right) u^2 dx, \tag{2.5}$$

$\mathcal{K} = \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2^*} dx = 1 \right\}$, and consider $I_1 = \inf_{u \in \mathcal{K}} Q(u)$.

Lemma 2.2. *Problem* (2.1) *has no positive solution in the following cases*:

(a) *If* $A + h(x) \ge 0$ *in some ball* $B_\delta(0)$ *and* $I_1 < 0$.

(b) *If h is a differentiable function such that $\langle h'(x), x\rangle$ has a fixed sign.*

Proof. We begin by proving nonexistence under hypothesis (a). Suppose that $I_1 < 0$, and let u be a positive solution to (2.1). By classical regularity results for elliptic equations we obtain that $u \in \mathcal{C}^\infty(\mathbb{R}^N \backslash \{0\})$. On the other hand, since $A + h(x) \geq 0$ in $B_\delta(0)$, we obtain that $-\Delta u \geq 0$ in $\mathcal{D}'(B_\delta(0))$. Therefore, since $u \geq 0$ and $u \neq 0$, by the strong maximum principle we obtain that $u(x) \geq c > 0$ in some ball $B_\eta(0) \subset\subset B_\delta(0)$. Let $\phi_n \in \mathcal{C}_0^\infty(\mathbb{R}^N)$, $\phi_n \geq 0$, $||\phi_n||_{2*} = 1$, be a minimizing sequence of I_1. By using $\frac{\phi_n^2}{u}$ as a test function in equation (2.1) we obtain

$$\int_{\mathbb{R}^N} \nabla\Big(\frac{\phi_n^2}{u}\Big)\nabla u = \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2}\phi_n^2 + \int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2}.$$

A direct computation gives

$$2\int_{\mathbb{R}^N} \frac{\phi_n}{u}\nabla\phi_n\nabla u\, dx - \int_{\mathbb{R}^N} \frac{\phi_n^2}{u^2}|\nabla u|^2 dx = \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2}\phi_n^2 + \int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2}, \tag{2.6}$$

and since

$$2\frac{\phi_n}{u}\nabla\phi_n\nabla u - \frac{\phi_n^2}{u^2}|\nabla u|^2 \leq |\nabla\phi_n|^2,$$

we conclude that

$$\int_{\mathbb{R}^N} |\nabla\phi_n|^2 dx \geq \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2}\phi_n^2 + \int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2}.$$

On the other hand, $I_1 < 0$ implies that we can find an integer n_0 such that if $n \geq n_0$,

$$\int_{\mathbb{R}^N} |\nabla\phi_n|^2 - \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2}\phi_n^2 < 0.$$

As a consequence $\int_{\mathbb{R}^N} \phi_n^2 u^{2^*-2} < 0$, for $n \geq n_0$, which contradicts the hypothesis $u > 0$.

Let us now prove (b). By using the Pohozaev multiplier $\langle x, \nabla u\rangle$, we obtain that if u is a positive solution to (2.1), then

$$\int_{\mathbb{R}^N} \frac{\langle h'(x), x\rangle}{|x|^2} u^2 dx = 0,$$

which is not possible if $\langle h'(x), x\rangle$ has a fixed sign and $u \not\equiv 0$.

Corollary 2.3. *Assume either*

i) $A > \Lambda_N$ *and* $h \geq 0$, *or*
ii) $A > \Lambda_N$ *and* $1 \leq \frac{4A}{(N-2)^2\|h\|_\infty}$;

then problem (2.1) *has no positive solution.*

2.2. **The local Palais-Smale condition: existence results.** To prove the existence results we will use a variational approach for the associated functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2}\int_{\mathbb{R}^N} \frac{A+h(x)}{|x|^2} u^2 dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx. \qquad (2.7)$$

We suppose that h satisfies the following hypotheses:

(h0) $A + h(0) > 0$.
(h1) $h \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.
(h2) For some $c_0 > 0$, $A + ||h||_\infty \le \Lambda_N - c_0$.

Critical points of J in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ are solutions to equation (2.1). We begin by proving a local Palais-Smale condition for J. Precisely, we prove the following theorem.

Theorem 2.4. *Suppose that* (h0), (h1), and (h2) *hold, and denote* $h(\infty) \equiv \limsup_{|x|\to\infty} h(x)$. *Let* $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ *be a Palais-Smale sequence for* J, *namely* $J(u_n) \to c < \infty$, $J'(u_n) \to 0$. *If*

$$c < c^* = \frac{1}{N} S^{\frac{N}{2}} \min\left\{ \left(1 - \frac{A+h(0)}{\Lambda_N}\right)^{\frac{N-1}{2}},\ \left(1 - \frac{A+h(\infty)}{\Lambda_N}\right)^{\frac{N-1}{2}} \right\},$$

then $\{u_n\}$ *has a converging subsequence.*

Proof. Let $\{u_n\}$ be a Palais-Smale sequence for J; then according to $(h2)$, $\{u_n\}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^N)$. Then, up to a subsequence, $i)$ $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$, $ii)$ $u_n \to u_0$ almost everywhere, and $iii)$ $u_n \to u_0$ in L^α_{loc}, $\alpha \in [1, 2^*)$. Therefore, by using the *concentration-compactness principle* by P.L. Lions (see [13] and [14]), there exists a subsequence (still denoted by $\{u_n\}$) which satisfies

(1) $|\nabla u_n|^2 \rightharpoonup d\mu \ge |\nabla u_0|^2 + \sum_{j\in\mathcal{J}} \mu_j \delta_{x_j} + \mu_0\delta_0$,
(2) $|u_n|^{2^*} \rightharpoonup d\nu = |u_0|^{2^*} + \sum_{j\in\mathcal{J}} \nu_j \delta_{x_j} + \nu_0\delta_0$,
(3) $S\nu_j^{\frac{2}{2^*}} \le \mu_j$ for all $j \in \mathcal{J} \cup \{0\}$, where $\mathcal{J}$ is at most countable,
(4) $\dfrac{u_n^2}{|x|^2} \rightharpoonup d\gamma = \dfrac{u_0^2}{|x|^2} + \gamma_0\delta_0$,
(5) $\Lambda_N \gamma_0 \le \mu_0$.

To study the concentration at infinity of the sequence we will also need to consider the following quantities:

$$\nu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |u_n|^{2^*} dx, \quad \mu_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} |\nabla u_n|^2 dx$$

and

$$\gamma_\infty = \lim_{R\to\infty} \limsup_{n\to\infty} \int_{|x|>R} \frac{u_n^2}{|x|^2} dx.$$

We claim that $\mathcal{J}$ is finite and for $j \in \mathcal{J}$, either $\nu_j = 0$ or $\nu_j \geq S^{N/2}$. We follow closely the arguments in [3]. Let $\varepsilon > 0$ and let ϕ be a smooth cut-off function centered at x_j, $0 \leq \phi(x) \leq 1$, such that

$$\phi(x) = \begin{cases} 1, & \text{if } |x - x_j| \leq \dfrac{\varepsilon}{2}, \\ 0, & \text{if } |x - x_j| \geq \varepsilon, \end{cases}$$

and $|\nabla \phi| \leq \dfrac{4}{\varepsilon}$. So we get

$$\begin{aligned} 0 &= \lim_{n\to\infty} \langle J'(u_n), u_n\phi\rangle \\ &= \lim_{n\to\infty} \Big(\int_{\mathbb{R}^N} |\nabla u_n|^2 \phi + \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \phi - \int_{\mathbb{R}^N} \frac{A + h(x)}{|x|^2} u_n^2 \phi - \int_{\mathbb{R}^N} \phi |u_n|^{2^*} \Big). \end{aligned}$$

From 1), 2), and 4) and since $0 \notin \operatorname{supp}(\phi)$ we find that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 \phi = \int_{\mathbb{R}^N} \phi d\mu, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} |u_n|^{2^*} \phi = \int_{\mathbb{R}^N} \phi d\nu,$$

and

$$\lim_{n\to\infty} \int_{B_\varepsilon(x_j)} \frac{A + h(x)}{|x|^2} u_n^2 \phi = \int_{B_\varepsilon(x_j)} \frac{A + h(x)}{|x|^2} u_0^2 \phi.$$

Taking limits as $\varepsilon \to 0$ we obtain

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \Big| \int_{\mathbb{R}^N} u_n \nabla u_n \nabla \phi \Big| \to 0.$$

Hence,

$$0 = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \langle J'(u_n), u_n\phi\rangle = \mu_j - \nu_j.$$

By 3) we have that $S\nu_j^{\frac{2}{2^*}} \leq \mu_j$; then we obtain that either $\nu_j = 0$ or $\nu_j \geq S^{N/2}$, which implies that $\mathcal{J}$ is finite. The claim is proved.

