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SECOND-BEST CONSTANT AND EXTREMAL FUNCTIONS IN SOBOLEV INEQUALITIES IN THE PRESENCE OF SYMMETRIES

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Abstract. We prove results on the existence of extremal functions for critical Sobolev inequalities on Riemannian manifolds when the functions are invariant under an isometry group. In order to get those results, we study precisely a concentration phenomenon around an orbit for a sequence of solutions of a nonlinear PDE invariant under the isometry group.

1. Introduction

In this article, we prove a result on the existence of extremal functions for critical Sobolev inequalities when the functions are invariant under an isometry group. This problem can be studied only once we have established the existence of optimal constants in the considered inequalities, and this article is the sequel to a previous one entitled "Optimal constants in critical Sobolev inequalities on Riemannian manifolds in the presence of symmetries" [12], which answers this question.

We note that, in the case of Sobolev inequalities that we study here, the existence of an extremal function (i.e., a nonzero function which satisfies the equality in the considered inequality) gives a nontrivial solution of the associated nonlinear PDE, a solution that we can not get through the usual variational method since it does not apply. The most simple example is given by the Sobolev inequality, which characterizes the Sobolev embedding of $H_1^2(M)$ in $L_{\frac{2n}{n-2}}(M)$, (M,g) being a compact Riemannian n-manifold: for any $u \in H_1^2(M)$,

$$||u||^2_{\frac{2n}{n-2}} \leq K^2(n,2)\left(||\nabla u||_2^2 + B_0||u||_2^2\right),$$

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where $K(n,2)$ is the best constant possible such that the above inequality remains true for any $u \in H_1^2(M)$. If there exists $u_0 \in H_1^2(M)$, $u_0 \neq 0$ such that the above inequality is in fact an equality, then $u_0 \in C^\infty(M)$ and can be chosen such that $u_0 > 0$ and $\Delta u_0 + B_0 u_0 = u_0^{\frac{n+2}{n-2}}$. This solution is a minimizer for the functional

$$I(u) = \frac{||\nabla u||_2^2 + B_0 ||u||_2^2}{||u||^2_{\frac{2n}{n-2}}}$$

on $H_1^2(M) \setminus \{0\}$, and since the minimum is precisely $K^{-2}(n,2)$ we know that standard variational techniques do not give the existence of such a solution (more precisely, it's the condition $\inf_{H_1^2(M)\setminus\{0\}} I(u) < K^{-2}(n,2)$ which prevents variational techniques from leading to a solution identically equal to zero).

For a more complete reference on variational techniques related to problems involving critical exponents, the reader may refer to Aubin, Brezis-Coron (for H-surface and harmonic maps), Brezis-Nirenberg, P.L. Lions, Schoen, etc.

The existence of extremal functions for critical inequalities is a classical problem, widely studied in past years (see for example [5], [13], and [19]). The proof uses a very subtle study of a concentration phenomenon for a sequence of solutions of a nonlinear PDE. When we add a condition of invariance on the considered functions, which is what we do in this article, the concentration phenomenon becomes a lot more complex, since the functions do not concentrate at a point but along an orbit for the isometry group. A first step is to prove that this concentration orbit is of minimal dimension and minimum volume, and a specific difficulty, with respect to the case when there is no invariance by an isometry group, is linked to the fact that one has to "control" the distance between the concentration orbit and the orbit where the functions are maximum.

In this article, we choose to give a fast and rather schematic presentation of the steps of the proof which are adaptations of those presented by Druet and Djadli in [5] (when there is no symmetry) or by Faget in [12]. We give a lot more detail in the parts dealing with new difficulties brought by the condition of invariance by an isometry group. A few steps call for a lot of very technical computations, and sometimes we prefer to study those problems first under ideal hypotheses to show more clearly the basic ideas of the proof, and then go further into details when those hypotheses aren't satisfied.

1.1. **Preliminary results.** Let (M,g) be a Riemannian n-manifold and $Is(M,g)$ be its isometry group.

When G is a subgroup of $Is(M,g)$, let

$$\mathcal{C}^\infty_G(M) = \{u \in \mathcal{C}^\infty(M) : \forall \sigma \in G, u \circ \sigma = u\},$$
$$\mathcal{C}^\infty_{0,G}(M) = \{u \in \mathcal{C}^\infty_0(M) : \forall \sigma \in G, u \circ \sigma = u\},$$

where $\mathcal{C}^\infty(M)$ is the set of $\mathcal{C}^\infty$ functions on M, and $\mathcal{C}^\infty_0(M)$ is the set of $\mathcal{C}^\infty$ functions on M with compact support. We also introduce, for any $p \geq 1$,

$$L^p_G(M) = \{u \in L^p(M) : \forall \sigma \in G, u \circ \sigma = u\},$$
$$H^p_{1,G}(M) = \{u \in H^p_1(M) : \forall \sigma \in G, u \circ \sigma = u\},$$

and

$$\mathring{H}^p_{1,G}(M) = \{u \in \mathring{H}^p_1(M) : \forall \sigma \in G, u \circ \sigma = u\},$$

where the Sobolev space $H^p_1(M)$ (respectively $\mathring{H}^p_1(M)$) is the completion of $\mathcal{C}^\infty(M)$ (respectively $\mathcal{C}^\infty_0(M)$) with respect to the following norm:

$$||u||_{H^p_1} = \Big(\int_M |\nabla u|^p \mathrm{dV}\Big)^{\frac{1}{p}} + \Big(\int_M |u|^p \mathrm{dV}\Big)^{\frac{1}{p}}.$$

When no confusion is possible, we write L^p_G, $\mathcal{C}^\infty_G$, $\mathcal{C}^\infty_{0,G}$, $H^p_{1,G}$, and $\mathring{H}^p_{1,G}$ instead of $L^p_G(M)$, $\mathcal{C}^\infty_G(M)$, $\mathcal{C}^\infty_{0,G}(M)$, $H^p_{1,G}(M)$, and $\mathring{H}^p_{1,G}(M)$. We now recall some standard results.

The space $\mathcal{C}^\infty_G(M)$ is dense in $H^p_{1,G}(M)$, and $\mathcal{C}^\infty_{0,G}(M)$ is dense in $\mathring{H}^p_{1,G}(M)$ (see for example [18]).

If M is compact, $Is(M,g)$ is a compact Lie group and if G is a subgroup of $Is(M,g)$, its closure $\overline{G}$ for the standard topology is a compact Lie subgroup of $Is(M,g)$. Since G-invariant functions are also $\overline{G}$-invariant functions, we can assume without loss of generality that G is compact. In that case, for any $x \in M$, $O_{x,G} = \{\sigma(x), \sigma \in G\}$, the G-orbit of x, is a compact submanifold of M. Furthermore, $S_{x,G} = \{\sigma \in G : \sigma(x) = x\}$, the isotropy group of x, is a Lie subgroup of G, and the quotient manifold $G/S_{x,G}$ is diffeomorphic to $O_{x,G}$.

A G-orbit $O_{x,G}$ is said to be principal if for any $y \in M$, $S_{y,G}$ possesses a subgroup which is conjugate to $S_{x,G}$. Principal orbits are of maximum dimension (but orbits of maximal dimension are not necessarily principal).

Let Ω be the union of all principal orbits. Then Ω is a dense open subset of M, and Ω/G is a quotient manifold. More precisely, if Π is the associated submersion, $(\Pi, \Omega, \Omega/G)$ is a fibration where each fiber is an orbit (for all these results we refer to [3]).

In the following, let $\mathrm{vol}\, O_{x,G}$ be the volume of the submanifold $O_{x,G}$ for the Riemannian metric induced on $O_{x,G}$. In the specific case where $O_{x,G}$ has finite cardinality, then $\mathrm{vol}\, O_{x,G} = \mathrm{Card}\, O_{x,G}$.

When no confusion is possible, we denote $O_{x,G}$ by O_x and $S_{x,G}$ by S_x.

We say that we choose a neighborhood of O_x when we choose $\delta > 0$ and we consider $O_{x,\delta} = \{y \in M : \mathrm{d}(y, O_x) < \delta\}$. We now recall the theorem which is the basis of this article. The proof of the following Theorem is presented in [12], in the more general case of $H^p_{1,G}(M)$.

Theorem [F] (Faget)**.** *Let (M, g) be a compact Riemannian n-manifold, G a compact subgroup of the isometry group $Is(M, g)$. Let k be the minimum orbit dimension of G, and A the minimum volume of orbits of dimension k. For sake of clarity, we write $N = n - k$. If at least one of the two following assumptions (H_1) or (H_2) holds, then there exists $B > 0$ such that for any $u \in H^2_{1,G}(M)$,*

$$||u||^2_{2^*} \leq \frac{K^2}{A^{2/N}} \left(||\nabla u||^2_2 + B||u||^2_2\right), \tag{1.1}$$

with

$$K^2 = K^2(N, 2) = \frac{4}{N(N-2)w_N^{\frac{2}{N}}},$$

where w_N stands for the volume of the standard unit sphere of dimension N, and $2^* = \frac{2N}{N-2}$.

(H_1) For any O_{x_0} a G-orbit of minimum dimension k and minimum volume A, there exists H a subgroup of the isometry group of M and $\gamma > 0$ such that

(1) *on $O_{x_0,\gamma} = \{x \in M : \mathrm{d}(x, O_{x_0}) < \gamma\}$, all H-orbits are principal,*
(2) *for any $x \in O_{x_0,\gamma}$, $O_{x,H} \subset O_{x,G}$ and $O_{x_0,G} = O_{x_0,H}$,*
(3) *for any $x \in O_{x_0,\gamma}$, $A = \mathrm{vol}\,(O_{x_0}) \leq \mathrm{vol}\,(O_{x,H})$.*

(H_2) For any O_{x_0} a G-orbit of minimum dimension k and minimum volume A, there exists H a **normal** *subgroup of G and $\gamma > 0$ such that*

(1) *on $O_{x_0,\gamma}$, all H-orbits are principal,*
(2) *$O_{x_0,H} = O_{x_0,G}$,*
(3) *for any $x \in O_{x_0,\gamma}$, $x \notin O_{x_0}$, $\dim O_{x,G} > k = \dim O_{x_0,G}$,*
(4) *for any $x \in O_{x_0}$, x is a critical point of the function $\mathfrak{v} = \mathrm{vol}\,(O_{x,H})$.*

The above inequality (1.1) is then said to be optimal with respect to the first constant. We now set

$$B_0(g) = \inf\{B \text{ s.t. } \forall u \in H^2_{1,G}(M), (1.1) \text{ holds with } B\}.$$

Clearly, (1.1) holds with $B_0(g)$, in the sense that for any $u \in H^2_{1,G}(M)$,

$$||u||^2_{2^*} \leq \frac{K^2}{A^{2/N}} \left(||\nabla u||^2_2 + B_0(g)||u||^2_2 \right). \qquad (I^2_{opt})$$

The inequality is then optimal with respect for the first and second constants; i.e., none of them can be improved. Proposition 1 below will give us a lower bound for $B_0(g)$. First, we recall some useful geometric results, and we give the notation that will be used in this article.

