Advances in Differential Equations Volume 9, Numbers 7-8, July/August 2004, Pages 797–828

BIFURCATION PROBLEMS ASSOCIATED WITH GENERALIZED LAPLACIANS

Klaus Schmitt and Inbo Sim[1]
Department of Mathematics, University of Utah
Salt Lake City, UT, 84112-0090

(Submitted by: Reza Aftabizadeh)

Abstract. This paper is concerned with bifurcation problems for nonlinear partial differential equations of the form

$$-\text{div}(a(|\nabla u|)\nabla u) = \lambda g(u)$$

which are subject to Dirichlet boundary conditions.

We show the existence of infinitely many nontrivial solutions of the eigenvalue problems in the case where $a(|t|) = |t|^{p-2}$ and $g(t) = |t|^{p-2}t$, $p > 1$. More general situations are also considered.

1. Introduction

This paper is concerned with bifurcation problems of the following boundary-value problem involving a partial differential equation in divergence form subject to Dirichlet boundary conditions

$$\begin{cases} -\text{div}(a(|\nabla u|)\nabla u) = \lambda g(u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$, with smooth boundary,

$$a \in C^1([0,\infty)),\ g \in C(\mathbb{R}),\ \lambda \in \mathbb{R},$$

and a is a strictly increasing function which satisfies a Δ'-condition (see [19]); i.e., there exists a positive constant C such that

$$a(st) \leq Ca(s)a(t), \forall s,t \in (0,\infty).$$

We define the homeomorphism $\varphi : \mathbb{R} \to \mathbb{R}$ by

$$\varphi(s) := a(|s|)s$$

Accepted for publication: June 2004.
AMS Subject Classifications: 35B45, 35B65, 35J60.

[1]Current address: Department of Mathematics and Statistics, Pusan National University, Pusan, 609-735, Republic of Korea

and

$$\Phi(t) := \int_0^t \varphi(s)ds.$$

It follows that φ satisfies a Δ' condition. Since a Δ' condition implies a Δ_2 condition, (Lemma 5.1; [19]), (i.e., there exists a positive constant C such that

$$\varphi(2t) \le C\varphi(t), \forall t \in (0,\infty)),$$

φ satisfies a Δ_2 condition. Hence Φ satisfies a Δ_2 condition (see [1]). We shall assume that $\overline{\Phi}$, the complementary function of Φ in the sense of Orlicz spaces, satisfies a Δ_2 condition (therefore, $W^1L_\Phi(\Omega)$, the Orlicz-Sobolev space determined by Φ (see [1, 19, 21, 33]), is reflexive, (see [1])).

We use Ljusternik-Schnirelmann theory to study problems of type (1.1), an approach similar to that of Coffman ([9]) and Amann ([2]), who have obtained infinitely many eigenvalues for $A(u) = \lambda B(u)$, where A is an odd potential operator which satisfies the $(S)_1$ condition and B is a strongly continuous odd potential operator. In particular we shall demonstrate the existence of an infinite number of bifurcation points under suitable assumptions on a and g, a result which, in some sense, extends the classical bifurcation theorem of Krasnosel'skii, which guarantees that all eigenvalues of the linearization of an operator having potential structure, regardless of their multiplicities, yield bifurcation points.

Our notation is such that when we consider $a(|t|) = |t|^{p-2}$ and $g(t) = |u|^{p-2}u$, $p > 1$, we have the nonlinear *eigenvalue* problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

In [5], Azorero-Alonzo have studied the above p-Laplacian eigenvalue problem and deduced the existence of infinitely many eigenvalues.

Anané and Lindqvist ([4, 27]) have proved that $\lambda_1(p)$, the principal (first, smallest) eigenvalue of all eigenvalues of (1.2) can be variationally characterized as

$$\lambda_1(p) = \inf\Big\{ \int_\Omega |\nabla u|^p dx : \ u \in W_0^{1,p}(\Omega), \ \int_\Omega |u|^p dx = 1 \Big\}$$

and $\lambda_1(p)$ is isolated and simple and has an associated eigenfunction that is strictly positive in Ω.

Del Pino and Manásevich ([11]) have shown that the pair $(\lambda_1(p), 0)$ is a bifurcation point of the perturbed equation

$$\begin{cases} -\operatorname{div}((|\nabla u|^{p-2})\nabla u) = \lambda(|u|^{p-2}u + h(u)), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

when $h : \mathbb{R} \to \mathbb{R}$ is odd and continuous and $th(t) > 0$ for $t \neq 0$ and satisfies following conditions:

- $h(t) = o(|t|^{p-1})$ as $t \to 0$,
- $|h(t)| \leq c + d|t|^{p-1}$, where c and d are constants.

Narukawa ([30]) showed that the pair $(\lambda_1(p), 0)$ is a bifurcation point of the perturbed equation

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = \lambda|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where $a(t) = t^{p-2} + o(t^{p-2})$ and $c_1 t^{p-2} \leq ta'(t) + a(t) \leq c_2 t^{p-2}$, near $t = 0$, where c_1 and c_2 are constants.

García-Huidobro, Manásevich, and Schmitt ([16]) have shown that when Ω is a ball and $\lim_{t\to 0} \frac{a(\sigma t)}{a(t)} = \sigma^{p-2}$, higher eigenvalues exist which are bifurcation points for nontrivial radial solutions of the perturbed eigenvalue problem

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = \lambda(a(|u| + k(|u|)), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where k satisfies conditions as h does in (1.3). We will restrict a throughout to the case where it satisfies an asymptotic homogeneity condition and employ homogenization procedures as in ([24]); that is, we assume that a satisfies

$$\lim_{t\to 0} \frac{a(\sigma t)}{a(t)} = \sigma^{p-2}, \text{ for all } \sigma \in \mathbb{R}_+, \text{ for some } p > 1. \tag{1.6}$$

2. Ljusternik-Schnirelmann theory

We shall next employ Ljusternik-Schnirelmann theory ([3, 9, 10, 35, 38]) to establish the existence of infinitely many eigenvalues of the eigenvalue problems mentioned.

Let X be a real Banach space, X^* its dual, and $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X. Let us consider smooth operators $F, G : X \to X^*$, and consider the associated eigenvalue problem

$$F'(u) = \lambda G'(u), \ \lambda \in \mathbb{R}, \ u \in X. \tag{2.1}$$

We make the following assumptions:

(H1) The space X is a real, reflexive, separable Banach space with $\dim X = \infty$, and $F, G : X \to \mathbb{R}$ are even functionals such that $F, G \in C^1(X, \mathbb{R})$ and $F(0) = G(0) = 0$.

(H2) The operator G' is strongly continuous; i.e.,

$$\text{if } u_n \rightharpoonup u, \text{ then } G'(u_n) \to G'(u),$$

and $G(u) \neq 0, u \in \overline{co}M(\alpha)$ implies $G'(u) \neq 0$, where $\overline{co}M(\alpha)$ is the closed convex hull of $M(\alpha) := \{u \in X : F(u) = \alpha\}$.

(H3) The operator F' is uniformly continuous on bounded sets and satisfies condition $(S)_1$; i.e.,

$$u_n \rightharpoonup u, \ F'(u_n) \to b \text{ implies } u_n \to u.$$

(H4) For each $\alpha > 0$, the level set $M(\alpha)$ is bounded and nonempty and

$$u \neq 0 \quad \text{implies} \quad \langle F'(u), u \rangle > 0, \quad \lim_{t \to +\infty} F(tu) = +\infty,$$

and

$$\inf_{u \in M(\alpha)} \langle F'(u), u \rangle > 0.$$

For fixed $\alpha > 0$, we consider the eigenvalue problem

$$F'(u) = \lambda G'(u), \ u \in M(\alpha), \ \lambda \in \mathbb{R}. \tag{2.2}$$

According to the Lagrange multiplier theorem (Proposition 43.21, [38]), $u \in M(\alpha)$ is a solution of (2.2) if and only if u is a critical point of G with respect to $M(\alpha)$.

Using the definition of the genus (Chapter 7, [32]), let

$$\beta_k := \begin{cases} 0, & \text{if } \mathcal{C}_k = \emptyset, \\ \sup_{C \in \mathcal{C}_k} \inf_{u \in C} |G(u)|, & \text{if } \mathcal{C}_k \neq \emptyset, \end{cases} \tag{2.3}$$

for $k = 1, 2, \ldots$. Here, $\mathcal{C}_k$ denotes the class of all compact symmetric subsets C of $M(\alpha)$ such that the genus of C, $\gamma(C) \geq k$. We shall show that β_k is a critical value of G on $M(\alpha)$. Furthermore, we define χ by

$$\chi := \begin{cases} \text{supremum over all } k \text{ such that } \beta_k > 0, \\ 0 \quad \text{if } \beta_1 = 0. \end{cases}$$

The following theorem (Theorem 2.1) is a modification of Theorem 44.A in [38].

Theorem 2.1. *With the assumptions* $(H1)$ *through* $(H4)$*, the following hold:*

(i) *If* $\beta_k > 0$ *for some* $k \in \mathbb{N}$, *then* (2.2) *possesses a nontrivial solution* u_k *corresponding to an eigenvalue* $\lambda_k \neq 0$ *and* $G(u_k) = \beta_k$. *If* F' *and* G' *are positive homogeneous, then* $\lambda_k = \frac{\alpha}{\beta_k}$.

(ii) *Suppose, for some* $k, p \in \mathbb{N}$, *we have*

$$\beta_k = \beta_{k+1} = \cdots = \beta_{k+p} > 0,$$

and denote by E_k *the set of all solutions* $u \in M(\alpha)$ *of* (2.2) *such that* $G(u) = \beta_k$. *Then*

$$\gamma(E_k) \geq p + 1.$$

(iii) $+\infty > \beta_1 \geq \beta_2 \geq \cdots \geq 0$ *and* $\beta_k \to 0$ *as* $k \to \infty$.

(iv) *If* $\chi = \infty$ *and* $G(u) = 0, u \in \overline{co}M(\alpha)$ *implies* $\langle G'(u), u\rangle = 0$, *then there exists an infinite sequence* $\{\lambda_k\}$ *of distinct eigenvalues for* (2.2) *and* $\lambda_k \to \infty$ *as* $k \to \infty$.

We shall use Theorem 2.1 to show the existence of a sequence of eigenvalues for the problem (2.4) below. J. García Azorero and I. Peral Alonso ([5]) also proved Theorem 2.2 by directly using Amann's theorem (Theorem A, [2]).

Theorem 2.2. *Let us consider eigenvalue problem*

$$\begin{cases} -\text{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.4}$$

$p > 1$.

If $\mathcal{C}_k = \{C \subset M(\alpha) : C$ *symmetric, compact,* $\gamma(C) \geq k\}$, *where*

$$M(\alpha) = \Big\{u \in W_0^{1,p}(\Omega) =: X \mid \frac{1}{p}\int_\Omega |\nabla u|^p dx = \alpha\Big\},\ \alpha > 0$$

and

$$\beta_k := \sup_{C\in\mathcal{C}_k} \inf_{u\in C} G(u),\ G(u) = \frac{1}{p}\int_\Omega |u|^p dx,\ F(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx,$$

then there are infinitely many distinct solutions $\{(\lambda_k, u_k)\}$ *of* (2.4), *such that* $G(u_k) = \beta_k$ *and* $\lambda_k = \frac{\alpha}{\beta_k}$ *with*

$$\lim_{k\to\infty} \lambda_k = \infty.$$

Proof. We need to check the assumptions $(H1)$ through $(H4)$ and that $\beta_k > 0$ for all $k \in \mathbb{N}$ (then $\chi = +\infty$) and $G(u) = 0, u \in \overline{co}M(\alpha)$ implies $\langle G'(u), u\rangle = 0$.

