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UNIVERSAL BOUNDS AT THE BLOW-UP TIME FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. We prove a priori supremum bounds for solutions to

$$u_t - \operatorname{div}\big(u^{m-1}|Du|^{\lambda-1}Du\big) = f(x)u^p\,,$$

as t approaches the time when u becomes unbounded. Such bounds are universal in the sense that they do not depend on u. Here f may become unbounded, or vanish, as $x \to 0$.

When $f \equiv 1$, we also prove a bound below, as well as uniform localization of the support, for subsolutions to the corresponding Cauchy problem.

1. Introduction

In this paper we give bounds for the blow-up rate of solutions to certain parabolic equations, of rather general structure. The bounds we find are universal, in the sense that they do not depend on the initial data corresponding to the solutions (if any), only on the structure of the equation, and on the blow-up time itself. More specifically, we investigate nonnegative solutions to

$$u_t - \operatorname{div}\big(u^{m-1}|Du|^{\lambda-1}Du\big) = f(x)u^p\,, \quad \text{in } Q_T = \Omega \times (0,T), \tag{1.1}$$

where $0 < T < \infty$. We always assume

$$p > m + \lambda - 1\,, \quad p > 1\,, \quad \lambda > 0\,.$$

Here f is a radial function in $C(\boldsymbol{R}^N \setminus \{0\})$, with $f(x) > 0$ if $x \neq 0$; we also always assume $f \in L^1_{\text{loc}}(\boldsymbol{R}^N)$, and write $f(x) = f(|x|)$. The domain Ω is any

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open subset of $\boldsymbol{R}^N$ such that $0 \in \Omega$. The accent here is on the behaviour of u near the point $(0,T)$, independent of the behaviour of f at $x = 0$.

We also prove a result of uniform localization over $(0,T)$ of the support of subsolutions to the Cauchy problem for (1.1), with initial data

$$u(x,0) = u_0(x) \geq 0\,, \quad x \in \boldsymbol{R}^N\,, \tag{1.2}$$

where $u_0 \in L^1(\boldsymbol{R}^N)$ and $\operatorname{supp} u_0$ is bounded, and we let $f \equiv 1$. In the same setting, we prove a bound below at the blow-up time.

For the sake of simplicity, we confine ourselves in this section to the case when

$$f(x) = |x|^{-\alpha}\,, \qquad -\infty < \alpha < \min(N, \lambda + 1)\,. \tag{1.3}$$

However, our methods carry over to the case when f is not a power function, under mild assumptions; we deal with the case of general f in Sections 4 and 5.

Let us introduce two characteristic constants (in fact, exponents) naturally appearing in this context:

$$\mathcal{H} = \frac{(p-1)(\lambda+1) - \alpha(m+\lambda-2)}{p-m-\lambda+1}\,, \quad \mathcal{B} = \frac{\lambda+1-\alpha}{(p-1)(\lambda+1) - \alpha(m+\lambda-2)}\,.$$

$\mathcal{H}$ gives the correct space-time scaling near the blow-up time T, and $\mathcal{B}$ gives the blow-up rate. As a general rule, we consider cases when $\mathcal{H}$ and $\mathcal{B}$ are positive, but for Propositions 1.5 and 1.7. See also Remark 1.4.

We denote the integral average by

$$\fint_E u(x)\,\mathrm{d}x = \frac{1}{|E|}\int_E u(x)\,\mathrm{d}x\,,$$

and let $B_\rho \subset \boldsymbol{R}^N$ be the ball of radius ρ, centered at $x = 0$.

Definition 1.1. *A local solution u to* (1.1) *is a nonnegative $u \in L^\infty_{\mathrm{loc}}(Q_T)$, such that $u \in C((0,T); L^2_{\mathrm{loc}}(\Omega))$, $|Du^\sigma|^{\lambda+1} \in L^1_{\mathrm{loc}}(Q_T)$, and $\sigma = (m+\lambda-1)/\lambda$, and satisfying for any $\eta \in C^1_0(Q_T)$*

$$\iint_{Q_T} \left\{ -u\eta_t + u^{m-1}|Du|^{\lambda-1}Du \cdot D\eta \right\} \mathrm{d}x\,\mathrm{d}t = \iint_{Q_T} f(x)u^p\eta\,\mathrm{d}x\,\mathrm{d}t\,.$$

In order to obtain the definition of a local subsolution, we only need replace the equality sign above with an inequality sign $\leq$, additionally assuming $\eta \geq 0$. Moreover, we define here a subsolution to (1.1)–(1.2) *as a local subsolution u such that $u(\cdot,t) \to u_0$ as $t \to 0$ in $L^1(\boldsymbol{R}^N)$.* □

In the following, we perform an a priori analysis of solutions to (1.1). However, a possible approach to a proof of existence of solutions to suitable initial-value/boundary-value problems could use the estimates of Section 3, and an approximation argument. We refer the reader to [8, 10, 18, 20, 26, 29] for more details on local bounds for solutions to nonlinear equations, as well as on the related literature.

In the terminology of [1], the case we consider in Theorem 1.2 below is subcritical, as $\alpha < \lambda + 1$; see also [9] for more information on the critical case.

In the following we denote by γ, γ_0, ..., generic positive constants depending only on N, m, λ, p, and α. The following two theorems, then, yield in fact universal bounds for u, since γ is independent of u itself.

Theorem 1.2. *The case $\alpha \geq 0$. Assume that $0 \leq \alpha < \min(N, \lambda + 1)$, and that*

$$0 < m + \lambda - 1 < p < m + \lambda - 1 + (\lambda + 1 - \alpha)/N \,. \tag{1.4}$$

Then any nonnegative solution to (1.1) *satisfies*

$$u(x,t) \leq \gamma (T-t)^{-\mathcal{B}} \,, \qquad |x| < (T-t)^{1/\mathcal{H}}/2 \,, \tag{1.5}$$

provided $(T-t)^{1/\mathcal{H}} < \operatorname{dist}(0, \partial\Omega)$ and $T/2 < t < T$.

Theorem 1.3. *The case $\alpha < 0$. Assume that*

$$0 < m + \lambda - 1 < p < m + \lambda - 1 + (\lambda + 1)/N \,, \tag{1.6}$$

and that $\alpha < 0$ satisfies

$$\alpha(m + \lambda - 2) < (p-1)(\lambda + 1) \,, \tag{1.7}$$

$$|\alpha| \leq \min\big\{ N(p-1) \,, N(p - m - \lambda + 1)/(m + \lambda - 1) \big\} \,. \tag{1.8}$$

Then (1.5) *is still in force, for the same range of t.*

Remark 1.4. The restriction (1.7) is imposed to guarantee that $\mathcal{H}$ and $\mathcal{B}$ are positive; it is obviously satisfied for all $\alpha < 0$ if $m + \lambda - 2 \geq 0$. The condition (1.8) is of a more technical nature; see Section 2.