1.2. **Geometric recollections and notation.** Let H be a subgroup of the isometry group of M, such that on $O_{x_0,\gamma}$ all H-orbits are principal. We denote by Π the submersion from $O_{x_0,\gamma}$ to $O_{x_0,\gamma}/H$. We can consider (see for example [18]) the quotient manifold $(O_{x_0,\gamma}/H, \tilde{g})$, where $\tilde{g}$ is the "quotient metric" induced by g for which Π becomes a Riemannian submersion. Any function u defined on $O_{x_0,\gamma}$ and H-invariant can be written $u = \tilde{u} \circ \Pi$, where $\tilde{u}$ is defined on $O_{x_0,\gamma}/H$, and for any function $\tilde{u}$ defined on $O_{x_0,\gamma}/H$, $u = \tilde{u} \circ \Pi$ is H-invariant and defined on $O_{x_0,\gamma}$. For any $x \in O_{x_0,\gamma}$, we denote by $\bar{x} = \Pi(O_x)$ ($\bar{x}_0 = \Pi(O_{x_0})$, $\bar{x}_\alpha = \Pi(O_{x_\alpha})$). This is the notation we will be using in all that follows. We denote by $\tilde{\mathfrak{v}}$ the function defined on $O_{x_0,\gamma}/H$ by $\tilde{\mathfrak{v}}(y) = \mathrm{vol}\,(\Pi^{-1}(y))$, and we denote by $\mathfrak{v}$ the function defined on $O_{x_0,\gamma}$ by $\mathfrak{v}(x) = \mathrm{vol}\,(O_x)$. If $\tilde{u} \in \mathcal{C}^0(O_{x_0,\gamma}/H)$ has a compact support we know, once again by [18], that

$$\int_{O_{x_0,\gamma}} \tilde{u} \circ \Pi \mathrm{dv}_g = \int_{O_{x_0,\gamma}/H} \tilde{u}\tilde{\mathfrak{v}} \mathrm{dv}_{\tilde{g}}.$$

Also, let us note that if $\tilde{\nabla}$ is the gradiant for the metric $\tilde{g}$ on $O_{x_0,\gamma}/H$, for any $\tilde{u}$ and $\tilde{w}$ defined on $O_{x_0,\gamma}/H$ we have

$$\left(\tilde{\nabla}\tilde{u}\tilde{\nabla}\tilde{w}\right) \circ \Pi = \nabla_g(\tilde{u} \circ \Pi)\nabla_g(\tilde{w} \circ \Pi).$$

This last result is in fact true in the more general case of a Riemannian submersion.

1.3. **A lower bound of the second-best constant $B_0(g)$.**

Proposition 1. *Let (M, g) be a Riemannian n-manifold, $n \geq 4$, and let G, k, A, and $N = n - k \geq 4$ be as in Theorem* [F]. *Assume that at least one of assumptions (H_1) or (H_2) from Theorem* [F] *holds. Then*

$$B_0(g) \geq \sup \left(\frac{A^{\frac{2}{N}}}{K^2} \mathrm{vol}\,(M)^{-\frac{2}{N}}, \frac{N-2}{4(N-1)} \left(\mathrm{Scal}_{\tilde{g}}(\bar{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{\tilde{\mathfrak{v}}}(\bar{x}_0) \right) \right).$$

Proof. Clearly, if we take $u \equiv 1$ in (I_{opt}^2), we get $B_0 \geq \frac{A^{\frac{2}{N}}}{K^2} \mathrm{vol}\,(M)^{-\frac{2}{N}}$. To get the second lower bound, we assume that either (H_1) or (H_2) holds for a subgroup H, and on $O_{x_0,\gamma}/H$, we consider, for $\varepsilon > 0$,

$$\tilde{u}_\varepsilon = (\varepsilon + r^2)^{1-\frac{N}{2}} - (\varepsilon + \gamma^2)^{1-\frac{N}{2}},$$

with $r = \mathrm{d}_{\tilde{g}}(\cdot, \bar{x}_0)$. With respect to the notation introduced above, we have

$$\int_M u_\varepsilon^2 \mathrm{dv}_g = \int_{O_{x_0,\gamma}/H} \tilde{\mathfrak{v}} \tilde{u}_\varepsilon^2 \mathrm{dv}_{\tilde{g}}, \quad \int_M u_\varepsilon^{2^*} \mathrm{dv}_g = \int_{O_{x_0,\gamma}/H} \tilde{\mathfrak{v}} \tilde{u}_\varepsilon^{2^*} \mathrm{dv}_{\tilde{g}},$$
$$\int_M |\nabla_g u_\varepsilon|^2 \mathrm{dv}_g = \int_{O_{x_0,\gamma}/H} \tilde{\mathfrak{v}} |\nabla_{\tilde{g}} \tilde{u}_\varepsilon|^2 \mathrm{dv}_{\tilde{g}}.$$

Standard computations lead to

$$||\nabla_g u_\varepsilon||_2^2 =$$
$$w_{N-1} \frac{(N-2)^2}{2} \varepsilon^{1-\frac{N}{2}} A I_N^{N/2} \Big(1 - \varepsilon \frac{N+2}{N-4} \Big(\frac{\mathrm{Scal}_{\tilde{g}}(\bar{x}_0)}{6N} + \frac{\Delta_{\tilde{g}} \tilde{\mathfrak{v}}(\bar{x}_0)}{2AN} \Big) + o(\varepsilon) \Big),$$
$$\text{if } N > 4$$
$$w_{N-1} \frac{(N-2)^2}{2} \varepsilon^{1-\frac{N}{2}} A \Big(I_N^{N/2} + \varepsilon \log(\varepsilon) \Big(\frac{\mathrm{Scal}_{\tilde{g}}(\bar{x}_0)}{6N} + \frac{\Delta_{\tilde{g}} \tilde{\mathfrak{v}}(\bar{x}_0)}{2AN} \Big) + o(\varepsilon \log(\varepsilon)) \Big),$$
$$\text{if } N = 4$$

$$\begin{aligned} ||u_\varepsilon||_2^2 &= \frac{2A w_{N-1}(N-1)(N-2)}{N(N-4)} \varepsilon^{2-N/2} I_N^{N/2} + o(\varepsilon), \quad && \text{if } N > 4 \\ &= \frac{-A w_{N-1}}{2} \log(\varepsilon) + o(\log(\varepsilon)), && \text{if } N = 4 \end{aligned}$$

and

$$||u_\varepsilon||_{2^*}^2 \geq$$
$$A^{\frac{N-2}{N}} K^2 \frac{(N-2)^2}{2} w_{N-1} I_N^{\frac{N}{2}} \varepsilon^{1-\frac{N}{2}} \left(1 - \varepsilon \Big(\frac{\mathrm{Scal}_{\tilde{g}}(\bar{x}_0)}{6(N-2)} + \frac{\Delta_{\tilde{g}} \tilde{\mathfrak{v}}(\bar{x}_0)}{2(N-2)A} \Big) + o(\varepsilon) \right)^{\frac{N-2}{N}},$$
$$\text{if } N > 4$$
$$A^{\frac{N-2}{N}} K^2 \frac{(N-2)^2}{2} w_{N-1} I_N^{N/2} \varepsilon^{1-N/2} \left(1 + o(\varepsilon \log(\varepsilon)\right), \quad \text{if } N = 4$$

where $I_p^q = \int_0^{+\infty}(1+t)^{-p}t^q \mathrm{dt}$, w_{N-1} stands for the volume of the standard unit sphere of dimension $N-1$, and $A = \tilde{\mathfrak{v}}(\bar{x}_0)$. Finally, we get

$$\frac{||\nabla_g u_\varepsilon||_2^2 + B||u_\varepsilon||_2^2}{||u_\varepsilon||_{2^*}^2} \leq \frac{A^{\frac{2}{N}}}{K^2}\Big[1 - \varepsilon\frac{N+2}{N-4}\Big(\frac{\mathrm{Scal}_{\tilde{g}}(\bar{x}_0)}{6N} + \frac{\Delta_{\tilde{g}}\tilde{\mathfrak{v}}(\bar{x}_0)}{2NA}\Big)$$
$$+ B\frac{4(N-1)\varepsilon}{N(N-2)(N-4)} + o(\varepsilon)\Big]\Big[1 + \varepsilon\Big(\frac{\mathrm{Scal}_{\tilde{g}}(\bar{x}_0)}{6N} + \frac{\Delta_{\tilde{g}}\tilde{\mathfrak{v}}(\bar{x}_0)}{2AN}\Big) + o(\varepsilon)\Big]$$
$$\leq \frac{A^{\frac{2}{N}}}{K^2}\Big[1 + \frac{\varepsilon}{N(N-4)}\Big(B\frac{4(N-1)}{N-2} - \Big(\mathrm{Scal}_{\tilde{g}}(\bar{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{\mathfrak{v}}(\bar{x}_0)}{A}\Big)\Big) + o(\varepsilon)\Big].$$

Since, with (I_{opt}^2),

$$\frac{||\nabla_g u_\varepsilon||_2^2 + B_0||u_\varepsilon||_2^2}{||u_\varepsilon||_{2^*}^2} \geq \frac{A^{\frac{2}{N}}}{K^2}$$

we get when ε goes to 0 that

$$B_0\frac{4(N-1)}{N-2} \geq \mathrm{Scal}_{\tilde{g}}(\bar{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{\mathfrak{v}}(\bar{x}_0)}{A}. \qquad \square$$

2. Extremal functions in the presence of symmetries

We are now interested in the existence of extremal functions for (I_{opt}^2), which means we are looking for $u_0 \in H_{1,G}^2(M)$, $u_0 \not\equiv 0$ such that (I_{opt}^2) is an equality. We prove the following theorem.