We compute (for more general cases, see [15])

$$\langle F'(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx, \quad \text{for all } v\in X$$

and

$$\langle G'(u), v\rangle = \int_\Omega |u|^{p-2}uv\,dx, \quad \text{for all } v\in X.$$

Further, F and G are $C^1(X,\mathbb{R})$ and $F(0)=G(0)=0$.

Secondly, we need to show the strong continuity of G'. Let $u_n \rightharpoonup u$ in X. By Hölder's inequality and the continuity of the Nemits'kii operators

$$u\mapsto |u|^{p-2}u$$

from $L^p(\Omega)$ into $L^{p'}(\Omega)$, $\frac{1}{p}+\frac{1}{p'}=1$, and the compact imbedding $X\hookrightarrow L^p(\Omega)$, it follows that

$$\begin{aligned}
&||G'(u_n)-G'(u)||_{X^*} = \sup_{||v||_X\le 1} |\langle G'(u_n)-G'(u), v\rangle| \\
&= \sup_{||v||_X\le 1}\Big|\int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)v\,dx\Big| \\
&\le \sup_{||v||_X\le 1}\Big(\int_\Omega \big||u_n|^{p-2}u_n-|u|^{p-2}u\big|^{\frac{p}{p-1}}\,dx\Big)^{\frac{p-1}{p}}\Big(\int_\Omega |v|^p dx\Big)^{\frac{1}{p}} \to 0.
\end{aligned}$$

Since G' is $(p-1)$ homogeneous, $G(u)=\frac{1}{p}\langle G'(u),u\rangle$. So, we get $G(u)\neq 0$, $u\in\overline{co}M(\alpha)$ implies $G'(u)\neq 0$.

The uniform continuity of F' follows from the continuity of the Nemits'kii operators and the inequality

$$||F'(u)-F'(v)||_{X^*}\le\Big[\int_\Omega \big||\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\big|^{\frac{p}{p-1}}\,dx\Big]^{\frac{p-1}{p}}.$$

It is well known (see [24]) that F' satisfies condition (S) in X; i.e.,

$$u_n\rightharpoonup u,\ \lim_{n\to\infty}\langle F'(u_n), u_n-u\rangle = 0 \text{ implies } u_n\to u,$$

and since (S) implies $(S)_1$ ([6]), we have proved (H3).

Using the definition of F, it is immediate that $(H4)$ holds.

Finally, since we can choose, for each $k\in\mathbb{N}$, $S\subset M(\alpha)$ symmetric and compact with $\gamma(S)=k$, and since G is continuous,

$$\beta_k := \sup_{C\in\mathcal{C}_k}\inf_{u\in C} G(u)\ge \inf_{u\in S} G(u) > 0,$$

and $G(u)=0, u\in\overline{co}M(\alpha)$ implies $u=0$. Therefore, $\langle G'(u),u\rangle = 0$. □

3. Regularity of solutions

In this section, we shall establish a uniform estimate on solutions of (1.1); i.e., we show that $||\nabla u||_{L^\infty(\Omega)}$ is bounded above by a constant depending on $||\nabla u||_{L_\Phi(\Omega)}$. This will be used in the proof of Theorem 4.8 in Section 4.

First, we shall obtain the uniform estimate of solutions of the following elliptic equations in divergence form:

$$Qu = \mathrm{div}A(u, \nabla u) + B(u, \nabla u) = 0 \text{ in } \Omega, \tag{3.1}$$

where Ω is an open, bounded subset of $\mathbb{R}^N$ $(N \geq 2)$ with a smooth boundary.

The coefficients A and B satisfy the following conditions:

$$\xi \cdot A(s, \xi) \geq |\xi|^p - a_0(s), \tag{3.2}$$

$$|A(s, \xi)| \leq a_1|\xi|^{p-1} - a_2(s), \tag{3.3}$$

$$|B(s, \xi)| \leq b_0|\xi|^p + b_1(s), \tag{3.4}$$

with constants $p > 1, a_1, b_0 \geq 0$, and appropriate nonnegative functions a_0, a_2, and b_1. Under these hypotheses, Ladyzhenskaya-Ural'tseva ([22]) showed that weak solutions of $Qu = 0$ are locally bounded if $b_0 = 0$ and that bounded weak solutions are Hölder continuous.

If, in addition, A is differentiable and satisfies the conditions

$$|A_\xi|(1 + |\xi|)^2 + |A_s|(1 + |\xi|) \leq \Lambda(|s|)(1 + |\xi|)^p, \tag{3.5}$$

$$a^{ij}\eta_i\eta_j \geq \lambda(|s|)(1 + |\xi|)^{p-2}|\eta|^2, \tag{3.6}$$

for $a^{ij} = \frac{\partial A^i}{\partial \xi_j}$, with Λ increasing, λ decreasing and positive, then bounded weak solutions have Hölder-continuous first derivatives. Recent work has shown that the latter result is true if $(1 + |\xi|)$ is replaced by $|\xi|$ in (3.5) and (3.6) ([12, 36]).

The regularity properties of a weak solution u of (3.1) are mainly investigated under the assumption that A satisfies a certain power growth condition, which means that $A(\cdot, |\xi|)$ grows like $|\xi|^{p-2}$ for some number $p > 1$ (see [12, 17, 22, 36]). For the general growth conditions (or nonstandard growth conditions), we refer to [7, 14, 25, 28, 29].

By adopting the results from [7], we shall infer the boundedness of solutions of (3.1).

We denote hereinafter by Φ an N-function and by Φ_1 an N-function such that $W^1L_\Phi(\Omega)$ is continuously imbedded in $E_{\Phi_1}(\Omega)$. (Throughout we shall follow the standard notation for Orlicz and Orlicz-Sobolev spaces like $E_\Phi(\Omega)$, $L_\Phi(\Omega)$, $W^1L_\Phi(\Omega)$, $W^1_0L_\Phi(\Omega)$, etc.)

Let $\overline{\Phi}$ and $\overline{\Phi}_1$ be N-functions complementary to the N-functions Φ and Φ_1. We suppose that the functions A^i and B are defined for $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, are measurable, and that we have the inequality

$$\overline{\Phi}_1\Big[\sum_{i=1}^{N} |A^i(s,\xi)|\Big] + \overline{\Phi}[B(s,\xi)] \le \Phi_1(\gamma|\xi|) + \Phi(\gamma s), \tag{3.7}$$

where $c \in L^1(\Omega)$ and γ is some constant.

Consider the following elliptic equation subject to Dirichlet boundary conditions:

$$\begin{cases} \sum_{i=1}^{N} \partial_i A^i(u,\nabla u) + B(u,\nabla u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{3.8}$$

where, $A^i,\ i = 1,\dots,N$, and B are Carathéodory functions from $\mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}$ satisfying growth conditions of the form

$$\sum_{i=1}^{N} A^i(s,\xi)\xi_i \ge C(|\xi|) - D(|s|), \tag{3.9}$$

$$|B(s,\xi)| \le E(|s|) + F(|s|)G(|\xi|), \tag{3.10}$$

for $|s| \ge s_0,\ \xi \in \mathbb{R}^N$, and almost every $x \in \Omega$, where s_0 is a positive number, C is an N-function, and D, E, F, and G are increasing functions from $[0,\infty)$ into $[0,\infty)$.

Definition 3.1. *A function $u \in W_0^1L_\Phi(\Omega)$ is called a* <u>*weak solution*</u> *for boundary-value problem* (3.8) *if for every function $\zeta \in W_0^1L_\Phi(\Omega)$ the following integral identity holds:*

$$\int_\Omega \sum_{i=1}^{N} A^i(u,\nabla u)\zeta_{x_i}(x)dx = \int_\Omega B(u,\nabla u)\zeta(x)dx. \tag{3.11}$$

If $u \in W_0^1L_\Phi(\Omega)$, by (3.7), then $A^i(u(\cdot),\nabla u(\cdot)) \in L_{\overline{\Phi}_1}(\Omega)$ and $B(u(\cdot), \nabla u(\cdot)) \in L_{\overline{\Phi}}(\Omega)$. Therefore, the integrals in (3.11) are meaningful.

Theorem 3.2. (Theorem 2, [7]) *Let $N \ge 2$, and let C be an N-function such that*

$$\int_1^\infty \Big(\frac{t}{C(t)}\Big)^{N'-1} dt = \infty, \tag{3.12}$$

where $N' = N/(N-1)$, the Hölder conjugate of N. Let C_N be the function defined by

$$C_N := C \circ H_N^{-1}, \tag{3.13}$$

where

$$H_N(r) := \Big(\int_0^r \Big(\frac{t}{C(t)}\Big)^{N'-1} dt\Big)^{1/N'} \ \textit{for } r \geq 0. \tag{3.14}$$

Assume that

(i) $C \circ G^{-1}$ *is an* N*-function;*
(ii) *there exist constants* $c > 0$ *and* $k > 1$ *such that*

$$D(s) \leq C_N(cs), \tag{3.15}$$

$$E(s) \leq \frac{1}{s} C_N(cs), \tag{3.16}$$

and

$$F(s) \leq \frac{1}{ks}((\overline{(C \circ G^{-1})})^{-1} \circ C_N)(cs), \tag{3.17}$$

for large s*, where* $\overline{C \circ G^{-1}}$ *is a complementary* N*-function of* $C \circ G^{-1}$*.*

Then any weak solution for boundary-value problem (3.8) *is bounded.*

Next, to get the local Hölder continuity of solutions of (3.1), we need to impose more conditions on A and B.

We assume that ψ is a C^1 function satisfying

$$c_1 \leq \frac{t\psi'(t)}{\psi(t)} \leq c_2, \text{ if } t > 0, \tag{3.18}$$

for some positive constants c_1 and c_2, and we define Ψ by

$$\Psi(t) = \int_0^t \psi(s) ds. \tag{3.19}$$

We write $W^{1,\Psi}(\Omega)$ for the class of measurable functions that are weakly differentiable in the open set Ω with

$$\int_\Omega \Psi(|\nabla u|) dx < \infty.$$

We simplify the assumptions in Corollary 1.5 ([26]) for our purpose.

Theorem 3.3. (Corollary 1.5, [26]) *Suppose that*

$$A(s,\xi) \cdot \xi \geq \psi(|\xi|)|\xi|, \tag{3.20}$$

$$|A(s,\xi)| \leq a_1 \psi(|\xi|), \tag{3.21}$$

$$|B(s,\xi)| \leq b_1 \psi(|\xi|)|\xi| + b_2, \tag{3.22}$$

where a_1, b_1, *and* b_2 *are nonnegative constants for* $x \in \Omega$, $|s| \leq M$, *and that* ψ *satisfies* (3.18). *If* B_ρ *and* B_R *are concentric balls in* Ω *with* $0 < \rho \leq R \leq 1$ *and* χ *is a positive constant such that*

$$b_2 R \leq \psi(\chi), \tag{3.23}$$

and $u \in L^\infty(\Omega) \cap W^{1,\Psi}(\Omega)$ *solves* $Qu = 0$ *in* Ω, *then* u *is locally Hölder continuous with*

$$\mathrm{osc}_{B_\rho} u \leq C \left(\frac{\rho}{R}\right)^\alpha (\mathrm{osc}_{B_R} u + \chi R)$$

for positive α *and* C *depending only on* a_1, b_1, M, c_1, c_2, *and* N.

