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UNIQUENESS AND PROFILE OF SOLUTIONS FOR A SUPERLINEAR ELLIPTIC EQUATION

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Abstract. The uniqueness and profile of solutions for a Matukuma-type equation on the whole space as the power of the nonlinear term tends to one are discussed. By using information on a linear eigenvalue problem and Lyapunov-Schmidt-like reduction, we obtain the uniqueness and asymptotic profile of solutions. We also consider the case of bounded domain in the same manner.

1. Introduction

The uniqueness of solutions for nonlinear elliptic equations is one of the main concerns in this field for decades. In this paper we study an elliptic problem of the form

$$\begin{cases} \Delta u + \lambda K(x) u^{1+\varepsilon} = 0 \quad \text{in } \mathbf{R}^n, \\ u > 0 \quad \text{in } \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 3$, $\lambda > 0$, and $\varepsilon > 0$ is a small parameter. The weight $K(x)$ is a function of $x \in \mathbf{R}^n$ with

(K.1) $$K \in C^1(\mathbf{R}^n), \quad K > 0 \ \text{ in } \ \mathbf{R}^n$$

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and is assumed to satisfy

(K.2) $$K(x) = |x|^{-\ell}(c_0 + c_1|x|^{-1} + O(|x|^{-\mu})) \quad \text{as } |x| \to \infty,$$

$c_0 > 0$, $c_1 \in \mathbf{R}$, $\mu > 1$, and $\ell > 2$, or

(K.3) $$\lim_{|x|\to\infty} |x|^t K(x) = 0$$

for any $t > 1$. We allow the case where $K(x) = e^{-|x|}$.

In order to seek a solution of (1.1), we define two function spaces. One is

$$\mathcal{D}(\mathbf{R}^n) := \Big\{ u : \int_{\mathbf{R}^n} K(x)u^2 \, dx < \infty \Big\}.$$

It is easy to see that $\mathcal{D}(\mathbf{R}^n)$ is a Hilbert space equipped with its inner product $\langle \cdot, \cdot \rangle$ defined by $\langle u, v \rangle := \int_{\mathbf{R}^n} K(x)u(x)v(x)\, dx$. We denote the norm of u in $\mathcal{D}(\mathbf{R}^n)$ by $\|u\| = \langle u, u \rangle^{1/2}$. We write $\|u\|_p = (\int_{\mathbf{R}^n} |u|^p \, dx)^{1/p}$. In the space $\mathcal{D}(\mathbf{R}^n)$, condition (K.2) or (K.3) enables us to treat (1.1) like a problem on a bounded domain.

The other is the space $\mathcal{D}_1$, the completion of $C_0^\infty(\mathbf{R}^n)$ with respect to the norm $\|\nabla \cdot \|_2$. Note that the embedding $\mathcal{D}_1 \hookrightarrow \mathcal{D}(\mathbf{R}^n)$ holds due to the Hölder and Sobolev inequalities. By elliptic regularity theory, solutions of (1.1) in $\mathcal{D}(\mathbf{R}^n)$ are indeed in $\mathcal{D}_1$.

The aim of this paper is to show the uniqueness of solutions of (1.1) in $\mathcal{D}(\mathbf{R}^n)$ for small $\varepsilon > 0$ and derive an asymptotic profile of solutions as $\varepsilon \downarrow 0$. Although variational methods are standard tools for this type of equation, we must be careful in treating (1.1), because the validity of the Palais-Smale condition is not obvious. However, we can also show that there exists a mountain-pass solution to (1.1) if $\varepsilon > 0$ is sufficiently small.

To obtain the uniqueness, we adopt a method which relies more on a bifurcation-like argument than the argument used in Lemma 3 of Lin [13] or Theorem 8.3 of Hulshof and van der Vorst [5] for a bounded domain. Both of their arguments are done by assuming the existence of two solutions and leading to a contradiction, and the precise profile is not obtained.

From now on, we say that λ_* is an eigenvalue if and only if the equation $-\Delta u = \lambda_* K(x)u$ has a nontrivial solution in $\mathcal{D}(\mathbf{R}^n)$. In this case, the nontrivial solution u is said to be an eigenfunction corresponding to λ_*. Our first result is stated as follows.

Theorem 1.1. *Suppose that* (K.1) *holds. If* $K(x)$ *satisfies* (K.2) *or* (K.3), *then there exists* $\varepsilon_0 > 0$ *such that* (1.1) *has a unique solution* $u_\varepsilon \in \mathcal{D}_1 \cap$

$C^2(\mathbf{R}^n)$ for any $\varepsilon \in (0, \varepsilon_0)$. Moreover, u_ε is expanded as

$$u_\varepsilon = \left(\frac{\lambda_1}{\lambda}\right)^{1/\varepsilon} \Big\{ (1 + \varepsilon C_1 + O(\varepsilon^2))\varphi + \varepsilon U_0 + \varepsilon^2 \tilde{U}_\varepsilon \Big\} \tag{1.2}$$

with $U_0 \perp \varphi$ and $\tilde{U}_\varepsilon \perp \varphi$ in $\mathcal{D}(\mathbf{R}^n)$ and $\limsup_{\varepsilon \downarrow 0} \|\tilde{U}_\varepsilon\| < \infty$, where λ_1 is the first eigenvalue and φ is the corresponding eigenfunction normalized as

$$\int_{\mathbf{R}^n} K(x)\varphi^2(x) \log \varphi(x)\, dx = 0, \quad \varphi > 0 \text{ in } \mathbf{R}^n,$$

$U_0 \in \mathcal{D}_1 \cap \mathcal{D}(\mathbf{R}^n)^\perp \cap C^2(\mathbf{R}^n)$ is a unique solution to

$$\Delta U_0 + \lambda_1 K(x) U_0 = -\lambda_1 K(x) \varphi \log \varphi, \tag{1.3}$$

and

$$C_1 = -\frac{\displaystyle\int_{\mathbf{R}^n} K(x) \Big\{ U_0 \varphi \log \varphi + \frac{1}{2}\varphi^2 (\log \varphi)^2 \Big\}\, dx}{\displaystyle\int_{\mathbf{R}^n} K(x)\varphi^2\, dx},$$

with $\mathcal{D}^\perp(\mathbf{R}^n) := \{u \in \mathcal{D}(\mathbf{R}^n) : \int_{\mathbf{R}^n} K(x) u \varphi\, dx = 0\}$. In addition, $\tilde{U}_\varepsilon$ converges to $\tilde{U}_0$ in $\mathcal{D}(\mathbf{R}^n)$, where $\tilde{U}_0 \in \mathcal{D}_1 \cap \mathcal{D}^\perp(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$ is a unique solution to

$$\Delta \tilde{U}_0 + \lambda_1 K(x) \tilde{U}_0 = -\lambda_1 K(x) \Big\{ (C_1 \varphi + U_0)(\log \varphi + 1) + \frac{1}{2} \varphi (\log \varphi)^2 \Big\}$$

in $\mathbf{R}^n$.

Remark 1.1. In the limiting case ($\varepsilon = 0$), Edelson and Rumbos [2] showed the existence of a complete orthogonal system in $\mathcal{D}(\mathbf{R}^n)$ under (K.1) with (K.2) or (K.1) with (K.3). See also Kabeya and Yanagida [8] for a related result. As is commented in [2], the first eigenvalue λ_1 is simple. Applying Theorems 2.9 and 2.32 of Li and Ni [10], we see that $\varphi \sim |x|^{2-n}$ and $u_\varepsilon \sim |x|^{2-n}$ at $|x| = \infty$. For $0 < \ell < 2$ and $\ell = 2$, asymptotic behaviors of solutions are discussed by Kabeya [6, 7].

To obtain the precise asymptotic profile as $\varepsilon \downarrow 0$, we need to show the existence and uniqueness of solutions to (1.1). Under (K.2) or (K.3), we can show that the Palais-Smale condition is satisfied. Once we obtain the existence of a solution, we can show that the solution is unique and is expanded as $u_\varepsilon = (1 + \varepsilon C_1 + O(\varepsilon^2))\varphi + \varepsilon U_0 + \varepsilon^2 \tilde{U}_\varepsilon$ by using an a priori estimate and the fixed-point theorem. We note that Proposition 6.2 of Yanagida and Yotsutani [14] gives the zeroth approximation of u_ε in the class of radial K and radial solutions.

As for the bounded domain case, Lin [13] considered the problem

$$\begin{cases} \Delta u + \lambda u^{1+\varepsilon} = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

and proved the uniqueness of solutions, where Ω is a bounded, smooth, convex domain in $\mathbf{R}^n$ ($n \geq 3$). A rough result about the asymptotic behavior of solutions as $\varepsilon \downarrow 0$ was obtained by Lee [9], who showed $\lim_{\varepsilon\downarrow 0} \min_G u(x) = \infty$ for any compact set $G \subset \Omega$ if $0 < \lambda < \lambda_1$, where λ_1 is the first eigenvalue of $-\Delta$ under the homogeneous Dirichlet boundary condition.

Our technique is also applicable to the case of a bounded domain. We will consider a slightly generalized problem

$$\begin{cases} \Delta u + \lambda K(x) u^{1+\varepsilon} = 0 \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $\lambda > 0$, $\varepsilon \geq 0$, and K is a function satisfying

(K.4) $$K \in C^1(\overline{\Omega}), \quad K > 0 \ \text{ in } \Omega.$$

According to de Figueiredo, Lions, and Nussbaum [1] (p. 51, condition (15)), we also impose the following condition on K near $\partial\Omega$:

(K.5) $$\begin{cases} \text{There exists } \delta > 0 \text{ such that } \theta \cdot \nabla K(x) \leq 0 \ \text{ near } \partial\Omega \\ \text{with any unit vector } \theta \text{ satisfying } |\theta - n(x)| < \delta, \\ \text{where } n(x) \text{ is the outward-unit-normal vector to } \partial\Omega. \end{cases}$$

We agree that, if $x \in \Omega$, then $n(x) = n(y)$, the outward-unit vector with $y \in \partial\Omega$ such that $|y - x| = \operatorname{dist}(x, \partial\Omega)$.

As in Theorem 1.1, we will show how the solutions blow up or vanish according to the value λ. It is standard to find a solution (1.5) via the mountain-pass theorem. The solution turns out to be a classical one by standard regularity theory. Then we will show the uniqueness of solutions to (1.5).

As before, we define $\mathcal{D}(\Omega) := \{u : \int_\Omega K(x) u^2 \, dx < \infty\}$. Then $\mathcal{D}(\Omega)$ is a Hilbert space equipped with its inner product $\langle \cdot, \cdot \rangle_\Omega$ defined by $\langle u, v \rangle_\Omega := \int_\Omega K(x) u(x) v(x) \, dx$. We denote the norm of u in $\mathcal{D}(\Omega)$ by $\|u\|_\Omega = \langle u, u \rangle_\Omega^{1/2}$. We write $\|u\|_{p,\Omega} = (\int_\Omega |u|^p \, dx)^{1/p}$.

The property of solutions to (1.5) is as follows, and the expansion is the same as in Theorem 1.1.