Theorem. *Let (M, g) be a smooth, compact Riemannian n-manifold, $n \geq 4$, G a compact subgroup of the isometry group $Is(M, g)$, k be the minimum orbit dimension of G, A the minimum volume of orbits of dimension k, and $N = n - k \geq 4$. Let $B_0(g)$ be as above. We assume that at least one of the following assumptions (H_1^+), (H_1^*), or (H_2) holds. Then if*

$$B_0(g)\frac{4(N-1)}{N-2} > \mathrm{Scal}_{\tilde{g}}(\bar{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{\tilde{\mathfrak{v}}}(\bar{x}_0),$$

there exist extremal functions for (I_{opt}^2).

(H_1^+) *For any O_{x_0} a G-orbit of minimum dimension k and minimum volume A, there exists H a subgroup of the isometry group of M and $\gamma > 0$ such that*

1. *on $O_{x_0,\gamma} = \{x \in M : \mathrm{d}(x, O_{x_0}) < \gamma\}$, all H-orbits are principal,*
2. *for any $x \in O_{x_0,\gamma}$, $O_{x,H} \subset O_{x,G}$ and $O_{x_0,G} = O_{x_0,H}$,*
3. *for any $x \in O_{x_0,\gamma}$, $A = \mathrm{vol}\,(O_{x_0}) \leq \mathrm{vol}\,(O_{x,H})$,*
4. *the function $\tilde{\mathfrak{v}}$ is nondegenerate in $\bar{x}_0$.*

(H_1^*) *Assumptions* (1) *to* (3) *from* (H_1^+) *hold, and only the last assumption changes to*

(4) $\Delta_{\tilde{g}}\tilde{\mathfrak{v}}(\bar{x}_0) = 0$.

(H_2) *For any* O_{x_0} *a* G*-orbit of minimum dimension* k *and minimum volume* A*, there exists* H *a normal subgroup of* G *and* $\gamma > 0$ *such that*

(1) *on* $O_{x_0,\gamma}$*, all* H*-orbits are principal,*
(2) $O_{x_0,H} = O_{x_0,G}$*,*
(3) *for any* $x \in O_{x_0,\gamma}$*,* $x \notin O_{x_0}$*,* $\dim O_{x,G} > k = \dim O_{x_0,G}$*,*
(4) *for any* $x \in O_{x_0}$*,* x *is a critical point of the function* $\mathfrak{v} = \mathrm{vol}\,(O_{x,H})$*.*

Remarks. (1) The reader has of course noticed that (H_1) from Theorem [F] is embedded in (H_1^+) and (H_1^*), and that (H_2) is the same in both theorems. This has to be, since when neither of (H_1) and (H_2) from Theorem [F] hold, we are not assured of the existence of $B_0(g)$.

(2) We will refer to $(H_1^{+,*})$ when assumption (4) is not specifically needed.

(3) We recall that according to Proposition 1,

$$B_0(g)\frac{4(N-1)}{N-2} \geq \mathrm{Scal}(\bar{x}_0) + \frac{3\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{\tilde{\mathfrak{v}}}(\bar{x}_0).$$

Proof. If $(H_1^{+,*})$ or (H_2) holds, then for any $u \in H_{1,G}^2(M)$ we have the following inequality:

$$\Big[\int_M u^{\frac{2N}{N-2}}\mathrm{dv}_g\Big]^{\frac{N-2}{N}} \leq \frac{K^2}{A^{\frac{2}{N}}}\Big[\int_M |\nabla_g u|^2\mathrm{dv}_g + B_0\int_M u^2\mathrm{dv}_g\Big] \qquad (I_{opt}^2)$$

with $B_0 = B_0(g)$ as above. For any $0 < \alpha < B_0$, there exists u_α such that

$$\Big[\int_M u_\alpha^{\frac{2N}{N-2}}\mathrm{dv}_g\Big]^{\frac{N-2}{N}} > \frac{K^2}{A^{\frac{2}{N}}}\Big[\int_M |\nabla_g u_\alpha|^2\mathrm{dv}_g + \alpha\int_M u_\alpha^2\mathrm{dv}_g\Big].$$

Let

$$I_\alpha(u) = \frac{\int_M |\nabla_g u|^2\mathrm{dv}_g + \alpha\int_M u^2\mathrm{dv}_g}{\Big[\int_M u^{\frac{2N}{N-2}}\mathrm{dv}_g\Big]^{\frac{N-2}{N}}}, \qquad \lambda_\alpha = \inf_{u\in H_{1,G}^2(M)} I_\alpha(u) < \frac{A^{\frac{2}{N}}}{K^2}.$$

Variational techniques presented in Faget [12] give the existence of a non-negative minimizer $v_\alpha \in \mathcal{C}_G^\infty$ of I_α, such that $\int_M v_\alpha^{\frac{2N}{N-2}}\mathrm{dv}_g = 1$ and

$$\Delta v_\alpha + \alpha v_\alpha = \lambda_\alpha v_\alpha^{\frac{N+2}{N-2}}. \tag{2.1}$$

The goal is now to study (v_α) when α goes to B_0 by lower values.

One easily checks that there exists $v \in H^2_{1,G}(M)$, $v \geq 0$ such that (v_α) converges weakly in $H^2_{1,G}(M)$ to v. If $v \not\equiv 0$, then v is a nonzero extremal function of (I^2_{opt}). We assume that $v \equiv 0$, and we prove that this option leads to

$$B_0 = \frac{N-2}{4(N-1)}\Big(\mathrm{Scal}_{\tilde g}(\bar x_0) + \frac{3\Delta_{\tilde g}\tilde{\mathfrak v}}{\tilde{\mathfrak v}}(\bar x_0)\Big).$$

We recall the definition of a concentration orbit: Set O_x a G-orbit of M; O_x is a concentration orbit of the sequence (u_α) if for any $\gamma > 0$,

$$\limsup_{\alpha\to B_0}\int_{O_{x,\gamma}} u_\alpha^{\frac{2N}{N-2}}\mathrm{dv}_g > 0,$$

where we recall that $O_{x,\gamma} = \{y \in M : \mathrm{d}(y, O_x) < \gamma\}$. Even though in our case α goes to B_0 and not to $+\infty$, all concentration phenomena studied in [12] remain true. We prove that, up to a subsequence, the sequence (v_α) has a unique concentration orbit, which is of dimension k and volume A, denoted by O_{x_0}.

In [12], we had to prove two fundamental inequalities involving v_α. The proof of this theorem requires a stronger estimate on v_α, but still we recall the two fundamental inequalities. We denote by O_{x_α} the orbit such that for any $x \in O_{x_\alpha}$, $u_\alpha(x) = \sup u_\alpha$. The (O_{x_α}) then go to O_{x_0} with α. We denote by $\bar x_\alpha = \Pi(O_{x_\alpha})$. If $(H_1^{+,*})$ or (H_2) holds, then the first fundamental inequality tells us that there exists $C > 0$ such that for any α and for any $x \in O_{x_0,\gamma}$, $\mathrm{d}_{\tilde g}(\bar x, \bar x_\alpha)^{\frac{N}{2}-1}\tilde u_\alpha(\bar x) \leq C$. If (H_2) holds, then the second fundamental inequality tells us that there exists $C > 0$ such that for any α, $\mathrm{d}_{\tilde g}(\bar x_0, \bar x_\alpha)^{\frac{N}{2}-1}\sup \tilde u_\alpha \leq C$.

Let O_{x_0} be the concentration orbit, and $\gamma > 0$ such that on $O_{x_0,\gamma}$ assumptions of $(H_1^{+,*})$ or (H_2) hold for a subgroup H.

Since $(H_1^{+,*})$ or (H_2) holds, we can work on $O_{x_0,\gamma}/H$ with the quotient metric $\tilde g$ induced by g on $O_{x_0,\gamma}/H$. With the notation introduced before, we denote by $\tilde v_\alpha$ the functions on $O_{x_0,\gamma}/H$, $\tilde{\mathfrak v}$ the volume, $\Pi(O_{x_0}) = \bar x_0$, $\Pi(O_{x_\alpha}) = \bar x_\alpha$, where Π is the submersion from O_{x_0} into $O_{x_0,\gamma}$. With (2.1), we get that the functions $\tilde v_\alpha$ satisfy

$$-\tilde\nabla(\tilde{\mathfrak v}\tilde\nabla\tilde v_\alpha) + \alpha\tilde{\mathfrak v}\tilde v_\alpha = \lambda_\alpha\tilde{\mathfrak v}\tilde v_\alpha^{\frac{N+2}{N-2}}.$$

For $\delta > 0$ small enough, we consider the exponential map at $\bar x_\alpha$, $\exp_{\bar x_\alpha} : \mathcal B_0(\delta) \subset \mathbb R^N \to B_{\bar x_\alpha}(\delta) \subset O_{x_0,\gamma}/H$. We keep the same notation as before, and we set for any $x \in \mathcal B_0(\delta)$,

$$u_\alpha(x) = \tilde v_\alpha(\exp_{\bar x_\alpha}(x)), \quad \mathfrak v(x) = \tilde{\mathfrak v}(\exp_{\bar x_\alpha}(x)), \quad g_\alpha = (\exp^*_{\bar x_\alpha}\tilde g).$$

First, we will work under ideal hypotheses. We assume that for any $\alpha < B_0$ $g_\alpha = \xi$ (Euclidian metric on $\mathcal{B}_0(\delta)$). We will see in the sequence that most of the difficulties which appear when we do not work under these hypotheses have been solved by Djadli and Druet [5]. We also assume that $\bar{x}_\alpha = \bar{x}_0$. We will see that when this assumption does not hold, some difficulties specific to our problem appear. The u_α satisfy

$$-\nabla_{g_\alpha}(\mathfrak{v}\nabla_{g_\alpha} u_\alpha) + \alpha \mathfrak{v} u_\alpha = \lambda_\alpha \mathfrak{v} u_\alpha^{\frac{N+2}{N-2}}, \tag{2.2}$$

and since we assumed that $\bar{x}_0 = \bar{x}_\alpha$, the sequence (u_α) concentrates on 0 when α goes to B_0 by lower values, and $u_\alpha(0) = \sup u_\alpha$.