The L^∞ and Hölder estimates for the gradient are proved under the hypotheses

$$a^{ij}\eta_i\eta_j \geq \frac{\psi(|\xi|)}{|\xi|}|\eta|^2, \tag{3.24}$$

$$|a^{ij}| \leq \Lambda \frac{\psi(|\xi|)}{|\xi|}, \tag{3.25}$$

$$|A(s,\xi) - A(t,\xi)| \leq \Lambda_1(1 + \psi(|\xi|))[|s-t|^\alpha], \tag{3.26}$$

$$|B(s,\xi)| \leq \Lambda_1(1 + \psi(|\xi|))|\xi| + b_2, \tag{3.27}$$

where $a^{ij} = \frac{\partial A^i}{\partial \xi_j}$ and b_2 is a nonnegative constant.

Theorem 3.4. (Theorem 1.7, [26]) *Suppose that conditions* (3.24)–(3.27) *hold for some positive constants* $\alpha \leq 1$, Λ, *and* Λ_1 *whenever* s *and* t *are in* $[-M, M]$ *for some positive constant* M, *and* $\xi \in \mathbb{R}^N$. *Then any* $W^{1,\Psi}(\Omega)$ *solution* u *of* $Qu = 0$ *in* Ω *with* $|u| \leq M$ *in* Ω *is in* $C^{1,\beta}(\Omega)$ *for some positive* β *depending on* α, Λ, c_1, c_2, *and* N, *and*

$$|u|_{1+\beta;\Omega'} \leq C(\alpha, \Lambda, c_1, c_2, N, \Lambda_1, \mathrm{dist}(\Omega', \partial\Omega), M), \tag{3.28}$$

where $|u|_{1+\beta;\Omega'}$ *is the norm for the Hölder space* $C^{1,\beta}(\Omega')$ (*for definitions, see* [20]), *for open* $\Omega' \subset\subset \Omega$.

We shall apply Lieberman's method, which reduces the L^∞ estimate to an integral estimate with $M = 0$ (Lemma 2.1, [25]). We assume that there exist a nonnegative constant b_1 and positive increasing functions ψ, f_1, and f_2 such that

$$A(s,\xi) \cdot \xi \geq |\xi|\psi(|\xi|) - f_1(|s|), \tag{3.29}$$

$$|B(s,\xi)| \leq b_1 \xi \cdot A(s,\xi) + f_2(|s|), \tag{3.30}$$

for all s.

Theorem 3.5. (Lemma 2.1, [25]) *If there are constants* $\theta \geq 0, a_1, R > 0$ *such that*

$$t\psi(t) \leq a_1 \Psi(t)^{1+\theta}, \tag{3.31}$$

$$f_1(Rt)\psi(t)t + f_2(Rt)\Psi(t) \leq a_1 \Psi(t)^{2+\theta}, \tag{3.32}$$

for $t > 0$, *then*

$$\sup_{\Omega} \Psi\Big(\frac{|u|}{R}\Big) \leq C(N)\Big[\frac{(1+a_1)(b_1+\theta+1)}{R}\Big]^N \int_{\Omega} \Psi\Big(\frac{|u|}{R}\Big)^{1+N\theta} dx. \tag{3.33}$$

We write $W^{1,\infty}(\Omega)$ to denote the function space consisting of bounded, measurable functions with all weak first derivatives bounded, with norm defined by

$$||u||_{W^{1,\infty}(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |u(x)| + \operatorname{ess\,sup}_{x\in\Omega} |\nabla u(x)|,$$

and, as usual, let

$$W_0^{1,\infty}(\Omega) := \overline{C_0^\infty(\Omega)}^{||\cdot||_{W^{1,\infty}(\Omega)}}$$

and adopt as an equivalent norm

$$||u||_{W_0^{1,\infty}(\Omega)} = \operatorname{ess\,sup}_{x\in\Omega} |\nabla u(x)|.$$

We shall obtain a uniform estimate on solutions of (1.1) by using Lieberman's ([26]) and Cianchi's ([7]) results.

Let us take the N-functions Φ and Φ_1 to be

$$\Phi(t) = \Phi_1(t) := \int_0^t a(|s|)s\, ds. \tag{3.34}$$

Lemma 3.6. *Let us assume that* Φ *satisfies* (3.12) *and there exists* $c > 1$ *such that*

$$\Phi(s) \leq \frac{1}{s}\Phi_N(cs), \tag{3.35}$$

and suppose that a is nonnegative, increasing, and satisfies a Δ_2 *condition. Let us fix a positive constant* $\overline{\lambda}$ *and suppose* $(\lambda, u) \in [-\overline{\lambda}, \overline{\lambda}] \times W_0^1 L_\Phi(\Omega)$ *is a solution of* (1.1). *Then u belongs to* $L^\infty(\Omega)$ *and* $||u||_{L^\infty(\Omega)}$ *is bounded by a constant depending on* $\overline{\lambda}$ *and* $||\nabla u||_{L_\Phi(\Omega)}$.

Proof. Let us take

$$A^i(s,\xi) := a(|\xi|)\xi_i,\ B(s,\xi) := \lambda g(s),$$

and

$$C = \Phi,\ D = 0,\ E = \overline{\lambda}\Phi,\ F = 0,\ G = I,$$

where I is the identity. Since $\Phi \circ I^{-1} = \Phi$ is an N-function, to apply Theorem 3.2, we need to check (3.7), (3.9), (3.10), and (3.15)–(3.17). (3.7) follows from the following inequality:

$$\overline{\Phi}(\varphi(t)) \le \Phi(2t).$$

To show (3.9), we note that

$$\sum_{i=1}^{N} A^i(s,\xi)\xi_i = a(|\xi|)|\xi|^2 \ge \Phi(|\xi|).$$

To show (3.10), we note

$$|B(s,\xi)| = |\lambda g(s)| \le \lambda\Phi(|s|) \le E(|s|).$$

Since we take $D = 0$ and $F = 0$, (3.15) and (3.17) are true. Finally, the assumption (3.35) implies (3.16).

Theorem 3.2 implies that

$$\text{ess sup}_\Omega |u(x)| \le C,$$

where C is a constant. If we choose $b_1 = \overline{\lambda}$, and f_1 and f_2 are positive constants in Theorem 3.5, it is not difficult to check (3.29) and (3.30). Since a satisfies a Δ_2 condition, (3.31) and (3.32) are true with $\theta = 0$, $a_1 = k(2)$, and $R = 1$. Using Theorem 3.5, we see that C depends on $\int_\Omega \Phi(|u|)dx$. Since Φ satisfies a Δ_2 condition, it follows (cf. Chapter 3, [21]) that

$$||u||_{L_\Phi(\Omega)} \to 0 \text{ if and only if } \int_\Omega \Phi(|u|)dx \to 0;$$

therefore, C depends on $||u||_{L_\Phi(\Omega)}$. Using the compact imbedding ($W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Phi(\Omega)$), we see that C depends on $||\nabla u||_{L_\Phi(\Omega)}$. □

Next, we shall obtain the Hölder continuity of solutions of (1.1). Put $\phi(\eta) = a(|\eta|)\eta$ for $\eta \in \mathbb{R}^N$, and we adopt the following notation:

$$\phi_{\eta_j} = (\phi^1_{\eta_j}, \ldots, \phi^N_{\eta_j}),\ \phi^i_{\eta_j} = \frac{\partial \phi^i}{\partial \eta_j},\ \text{for } 1 \le i,j \le N.$$

Then,

$$\phi^i_{\eta_j}(\eta) = a'(|\eta|)\frac{\eta_i\eta_j}{|\eta|} + a(|\eta|)\delta_{ij}.$$

Therefore, if we assume

$$c_1 a(t) \le a'(t)t + a(t) \le c_2 a(t), \forall t > 0, \tag{3.36}$$

we then have a growth condition

$$|\phi^i_{\eta_j}(\eta)| \le c_3 a(|\eta|),\ 1 \le i,j \le N,\ \eta \in \mathbb{R}^N, \tag{3.37}$$

and an ellipticity condition

$$\sum_{i,j=1}^{N} \phi^i_{\eta_j}(\eta)\xi_i\xi_j \geq c_4 a(|\eta|)|\xi|^2,\ \xi \in \mathbb{R}^N. \tag{3.38}$$

Remark 3.7. We note that (3.36) implies (3.18) with $\varphi(t) = a(|t|)t$, and we have $W^1_0 L_\Phi(\Omega) \subset W^{1,\Phi}(\Omega)$.

Lemma 3.8. *Suppose that assumption* (3.36) *holds and a is nonnegative, increasing, and satisfies a Δ_2 condition, and Φ is an N-function as stated in Lemma* 3.6. *Let (λ, u) be a solution as in Lemma* 3.6. *Then, u is Hölder continuous and there exist constants $0 < \theta < 1$ and $C > 0$ depending on $\overline{\lambda}$ and $||\nabla u||_{L_\Phi(\Omega)}$ such that the inequality*

$$|u(x) - u(y)| \leq C|x - y|^\theta \tag{3.39}$$

holds for any $x, y \in \Omega$.

Proof. By Lemma 3.6, $u \in L^\infty(\Omega)$ and (3.36) implies (3.18). It is not hard to check that conditions (3.20)–(3.22) hold with $\varphi(t) = a(|t|)t$. Since $u \in L^\infty(\Omega) \cap W^{1,\Phi}(\Omega)$ solves

$$Qu = \operatorname{div}(a(|\nabla u|)\nabla u) + \lambda g(u) = 0,$$

by Theorem 3.3, u is locally Hölder continuous with

$$\operatorname{osc}_{B_\rho} u \leq C\left(\frac{\rho}{R}\right)^\alpha (\operatorname{osc}_{B_R} u + \chi R) \tag{3.40}$$

for positive α and C depending on $M := \operatorname{ess\,sup}_{x\in\Omega}|u(x)|$, c_2, N, and c_1, whenever B_ρ and B_R are concentric balls in Ω with $0 < \rho \leq R \leq 1$ and χ is any positive number.

Let $x_0 \in \partial\Omega$; then by a standard argument (see Theorem 4.2 in [37]), we have, for $\rho_1 \leq R_1$ and $\gamma < 1$,

$$\operatorname{osc}_{B_{\rho_1}(x_0)} u \leq C\Big\{\left(\frac{\rho_1}{R_1}\right)^{\delta(1-\gamma)} + \epsilon(\rho_1)\Big\}, \tag{3.41}$$

where C and δ depend on α, N, $\sup_{B_{\rho_1}} u, \theta$ and ρ^* (θ and ρ^* are related to the uniform regularity of $\partial\Omega$) and

$$\epsilon(\rho_1) = \underset{\partial\Omega\cap B_{\rho_1^\gamma R_1^{1-\gamma}}}{\operatorname{osc}} u.$$

Combining (3.40) and (3.41) and Lemma 3.6, we have the global Hölder continuity of a solution u of (1.1) depending on $\overline{\lambda}$ and $||\nabla u||_{L_\Phi(\Omega)}$. □

We cannot apply Theorem 3.4 to equation (1.1) directly to get the local boundedness of the gradient of solutions of (1.1) because the right-hand side of (1.1) depends on $u(x)$. To overcome this difficulty, we shall follow the methods given by DiBenedetto ([12]) and Fukagai-Narukawa ([13]).

Let us consider the following elliptic equation,

$$-\text{div}(a_\epsilon(|\nabla v|)\nabla v) = h(x), \tag{3.42}$$

for a uniformly bounded, Hölder-continuous function $h(x)$. Here

$$a_\epsilon(t) := a(\sqrt{\epsilon + t^2}),\ 0 < \epsilon < 1. \tag{3.43}$$

Before we state the result on the local boundedness of the gradient, we shall apply Theorem 1.1 ([14]) to (3.42) to get the local boundedness and local Hölder continuity of solutions of (3.42).