Let us now study the possibility of concentration at $x = 0$ and at ∞. Let ψ be a regular function such that $0 \leq \psi(x) \leq 1$,

$$\psi(x) = \begin{cases} 1, & \text{if } |x| > R + 1 \\ 0, & \text{if } |x| < R, \end{cases}$$

and $|\nabla\psi| \le \frac{4}{R}$. From (2.4) we obtain that

$$\frac{\displaystyle\int_{\mathbb{R}^N} |\nabla(u_n\psi)|^2 dx - (A+h(\infty))\int_{\mathbb{R}^N} \frac{\psi^2 u_n^2}{|x|^2}dx}{\Big(\displaystyle\int_{\mathbb{R}^N} |\psi u_n|^{2^*}\Big)^{2/2^*}} \ge S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}. \tag{2.8}$$

Hence

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla(u_n\psi)|^2 dx - (A+h(\infty))\int_{\mathbb{R}^N} \frac{\psi^2 u_n^2}{|x|^2}dx\\ &\ge S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}\Big(\int_{\mathbb{R}^N} |\psi u_n|^{2^*}\Big)^{2/2^*}.\end{aligned}$$

Therefore, we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^N} \psi^2|\nabla u_n|^2 dx + \int_{\mathbb{R}^N} u_n^2|\nabla\psi|^2 dx + 2\int_{\mathbb{R}^N} u_n\psi\nabla u_n\nabla\psi dx\\ &\ge (A+h(\infty))\int_{\mathbb{R}^N}\frac{\psi^2u_n^2}{|x|^2}dx + S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}\Big(\int_{\mathbb{R}^N} |\psi u_n|^{2^*}\Big)^{2/2^*}.\end{aligned}$$

We claim that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\Big\{\int_{\mathbb{R}^N} u_n^2|\nabla\psi|^2dx + 2\int_{\mathbb{R}^N}|u_n|\psi|\nabla u_n||\nabla\psi|dx\Big\} = 0.$$

Using Hölder's inequality we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N}|u_n|\psi|\nabla u_n||\nabla\psi|dx\\ &\le \Big(\int_{R<|x|<R+1}|u_n|^2|\nabla\psi|^2dx\Big)^{1/2}\Big(\int_{R<|x|<R+1}|\nabla u_n|^2dx\Big)^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}\lim_{n\to\infty}\int_{\mathbb{R}^N}|u_n|\psi|\nabla u_n||\nabla\psi|dx &\le C\Big(\int_{R<|x|<R+1}|u_0|^2|\nabla\psi|^2dx\Big)^{1/2}\\ &\le C\Big(\int_{R<|x|<R+1}|u_0|^{2^*}dx\Big)^{2/2^*}\Big(\int_{R<|x|<R+1}|\nabla\psi|^N dx\Big)^{2/N}\\ &\le \overline{C}\Big(\int_{R<|x|<R+1}|u_0|^{2^*}dx\Big)^{2/2^*}.\end{aligned}$$

Therefore, we conclude that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n|\psi|\nabla u_n||\nabla\psi|dx \le \overline{C}\lim_{R\to\infty}\Big(\int_{R<|x|<R+1}|u_0|^{2^*}dx\Big)^{2/2^*}=0.$$

Using the same argument we can prove that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^N}u_n^2|\nabla\psi|^2=0.$$

Then we get

$$\mu_\infty-(A+h(\infty))\gamma_\infty \ge S\Big(1-\frac{A+h(\infty)}{\Lambda_N}\Big)^{\frac{N-1}{N}}\nu_\infty^{2/2^*}. \tag{2.9}$$

Since $\lim\limits_{R\to\infty}\lim\limits_{n\to\infty}\langle J'(u_n),u_n\psi\rangle=0$, we obtain that $\mu_\infty-(A+h(\infty))\gamma_\infty\le\nu_\infty$. Therefore, we conclude that either $\nu_\infty=0$ or $\nu_\infty\ge S^{\frac{N}{2}}\big(1-\frac{A+h(\infty)}{\Lambda_N}\big)^{\frac{N-1}{2}}$.

The same holds for the concentration in $x_0=0$, namely that either

$$\nu_0=0 \text{ or } \nu_0\ge S^{\frac{N}{2}}\Big(1-\frac{A+h(0)}{\Lambda_N}\Big)^{\frac{N-1}{2}}.$$

As a conclusion we obtain

$$\begin{aligned} c &= J(u_n)-\frac{1}{2}\langle J'(u_n),u_n\rangle+o(1)\\ &= \frac{1}{N}\int_{\mathbb{R}^N}|u_n|^{2^*}dx+o(1)=\frac{1}{N}\Big\{\int_{\mathbb{R}^N}|u_0|^{2^*}dx+\nu_0+\nu_\infty+\sum_{j\in\mathcal{J}}\nu_j\Big\}. \end{aligned}$$

If we assume the existence of $j\in\mathcal{J}\cup\{0,\infty\}$ such that $\nu_j\ne0$, then we obtain that $c\ge c^*$, a contradiction with the hypothesis; then up to a subsequence $u_n\to u_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

To find solutions requires considering some path in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ along which the maximum of $J(\gamma(t))$ is less than c^*. To do that, for $H=\max\{h(0),h(\infty)\}$, we consider $\{w_\mu\}$, the one-parameter family of minimizers to problem (2.4) where A is replaced by $A+H$. Then we have the following result.

Theorem 2.5. *Suppose that* (h0), (h1), *and* (h2) *hold. Assume the existence of* $\mu_0>0$ *such that*

$$\int_{\mathbb{R}^N}h(x)\frac{w_{\mu_0}^2(x)}{|x|^2}dx > H\int_{\mathbb{R}^N}\frac{w_{\mu_0}^2(x)}{|x|^2}dx; \tag{2.10}$$

then (2.1) *has at least a positive solution.*

Proof. Let μ_0 be as in the hypothesis; then if we set

$$f(t) = J(tw_{\mu_0}) = \frac{t^2}{2}\Big(\int_{\mathbb{R}^N} |\nabla w_{\mu_0}|^2 dx - \int_{\mathbb{R}^N} \frac{A+h(x)}{|x|^2} w_{\mu_0}^2 dx\Big) - \frac{t^{2^*}}{2^*}\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*} dx,\ t \geq 0$$

we can see easily that f achieves its maximum at some $t_0 > 0$ and we can prove the existence of $\rho > 0$ such that $J(tw_{\mu_0}) < 0$ if $||tw_{\mu_0}|| \geq \rho$. By a simple calculation we obtain that

$$t_0^{2^*-2} = \frac{\int_{\mathbb{R}^N} |\nabla w_{\mu_0}|^2 dx - \int_{\mathbb{R}^N} \frac{A+h(x)}{|x|^2} w_{\mu_0}^2 dx}{\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*} dx},$$

and

$$J(t_0 w_{\mu_0}) = \max_{t\geq 0} J(tw_{\mu_0}) = \frac{1}{N}\Big(\frac{\int_{\mathbb{R}^N} |\nabla w_{\mu_0}|^2 dx - \int_{\mathbb{R}^N} \frac{A+h(x)}{|x|^2} w_{\mu_0}^2 dx}{\big(\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*} dx\big)^{2/2^*}}\Big)^{N/2}. \tag{2.11}$$

Using (2.10) we obtain that

$$J(t_0 w_{\mu_0}) < \frac{1}{N}\Big(\frac{\int_{\mathbb{R}^N} |\nabla w_{\mu_0}|^2 dx - (A+H)\int_{\mathbb{R}^N} \frac{w_{\mu_0}^2}{|x|^2} dx}{\big(\int_{\mathbb{R}^N} |w_{\mu_0}|^{2^*} dx\big)^{2/2^*}}\Big)^{N/2}$$
$$= \frac{1}{N} S^{\frac{N}{2}}\Big(1 - \frac{A+H}{\Lambda_N}\Big)^{\frac{N-1}{2}} \leq c^*. \tag{2.12}$$

We set $\Gamma = \{\gamma \in C([0,1], \mathcal{D}^{1,2}(\mathbb{R}^N)) :\ \gamma(0) = 0 \text{ and } J(\gamma(1)) < 0\}$. Let

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

Since $J(t_0 w_{\mu_0}) < c^*$, we get a mountain-pass critical point u_0. Then we have just to prove that we can choose $u_0 \geq 0$. We give two different proofs.

First proof. Consider the Nehari manifold,

$$M \equiv \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) :\ u \neq 0 \text{ and } \langle J'(u), u\rangle = 0\}$$
$$= \Big\{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) :\ u \neq 0 \text{ and } \int_\Omega |\nabla u|^2 dx = \int_\Omega \frac{A+h(x)}{|x|^2} u^2 dx + \int_\Omega |u|^{2^*} dx\Big\}.$$

Notice that $u_0, |u_0| \in M$. Since u_0 is a mountain-pass solution to problem (2.1), one can prove easily that $c \equiv J(u_0) = \min_{u\in M} J(u)$ (see [19]). Moreover, as $J(|u_0|) = \min_{u\in M} J(u)$, then $|u_0|$ is also a critical point of J.