Note that also in Theorem 1.2, where $0 \leq \alpha < \lambda + 1$, we have $\mathcal{H} > 0$ and $\mathcal{B} > 0$, since $p > m + \lambda - 1$. □

One of the essential ingredients of the proof of the supremum estimates of Theorems 1.2 and 1.3 is the following integral estimate, actually valid under weaker assumptions.

Proposition 1.5. *Let $N > \alpha \geq 0$, $m + \lambda - 1 > 0$, and $0 < \theta < \min(1, m + \lambda - 1)$, and let u be a nonnegative supersolution to* (1.1)*. Then for $0 < t < T$, $0 < \rho^{\mathcal{H}} \leq (T - t)$, $2\rho < \operatorname{dist}(0, \partial\Omega)$, we have*

$$\Big(\fint_{B_\rho} u(x,t)^{1-\theta} \, \mathrm{d}x \Big)^{\frac{1}{1-\theta}} \leq \gamma \rho^{-\mathcal{H}\mathcal{B}} \,, \tag{1.9}$$

where γ depends on θ too.

If $\alpha < 0$, inequality (1.9) *is still in force, provided* (1.8) *is also satisfied.*

The other ingredient of the proof of the supremum estimate is a suitable kind of local L^1-L^∞ estimate for subsolutions (Lemma 3.3); when used in conjunction with the integral estimate (1.9), it yields at once the sought-after bound.

Remark 1.6. The blow-up rate for solutions to nonlinear parabolic equations has been investigated in many papers; see, e.g., [3, 7, 11, 13, 14, 15, 16, 24, 30], and the references therein.

The existence of universal bounds has been addressed in [22, 23, 25, 27, 28], in the semilinear case when $f(x) \equiv 1$, $m = 1$, and $\lambda = 1$. However, to the best of our knowledge, the blow-up rate (1.5) is proved here for the first time for equations of the general structure (1.1).

We should remark that in the semilinear case the blow-up rate $(T - t)^{-1/(p-1)}$ is expected to be valid, for all solutions, in the range $p(N - 2) < N + 2$. Our assumption (1.4) on p is more restrictive, since it corresponds to $p < 1 + 2/N$ in that case. On the other hand, we do not need extra assumptions on the structure of the solution, like, e.g., radial symmetry or monotonicity in time, which have been often employed to estimate blow-up rates. Our methods are different from those employed to obtain universal bounds, in the semilinear case, in [3], where systems were considered in the case corresponding to $p < 1 + 2/N$, and in [7], where bounds for all solutions were obtained when $p(N - 1)^2 < N(N + 2)$. In [23, 28], universal bounds are derived for more general p under extra structure assumptions (e.g., for $p(N-2) < N+2$ and $N \leq 4$). See [23] for a discussion of the range of validity of universal bounds, as well as of their implications in blow-up problems. □

The case when $\mathcal{H} < 0$ is essentially out of the scope of this paper; however, our approach does yield some results in that case too. We give below an example in this direction; the proof is sketched in Remark 5.3.

Proposition 1.7. *Let $f(x) = (1 + |x|)^{-\alpha}$, with*

$$\alpha < 0 \,, \quad m + \lambda - 2 < 0 \,, \quad (p - 1)(\lambda + 1) < \alpha(m + \lambda - 2)$$

(*so that* $\mathcal{H}$, $\mathcal{B} < 0$). *Assume that* (1.6) *is satisfied. Then the only nonnegative solution to* (1.1) *defined in the strip* $\boldsymbol{R}^N \times (0,T)$ *is the trivial one* $u \equiv 0$.

The case treated in Proposition 1.7 was not considered in [2, 19], where it was assumed, in our notation, that $m + \lambda \geq 2$.

1.1. **Two results for subsolutions in $\boldsymbol{R}^N$ when $f \equiv 1$.** Next we look at subsolutions defined in $\boldsymbol{R}^N \times (0,T)$. First we prove a bound below at blow-up time, i.e., that, under appropriate assumptions, a subsolution blows up at the expected rate. In next result (different than the previous ones) we need that T is the blow-up time, i.e., that $\|u(\cdot,t)\|_\infty$ becomes unbounded as $t \to T$.

Theorem 1.8. *Let* u *be a subsolution to* (1.1) *defined in* $\boldsymbol{R}^N \times (0,T)$. *Assume that* $f \equiv 1$, *and that* u *blows up at time* $\infty > T > 0$. *Then*

$$(T-t)^{\frac{1}{p-1}} \|u(\cdot,t)\|_{\infty,\boldsymbol{R}^N} \geq (p-1)^{-\frac{1}{p-1}}\,, \qquad 0 < t < T\,. \tag{1.10}$$

Remark 1.9. The bound below in (1.10) is well known when $m = \lambda = 1$ [17]; see [12] for the case $m > 1$ and $\lambda = 1$. Note that the constant on the right-hand side of (1.10) is the same as in the case $m = \lambda = 1$, and it is known to be optimal [24]. □

Finally we prove that compactly supported subsolutions have uniformly bounded support, for all $0 < t < T$; define

$$Z(t) = \inf\{r > 0 \mid u(x,t) = 0 \quad \text{a.e. } |x| > r\}\,. \tag{1.11}$$

Theorem 1.10. *Let* u *be a nonnegative subsolution to the Cauchy problem* (1.1)–(1.2). *Assume that* $f \equiv 1$, $m + \lambda - 2 > 0$, *that*

$$u(x,t) \leq c(T-t)^{-\frac{1}{p-1}}\,, \qquad x \in \boldsymbol{R}^N\,, 0 < t < T\,, \tag{1.12}$$

and that $\operatorname{supp} u_0$ *is bounded. Then there exists a finite constant* C *such that*

$$Z(t) \leq C\,, \qquad 0 < t < T\,. \tag{1.13}$$

Remark 1.11. We employ here a technique (partially building on our previous work [4, 5, 6]) suitable for rather general structures (see also Remark 1.12).

Note that we hypothesize the bound (1.12), since this is the piece of information we really need (besides of course the degeneracy of the equation, i.e., $m + \lambda - 2 > 0$). However, Theorem 1.2 can be used to prove such a bound, at least under the assumptions stipulated there (indeed, $\mathcal{B} = 1/(p-1)$ if $\alpha = 0$). □

Remark 1.12. The techniques employed in this paper rely only on energetic arguments, so that they hold for functions satisfying conditions more general than (1.1). For example, our supremum estimates hold for functions satisfying (in a standard weak sense)

$$\gamma_0 f(x,t) u^p \leq u_t - \operatorname{div} \boldsymbol{a}(x,t,u,Du) \leq \gamma f(x,t) u^p \,, \tag{1.14}$$

under suitable structure assumptions on f and $\boldsymbol{a}$ modelled after the operator in (1.1). Proposition 1.5 requires only that the first inequality in (1.14) be in force, and Theorems 1.8 and 1.10 require only the second one. □

1.2. **Content of the paper.** First we give in Section 2 the proof of Proposition 1.5; the proof exploits suitable ordinary differential inequalities satisfied by certain integral norms of u. Essentially, these inequalities would imply that u blows up *before* $t = T$, if (1.9) were not satisfied.