Lemma 3.9. *Let* (3.43) *hold and* $h(x)$ *be a uniformly bounded, Hölder-continuous function on* Ω. *Then, any weak solution* $v_\epsilon \in W^1L_\Phi(\Omega)$ *of* (3.42) *is in* $L^\infty(\Omega)$, *and therefore* $v_\epsilon \in W^1L_\Phi(\Omega) \cap L^\infty(\Omega)$ *is Hölder continuous in* Ω *with exponent* $0 < \theta < 1$ *depending only on* $||v_\epsilon||_{L^\infty(\Omega)}$ *and* $||h||_{L^\infty(\Omega)}$. *Furthermore, for any domain* $\Omega' \subset\subset \Omega$, *there exists a constant* $M_1 > 0$ *depending only on* $||v_\epsilon||_{L^\infty(\Omega)}$ *and* $||h||_{L^\infty(\Omega)}$ *and* $\text{dist}(\Omega', \partial\Omega)$ *but independent of* $\epsilon \in (0,1)$ *explicitly such that the inequality*

$$|v_\epsilon(x) - v_\epsilon(y)| \leq M_1|x-y|^\theta$$

holds for any $x, y \in \Omega'$.

Proof. Let v_ϵ be a solution of (3.42). If we put $\varphi_\epsilon(t) := a_\epsilon(|t|)t$, then φ_ϵ has the following properties:

(i) $\varphi_\epsilon(0) = 0,\ \varphi_\epsilon(t) > 0$ for $t0,\ \lim_{t\to\infty}\varphi_\epsilon(t) = \infty$;
(ii) φ_ϵ is increasing;
(iii) φ_ϵ is continuous;
(iv) the N-function Φ_ϵ given by

$$\Phi_\epsilon(t) := \int_0^t \varphi_\epsilon(s)ds,\ t \geq 0,$$

satisfies a Δ_2 condition near infinity (because a satisfies a Δ_2 condition).

Then by Theorem 1.1 and Remark 1.1, [14], a weak solution $v_\epsilon \in W^1L_\Phi(\Omega)$ of (3.16) is bounded by a finite constant depending on $||u||_{L^1(\Omega)}$ (we take $k_0 = 0$ and $\alpha < 1 + \frac{1}{n}$); therefore, by the continuous imbedding $W^1L_\Phi(\Omega) \hookrightarrow L^1(\Omega)$, it is bounded by $||u||_{W^1L_\Phi(\Omega)}$.

To show the local Hölder continuity of bounded solutions, we need to check (3.36). We compute

$$a(\sqrt{\epsilon+t^2}) \le a'_\epsilon(t)t + a_\epsilon(t) = \frac{a'(\sqrt{\epsilon+t^2})t^2}{\sqrt{\epsilon+t^2}} + a(\sqrt{\epsilon+t^2})$$
$$\le a'(\sqrt{\epsilon+t^2})\sqrt{\epsilon+t^2} + a(\sqrt{\epsilon+t^2}) \le c_2 a(\sqrt{\epsilon+t^2}).$$

By Lemma 3.8, v_ϵ is Hölder continuous in Ω with exponent $0 < \theta < 1$ depending only on $||v_\epsilon||_{L^\infty(\Omega)}$. Furthermore, for any domain $\Omega' \subset\subset \Omega$, there exists a constant $M_1 > 0$ depending only on $||v_\epsilon||_{L^\infty(\Omega)}$ and $||h||_{L^\infty(\Omega)}$ and $\operatorname{dist}(\Omega', \partial\Omega)$, but independent of $\epsilon \in (0,1)$ explicitly, such that the inequality

$$|v_\epsilon(x) - v_\epsilon(y)| \le M_1|x-y|^\theta$$

holds for any $x, y \in \Omega'$. □

Therefore, we obtain the local boundedness of the gradient from Theorem 3.4 and the global uniform estimate on the gradient of solutions on Ω as a sequence of local estimates by using a local reflection method (pp. 190–192, [13]).

Now, we are ready to state our main theorem of this section. We shall omit the proof of Theorem 3.10, since it is very similar to the proof of Proposition 3.8 in [13].

Theorem 3.10. *Let* (3.36) *hold, and suppose a is nonnegative, increasing, and satisfies a Δ_2 condition, and Φ is an N-function stated in Lemma* 3.6. *Let $\overline{\lambda}$ be a positive constant and $(\lambda, u) \in [-\overline{\lambda}, \overline{\lambda}] \times W_0^1 L_\Phi(\Omega)$ be a solution of* (1.1). *Then, u belongs to $W_0^{1,\infty}(\Omega)$ and $||\nabla u||_{L^\infty(\Omega)}$ is bounded above by constants depending only on $\overline{\lambda}$ and $||\nabla u||_{L_\Phi(\Omega)}$.*

4. The main theorems

In this section, we consider bifurcation problems for the following eigenvalue problem:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = \lambda(a(|u|)u + k(u)), & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}^N$ is an open and bounded subset with a smooth boundary,

$$a \in C^1([0,\infty)),\ k \in C(\mathbb{R}),\ \lambda \in \mathbb{R} \tag{4.2}$$

and we assume that a is a strictly increasing function that satisfies a Δ' condition, (1.6), and (3.36), and that $\overline{\Phi}$, the complementary function of Φ, satisfies a Δ_2 condition and (3.35).

Before stating the main results, we establish the continuity of the solution operator, for the following problem.

Let us consider the boundary-value problem

$$\begin{cases} -\mathrm{div}(a(|\nabla u|)\nabla u) = h, & x \in \Omega \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{4.3}$$

where $a : [0,\infty) \to [0,\infty)$ is an increasing function.

We let ϕ and Φ be defined as before.

It is well-known that for each $h \in (W_0^1 L_\Phi(\Omega))^*$, (4.3) has a unique solution u_h (Theorem 1, [23]).

This defines an operator $P_T : (W_0^1 L_\Phi(\Omega))^* \to W_0^1 L_\Phi(\Omega)$ by

$$u_h := P_T(h), \tag{4.4}$$

where $\langle Tu, v\rangle := \int_\Omega a(|\nabla u|)\nabla u \cdot \nabla v\, dx,\ \forall v \in W_0^1 L_\Phi(\Omega) =: X$.

Let us define operators B and $K : X \to X^*$ by

$$\begin{aligned} \langle B(\lambda, u), v\rangle :=& \lambda \int_\Omega a(|u|)uv\, dx,\ \forall u, v \in X, \\ \langle K(\lambda, u), v\rangle :=& \lambda \int_\Omega k(u)v\, dx,\ \forall u, v \in X, \end{aligned} \tag{4.5}$$

where k satisfies following conditions:

- $k(t) = o(|\varphi(t)|)$ as $t \to 0$,
- $|k(t)| \le c_1 + c_2|\varphi(t)|, \forall t \in \mathbb{R}$, where c_1 and c_2 are constant.

Since

$$\overline{\Phi}(\varphi(t)) \le t\varphi(t) \le \int_t^{2t} \varphi(s)ds \le \Phi(2t), \tag{4.6}$$

and $L_\Phi = \tilde{L}_\Phi = E_\Phi$ (because Φ satisfies a Δ_2 condition), the operators T, B, and K are well-defined.

Lemma 4.1. *The operators T, B, and K are continuous operators, and B and K are compact operators.*

Proof. In Lemma 3.4, [15], the continuity of T, and in Lemma 2.1, [8], the continuity of B were proved. We similarly can prove the continuity of K.

Let $u_n \rightharpoonup u$ in X; then $u_n \to u$ in X, as follows from the compactness of the imbedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Phi(\Omega)$. Due to Hölder's inequality and the continuity of the Nemits'kii operator from $L_\Phi(\Omega)$ to $L_{\overline{\Phi}}(\Omega)$ ([23]) given by

$$u \mapsto a(|u|)u, \tag{4.7}$$

we have

$$\begin{aligned}|\langle B(\cdot,u_n)-B(\cdot,u),v\rangle| &= |\lambda|\Big|\int_\Omega \Big(a(|u_n|)u_n-a(|u|)u\Big)v\,dx\Big| \\ &\le |\lambda|\Big\|a(|u_n|)u_n-a(|u|)u\Big\|_{L_{\overline{\Phi}}(\Omega)}\|v\|_{L_\Phi(\Omega)}\to 0.\end{aligned}$$

Similarly, we can prove the compactness of K. □

Since $\overline{\Phi}$ is increasing and φ satisfies a Δ' condition, we have

$$\overline{\Phi}\Big(\frac{1}{C}\frac{\varphi(u)}{\varphi(C_1||u||_{L_\Phi(\Omega)})}\Big)\le\overline{\Phi}\Big(\varphi\Big(\frac{u}{C_1||u||_{L_\Phi(\Omega)}}\Big)\Big),$$

where C is the positive constant implied by the Δ' condition for φ and $C_1\ge 2$. Using (4.6) and the monotonicity of Φ, we have

$$\overline{\Phi}\Big(\varphi\Big(\frac{u}{C_1||u||_{L_\Phi(\Omega)}}\Big)\Big)\le\Phi\Big(\frac{2}{C_1}\frac{u}{||u||_{L_\Phi(\Omega)}}\Big)\le\Phi\Big(\frac{u}{||u||_{L_\Phi(\Omega)}}\Big).$$

Therefore, using the definition of the Luxemburg norm, we have

$$||\varphi(u)||_{L_{\overline{\Phi}}(\Omega)}\le C\varphi(C_1||u||_{L_\Phi(\Omega)}).$$

Using the compactness of the imbedding

$$W_0^1L_\Phi(\Omega)\hookrightarrow L_\Phi(\Omega),$$

and the monotonicity of φ, we have

$$||\varphi(u)||_{L_{\overline{\Phi}}(\Omega)}\le C\varphi(C_1||u||_{L_\Phi(\Omega)})\le C\varphi(C_2||u||_X), \tag{4.8}$$

where C_2 is a positive constant. If $C_2>1$, since φ satisfies a Δ_2 condition, then we have

$$\varphi(C_2||u||_X)\le C_3\varphi(||u||_X),$$

or if $C_2\le 1$, since φ is increasing, then we have

$$\varphi(C_2||u||_X)\le\varphi(||u||_X).$$

Hence we have

$$||\varphi(u)||_{L_{\overline{\Phi}}(\Omega)}\le C_4\varphi(||u||_X), \tag{4.9}$$

where C_4 is a positive constant.