Second proof. Here we use a variation of the deformation lemma. Since u_0 is a mountain-pass critical point of J, which is even, we have

$$c = J(u_0) = J(|u_0|) = \max_{t>0} J(t|u_0|).$$

Let $t_1 > 0$ be such that $J(t_1|u_0|) < 0$. We set $\gamma_0(t) = t(t_1|u_0|)$ for $t \in [0,1]$. Notice that $\gamma_0 \in \Gamma$ and

$$c = J(|u_0|) = \max_{t\in[0,1]} J(\gamma_0(t)).$$

If $|u_0|$ is a critical point of J, then we are done. If not, then using Lemma 3.7 of [10] we obtain that γ_0 can be deformed to a path $\gamma_1 \in \Gamma$ with $\max_{t\in[0,1]} J(\gamma_1(t)) < c$, a contradiction with the definition of c as a *min-max value*.

Hence, we have nonnegative solution to problem (2.1). The positivity of the solution u_0 is an application of the strong maximum principle by using hypotheses (h0) and (h1).

We give now some sufficient conditions on h to have hypothesis (2.10).

Lemma 2.6. *Suppose one of the following hypotheses holds:*

(1) $h(x) \geq h(0)+c_1|x|^{\nu_{A+H}(N-2)}$ *for* $|x|$ *small and* $c_1 > 0$ *if* $h(0) \geq h(\infty)$, *or*

(2) $h(x) \geq h(\infty) + c_2|x|^{-\nu_{A+H}(N-2)}$ *for* $|x|$ *large and* $c_2 > 0$ *if* $h(\infty) \geq h(0)$;

then there exists $\mu_0 > 0$ *such that* (2.10) *holds.*

Proof. Let $\delta > 0$ be small such that if $|x| < \delta$, then $h(x) \geq h(0) + c_1|x|^{\nu_{A+H}(N-2)}$. For simplicity of notation we set $\nu_{A+H} = \nu$. Let

$$I_{\delta,\mu} = \int_{|x|<\delta} (h(x) - H)\frac{dx}{|x|^{(1-\nu)N+2\nu}(\mu^{2\nu} + |x|^{2\nu})^{N-2}};$$

then

$$I_{\delta,\mu} \geq c_1 \int_{|x|<\delta} \frac{|x|^{\nu(N-2)}dx}{|x|^{(1-\nu)N+2\nu}(\mu^{2\nu} + |x|^{2\nu})^{N-2}}.$$

Since $\nu(N-2) - [(1-\nu)N + 2\nu + 2\nu(N-2)] = -N$, we conclude that $I_{\delta,\mu} \to \infty$ as $\mu \to 0$. On the other hand

$$\int_{|x|\geq\delta} |h(x)-H|\frac{dx}{|x|^{(1-\nu)N+2\nu}(\mu^{2\nu} + |x|^{2\nu})^{N-2}} \leq C \int_{|x|\geq\delta} \frac{dx}{|x|^{(1+\nu)N-2\nu}} \leq C(\delta).$$

Therefore, we get the existence of $\mu_0 > 0$ such that

$$\int_{\mathbb{R}^N} (h(x) - H)\frac{w^2_{\mu_0}(x)}{|x|^2}dx$$

$$\geq \int_{|x|<\delta} (h(x) - H) \frac{w_{\mu_0}^2(x)}{|x|^2} dx - \int_{|x|\geq\delta} |h(x) - H| \frac{w_{\mu_0}^2(x)}{|x|^2} dx > 0.$$

Then the result follows.

The second case follows by using the same argument near infinity.

3. Perturbation of the nonlinear term: Multiplicity of positive solutions

In this section we deal with the following problem:

$$\begin{cases} -\Delta u = \dfrac{\lambda}{|x|^2} u + k(x) u^{2^*-1}, \ x \in \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, \text{ and } u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{cases} \tag{3.1}$$

where $N \geq 3,\ 0 < \lambda < \Lambda_N$, and k is a positive function.

3.1. **Existence.** Assume that k satisfies the hypothesis

(K0) $k \in L^\infty(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ and $||k||_\infty > \max\{k(0), k(\infty)\}$, where $k(\infty) \equiv \limsup_{|x|\to\infty} k(x)$.

We associate to problem (3.1) the following functional:

$$J_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2} \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - \frac{1}{2^*} \int_{\mathbb{R}^N} k(x) |u|^{2^*} dx. \tag{3.2}$$

As in the first section we have the following lemma.

Lemma 3.1. *Let $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ be a Palais-Smale sequence for J_λ, namely $J_\lambda(u_n) \to c < \infty,\ J'_\lambda(u_n) \to 0$. If*

$$c < \tilde{c}(\lambda) = \frac{1}{N} S^{\frac{N}{2}} \min \Big\{ ||k||_\infty^{-\frac{N-2}{2}},\ (k(0))^{-\frac{N-2}{2}} \Big(1 - \frac{\lambda}{\Lambda_N}\Big)^{\frac{N-1}{2}}, \\ (k(\infty))^{-\frac{N-2}{2}} \Big(1 - \frac{\lambda}{\Lambda_N}\Big)^{\frac{N-1}{2}} \Big\},$$

then $\{u_n\}$ has a converging subsequence.

The proof is similar to the proof of Theorem 2.4.

In the case in which k is a radial positive function, we can prove the following improved Palais-Smale condition.

Lemma 3.2. *Define*

$$\tilde{c}_1(\lambda) = \frac{1}{N} S^{\frac{N}{2}} \Big(1 - \frac{\lambda}{\Lambda_N}\Big)^{\frac{N-1}{2}} \min \big\{ (k(0))^{-\frac{N-2}{2}},\ (k(\infty))^{-\frac{N-2}{2}} \big\}.$$

If $\{u_n\} \subset \mathcal{D}^{1,2}(\mathbb{R}^N)$ *is a Palais-Smale sequence for* J_λ*, namely* $J_\lambda(u_n) \to c$, $J'_\lambda(u_n) \to 0$*, and* $c < \tilde{c}_1$*, then* $\{u_n\}$ *has a converging subsequence.*

Remark 3.3. This follows from the fact that the inclusion of $H^1_r(\Omega) \equiv \{u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega),\ u \text{ radial}\}$, where $\Omega = \{x \in \mathbb{R}^N : R_1 < |x| < R_2\}$, in $L^q(\Omega)$ is compact for all $1 \le q < \infty$ and in particular for $q = 2^*$; see [12].

As a consequence we obtain the following existence result.

Theorem 3.4. *Let* k *be a positive function such that* (K0) *is satisfied. Assume that there exists* $\mu_0 > 0$ *such that*

$$\int_{\mathbb{R}^N} k(x) w^{2^*}_{\mu_0}(x)dx > \max\{k(0), k(\infty)\} \int_{\mathbb{R}^N} w^{2^*}_{\mu_0}(x)dx, \qquad (3.3)$$

where w_{μ_0} *is a solution to the problem*

$$\begin{cases} -\Delta w = \dfrac{\lambda}{|x|^2} w + w^{2^*-1},\ x \in \mathbb{R}^N, \\ w > 0 \text{ in } \mathbb{R}^N,\ \text{and } w \in \mathcal{D}^{1,2}(\mathbb{R}^N). \end{cases}$$

Then (3.1) *has at least a positive solution.*

Proof. Since the proof is similar to the proof of Theorem 2.5, we omit it.

Remark 3.5. Assume that one of the following hypotheses holds:

(1) $k(x) \ge k(0) + c_1|x|^{2\nu_\lambda}$ for $|x|$ small and $c_1 > 0$ if $k(0) \ge k(\infty)$, or
(2) $k(x) \ge k(\infty) + c_2|x|^{-2\nu_\lambda}$ for $|x|$ large and $c_2 > 0$ if $k(\infty) \ge k(0)$.

Then there exists $\mu_0 > 0$ such that (3.3) holds.

Let us set

$$b(\lambda) \equiv \begin{cases} +\infty \qquad \text{if } k(0) = k(\infty) = 0 \\ \min\left\{(k(0))^{-\frac{N-2}{2}}\left(1 - \dfrac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}}, (k(\infty))^{-\frac{N-2}{2}}\left(1 - \dfrac{\lambda}{\Lambda_N}\right)^{\frac{N-1}{2}}\right\} \\ \qquad \text{otherwise.} \end{cases}$$

Lemma 3.6. *If* (K0) *holds, there exists* $\varepsilon_0 > 0$ *such that* $||k||_\infty^{-\frac{N-2}{2}} \le b(\varepsilon_0)$ *and*

$$\tilde{c}(\lambda) = \tilde{c} \equiv \frac{1}{N} S^{N/2} ||k||_\infty^{-\frac{N-2}{2}} \qquad (3.4)$$

for any $0 < \lambda \le \varepsilon_0$.