We denote by $x_{0,\alpha} = \exp_{\bar{x}_\alpha}^{-1}(\bar{x}_0)$. When $\bar{x}_\alpha = \bar{x}_0$, we get that $x_{0,\alpha} = 0$, but this notation will be useful when not working under ideal hypotheses.

With (2.2) and the assumption that $g_\alpha = \xi$, we get

$$-\nabla(\mathfrak{v}\nabla u_\alpha) + \alpha \mathfrak{v} u_\alpha = \lambda_\alpha \mathfrak{v} u_\alpha^{\frac{N+2}{N-2}}. \tag{E_α}$$

We consider a cut-off function $\eta \in \mathcal{C}^\infty(\mathcal{B}_0(\gamma))$, $\eta = 1$ on $\mathcal{B}_0(\delta/2)$, $|\nabla \eta| \le C\delta^{-1}$ on $\mathcal{B}_0(\delta)$, $\eta = 0$ on $C_0(\gamma) = \mathcal{B}_0(\gamma) \setminus \mathcal{B}_0(\delta)$. We get

$$\int_{\mathcal{B}_0(\delta)} |\nabla(\mathfrak{v}^{1/2}\eta u_\alpha)|^2 \mathrm{d}\xi = \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta^2(|\nabla u_\alpha|^2 + 2u_\alpha \eta \nabla u_\alpha \nabla \eta + |\nabla \eta|^2 u_\alpha^2))$$
$$+ u_\alpha^2 \left(\frac{1}{4}\eta^2 \mathfrak{v}^{-1}|\nabla \mathfrak{v}|^2 + \eta \nabla \eta \nabla \mathfrak{v}\right) + \eta^2 u_\alpha \nabla \mathfrak{v} \nabla u_\alpha \mathrm{d}\xi. \tag{2.3}$$

On the other hand,

$$-\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla(\mathfrak{v}\nabla u_\alpha)\mathrm{d}\xi = \int_{\mathcal{B}_0(\delta)} \mathfrak{v}\nabla u_\alpha \nabla(\eta^2 u_\alpha)\mathrm{d}\xi$$
$$= \int_{\mathcal{B}_0(\delta)} \mathfrak{v}\left(|\nabla u_\alpha|^2\eta^2 + 2\eta u_\alpha \nabla \eta \nabla u_\alpha\right)\mathrm{d}\xi. \tag{2.4}$$

We compute (2.4) into (2.3), which leads to

$$-\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla(\mathfrak{v}\nabla u_\alpha)\mathrm{d}\xi = \int_{\mathcal{B}_0(\delta)} |\nabla(\mathfrak{v}^{1/2}\eta u_\alpha)|^2 \mathrm{d}\xi - \int_{\mathcal{B}_0(\delta)} \mathfrak{v}|\nabla \eta|^2 u_\alpha^2 \mathrm{d}\xi$$
$$-\int_{\mathcal{B}_0(\delta)} u_\alpha^2 \left(\frac{1}{4}\eta^2 \mathfrak{v}^{-1}|\nabla \mathfrak{v}|^2 + \eta \nabla \eta \nabla \mathfrak{v}\right)\mathrm{d}\xi - \int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla \mathfrak{v} \nabla u_\alpha \mathrm{d}\xi.$$

We then get

$$\int_{\mathcal{B}_0(\delta)} |\nabla(\mathfrak{v}^{\frac{1}{2}}\eta u_\alpha)|^2 \mathrm{d}\xi \le -\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla(\mathfrak{v}\nabla u_\alpha)\mathrm{d}\xi + \frac{C}{\delta^2}\int_{C(\delta)} u_\alpha^2 \mathfrak{v}\mathrm{d}\xi$$

$$+\underbrace{\varepsilon_\delta \int_{\mathcal{B}_0(\delta)} u_\alpha^2 \mathrm{d}\xi}_{a} + \underbrace{\frac{1}{2}\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha^2 \Delta \mathfrak{v} \mathrm{d}\xi}_{b}, \tag{2.5}$$

where $C(\delta) = \mathcal{B}_0(\delta) \setminus \mathcal{B}_0(\delta/2)$, ε_δ goes to 0 with δ, and a and b are obtained as follows.

Since we assumed that $\bar{x}_0 = \bar{x}_\alpha$, 0 is a critical point for $\mathfrak{v}$, and then $\sup|\nabla \mathfrak{v}|$ on $\mathcal{B}_0(\delta)$ goes to 0 with δ, which justifies a.

On the other hand, integrating by parts,

$$\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla \mathfrak{v} \nabla u_\alpha \mathrm{d}\xi = -\int_{\mathcal{B}_0(\delta)} u_\alpha \nabla(\eta^2 u_\alpha \nabla \mathfrak{v}) \mathrm{d}\xi$$

$$= -\int_{\mathcal{B}_0(\delta)} u_\alpha^2 \eta^2 \nabla(\nabla \mathfrak{v}) \mathrm{d}\xi - \int_{\mathcal{B}_0(\delta)} u_\alpha \eta^2 \nabla u_\alpha \nabla \mathfrak{v} \mathrm{d}\xi - \int_{\mathcal{B}_0(\delta)} u_\alpha^2 |\nabla \eta^2||\nabla \mathfrak{v}| \mathrm{d}\xi,$$

which leads to

$$2\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla \mathfrak{v} \nabla u_\alpha \mathrm{d}\xi = \int_{\mathcal{B}_0(\delta)} u_\alpha^2 \eta^2 \Delta \mathfrak{v} \mathrm{d}\xi - \int_{\mathcal{B}_0(\delta)} u_\alpha^2 |\nabla \eta^2||\nabla \mathfrak{v}| \mathrm{d}\xi$$

and

$$\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla \mathfrak{v} \nabla u_\alpha \mathrm{d}\xi \leq \frac{1}{2}\int_{\mathcal{B}_0(\delta)} u_\alpha^2 \eta^2 \Delta \mathfrak{v} \mathrm{d}\xi + \varepsilon_\delta \int_{\mathcal{B}_0(\delta)} u_\alpha^2 \mathrm{d}\xi,$$

where ε_δ goes to 0 with δ, which justifies b. Since $\mathfrak{v}^{1/2}\eta u_\alpha$ has compact support in $\mathbb{R}^N$, we have the following inequality:

$$\Big[\int_{\mathcal{B}_0(\delta)} (\mathfrak{v}^{\frac{1}{2}} \eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}} \leq K^2 \int_{\mathcal{B}_0(\delta)} |\nabla(\mathfrak{v}^{\frac{1}{2}} \eta u_\alpha)|^2 \mathrm{d}\xi.$$

Getting back to (2.5), we find

$$\frac{1}{K^2}\Big[\int_{\mathcal{B}_0(\delta)} (\mathfrak{v}^{\frac{1}{2}} \eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}} \leq -\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \nabla(\mathfrak{v} \nabla u_\alpha) \mathrm{d}\xi$$

$$+\frac{C}{\delta^2}\int_{C(\delta)} u_\alpha^2 \mathfrak{v} \mathrm{d}\xi + \frac{1}{2}\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha^2 \Delta \mathfrak{v} \mathrm{d}\xi + \varepsilon_\delta \int_{\mathcal{B}_0(\delta)} u_\alpha^2 \mathrm{d}\xi.$$

With (E_α), we get

$$\alpha \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^2 \mathrm{d}\xi \leq \lambda_\alpha \int_{\mathcal{B}_0(\delta)} \mathfrak{v} \eta^2 u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi - \frac{1}{K^2}\Big[\int_{\mathcal{B}_0(\delta)} (\mathfrak{v}^{\frac{1}{2}} \eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}}$$

$$+\frac{1}{2}\int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha^2 \Delta \mathfrak{v} \mathrm{d}\xi + \frac{C}{\delta^2}\int_{C(\delta)} u_\alpha^2 \mathfrak{v} \mathrm{d}\xi + \varepsilon_\delta \int_{\mathcal{B}_0(\delta)} u_\alpha^2 \mathrm{d}\xi, \tag{2.6}$$

where we recall on one side that $\lambda_\alpha \leq \frac{A^{\frac{2}{N}}}{K^2}$, on the other side that

$$\int_M v_\alpha^{\frac{2N}{N-2}} \mathrm{dv}_g = 1,$$

which means that

$$\int_{\mathcal{B}_0(\delta)} \mathfrak{v} u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi \leq 1. \tag{2.7}$$

We now want to give a subtle upper bound of

$$A_\alpha = \lambda_\alpha \int_{\mathcal{B}_0(\delta)} \mathfrak{v}\eta^2 u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi - \frac{1}{K^2}\Big[\int_{\mathcal{B}_0(\delta)} (\mathfrak{v}^{1/2}\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}}.$$

We write that

$$A_\alpha \leq \frac{1}{K^2}\Big(A^{\frac{2}{N}} \int_{\mathcal{B}_0(\delta)} \mathfrak{v}\eta^2 u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi - \Big[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}^{\frac{N}{N-2}} (\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}}\Big). \tag{2.8}$$

By Hölder's inequality we have that

$$\int_{\mathcal{B}_0(\delta)} \mathfrak{v}\eta^2 u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi \leq \Big[\int_{\mathcal{B}_0(\delta)} \mathfrak{v} u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{2}{N}} \Big[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}}. \tag{2.9}$$

Moreover, we recall that the Taylor expansion in $x_{0,\alpha}$ of $\mathfrak{v}$ is, for any $x \in \mathcal{B}_0(\delta)$,

$$\mathfrak{v}(x) = A + a_{ij\alpha}(x^i - x^i_{0,\alpha})(x^j - x^j_{0,\alpha}) + O(||x - x_{0,\alpha}||^3), \tag{2.10}$$

where $a_{ij\alpha} = \frac{1}{2}\frac{\partial^2_{ij}\mathfrak{v}}{\partial x^i \partial x^j}(x_{0,\alpha})$. Since we have assumed that $\bar{x}_\alpha = \bar{x}_0$, (2.10) is in fact easier to write in this part of the proof, but we will get back to it in part 2.2. We get, for any $x \in \mathcal{B}_0(\delta)$, $\mathfrak{v}(x) = A + a_{ij}x^i x^j + O(||x^3||)$, where $a_{ij} = \frac{1}{2}\frac{\partial^2_{ij}\mathfrak{v}}{\partial x^i \partial x^j}(0)$, which leads to

$$\begin{aligned}\int_{\mathcal{B}_0(\delta)} \mathfrak{v}^{\frac{N}{N-2}} (\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi &= \int_{\mathcal{B}_0(\delta)} \mathfrak{v}\mathfrak{v}^{\frac{2}{N-2}} (\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \\ &= \int_{\mathcal{B}_0(\delta)} \big(A + a_{ij}x^i x^j + O(||x||^3)\big)^{\frac{2}{N-2}} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi.\end{aligned}$$