Next, we complete the proof of Theorem 1.2 in Section 3. As we explained above, this is achieved by means of a local a priori L^1-L^∞ estimate. In turn, the proof of this result, Lemma 3.3, follows by a new process of iteration we introduce in order to be able, roughly speaking, to absorb the nonlinearity on the right-hand side. This is the place where the upper bound on p is used. As a side remark, we mention that the local estimate is proven essentially by a careful use of the standard elliptic Sobolev-Gagliardo-Nirenberg embedding, and by a modification of the DeGiorgi-DiBenedetto techniques, which may be of independent interest.

In our opinion one of the points of interest of our approach is that it covers, essentially without changes, both the cases of decreasing and increasing $\rho \mapsto f(\rho)$, as well as the cases of power-like or more general f. However, for the sake of clarity, we consider in Section 3 only the case (1.3) with $\alpha \geq 0$. The few changes needed in the case of general and nonincreasing $f(\rho)$ are given in detail in Section 4.

The case (1.3) with $\alpha < 0$ is then treated in Section 5, together with the case of general and nondecreasing f. Again, only a few minor changes have to be introduced in the approach already laid out, in order to prove Theorem 1.3 and Proposition 1.7.

Finally, the proofs of Theorems 1.8 and 1.10 are given in Sections 6 and 7. Both of them rely on the change of variables (here u is a subsolution to (1.1) as specified in Subsection 1.1)

$$w(x,\tau) = (T-t)^{\frac{1}{p-1}} u(x,t) \,, \qquad \tau = -\ln(T-t) \,. \tag{1.15}$$

The function w satisfies for $x \in \boldsymbol{R}^N$ and $-\ln T < \tau < \infty$,

$$w_\tau + \frac{w}{p-1} \le e^{-\eta\tau} \operatorname{div}\left(w^{m-1}|Dw|^{\lambda-1}Dw\right) + w^p\,, \tag{1.16}$$

with $\eta = (p-m-\lambda+1)/(p-1)$.

2. Integral estimates near the blow-up time

Proposition 1.5 follows as an immediate corollary of Lemma 2.1 below; the extra requirement (1.8) in the case $\alpha < 0$ is needed to ensure that the function $L(\rho)$ there is well defined.

We prove the lemma for general f, for use in Sections 4 and 5; actually we may even dispense with the assumption that f be radial.

Lemma 2.1. *Let $p > m+\lambda-1 > 0$ and $0 < \theta < \min(1, m+\lambda-1)$, and let u be a nonnegative supersolution to* (1.1)*. Assume that the function*

$$L(\rho) = \rho^{\frac{(\lambda+1)(p-1)}{p-m-\lambda+1}} \left(\frac{⨍ B_\rho f(x)^{-\frac{1-\theta}{p-1}}\,\mathrm{d}x}{⨍ B_\rho f(x)^{-\frac{m+\lambda-\theta-1}{p-m-\lambda+1}}\,\mathrm{d}x} \right)^{\frac{p-1}{p-\theta}}$$

is well defined (i.e., the integrals above are finite) for $0 < 2\rho < \operatorname{dist}(0, \partial\Omega)$. Then for all ρ such that, in addition, $L(\rho) \le (T-t)$, we have

$$\left(⨍_{B_\rho} u(x,t)^{1-\theta}\,\mathrm{d}x \right)^{\frac{1}{1-\theta}} \tag{2.1}$$
$$\le \gamma \rho^{-\frac{\lambda+1}{p-m-\lambda+1}} \left(⨍_{B_{2\rho}} f(x)^{-\frac{m+\lambda-\theta-1}{p-m-\lambda+1}}\,\mathrm{d}x \right)^{\frac{1}{p-\theta}} \left(⨍_{B_{2\rho}} f(x)^{-\frac{1-\theta}{p-1}}\,\mathrm{d}x \right)^{\frac{p-1}{(p-\theta)(1-\theta)}}.$$

Here γ depends on θ.

Remark 2.2. No assumption is made on the behaviour of $L(\rho)$ as $\rho \to 0$. The estimate is valid for any given ρ and time level t satisfying the assumptions in the statement. □

Proof. Fix $0 < \theta < \min(1, m+\lambda-1)$, and let $\zeta \in C^1(\boldsymbol{R}^N)$ such that $\zeta(x) = 1$ $x \in B_\rho$, $\zeta(x) = 0$ $x \notin B_{2\rho}$, and $|D\zeta| \le 2/\rho$. We choose as a testing function $u^{-\theta}\zeta^s$, $s > \lambda+1$, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \frac{1}{1-\theta} \int_{\boldsymbol{R}^N} \zeta^s u^{1-\theta}\,\mathrm{d}x - \theta \int_{\boldsymbol{R}^N} \zeta^s u^{m-\theta-2} |Du|^{\lambda+1}\,\mathrm{d}x$$
$$+ s \int_{\boldsymbol{R}^N} \zeta^{s-1} u^{m-\theta-1} |Du|^{\lambda-1} Du \cdot D\zeta\,\mathrm{d}x \ge \int_{\boldsymbol{R}^N} \zeta^s f(x) u^{p-\theta}\,\mathrm{d}x\,.$$

Hence, a standard application of Young's inequality yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\boldsymbol{R}^N}\zeta^s u^{1-\theta}\,\mathrm{d}x \tag{2.2}$$
$$\geq -\frac{\gamma}{\rho^{\lambda+1}}\int_{\boldsymbol{R}^N}\zeta^{s-\lambda-1}u^{m+\lambda-\theta-1}\,\mathrm{d}x + (1-\theta)\int_{\boldsymbol{R}^N}\zeta^s f(x)u^{p-\theta}\,\mathrm{d}x\,,$$

where γ depends on θ. Next we partially absorb the first term on the right-hand side of (2.2) in the last integral. To this end, choose $s=(\lambda+1)(p-\theta)/(p-m-\lambda+1)>\lambda+1$. Then, by Hölder's and Young's inequalities

$$\frac{1}{\rho^{\lambda+1}}\int_{\boldsymbol{R}^N}\zeta^{s-\lambda-1}u^{m+\lambda-\theta-1}\,\mathrm{d}x$$
$$\leq \frac{\gamma}{\rho^{\lambda+1}}\Big(\int_{\boldsymbol{R}^N}\zeta^s f(x)u^{p-\theta}\,\mathrm{d}x\Big)^{\frac{m+\lambda-\theta-1}{p-\theta}}(\rho^N J_1)^{\frac{p-m-\lambda+1}{p-\theta}}$$
$$\leq \varepsilon\int_{\boldsymbol{R}^N}\zeta^s f(x)u^{p-\theta}\,\mathrm{d}x + C_\varepsilon\rho^{N-\frac{(\lambda+1)(p-\theta)}{p-m-\lambda+1}}J_1\,, \tag{2.3}$$

where

$$J_1 = \fint_{B_{2\rho}} f(x)^{-\frac{m+\lambda-\theta-1}{p-m-\lambda+1}}\,\mathrm{d}x\,.$$