Proof. From $(K0)$ it follows that if ε_0 is sufficiently small then $||k||_\infty^{-\frac{N-2}{2}} \le b(\varepsilon_0)$, and hence from the definition of $\tilde{c}(\lambda)$ we obtain the result.

3.2. **Multiplicity.** To find multiplicity results for problem (3.1) we need the following extra hypotheses on k:

(K1) the set $\mathcal{C}(k) = \Big\{a \in \mathbb{R}^N : k(a) = \max_{x\in\mathbb{R}^N} k(x)\Big\}$ is finite, say $\mathcal{C}(k) = \big\{a_j : 1 \le j \le \mathrm{Card}\,(\mathcal{C}(k))\big\}$;

(K2) there exists $2 < \theta < N$ such that if $a_j \in \mathcal{C}(k)$, then $k(a_j) - k(x) = o(|x-a_j|)^\theta$ as $x \to a_j$.

Consider $0 < r_0 \ll 1$ such that $B_{r_0}(a_j) \cap B_{r_0}(a_i) = \emptyset$ for $i \ne j$, $1 \le i, j \le \mathrm{Card}\,(\mathcal{C}(k))$. Let $\delta = \frac{r_0}{3}$ and for any $1 \le j \le \mathrm{Card}\,(\mathcal{C}(k))$ define the following function:

$$T_j(u) = \frac{\int_{\mathbb{R}^N} \psi_j(x)|\nabla u|^2 dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx}, \quad \text{where} \quad \psi_j(x) = \min\{1, |x-a_j|\}. \tag{3.5}$$

Notice that if $u \not\equiv 0$ and $T_j(u) \le \delta$, then

$$\begin{aligned} r_0 \int_{\mathbb{R}^N\setminus B_{r_0}(a_j)} |\nabla u|^2 dx &\le \int_{\mathbb{R}^N\setminus B_{r_0}(a_j)} \psi_j(x)|\nabla u|^2 dx \\ &\le \int_{\mathbb{R}^N} \psi_j(x)|\nabla u|^2 dx \le \delta \int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{r_0}{3}\int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned}$$

Hence we have the following property.

Lemma 3.7. *Let $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ be such that $T_j(u) \le \delta$; then*

$$\int_{\mathbb{R}^N} |\nabla u|^2 \ge 3 \int_{\mathbb{R}^N\setminus B_{r_0}(a_j)} |\nabla u|^2 dx.$$

As a consequence we obtain the following separation result.

Corollary 3.8. *Consider $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \not\equiv 0$, such that $T_i(u) \le \delta$ and $T_j(u) \le \delta$; then $i = j$.*

Proof. By Lemma 3.7 we obtain that

$$2\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge 3\Big(\int_{\mathbb{R}^N\setminus B_{r_0}(a_i)} |\nabla u|^2 dx + \int_{\mathbb{R}^N\setminus B_{r_0}(a_j)} |\nabla u|^2 dx\Big).$$

If $i \ne j$, we find that

$$2\int_{\mathbb{R}^N} |\nabla u|^2 dx \ge 3\int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

a contradiction if $u \not\equiv 0$.

Consider the Nehari manifold,

$$M(\lambda) = \{u \in \mathcal{D}^{1,2}(\mathbb{R}^N) : \ u \not\equiv 0 \text{ and } \langle J'_\lambda(u), u\rangle = 0\}. \tag{3.6}$$

Therefore, if $u \in M(\lambda)$,

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx = \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx.$$

Notice that for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, $u \not\equiv 0$, there exists $t > 0$ such that $tu \in M(\lambda)$ and for all $u \in M(\lambda)$ we have

$$\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx < (2^* - 1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx;$$

hence, there exists $c_1 > 0$ such that $\forall u \in M(\lambda)$, $||u||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \geq c_1$.

Definition 3.9. *For any* $0 < \lambda < \Lambda_N$ *and* $1 \leq j \leq \operatorname{Card}(\mathcal{C}(k))$, *let us consider* $M_j(\lambda) = \{u \in M(\lambda) : T_j(u) < \delta\}$ *and its boundary* $\Gamma_j(\lambda) = \{u \in M(\lambda) :\ T_j(u) = \delta\}$. *We define*

$$m_j(\lambda) = \inf\{J_\lambda(u) :\ u \in M_j(\lambda)\} \text{ and } \eta_j(\lambda) = \inf\{J_\lambda(u) :\ u \in \Gamma_j(\lambda)\}.$$

The following two lemmas give the behavior of the functional with respect to the critical level $\tilde{c}$.

Lemma 3.10. *Suppose that* (K0), (K1), *and* (K2) *hold; then* $M_j(\lambda) \neq \emptyset$ *and there exists* $\varepsilon_1 > 0$ *such that*

$$m_j(\lambda) < \tilde{c} \quad \text{for all } 0 < \lambda \leq \varepsilon_1 \text{ and } 1 \leq j \leq \operatorname{Card}(\mathcal{C}(k)). \tag{3.7}$$

Proof. We set

$$v_{\mu,j}(x) = \frac{1}{(\mu^2 + |x - a_j|^2)^{\frac{N-2}{2}}} \quad \text{and} \quad u_{\mu,j} = \frac{v_{\mu,j}}{||v_{\mu,j}||_{2^*}}; \tag{3.8}$$

then $||u_{\mu,j}||_{2^*} = 1$ and $\int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx = S$. If

$$t_{\mu,j}(\lambda) = \Big(\frac{\int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx - \lambda \int_{\mathbb{R}^N} |x|^{-2} u_{\mu,j}^2 dx}{\int_{\mathbb{R}^N} k(x)|u_{\mu,j}|^{2^*} dx} \Big)^{\frac{N-2}{4}},$$

then $t_{\mu,j}(\lambda) u_{\mu,j} \in M(\lambda)$. Making the change of variable $x - a_j = \mu y$, we obtain

$$T_j(t_{\mu,j}(\lambda) u_{\mu,j}) = \frac{\int_{\mathbb{R}^N} \psi_j(x) |\nabla u_{\mu,j}|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx} = \frac{\int_{\mathbb{R}^N} \psi_j(a_j + \mu y) |\nabla u_0(y)|^2 dy}{\int_{\mathbb{R}^N} |\nabla u_0(y)|^2 dy},$$

where $u_0(x)$ is $u_{\mu,j}$ to scale $\mu = 1$ and concentrated at the origin. Then

$$\lim_{\mu \to 0} T_j(t_{\mu,j}(\lambda) u_{\mu,j}) = \frac{\int_{\mathbb{R}^N} \psi_j(a_j) |\nabla u_0(y)|^2 dy}{\int_{\mathbb{R}^N} |\nabla u_0(y)|^2 dy} = \psi_j(a_j) = 0,$$

uniformly in λ. Hence we get the existence of μ_0 independent of λ such that if $\mu < \mu_0$, then $t_{\mu,j}(\lambda)u_{\mu,j} \in M_j(\lambda)$. Notice that

$$t_{\mu,j}(\lambda) \geq t_1(\lambda) \equiv \Big(\|k\|_\infty^{-1}\Big(1 - \frac{\lambda}{\Lambda_N}\Big)S\Big)^{\frac{N-2}{4}}.$$

In order to prove (3.7), it is sufficient to show the existence of $\mu < \mu_0$ such that if $0 < \lambda < \varepsilon_1$, then

$$\max_{t \geq t_1(\lambda)} J_\lambda(tu_{\mu,j}) = J_\lambda(t_{\mu,j}(\lambda)u_{\mu,j}) < \tilde{c}.$$

We have

$$\begin{aligned}
&\max_{t \geq t_1(\lambda)} J_\lambda(tu_{\mu,j}) \\
&\leq \max_{t>0} \Big\{ \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} k(x)|u_{\mu,j}|^{2^*} dx \Big\} - \frac{1}{2}\lambda t_1^2(\lambda) \int_{\mathbb{R}^N} \frac{u_{\mu,j}^2}{|x|^2} dx
\end{aligned}$$

and

$$\begin{aligned}
&\max_{t>0} \Big\{ \frac{t^2}{2} \int_{\mathbb{R}^N} |\nabla u_{\mu,j}|^2 dx - \frac{t^{2^*}}{2^*} \int_{\mathbb{R}^N} k(x)|u_{\mu,j}|^{2^*} dx \Big\} \\
&= \frac{1}{N}\Big(\frac{S}{\big(\int_{\mathbb{R}^N} k(x)|u_{\mu,j}|^{2^*} dx\big)^{2/2^*}} \Big)^{N/2}.
\end{aligned}$$