Theorem 1.2. *Suppose that* $K(x)$ *satisfies* (K.4) *and* (K.5). *Then there exists* $\varepsilon_0 > 0$ *such that* (1.5) *has a unique solution* $u_\varepsilon \in C^2(\Omega) \cap C^1(\overline{\Omega})$ *for any* $\varepsilon \in (0, \varepsilon_0)$. *Moreover,* u_ε *is expanded as*

$$u_\varepsilon = \Big(\frac{\lambda_1}{\lambda}\Big)^{1/\varepsilon} \Big\{(1 + \varepsilon C_1 + O(\varepsilon^2))\psi + \varepsilon U_0 + \varepsilon^2 \tilde{U}_\varepsilon\Big\} \tag{1.6}$$

with $U_0 \perp \psi$ *and* $\tilde{U}_\varepsilon \perp \psi$ *in* $\mathcal{D}(\Omega)$ *and* $\limsup_{\varepsilon \downarrow 0} \|\tilde{U}_\varepsilon\|_\Omega < \infty$, *where* λ_1 *is the first eigenvalue and* ψ *is the corresponding eigenfunction normalized as*

$$\int_\Omega K(x)\psi^2(x) \log \psi(x)\, dx = 0, \quad \psi > 0 \ \text{ in } \Omega,$$

$U_0 \in \mathcal{D}^\perp(\Omega) \cap C^2(\Omega) \cap C(\overline{\Omega})$ *is a unique solution to*

$$\Delta U_0 + \lambda_1 K(x) U_0 = -\lambda_1 K(x)\psi \log \psi \tag{1.7}$$

with $U_0 = 0$ *on* $\partial\Omega$, *and*

$$C_1 = -\frac{\displaystyle\int_\Omega K(x)\Big\{U_0\psi \log \psi + \frac{1}{2}\psi^2(\log \psi)^2\Big\}\, dx}{\displaystyle\int_\Omega K(x)\psi^2\, dx},$$

where $\mathcal{D}^\perp(\Omega) = \{u \in \mathcal{D}(\Omega) : \int_\Omega K(x) u \psi\, dx = 0\}$. *In addition,* $\tilde{U}_\varepsilon$ *converges to* $\tilde{U}_0$ *in* $\mathcal{D}(\Omega)$, *with* $\tilde{U}_0 \in \mathcal{D}^\perp(\Omega) \cap C^2(\Omega) \cap C(\overline{\Omega})$ *a unique solution to*

$$\Delta \tilde{U}_0 + \lambda_1 K(x)\tilde{U}_0 = -\lambda_1 K(x)\Big\{(C_1\psi + U_0)(\log \psi + 1) + \frac{1}{2}\psi(\log \psi)^2\Big\}$$

in Ω *with* $\tilde{U}_0 = 0$ *on* $\partial\Omega$.

We note that λ_1 is simple.

This paper is organized as follows. In Section 2, we prove the existence of a solution to (1.1). Proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4, respectively. Section 5 is devoted to a proof of the proposition used in the proof of Theorem 1.1.

2. Existence of a solution

In this section, we will show the existence of a solution to (1.1) under (K.2) or (K.3) for $0 < \varepsilon < 4/(n-2)$.

Proposition 2.1. *Suppose that* (K.1) *with* (K.2) *or* (K.1) *with* (K.3) *holds. Then* (1.1) *has a solution* $u_\varepsilon \in \mathcal{D}_1$ *for any* $\lambda > 0$ *and* $\varepsilon \in (0, 4/(n-2))$. *Moreover,* $u_\varepsilon > 0$ *in* $\mathbf{R}^n$.

Proof. Without loss of generality, we have only to study

$$\Delta u + K(x)u^{1+\varepsilon} = 0 \quad \text{in } \mathcal{D}_1. \tag{2.1}$$

Let us define

$$I_\varepsilon := \inf_{u\in\mathcal{D}_1,\ u\not\equiv 0} \frac{\displaystyle\int_{\mathbf{R}^n} |\nabla u|^2\,dx}{\Big(\displaystyle\int_{\mathbf{R}^n} K(x)|u|^{2+\varepsilon}\,dx\Big)^{2/(2+\varepsilon)}}. \tag{2.2}$$

First note that

$$\begin{aligned}&\int_{\mathbf{R}^n} K(x)|u|^{2+\varepsilon}\,dx \qquad (2.3)\\ &\le \Big(\int_{\mathbf{R}^n} K(x)^{2n/\{4-(n-2)\varepsilon\}}\,dx\Big)^{\frac{4-(n-2)\varepsilon}{2n}} \Big(\int_{\mathbf{R}^n} |u|^{2n/(n-2)}\,dx\Big)^{\frac{(n-2)(2+\varepsilon)}{2n}} < \infty\end{aligned}$$

for $u \in \mathcal{D}_1$ since

$$\int_{\mathbf{R}^n\setminus B_{R_0}} K(x)^{2n/\{2n-(2+\varepsilon)(n-2)\}}\,dx \le \tilde{C} R_0^{n-2n\ell/\{2n-(2+\varepsilon)(n-2)\}} < \infty, \tag{2.4}$$

where $B_{R_0} := \{x \in \mathbb{R}^n : |x| < R_0\}$, with a constant $\tilde{C} > 0$ in view of $\ell > 2$ from (K.2) or (K.3) (in this case, we can take $\ell > 2$ arbitrarily).

Next, we prove that $I_\varepsilon > 0$. Indeed, from (2.3) and the Sobolev inequality, we have

$$S^2\Big(\int_{\mathbf{R}^n} K(x)|u|^{2+\varepsilon}\,dx\Big)^{\frac{2}{2+\varepsilon}} \le \Big(\int_{\mathbf{R}^n} K(x)^{\frac{2n}{4-(n-2)\varepsilon}}\,dx\Big)^{\frac{4-(n-2)\varepsilon}{n(2+\varepsilon)}} \int_{\mathbf{R}^n} |\nabla u|^2\,dx,$$

where S is the best Sobolev constant. Thus we see that

$$\frac{\displaystyle\int_{\mathbf{R}^n} |\nabla u|^2\,dx}{\Big(\displaystyle\int_{\mathbf{R}^n} K(x)|u|^{2+\varepsilon}\,dx\Big)^{2/(2+\varepsilon)}} \ge S^2\Big(\int_{\mathbf{R}^n} K(x)^{\frac{2n}{4-(n-2)\varepsilon}}\,dx\Big)^{-\frac{4-(n-2)\varepsilon}{n(2+\varepsilon)}} > 0;$$

i.e., $I_\varepsilon > 0$. Hence, if we find a minimizer, it corresponds to a nontrivial positive solution.

We may take a minimizing sequence $\{u_j\}$ such that

$$\int_{\mathbf{R}^n} K(x)u_j^{2+\varepsilon}\,dx = 1 \quad \text{and} \quad \int_{\mathbf{R}^n} |\nabla u_j|^2\,dx \to I_\varepsilon$$

as $j \to \infty$. Thus we can choose a subsequence (still denoted by $\{u_j\}$) such that

$$\begin{cases} u_j \rightharpoonup u_\infty & \text{weakly in } \mathcal{D}_1, \\ u_j \to u_\infty & \text{locally strongly in } L^{2+\varepsilon}(\mathbf{R}^n) \text{ and a.e. in } \mathbf{R}^n. \end{cases}$$

Moreover, again by the Hölder and Sobolev inequalities, we have

$$\begin{aligned} \int_{\mathbf{R}^n \setminus B_{R_0}} K(x)|u_j|^{2+\varepsilon}\,dx &\le \Big(\int_{\mathbf{R}^n \setminus B_{R_0}} K(x)^{\frac{2n}{2n-(2+\varepsilon)(n-2)}}\,dx \Big)^{\frac{2n-(2+\varepsilon)(n-2)}{2n}} \\ &\quad \times \Big(\int_{\mathbf{R}^n \setminus B_{R_0}} |u_j|^{\frac{2n}{n-2}}\,dx \Big)^{\frac{(2+\varepsilon)(n-2)}{2n}} \\ &\le S^{-(2+\varepsilon)} \Big(\int_{\mathbf{R}^n \setminus B_{R_0}} K(x)^{\frac{2n}{2n-(2+\varepsilon)(n-2)}}\,dx \Big)^{\frac{2n-(2+\varepsilon)(n-2)}{2n}} \Big(\int_{\mathbf{R}^n} |\nabla u_j|^2\,dx \Big)^{\frac{2+\varepsilon}{2}}. \end{aligned}$$

Hence, we see that $\int_{\mathbf{R}^n \setminus B_{R_0}} K(x) u_j^{2+\varepsilon}\,dx < \eta$ for arbitrarily given $\eta > 0$, choosing $R_0 > 0$ sufficiently large in view of (2.4).

Since u_j converges to u_∞ in $L^{2+\varepsilon}(B_R)$, we see that

$$\int_{B_R} K(x) u_j^{2+\varepsilon}\,dx - \eta < \int_{\mathbf{R}^n} K(x) u_j^{2+\varepsilon}\,dx < \int_{B_R} K(x) u_j^{2+\varepsilon}\,dx + \eta,$$

and hence

$$1 = \lim_{j\to\infty} \int_{\mathbf{R}^n} K(x) u_j^{2+\varepsilon}\,dx = \int_{\mathbf{R}^n} K(x) u_\infty^{2+\varepsilon}\,dx. \tag{2.5}$$

Since u_j also converges to u_∞ weakly in $\mathcal{D}_1$, we have

$$\int_{\mathbf{R}^n} |\nabla u_\infty|^2\,dx \le \liminf_{j\to\infty} \int_{\mathbf{R}^n} |\nabla u_j|^2\,dx = I_\varepsilon.$$

On the other hand, from (2.5) we obtain

$$I_\varepsilon \le \int_{\mathbf{R}^n} |\nabla u_\infty|^2\,dx.$$

This implies the strong convergence of u_j in $\mathcal{D}_1$ and the existence of a minimizer of (2.2). By the standard regularity argument, we see $v \in C^2(\mathbf{R}^n)$, and by the maximum principle, $v > 0$ in $\mathbf{R}^n$. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Note that the assumptions (K.1) together with (K.2) or (K.3) are always assumed in this section. Using the scaling $u = (\lambda_1/\lambda)^{1/\varepsilon} w$, we have only to consider the problem

$$\begin{cases} \Delta w + \lambda_1 K(x) w^{1+\varepsilon} = 0 \quad \text{in } \mathbf{R}^n, \\ w > 0 \quad \text{in } \mathbf{R}^n, \\ w \in \mathcal{D}_1. \end{cases} \tag{3.1}$$

We prove a precise expansion step by step. The following is on the behavior of the normalized solution. Also note that the simplicity of the first eigenvalue allows us to express w_ε in the form $w_\varepsilon = C(\varepsilon)\varphi + U_\varepsilon$ with $\varphi \perp U_\varepsilon$ (in $\mathcal{D}(\mathbf{R}^n)$).