Collecting (2.2) and (2.3) we get, on choosing a small enough $\varepsilon>0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\boldsymbol{R}^N}\zeta^s u^{1-\theta}\,\mathrm{d}x \geq \frac{1-\theta}{2}\int_{\boldsymbol{R}^N}\zeta^s f(x)u^{p-\theta}\,\mathrm{d}x - \gamma\rho^{N-\frac{(\lambda+1)(p-\theta)}{p-m-\lambda+1}}J_1\,,$$

whence, by means of a further application of Hölder's inequality,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_{\boldsymbol{R}^N}\zeta^s u^{1-\theta}\,\mathrm{d}x \geq \gamma_0(\rho^N J_2)^{-\frac{p-1}{1-\theta}}\Big(\int_{\boldsymbol{R}^N}\zeta^s u^{1-\theta}\,\mathrm{d}x\Big)^{\frac{p-\theta}{1-\theta}} - \gamma_1\rho^{N-\frac{(\lambda+1)(p-\theta)}{p-m-\lambda+1}}J_1\,,$$

with

$$J_2 = \fint_{B_{2\rho}} f(x)^{-\frac{1-\theta}{p-1}}\,\mathrm{d}x\,.$$

On setting

$$y(\tau) = \frac{1}{\rho^N}\int_{\boldsymbol{R}^N}\zeta(x)^s u(x,\tau)^{1-\theta}\,\mathrm{d}x\,,$$

we finally obtain for two constants γ_0 and γ_1, which stay fixed for the rest of this proof,

$$y' \geq \gamma_0 J_2^{-\frac{p-1}{1-\theta}} y^{\frac{p-\theta}{1-\theta}} - \gamma_1\rho^{-\frac{(\lambda+1)(p-\theta)}{p-m-\lambda+1}}J_1\,. \tag{2.4}$$

Fix $t\in(0,T)$ and consider the alternative

i) $\gamma_0 J_2^{-\frac{p-1}{1-\theta}} y(t)^{\frac{p-\theta}{1-\theta}} \leq 2\gamma_1\rho^{-\frac{(\lambda+1)(p-\theta)}{p-m-\lambda+1}}J_1\,;$

ii) $\gamma_0 J_2^{-\frac{p-1}{1-\theta}} y(t)^{\frac{p-\theta}{1-\theta}} > 2\gamma_1 \rho^{-\frac{(\lambda+1)(p-\theta)}{p-m-\lambda+1}} J_1$.

The alternative i) immediately implies

$$y(t) \le \gamma \rho^{-\frac{(\lambda+1)(1-\theta)}{p-m-\lambda+1}} J_1^{\frac{1-\theta}{p-\theta}} J_2^{\frac{p-1}{p-\theta}} . \tag{2.5}$$

If ii) holds, we infer from (2.4) that

$$y' \ge \frac{\gamma_0}{2} J_2^{-\frac{p-1}{1-\theta}} y^{\frac{p-\theta}{1-\theta}} , \tag{2.6}$$

at least in a right neighbourhood of t. Actually, (2.6) implies that y is increasing, and therefore that i) and (2.6) itself are in force over the whole interval (t, T). An elementary analysis then proves

$$y(t) \le \gamma J_2 (T-t)^{-\frac{1-\theta}{p-1}} , \tag{2.7}$$

since y may not blow up before $\tau = T$.

Next we impose that the estimate (2.7) implies (2.5), requiring

$$J_2 (T-t)^{-\frac{1-\theta}{p-1}} \le \rho^{-\frac{(\lambda+1)(1-\theta)}{p-m-\lambda+1}} J_1^{\frac{1-\theta}{p-\theta}} J_2^{\frac{p-1}{p-\theta}} .$$

The proof is therefore concluded, as this last requirement coincides with $L(\rho) \le T - t$. □

3. Supremum estimate

We divide the proof into several lemmas. We always understand that the involved spatial domains are contained in Ω, and that u is a nonnegative subsolution to (1.1).

Define for $r < s$, $\theta > \tau$, and $h > k \ge 0$, the quantities $C(h,k) = h/(h-k) \ge 1$, and

$$\mathcal{F}(r,s,\theta,\tau,h,k) = \Big\{ \frac{(h-k)^{-p+1}}{\theta-\tau} s^\alpha + \frac{(h-k)^{-p+m+\lambda-1}}{(s-r)^{\lambda+1}} s^\alpha C(h,k)^{(m-1)_+} + C(h,k)^p \Big\} C(h,k)^{(1-m)_+} .$$

Our first lemma is a precise version of energy inequality.

Lemma 3.1. *Let $N > \alpha \ge 0$. Define $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_1 \subset \mathcal{C}_2$, as*

$$\mathcal{C}_i = B_{r_i} \times (t_i, t) , \quad 0 < r_1 < r_2 , \quad 0 < t_2 < t_1 < t .$$

Then for all $h_1 > h_2 > 0$ we have for $\omega > \max(1, 2-m)$

$$\sup_{t_1<\tau<t} \int_{B_{r_1}(\tau)} (u-h_1)_+^{\omega+1} \,\mathrm{d}x + \iint_{\mathcal{C}_1} \Big| D(u-h_1)_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} \Big|^{\lambda+1} \,\mathrm{d}x\,\mathrm{d}\tau$$

$$\leq \gamma F_0 \iint_{\mathcal{C}_2} |x|^{-\alpha}(u-h_2)_+^{\omega+p}\,\mathrm{d}x\,\mathrm{d}\tau\,. \quad (3.1)$$

Here $F_0 = \mathcal{F}(r_1, r_2, t_1, t_2, h_1, h_2)$; γ *depends also on* ω.