In view of assumption (K2) we have that for some positive constants $\bar{c}_1$ and $\bar{c}_2$

$$\begin{aligned}
&\int_{\mathbb{R}^N} k(x)|u_{\mu,j}|^{2^*} dx = ||k||_\infty - \int_{\mathbb{R}^N} (k(a_j) - k(x))|u_{\mu,j}|^{2^*} dx \\
&= ||k||_\infty - \bar{c}_1 \mu^N \int_{\mathbb{R}^N} \frac{k(a_j) - k(x)}{(\mu^2 + |x - a_j|^2)^N} dx \\
&\geq ||k||_\infty - \bar{c}_1 \mu^N \Big\{ \int_{B_\delta(a_j)} \frac{\bar{c}_2 |x - a_j|^\theta dx}{(\mu^2 + |x - a_j|^2)^N} \\
&\qquad\qquad + 2\|k\|_\infty \int_{\mathbb{R}^N \setminus B_\delta(a_j)} \frac{dx}{(\mu^2 + |x - a_j|^2)^N} \Big\} \\
&\geq ||k||_\infty - \bar{c}_1 \mu^N \Big\{ \mu^{\theta - N} \bar{c}_2 \int_{\mathbb{R}^N} \frac{|y|^\theta dy}{(1 + |y|^2)^N} + 2\|k\|_\infty \int_{|y| \geq \delta} \frac{dy}{|y|^{2N}} \Big\} \\
&= ||k||_\infty + O(\mu^\theta).
\end{aligned}$$

Then we obtain that

$$\begin{aligned}\max_{t\geq t_1(\lambda)} J(tu_{\mu,j}) &\leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}+O(\mu^\theta)} - \frac{1}{2}\lambda t_1^2(\lambda)\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx\\ &\leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu^\theta) - \frac{1}{2}\lambda t_1^2(\lambda)\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx.\end{aligned}$$

Using estimate A.6 from [16] we obtain that for some positive constant c

$$\int_{\mathbb{R}^N}\frac{u_{\mu,j}^2}{|x|^2}dx \geq c\mu^2 \text{ as } \mu \to 0.$$

Therefore we get

$$\begin{aligned}\max_{t\geq t_1(\lambda)} J(tu_{\mu,j}) &\leq \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu^\theta) - \frac{1}{2}c\lambda t_1^2(\lambda)\mu^2\\ &\leq \tilde{c} + \bar{c}_3\mu^\theta - \frac{1}{2}c\lambda t_1^2(\lambda)\mu^2,\end{aligned}$$

where $\bar{c}_3$ is a positive constant. Since from (K2) we have $2 < \theta < N$, we get the existence of ε_1 and μ_0 such that if $\mu < \mu_0$ and $0 < \lambda < \varepsilon_1$, then $\max_{t\geq t_1(\lambda)} J(tu_{\mu,j}) < \tilde{c}$ and the result follows.

We prove now the next result.

Lemma 3.11. *Suppose that* (K0), (K1), *and* (K2) *are satisfied; then there exists* ε_2 *such that for all* $0 < \lambda < \varepsilon_2$ *we have* $\tilde{c} < \eta_j(\lambda)$.

Proof. We argue by contradiction. We assume the existence of $\lambda_n \to 0$ and $\{u_n\}$ such that $u_n \in \Gamma_j(\lambda_n)$ and $J_{\lambda_n}(u_n) \to c \leq \tilde{c} = \frac{1}{N}S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$. We can easily prove that $\{u_n\}$ is bounded. Then up to a subsequence we get the existence of $l > 0$ such that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^2dx = \lim_{n\to\infty}\int_{\mathbb{R}^N}k(x)|u_n|^{2^*}dx = l.$$

Notice that $l \geq S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$. On the other hand, by the definition of $\{u_n\}$ we have

$$\begin{aligned}\frac{1}{N}l + o(1) &= J_{\lambda_n}(u_n)\\ &= \frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_n|^2dx - \frac{\lambda_n}{2}\int_{\mathbb{R}^N}\frac{u_n^2}{|x|^2} - \frac{1}{2^*}\int_{\mathbb{R}^N}k(x)|u_n|^{2^*}dx\end{aligned}$$

$$\leq \frac{1}{N}S^{N/2}||k||_\infty^{-\frac{N-2}{2}} + o(1).$$

Then we conclude that $l = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$, and therefore

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\big(||k||_\infty - k(x)\big)|u_n|^{2^*}dx = 0. \tag{3.9}$$

We set $w_n = \dfrac{u_n}{||u_n||_{2^*}}$; then $||w_n||_{2^*} = 1$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla w_n|^2 dx = S.$$

Hence by using the concentration compactness arguments by P.L. Lions (see also Propositions 5.1 and 5.2 in [18]), we get the existence of $w_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that w_n converges to w_0 weakly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ (up to a subsequence) and one of the following alternatives holds:

(1) $w_0 \not\equiv 0$ and $w_n \to w_0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.

(2) $w_0 \equiv 0$ and either

i) $|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0}$ and $|w_n|^{2^*} \to d\nu = \delta_{x_0}$

or

ii) $|\nabla w_n|^2 \rightharpoonup d\mu_\infty = S\delta_\infty$ and $|w_n|^2 \rightharpoonup d\nu_\infty = \delta_\infty$.

The last case means that

$$\nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|w_n|^{2^*}dx = 1 \quad \text{ and}$$

$$\mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int_{|x|>R}|\nabla w_n|^{2^*}dx = S.$$

If the first alternative holds, from (3.9) we obtain that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\big(||k||_\infty - k(x)\big)w_n^{2^*}dx = \int_{\mathbb{R}^N}\big(||k||_\infty - k(x)\big)w_0^{2^*}dx = 0,$$

a contradiction of the fact that k is not a constant.

Assume that we have the alternative 2 i); then since $T_j(w_n) = T_j(u_n) = \delta$, we conclude that

$$\delta = T_j(w_n) = \lim_{n\to\infty}\frac{\int_{\mathbb{R}^N}\psi_j(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N}|\nabla w_n|^2 dx} = \psi_j(x_0).$$

Hence the concentration is impossible at any point $a_j \in \mathcal{C}(k)$. On the other hand, from (3.9) we obtain that

$$0 = \lim_{n\to\infty}\int_{\mathbb{R}^N}\big(||k||_\infty - k(x)\big)w_n^{2^*}dx = ||k||_\infty - k(x_0),$$

a contradiction.

To analyze concentration at ∞, consider a regular function ξ satisfying

$$\xi(x) = \begin{cases} 1, & \text{if } |x| > R+1 \\ 0, & \text{if } |x| < R, \end{cases}$$

where R is chosen in a such way that $|a_j| < R - 1$ for all j. Then we have

$$\begin{aligned} \delta = T_j(w_n) &= \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} \psi_j(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx} \\ &= \lim_{R\to\infty} \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla w_n|^2 dx + \int_{\mathbb{R}^N} (1-\xi(x))\psi_j(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx}. \end{aligned}$$

Since $\lim\limits_{n\to\infty} \int_{\mathbb{R}^N} (1-\xi(x))\psi_j(x)|\nabla w_n|^2 dx = 0$, we conclude that

$$\delta = \lim_{R\to\infty} \lim_{n\to\infty} \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla w_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla w_n|^2 dx} = 1,$$

a contradiction if we choose $\delta < 1$. So we conclude.

We need now the following lemma, which is suggested by the work of Tarantello [17]. See also [6].

Lemma 3.12. *Assume that* $0 < \lambda < \min\{\varepsilon_1, \varepsilon_2\}$ *where* ε_1 *and* ε_2 *are given by Lemmas* 3.10 *and* 3.11. *Then for all* $u \in M_j(\lambda)$ *there exists* $\rho_u > 0$ *and a differentiable function* $f : B(0, \rho_u) \subset \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ *such that* $f(0) = 1$ *and for all* $w \in B(0, \rho_u)$ *we have* $f(w)(u - w) \in M_j(\lambda)$. *Moreover, for all* $v \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *we have*

$$\langle f'(0), v\rangle = -\frac{2\displaystyle\int_{\mathbb{R}^N} \nabla u \nabla v dx - 2\lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^2} dx - 2^* \int_{\mathbb{R}^N} k(x)|u|^{2^*-2} uv dx}{\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - (2^*-1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx}. \tag{3.10}$$

Proof. Let $u \in M_j(\lambda)$ and let $G : \mathbb{R} \times \mathcal{D}^{1,2}(\mathbb{R}^N) \to \mathbb{R}$ be the function defined by

$$G(t, w) = t\Big(\int_{\mathbb{R}^N} |\nabla(u-w)|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{(u-w)^2}{|x|^2} dx \Big) - t^{2^*-1} \int_{\mathbb{R}^N} k(x)|u-w|^{2^*} dx.$$

Then $G(1, 0) = 0$ and

$$G_t(1, 0) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx - (2^*-1) \int_{\mathbb{R}^N} k(x)|u|^{2^*} dx \neq 0$$

(since $u \in M_j(\lambda)$). Then by using the implicit function theorem we get the existence of $\rho_u > 0$ small enough and of a differentiable function f satisfying the required property. Moreover, notice that

$$\begin{aligned}\langle f'(0), v\rangle &= -\frac{\langle G_w(1,0), v\rangle}{G_t(1,0)}\\ &= -\frac{2\int_{\mathbb{R}^N} \nabla u \nabla v\,dx - 2\lambda \int_{\mathbb{R}^N} \frac{uv}{|x|^2}dx - 2^* \int_{\mathbb{R}^N} k(x)|u|^{2^*-2}uv\,dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}dx - (2^*-1)\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx}.\end{aligned}$$

We are now in position to prove the main result.