Lemma 4.2. *The operator K satisfies*

$$\lim_{||u||_X\to 0}\frac{||K(\cdot,u)||_{X^*}}{\varphi(||u||_X)}=0.$$

Proof. By definition of the norm in X^* we have

$$\begin{aligned}\lim_{||u||_X\to 0}\frac{||K(\cdot,u)||_{X^*}}{\varphi(||u||_X)} &= \lim_{||u||_X\to 0}\sup_{||v||_X\le 1}\frac{1}{\varphi(||u||_X)}|\langle K(\cdot,u),v\rangle| \\ &= \lim_{||u||_X\to 0}\sup_{||v||_X\le 1}\frac{|\lambda|}{\varphi(||u||_X)}\left|\int_\Omega k(u)v\,dx\right| \qquad (4.10)\\ &\le |\lambda|\lim_{||u||_X\to 0}\sup_{||v||_X\le 1}\int_\Omega\frac{|k(u)|}{|\varphi(u)|}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx.\end{aligned}$$

Define, for $\delta>0$, the set $\Omega_\delta(u):=\{x\in\Omega:|\varphi(u(x))|\ge\delta\}$. Note that $|\Omega_\delta(u)|\to 0$, as $||u||_X\to 0$, where $|\Omega_\delta(u)|$ is the measure of the set $\Omega_\delta(u)$. Then, for any given $\epsilon>0$, there exists $\delta>0$ such that

$$\frac{|k(u)|}{|\varphi(u)|}\le\epsilon,$$

for $|\varphi(u)|<\delta$. Thus we can split the last integral in (4.10) into two integrals, over $\Omega\backslash\Omega_\delta(u)$ and $\Omega_\delta(u)$, respectively. For these integrals we have the following estimates: For the set $\Omega\backslash\Omega_\delta(u)$, it follows from Hölder's inequality, the fact that $E_{\overline{\Phi}}(\Omega)=\tilde{L}_{\overline{\Phi}}(\Omega)=L_{\overline{\Phi}}(\Omega)$ (since $\overline{\Phi}$ satisfies a Δ_2 condition) and $\overline{\Phi}(\varphi(u))\le u\varphi(u)\le\Phi(2u)$, and (4.9), that

$$\begin{aligned}&\int_{\Omega\backslash\Omega_\delta(u)}\frac{|k(u)|}{|\varphi(u)|}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx\le\epsilon\int_{\Omega\backslash\Omega_\delta(u)}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx \qquad (4.11)\\ &\le\epsilon\int_\Omega\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx\le\epsilon\Big\|\frac{|\varphi(u)|}{\varphi(||u||_X)}\Big\|_{L_{\overline{\Phi}}(\Omega)}||v||_{L_\Phi(\Omega)}\le\epsilon C,\end{aligned}$$

where C is some constant. For the set $\Omega_\delta(u)$, we get

$$\begin{aligned}&\int_{\Omega_\delta(u)}\frac{|k(u)|}{|\varphi(u)|}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx\le\int_{\Omega_\delta(u)}\frac{c_1+c_2|\varphi(u)|}{|\varphi(u)|}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx\\ &\le\frac{c_1}{\delta}\int_{\Omega_\delta(u)}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx+\int_{\Omega_\delta(u)}\frac{c_2|\varphi(u)|}{\varphi(||u||_X)}|v|dx \qquad (4.12)\\ &\le\Big(\frac{c_1}{\delta}+c_2\Big)\int_{\Omega_\delta(u)}\frac{|\varphi(u)|}{\varphi(||u||_X)}|v|dx.\end{aligned}$$

Since $1\in L_\Phi(\Omega)$ and

$$||1||_{L_\Phi(\Omega_\delta(u))}=\inf\Big\{k:\int_{\Omega_\delta(u)}\Phi(\frac{1}{k})dx\le 1\Big\}=\inf\Big\{k:\Phi(\frac{1}{k})\le\frac{1}{|\Omega_\delta(u)|}\Big\}, \qquad (4.13)$$

it follows that $||1||_{L_\Phi(\Omega_\delta(u))} \to 0$ as $||u||_X \to 0$. By the above argument, we get

$$\int_\Omega \frac{|\varphi(u)|}{\varphi(||u||_X)}|v|\ dx < \infty,$$

and since Φ satisfies a Δ_2 condition, by the generalized Hölder's inequality (p. 118, [19]) and (4.9), we have

$$\left(\frac{c_1}{\delta} + c_2\right) \int_{\Omega_\delta(u)} \frac{|\varphi(u)|}{\varphi(||u||_X)}|v|\ dx \le C||v||_X||1||_{L_\Phi(\Omega_\delta(u))} \to 0, \tag{4.14}$$

where C is an imbedding constant. □

We need the following two additional lemmas.

Lemma 4.3. *If a satisfies a strict monotonicity condition, then T satisfies condition (S).*

Proof. Let $u_n \rightharpoonup u$ and

$$\lim_{n\to\infty} \langle T(u_n), u_n - u\rangle = 0.$$

Since a satisfies a monotonicity condition, that is, for all $\xi, \eta \in \mathbb{R}^N$,

$$[a(|\xi|)\xi - a(|\eta|)\eta] \cdot (\xi - \eta) \ge [a(|\xi|)|\xi| - a(|\eta|)|\eta|](|\xi| - |\eta|) \ge 0,$$

we have

$$\langle T(u_n) - T(u), u_n - u\rangle = \int_\Omega [a(|\nabla u_n|)\nabla u_n - a(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) dx \ge 0;$$

therefore, since $\lim_{n\to\infty}\langle T(u_n), u_n - u\rangle = 0$ and T is bounded, we have

$$\int_\Omega f_n(x) dx \to 0,$$

where $f_n(x) := [a(|\nabla u_n|\nabla u_n) - a(|\nabla u|)\nabla u] \cdot (\nabla u_n - \nabla u) \ge 0$. Hence, we have

$$f_n(x) \to 0,\ \text{for almost all } x \in \Omega.$$

On the other hand, by the compact imbedding $W_0^1 L_\Phi(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$u_n(x) \to u(x),\ \text{for almost all } x \in \Omega.$$

Therefore, there exists $M \subset \Omega$, such that $|\Omega\backslash M| = 0$ and

$$f_n(x) \to 0 \text{ and } u_n(x) \to u(x) \text{ in } M.$$

Since $|\nabla u(x)|$ is finite almost everywhere, we can also suppose that $|\nabla u(x)|$ is finite on M (we can change M by a set of measure zero, if necessary). We assert that we have

$$\lim_{n\to\infty} \nabla u_n(x) = \nabla u(x), \text{ for } x \in M. \tag{4.15}$$

It follows from $a(|\xi|)\xi \cdot \xi \geq \Phi(|\xi|)$ that

$$\begin{aligned} f_n(x) &= [a(|\nabla u_n(x)|)\nabla u_n(x) - a(|\nabla u(x)|)\nabla u(x)] \cdot (\nabla u_n(x) - \nabla u(x)) \\ &= a(|\nabla u_n(x)|)|\nabla u_n(x)|^2 + a(|\nabla u(x)|)|\nabla u(x)|^2 \\ &- a(|\nabla u_n(x)|)\nabla u_n(x) \cdot \nabla u(x) - a(|\nabla u(x)|)\nabla u(x) \cdot \nabla u_n(x) \\ &\geq \Phi(|\nabla u_n(x)|) + \Phi(|\nabla u(x)|) \\ &- a(|\nabla u_n(x)|)|\nabla u_n(x)||\nabla u(x)| - a(|\nabla u(x)|)|\nabla u(x)||\nabla u_n(x)|. \end{aligned} \tag{4.16}$$

It follows from (4.16) that $|\nabla u_n(x)|$ is bounded, since $\Phi(|\xi|)$ is increasing faster than $a(|\xi|)|\xi|$. From every subsequence, we can then extract a further subsequence which is convergent for $x \in M$ (once again, we can change M by a set of measure zero, if necessary). But, by virtue of the strict monotonicity condition, namely,

$$[a(|\xi'|)\xi' - a(|\xi|)\xi] \cdot (\xi' - \xi) > 0, \text{ for } \xi' \neq \xi,$$

such a limit is $\nabla u(x)$ (note:

$$0 < [a(|\nabla u_n(x)|)\nabla u_n(x) - a(|\nabla u(x)|)\nabla u(x)] \cdot (\nabla u_n(x) - \nabla u(x)) = f_n(x) \to 0,$$

for $x \in M$); hence, assertion (4.15) holds. Therefore, we have

$$\nabla u_n(x) \to \nabla u(x), \text{ a.e. on } \Omega.$$

By Egoroff's theorem (see [34]), for each $\epsilon > 0$, there exists $M' \subset \Omega$ such that $|\Omega \backslash M'| \leq \epsilon$ and $\nabla u_n(x) \to \nabla u(x)$ uniformly on M'. Since $|\nabla u_n(x)|$ and $|\nabla u(x)|$ are bounded, we compute

$$\begin{aligned} &\int_\Omega \Phi(|\nabla u_n(x) - \nabla u(x)|)dx \\ &= \int_{\Omega \backslash M'} \Phi(|\nabla u_n(x) - \nabla u(x)|)dx + \int_{M'} \Phi(|\nabla u_n(x) - \nabla u(x)|)dx \\ &= \int_{M \backslash M'} \Phi(|\nabla u_n(x) - \nabla u(x)|)dx + \int_{M'} \Phi(|\nabla u_n(x) - \nabla u(x)|)dx \\ &\leq c \cdot \epsilon + |M'|\Phi(\epsilon). \end{aligned}$$

Since Φ satisfies a Δ_2 condition, it follows (cf. Chapter 3, [21]) that

$$||\nabla u_n - \nabla u||_{L_\Phi} \to 0 \text{ if and only if } \int_\Omega \Phi(|\nabla u_n - \nabla u|)dx \to 0;$$

therefore, we have $u_n \to u$ in $W_0^1 L_\Phi(\Omega)$. □

Let us define

$$\begin{aligned} \langle J(u), v\rangle &:= \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v\, dx, \\ \langle f(\lambda, u), v\rangle &:= \lambda \int_\Omega |u|^{p-2} uv\, dx, \end{aligned} \tag{4.17}$$

and

$$T_\sigma(u) := \begin{cases} \frac{1}{\varphi(\sigma)} T(\sigma u), & \text{if } \sigma \in (0,1], \\ J(u), & \text{if } \sigma = 0, \end{cases} \tag{4.18}$$

and

$$f_\sigma(\lambda, u) := \begin{cases} \frac{1}{\varphi(\sigma)}(B(\lambda, \sigma u) + K(\lambda, \sigma u)), & \text{if } \sigma \in (0,1], \\ f(\lambda, u), & \text{if } \sigma = 0. \end{cases} \tag{4.19}$$

The proof of the following lemma is very similar to the proof of Theorem 6.4 in ([24]). The result will be important for our purposes in establishing necessary conditions for bifurcation; see e.g. the proof of Theorem 4.7.

Lemma 4.4. *If a satisfies a strict monotonicity condition and* (1.6), *and* $\{\sigma_n\} \subset (0,1]$ *is such that* $\sigma_n \to \sigma$, *then* T_{σ_n} *satisfies condition* (S) *uniformly; i.e., if* $\{n_k\} \subset \mathbb{N}$ *and* $v_{n_k} \rightharpoonup v$ *and* $w_{n_k} \to v$ *in* X, *such that*

$$\lim_{k\to\infty} \langle T_{\sigma_{n_k}}(v_{n_k}), v_{n_k} - w_{n_k}\rangle = 0,$$

then $v_{n_k} \to v$ *in* $W_0^1 L_\Phi(\Omega)$.

We also have the following stability result for the solution operator P_T of (4.4).

Theorem 4.5. *With the above definitions* (4.17)–(4.19), *we have*

$$P_{T_{\sigma_n}}(f_{\sigma_n}) \to P_{T_\sigma}(f_\sigma) \text{ in } X,$$

whenever $\sigma_n \to \sigma$, *and* $f_{\sigma_n} \to f_\sigma$.

Proof. This was proved in Lemma 6.1, [24], with the following assumptions on T_σ and f_σ:

(i) T_σ is continuous, bounded, strictly monotone, and satisfies condition (S) and $T_\sigma(0) = 0, \forall \sigma \in [0,1]$.

(ii) $f_{\sigma_n} \to f_\sigma$ in X^* (in the norm topology), $\forall \sigma \in [0,1]$.