Lemma 3.1. *For any solution w_ε to (3.1), $\|w_\varepsilon\|_2^\varepsilon \to 1$ and $w_\varepsilon/\|\nabla w_\varepsilon\|_2 \to c\varphi$ in $\mathcal{D}_1$ as $\varepsilon \to 0$ with $c = (\lambda_1 \int_{\mathbf{R}^n} K(x)\varphi^2\,dx)^{-1/2}$.*

Proof. As in the proof of Lemma 3 of [13], we rescale (3.1) by

$$v_\varepsilon(y) = \frac{w_\varepsilon(x)}{\|w_\varepsilon\|_\infty}, \qquad y := \|w_\varepsilon\|_\infty^{\varepsilon/2}(x - Q_\varepsilon),$$

where Q_ε is the maximum point of w_ε. By this scaling, we see that $v_\varepsilon(y)$ is a solution to

$$\Delta_y v_\varepsilon + \lambda_1 K\Big(\frac{y}{\|w_\varepsilon\|_\infty^{\varepsilon/2}} + Q_\varepsilon\Big) v_\varepsilon^{1+\varepsilon} = 0. \tag{3.2}$$

Suppose that $\|w_\varepsilon\|_\infty^\varepsilon \to \infty$ as $\varepsilon \downarrow 0$. By Proposition 5.1 in the Appendix, we see that $\{Q_\varepsilon\}$ is uniformly bounded in $\mathbf{R}^n$. Then, since $\|v_\varepsilon\|_\infty = 1$, we can choose a subsequence $\{\varepsilon_j\}$ ($\varepsilon_j \to 0$ as $j \to \infty$) such that $Q_{\varepsilon_j} \to Q_0$ as $j \to \infty$ and $v_{\varepsilon_j}(y)(> 0) \to V(y)$ locally uniformly in $\mathbf{R}^n$, where V is a solution to

$$\Delta_y V + \lambda_1 K(Q_0) V = 0 \quad \text{in } \mathbf{R}^n. \tag{3.3}$$

Hence any solution to (3.3) must have a finite zero; however, v_{ε_j} is positive in $\mathbf{R}^n$. This is a contradiction. Thus $\|w_\varepsilon\|_\infty^\varepsilon$ is uniformly bounded.

Setting $\tilde{v}_\varepsilon(x) := w_\varepsilon/\|w_\varepsilon\|_\infty$, we see that

$$\int_{\mathbf{R}^n} |\nabla \tilde{v}_\varepsilon|^2\,dx \le \lambda_1 \|w_\varepsilon\|_\infty^\varepsilon \int_{\mathbf{R}^n} K(x)\tilde{v}_\varepsilon^2\,dx.$$

Now we put $V_\varepsilon := \tilde{v}_\varepsilon/(\int_{\mathbf{R}^n} K(x)\tilde{v}_\varepsilon^2\,dx)^{1/2}$. Then $\int_{\mathbf{R}^n} K(x)V_\varepsilon^2\,dx = 1$ and $\int_{\mathbf{R}^n} |\nabla V_\varepsilon|^2\,dx \le \lambda_1\|w_\varepsilon\|_\infty^\varepsilon$. Since the maximum point of V_ε is uniformly bounded, there exists $V_0(\not\equiv 0) \in \mathcal{D}_1$ such that V_ε converges indeed strongly to V_0 in $\mathcal{D}_1$ as $\varepsilon \downarrow 0$ (at least along a subsequence—the strong convergence is deduced from an argument similar to that in Section 2). Suppose that we can choose a sequence such that $\|w_\varepsilon\|_\infty^\varepsilon \to c_0 \neq 1$. Then we obtain $c_0\lambda_1 \geq \int_{\mathbf{R}^n} |\nabla V_0|^2\,dx$. This implies that V_0 must be an eigenfunction corresponding to the eigenvalue not greater than $c_0\lambda_1$. However, if $c_0 < 1$, then $c_0\lambda_1$ must be the smallest eigenvalue, which is absurd. If $c_0 > 1$, and if V_0 is an eigenfunction corresponding to a larger eigenvalue, V_0 must be a sign-changing solution, which violates the positivity of V_ε.

Thus we see that $V_\varepsilon \to \tilde{c}\varphi$ in $\mathcal{D}_1$ as $\varepsilon \downarrow 0$ with $\tilde{c} = (\int_{\mathbf{R}^n} K(x)\varphi^2\,dx)^{-1/2}$ by the simplicity of λ_1. Also we see that $\|w_\varepsilon\|_\infty^\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Hence, in the expression $w_\varepsilon = C(\varepsilon)\varphi + U_\varepsilon$, $C(\varepsilon)$ is dominant and $w_\varepsilon/\|\nabla w_\varepsilon\|_2$ converges to $c\varphi$ in $\mathcal{D}_1$ with $c = (\lambda_1 \int_{\mathbf{R}^n} K\varphi^2\,dx)^{-1/2}$ as $\varepsilon \downarrow 0$. Consequently, we see that $\|\nabla w_\varepsilon\|_2^\varepsilon \to 1$ as $\varepsilon \downarrow 0$. □

The following shows the ratio of the magnitude of w_ε and the remainder term.

Lemma 3.2. *Let $w_\varepsilon = C(\varepsilon)\varphi + U_\varepsilon$ with $\varphi \perp U_\varepsilon$ (in $\mathcal{D}(\mathbf{R}^n)$) be a solution to* (3.1). *Then U_ε satisfies $\limsup_{\varepsilon\to 0} \|\nabla U_\varepsilon\|_2/(C(\varepsilon)\varepsilon) < \infty$.*

Proof. In view of Proposition 2.1, we see that U_ε and $C(\varepsilon)$ exist. Moreover, by Lemma 3.1, we also have $\lim_{\varepsilon\downarrow 0} C(\varepsilon)^{-1}\|\nabla U_\varepsilon\|_2 = 0$. Thus, we can expand $(C(\varepsilon)\varphi + U_\varepsilon)^{1+\varepsilon}$ as

$$\begin{aligned}(C(\varepsilon)\varphi + U_\varepsilon)^{1+\varepsilon} &= (C(\varepsilon)\varphi + U_\varepsilon)\exp\Big\{\varepsilon\log(C(\varepsilon)\varphi + U_\varepsilon)\Big\} \\ &= (C(\varepsilon)\varphi + U_\varepsilon)\Big\{1 + \varepsilon\log(C(\varepsilon)\varphi + U_\varepsilon) + R_1(\varepsilon, U_\varepsilon)\Big\} \\ &= C(\varepsilon)\varphi + U_\varepsilon + C(\varepsilon)\varepsilon\varphi\log(C(\varepsilon)\varphi + U_\varepsilon) + R_2(\varepsilon, U_\varepsilon),\end{aligned} \tag{3.4}$$

where

$$\|\nabla R_1(\varepsilon, U_\varepsilon)\|_2 = O(\varepsilon^2 + \varepsilon^2\|\nabla U_\varepsilon\|_2), \quad \|\nabla R_2(\varepsilon, U_\varepsilon)\|_2 = O(\varepsilon^2 + \varepsilon\|\nabla U_\varepsilon\|_2).$$

Now we substitute $w_\varepsilon = C(\varepsilon)\varphi + U_\varepsilon$ into (3.1), multiply both sides by U_ε, and integrate it over $\mathbf{R}^n$. Then we have

$$\int_{\mathbf{R}^n} |\nabla U_\varepsilon|^2\,dx = \lambda_1 \int_{\mathbf{R}^n} K(x)(C(\varepsilon)\varphi + U_\varepsilon)^{1+\varepsilon} U_\varepsilon\,dx.$$

Letting $W_\varepsilon := U_\varepsilon/(C(\varepsilon)\varepsilon)$, we obtain

$$\int_{\mathbf{R}^n} |\nabla W_\varepsilon|^2\,dx = \lambda_1 C(\varepsilon)^\varepsilon \int_{\mathbf{R}^n} K(x)\frac{(\varphi + \varepsilon W_\varepsilon)^{1+\varepsilon}}{\varepsilon} W_\varepsilon\,dx.$$

Since $\int_{\mathbf{R}^n} K(x)\varphi W_\varepsilon\,dx = 0$, we have

$$\int_{\mathbf{R}^n} K(x)\frac{(\varphi + \varepsilon W_\varepsilon)^{1+\varepsilon}}{\varepsilon} W_\varepsilon\,dx = \int_{\mathbf{R}^n} K(x)\frac{(\varphi + \varepsilon W_\varepsilon)^{1+\varepsilon} - \varphi}{\varepsilon} W_\varepsilon\,dx. \tag{3.5}$$

Using the expansion in (3.4) (regarding $C(\varepsilon) = 1$ and $U_\varepsilon = \varepsilon W_\varepsilon$), we see that the right-hand side of (3.5) yields

$$\int_{\mathbf{R}^n} K(x)\Big\{W_\varepsilon + \varphi \log(\varphi + \varepsilon W_\varepsilon) + R_3(\varepsilon, W_\varepsilon)\Big\}W_\varepsilon\,dx$$

with $\|\nabla R_3(\varepsilon, W_\varepsilon)\|_2 = O(\varepsilon + \varepsilon\|\nabla W_\varepsilon\|_2)$. Thus we have

$$\begin{aligned}&\int_{\mathbf{R}^n} |\nabla W_\varepsilon|^2\,dx \\ &\quad = \lambda_1 C(\varepsilon)^\varepsilon \int_{\mathbf{R}^n} K(x)\Big\{W_\varepsilon + \varphi \log(\varphi + \varepsilon W_\varepsilon) + R_3(\varepsilon, W_\varepsilon)\Big\}W_\varepsilon\,dx.\end{aligned} \tag{3.6}$$

Let $\chi_\varepsilon := W_\varepsilon/\|\nabla W_\varepsilon\|_2$. Then (3.6) yields

$$\begin{aligned}1 = \lambda_1 C(\varepsilon)^\varepsilon \int_{\mathbf{R}^n} K\chi_\varepsilon^2\,dx + \frac{\lambda_1 C(\varepsilon)^\varepsilon}{\|\nabla W_\varepsilon\|_2}\int_{\mathbf{R}^n} K\varphi\Big\{\log(\varphi + \varepsilon\|\nabla W_\varepsilon\|_2\chi_\varepsilon)\Big\}\chi_\varepsilon\,dx \\ + \frac{\lambda_1 C(\varepsilon)^\varepsilon}{\|\nabla W_\varepsilon\|_2}\int_{\mathbf{R}^n} KR_3(\varepsilon, W_\varepsilon)\chi_\varepsilon\,dx.\end{aligned} \tag{3.7}$$

Suppose that $\limsup_{\varepsilon\downarrow 0}\|\nabla W_\varepsilon\|_2 = \infty$. Then the second and third terms in the right-hand side of (3.7) converge to 0 along (at least) a subsequence since $\log\{C(\varepsilon)\varphi + \varepsilon\|\nabla W_\varepsilon\|_2\chi_\varepsilon\} = o(\|\nabla W_\varepsilon\|_2)$. Moreover, as in the proof of Lemma 3.1, we see that $C(\varepsilon)^\varepsilon \to 1$ as $\varepsilon \downarrow 0$. Hence, $\{\chi_\varepsilon\}$ has a subsequence which converges to a constant multiple of the eigenfunction φ by an argument similar to the proof of Proposition 2.1. However, since $\int_{\mathbf{R}^n} K(x)\chi_\varepsilon\varphi\,dx = 0$, this is a contradiction. Thus we see that $\|\nabla U_\varepsilon\|_2/C(\varepsilon)\varepsilon$ is bounded. □

Next, as long as $w_\varepsilon = C(\varepsilon)\varphi + U_\varepsilon$ is a solution to (3.1), we can determine the value of $C(0)$.