Theorem 3.13. *Assume that* (K0), (K1), *and* (K2) *hold; then there exists* ε_3 *small such that for all* $0 < \lambda < \varepsilon_3$ *equation* (3.1) *has* $\operatorname{Card}(\mathcal{C}(k))$ *positive solutions* $u_{j,\lambda}$ *such that*

$$|\nabla u_{j,\lambda}|^2 \to S^{N/2}||k||_\infty^{-(N-2)/2}\delta_{a_j} \text{ and } |u_{j,\lambda}|^{2^*} \to S^{N/2}||k||_\infty^{-N/2}\delta_{a_j} \text{ as } \lambda \to 0. \tag{3.11}$$

Proof. Assume that $0 < \lambda < \varepsilon_3 = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, where ε_0, ε_1, and ε_2 are given by Lemmas 3.6, 3.10, and 3.11. Let $\{u_n\}$ be a minimizing sequence to J_λ in $M_j(\lambda)$; that is, $u_n \in M_j(\lambda)$ and $J_\lambda(u_n) \to m_j(\lambda)$ as $n \to \infty$. Since $J_\lambda(u_n) = J_\lambda(|u_n|)$, we can choose $u_n \geq 0$. Notice that we can prove the existence of c_1 and c_2 such that $c_1 \leq ||u_n||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \leq c_2$. By the Ekeland variational principle we get the existence of a subsequence, denoted also by $\{u_n\}$, such that

$$J_\lambda(u_n) \leq m_j(\lambda) + \frac{1}{n} \text{ and } J_\lambda(w) \geq J_\lambda(u_n) - \frac{1}{n}||w - u_n|| \text{ for all } w \in M_j(\lambda).$$

Let $0 < \rho < \rho_n \equiv \rho_{u_n}$ and $f_n \equiv f_{u_n}$, where ρ_{u_n} and f_{u_n} are given by Lemma 3.12. We set $v_\rho = \rho v$, where $||v||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = 1$; then $v_\rho \in B(0, \rho_n)$ and we can apply Lemma 3.12 to obtain that $w_\rho = f_n(v_\rho)(u_n - v_\rho) \in M_j(\lambda)$.

Therefore, we get

$$\begin{aligned}\frac{1}{n}||w_\rho - u_n|| &\geq J_\lambda(u_n) - J_\lambda(w_\rho) = \langle J'_\lambda(u_n), u_n - w_\rho\rangle + o(||u_n - w_\rho||)\\ &\geq \rho f_n(\rho v)\langle J'_\lambda(u_n), v\rangle + o(||u_n - w_\rho||).\end{aligned}$$

Hence we conclude that

$$\langle J'_\lambda(u_n), v\rangle \leq \frac{1}{n}\frac{||w_\rho - u_n||}{\rho f_n(\rho v)}(1 + o(1)) \text{ as } \rho \to 0.$$

Since $|f_n(\rho v)| \to |f_n(0)| \geq c$ as $\rho \to 0$ and

$$\frac{||w_\rho - u_n||}{\rho} = \frac{||f_n(0)u_n - f_n(\rho v)(u_n - \rho v)||}{\rho}$$
$$\leq \frac{||u_n|||f_n(0) - f_n(\rho v)| + |\rho||f_n(\rho v)|}{\rho} \leq C|f_n'(0)|||v|| + c_3 \leq c,$$

we conclude that $J_\lambda'(u_n) \to 0$ as $n \to \infty$. Hence $\{u_n\}$ is a Palais-Smale sequence for J_λ. Since $m_j(\lambda) < \tilde{c}$ and $\tilde{c} = \tilde{c}(\lambda)$ for $\lambda \leq \varepsilon_0$, from Lemma 3.1 we get the existence result.

To prove (3.11) we follow the proof of Lemma 3.11. Assume $\lambda_n \to 0$ as $n \to \infty$ and let $u_n \equiv u_{j_0,\lambda_n} \in M_{j_0}(\lambda_n)$ be a solution to problem (3.1) with $\lambda = \lambda_n$. Then up to a subsequence we get the existence of $l_1 > 0$ such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)|u_n|^{2^*} dx = l_1.$$

Therefore, as in the proof of Lemma 3.11 we obtain that $l_1 = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (||k||_\infty - k(x))u_n^{2^*} dx = 0.$$

We set $w_n = \dfrac{u_n}{||u_n||_{2^*}}$; then $||w_n||_{2^*} = 1$ and $\lim\limits_{n\to\infty} ||w_n||^2_{\mathcal{D}^{1,2}(\mathbb{R}^N)} = S$. Hence we get the existence of $w_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that one of the following alternatives holds:

(1) $w_0 \not\equiv 0$ and $w_n \to w_0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^N)$.
(2) $w_0 \equiv 0$ and either
    i) $|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0}$ and $|w_n|^{2^*} \to d\nu = \delta_{x_0}$
    or
    ii) $|\nabla w_n|^2 \rightharpoonup d\mu_\infty = S\delta_\infty$ and $|w_n|^{2^*} \rightharpoonup d\nu_\infty = \delta_\infty$.

As in Lemma 3.11, alternative 1 and alternative 2 ii) do not hold. Then we conclude that the unique possible behavior is the alternative 2 i); namely, we get the existence of $x_0 \in \mathbb{R}^N$ such that

$$|\nabla w_n|^2 \rightharpoonup d\mu = S\delta_{x_0} \text{ and } |w_n|^{2^*} \rightharpoonup d\nu = \delta_{x_0}.$$

Since

$$\int_{\mathbb{R}^N} |\nabla w_n|^2 dx = S + o(1) = S\int_{\mathbb{R}^N} |w_n|^{2^*} dx + o(1)$$
$$= \frac{S}{||k||_\infty} \int_{\mathbb{R}^N} k(x)|w_n|^{2^*} dx + o(1) = \frac{S}{||k||_\infty} k(x_0) + o(1),$$

we obtain that $x_0 \in \mathcal{C}(k)$. Using Corollary 3.8, we conclude that $x_0 = a_{j_0}$, and the result follows.

Remark 3.14. As in [4], we can prove the same kind of result under a more general condition on k. For instance, we can assume that k changes sign and the following conditions hold:

(K′1) $\max_{x\in\mathbb{R}^N} k(x) > 0$ and $\mathcal{C}'(k) = \{a \in \mathbb{R}^N : k(a) = \max_{x\in\mathbb{R}^N} k(x)\}$ is a finite set.

(K′2) (K2) holds.

In this case the level at which the Palais-Smale conditions fails becomes

$$\hat{c}(\lambda) = \frac{1}{N} S^{\frac{N}{2}} \min\Big\{ ||k_+||_\infty^{-\frac{N-2}{2}},\ (k_+(0))^{-\frac{N-2}{2}} \Big(1-\frac{\lambda}{\Lambda_N}\Big)^{\frac{N-1}{2}}, \\ (k_+(\infty))^{-\frac{N-2}{2}} \Big(1-\frac{\lambda}{\Lambda_N}\Big)^{\frac{N-1}{2}} \Big\}.$$

4. Category setting

In this section we use the Lusternik-Schnirelman category theory to get multiplicity results for problem (3.1); we refer to [1] for a complete discussion. We follow the argument by Musina in [15]. We assume that k is a nonnegative function and that $0 < \lambda < \overline{\varepsilon}_0$, where $\overline{\varepsilon}_0$ is chosen in a such way that $\big(1 - \frac{\overline{\varepsilon}_0}{\Lambda_N}\big)^{\frac{N-1}{2}} > \frac{1}{2}$ and $\overline{\varepsilon}_0 \le \varepsilon_0$, ε_0 being given in Lemma 3.6. We set for $\delta > 0$

$$\mathcal{C}(k) = \{a \in \mathbb{R}^N : k(a) = ||k(x)||_\infty\}$$

and

$$\mathcal{C}_\delta(k) = \{x \in \mathbb{R}^N : dist(x, \mathcal{C}(k)) \le \delta\}.$$

We suppose that (K2) holds and

(K3) there exist R_0, $d_0 > 0$ such that $\sup_{|x|>R_0} |k(x)| \le ||k||_\infty - d_0$.

Let $M(\lambda)$ be defined by (3.6). Consider $\tilde{M}(\lambda) \equiv \{u \in M(\lambda) :\ J_\lambda(u) < \tilde{c}\}$. Then we have the following local Palais-Smale condition.