(iii) (a) If $\{n_k\} \subset \mathbb{N}$ and $v_{n_k} \to v$ in X, then

$$T_{\sigma_{n_k}}(v_{n_k}) \to T_\sigma(v) \text{ in } X^*.$$

(b) T_{σ_n} satisfies the coercivity condition with the same constant C in J, and T_{σ_n} satisfies condition (S) uniformly; i.e., if $\{n_k\} \subset \mathbb{N}$, and $v_{n_k} \rightharpoonup v$ and $w_{n_k} \to v$ in X, such that

$$\lim_{k\to\infty} \langle T_{\sigma_{n_k}}(v_{n_k}), v_{n_k} - w_{n_k}\rangle = 0,$$

then

$$v_{n_k} \to v \text{ in } X.$$

Condition (i), in the case $\sigma = 0$ $(T_0 = J)$, was proved in [24], pp. 200, 110–111; it follows from the continuity, boundedness, coercivity, strict monotonicity of the p-Laplacian J, defined by

$$\langle J(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx,$$

the fact that J satisfies condition (S), and that $J(0) = 0$.

For the case $\sigma \in (0, 1]$, the continuity of T_σ follows from the continuity of T (Lemma 4.1). Using the compact imbedding $(W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Phi(\Omega))$ and (4.9), we compute

$$\begin{aligned}
|\langle T_\sigma(u), v\rangle| &= \Big|\Big\langle \frac{1}{\varphi(\sigma)} T(\sigma u), v\Big\rangle\Big| = \Big|\frac{1}{\varphi(\sigma)} \int_\Omega a(|\nabla \sigma u|)\nabla \sigma u \cdot \nabla v \, dx\Big| \\
&\le \frac{1}{\varphi(\sigma)} ||\varphi(\sigma \nabla u)||_{L_{\overline{\Phi}}(\Omega)} ||\nabla v||_{L_\Phi(\Omega)} = \frac{C_1}{\varphi(\sigma)} ||\varphi(\nabla u)||_{L_{\overline{\Phi}}(\Omega)} ||v||_X \\
&\le \frac{C_2}{\varphi(\sigma)} \varphi(||\nabla u||_{L_\Phi(\Omega)}) ||v||_X = \frac{C_2}{\varphi(\sigma)} \varphi(||u||_X) ||v||_X.
\end{aligned}$$

Therefore we have

$$||T_\sigma(u)||_{X^*} \le \frac{C_2}{\varphi(\sigma)} \varphi(||u||_X).$$

Hence, T_σ is bounded. Moreover, the strict monotonicity of T_σ on X follows from the strict monotonicity of φ. To show that T_σ satisfies condition (S), once again we use the argument in [24]. By definition of T_σ, $T_\sigma(0) = 0$.

We now prove that $f_{\sigma_n} \to f_\sigma$ in condition (ii). First, we consider the case $\sigma > 0$. In this case $\sigma_n > 0$, for all n sufficiently large. If $\lambda_n \to \lambda$ in $\mathbb{R}$ and $u_n \to u$ in X, then, by the continuity of B, K, and φ,

$$f_{\sigma_n}(u_n) = \frac{B(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)} - \frac{K(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)} \to \frac{B(\lambda, \sigma u)}{\varphi(\sigma)} - \frac{K(\lambda, \sigma u)}{\varphi(\sigma)} = f_\sigma(u).$$

In the case $\sigma = 0$, we need the following inequalities with $\sigma_n = ||u_n||_X$ and $v_n = \sigma_n u_n$ for σ_n is sufficiently small. We have

$$\Big|\Big\langle \frac{K(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)}, v\Big\rangle\Big| = \Big|\Big\langle \frac{K(\lambda_n, v_n)}{\varphi(\sqrt{||v_n||_X})}, v\Big\rangle\Big| \leq \Big|\Big\langle \frac{K(\lambda_n, v_n)}{\varphi(||v_n||_X)}, v\Big\rangle\Big| \to 0,$$

for all $v \in X$. Using the continuity of φ and Lemma 4.2 and (1.6) and the above fact, we compute

$$\begin{aligned}
&|\langle f_{\sigma_n}(\lambda_n, \sigma_n u_n) - f(\lambda, u), v\rangle| \\
&\leq \Big|\Big\langle \frac{B(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)} - \frac{K(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)} - f(\lambda, u), v\Big\rangle\Big| \\
&\leq \Big|\Big\langle \frac{B(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)} - -f(\lambda, u), v\Big\rangle\Big| + \Big|\Big\langle \frac{K(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)}, v\Big\rangle\Big| \\
&= \Big|\int_\Omega \Big(\lambda_n \frac{\varphi(\sigma_n u_n)}{\varphi(\sigma_n)} - \lambda|u|^{p-2}u\Big) v\, dx\Big| + \Big|\Big\langle \frac{K(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)}, v\Big\rangle\Big| \\
&\leq \Big\|\lambda_n \frac{1}{\varphi(\sigma_n)} \varphi(\sigma_n u_n) - \lambda|u|^{p-2}u\Big\|_{L_{\overline{\Phi}}(\Omega)} ||v||_{L_\Phi(\Omega)} + \Big|\Big\langle \frac{K(\lambda_n, \sigma_n u_n)}{\varphi(\sigma_n)}, v\Big\rangle\Big| \to 0.
\end{aligned}$$

For condition (iii), as above, we consider two cases. In the case $\sigma > 0$, we can use an argument similar to (ii) to get part (a). In the case $\sigma = 0$, for all $v \in X$, by the continuity of φ and (1.6),

$$\begin{aligned}
&\Big|\Big\langle \frac{T(\sigma_n u_n)}{\varphi(\sigma_n)} - J(u), v\Big\rangle\Big| \\
&\qquad\leq \Big\|\frac{1}{\varphi(\sigma_n)} \varphi(\sigma_n \nabla u_n) - |\nabla u|^{p-2}\nabla u\Big\|_{L_{\overline{\Phi}}(\Omega)} ||v||_{L_\Phi(\Omega)} \to 0.
\end{aligned}$$

In Lemma 6.1, [24], only part (b) of (iii) was used (i.e., T_{σ_n} satisfies the coercivity with the same constant C in J) to show the boundedness of $\{u_n\}$, where $u_n = P_{T_{\sigma_n}}(f_n)$. However, $\{u_n\}$ is bounded in X since T_{σ_n} satisfies condition (iii) in Section 1 ([18]). The other condition of part (b) was proved in Lemma 4.4. □

As an immediate consequence of this lemma, we have the following corollary:

Corollary 4.6. *The mapping P defined by* (4.4) *is continuous from X^* to X.*

From Theorem 4.5, we see that $P_{T_{\sigma_n}}(f_{\sigma_n}) \to P_{T_\sigma}(f_\sigma)$ in X, whenever $\sigma_n \to \sigma$ in $[0, 1]$ and $f_n \to f$ in X^*. Applying Theorem 4.5 with

$$f_n = B_{\sigma_n}(\lambda_n, u_n),\ f = B_\sigma(\lambda, u),$$

we have that

$$(\sigma, \lambda, u) \mapsto P_{T_\sigma}[f_\sigma(\lambda, u)] \tag{4.20}$$

is completely continuous on $[0,1] \times \mathbb{R} \times X$.

By the preceding lemmas and Theorem 3.10, we get a necessary condition for the existence of a bifurcation point.

Theorem 4.7. *If* $(\overline{\lambda}, 0)$ *is a bifurcation point of* (4.1) *and a satisfies* (1.6), *then* $\overline{\lambda}$ *is an eigenvalue of* (2.4).

Proof. Suppose that $(\overline{\lambda}, 0)$ is a bifurcation point of (4.1). Then there is a sequence $\{(\lambda_n, u_n)\}_{n=1}^{\infty}$ of nontrivial solutions of (4.1) such that $\lambda_n \to \overline{\lambda}$ in $\mathbb{R}$ and $u_n \to 0$ in $W_0^1 L_\Phi(\Omega)$. Also, since (λ_n, u_n) satisfies (4.1) we have

$$\langle T(u_n) - B(\lambda_n, u_n) - K(\lambda_n, u_n), v \rangle = 0, \forall v \in X. \tag{4.21}$$

Dividing (4.21) by $\varphi(||u_n||_X)$ (note that $\varphi(||u_n||_X) \neq 0, \forall n$), we have

$$\Big\langle \frac{T(u_n)}{\varphi(||u_n||_X)} - \frac{B(\lambda_n, u_n)}{\varphi(||u_n||_X)} - \frac{K(\lambda_n, u_n)}{\varphi(||u_n||_X)}, v \Big\rangle = 0, \forall v \in X. \tag{4.22}$$

By (4.22) and Lemma 4.2, we thus conclude

$$\sup_{||v||_X \le 1} \Big| \Big\langle \frac{T(u_n)}{\varphi(||u_n||_X)} - \frac{B(\lambda_n, u_n)}{\varphi(||u_n||_X)}, v \Big\rangle \Big| = \sup_{||v||_X \le 1} \Big| \Big\langle \frac{K(u_n)}{\varphi(||u_n||_X)}, v \Big\rangle \Big| \to 0;$$

i.e.,

$$\frac{T(u_n)}{\varphi(||u_n||_X)} - \frac{B(\lambda_n, u_n)}{\varphi(||u_n||_X)} \to 0 \text{ in } X^*.$$

Since $v_n := \frac{u_n}{||u_n||_X}$ is bounded in X, we can assume that $v_n \rightharpoonup v_0$ in X for some v_0.

Next, we claim that $\frac{B(\lambda_n, u_n)}{\varphi(||u_n||_X)} \to f(\overline{\lambda}, v_0)$ in X^*. Due to (1.6) and the compact imbedding $(W_0^1 L_\Phi(\Omega) \hookrightarrow L_\Phi(\Omega))$, and the continuity of the Nemits'kii operator from L_Φ into $L_{\overline{\Phi}}$, which is defined by $u \mapsto \varphi(u)$, we obtain

$$\Big\| \frac{B(\lambda_n, u_n)}{\varphi(||u_n||_X)} - f(\overline{\lambda}, v_0) \Big\|_{X^*} = \sup_{||v||_X \le 1} \Big| \int_\Omega \Big(\frac{\lambda_n \varphi(u_n)}{\varphi(||u_n||_X)} \frac{u_n}{|u_n|} - \overline{\lambda} |v_0|^{p-2} v_0 \Big) v \, dx \Big|$$

$$\le \Big\| \frac{\lambda_n |\varphi(u_n)|}{\varphi(||u_n||_X)} \frac{u_n}{|u_n|} - \overline{\lambda} |v_0|^{p-2} v_0 \Big\|_{L_{\overline{\Phi}}(\Omega)} ||v||_{L_\Phi(\Omega)}$$

$$= \Big\| \frac{\lambda_n |\varphi(v_n ||u_n||_X)|}{\varphi(||u_n||_X)} \frac{v_n}{|v_n|} - \overline{\lambda} |v_0|^{p-2} v_0 \Big\|_{L_{\overline{\Phi}}(\Omega)} ||v||_{L_\Phi(\Omega)} \to 0.$$

We thus infer that

$$\frac{T(u_n)}{\varphi(||u_n||_X)} \to f(\overline{\lambda}, v_0) \text{ in } X^*. \tag{4.23}$$

It follows from this strong convergence and from $v_n \rightharpoonup v_0$ in X that

$$\begin{aligned}&\Big|\Big\langle \frac{T(v_n||u_n||_X)}{\varphi(||u_n||_X)}, v_n - v_0\Big\rangle\Big|\\ &= \Big|\Big\langle \frac{T(v_n||u_n||_X)}{\varphi(||u_n||_X)} - f(\overline{\lambda}, v_0) + f(\overline{\lambda}, v_0), v_n - v_0\Big\rangle\Big| \to 0.\end{aligned}$$

This implies

$$\lim_{n\to\infty} \langle T_{||u_n||_X}(v_n), v_n - v_0\rangle = 0.$$

Since T_σ satisfies condition (S), we have $v_n \to v_0$ in X. Therefore

$$\frac{T(u_n)}{\varphi(||u_n||_X)} \to J(v_0) \text{ in } X. \tag{4.24}$$

In particular, $||v_0||_X = 1$, and it follows from (4.23)–(4.24) that

$$J(v_0) - f(\overline{\lambda}, v_0) = 0.$$

This means that $\overline{\lambda}$ is an eigenvalue of (2.4). □

Using Theorem 2.1, we may establish the existence of infinitely many nontrivial solutions $(\overline{\lambda}_k, \overline{u}_k)$ of (4.1) with $\lim \overline{\lambda}_k = \infty$. This, in turn, may be used to obtain bifurcation results at the higher eigenvalues of (2.4).