Proof. Let ζ be a smooth, nonnegative cutoff function such that

$$\zeta \equiv 1 \quad \text{in } \mathcal{C}_1, \qquad \zeta \equiv 0 \quad \text{outside of } \mathcal{C}_2,$$
$$|D\zeta| \leq 2/(r_2 - r_1)\,, \qquad 0 \leq \zeta_\tau \leq 2/(t_1 - t_2)\,.$$

We use as a testing function $(u-\widetilde{h})_+^{\omega}\zeta^{\lambda+1}$, where $\widetilde{h} = (h_1+h_2)/2$. Standard calculations yield

$$\sup_{t_2<\tau<t} \int_{B_{r_2}(\tau)} (u-\widetilde{h})_+^{\omega+1}\zeta^{\lambda+1}\,\mathrm{d}x + \iint_{\mathcal{C}_2} |Du|^{\lambda+1}u^{m-1}(u-\widetilde{h})_+^{\omega-1}\zeta^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq \gamma\|\zeta_\tau\|_\infty \iint_{\mathcal{C}_2} (u-\widetilde{h})_+^{\omega+1}\,\mathrm{d}x\,\mathrm{d}\tau + \gamma\|D\zeta\|_\infty^{\lambda+1} \iint_{\mathcal{C}_2} u^{m-1}(u-\widetilde{h})_+^{\omega+\lambda}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$+ \gamma \iint_{\mathcal{C}_2} |x|^{-\alpha}u^p(u-\widetilde{h})_+^{\omega}\,\mathrm{d}x\,\mathrm{d}\tau =: I_1 + I_2 + I_3\,. \quad (3.2)$$

Next we estimate the three integrals I_j essentially in terms of the same quantity, i.e., of the right-hand side of (3.1). Since

$$\frac{u-h_2}{\widetilde{h}-h_2} > 1\,, \qquad \text{where } u > \widetilde{h}, \quad (3.3)$$

we have

$$I_1 \leq \frac{\gamma}{t_1 - t_2} \iint_{\mathcal{C}_2} |x|^{\alpha}|x|^{-\alpha}\frac{(u-h_2)^{p-1}}{(\widetilde{h}-h_2)^{p-1}}(u-\widetilde{h})_+^{\omega+1}\,\mathrm{d}x\,\mathrm{d}\tau$$
$$\leq \gamma\frac{(h_1-h_2)^{-p+1}}{t_1-t_2}r_2^{\alpha} \iint_{\mathcal{C}_2} |x|^{-\alpha}(u-h_2)_+^{\omega+p}\,\mathrm{d}x\,\mathrm{d}\tau\,. \quad (3.4)$$

Next, note that, by the same token, if $u > \widetilde{h} > h_2$, then

$$u = u-h_2+h_2 < u-h_2+\frac{u-h_2}{\widetilde{h}-h_2}h_2 \leq C(\widetilde{h}, h_2)(u-h_2) = 2C(h_1, h_2)(u-h_2)\,, \quad (3.5)$$

when we recall that $C(h_1, h_2) = h_1/(h_1 - h_2)$. Thus

$$u^{m-1} \leq 2^{(m-1)_+}C(h_1, h_2)^{(m-1)_+}(u-h_2)_+^{m-1}\,, \qquad \text{where } u > \widetilde{h},$$

in both cases $m \geq 1$ and $m < 1$. Therefore, reasoning as in (3.4),

$$\begin{aligned} I_2 &\leq \gamma \frac{C(h_1,h_2)^{(m-1)_+}}{(r_2-r_1)^{\lambda+1}} \iint_{\{u>\widetilde{h}\}\cap\mathcal{C}_2} (u-h_2)_+^{\omega+m+\lambda-1}\,\mathrm{d}x\,\mathrm{d}\tau \qquad (3.6)\\ &\leq \gamma \frac{C(h_1,h_2)^{(m-1)_+}}{(r_2-r_1)^{\lambda+1}} \iint_{\{u>\widetilde{h}\}\cap\mathcal{C}_2} \frac{(u-h_2)^{p-m-\lambda+1}}{(\widetilde{h}-h_2)^{p-m-\lambda+1}}(u-h_2)_+^{\omega+m+\lambda-1}\,\mathrm{d}x\,\mathrm{d}\tau\\ &\leq \gamma C(h_1,h_2)^{(m-1)_+} \frac{(h_1-h_2)^{-p+m+\lambda-1}}{(r_2-r_1)^{\lambda+1}} r_2^{\alpha} \iint_{\mathcal{C}_2} |x|^{-\alpha}(u-h_2)_+^{\omega+p}\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

According to (3.5), I_3 can be majorized by

$$I_3 \leq \gamma C(h_1,h_2)^p \iint_{\mathcal{C}_2} |x|^{-\alpha}(u-h_2)_+^{\omega+p}\,\mathrm{d}x\,\mathrm{d}\tau\,. \qquad (3.7)$$

The right-hand side of (3.2) is therefore bounded above by the right-hand side of (3.1). To conclude the proof, if $m \geq 1$, we need only observe that

$$\begin{aligned} |Du|^{\lambda+1}u^{m-1}(u-\widetilde{h})_+^{\omega-1} &\geq |Du|^{\lambda+1}(u-h_1)_+^{\omega+m-2}\\ &= \Big(\frac{\lambda+1}{\omega+m+\lambda-1}\Big)^{\lambda+1}\left|D(u-h_1)_+^{\frac{\omega+m+\lambda-1}{\lambda+1}}\right|^{\lambda+1}. \end{aligned}$$

A slightly different argument is needed if $m < 1$; we reason as in (3.5), and calculate

$$\begin{aligned} &\iint_{\mathcal{C}_1} |Du|^{\lambda+1}u^{m-1}(u-\widetilde{h})_+^{\omega-1}\,\mathrm{d}x\,\mathrm{d}\tau\\ &\qquad \geq \iint_{\{u>h_1\}\cap\mathcal{C}_1} |Du|^{\lambda+1}C(h_1,\widetilde{h})^{m-1}(u-\widetilde{h})_+^{\omega+m-2}\,\mathrm{d}x\,\mathrm{d}\tau\\ &\qquad \geq 2^{m-1}C(h_1,h_2)^{m-1}\iint_{\mathcal{C}_1} |Du|^{\lambda+1}(u-h_1)_+^{\omega+m-2}\,\mathrm{d}x\,\mathrm{d}\tau\\ &= 2^{m-1}C(h_1,h_2)^{m-1}\Big(\frac{\lambda+1}{\omega+m+\lambda-1}\Big)^{\lambda+1}\iint_{\mathcal{C}_1}\left|D(u-h_1)_+^{\frac{\omega+m+\lambda-1}{\lambda+1}}\right|^{\lambda+1}. \end{aligned}$$

□

Our second lemma is an integral estimate preparing the way for the supremum estimate of Lemma 3.3. The constants M_1 and M_2 appearing in the statement are actually optimal.