We know that

$$\big(A + a_{ij}x^i x^j + O(||x||^3)\big)^{\frac{2}{N-2}} = A^{\frac{2}{N-2}}\Big(1 + \frac{2}{A(N-2)} a_{ij}x^i x^j + O(||x||^3)\Big).$$

This way

$$
\begin{aligned}
&\int_{\mathcal{B}_0(\delta)} \mathfrak{v}^{\frac{N}{N-2}} (\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \\
&= A^{\frac{2}{N-2}} \int_{\mathcal{B}_0(\delta)} \Big(1 + \frac{2}{A(N-2)} a_{ij} x^i x^j + O(||x||^3)\Big) \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \\
&= A^{\frac{2}{N-2}} \Big(\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi + \frac{2}{(N-2)A} a_{ij} \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} x^i x^j \mathrm{d}\xi \\
&+ \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} O(||x||^3) \mathrm{d}\xi \Big).
\end{aligned}
$$

We observe that

$$
\lim_{\alpha \to B_0} \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi = 1,
$$

and

$$
\lim_{\alpha \to B_0} \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} x^i x^j \mathrm{d}\xi = 0, \quad \lim_{\alpha \to B_0} \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} O(||x||^3) \mathrm{d}\xi = 0.
$$

This way,

$$
\begin{aligned}
&\Big[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}^{\frac{N}{N-2}} (\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \Big]^{\frac{N-2}{N}} \\
&= A^{\frac{2}{N}} \Big[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \Big]^{\frac{N-2}{N}} \Big(1 + \frac{2a_{ij}}{(N-2)A} \frac{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} x^i x^j \mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi} \\
&+ \frac{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} O(||x||^3) \mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi} \Big)^{\frac{N-2}{N}} \\
&= A^{\frac{2}{N}} \Big[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \Big]^{\frac{N-2}{N}} \Big(1 + \frac{2a_{ij}}{NA} \frac{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} x^i x^j \mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi} \\
&+ \frac{N-2}{N} \frac{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} O(||x||^3) \mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi} \Big).
\end{aligned} \tag{2.11}
$$

Getting back to (2.8), by (2.9) and (2.11) we then get that

$$
A_\alpha \le \frac{A^{\frac{2}{N}}}{K^2} \Big(\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi \Big)^{\frac{N-2}{N}} \Big[\Big(\int_{\mathcal{B}_0(\delta)} \mathfrak{v} u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi \Big)^{\frac{2}{N}} - 1 \tag{2.12}
$$

$$-\frac{2a_{ij}}{NA}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi}-\frac{N-2}{N}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x||^3)\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi}\Big].$$

Getting back to (2.6), we find

$$\begin{aligned}&\alpha\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi\\ &\le\frac{A^{\frac{2}{N}}}{K^2}\Big(\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\Big)^{\frac{N-2}{N}}\Big[\Big(\int_{\mathcal{B}_0(\delta)}\mathfrak{v}u_\alpha^{\frac{2N}{N-2}}\mathrm{d}\xi\Big)^{\frac{2}{N}}-1\\ &-\frac{2a_{ij}}{NA}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi}-\frac{N-2}{N}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x||^3)\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi}\Big]\\ &+\frac{1}{2}\int_{\mathcal{B}_0(\delta)}\eta^2u_\alpha^2\Delta\mathfrak{v}\mathrm{d}\xi+\frac{C}{\delta^2}\int_{C(\delta)}u_\alpha^2\mathfrak{v}\mathrm{d}\xi+\varepsilon_\delta\int_{\mathcal{B}_0(\delta)}u_\alpha^2\mathrm{d}\xi.\end{aligned}$$

By (2.7), $\Big(\int_{\mathcal{B}_0(\delta)}\mathfrak{v}u_\alpha^{\frac{2N}{N-2}}\mathrm{d}\xi\Big)^{\frac{2}{N}}-1\le 0$; this way

$$\begin{aligned}\alpha\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi&\le\frac{-2a_{ij}A^{\frac{2-N}{N}}}{NK^2}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\Big[\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\Big]^{\frac{2}{N}}}\\ &-\frac{(N-2)A^{\frac{2}{N}}}{NK^2}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x||^3)\mathrm{d}\xi}{\Big[\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\Big]^{\frac{2}{N}}}\\ &+\frac{1}{2}\int_{\mathcal{B}_0(\delta)}\eta^2u_\alpha^2\Delta\mathfrak{v}\mathrm{d}\xi+\frac{C}{\delta^2}\int_{C(\delta)}u_\alpha^2\mathfrak{v}\mathrm{d}\xi+\varepsilon_\delta\int_{\mathcal{B}_0(\delta)}u_\alpha^2\mathrm{d}\xi,\end{aligned}$$

and

$$\begin{aligned}\alpha\le&\frac{-2A^{\frac{2-N}{N}}}{NK^2\Big[\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\Big]^{\frac{2}{N}}}a_{ij}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}\\ &-\frac{(N-2)A^{\frac{2}{N}}}{NK^2}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x||^3)\mathrm{d}\xi}{\Big[\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\Big]^{\frac{2}{N}}\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}\\ &+\frac{1}{2}\frac{\int_{\mathcal{B}_0(\delta)}\eta^2u_\alpha^2\Delta\mathfrak{v}\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}+\frac{C}{\delta^2}\frac{\int_{C(\delta)}\mathfrak{v}u_\alpha^2\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}+\varepsilon_\delta\frac{\int_{\mathcal{B}_0(\delta)}u_\alpha^2\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}.\end{aligned}\tag{2.13}$$

The sequence of this first part of the proof is devoted to the study of

$$\lim_{\alpha\to B_0} a_{ij}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}.$$

In order to estimate this limit, we will use a technique of "blow up," and this is where our ideal assumption $\bar{x}_0=\bar{x}_\alpha$ will be crucial. Indeed, we recall that when not working with ideal hypotheses, we have to study

$$\lim_{\alpha\to B_0} a_{ij\alpha}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}(x^i-x^i_{0,\alpha})(x^j-x^j_{0,\alpha})\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}.$$

This will be done in part 2.2. Regarding the other terms of inequality (2.13), we easily show that

$$\lim_{\alpha\to B_0}\frac{1}{2}\frac{\int_{\mathcal{B}_0(\delta)}\eta^2\Delta\mathfrak{v}u_\alpha^2\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}=\frac{1}{2}\frac{\Delta\mathfrak{v}(0)}{A}+\varepsilon_\delta$$

and, the same way as in Djadli-Druet [5],

$$\lim_{\alpha\to B_0}-\frac{(N-2)A^{\frac{2}{N}}}{NK^2}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x||^3)\mathrm{d}\xi}{\left[\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\right]^{\frac{2}{N}}\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}$$
$$+\frac{C}{\delta^2}\frac{\int_{C(\delta)}u_\alpha^2\mathfrak{v}\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}+\varepsilon_\delta\frac{\int_{\mathcal{B}_0(\delta)}u_\alpha^2\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}=\varepsilon(\delta).$$

We bring to the reader's attention that under the assumption $\bar{x}_\alpha=\bar{x}_0$,

$$\lim_{\alpha\to B_0}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x||^3)\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}=0$$

is easily obtained thanks to the first fundamental inequality. However, when we are not in this case, we have to study

$$\lim_{\alpha\to B_0}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x-x_{0,\alpha}||^3)\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}.$$

This will be done at the end of part 2.2.

2.1. **Study of** $\lim_{\alpha\to B_0} a_{ij}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}$ **under ideal assumptions.**
We set $\mu_\alpha^{1-N/2}=||\tilde{v}_\alpha||_\infty$, $\tilde{v}_\alpha$ as above. When δ is small enough, we consider the exponential mapping

$$\exp_{\bar{x}_\alpha}:\mathcal{B}_0(\delta)\subset\mathbb{R}^N\to B_{\bar{x}_\alpha}(\delta)\subset O_{x_0,\gamma}/H.$$

We do a "blow up" in $\bar{x}_\alpha$, and we recall that with our ideal hypotheses, $\bar{x}_\alpha=\bar{x}_0$. For any $y\in\mathcal{B}_0(\delta\mu_\alpha^{-1})\subset\mathbb{R}^N$ we set

$$\tilde{g}_\alpha=\mu_\alpha^{-2}\exp^*_{\bar{x}_\alpha}g_\alpha.$$

Under our assumption, $g_\alpha=\xi$, but even when this does not hold, since μ_α goes to 0 when α goes to B_0, then $\tilde{g}_\alpha$ goes to ξ with α in $C^2(\Omega)$, Ω any compact regular subset of $\mathbb{R}^N$. Also, we set

$$\Phi_\alpha(y)=\mu_\alpha^{N/2-1}\tilde{v}_\alpha(\exp_{\bar{x}_\alpha}(\mu_\alpha y)),\qquad \mathfrak{v}_\alpha(y)=\mathfrak{v}(\exp_{\bar{x}_\alpha}(\mu_\alpha y)).$$

We check that for any α

$$-\nabla(\mathfrak{v}_\alpha\nabla\Phi_\alpha)+\alpha\mathfrak{v}_\alpha\mu_\alpha^2\Phi_\alpha=\lambda_\alpha\mathfrak{v}_\alpha\Phi_\alpha^{\frac{N+2}{N-2}}$$

and $\Phi_\alpha(0)=1$. Furthermore,

$$\lim_{\alpha\to B_0}\Phi_\alpha=\Phi$$

in $L^\infty(\Omega)$, Ω any compact regular subset of $\mathbb{R}^N$, and $\Phi(0)=1$; this way $\Phi\neq 0$ and Φ is a solution of

$$-\nabla(\mathfrak{v}(0)\nabla\Phi)=\frac{A^{\frac{2}{N}}}{K^2}\mathfrak{v}(0)\Phi^{\frac{N+2}{N-2}};$$

i.e., on $\mathbb{R}^N$

$$\Delta\Phi=\frac{A^{\frac{2}{N}}}{K^2}\Phi^{\frac{N+2}{N-2}},$$

which leads to (classical result)

$$\Phi(y)=\Big(1+\frac{A^{\frac{2}{N}}||y||^2}{N(N-2)K^2}\Big)^{\frac{2-N}{2}}. \tag{2.14}$$