Lemma 3.3. *For a solution $w_\varepsilon = C(\varepsilon)\varphi + U_\varepsilon$ to* (3.1), *the coefficient $C(\varepsilon)$ satisfies $\lim_{\varepsilon\downarrow 0} C(\varepsilon) = 1$.*

Proof. By Lemma 3.2, we may assume that any solution w to (3.1) is of the form

$$w = C(\varepsilon)(\varphi + \varepsilon V_\varepsilon) \tag{3.8}$$

with $\varphi \perp V_\varepsilon$ in $\mathcal{D}(\mathbf{R}^n)$ and $\limsup_{\varepsilon\downarrow 0}\|\nabla V_\varepsilon\|_2 < \infty$. Substituting (3.8) for (3.1), we have

$$\Delta w + \lambda_1 K(x) w^{1+\varepsilon} = C(\varepsilon)\varepsilon\Delta V_\varepsilon + \lambda_1 K(x)\Big[\Big\{C(\varepsilon)(\varphi + \varepsilon V_\varepsilon)\Big\}^{1+\varepsilon} - C(\varepsilon)\varphi\Big] = 0.$$

Thus we obtain

$$\Delta V_\varepsilon+\lambda_1 K(x)V_\varepsilon=-\lambda_1 K(x)\frac{\left[\left\{C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\right\}^{1+\varepsilon}-C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\right]}{C(\varepsilon)\varepsilon}. \tag{3.9}$$

Since (3.1) has a solution, (3.9) must have a solution. Regarding the right-hand side of (3.9) as an inhomogeneous term, we see that

$$\int_{\mathbf{R}^n} K(x)\Big[\frac{\left\{C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\right\}^{1+\varepsilon}-C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)}{C(\varepsilon)\varepsilon}\Big]\varphi(x)\,dx=0 \tag{3.10}$$

is a necessary condition for (3.9) to have a solution. Thus (3.10) holds for any $\varepsilon>0$. Hence we can take the limiting process as $\varepsilon\downarrow 0$ to show

$$\lim_{\varepsilon\downarrow 0}\int_{\mathbf{R}^n} K(x)(\varphi+\varepsilon V_\varepsilon)\frac{\left\{C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\right\}^{\varepsilon}-1}{\varepsilon}\varphi\,dx=0.$$

Since

$$\begin{aligned}&\left\{C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\right\}^{\varepsilon} \qquad (3.11)\\ &=1+\varepsilon\log\{C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\}+\frac{\varepsilon^2}{2}\big(\log\{C(\varepsilon)(\varphi+\varepsilon V_\varepsilon)\}\big)^2+R_4(\varepsilon,V_\varepsilon),\end{aligned}$$

where $\|\nabla R_4(\varepsilon,V_\varepsilon)\|_2=O(\varepsilon^3(\log C(\varepsilon))^3+\varepsilon^3\|\nabla V_\varepsilon\|_2)$ by (3.4), and since $\varepsilon V_\varepsilon$ is a higher-order term, in view of Lemma 3.2, we have

$$\int_{\mathbf{R}^n} K(x)\varphi^2\log(C(0)\varphi)\,dx=0;\ \text{i.e., } C(0)=1$$

by $\int_{\mathbf{R}^n} K(x)\varphi^2\log\varphi\,dx=0$ and $C(\varepsilon)^\varepsilon\to 1$ as $\varepsilon\downarrow 0$. □

As we have seen above, a solution w to (3.1) must be of the form

$$w=C(\varepsilon)\varphi+\varepsilon V_\varepsilon \tag{3.12}$$

with $C(0)=1$, $\varphi\perp V_\varepsilon$ in $\mathcal{D}(\mathbf{R}^n)$, and $\|\nabla V_\varepsilon\|_2$ is uniformly bounded. We may assume that $C(\varepsilon)=1+D(\varepsilon)$ with $D(\varepsilon)\to 0$ (as $\varepsilon\downarrow 0$) and $V_\varepsilon=U_0+\Phi_\varepsilon$, with $\limsup_{\varepsilon\downarrow 0}\|\nabla\Phi_\varepsilon\|_2<\infty$. Here U_0 is a solution to

$$\begin{cases}\Delta U_0+\lambda_1 K(x)U_0=-\lambda_1 K(x)\varphi(x)\log\varphi(x) \quad \text{in } \mathbf{R}^n,\\ U_0\in\mathcal{D}_1\cap\mathcal{D}^\perp(\mathbf{R}^n).\end{cases} \tag{3.13}$$

We note that (3.13) has a unique solution in $C^2(\mathbf{R}^n)\cap\mathcal{D}^\perp(\mathbf{R}^n)\cap\mathcal{D}_1$ in view of the invertibility of $(-\Delta-\lambda_1 K(x))$ in $\mathcal{D}^\perp(\mathbf{R}^n)$ and $\int_{\mathbf{R}^n} K(x)\varphi^2\log\varphi\,dx=0$ together with the regularity theory since $K\varphi\log\varphi\in L^\infty(\mathbf{R}^n)\cap C^1(\mathbf{R}^n)$. Thus we may set

$$w=(1+D(\varepsilon))\varphi+\varepsilon U_0+\varepsilon\Phi_\varepsilon. \tag{3.14}$$

Lemma 3.4. *For a solution w to* (3.1), *there holds* $\limsup_{\varepsilon\downarrow 0}\varepsilon^{-1}\|\nabla\Phi_\varepsilon\|_2 < \infty$ *and* $\limsup_{\varepsilon\downarrow 0}\varepsilon^{-1}D(\varepsilon) < \infty$.

Proof. Since w_ε is a solution to (3.1) and U_0 is a solution to (3.13), we see that Φ_ε exists and is given as a solution to

$$\varepsilon(\Delta\Phi_\varepsilon + \lambda_1 K(x)\Phi_\varepsilon) + \lambda_1 K(x)\Big[\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\}^{1+\varepsilon} - C(\varepsilon)\varphi - \varepsilon(U_0+\Phi_\varepsilon) - \varepsilon\varphi\log\varphi\Big] = 0. \tag{3.15}$$

We need to expand the right-hand side in the Taylor series as in (3.11). Since

$$\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\}^{\varepsilon} = 1 + \varepsilon\log\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\} + \frac{\varepsilon^2}{2}\Big[\log\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\}\Big]^2 + R_5(\varepsilon,\Phi_\varepsilon), \tag{3.16}$$

we have

$$\begin{aligned}&\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\}^{1+\varepsilon}\\ &= C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon) + \varepsilon\big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\big\}\log\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\}\\ &+\frac{1}{2}\varepsilon^2\varphi\Big(\log\varphi\Big)^2 + R_6(\varepsilon,\Phi_\varepsilon)\end{aligned}$$

by $C(0)=1$, where $\|\nabla R_5(\varepsilon,\Phi_\varepsilon)\|_2 = O(\varepsilon^3+\varepsilon^4\|\nabla\Phi_\varepsilon\|_2)$ and $\|\nabla R_6(\varepsilon,\Phi_\varepsilon)\|_2 = O(\varepsilon^3+\varepsilon^4\|\nabla\Phi_\varepsilon\|_2)$. Moreover, we see that

$$\log\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\} = \log\Big(C(\varepsilon)\varphi\Big) + \frac{\varepsilon(U_0+\Phi_\varepsilon)}{C(\varepsilon)\varphi} + R_7(\varepsilon,\Phi_\varepsilon),$$

where $\|\nabla R_7(\varepsilon,\Phi_\varepsilon)\|_2 = O(\varepsilon^2+\varepsilon^2\|\nabla\Phi_\varepsilon\|_2)$. Hence we have

$$\begin{aligned}&\Big\{C(\varepsilon)\varphi + \varepsilon(U_0+\Phi_\varepsilon)\Big\}^{1+\varepsilon} - C(\varepsilon)\varphi - \varepsilon(U_0+\Phi_\varepsilon) - \varepsilon\varphi\log\varphi\\ &= D(\varepsilon)\varepsilon\varphi + D(\varepsilon)\varepsilon\varphi\log\varphi\\ &\quad+\varepsilon^2(U_0+\Phi_\varepsilon) + \varepsilon^2(U_0+\Phi_\varepsilon)\log\varphi + \frac{1}{2}\varepsilon^2\varphi(\log\varphi)^2 + R_8(\varepsilon,D(\varepsilon),\Phi_\varepsilon)\end{aligned} \tag{3.17}$$

by $C(\varepsilon) = 1 + D(\varepsilon)$ and $\log(1 + D(\varepsilon)) = D(\varepsilon) + O(D(\varepsilon)^2)$, where $\|\nabla R_8\|_2 = O(\varepsilon(D(\varepsilon))^2 + \varepsilon^3 + \varepsilon^3\|\nabla\Phi_\varepsilon\|_2)$. Thus we obtain

$$\Delta\Phi_\varepsilon + \lambda_1 K(x)\Phi_\varepsilon = -\lambda_1 K(x)\bigg\{ D(\varepsilon)\varphi(\log\varphi(x) + 1) + \varepsilon(U_0 + \Phi_\varepsilon) + \varepsilon(U_0 + \Phi_\varepsilon)\log\varphi + \frac{1}{2}\varepsilon\varphi(\log\varphi)^2 + \frac{1}{\varepsilon}R_8(\varepsilon, D(\varepsilon), \Phi_\varepsilon)\bigg\}. \tag{3.18}$$

If $\lim_{\varepsilon\downarrow 0}|D(\varepsilon)|/\varepsilon = 0$, then we are done. If $\limsup_{\varepsilon\downarrow 0}|D(\varepsilon)|/\varepsilon > 0$ (might be infinite), then we see that $\limsup_{\varepsilon\downarrow 0}\|\nabla\Phi_\varepsilon\|_2/D(\varepsilon) < \infty$.