Lemma 3.2. *Assume here* $0 \le \alpha < \min(N, \lambda+1)$, *and*

$$p < m + \lambda - 1 + \nu \frac{\lambda + 1 - \alpha}{N} , \tag{3.8}$$

for a given $0 < \nu < \omega + p$. *Let also*

$$a_1 > a_2 > 0 , \quad R_2 > R_1 > 0 , \quad t > \tau_1 > \tau_2 > 0 .$$

Then,

$$\sup_{\tau_1 < \tau < t} \int_{B_{R_1}(\tau)} (u - a_1)_+^{\omega+1} \,\mathrm{d}x \le \gamma (t - \tau_2) R_2^{(-\alpha + \frac{N}{s}) M_1} F_1^{M_1} \Big(\sup_{\tau_2 < \tau < t} \int_{B_{R_2}(\tau)} (u - a_2)_+^{\nu} \,\mathrm{d}x \Big)^{M_2} , \tag{3.9}$$

where $F_1 = \mathcal{F}(R_1, R_2, \tau_1, \tau_2, a_1, a_2)$ *and*

$$M_1 = \frac{m + \lambda - 1 + \nu \frac{\lambda+1}{N} - (\nu - \omega)}{m + \lambda - 1 + \nu \big(\frac{\lambda+1}{N} - \frac{1}{s} \big) - p} > 1 ,$$

$$M_2 = \frac{m + \lambda - 1 + (\omega + p) \frac{\lambda+1}{N} - p - \frac{1}{s} (\omega + m + \lambda - 1)}{m + \lambda - 1 + \nu \big(\frac{\lambda+1}{N} - \frac{1}{s} \big) - p} > 1 .$$

Here (*exploiting* (3.8)), *we select* $\max\{1, N/(\lambda+1)\} < s < N/\alpha$ *so that the denominator of* M_1 *and* M_2 *is positive, if* $\alpha > 0$. *Moreover,* ω *must be chosen so large that* (3.17) *below is satisfied* (*when* $N > \lambda + 1$), *as well as* $\omega > \max(1, 2 - m)$.

If $\alpha = 0$ *we may take* $s = \infty$ (*i.e., we let formally* $1/s = 0$ *above*).

For s as in the statement, the bounds $M_1 > 1$ and $M_2 > 1$ follow from $\nu < \omega + p$ and $p > m + \lambda - 1$. The introduction of the parameters s, ν, and ω seems to be technically unavoidable. However, they will not enter the final estimate of Theorem 1.2.

Proof. Introduce the sequences

$$\rho_n = R_2 - (R_2 - R_1) 2^{-n} , \quad \vartheta_n = \tau_2 + (\tau_1 - \tau_2) 2^{-n} , \quad k_n = a_2 + (a_1 - a_2) 2^{-n} ,$$

$n \ge 0$. Note that ρ_n is increasing, while ϑ_n and k_n are decreasing. We apply Lemma 3.1 with

$$r_1 = \rho_n, \quad r_2 = \rho_{n+1}; \qquad t_1 = \vartheta_n, \quad t_2 = \vartheta_{n+1}; \qquad h_1 = k_n, \quad h_2 = k_{n+1}.$$

It follows from direct inspection that

$$\mathcal{F}(\rho_n, \rho_{n+1}, \vartheta_n, \vartheta_{n+1}, k_n, k_{n+1}) \le 2^{A(n+1)} F_1, \quad A = p + 1 + 2(1 - m)_+, \tag{3.10}$$

where F_1 has been defined in the statement. Therefore, denoting $E_n = B_{\rho_n} \times (\vartheta_n, t)$, we obtain from Lemma 3.1,

$$\sup_{\vartheta_{2n+1}<\tau<t} \int_{B_{\rho_{2n+1}}(\tau)} (u-k_{2n+1})_+^{\omega+1} \, \mathrm{d}x + \iint_{E_{2n+1}} \left| D(u-k_{2n+1})_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} \right|^{\lambda+1} \mathrm{d}x \, \mathrm{d}\tau$$
$$\le \gamma 2^{2An} F_1 \iint_{E_{2n+2}} |x|^{-\alpha} (u-k_{2n+2})_+^{\omega+p} \, \mathrm{d}x \, \mathrm{d}\tau \,. \tag{3.11}$$

We have singled out the E_n with odd index, since for technical reasons the iteration procedure will actually involve only those domains. Let z_n be a smooth cut-off function such that

$$0 \le z_n \le 1 \,, \qquad z_n \equiv 1 \quad \text{in } E_{2n}, \qquad z_n \equiv 0 \quad \text{outside of } E_{2n+1};$$
$$|Dz_n| \le 2^{2n+1}(R_2-R_1)^{-1} \,, \qquad 0 \le z_{n\tau} \le 2^{2n+1}(\tau_1-\tau_2)^{-1} \,.$$

Recalling that $(\omega+m+\lambda-1) > (\lambda+1)$ (as a consequence of $\omega > 2-m$), we can check that

$$\iint_{E_{2n+1}} \left| D\big[(u-k_{2n})_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} z_n\big] \right|^{\lambda+1}$$
$$\le 2^{\lambda+1} \iint_{E_{2n+1}} \left| D(u-k_{2n+1})_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} \right|^{\lambda+1} \mathrm{d}x \, \mathrm{d}\tau$$
$$+ 2^{\lambda+1} \iint_{E_{2n+1}} |Dz_n|^{\lambda+1} (u-k_{2n+1})_+^{\omega+m+\lambda-1} \, \mathrm{d}x \, \mathrm{d}\tau \,. \tag{3.12}$$

The same argument we already employed in (3.6) shows that the last integral can be bounded above by

$$\frac{\gamma 2^{2n(\lambda+1)} R_2^{\alpha}}{(R_2-R_1)^{\lambda+1}(k_{2n+1}-k_{2n+2})^{p-m-\lambda+1}} \iint_{E_{2n+1}} |x|^{-\alpha} (u-k_{2n+2})_+^{\omega+p} \, \mathrm{d}x \, \mathrm{d}\tau$$
$$\le \gamma 2^{2n(p+1)} F_1 \iint_{E_{2n+1}} |x|^{-\alpha} (u-k_{2n+2})_+^{\omega+p} \, \mathrm{d}x \, \mathrm{d}\tau \,. \tag{3.13}$$

Next define

$$v_n = (u-k_{2n})_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} z_n \,.$$

Collecting (3.11)–(3.13) we get for $n \ge 0$

$$\sup_{\vartheta_{2n+1}<\tau<t} \int_{B_{\rho_{2n+1}}(\tau)} v_n^q \, \mathrm{d}x + \iint_{E_{2n+1}} |Dv_n|^{\lambda+1} \, \mathrm{d}x \, \mathrm{d}\tau$$

$$\leq \gamma 2^{2An} F_1 \iint_{E_{2n+2}} |x|^{-\alpha} (u - k_{2n+2})_+^{\omega+p} \,\mathrm{d}x \,\mathrm{d}\tau$$
$$\leq \gamma 2^{2An} F_1 \iint_{E_{2n+3}} |x|^{-\alpha} v_{n+1}^{\beta} \,\mathrm{d}x \,\mathrm{d}\tau \,, \quad (3.14)$$

where

$$q = \frac{(\omega+1)(\lambda+1)}{(\omega+m+\lambda-1)}\,, \qquad \beta = \frac{(\omega+p)(\lambda+1)}{\omega+m+\lambda-1}\,. \quad (3.15)$$