One can write

$$a_{ij}\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi=\mu_\alpha^2 a_{ij}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha\left(\eta_\alpha\Phi_\alpha\right)^{\frac{2N}{N-2}}y^iy^j\mathrm{d}\xi$$

and

$$\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi=\mu_\alpha^2\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^2\mathrm{d}\xi.$$

In order to study

$$\lim_{\alpha\to B_0}\frac{\mu_\alpha^2 a_{ij}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha\,(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\,y^iy^j\mathrm{d}\xi}{\mu_\alpha^2\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^2\mathrm{d}\xi}$$

we need the following strong estimate on v_α: there exists C such that for any $O_x\in O_{x_0,\gamma}$

$$v_\alpha\mu_\alpha^{1-N/2}\mathrm{d}(O_{x_\alpha},O_x)^{N-2}\le C, \tag{2.15}$$

where we recall that $\mu_\alpha^{1-N/2}=||\tilde{v}_\alpha||_\infty=||v_\alpha||_\infty$. We notice that this inequality is stronger than the first fundamental inequality.

The proof of inequality (2.15) uses the notion of G-invariant Green function, and in order to make this article more readable, it is presented in a different part. This inequality is true on M, but since we will work mainly in $O_{x_0,\gamma}/H$, we say "for α big enough" when we need that O_{x_α} is in $O_{x_0,\gamma}$.

After quotient and blow-up, (2.15) becomes the following: there exists C such that for α big enough, for any $y\in\mathcal{B}_0(\delta\mu_\alpha^{-1})$,

$$\mathrm{d}(y,y_\alpha)^{N-2}\Phi_\alpha\le C. \tag{2.16}$$

With (2.16) we get that

$$\Phi_\alpha^{(2N)/(N-2)}||y||^2\le C||y||^{-2(N-1)}\quad\text{and}\quad\Phi_\alpha^2\le C||y||^{-2(N-2)}.$$

For $N>4$, we have that Φ_α goes to Φ and $\mathfrak{v}_\alpha$ goes to A almost everywhere with α. Since $y^iy^j\le||y||^2$ we easily get with Lebesgue's theorem

$$\lim_{\alpha\to B_0}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha\,(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\,y^iy^j\mathrm{d}\xi=A\int_{\mathbb{R}^N}\Phi^{(2N)/(N-2)}y^iy^j\mathrm{d}\xi$$

and

$$\lim_{\alpha\to B_0}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^2\mathrm{d}\xi=A\int_{\mathbb{R}^N}\Phi^2\mathrm{d}\xi.$$

For $N=4$, we get that

$$\lim_{\alpha\to B_0}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^2\mathrm{d}\xi=\infty$$

(which does not affect our result). This way when α goes to B_0,

$$\lim_{\alpha\to B_0}a_{ij}\frac{\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}y^iy^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha\Phi_\alpha^2\mathrm{d}\xi}=-\frac{1}{2}\Delta_{\tilde{g}}\tilde{\mathfrak{v}}(\bar{x}_0)\frac{\int_{\mathbb{R}^N}\Phi^{\frac{2N}{N-2}}y^iy^j\mathrm{d}\xi}{N\int_{\mathbb{R}^N}\Phi^2\mathrm{d}\xi}, \tag{2.17}$$

when $N>4$, and since by (2.14) Φ is known (radial) we get that

$$(2.17) = -\frac{1}{2}\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)\frac{\int_{\mathbb{R}^N}\Phi^{\frac{2N}{N-2}}r^2\mathrm{d}\xi}{N\int_{\mathbb{R}^N}\Phi^2\mathrm{d}\xi} = \frac{(N-4)NK^2}{4(N-1)A^{\frac{2}{N}}}\Big(-\frac{1}{2}\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)\Big) \text{ if } N>4$$
$$= 0 \text{ if } N=4.$$

Finally, getting back to (2.13), and since $\Big(\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}\mathrm{d}\xi\Big)^{\frac{2}{N}}$ goes to 1 when α goes to B_0, we find

$$\alpha \le \frac{-2A^{\frac{2-N}{N}}}{NK^2}\frac{K^2}{A^{\frac{2}{N}}}\frac{(N-4)N}{4(N-1)}\Big(-\frac{1}{2}\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)\Big)+\frac{1}{2}\frac{\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)}{A}$$
$$= \frac{(N-4)\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)}{4(N-1)A}+\frac{1}{2}\frac{\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)}{A} = \frac{3(N-2)\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)}{4(N-1)A}.$$

Since by Proposition 1, under the hypothesis that $g_\alpha=\xi$ we have $B_0 \ge \frac{3(N-2)\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)}{4(N-1)A}$ and α goes to B_0, we have $B_0 = \frac{3(N-2)\Delta_{\tilde g}\tilde{\mathfrak{v}}(\bar x_0)}{4(N-1)A}$ when $v\equiv 0$, which is the result we were looking for.

2.2. **Study of** $\lim_{\alpha\to B_0} a_{ij}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}x^ix^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}$ **without ideal assumptions.** We assume now that we are not in the case where $\bar x_\alpha=\bar x_0$. We get back to 2.1, only this time we have to study

$$\lim_{\alpha\to B_0} a_{ij\alpha}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}(x^i-x^i_{0,\alpha})(x^j-x^j_{0,\alpha})\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}.$$

Once again we use the technique of the blow up. Precisely, we have the exponential mapping $\exp_{\bar x_\alpha}:\mathcal{B}_0(\delta)\subset\mathbb{R}^N\to B_{\bar x_\alpha}(\delta)\subset O_{\bar x_0,\gamma}/H$. On $\mathcal{B}_0(\delta\mu_\alpha^{-1})$, let $\tilde g_\alpha=\mu_\alpha^{-2}\exp^*_{\bar x_\alpha}g_\alpha$, and, as in part 2.1 we set for any $y\in\mathcal{B}_0(\delta\mu_\alpha^{-1})$

$$\Phi_\alpha(y)=\mu_\alpha^{N/2-1}\tilde v_\alpha(\exp_{\bar x_\alpha}(\mu_\alpha y)),$$
$$\mathfrak{v}_\alpha(y)=\mathfrak{v}(\exp_{\bar x_\alpha}(\mu_\alpha y)), \quad\text{and}\quad \eta_\alpha=\eta(\exp_{\bar x_\alpha}(\mu_\alpha y)).$$

We still have the strong estimate, which states that there exists C such that for any $O_x\in O_{x_0,\gamma}$

$$v_\alpha\mu^{1-N/2}\mathrm{d}(O_{x_\alpha},O_x)^{N-2}\le C. \tag{2.18}$$

After quotient and blow up, we get that there exists C such that for α big enough, for any $y\in\mathcal{B}_0(\delta\mu_\alpha^{-1})$

$$\mathrm{d}(y,y_\alpha)^{N-2}\Phi_\alpha\le C. \tag{2.19}$$

We get

$$
\begin{aligned}
&a_{ij\alpha}\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}(x^i-x^i_{0,\alpha})(x^j-x^j_{0,\alpha})\mathrm{d}\xi \qquad (2.20)\\
&=\mu_\alpha^2 a_{ij\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}(y^i-y^i_{0,\alpha})(y^j-y^j_{0,\alpha})\mathrm{d}\xi\\
&=\mu_\alpha^2 a_{ij\alpha}\Big(\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}y^iy^j\mathrm{d}\xi\\
&\quad-\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}(y^i_{0,\alpha}y^j+y^j_{0,\alpha}y^i-y^i_{0,\alpha}y^j_{0,\alpha})\mathrm{d}\xi\Big).
\end{aligned}
$$

The first term has been studied in 2.1. We look at the last part of the equality. We will use a principle developed by Collion [4].

$$
\begin{aligned}
&-a_{ij\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha(y^i_{0,\alpha}y^j+y^j_{0,\alpha}y^i-y^i_{0,\alpha}y^j_{0,\alpha})\mathrm{d}\xi\\
&=-a_{ij\alpha}\Big(y^i_{0,\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^j\mathrm{d}\xi\\
&+y^j_{0,\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^i\mathrm{d}\xi-y^i_{0,\alpha}y^j_{0,\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha\mathrm{d}\xi\Big)\\
&=a_{ij\alpha}B_\alpha(-y^i_{0,\alpha}A^j_\alpha-y^j_{0,\alpha}A^i_\alpha+y^i_{0,\alpha}y^j_{0,\alpha})\\
&=a_{ij\alpha}B_\alpha\left((A^i_\alpha-y^i_{0,\alpha})(A^j_\alpha-y^j_{0,\alpha})-A^i_\alpha A^j_\alpha\right),
\end{aligned}
$$

where

$$B_\alpha=\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha\mathrm{d}\xi$$

and

$$A^i_\alpha=\Big(\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^i\mathrm{d}\xi\Big)B_\alpha^{-1}.$$

Since $a_{ij\alpha}$ is the Hessian matrix of $\tilde{\mathfrak{v}}$ in $\bar{x}_0$ (the point where $\tilde{\mathfrak{v}}$ is minimum), we get that $a_{ij\alpha}(A^i_\alpha-y^i_{0,\alpha})(A^j_\alpha-y^j_{0,\alpha})\geq 0$. We now study $a_{ij\alpha}B_\alpha A^i_\alpha A^j_\alpha$.

$$
\begin{aligned}
&a_{ij\alpha}B_\alpha A^i_\alpha A^j_\alpha\\
&=\frac{a_{ij\alpha}}{B_\alpha}\Big(\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^i\mathrm{d}\xi\Big)\Big(\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^j\mathrm{d}\xi\Big).
\end{aligned}
$$