Theorem 4.8. *Let a be an increasing C^1 function on $[0,\infty)$ which satisfies* (1.6), *and a strict monotonicity and a Δ' condition; then for $k = 1, 2, \ldots$, each trivial solution $(\lambda_k, 0)$ of* (2.4) *is a bifurcation point of* (4.1).

Proof. We shall establish the existence of solutions $(\tilde{\lambda}_k, v_k)$ of (4.1) such that for each k,

$$||v_k(\alpha)||_X \to 0 \text{ as } \alpha \to 0 \tag{4.25}$$

and

$$\lim_{\alpha\to 0} \tilde{\lambda}_k(\alpha) = \lambda_k. \tag{4.26}$$

Let $0 < \epsilon < 1$ be fixed but arbitrary. For any constant $\delta > 0$, modify $\psi(t) = \frac{\varphi(\delta t)}{\varphi(\delta)}$ as follows:

$$\overline{\psi}(t) = \chi_\delta(t)\psi(t) + (1+\epsilon)(1 - \chi_\delta(t))t^{p-1} \quad \text{for } t \geq 0, \tag{4.27}$$

where $\chi_\delta(t)$ is a smooth, nonnegative, nonincreasing function satisfying

$$\chi_\delta(t) := \begin{cases} 1 & \text{for } 0 \leq t \leq \delta,\\ 0 & \text{for } t \geq 2\delta.\end{cases}$$

Using (4.27), for sufficiently small $\delta > 0$, we get

$$(1-\epsilon)t^{p-1} \le \overline{\psi}(t) \le (1+\epsilon)t^{p-1}. \tag{4.28}$$

Putting $\overline{\psi}(t) = \overline{\Psi}'(t)$, and integrating (4.28), we see that

$$\frac{(1-\epsilon)}{p}t^p \le \overline{\Psi}(t) \le \frac{(1+\epsilon)}{p}t^p \tag{4.29}$$

holds for sufficiently small $\delta > 0$.

Let us define a mapping $h : M_{\overline{\Psi}}(\alpha) \to M(\alpha)$ by

$$h(u) := \Big(\int_\Omega \overline{\Psi}(|\nabla u|)dx \Big/ \frac{1}{p}\int_\Omega |\nabla u|^p dx \Big)^{1/p} u,$$

where

$$M_{\overline{\Psi}}(\alpha) = \Big\{ u \in W_0^1 L_{\overline{\Psi}}(\Omega) : \int_\Omega \overline{\Psi}(|\nabla u|)dx = \alpha \Big\}$$

and

$$M(\alpha) = \Big\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |\nabla u|^p dx = \alpha \Big\}.$$

Then h is an odd homeomorphism. Taking $\delta > 0$, smaller still, if necessary, by (4.29), we have

$$(1-\epsilon)^{1/p}|u(x)| \le h(|u(x)|) = |h(u)(x)| \le (1+\epsilon)^{1/p}|u(x)|, \tag{4.30}$$

almost everywhere on Ω.

Let $s_k(\alpha)$ be the constant given by

$$s_k(\alpha) := \sup_{A \in \mathcal{C}_{\overline{\Psi},k}(\alpha)} \inf_{u \in A} \int_\Omega \overline{\Psi}(u)dx, \tag{4.31}$$

where $\mathcal{C}_{\overline{\Psi},k}(\alpha) := \{C \subset M_{\overline{\Psi}}(\alpha) :\ C$ symmetric, compact, $\gamma(C) \ge k\}$. Using (4.29) and (4.30), the inequalities

$$\begin{aligned} s_k(\alpha) &\ge \inf_{u \in A} \int_\Omega \overline{\Psi}(u)dx \ge \inf_{u\in A}\int_\Omega \frac{1-\epsilon}{p}|u|^p dx \\ &= \inf_{v \in h(A)} \int_\Omega \frac{1-\epsilon}{p}|h^{-1}(v)|^p dx \ge \inf_{v\in h(A)} \int_\Omega \frac{1}{p}\frac{(1-\epsilon)}{(1+\epsilon)}|v|^p dx \end{aligned}$$

follow for any $A \in \mathcal{C}_{\overline{\Psi},k}(\alpha)$. Since $h : \mathcal{C}_{\overline{\Psi},k} \to \mathcal{C}_k$ is a 1-1 mapping (see Chapter 7, [32]), we have

$$s_k(\alpha) \ge \frac{1-\epsilon}{1+\epsilon}\beta_k(\alpha). \tag{4.32}$$

Similarly, we have the reverse inequality:

$$s_k(\alpha) \le \frac{1+\epsilon}{1-\epsilon}\beta_k(\alpha). \tag{4.33}$$

It follows from Theorem 2.1 that there exists a critical point $w_k(\alpha)$ corresponding to the critical value $s_k(\alpha)$ and the Lagrange multiplier $\mu_k(\alpha)$. They satisfy the modified equation

$$\begin{cases} -\mathrm{div}\left(\overline{\psi}(|\nabla w_k(\alpha)|)\right) = \mu_k(\alpha)\left(\overline{\psi}(w_k(\alpha))\right), & x \in \Omega \\ w_k(\alpha) = 0, & x \in \partial\Omega. \end{cases} \tag{4.34}$$

Thus, from equation (4.34) and the inequalities (4.29) and (4.32), we have

$$\begin{aligned} \mu_k(\alpha) &= \int_\Omega \overline{\psi}(|\nabla w_k(\alpha)|)|\nabla w_k(\alpha)|dx \Big/ \int_\Omega \overline{\psi}(w_k(\alpha))w_k(\alpha)dx \\ &\le \frac{1+\epsilon}{1-\epsilon}\int_\Omega p\overline{\Psi}(|\nabla w_k(\alpha)|)dx \Big/ \frac{1-\epsilon}{1+\epsilon}\int_\Omega p\overline{\Psi}(|w_k(\alpha)|)dx \\ &= \left(\frac{1+\epsilon}{1-\epsilon}\right)^2 \frac{\alpha}{s_k(\alpha)} \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^3 \frac{\alpha}{\beta_k(\alpha)} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^3 \lambda_k. \end{aligned}$$

We similarly get the reverse inequality, and therefore we have

$$\left(\frac{1-\epsilon}{1+\epsilon}\right)^3 \lambda_k \le \mu_k(\alpha) \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^3 \lambda_k. \tag{4.35}$$

By Theorem 3.10, applying the a priori estimate on solutions of (4.34), we see that the inequalities

$$||\nabla w_k(\alpha)||_{L^\infty(\Omega)} < \delta,\ ||w_k(\alpha)||_{L^\infty(\Omega)} < \delta$$

hold if $||\nabla w_k(\alpha)||_{L_{\overline{\Psi}}(\Omega)}$ is sufficiently small. On the other hand, since $\overline{\Psi}$ satisfies a Δ_2 condition, it follows (cf. Chapter 3, [21]) that

$$||\nabla w_k||_{L_{\overline{\Psi}}} \to 0 \text{ if and only if } \int_\Omega \overline{\Psi}(|\nabla w_k|)dx \to 0;$$

therefore,

$$||\nabla w_k(\alpha)||_{L_{\overline{\Psi}}(\Omega)} \to 0 \quad \text{as} \quad \alpha = \int_\Omega \overline{\Psi}(|\nabla w_k(\alpha)|)dx \to 0.$$

Using (4.27), since $\overline{\psi}(t) = \psi(t)$ when $0 \le t \le \delta$, if α is sufficiently small, then $w_k(\alpha)$ solves the equation (4.1) and $\mu_k(\alpha) = \tilde{\lambda}_k(\alpha)$. Since $\psi(t) = \frac{\varphi(\delta t)}{\varphi(\delta)}$, we know that $\delta w_k(\alpha)$ solves the original equation (4.1).

Letting $\alpha \to 0$, it follows from inequality (4.35) that

$$\limsup_{\alpha \to 0} \tilde{\lambda}_k(\alpha) \le \left(\frac{1+\epsilon}{1-\epsilon}\right)^3 \lambda_k.$$

Since $\epsilon > 0$ is arbitrary,

$$\limsup_{\alpha \to 0} \tilde{\lambda}_k(\alpha) \le \lambda_k.$$

Similarly, we get

$$\liminf_{\alpha \to 0} \tilde{\lambda}_k(\alpha) \ge \lambda_k.$$

Consequently,

$$\lim_{\alpha \to 0} \tilde{\lambda}_k(\alpha) = \lambda_k.$$

□

Finally, we shall prove that the first eigenvalue of (2.4) is a global bifurcation point of (4.1) in the sense of Rabinowitz ([31]).

If we define $F : \mathbb{R} \times X \to X^*$ by

$$F(\lambda, u) := B(\lambda, u) + K(\lambda, u),$$

then $F(\lambda, 0) = 0$, $\forall \lambda \in \mathbb{R}$, by Lemma 4.1, and since F is completely continuous and P_T is continuous, where P_T is the solution operator in (4.4), $P_T \circ F$ is a completely continuous mapping. Hence, the topological degree $\mathrm{d}(I - P_T[F(\lambda, \cdot)], U, 0)$ is defined for all open, bounded subsets $U \ni 0$ of X provided

$$u \ne P_T[F(\lambda, u)], \ \forall u \in \partial U.$$

It is well known that for all $g_0 \in (W_0^{1,p}(\Omega))^*$,

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = g_0, & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$$

has a unique solution

$$u = u_{g_0} = P_J(g_0) := P_0(g_0). \tag{4.36}$$

We modify the proof in [24] to establish the global bifurcation result.

Since it is known that the principal eigenvalue λ_1 is isolated, we may choose $\epsilon > 0$ such that the interval $[\lambda_1 - \epsilon, \lambda_1 + \epsilon]$ contains no other eigenvalues. For such $\epsilon > 0$, the following result will be valid.

Theorem 4.9. $(\lambda_1, 0)$ *is a bifurcation point of* (4.1). *Let*

$$\mathcal{S} = \overline{\{(\lambda, u) : \text{ is a solution of } (4.1) \text{ with } u \ne 0\}} \cup ([\lambda_1 - \epsilon, \lambda_1 + \epsilon] \times \{0\}), \tag{4.37}$$

and let $\mathcal{C}$ *be the connected component of* $\mathcal{S}$ *containing* $[\lambda_1-\epsilon,\lambda_1+\epsilon]\times\{0\}$. *Then, either*

(i) $\mathcal{C}$ *is unbounded in* $\mathbb{R}\times X$, *or*

(ii) $(\lambda,0)\in\mathcal{C}$ *for some other eigenvalue* λ *of* (2.4).