Multiplying both sides of (3.18) by φ and integrating it over $\mathbf{R}^n$, we have

$$D(\varepsilon)\int_{\mathbf{R}^n} K\varphi^2\,dx + \varepsilon\int_{\mathbf{R}^n} K\varphi(U_0 + \Phi_\varepsilon)\log\varphi\,dx + \frac{\varepsilon}{2}\int_{\mathbf{R}^n} K\varphi^2(\log\varphi)^2\,dx + \frac{1}{\varepsilon}\int_{\mathbf{R}^n} K\varphi R_8\,dx = 0,$$

since $\varphi \perp U_0$, $\varphi \perp \Phi_\varepsilon$ in $\mathcal{D}(\mathbf{R}^n)$, and $\int_{\mathbf{R}^n} K\varphi^2\log\varphi\,dx = 0$. In view of the order of R_8, we see that $D(\varepsilon) = O(\varepsilon)$. Hence we obtain $\limsup_{\varepsilon\downarrow 0}\varepsilon^{-1}\|\nabla\Phi_\varepsilon\|_2 < \infty$ and $\limsup_{\varepsilon\downarrow 0}\varepsilon^{-1}D(\varepsilon) < \infty$. □

In view of Lemma 3.4, we may further assume that $\Phi_\varepsilon = \varepsilon\Psi_\varepsilon$ with $\limsup_{\varepsilon\downarrow 0}\|\nabla\Psi_\varepsilon\|_2 < \infty$. By (3.15), we see

$$\varepsilon^2(\Delta\Psi_\varepsilon + \lambda_1 K(x)\Psi_\varepsilon) + \lambda_1 K(x)\Big[\Big(C(\varepsilon)\varphi + \varepsilon(U_0 + \varepsilon\Psi_\varepsilon)\Big)^{1+\varepsilon} - C(\varepsilon)\varphi - \varepsilon(U_0 + \varepsilon\Psi_\varepsilon) - \varepsilon\varphi\log\varphi\Big] = 0. \tag{3.19}$$

We have only to find a unique solution Ψ_ε to (3.19) with unique $C(\varepsilon)$ by the contraction-mapping principle.

Proof of Theorem 1.1. Let us denote the operator $(-\Delta - \lambda_1 K(x))^{-1}$ in $\mathcal{D}^\perp(\mathbf{R}^n)$ by L. Note that L is a bounded linear operator from $\mathcal{D}^\perp(\mathbf{R}^n)$ into itself. If the above Ψ_ε exists, it satisfies

$$\Psi_\varepsilon = \frac{1}{\varepsilon^2}L\Big[\lambda_1 K(x)\Big\{\Big(C(\varepsilon)\varphi + \varepsilon(U_0 + \varepsilon\Psi_\varepsilon)\Big)^{1+\varepsilon} - C(\varepsilon)\varphi - \varepsilon(U_0 + \varepsilon\Psi_\varepsilon) - \varepsilon\varphi\log\varphi\Big\}\Big].$$

In view of the expansion (3.18), the desired Ψ_ε must satisfy

$$\Psi_\varepsilon = \lambda_1 L\Big[K(x)\Big\{(\frac{D(\varepsilon)}{\varepsilon}\varphi + U_0)(\log\varphi + 1) + \frac{1}{2}\varphi(\log\varphi)^2 + \frac{R_9(\varepsilon, D(\varepsilon), \Psi_\varepsilon)}{\varepsilon^2}\Big\}\Big], \tag{3.20}$$

where $\|R_9(\varepsilon, D(\varepsilon), \Psi_\varepsilon)\| = O(\varepsilon^3 + \varepsilon(D(\varepsilon))^2 + \varepsilon^3\|\Psi_\varepsilon\|)$ for $\Psi \in \mathcal{D}^\perp(\mathbf{R}^n)$. We note that $\varphi, \varphi\log\varphi, \varphi(\log\varphi)^2, U_0 \in \mathcal{D}(\mathbf{R}^n)$ in view of (K.2) or (K.3) and due to the decay rate of φ. Moreover, we have $U_0\log\varphi \in \mathcal{D}(\mathbf{R}^n)$ since $\varphi \sim |x|^{2-n}$ at $|x| = \infty$ and $U_0 \in \mathcal{D}_1$. Indeed, we obtain

$$\int_{\mathbf{R}^n} K(x)U_0^2(\log\varphi)^2\,dx \le \Big(\int_{\mathbf{R}^n}(K(x)(\log\varphi)^2)^{n/2}\,dx\Big)^{2/n}\Big(\int_{\mathbf{R}^n} U_0^{2n/(n-2)}\,dx\Big)^{(n-2)/n},$$

and the right-hand side is bounded by (K.2) or (K.3).

First, we show the existence of the limit $C_1 := \lim_{\varepsilon\downarrow 0} D(\varepsilon)/\varepsilon$ and find its value. Note that there exists at least one $\Psi_\varepsilon \in \mathcal{D}^\perp(\mathbf{R}^n)$ satisfying (3.20) since we have shown the existence of a solution to (3.1). By (3.20), we have

$$\Delta\Psi_\varepsilon + \lambda_1 K(x)\Psi_\varepsilon = -\lambda_1 K(x)\Big\{(\frac{D(\varepsilon)}{\varepsilon}\varphi + U_0)(\log\varphi + 1) + \frac{1}{2}\varphi(\log\varphi)^2 + \frac{R_9(\varepsilon, D(\varepsilon), \Psi_\varepsilon)}{\varepsilon^2}\Big\}. \tag{3.21}$$

Multiplying both sides of (3.21) by φ, integrating it over $\mathbf{R}^n$, and letting $\varepsilon \downarrow 0$, we see that the limit C_1 exists and we have

$$\int_{\mathbf{R}^n} K(x)\Big\{C_1\varphi^2 + C_1\varphi^2\log\varphi + U_0\varphi + U_0\varphi\log\varphi + \frac{1}{2}\varphi^2(\log\varphi)^2\Big\}\,dx = 0.$$

Since $\int_{\mathbf{R}^n} K(x)\varphi^2\log\varphi\,dx = 0$ and since $\int_{\mathbf{R}^n} K(x)U_0\varphi\,dx = 0$, we obtain

$$C_1 = -\frac{\displaystyle\int_{\mathbf{R}^n} K(x)\Big\{U_0\varphi\log\varphi + \frac{1}{2}\varphi^2(\log\varphi)^2\Big\}\,dx}{\displaystyle\int_{\mathbf{R}^n} K(x)\varphi^2\,dx}.$$

Thus we may assume that $D(\varepsilon) = \varepsilon(C_1 + C_2(\varepsilon))$ with $C_2(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

To prove the uniqueness, we apply the contraction-mapping principle. Let $M_1 > 0$ and $M_2 > 0$, and define the space

$$\mathcal{S} := \{(\Psi, c) \in \mathcal{D}^\perp(\mathbf{R}^n) \times \mathbf{R} : \int_{\mathbf{R}^n} K(x)\Psi^2\,dx < M_1, |c| \le M_2\}$$

with its metric

$$d((\Psi_1, c_1), (\Psi_2, c_2)) := \|\Psi_1 - \Psi_2\| + |c_1 - c_2|.$$

Let us define an operator $T_1 : \mathcal{D}(\mathbf{R}^n) \mapsto \mathcal{D}(\mathbf{R}^n)$ by

$$T_1(\Psi, c) := L\Big[\lambda_1 K(x)\Big\{(\frac{D(\varepsilon)}{\varepsilon}\varphi + U_0)(\log\varphi + 1) + \frac{1}{2}\varphi(\log\varphi)^2 + \frac{R_9(\varepsilon, c, \Psi)}{\varepsilon^2}\Big\}\Big].$$

Then we have

$$\|T_1(U, c)\| \le \lambda_1 \|L\|_{\mathcal{D}^\perp \to \mathcal{D}^\perp}\Big\{\frac{D(\varepsilon)}{\varepsilon}\Big(\|\varphi\| + \|\varphi\log\varphi\|\Big) + \|U_0\| + \|U_0 \log\varphi\| + \frac{1}{2}\|\varphi(\log\varphi)^2\| + O(\varepsilon)\|U\| \Big\}. \tag{3.22}$$

The leading term of the right-hand side is independent of U if we take $\varepsilon > 0$ small, since $D(\varepsilon) = \varepsilon(C_1 + C_2(\varepsilon))$ with $C_2(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$. Moreover, we see

$$\|T_1(\Psi_1, c_1) - T_1(\Psi_2, c_2)\| \le O(\varepsilon)\|\Psi_1 - \Psi_2\|. \tag{3.23}$$

We need to find a fixed point in $\mathcal{S}$. For this purpose, we define another contraction map defined on $\mathbf{R}$. As we have seen above, $C_2(\varepsilon)$ satisfies

$$\begin{aligned} &(C_1 + C_2(\varepsilon)) \int_{\mathbf{R}^n} K\varphi^2\, dx \\ &= -\int_{\mathbf{R}^n} KU_0 \varphi \log\varphi\, dx - \frac{1}{2}\int_{\mathbf{R}^n} K\varphi^2(\log\varphi)^2\, dx \\ &\qquad - \frac{1}{\varepsilon^2}\int_{\mathbf{R}^n} KR_9(\varepsilon, \varepsilon(C_1 + C_2(\varepsilon)), \Psi_\varepsilon)\varphi\, dx. \end{aligned}$$

Thus we define a map T_2 by

$$\begin{aligned} &T_2(\Psi, c) \\ &:= -C_1 - \frac{1}{Q}\Big\{\int_{\mathbf{R}^n} KU_0\varphi\log\varphi\, dx + \frac{1}{2}\int_{\mathbf{R}^n} K\varphi^2(\log\varphi)^2\, dx \\ &\qquad + \frac{1}{\varepsilon^2}\int_{\mathbf{R}^n} KR_9(\varepsilon, \varepsilon(C_1 + c), \Psi)\varphi\, dx\Big\} \qquad (3.24) \\ &= -C_1 - \frac{1}{Q}\Big\{\int_{\mathbf{R}^n} KU_0\varphi\log\varphi\, dx + \frac{1}{2}\int_{\mathbf{R}^n} K\varphi^2(\log\varphi)^2\, dx\Big\} \\ &\qquad + O(\varepsilon + \varepsilon\|\Psi\|), \end{aligned}$$

by letting $Q := \int_{\mathbf{R}^n} K\varphi^2\,dx$. Note that T_2 is well-defined on $\mathcal{D}^\perp(\mathbf{R}^n)$ by

$$\int_{\mathbf{R}^n} K|U_0\varphi\log\varphi|\,dx \le \Big(\int_{\mathbf{R}^n} KU_0^2\,dx\Big)^{\frac12}\Big(\int_{\mathbf{R}^n} K(\varphi\log\varphi)^2\,dx\Big)^{\frac12} < \infty$$

due to $\varphi\log\varphi \in \mathcal{D}(\mathbf{R}^n)$ and

$$\int_{\mathbf{R}^n} K|R_9|\varphi\,dx \le \Big(\int_{\mathbf{R}^n} KR_9^2\,dx\Big)^{\frac12}\Big(\int_{\mathbf{R}^n} K\varphi^2\,dx\Big)^{\frac12} < \infty.$$

In addition, in view of the definition of C_1, we have $T_2(\Psi, c) = O(\varepsilon + \varepsilon\|\Psi\|)$.