Once again with (2.19) we get

$$\Phi_\alpha^{2N/(N-2)}y^i\leq C||y||^{-2N+1}.$$

As in part 2.1, we have

$$\lim_{\alpha\to B_0}\Phi_\alpha=\Phi=\Big(1+\frac{A^{\frac{2}{N}}||y||^2}{K^2N(N-2)}\Big)^{\frac{2-N}{N}},$$

and we get that

$$\lim_{\alpha\to B_0}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{2N/(N-2)}\mathfrak{v}_\alpha y^i\mathrm{d}\xi=A\int_{\mathbb{R}^N}\Phi^{2N/(N-2)}y^i\mathrm{d}\xi.$$

Since Φ is radial, we get

$$\lim_{\alpha\to B_0}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^i\mathrm{d}\xi=0.$$

Then $\lim_{\alpha\to B_0}a_{ij\alpha}B_\alpha A^i_\alpha A^j_\alpha=0$, which leads to

$$\lim_{\alpha\to B_0}a_{ij\alpha}B_\alpha\left((A^i_\alpha-y^i_{0,\alpha})(A^j_\alpha-y^j_{0,\alpha})-A^i_\alpha A^j_\alpha\right)\geq 0.$$

Finally,

$$\begin{aligned}&\lim_{\alpha\to B_0}\frac{-a_{ij\alpha}\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}(x^i-x^i_{0,\alpha})(x^j-x^j_{0,\alpha})\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}\\&=\lim_{\alpha\to B_0}\frac{-a_{ij\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}(\eta_\alpha\Phi_\alpha)^{\frac{2N}{N-2}}\mathfrak{v}_\alpha(y^i-y^i_{0,\alpha})(y^j-y^j_{0,\alpha})\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^2\mathrm{d}\xi}\\&\leq\lim_{\alpha\to B_0}\frac{-a_{ij\alpha}\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\Phi_\alpha^{\frac{2N}{N-2}}\mathfrak{v}_\alpha y^iy^j\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta\mu_\alpha^{-1})}\mathfrak{v}_\alpha(\eta_\alpha\Phi_\alpha)^2\mathrm{d}\xi}=\left(\frac{1}{2}\Delta\mathfrak{v}(0)\right)\frac{(N-4)NK^2}{4(N-1)A^{2/N}}.\end{aligned}$$

We get back to (2.13). To get the estimate we are looking for, we still have to prove that

$$\lim_{\alpha\to B_0}\frac{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}}O(||x-x_{0,\alpha}||^3)\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2\mathrm{d}\xi}=0. \tag{2.21}$$

When (H_2) holds, we easily get (2.21) using both first and second fundamental inequalities. When working with (H_1^+), we actually don't prove (2.21), but the condition that $\tilde{\mathfrak{v}}$ is nondegenerate in $\bar{x}_0$ gives us the result nonetheless.

Indeed, if $\tilde{\mathfrak{v}}$ is nondegenerate in $\bar{x}_0$, then there exists $C>0$ such that

$$a_{ij\alpha}(x^i-x^i_{0,\alpha})(x^j-x^j_{0,\alpha})\geq C||x-x_{0,\alpha}||^2.$$

On the other hand,

$$O(||x-x_{0,\alpha}||^3) = ||x-x_{0,\alpha}||O(||x-x_{0,\alpha}||^2) < \delta O(||x-x_{0,\alpha}||^2).$$

For δ small enough, this last term is "absorbed" by the first one in the final limit, and we get the result.

Getting back to (2.13) we find that

$$\lim_{\alpha\to B_0} \alpha \leq \frac{3(N-2)\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{4(N-1)\tilde{\mathfrak{v}}}(\bar{x}_0).$$

By Proposition 1, when $g_\alpha = \xi$, we get that

$$B_0 = \frac{3(N-2)\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{4(N-1)\tilde{\mathfrak{v}}}(\bar{x}_0)$$

when $v \equiv 0$, which is the result we were looking for.

2.3. **Last part of the proof without ideal assumptions.** From now on, we don't assume anymore that $g_\alpha = \xi$. Equation (E_α) is then

$$-\nabla_{g_\alpha}(\mathfrak{v}\nabla_{g_\alpha}u_\alpha) + \alpha\mathfrak{v}u_\alpha = \lambda_\alpha \mathfrak{v}u_\alpha^{\frac{N+2}{N-2}},$$

and since

$$\Delta_\xi u_\alpha = \Delta_{g_\alpha}u_\alpha + (g_\alpha^{ij}-\delta^{ij})\partial_{ij}u_\alpha - g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k\partial_k u_\alpha$$

we get

$$\begin{aligned}-\nabla_\xi(\mathfrak{v}\nabla_\xi u_\alpha) &= \lambda_\alpha\mathfrak{v}u_\alpha^{\frac{N+2}{N-2}} - \alpha\mathfrak{v}u_\alpha + (\nabla_{g_\alpha}\mathfrak{v}\nabla_{g_\alpha}u_\alpha)\\ &\quad + \mathfrak{v}\left((g_\alpha^{ij}-\delta^{ij})\partial_{ij}u_\alpha - g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k\partial_k u_\alpha\right) - \nabla_\xi\mathfrak{v}\nabla_\xi u_\alpha.\end{aligned}$$

Getting back to (2.6), with the convention that $\nabla_\xi = \nabla$, we get

$$\begin{aligned}&\alpha\int_{\mathcal{B}_0(\delta)}\mathfrak{v}(\eta u_\alpha)^2 d\xi \leq \lambda_\alpha\int_{\mathcal{B}_0(\delta)}\mathfrak{v}\eta^2 u_\alpha^{\frac{2N}{N-2}}d\xi - \frac{1}{K^2}\Big[\int_{\mathcal{B}_0(\delta)}(\mathfrak{v}^{\frac{1}{2}}\eta u_\alpha)^{\frac{2N}{N-2}}d\xi\Big]^{\frac{N-2}{N}}\\ &+\frac{C}{\delta^2}\int_{C(\delta)}\mathfrak{v}u_\alpha^2 d\xi + \varepsilon_\delta\int_{\mathcal{B}_0(\delta)}u_\alpha^2 d\xi\\ &+\int_{\mathcal{B}_0(\delta)}\eta^2 u_\alpha\mathfrak{v}\left[(g_\alpha^{ij}-\delta^{ij})\partial_{ij}u_\alpha - g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k\partial_k u_\alpha\right]d\xi\\ &\underbrace{+\frac{1}{2}\int_{\mathcal{B}_0(\delta)}(\eta u_\alpha)^2\Delta_\xi\mathfrak{v}d\xi + \int_{\mathcal{B}_0(\delta)}\eta^2 u_\alpha\left(\nabla_{g_\alpha}\mathfrak{v}\nabla_{g_\alpha}u_\alpha - \nabla\mathfrak{v}\nabla u_\alpha\right)d\xi}_{a}.\end{aligned}$$

In fact, we get that

$$\begin{aligned} a &= \frac{1}{2}\int_{\mathcal{B}_0(\delta)} (\eta u_\alpha)^2 \Delta_{g_\alpha}\mathfrak{v}\mathrm{d}\xi + \frac{1}{2}\int_{\mathcal{B}_0(\delta)} u_\alpha^2 \left(\nabla\mathfrak{v}\nabla(\eta^2) - \nabla_{g_\alpha}\mathfrak{v}\nabla_{g_\alpha}(\eta^2)\right)\mathrm{d}\xi \\ &\le \frac{1}{2}\int_{B_0(\delta)} (\eta u_\alpha)^2 \Delta_{g_\alpha}\mathfrak{v}\mathrm{d}\xi + \frac{C}{\delta^2}\int_{C(\delta)} u_\alpha^2 \mathrm{d}\xi. \end{aligned}$$

Finally,

$$\alpha \le \frac{A_\alpha}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^2 \mathrm{d}\xi} + \frac{B_\alpha}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^2 \mathrm{d}\xi} + \frac{1}{2}\frac{\int_{\mathcal{B}_0(\delta)} (\eta u_\alpha)^2 \Delta_{g_\alpha}\mathfrak{v}\mathrm{d}\xi}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^2 \mathrm{d}\xi} + C_\alpha,$$

where

$$A_\alpha = \lambda_\alpha \int_{\mathcal{B}_0(\delta)} \mathfrak{v}\eta^2 u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi - \frac{1}{K^2}\Big[\int_{\mathcal{B}_0(\delta)} (\mathfrak{v}^{\frac{1}{2}}\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big]^{\frac{N-2}{N}},$$

$$B_\alpha = \int_{\mathcal{B}_0(\delta)} \eta^2 u_\alpha \mathfrak{v}\left[(g_\alpha^{ij} - \delta^{ij})\partial_{ij} u_\alpha - g_\alpha^{ij}\Gamma(g_\alpha)_{ij}^k \partial_k u_\alpha\right]\mathrm{d}\xi$$

and $\lim_{\alpha\to B_0} C_\alpha = 0$. In a similar way as done in Djadli-Druet [5], we get that

$$\lim_{\alpha\to B_0} \frac{B_\alpha}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^2 \mathrm{d}\xi} = \frac{1}{6} Scal_{\tilde{g}}(\bar{x}_0) + \varepsilon_\delta.$$

In order to bound A_α from above, we get back to (2.12).

$$\begin{aligned} A_\alpha \le{}& \frac{A^{\frac{2}{N}}}{K^2}\Big(\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\Big)^{\frac{N-2}{N}} \Big[\Big(\int_{\mathcal{B}_0(\delta)} \mathfrak{v} u_\alpha^{\frac{2N}{N-2}} \mathrm{d}\xi\Big)^{\frac{2}{N}} - 1\Big] \\ &- \frac{2A^{\frac{2-N}{N}} a_{ij}}{NK^2} \frac{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} x^i x^j \mathrm{d}\xi}{\left[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\right]^{\frac{2}{N}}} \\ &- \frac{(N-2)A^{\frac{2}{N}} \int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} O(||x||^3)\mathrm{d}\xi}{NK^2\left[\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^{\frac{2N}{N-2}} \mathrm{d}\xi\right]^{\frac{2}{N}}}. \end{aligned}$$