Proof. Put $\lambda_1-\epsilon := a$ and $\lambda_1+\epsilon := b$. Then we note that, since a and b are not eigenvalues of (4.22) (see Proposition 2.2, [11]), then 0 is the only zero of $I-P_0[f(a,\cdot)]$ and $I-P_0[f(b,\cdot)]$ in X. Therefore the degrees in (4.38) and (4.39) for these two mappings are defined for all $r>0$ (and are independent of r). For $\sigma\in[0,1]$, $u\in X$, and $\lambda\in\mathbb{R}$, we define

$$T_{1,\sigma}(u) := \begin{cases} \frac{\varphi(1)}{\varphi(\sigma)}T(\sigma u), & \text{if } \sigma\in(0,1],\\ \varphi(1)J(\lambda,u), & \text{if } \sigma=0,\end{cases}$$

and

$$f_{1,\sigma}(\lambda,u) := \begin{cases} \frac{\varphi(1)}{\varphi(\sigma)}F(\lambda,\sigma u) = \frac{\varphi(1)}{\varphi(\sigma)}(B(\lambda,\sigma u)+K(\lambda,\sigma u)), & \text{if } \sigma\in(0,1],\\ \varphi(1)f(\lambda,u), & \text{if } \sigma=0.\end{cases}$$

It suffices to prove that 0 is an isolated solution of (4.1) for $\lambda=a,b$ and, for $r>0$ sufficiently small,

$$\mathrm{d}(I-P_T[F(a,\cdot)],B_r(0),0) = \mathrm{d}(I-P_0[f(a,\cdot)],B_r(0),0) \tag{4.38}$$

and

$$\mathrm{d}(I-P_T[F(b,\cdot)],B_r(0),0) = \mathrm{d}(I-P_0[f(b,\cdot)],B_r(0),0). \tag{4.39}$$

To prove (4.38), we need only show that there exists $r>0$ sufficiently small, such that, for all $\sigma\in[0,1]$, the equation

$$u - P_{T_\sigma}[f_\sigma(a,u)] = 0 \tag{4.40}$$

has no nontrivial solutions in $\overline{B_r(0)}$. Suppose that this is not the case and there exist sequences $\{u_n\}\subset X$ and $\{\sigma_n\}\subset(0,1]$, such that

$$||u_n||_X \neq 0,\ \forall n,\ ||u_n||_X \to 0\ (n\to\infty),$$

and

$$u_n = P_{T_{\sigma_n}}[f_{\sigma_n}(a,u_n)], \forall n.$$

This equation has the following form:

$$\langle T_{\sigma_n}(u_n) - f_{\sigma_n}(a,u_n), w\rangle = 0, \forall w\in X.$$

By the definitions of T_σ and f_σ, this is equivalent to

$$\frac{\varphi(1)}{\varphi(\sigma_n)}\langle T(\sigma_n u_n) - B(a,\sigma_n u_n) - K(a,\sigma_n u_n), w\rangle = 0,\ \forall w\in X.$$

As before, by setting $v_n := \frac{u_n}{||u_n||_X}$ and dividing this by $\varphi(\sigma_n||u_n||_X)$, we get the following equality:

$$\Big\langle \frac{T(\sigma_n||u_n||_X v_n)}{\varphi(\sigma_n||u_n||_X)} - \frac{B(a,\sigma_n||u_n||_X v_n)}{\varphi(\sigma_n||u_n||_X)} - \frac{K(a,\sigma_n||u_n||_X v_n)}{\varphi(\sigma_n||u_n||_X)}, w \Big\rangle = 0, \ \forall w \in X.$$

This is equivalent to

$$\big\langle T_{\sigma_n||u_n||_X}(v_n) - f_{\sigma_n||u_n||_X}(a,v_n), w \big\rangle = 0, \ \forall w \in X.$$

Therefore,

$$v_n = P_{T_{\sigma_n||u_n||_X}}[f_{\sigma_n||u_n||_X}(a,v_n)]. \tag{4.41}$$

By assuming $v_n \rightharpoonup v$ and using the complete continuity of the mapping in (4.20) and the fact that $\sigma_n||u_n||_X \to 0$, we see that

$$P_{T_{\sigma_n||u_n||_X}}[f_{\sigma_n||u_n||_X}(a,v_n)] \to P_{\varphi(1)J}[\varphi(1)f(a,v)] = P_0[f(a,v)] \text{ in } X.$$

Hence (4.41) implies that $v_n \to v$ and $v = P_0[f(a,v)]$. Because $||v||_X = 1$, this means that a is an eigenvalue of (4.22). This contradiction proves that there exists $r > 0$, such that (4.40) has no solutions in $\overline{B_r(0)}\backslash\{0\}$.

Next, we observe that $\{I - P_{T_\sigma}[f_\sigma(a,\cdot)] : \sigma \in [0,1]\}$ is a family of compact perturbations of the identity on $\overline{B_r(0)}$. Moreover, by the above proof,

$$u - P_{T_\sigma}[f_\sigma(a,u)] \neq 0,$$

for all $u \in \partial B_r(0)$ and all $\sigma \in [0,1]$.

By (4.36) (for the first equality), (4.18) and (4.19) (for the second equality), the definitions of $P_{T_0}[f_0(a,\cdot)]$ and $P_{T_{1,0}}[f_{1,0}(a,\cdot)]$ (for the third equality), the homotopy-invariance property of the Leray-Schauder degree (for the fourth equality), the definitions of $P_{T_1}[f_1(a,\cdot)]$ and $P_{T_{1,1}}[f_{1,1}(a,\cdot)]$ (for the fifth equality), and (4.18) and (4.19) (for the sixth equality),

$$\begin{aligned} \mathrm{d}(I - P_0[f(a,\cdot)], B_r(0), 0) &= \mathrm{d}(I - P_J[f(a,\cdot)], B_r(0), 0) \\ &= \mathrm{d}(I - P_{T_0}[f_0(a,\cdot)], B_r(0), 0) \\ &= \mathrm{d}(I - P_{T_{1,0}}[f_{1,0}(a,\cdot)], B_r(0), 0) \\ &= \mathrm{d}(I - P_{T_{1,1}}[f_{1,1}(a,\cdot)], B_r(0), 0) \\ &= \mathrm{d}(I - P_{T_1}[f_1(a,\cdot)], B_r(0), 0) \\ &= \mathrm{d}(I - P[F(a,\cdot)], B_r(0), 0); \end{aligned}$$

i.e., (4.38) holds. (4.39) follows in a similar manner. Since

$$\mathrm{d}(I - P_0[f(a,\cdot)], B_r(0), 0) \neq \mathrm{d}(I - P_0[f(b,\cdot)], B_r(0), 0)$$

(see Proposition 2.2, [11]), using the properties of the Leray-Schauder degree in [31], we obtain the alternatives, as stated. □

REFERENCES

[1] R. Adams, "Sobolev Spaces," Academic Press, New York, 1975.

[2] H. Amann, *Liusternik-Schnirelman theory and non-linear eigenvalue problems*, Math. Ann., 199 (1975), 55–72.

[3] H. Amann, "Lectures on Some Fixed Point Theorems, Monografias de Matematica," Technical Report, IMPA, Rio de Janeiro, 1974.

[4] A. Anané, *Simplicité et isolation de la première valeur propre du p-Laplacien,* C. R. Acad. Sci Paris, 305 (1987), 725–728.

[5] J. García Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian: Nonlinear eigenvalues,* Comm. Partial Differential Equations, 12 (1987), 1389–1430.

[6] F. Browder, *Fixed point theory and nonlinear problems,* Bull. Amer. Math. Soc., 9 (1983), 1–39.

[7] A. Cianchi, *Boundedness of solutions to variational problems under general growth conditions,* Comm. Partial Differential Equations, 22 (1997), 1629–1646.

[8] P. Clément, M. García-Huidobro, R. Manásevich, and K. Schmitt, *Mountain pass type solutions for quasilinear elliptic equations,* Calc. Variations and PDE, 11 (2000), 33–62.

[9] C.V. Coffman, *A minimum-maximum principle for a class of nonlinear integral equations,* J. Analyse Math., 22 (1969), 391–419.

[10] C.V. Coffman, *Lyusternik-Schnirelman theory and eigenvalue problems for monotone potential operators,* J. Funct. Anal., 14 (1973), 237–252.

[11] M.A. del Pino and Raúl F. Manásevich, *Global bifurcation from the eigenvalues of the p-Laplacian,* J. Differential Equations, 92 (1991), 226–251.

[12] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations,* Nonlinear Analysis, TMA, 7 (1983), 827–850.

[13] N. Fukagai and K. Narukawa, *Bifurcation phenomena associated to a class of p-Laplacian like operators,* Manuscripta Math., 109 (2002), 175–201.

[14] M. Fuchs and L. Gongbao, L^∞*-bounds for elliptic equations on Orlicz-Sobolev spaces,* Archiv der Mathematik, 72 (1999), 293–297.

[15] M. García-Huidobro, V. Le, R. Manásevich, and K. Schmitt, *On principal eigenvalues for quasilinear elliptic differential operators: an Orlicz-Sobolev space setting,* Nonl. Differential Equations and Applications, 6 (1999), 207–225.

[16] M. García-Huidobro, R. Manásevich, and K. Schmitt, *Some bifurcation results for a class of p-Laplacian like operators,* Differential and Integral Equations, 10 (1997), 51–66.

[17] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer Verlag, Berlin, 1983.

[18] J. Gossez and V. Mustonen, *Variational inequalities in Orlicz-Sobolev spaces,* Nonlinear Analysis, TMA, 11 (1987), 379–392.

[19] M.A. Krasnosel'skii and Y.B. Rutickii, "Convex Functions and Orlicz Spaces," Noordhoff, Groningen, 1961.

[20] N.V. Krylov, "Lectures on Elliptic and Parabolic Equations in Hölder Spaces," Amer. Math. Soc., Providence, RI, 1996.

[21] A. Kufner, O. John, and S. Fučik, "Function Spaces," Noordhoff, Leyden, 1977.

[22] O.A. Ladyzhenskaya and N.N. Ural'tseva, "Linear and Quasilinear Elliptic Equations," volume 46 of Math. in Sci. and Eng., Academic Press, New York, 1968.

[23] V. Le and K. Schmitt, *Quasilinear elliptic equations and inequalities with rapidly growing coefficients,* J. London Math. Soc., 62 (2000), 852–872.

[24] V.K. Le and K. Schmitt, "Global Bifurcation in Variational Inequalities: Applications to Obstacle and Unilateral Problems," Springer-Verlag, New York, 1997.

[25] G. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations,* Comm. Partial Differential Equations, 16 (1991), 331–361.

[26] G. Lieberman, *Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures,* Comm. Partial Differential Equations, 18 (1993), 1191–1212.

[27] P. Lindqvist, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0,$ Proc. Amer. Math. Soc., 109 (1990), 157–164.

[28] P. Marcellini, *Regularity and existence of solutions of elliptic equations with p, q-growth conditions,* J. Differential Equations, 90 (1991), 1–30.

[29] P. Marcellini, *Regularity for elliptic equations with general growth conditions,* J. Differential Equations, 105 (1993), 296–333.

[30] K. Narukawa, *Global bifurcation for quasilinear elliptic equations,* Nonlinear Analysis, TMA, 30 (1997), 5241–5249.

[31] P. Rabinowitz, *Some aspects of nonlinear eigenvalue problems,* Rocky Mountain J. Math., 3 (1973), 162–202.

[32] P. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," CBMS Regional Conf. no. 65, AMS, Providence, 1986.

[33] M. Rao and Z. Ren, "Theory of Orlicz Spaces," Marcel Dekker, New York, 1985.

[34] W. Rudin, "Real and Complex Analysis," McGraw Hill, New York, 1966.

[35] M. Tienari, *Ljusternik-Schnirelmann theorem for the generalized Laplacian,* J. Differential Equations, 161 (2000), 174–190.

[36] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations,* J. Differential Equations, 51 (1984), 126–150.

[37] N.S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations,* Comm. Pure Appl. Math., 20 (1967), 721–747.

[38] E. Zeidler, "Nonlinear Functional Analysis," volume III. Springer Verlag, New York, 1989.