Lemma 4.1. *Let $\{v_n\} \subset M(\lambda)$ be such that*

$$J_\lambda(v_n) \to c < \tilde{c} \text{ and } J'_{\lambda|_{M(\lambda)}}(v_n) \to 0; \tag{4.1}$$

then $\{v_n\}$ contains a converging subsequence.

Proof. Assume that $\{v_n\}$ satisfies (4.1); then there exists $\{\alpha_n\} \subset \mathbb{R}$ such that

$$J'_\lambda(v_n) - \alpha_n G'_\lambda(v_n) \to 0 \text{ as } n \to \infty \text{ in } \mathcal{D}^{-1,2}(\mathbb{R}^N), \tag{4.2}$$

where $G_\lambda(u) = \langle J'_\lambda(u), u\rangle$. Since $\{v_n\} \subset M(\lambda)$ and $J_\lambda(v_n) \le \tilde{c}$, we have $r_1 \le ||v_n||_{\mathcal{D}^{1,2}(\mathbb{R}^N)} \le r_2$ for some constants $r_1, r_2 > 0$. Using v_n as a test function in (4.2) we conclude that $\alpha_n \to 0$ as $n \to \infty$. Hence, $\{v_n\}$ is a Palais-Smale sequence for J_λ at the level $c < \tilde{c}$, and then the result follows by using Lemma 3.1.

To prove that $\tilde{M}(\lambda) \neq \emptyset$ we give the next result.

Lemma 4.2. *There exists $\overline{\varepsilon}_1 > 0$ such that if $0 < \lambda < \lambda_0 := \min\{\overline{\varepsilon}_0, \overline{\varepsilon}_1\}$, then $\tilde{M}(\lambda) \neq \emptyset$. Moreover, for any $\{\lambda_n\} \subset \mathbb{R}_+$ such that $\lambda_n \to 0$ as $n \to \infty$ and $\{v_n\} \subset \tilde{M}(\lambda_n)$, there exist $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \mathbb{R}_+$ such that $x_n \to x_0 \in \mathcal{C}(k), r_n \to 0$ as $n \to \infty$, and*

$$v_n - \left(\frac{S}{||k||_\infty}\right)^{\frac{N-2}{4}} u_{r_n}(\cdot - x_n) \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{4.3}$$

where

$$u_r(x) = \frac{C_r}{(r^2 + |x|^2)^{\frac{N-2}{2}}} \tag{4.4}$$

and C_r is the normalizing constant to be $||u_r||_{2^*} = 1$.

Proof. The first assertion follows using the same argument as in Lemma 3.10 since we have

$$\max_{t>0} J_\lambda(tw_{\mu,x}) \le \frac{1}{N}\frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu^\theta) - c\lambda\mu^2 < \tilde{c} \text{ for } \mu \text{ small and } 2 < \theta < N,$$

where $w_{\mu,x}(y) = \dfrac{C}{(\mu^2 + |y-x|^2)^{\frac{N-2}{2}}}$, $x \in \mathcal{C}(k)$, and C is the normalizing constant such that $||w_{\mu,x}||_{2^*} = 1$ (see also [5]). As a consequence, there exists λ_0 such that for all $0 < \lambda < \lambda_0$ the set $\tilde{M}(\lambda)$ is not empty. To prove the second part of the lemma, eventually passing to a subsequence we set

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla v_n|^2 dx = \lim_{n\to\infty} \int_{\mathbb{R}^N} k(x)|v_n|^{2^*} dx = l.$$

Then as in Lemma 3.11 we can prove that $l = S^{N/2}||k||_\infty^{-\frac{N-2}{2}}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (||k||_\infty - k(x)) v_n^{2^*} dx = 0. \tag{4.5}$$

Consider the normalized function $w_n = \dfrac{v_n}{||v_n||_{2^*}}$ and

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} |\nabla w_n|^2 dx = S.$$

Using the concentration-compactness arguments by P.L. Lions, we obtain the existence of $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \mathbb{R}_+$ such that

$$w_n - u_{r_n}(\cdot - x_n) \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{4.6}$$

and $w_n \rightharpoonup w_0 \in \mathcal{D}^{1,2}(\mathbb{R}^N)$. Moreover, by the same argument as in the proof of Lemma 3.11 the weak limit is $w_0 = 0$. We will show now that the concentration at infinity is not possible. Indeed, if concentration at ∞ occurs, by using (4.5) and (K3) we obtain

$$\begin{aligned}||k||_\infty &= \int_{\mathbb{R}^N} k(x) w_n^{2^*} dx + o(1) = \int_{\mathbb{R}^N\setminus B_{R_0}(0)} k(x) w_n^{2^*} dx + o(1)\\ &\le \sup_{|x|>R_0} |k(x)| \int_{\mathbb{R}^N\setminus B_{R_0}(0)} w_n^{2^*} dx + o(1) \le (||k||_\infty - d_0) + o(1),\end{aligned}$$

which is a contradiction. Then the unique possible concentration is at some point $x_0 \in \mathbb{R}^N$. Hence we conclude that, up to a subsequence, $r_n \to 0$ and

$$|\nabla u_{r_n}(x - x_n)|^2 \rightharpoonup S\delta_{x_0}.$$

Using (4.5) it is easy to obtain that $x_0 \in \mathcal{C}(k)$.

Remark 4.3. Notice that as a consequence of the above lemmas we obtain the existence of at least $cat(\tilde{M}(\lambda))$ solutions that eventually can change sign.

Hereafter we concentrate our study on the analysis of $cat(\tilde{M}(\lambda))$, the behavior of the energy, and the positivity of solutions.

If R_0 is as in hypothesis (K3), we define

$$\xi(x) = \begin{cases} x & \text{if } |x| \le R_0,\\ R_0 \dfrac{x}{|x|} & \text{if } |x| \ge R_0,\end{cases}$$

and for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u \neq 0$ we set

$$\Xi(u) = \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla u|^2 dx}{\int_{\mathbb{R}^N} |\nabla u|^2 dx}. \tag{4.7}$$

We recall that for $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ such that $u \neq 0$ we have $t_\lambda(u)u \in M(\lambda)$, where $t_\lambda(u)$ is given by

$$t_\lambda(u) = \Big(\frac{\int_{\mathbb{R}^N} |\nabla u|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx}{\int_{\mathbb{R}^N} k(x)|u|^{2^*} dx} \Big)^{\frac{N-2}{4}}.$$

Let $\Psi_\lambda : \mathbb{R}^N \to \mathcal{D}^{1,2}(\mathbb{R}^N)$ be given by $\Psi_\lambda(x) = t_\lambda(u_{\mu_\lambda}(\cdot - x))u_{\mu_\lambda}(\cdot - x)$, where u_{μ_λ} is given by (3.8), $\mu_\lambda \equiv g(\lambda)$ such that $g(\lambda) \to 0$ as $\lambda \to 0$. Notice that if $x \in \mathcal{C}(k)$ and λ is sufficiently small, then

$$J_\lambda(\Psi_\lambda(x)) = \max_{t>0} J_\lambda(tu_{\mu_\lambda}(\cdot - x)) \leq \frac{1}{N} \frac{S^{N/2}}{||k||_\infty^{\frac{N-2}{2}}} + O(\mu_\lambda^\theta) - c\lambda\mu_\lambda^2 < \tilde{c}. \quad (4.8)$$

Then we can prove the existence of $\lambda_0, c_1, c_2 > 0$ such that for all $0 < \lambda < \lambda_0$ we have $\Psi_\lambda(x) \in \tilde{M}(\lambda)$, $J_\lambda(\Psi_\lambda(x)) = \tilde{c} + o(1)$ as $\lambda \to 0$, and $c_1 < t_\lambda(u_{\mu_\lambda}(\cdot - x)) < c_2$ for all $x \in \mathcal{C}(k)$. As a consequence, taking limits for $\lambda \to 0$ we obtain by Lemma 4.2 that for any $x \in \mathcal{C}(k)$

$$|\nabla \Psi_\lambda(x)|^2 \rightharpoonup d\mu = S^{N/2}||k||_\infty^{-\frac{N-2}{2}} \delta_x \text{ and } |\Psi(x)|^{2^*} \rightharpoonup d\nu = (S||k||_\infty^{-1})^{N/2}\delta_x. \quad (4.9)$$

We prove now the next result.

Lemma 4.4. *For $\lambda \to 0$ we have*

(1) $\Xi(\Psi_\lambda(x)) = x + o(1)$ *uniformly for* $x \in B_{R_0}(0)$.
(2) $\sup\{dist(\Xi(u), \mathcal{C}(k)) : u \in \tilde{M}(\lambda)\} \to 0$.