Now consider the map $T_1 \times T_2 : \mathcal{D}^\perp(\mathbf{R}^n)\times\mathbf{R} \mapsto \mathcal{D}^\perp(\mathbf{R}^n)\times\mathbf{R}$. Let $(\Psi_i, c_i) \in \mathcal{S}$ with $i = 1, 2$. By (3.22) and (3.24), taking $M_1, M_2 > 0$ sufficiently large, we see that $T_1 \times T_2$ is a map from $\mathcal{S}$ into itself. Moreover, since $\|R_9(\varepsilon, \varepsilon(C_1 + c), \Psi)\| = O(\varepsilon^3(1+c)^2 + \varepsilon^3\|\Psi\|)$, we see that

$$|T_2(\Psi_1, c_1) - T_2(\Psi_2, c_2)| \le O(\varepsilon)|c_1 - c_2|. \tag{3.25}$$

Thus, if $\varepsilon > 0$ is sufficiently small, from (3.23) and (3.25), we obtain

$$d((T_1\times T_2)(\Psi_1, c_1), (T_1\times T_2)(\Psi_2, c_2)) \le O(\varepsilon)d((\Psi_1, c_1), (\Psi_2, c_2)).$$

Hence there exists a unique fixed point $(\tilde U_\varepsilon, C_2(\varepsilon))$ in $\mathcal{S}$ with $C_2(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$.

As for the behavior of $\tilde U_\varepsilon$ as $\varepsilon \downarrow 0$, in view of the decay rate of $K(x)$ and the fact that $\varphi, U_0 \in \mathcal{D}$, we have

$$\begin{aligned}
&\int_{\mathbf{R}^n} K(x)\Big\{(\frac{D(\varepsilon)}{\varepsilon}\varphi + U_0)(\log\varphi + 1) + \frac12\varphi(\log\varphi)^2 + \frac{R_9(\varepsilon, D(\varepsilon), \Psi_\varepsilon)}{\varepsilon^2} \\
&\qquad\qquad - (C_1\varphi + U_0)(\log\varphi + 1) - \frac12\varphi(\log\varphi)^2\Big\}^2\,dx \\
&= \int_{\mathbf{R}^n} K(x)\Big\{C_2(\varepsilon)\varphi(\log\varphi + 1) + \frac{1}{\varepsilon^2}R_9\Big\}^2\,dx \to 0
\end{aligned}$$

as $\varepsilon \downarrow 0$. Thus, the terms in the braces on the right-hand side of (3.21) converge in $\mathcal{D}(\mathbf{R}^n)$ to $(C_1\varphi + U_0)(\log\varphi + 1) + \frac12\varphi(\log\varphi)^2$. Thus we also see that $\tilde U_\varepsilon \to \tilde U_0$ in $\mathcal{D}(\mathbf{R}^n)$. Here $\tilde U_0 \in \mathcal{D}_1 \cap \mathcal{D}^\perp(\mathbf{R}^n) \cap C^2(\mathbf{R}^n)$ is a unique solution to

$$\begin{aligned}
&\Delta\tilde U_0 + \lambda_1 K(x)\tilde U_0 \\
&\quad = -\lambda_1 K(x)\Big\{(C_1\varphi + U_0)(\log\varphi + 1) + \frac12\varphi(\log\varphi)^2\Big\},
\end{aligned}\tag{3.26}$$

in $\mathbf{R}^n$. The existence and uniqueness of $\tilde{U}_0$ is ensured since the right-hand side is orthogonal to φ in $\mathcal{D}(\mathbf{R}^n)$ by the definition of C_1 and the invertibility of $(-\Delta - \lambda_1 K(x))$ on $\mathcal{D}^{\perp}(\mathbf{R}^n)$. □

4. Proof of Theorem 1.2.

In this section, we prove Theorem 1.2. As in the previous section, we consider

$$\begin{cases} \Delta w + \lambda_1 K(x) w^{1+\varepsilon} = 0 \quad \text{in } \Omega, \\ w > 0 \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

In this section, the reader may regard $K(x) \equiv 1$. There is no technical difference between $K(x) \equiv 1$ and $K(x) \not\equiv 1$.

First, we give a basic property of w_ε.

Lemma 4.1. *Suppose that* (K.4) *and* (K.5) *hold. Then there exists a solution w_ε to* (4.1) *for any $\varepsilon \in (0, 4/(n-2))$. Moreover, the maximum point of w_ε is uniformly away from $\partial\Omega$ as $\varepsilon \downarrow 0$.*

Proof. The existence of a solution to (4.1) is standard.

For the awayness of the maximum point of w_ε from the boundary, following Gidas, Ni, and Nirenberg [3], we introduce specific notation which is used in the method of the moving plane. Without loss of generality, we may assume that $\xi = (1, 0, \ldots, 0) \in \partial\Omega$ and an outward-normal vector at this point is given by $(1, 0, \ldots, 0)$. Let $T_\sigma := \{(\sigma, x_2, \ldots, x_n)\}$ be a hyperplane perpendicular to the x_1 axis and $x^\sigma = (2\sigma - x_1, x_2, \ldots, x_n)$ be the reflection point of x with respect to T_σ. Let Σ_σ be a subset of Ω defined by

$$\Sigma_\sigma := \{(t, x_2, \ldots, x_n) \in \Omega : \sigma < t < 1\},$$

and Σ'_σ be the reflection of Σ_σ with respect to T_σ. We define

$$\sigma^* := \inf\{\sigma : \Sigma'_\sigma \subset \Omega\}.$$

If Ω is convex, we have $\sigma^* < 1$.

Let $V_\varepsilon(x) := w_\varepsilon(x^\sigma) - w_\varepsilon(x)$ for $x \in \Sigma_\sigma$. By the clever use of the Hopf boundary-point lemma and the maximum principle, it was shown in Theorem 2.1 of [3] (condition (K.5), i.e., $\theta \cdot \nabla K(x) \le 0$ for $|\theta - n(x)| < \delta$, does not violate the applicability of this theorem—see also Remark 1.5 of de Figueiredo, Lions, and Nussbaum [1]) that $w_\varepsilon(x)$ is positive in Σ_σ for any $\sigma \in (\sigma^*, 1)$. This implies that $w_\varepsilon(x)$ is monotone decreasing in $x_1 \in (\sigma^*, 1)$, and hence the maximal point cannot lie in Σ_σ. Since the same argument applies to every point on $\partial\Omega$, the maximal point must be uniformly away from $\partial\Omega$.

Next suppose that Ω is not convex. Although (4.1) is dependent on x, we may assume that the point $x_0 := (-r_0, 0, \ldots, 0)$ $(r_0 > 0)$ is on $\partial\Omega$ and that the ball $B_{r_0} := \{x \in \mathbf{R}^n : |x| < r_0\}$, which has empty intersection with $\partial\Omega$, is circumscribed to $\partial\Omega$ in view of the smoothness of $\partial\Omega$ (Ω is concave at this point).

Now as in [1], we use the Kelvin transformation, which maps an open neighborhood U of a boundary portion containing x_0 into B_{r_0}. Let

$$\mathcal{K} : x \mapsto y = \frac{r_0^2 x}{|x|^2} \ (x \in U) \quad \text{and} \quad v(y) := \left(\frac{r_0^2}{|y|}\right)^{n-2} w(x).$$

Note that $\mathcal{K}(-r_0, 0, \ldots, 0) = (-r_0, 0, \ldots, 0)$ and $\mathcal{K}(U) \subset B_{r_0}$. Moreover, since the Kelvin transformation is a kind of "mirror reflection" with respect to ∂B_{r_0}, the convexity of B_{r_0} makes $\mathcal{K}(U)$ convex near $(-r_0, 0, \ldots, 0)$.

Since $w(x)$ is a solution to (4.1), $v(y)$ is a solution to

$$\Delta_y v(y) + \lambda_1 r_0^{2n-(n-2)(1+\varepsilon)} |y|^{(n-2)(1+\varepsilon)-(n+2)} K(x) v^p = 0, \tag{4.2}$$

where Δ_y is the Laplacian with respect to y. Let us define a function by $\tilde{K}(y) := |y|^{(n-2)(1+\varepsilon)-(n+2)} K(x)$. Note that $\tilde{K}(y)$ satisfies (K.5) for y near $(-r_0, 0, \ldots, 0)$ and $\varepsilon \in (0, 4/(n-2))$, as $K(x)$ does near the boundary. Indeed, for $\theta = (\theta_1, \ldots, \theta_n)$, we have

$$\begin{aligned}
&\theta \cdot \nabla_y \tilde{K}(y) \\
&= |y|^{(n-2)(1+\varepsilon)-(n+4)} \Big[\Big\{ (n-2)(1+\varepsilon) - (n+2) \Big\} (\theta \cdot y) K(x) \\
&\qquad + |r_0|^2 \Big\{ \theta \cdot \nabla_x K(x_0) - \frac{2}{|y|^2} (\theta \cdot y) \big(y \cdot \nabla_x K(x_0) \big) \Big\} \Big].
\end{aligned}$$

In this case, we can take $\theta = (-1, 0, \ldots, 0)$ and obtain

$$\begin{aligned}
&\theta \cdot \nabla_y \tilde{K}(y_0) \\
&= |r_0|^{(n-2)(1+\varepsilon)-(n+4)} \Big[\{ (n-2)(1+\varepsilon) - (n+2) \} r_0 K(x_0) + r_0^2 \frac{\partial K}{\partial x_1}(x_0) \Big] \\
&< 0
\end{aligned}$$

for $y_0 = \mathcal{K}(x_0) = (-r_0, 0, \ldots, 0)$ by $\partial K(x_0)/\partial x_1 \le 0$ in view of (K.5). Hence we see that any θ close to $(-1, 0, \ldots, 0)$ satisfies this inequality and that (K.5) holds for $\tilde{K}$. Thus we can apply the moving-plane method to (4.2) in $\mathcal{K}(U)$. Since $\mathcal{K}(U)$ is convex near $(-r_0, 0, \ldots, 0)$, the conclusion follows from the convex case. □

Next, we give the rough asymptotic behavior of a solution to (4.1) identical to Lemma 3.1.

Lemma 4.2. *Suppose that* (K.4) *and* (K.5) *hold. Then, for any solution* w_ε *to* (4.1), $\|\nabla w_2\|_{2,\Omega}^{\varepsilon} \to 1$ *and* $w_\varepsilon/\|\nabla w_\varepsilon\|_{2,\Omega} \to c\psi$ *in* $H_0^1(\Omega)$ *and thus in* $\mathcal{D}(\Omega)$ *as* $\varepsilon \to 0$ *with* $c = (\lambda_1 \int_\Omega K(x)\psi^2\,dx)^{-1/2}$.