As a consequence of Hölder's inequality, for s as in the statement,

$$\int_{B_{\rho_{2n+3}}} |x|^{-\alpha} v_{n+1}^{\beta} \,\mathrm{d}x \leq \Big(\int_{B_{\rho_{2n+3}}} |x|^{-\alpha s} \,\mathrm{d}x \Big)^{\frac{1}{s}} \Big(\int_{B_{\rho_{2n+3}}} v_{n+1}^{\frac{\beta s}{s-1}} \,\mathrm{d}x \Big)^{1-\frac{1}{s}}$$
$$\leq \gamma R_2^{-\alpha+\frac{N}{s}} \Big(\int_{B_{\rho_{2n+3}}} v_{n+1}^{\frac{\beta s}{s-1}} \,\mathrm{d}x \Big)^{1-\frac{1}{s}} . \quad (3.16)$$

Note that, once s has been fixed, we may assume that ω is so large that

$$\beta s/(s-1) < N(\lambda+1)/(N-\lambda-1)\,, \qquad \text{if } N > \lambda+1. \quad (3.17)$$

Indeed, $\beta \to \lambda+1$ as $\omega \to \infty$, while $s > N/(\lambda+1)$ (see (3.15)). Then we may invoke the Nirenberg-Gagliardo embedding

$$\Big(\int v_{n+1}^{\frac{\beta s}{s-1}} \,\mathrm{d}x \Big)^{1-\frac{1}{s}} \leq \gamma \Big(\int |Dv_{n+1}|^{\lambda+1} \,\mathrm{d}x \Big)^{\frac{\kappa\beta}{\lambda+1}} \Big(\int v_{n+1}^{\mu} \,\mathrm{d}x \Big)^{\frac{(1-\kappa)\beta}{\mu}} \,, \quad (3.18)$$

where γ depends on N, m, λ, p, ω, s, and μ, and

$$\kappa = \frac{N(\lambda+1)}{\beta} \frac{\beta - \mu\big(1-\frac{1}{s}\big)}{N(\lambda+1-\mu)+(\lambda+1)\mu} \in (0,1)\,;$$

here we select $0 < \mu < \beta$ in terms of the given constant ν, as

$$\mu = \nu \frac{\lambda+1}{\omega+m+\lambda-1}\,, \qquad 0 < \nu < \omega+p\,.$$

Indeed, one can easily check that

$$\frac{\kappa\beta}{\lambda+1} < 1\,, \qquad \text{since} \quad p < m+\lambda-1+\nu\Big(\frac{\lambda+1}{N} - \frac{1}{s}\Big)\,.$$

Note that last inequality follows from our assumptions on s. From (3.16) and (3.18) we obtain

$$\int_{B_{\rho_{2n+3}}} |x|^{-\alpha} v_{n+1}^{\beta} \,\mathrm{d}x \leq R_2^{-\alpha+\frac{N}{s}} \Big(\int |Dv_{n+1}|^{\lambda+1} \,\mathrm{d}x \Big)^{\frac{\kappa\beta}{\lambda+1}} \Big(\int v_{n+1}^{\mu} \,\mathrm{d}x \Big)^{\frac{(1-\kappa)\beta}{\mu}} . \quad (3.19)$$

Integrating (3.19) in time and applying Hölder's inequality yields the estimate

$$\iint |x|^{-\alpha} v_{n+1}^{\beta} \,\mathrm{d}x\,\mathrm{d}\tau \le \gamma (t-\vartheta_{2n+3})^{1-\frac{\kappa\beta}{\lambda+1}} R_2^{-\alpha+\frac{N}{s}} \times \Big(\iint |Dv_{n+1}|^{\lambda+1} \,\mathrm{d}x\,\mathrm{d}\tau \Big)^{\frac{\kappa\beta}{\lambda+1}} \Big(\sup_{\vartheta_{2n+3}<\tau<t} \int v_{n+1}(x,\tau)^{\mu} \,\mathrm{d}x \Big)^{\frac{(1-\kappa)\beta}{\mu}} ,$$

which, upon substitution in (3.14), yields, after a further application of Young's inequality,

$$\sup_{\vartheta_{2n+1}<\tau<t} \int_{B_{\rho_{2n+1}}(\tau)} v_n^q \,\mathrm{d}x + \iint_{E_{2n+1}} |Dv_n|^{\lambda+1} \,\mathrm{d}x\,\mathrm{d}\tau \le \delta \iint_{E_{2n+3}} |Dv_{n+1}|^{\lambda+1} \,\mathrm{d}x\,\mathrm{d}\tau + b^n C_\delta (t-\vartheta_{2n+3}) R_2^{(-\alpha+\frac{N}{s})M_1} F_1^{M_1} \Big(\sup_{\vartheta_{2n+3}<\tau<t} \int_{B_{\rho_{2n+3}}(\tau)} v_{n+1}^{\mu} \,\mathrm{d}x \Big)^{M_2} , \tag{3.20}$$

for $b = 2^{2AM_1}$, where $\delta > 0$ is to be chosen, and $M_1 = \big[1-\kappa\beta/(\lambda+1)\big]^{-1}$ and $M_2 = (1-\kappa)\beta M_1/\mu$. We recall that for all $n \ge 0$,

$$(u-a_1)_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} \chi_{B_{R_1}\times(\tau_1,t)} \le v_n \le (u-a_2)_+^{\frac{\omega+m+\lambda-1}{\lambda+1}} \chi_{B_{R_2}\times(\tau_2,t)} .$$

Thus, on applying (3.20) recursively in order to bound the integral containing the gradient, we obtain for $i \ge 1$

$$\sup_{\tau_1<\tau<t} \int_{B_{R_1}(\tau)} (u-a_1)_+^{\omega+1} \,\mathrm{d}x \le \delta^i \iint_{E_{2i+1}} |Dv_i|^{\lambda+1} \,\mathrm{d}x\,\mathrm{d}\tau + \Big(\sum_{n=0}^{i} b^n \delta^n \Big) C_\delta (t-\tau_2) R_2^{(-\alpha+\frac{N}{s})M_1} F_1^{M_1} \Big(\sup_{\tau_2<\tau<t} \int_{B_{R_2}(\tau)} (u-a_2)_+^{\nu} \,\mathrm{d}x \Big)^{M_2} . \tag{3.21}$$

Clearly the series above converges for $\delta < 1/b$. The first term on the right-hand side of (3.21) can be majorized by

$$\delta^i 2^{2i(\lambda+1)} \cdot \text{constant} ,$$

owing to the definition of v_n and of z_n, so that it tends to zero if $\delta < 4^{-\lambda-1}$. Thus, on letting $i \to \infty$ we obtain the sought-after estimate, once we make M_1 and M_2 explicit, using the definitions of κ, β, and μ. □

Lemma 3.3. *Under the same assumptions as in Lemma* 3.2, *where we additionally stipulate* $0<\nu<\omega+1$, *we have*

$$\|u\|_{\infty,B_{d_1}\times(T_1,t)} \le d_2^{\frac{\alpha}{p-1}}(T_1-T_2)^{-\frac{1}{p-1}} + d_2^{\frac{\alpha}{p-m-\lambda+1}}(d_2-d_1)^{-\frac{\lambda+1}{p-m-\lambda+1}} + \gamma(t-T_2)^{\frac{1}{\omega+1-\nu}} d_2^{(-\alpha+\frac{N}{s})\frac{M_1}{\omega+1-\nu}} \Big(\sup_{T_2<\tau<t}\int_{B_{d_2}(\tau)} u^\nu \,\mathrm{d}x\Big)^{\frac{M_2-1}{\omega+1-\nu}}, \quad (3.22)$$

for all $0<T_2<T_1<t<T$ *and* $0<d_1<d_2$. *Here* γ *depends also on* ω *and on* ν; M_1 *and* M_2 *have been introduced in Lemma* 3.2.