The last two terms have been studied in parts 2.1 and 2.2. As for the first term, by Cartan's expansion of $\tilde{g}$ in geodesic normal coordinates, we get

$$\mathrm{d}\xi = \Big(1 + \frac{1}{6} Ric_{\tilde{g}}(\bar{x}_\alpha)_{ij} x^i x^j + O(||x||^3)\Big)\mathrm{dv}_{g_\alpha},$$

where $Ric_{\tilde{g}}$ denotes the Ricci curvature of $\tilde{g}$. As was done in [5], we get

$$\lim_{\alpha\to B_0} \frac{A_\alpha}{\int_{\mathcal{B}_0(\delta)} \mathfrak{v}(\eta u_\alpha)^2 \mathrm{d}\xi} = \frac{N-4}{12(N-1)} Scal_{\tilde{g}}(\bar{x}_0) + \frac{N-4}{4(N-1)} \frac{\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{\tilde{\mathfrak{v}}}(\bar{x}_0),$$

where we recall that $\tilde{\mathfrak{v}}(\bar{x}_0) = A$. Finally,

$$\begin{aligned}\lim_{\alpha\to B_0} \alpha &\leq \frac{N-4}{12(N-1)} Scal_{\tilde{g}}(\bar{x}_0) + \frac{N-4}{4(N-1)} \frac{\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{\tilde{\mathfrak{v}}}(\bar{x}_0) \\ &\quad + \frac{\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{2\tilde{\mathfrak{v}}}(\bar{x}_0) + \frac{1}{6} Scal_{\tilde{g}}(\bar{x}_0) + \varepsilon \\ &\leq \frac{N-2}{4(N-1)} Scal_{\tilde{g}}(\bar{x}_0) + \frac{3(N-2)\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{4(N-1)\tilde{\mathfrak{v}}}(\bar{x}_0) + \varepsilon.\end{aligned}$$

By Proposition 1,

$$B_0 = \frac{N-2}{4(N-1)} Scal_{\tilde{g}}(\bar{x}_0) + \frac{3(N-2)\Delta_{\tilde{g}}\tilde{\mathfrak{v}}}{4(N-1)\tilde{\mathfrak{v}}}(\bar{x}_0)$$

when $v \equiv 0$, which is the result we were looking for. □

2.4. **A strong estimate on v_α.** We will now prove a strong estimate on v_α. This result was first established by Druet and Robert [10] when there is no symmetry. The proofs are in many aspects alike, but we choose to develop here the points which are not as usual as in the case studied by Druet-Robert. Precisely, this proof uses the notion of a G-invariant Green's function, which in itself is interesting.

We prove that there exists C such that for any α

$$v_\alpha(x)\mu_\alpha^{1-N/2} \mathrm{d}(O_{x_\alpha}, O_x)^{N-2} \leq C, \tag{2.22}$$

where we recall that $\mu_\alpha^{1-N/2} = ||v_\alpha||_\infty$ and O_{x_α} is such that for any $x \in O_{x_\alpha}$, $v_\alpha(x) = ||v_\alpha||_\infty$.

In fact, we need a little bit less than this result for the main proof of this article; i.e., the following estimate would be sufficient to get our result: for any $0 < \nu < 1$, there exists C_ν such that for any α

$$v_\alpha(x)\mu_\alpha^{(1-N/2)(1-2\nu)} \mathrm{d}(O_{x_\alpha}, O_x)^{(N-2)(1-\nu)} \leq C_\nu. \tag{2.23}$$

But once we proved (2.23), it does not cost much to get (2.22), which is a prettier result. Anyhow, we insist on the proof of (2.23), and we refer to Druet-Robert [10] and Collion [4] for (2.22).

Proof. Let $v_\alpha > 0$ be the G-invariant solution of

$$\Delta_g v_\alpha + \alpha v_\alpha = \lambda_\alpha v_\alpha^{(N+2)/(N-2)}$$

and, with the notation introduced in 1.2, we get $\tilde{v}_\alpha$ the solution of

$$-\tilde{\nabla}_{\tilde{g}}(\tilde{\mathfrak{v}}\tilde{\nabla}_{\tilde{g}}\tilde{v}_\alpha) + \alpha\tilde{\mathfrak{v}}\tilde{v}_\alpha = \lambda_\alpha\tilde{\mathfrak{v}}\tilde{v}_\alpha^{(N+2)/(N-2)}$$

on $O_{x_0,\gamma}/H$. Since $B_0 > 0$, we easily notice that $\Delta_g + B_0$ is a coercive operator. We consider the operator L_α such that for any $\Phi \in \mathcal{C}^\infty_G(M)$

$$L_\alpha\Phi = \Delta_g\Phi + \alpha\Phi - \lambda_\alpha v_\alpha^{4/(N-2)}\Phi.$$

It is clear that $L_\alpha(v_\alpha) = 0$, and this way L_α satisfies the maximum principle on M. We then define the G-invariant Green's function of the operator $\Delta_g + B_0$ in x_α to be the unique function $\hat{G}_{x_\alpha}$ such that for any $\Phi \in \mathcal{C}^\infty_G(M)$,

$$\int_M \hat{G}_{x_\alpha}(x)\left(\Delta_g\Phi + B_0\Phi\right)(x)\mathrm{dv}_g = \Phi(x_\alpha).$$

This function satisfies the usual properties of a Green's function: there exist C_1, C_2, C_3, and C_4, four positive constants which do not depend on α, such that

$$C_1\mathrm{d}(O_{x_\alpha}, O_x)^{2-N} \le \hat{G}_{x_\alpha}(x) \le C_2\mathrm{d}(O_{x_\alpha}, O_x)^{2-N} \tag{2.24}$$

and

$$C_3\mathrm{d}(O_{x_\alpha}, O_x)^{1-N} \le |\nabla\hat{G}_{x_\alpha}(x)| \le C_4\mathrm{d}(O_{x_\alpha}, O_x)^{1-N}. \tag{2.25}$$

We get the existence and properties of $\hat{G}_{x_\alpha}(x)$ as follows: Let $h_\alpha \in \mathcal{C}^\infty_G(M \setminus O_{x_\alpha})$ be the function defined by

$$h_\alpha = \eta\,\mathrm{d}(O_{x_\alpha}, O_x)^{2-N}\left(w_{N-1}(N-2)\right)^{-1},$$

where η is a G-invariant cut-off function with compact support in $O_{x_0,\gamma}$. With the same notation as before: $h_\alpha = \tilde{h}_\alpha \circ \Pi$, where $\tilde{h}_\alpha \in \mathcal{C}^\infty(O_{x_0,\gamma}/H \setminus \bar{x}_\alpha)$

$$\tilde{h}_\alpha = \tilde{\eta} r^{2-N}(w_{N-1}(N-2))^{-1},$$

$r = \mathrm{d}(\bar{x}_\alpha, \bar{x})$, $\tilde{\eta}$ a cut-off function. We get that $|\tilde{\nabla}(\tilde{\mathfrak{v}}\tilde{\nabla}\tilde{h}_\alpha)| \le Cr^{2-N}$. Now let H_{x_α} be the weak G-invariant solution on M of

$$\Delta_g H_{x_\alpha} + B_0 H_{x_\alpha} = -\Delta_g h_\alpha - B_0 h_\alpha.$$

We then get $\tilde{H}_{x_\alpha}$ the weak solution on $O_{x_0,\gamma}/H$ of

$$-\tilde{\nabla}(\tilde{\mathfrak{v}}\tilde{\nabla}\tilde{H}_{x_\alpha}) + B_0\tilde{\mathfrak{v}}\tilde{H}_{x_\alpha} = \tilde{\nabla}(\tilde{\mathfrak{v}}\nabla\tilde{h}_\alpha) - B_0\tilde{\mathfrak{v}}\tilde{h}_\alpha.$$

A proof by induction gives that there exists C such that for any $0 < \varepsilon < 1$

$$|\tilde{H}_{x_\alpha}| \leq \frac{C}{r^{N-3+\varepsilon}} \text{ and } |\tilde{\nabla}\tilde{H}_{x_\alpha}| \leq \frac{C}{r^{N-2+\varepsilon}}. \tag{2.26}$$

The idea of this proof is to multiply $\tilde{H}_{x_\alpha}$ by r^p for $2 \leq p < N-2$ and study the equation satisfied by $\tilde{\nabla}(\tilde{\mathfrak{v}}\tilde{\nabla}(r^p\tilde{H}_{x_\alpha}))$. This gives that for certain q and s, there exists $C > 0$ independent of α such that $||r^p\tilde{H}_{x_\alpha}||_q \leq C$ and $||\tilde{\nabla}(r^p\tilde{H}_{x_\alpha})||_s \leq C$. Then we study $r^{N-3+\varepsilon}\tilde{H}_{x_\alpha}$, and we get that $||r^{N-3+\varepsilon}\tilde{H}_{x_\alpha}||_{C^0} \leq C$ and $||\tilde{\nabla}(r^{N-2+\varepsilon}\tilde{H}_{x_\alpha})||_{C^0} \leq C$, which gives us the estimates on $\tilde{H}_{x_\alpha}$ and $|\tilde{\nabla}\tilde{H}_{x_\alpha}|$.

We take $\hat{G}_{x_\alpha} = h_\alpha + H_{x_\alpha}$. We verify that for any $\Phi \in \mathcal{C}_G^\infty$, we have

$$\int_M \hat{G}_{x_\alpha}(x)\left(\Delta_g\Phi + B_0\Phi\right)(x)\mathrm{dv}_g = \Phi(x_\alpha),$$

and the two fundamental properties of a Green's function (2.24) and (2.25) (well known when there are no symmetries) are a consequence of the choice of h_α and (2.26). Once we have the existence and properties of the G-invariant Green's function, we can follow the proof's scheme presented in [10]. Computations give for $0 < \nu < 1$ and α big enough

$$L_\alpha\hat{G}_{x_\alpha}(x)^{1-\nu} \geq 0 = L_\alpha v_\alpha$$

on $M \setminus O_{x_\alpha, R\mu_\alpha}$ and

$$v_\alpha \leq C\mu_\alpha^{(N/2-1)(1-2\nu)}G_{x_\alpha}(x)^{1-\nu}$$

on $\partial(M \setminus O_{x_\alpha, R\mu_\alpha})$. By application of the maximum principle, we get that

$$v_\alpha \leq C\mu_\alpha^{(N/2-1)(1-2\nu)}G_{x_\alpha}(x)^{1-\nu}$$

on M, which, combined with (2.24), leads to (2.23). □

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