Proof. As in the proof of Lemma 3.1, we rescale (4.1) by

$$v_\varepsilon(y) = \frac{w_\varepsilon(x)}{\|w_\varepsilon\|_{\infty,\Omega}}, \qquad y := \|w_\varepsilon\|_{\infty,\Omega}^{\varepsilon/2}(x - Q_\varepsilon),$$

where Q_ε is the maximum point of w_ε. By this scaling, we see that $v_\varepsilon(y)$ is a solution to

$$\Delta_y v_\varepsilon + \lambda_1 K(\frac{y}{\|w_\varepsilon\|_{\infty,\Omega}^{\varepsilon/2}} + Q_\varepsilon)v_\varepsilon^{1+\varepsilon} = 0. \tag{4.3}$$

Suppose that $\|w_\varepsilon\|_{\infty,\Omega}^{\varepsilon} \to \infty$ as $\varepsilon \downarrow 0$. Then, since $\|v_\varepsilon\|_{\infty,\Omega} = 1$, we can choose a subsequence $\{\varepsilon_j\}$ ($\varepsilon_j \to 0$ as $j \to \infty$) such that $Q_{\varepsilon_j} \to Q_0$ as $j \to \infty$ and $v_{\varepsilon_j}(y)(> 0) \to V(y)$ locally uniformly in $\mathbf{R}^n$, where V is a solution to

$$\Delta_y V + \lambda_1 K(Q_0)V = 0 \quad \text{in } \mathbf{R}^n. \tag{4.4}$$

By Lemma 4.1, we see that Q_0 is an interior point of Ω. Hence any solution to (4.4) must have a finite zero; however, v_{ε_j} is positive in Ω. This is a contradiction. Thus $\|w_\varepsilon\|_{\infty,\Omega}^{\varepsilon}$ is uniformly bounded.

For the remaining part of the proof, we exactly follow the proof of Lemma 3.1, and we omit the precise proof. □

The following is on the ratio of the coefficient of the eigenfunction space and that of its perpendicular direction identical to Lemma 3.2.

Lemma 4.3. *Let* $w_\varepsilon = C(\varepsilon)\psi + U_\varepsilon$ *with* $\psi \perp U_\varepsilon$ *(in* $\mathcal{D}(\Omega)$*) be a solution to* (4.1). *Then* U_ε *satisfies* $\limsup_{\varepsilon\to 0}\|\nabla U_\varepsilon\|_{2,\Omega}/(C(\varepsilon)\varepsilon) < \infty$.

Proof. By Lemma 4.2, we see that U_ε and $C(\varepsilon)$ exist. Moreover, we also have $\lim_{\varepsilon\downarrow 0}(C(\varepsilon))^{-1}\|\nabla U_\varepsilon\|_{2,\Omega} = 0$. Thus, the rest of the proof is identical to that of Lemma 3.2. □

Similar to Lemma 3.3, we have $C(0) = 1$.

From the above consideration, we see that there exists a unique solution U_0 to

$$\begin{cases} \Delta U_0 + \lambda_1 K(x)U_0 = -\lambda_1 K(x)\psi(x)\log\psi(x) & \text{in } \Omega, \\ U_0 = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

in $\mathcal{D}^{\perp}(\Omega) \cap C^2(\Omega) \cap C^1(\overline{\Omega})$. Indeed, to see this, we use the invertibility of $(-\Delta - \lambda_1 K(x))$ in $\mathcal{D}^{\perp}(\Omega)$ and $\int_\Omega K(x)\psi^2 \log\psi\, dx = 0$ with the regularity theory since $K\psi\log\psi \in C^\alpha(\overline{\Omega}) \cap C^1(\Omega)$ $(0 < \alpha < 1)$.

Thus, as in the whole-space case, we may assume that $C(\varepsilon) = 1 + D(\varepsilon)$ $(D(\varepsilon) \to 0)$ as $\varepsilon \downarrow 0$ and $V_\varepsilon = U_0 + \Phi_\varepsilon$; i.e., $U_\varepsilon = \varepsilon U_0 + \varepsilon\Phi_\varepsilon$. Since (4.1) has a solution, we see that there exists Φ_ε. Next, we estimate the vanishing order of Φ_ε in a way identical to Lemma 3.4.

Lemma 4.4. $\limsup_{\varepsilon\downarrow 0} \varepsilon^{-1}\|\nabla\Phi_\varepsilon\|_{2,\Omega} < \infty$ *and* $\limsup_{\varepsilon\downarrow 0} \varepsilon^{-1} D(\varepsilon) < \infty$.

Proof. Almost all parts of the proof are identical to those of Lemma 3.4. The key point is the expansion of $(C(\varepsilon)\psi + \varepsilon(U_0 + \Phi_\varepsilon))^{1+\varepsilon}$, which is exactly the same as in the proof of Lemma 3.4. So we omit the proof. □

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. The uniqueness is proved in a manner identical to the proof of Theorem 1.1. So we omit the details. Only as for the limiting behavior, we also see that $\tilde{U}_\varepsilon \to \tilde{U}_0$ in $\mathcal{D}(\Omega)$ as $\varepsilon \downarrow 0$. Here $\tilde{U}_0 \in \mathcal{D}(\Omega) \cap C^2(\Omega) \cap C(\overline{\Omega})$ with $\tilde{U}_0$ on $\partial\Omega$ is a solution to

$$\begin{aligned} &\Delta\tilde{U}_0 + \lambda_1 K(x)\tilde{U}_0 = \\ &\qquad -\lambda_1 K(x)\Big\{(C_1\psi + U_0)(\log\psi + 1) + \frac{1}{2}\psi(\log\psi)^2\Big\}. \end{aligned} \tag{4.6}$$

Since our original problem is (1.5), the desired solution requires the coefficient $(\lambda_1/\lambda)^{1/\varepsilon}$ as in (1.6). □

5. Appendix

In this section, we prove that the maximum point of a solution to (1.1) is uniformly bounded. The precise decay order of K is needed here to prove the boundedness of the maximum point. As we have seen in the proof of Theorem 1.1, the decay rate of v_ε in Lemma 3.1 is known from Li and Ni [10] [11]. Here we prove that v_ε decays uniformly at infinity.

Proposition 5.1. *Suppose that* (K.1) *with* (K.2) *or* (K.1) *with* (K.3) *holds. Then the maximum point of a solution* v_ε *to* (1.1) *is uniformly bounded as* $\varepsilon \downarrow 0$.

Proof. We deduce the decay order of w_ε for a fixed exterior domain. We assume (K.1) with (K.2) only (taking arbitrary $\ell > 2$ in case of (K.3)).

Without loss of generality, we may assume $\lambda = 1$. Fix $\varepsilon > 0$. Using the Green's function of $-\Delta$ in $\mathbf{R}^n$, we have

$$v_\varepsilon(x) = \frac{1}{(n-2)\omega_n}\int_{\mathbf{R}^n} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy, \tag{5.1}$$

where ω_n is the area of the unit sphere S^{n-1}. By the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbf{R}^n} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy &\le \int_{|x-y|\le 1} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy \\ + &\quad \Big\{\int_{|x-y|\ge 1}\Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\,dy\Big\}^{\frac{n+2-(n-2)\varepsilon}{2n}} \\ &\qquad\qquad \times\Big\{\int_{|x-y|\ge 1} v_\varepsilon^{2n/(n-2)}\,dy\Big\}^{\frac{(n-2)(1+\varepsilon)}{2n}}.\end{aligned} \tag{5.2}$$

Since we may assume $K(y) \le c_*|y|^{-\ell}$ on $|x| \ge R_*$ with some $c_* > 0$, since $\|v_\varepsilon\|_\infty$ is finite and since $v_\varepsilon \in \mathcal{D}_1$, the right-hand side is finite.

From now on, $R_0 > 0$ is supposed to be large so that

$$|x|^\ell K(x) \le 2c_0 \tag{5.3}$$

on $|x| \ge R_0$, we take x with $|x| \ge R_1 := \max\{2R_0, R_0+1\}$, and C and C_ε denote generic constants independent of, and dependent on, ε, respectively. For $|x| \ge R_1$, first we note that

$$\int_{|x-y|\le 1} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy \le C\|v_\varepsilon\|_\infty^{1+\varepsilon}|x|^{-\ell}\int_{|x-y|\le 1}\frac{1}{|x-y|^{n-2}}\,dy \le C_\varepsilon|x|^{-\ell}$$

in view of (5.3) and $|x-y| \le 1$. Next, we decompose

$$\int_{|x-y|\ge 1}\Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\,dy = I_1 + I_2 + I_3 + I_4$$

with

$$\begin{aligned} I_1 &:= \int_{1\le|x-y|\le|x|/2}\Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\,dy, \\ I_2 &:= \int_{|x|/2\le|x-y|\le 2|x|}\Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\,dy, \\ I_3 &:= \int_{\substack{|x-y|\ge 2|x| \\ 2|x|\ge|y|\ge|x|}}\Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\,dy,\end{aligned}$$

$$I_4 := \int_{\substack{|x-y|\ge 2|x| \\ |y|\ge 2|x|}} \Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\, dy.$$

In the following calculations, we use (5.3) without mentioning it.

On $1 \le |x-y| \le |x|/2$, we see $|y| \ge |x|/2 \ge R_0$ and have

$$\begin{aligned} I_1 &\le C_\varepsilon |x|^{-2\ell n/\{n+2-(n-2)\varepsilon\}} \int_1^{|x|/2} r^{-\frac{2n(n-2)}{n+2-(n-2)\varepsilon}+n-1}\, dr \\ &\le C_\varepsilon (|x|^{-\frac{2\ell n}{n+2-(n-2)\varepsilon}} + |x|^{-\frac{n\{(n-2)(1+\varepsilon)+2(\ell-2)\}}{n+2-(n-2)\varepsilon}}). \end{aligned}$$

For I_2, we have

$$\begin{aligned} I_2 &\le C_\varepsilon |x|^{-\frac{2n(n-2)}{n+2-(n-2)\varepsilon}} \Big(\int_{|y|\le R_0} + \int_{R_0\le |y|\le 3|x|}\Big) K(y)^{\frac{2n}{n+2-(n-2)\varepsilon}}\, dy \\ &\le C_\varepsilon (|x|^{-\frac{2n(n-2)}{n+2-(n-2)\varepsilon}} + |x|^{-\frac{n\{(n-2)(1+\varepsilon)+2(\ell-2)\}}{n+2-(n-2)\varepsilon}}), \end{aligned}$$

since $|x-y| \le 2|x|$ implies $|y| \le 3|x|$. Similarly, for I_3, we obtain

$$\begin{aligned} I_3 &\le C_\varepsilon |x|^{-\frac{2n(n-2)}{n+2-(n-2)\varepsilon}} |x|^{-\frac{2\ell n}{n+2-(n-2)\varepsilon}} \int_{|x|\le |y|\le 2|x|} dy \\ &\le C_\varepsilon |x|^{-\frac{n\{(n-2)(1+\varepsilon)+2(\ell-2)\}}{n+2-(n-2)\varepsilon}}. \end{aligned}$$

Finally, for I_4, we note that $|x-y| \ge 2|x|$ with $|y| \ge 2|x|$ implies $|x-y| \ge |y|/2$ (indeed, $|x-y| \ge |y| - |x| \ge |y| - |y|/2$). Thus we have

$$I_4 \le C_\varepsilon \int_{\substack{|x-y|\ge 2|x| \\ |y|\ge 2|x|}} |y|^{-\frac{2n(n-2+\ell)}{n+2-(n-2)\varepsilon}}\, dy \le C_\varepsilon |x|^{-\frac{n\{(n-2)(1+\varepsilon)+2(\ell-2)\}}{n+2-(n-2)\varepsilon}}.$$

Hence we obtain

$$\begin{aligned} &\Big\{ \int_{|x-y|\ge 1} \Big(\frac{K(y)}{|x-y|^{n-2}}\Big)^{\frac{2n}{n+2-(n-2)\varepsilon}}\, dy \Big\}^{\frac{n+2-(n-2)\varepsilon}{2n}} \le (I_1+I_2+I_3+I_4)^{\frac{n+2-(n-2)\varepsilon}{2n}} \\ &\le C_\varepsilon (|x|^{-\ell} + |x|^{-\{(n-2)(1+\varepsilon)+2(\ell-2)\}/2} + |x|^{-(n-2)}). \end{aligned}$$

Substituting the above inequalities into (5.2), in view of (5.1), we have

$$v_\varepsilon(x) \le C_\varepsilon |x|^{-\ell} + C_\varepsilon |x|^{-\min\{\ell, \{(n-2)(1+\varepsilon)+2(\ell-2)\}/2\}} \le C_\varepsilon |x|^{-p_\varepsilon} \tag{5.4}$$

for $|x| \ge R_1$ with $p_\varepsilon := \min\{\ell, \{(n-2)(1+\varepsilon)+2(\ell-2)\}/2\}$. Thus we have obtained an upper bound of the decay order of v_ε.