Proof. Let $x \in B_{R_0}(0)$; then by (4.9) we obtain that, as $\lambda \to 0$,

$$\Xi(\Psi_\lambda(x)) = \frac{\int_{\mathbb{R}^N} \xi(y)|\nabla \Psi_\lambda(x)|^2 dy}{\int_{\mathbb{R}^N} |\nabla \Psi_\lambda(x)|^2 dy} = \frac{\int_{\mathbb{R}^N} \xi(y) d\mu}{\int_{\mathbb{R}^N} d\mu} + o(1) = x + o(1).$$

To prove the second assertion we take $\lambda_n \to 0$ and let $v_n \in \tilde{M}(\lambda_n)$; then by Lemma 4.2 we get the existence of $\{x_n\} \subset \mathbb{R}^N$ and $\{r_n\} \subset \mathbb{R}_+$ such that $x_n \to x_0 \in \mathcal{C}(k), r_n \to 0$ as $n \to \infty$, and

$$v_n - \Big(\frac{S}{||k||_\infty} \Big)^{\frac{N-2}{4}} u_{r_n}(\cdot - x_n) \to 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^N).$$

Since Ξ is a continuous function, we obtain that

$$\Xi(v_n) = \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla v_n|^2 dx}{\int_{\mathbb{R}^N} |\nabla v_n|^2 dx} = \frac{\int_{\mathbb{R}^N} \xi(x)|\nabla u_{r_n}(\cdot - x_n)|^2 dx}{\int_{\mathbb{R}^N} |\nabla u_{r_n}(\cdot - x_n)|^2 dx} + o(1) = \xi(x_0) + o(1).$$

Since $x_0 \in \mathcal{C}(k) \subset B_{R_0}(0)$, we conclude that $\xi(x_0) = x_0$, and the result follows.

We are now able to prove the main result.

Theorem 4.5. *Assume that hypotheses* (K0), (K2), *and* (K3) *hold, and let* $\delta > 0$. *Then there exists* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$, *equation* (3.1) *has at least* $cat_{\mathcal{C}_\delta(k)}\mathcal{C}(k)$ *solutions.*

Proof. Given $\delta > 0$ there exists $\lambda_0(\delta) > 0$ such that by Lemma 4.4 and (4.8), for $0 < \lambda < \lambda_0(\delta)$ we have that $\Psi_\lambda(x) \in \tilde{M}(\lambda)$ for any $x \in \mathcal{C}(k)$, and

$$|\Xi(\Psi_\lambda(x)) - x| < \delta \text{ for all } x \in B_{R_0}(0) \text{ and } \Xi(u) \in \mathcal{C}_\delta(k) \text{ for all } u \in \tilde{M}(\lambda).$$

Let $\mathcal{H}(t,x) = x + t(\Xi(\Psi_\lambda(x)) - x)$, where $(t,x) \in [0,1] \times \mathcal{C}(k)$; then $\mathcal{H}$ is a continuous function and $dist(\mathcal{H}(t,x), \mathcal{C}(k)) \leq \delta$ for all $(t,x) \in [0,1] \times \mathcal{C}(k)$. Hence $\mathcal{H}([0,1] \times \mathcal{C}(k)) \subset \mathcal{C}_\delta(k)$.

Since $\mathcal{H}(0,x) = x$ and $\mathcal{H}(1,x) = \Xi(\Psi_\lambda(x))$, we conclude that $\Xi \circ \Psi_\lambda$ is homotopic to the inclusion $\mathcal{C}(k) \hookrightarrow \mathcal{C}_\delta(k)$. Since J_λ satisfies the Palais-Smale condition below the level $\tilde{c}$, to prove the theorem we need just to prove that $cat(\tilde{M}(\lambda)) \geq cat_{\mathcal{C}_\delta(k)}\mathcal{C}(k)$.

Suppose that $\{M_i\}$, $i = 1, \dots, n_0$, is a closed covering of $\tilde{M}(\lambda)$; then for any $i = 1, \dots, n_0$ there exists a homotopy $\mathcal{H}_i : [0,1] \times M_i \to \tilde{M}(\lambda)$ such that

$$\mathcal{H}_i(0,u) = u \text{ for all } u \in M_i \text{ and } \mathcal{H}_i(1,\cdot) = \text{constant for } i = 1, \dots, n_0.$$

Notice that from (4.8), we obtain that $\Psi_\lambda(\mathcal{C}(k)) \subset \tilde{M}(\lambda)$. We set $\mathcal{C}_i = \Psi_\lambda^{-1}(M_i)$; then $\mathcal{C}_i$ is closed in $\mathcal{C}_\delta(k)$ and $\mathcal{C}(k) \subset \cup_i \mathcal{C}_i \subset \mathcal{C}_\delta(k)$. Then we have only to show that the $\mathcal{C}_i$ are contractible in $\mathcal{C}_\delta(k)$. We set $\mathcal{G}_i : [0,1] \times \mathcal{C}_i \to \mathcal{C}_\delta(k)$, where $\mathcal{G}_i(t,x) = \Xi(\mathcal{H}_i(t, \Psi_\lambda(x)))$. Then

$$\mathcal{G}_i(0,x) = \Xi \circ \Psi_\lambda(x) \text{ for all } x \in \mathcal{C}_i \text{ and } \mathcal{G}_i(1,\cdot) = \text{ constant for } i = 1, \dots, n_0.$$

Since $\Xi \circ \Psi_\lambda$ is homotopic to the inclusion $\mathcal{C}(k) \hookrightarrow \mathcal{C}_\delta(k)$, we have that the $\mathcal{C}_i$ are contractible in $\mathcal{C}_\delta(k)$. To complete the proof it remains to prove that any solution has a fixed sign. We follow the argument used in [8]. Assume that $u = u^+ - u^-$ with $u^+ \geq 0, u^- \geq 0$ and $u^+ \not\equiv 0, u^- \not\equiv 0$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u^\pm|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{|u^\pm|^2}{|x|^2} dx &\geq S(1 - \frac{\lambda}{\Lambda_N})^{\frac{N-1}{N}} \Big(\int_{\mathbb{R}^N} |u^\pm|^{2^*} dx \Big)^{2/2^*} \\ &\geq S(1 - \frac{\lambda}{\Lambda_N})^{\frac{N-1}{N}} ||k||_\infty^{-\frac{2}{2^*}} \Big(\int_{\mathbb{R}^N} k(x)|u^\pm|^{2^*} dx \Big)^{2/2^*}. \end{aligned} \tag{4.10}$$

Since u is a solution to problem (3.1), we obtain that

$$\int_{\mathbb{R}^N} |\nabla u^\pm|^2 dx - \lambda \int_{\mathbb{R}^N} \frac{|u^\pm|^2}{|x|^2} dx = \int_{\mathbb{R}^N} k(x)|u^\pm|^{2^*} dx. \tag{4.11}$$

Therefore, we conclude that

$$\tilde{c} > J_\lambda(u) = \frac{1}{N}\int_{\mathbb{R}^N} k(x)|u|^{2^*}dx = \frac{1}{N}\Big\{\int_{\mathbb{R}^N} k(x)|u^+|^{2^*}dx + \int_{\mathbb{R}^N} k(x)|u^-|^{2^*}dx\Big\}$$
$$\geq \frac{2S^{\frac{N}{2}}}{N}\Big(1-\frac{\lambda}{\Lambda_N}\Big)^{\frac{N-1}{2}}||k||_\infty^{-\frac{N-2}{2}}.$$

Hence we obtain $2(1-\frac{\lambda}{\Lambda_N})^{\frac{N-1}{2}} \leq 1$, which contradicts the choice of λ.

Remark 4.6. i) If $\mathcal{C}(k)$ is finite, then for λ small, equation (3.1) has at least $\mathrm{Card}(\mathcal{C}(k))$ solutions.

ii) We give now a typical example where equation (3.1) has infinitely many solutions. Let $\eta : \mathbb{R} \to \mathbb{R}_+$ such that η is regular, $\eta(0) = 0$, and $\eta(r) = 1$ for $r \geq \frac{1}{2}$. We define k_1 on $[0, 1] \subset \mathbb{R}$ by

$$k_1(r) = \begin{cases} 0 & \text{if } r = 0, \\ 1 - \eta(r)|\sin(\frac{1}{r})|^\theta & \text{if } 0 < r \leq 1, \end{cases}$$

where $2 < \theta < N$. Notice that k_1 has infinitely many global maximums archived on the set

$$\mathcal{C}(k_1) = \Big\{r_n = \frac{1}{n\pi} \text{ for } n \geq 1\Big\}.$$

Now we define k to be any continuous, bounded function such that $k(x) = k_1(|x|)$ if $|x| \leq 1$, $||k||_\infty \leq 1$, and $\lim_{|x|\to\infty} k(x) = 0$. Since for all $m \in \mathbb{N}$, there exists $\delta(m)$ such that $\mathrm{cat}_{\mathcal{C}_\delta(k)}(\mathcal{C}(k)) = m$, then we conclude that equation (3.1) has at least m solutions for $0 < \lambda < \lambda(\delta)$.

iii) Let us note that if δ becomes larger, then $\mathrm{cat}_{\mathcal{C}_\delta(k)}(\mathcal{C}(k))$ decreases, so that Theorem 4.5 is interesting for δ small.

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