If $p_\varepsilon \ge n-2$, then the proof is done. If $p_\varepsilon < n-2$, then, by (5.4), we can estimate the right-hand side of (5.1) directly (without using the Hölder inequality) to obtain a better decay estimate than (5.4).

Following the previous process, we again decompose the integral of (5.1) as

$$\int_{\mathbf{R}^n} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy = J_1 + J_2 + J_3 + J_4 + J_5$$

with

$$J_1 := \int_{|x-y|\le 1} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy, \qquad J_2 := \int_{1\le|x-y|\le|x|/2} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy,$$

$$J_3 := \int_{|x|/2\le|x-y|\le 2|x|} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy, \quad J_4 := \int_{\substack{|x-y|\ge 2|x|\\ 2|x|\ge|y|\ge|x|}} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy,$$

$$J_5 := \int_{\substack{|x-y|\ge 2|x|\\ |y|\ge 2|x|}} \frac{K(y)v_\varepsilon(y)^{1+\varepsilon}}{|x-y|^{n-2}}\,dy.$$

Then, denoting a generic constant (which may depend on ε) by C_ε, we have, via a step similar to the previous one,

$$\begin{aligned}
J_1 &\le C_\varepsilon|x|^{-\ell-p_\varepsilon(1+\varepsilon)}\int_{|x-y|\le 1} dy \le C_\varepsilon|x|^{-\ell-p_\varepsilon(1+\varepsilon)},\\
J_2 &\le C_\varepsilon|x|^{-\ell-p_\varepsilon(1+\varepsilon)}\int_1^{|x|/2} r\,dr \le C_\varepsilon|x|^{2-\ell-p_\varepsilon(1+\varepsilon)},\\
J_3 &\le C_\varepsilon|x|^{-(n-2)}\Big(\int_{|y|\le R_0} + \int_{R_0\le|y|\le 3|x|}\Big)K(y)v_\varepsilon(y)^{1+\varepsilon}\,dy\\
&\le C_\varepsilon\big(|x|^{-(n-2)} + |x|^{2-\ell-p_\varepsilon(1+\varepsilon)}\big),\\
J_4 &\le C_\varepsilon|x|^{-(n-2)}|x|^{-\ell-p_\varepsilon(1+\varepsilon)}\int_{2|x|\ge|y|\ge|x|} dy \le C_\varepsilon|x|^{2-\ell-p_\varepsilon(1+\varepsilon)},\\
J_5 &\le C_\varepsilon\int_{|x|}^\infty |y|^{-p_\varepsilon(1+\varepsilon)+(1-\ell)}\,dy \le C_\varepsilon|x|^{2-\ell-p_\varepsilon(1+\varepsilon)}.
\end{aligned}$$

Thus we obtain $v_\varepsilon(x) \le C_\varepsilon|x|^{2-\ell-p_\varepsilon(1+\varepsilon)}$ for $|x| \ge R_1$, which is a better decay estimate than (5.4). For fixed $\varepsilon > 0$, repeating this process finitely many times, we have $v_\varepsilon(x) \le C_\varepsilon|x|^{-(n-2)}$ for $|x| \ge R_1$.

Then as in Theorem 2.8 or 2.16 of Li and Ni [11], we obtain

$$v_\varepsilon(x) = \tilde{C}_\varepsilon|x|^{-(n-2)} + h_\varepsilon(x) \tag{5.5}$$

for $|x| \ge R_1$ with

$$\tilde{C}_\varepsilon := \lim_{|x|\to\infty}|x|^{n-2}v_\varepsilon(x) = (n-2)^{-1}\omega_n^{-1}\int_{\mathbf{R}^n} K(x)v_\varepsilon^{1+\varepsilon}\,dx,$$

and $h_\varepsilon(x)$ being a higher-order term.

To prove Proposition 5.1, we need to deduce a more accurate expansion on v_ε. In view of Theorem 2.8 or Theorem 2.16 of [11], $h_\varepsilon(x)$ is expressed as follows: in case of (K.3),

$$h_\varepsilon(x) = \frac{a_\varepsilon \cdot x}{|x|^n} + R_\varepsilon\left(\frac{x}{|x|^2}\right)\frac{1}{|x|^{n-1}}, \tag{5.6}$$

in case of (K.2) with $\ell \geq 3$,

$$h_\varepsilon(x) = \frac{a_\varepsilon \cdot x}{|x|^n} + \frac{C_{1,\varepsilon}}{|x|^{n-2+\gamma_\varepsilon}} + R_\varepsilon\left(\frac{x}{|x|^2}\right)\frac{1}{|x|^{n-1}}, \tag{5.7}$$

and in case of (K.2) with $2 < \ell < 3$,

$$\begin{aligned} h_\varepsilon(x) = &\sum_{m=1}^{2k_\varepsilon+1} \frac{C_{1,m,\varepsilon}}{|x|^{n-2+m\gamma_\varepsilon}} \\ &+\sum_{m=1}^{k_\varepsilon} \frac{C_{2,m,\varepsilon}}{|x|^{n-1+m\gamma_\varepsilon}} + \frac{a_\varepsilon \cdot x}{|x|^n}\Big(1 + \sum_{m=1}^{k_\varepsilon} \frac{C_{3,m,\varepsilon}}{|x|^{m\gamma_\varepsilon}}\Big) + R_\varepsilon\left(\frac{x}{|x|^2}\right)\frac{1}{|x|^{n-1}}, \end{aligned} \tag{5.8}$$

where $a_\varepsilon \in \mathbf{R}^n$ is a constant vector, $C_{1,\varepsilon}$ and $C_{i,m,\varepsilon}$ $(i = 1,\ 2,\ 3)$ are constants, $\gamma_\varepsilon = (n-2)(1+\varepsilon) - (n-\ell)$, k_ε is a natural number such that $k_\varepsilon\gamma_\varepsilon \leq 1 < (k_\varepsilon+1)\gamma_\varepsilon$, and $R_\varepsilon(t)$ is a Lipschitz-continuous function near $t = 0$ with $R_\varepsilon(0) = 0$. Note that $\gamma_\varepsilon \to \ell - 2 > 0$ as $\varepsilon \downarrow 0$. Thus there exists a natural number n_0 such that $k_\varepsilon \leq n_0$ for any $\varepsilon > 0$.

As in the proof of Theorem 2.16 of [11] (see (2.21) or (2.22) on p. 203), $C_{1,\varepsilon} = C_{1,1,\varepsilon} = -c_0\tilde{C}_\varepsilon^{1+\varepsilon}/\{\gamma_\varepsilon(n-2+\gamma_\varepsilon)\}$. Indeed, in case of (K.2) for $|x| \geq R_1$, v_ε satisfies

$$\Delta v_\varepsilon + |x|^{-\ell}(c_0 + c_1|x|^{-1} + O(|x|^{-\mu}))(\tilde{C}_\varepsilon^{1+\varepsilon}|x|^{-(n-2)(1+\varepsilon)} + v_{1,\varepsilon}) = 0$$

with $v_{1,\varepsilon}$ being the remainder term. Then according to the deduction of (2.21) or (2.22) in the proof of Theorem 2.16 of [11], we have

$$v_\varepsilon(x) = \tilde{C}_\varepsilon|x|^{-(n-2)} - \frac{c_0\tilde{C}_\varepsilon^{1+\varepsilon}}{\gamma_\varepsilon(\gamma_\varepsilon+n-2)}|x|^{-(n-2)-\gamma_\varepsilon} + \frac{a_\varepsilon \cdot x}{|x|^n} + v_{2,\varepsilon}(x)$$

for $\ell \geq 3$ and

$$v_\varepsilon(x) = \tilde{C}_\varepsilon|x|^{-(n-2)} - \frac{c_0\tilde{C}_\varepsilon^{1+\varepsilon}}{\gamma_\varepsilon(\gamma_\varepsilon+n-2)}|x|^{-(n-2)-\gamma_\varepsilon} + v_{2,\varepsilon}(x)$$

for $2 < \ell < 3$, where $v_{2,\varepsilon}(x)$ is a remainder term. In the latter case, we can repeat the above process to obtain (5.8). Then the remainder term can be expressed as in (5.7) or (5.8).

In the case of (K.3), in view of the decay property of K, h_ε is expressed as (5.6).

Suppose that the maximum point x_ε of v_ε goes to infinity along a sequence $\{\varepsilon_j\}$ ($\varepsilon_j \to 0$ as $j \to \infty$). Let us fix x_0 so that $|x_0| \geq R_1$. Then we have

$$\frac{v_{\varepsilon_j}(x_{\varepsilon_j})}{v_{\varepsilon_j}(x_0)} = (\frac{x_{\varepsilon_j}}{x_0})^{-(n-2)}\Big(\frac{C_{\varepsilon_j} + |x_{\varepsilon_j}|^{n-2}h_{\varepsilon_j}(x_{\varepsilon_j})}{C_{\varepsilon_j} + |x_0|^{n-2}h_{\varepsilon_j}(x_0)}\Big) > 1. \tag{5.9}$$

Let us decompose

$$h_\varepsilon(x) = h_{1,\varepsilon}(x) + R_\varepsilon(\frac{x}{|x|^2})\frac{1}{|x|^{n-1}}.$$

In $h_{1,\varepsilon}(x)$, all the coefficients are functions of C_ε and so are the Lipschitz constants. $h_\varepsilon(x)$ is a higher-order term compared with $|x|^{-(n-2)}$. If C_{ε_j} is dominant, then it is easy to see that (5.9) does not hold. Since the same function is in both the numerator and the denominator, even if h_{ε_j} is dominant and if $x_{\varepsilon_j} \to \infty$, then the inequality (5.9) cannot hold. Hence, the maximum point is uniformly bounded. □

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