Proof. We introduce the sequences

$$r_n = d_1 + (d_2-d_1)2^{-n}, \quad t_n = T_1 - (T_1-T_2)2^{-n},$$
$$h_n = k(1-2^{-n-1}), \quad \widetilde{h}_n = (h_n+h_{n+1})/2, \qquad n\ge 0;$$

$\{r_n\}$ is decreasing, while $\{t_n\}$ and $\{h_n\}$ are increasing. The constant k will be selected below. We write (3.9) for the choice

$$R_1 = r_{n+1}, \quad R_2 = r_n, \quad \tau_1 = t_{n+1}, \quad \tau_2 = t_n, \quad a_1 = \widetilde{h}_n, \quad a_2 = h_n.$$

Keeping in mind that, for A as in (3.10),

$$\mathcal{F}(r_{n+1}, r_n, t_{n+1}, t_n, \widetilde{h}_n, h_n) \le \gamma 2^{An} F_2,$$
$$F_2 := \mathcal{F}(d_1,d_2,T_1,T_2,k,0) = \frac{d_2^\alpha}{(T_1-T_2)k^{p-1}} + \frac{d_2^\alpha}{(d_2-d_1)^{\lambda+1}k^{p-m-\lambda+1}} + 1,$$

we obtain from (3.9)

$$\sup_{t_{n+1}<\tau<t}\int_{B_{r_{n+1}}(\tau)} (u-\widetilde{h}_n)_+^{\omega+1}\,\mathrm{d}x \le \gamma(t-t_n)2^{AM_1n} d_2^{(-\alpha+\frac{N}{s})M_1} F_2^{M_1} \Big(\sup_{t_n<\tau<t}\int_{B_{r_n}(\tau)} (u-h_n)_+^\nu\,\mathrm{d}x\Big)^{M_2}.$$

Next we observe that, by reasoning as in (3.3)–(3.4),

$$\int_{B_{r_{n+1}}(\tau)} (u-h_{n+1})_+^\nu\,\mathrm{d}x \le (h_{n+1}-\widetilde{h}_n)^{\nu-\omega-1}\int_{B_{r_{n+1}}(\tau)} (u-\widetilde{h}_n)_+^{\omega+1}\,\mathrm{d}x.$$

Collecting the previous estimates, we get the recursive inequality

$$Y_{n+1} := \sup_{t_{n+1}<\tau<t}\int_{B_{r_{n+1}}(\tau)} (u-h_{n+1})_+^\nu\,\mathrm{d}x \le \gamma b^n (t-T_2) d_2^{(-\alpha+\frac{N}{s})M_1} F_2^{M_1} k^{\nu-\omega-1} Y_n^{M_2},$$

where $b = 2^{A+\omega+1-\nu}$. Since $M_2 > 1$, we may apply the standard Lemma 5.6, Chapter II, of [21] to infer that $Y_n \to 0$ provided

$$(t - T_2) d_2^{(-\alpha+\frac{N}{s})M_1} F_2^{M_1} k^{\nu-\omega-1} Y_0^{M_2-1} \le \gamma_0 \,, \tag{3.23}$$

for a suitable $\gamma_0 > 0$. This in turn implies that

$$\|u\|_{\infty, B_{d_1}\times(T_1,t)} \le k \,, \tag{3.24}$$

by definition of Y_n. We choose k from

$$k = \Big[\gamma_0^{-1} (t - T_2) d_2^{(-\alpha+\frac{N}{s})M_1} F_2^{M_1} \Big(\sup_{T_2<\tau<t} \int_{B_{d_2}(\tau)} u^\nu \,\mathrm{d}x \Big)^{M_2-1} \Big]^{\frac{1}{\omega+1-\nu}} ,$$

so that (3.23) holds. Such a k is uniquely defined, as the function $k \mapsto k^{\nu-\omega-1} F_2^{M_1}$ is strictly decreasing, due to the definition of F_2 and to our assumption $\nu < \omega + 1$. We conclude the proof by considering the alternative

$$\text{i)} \quad F_2 > 3 \,; \qquad \text{ii)} \quad F_2 \le 3 \,. \tag{3.25}$$

When i) holds, it follows at once from the definition of F_2 and from (3.24) that at least one of the two estimates

$$\begin{aligned} \|u\|_{\infty, B_{d_1}\times(T_1,t)} &\le d_2^{\frac{\alpha}{p-1}} (T_1 - T_2)^{-\frac{1}{p-1}} \,, \\ \|u\|_{\infty, B_{d_1}\times(T_1,t)} &\le d_2^{\frac{\alpha}{p-m-\lambda+1}} (d_2 - d_1)^{-\frac{\lambda+1}{p-m-\lambda+1}} \end{aligned} \tag{3.26}$$

must be valid. If instead ii) is in force, it follows from the definition of k, and from (3.24) again, that

$$\|u\|_{\infty, B_{d_1}\times(T_1,t)} \le \gamma (t - T_2)^{\frac{1}{\omega+1-\nu}} d_2^{(-\alpha+\frac{N}{s})\frac{M_1}{\omega+1-\nu}} \Big(\sup_{T_2<\tau<t} \int_{B_{d_2}(\tau)} u^\nu \,\mathrm{d}x \Big)^{\frac{M_2-1}{\omega+1-\nu}} . \tag{3.27}$$

Finally, on combining (3.26) with (3.27) we obtain (3.22). □

Proof of Theorem 1.2. We select $0 < \nu < 1$, so that (3.8) is in force. This is possible, for a suitable ν, since $p < m + \lambda - 1 + (\lambda + 1 - \alpha)/N$ by assumption. Then we fix an s meeting the requirements in Lemma 3.2. The (large) constant ω must satisfy the assumptions stipulated in the statements of Lemmas 3.1–3.2, and is selected after the choice of ν and s.

Let us choose in (3.22)

$$d_1 = (T-t)^{1/\mathcal{H}}/2 \,, \quad d_2 = (T-t)^{1/\mathcal{H}} \,, \quad T_1 = t-(T-t)/2 \,, \quad T_2 = t-(T-t) \,,$$

so that all three terms on the right-hand side of (3.22) reduce to constant multiples of $(T - t)^{-\mathcal{B}}$. In fact in the third term there we use the integral average estimate (1.9), where we let $1-\theta=\nu$, so that the factor $(T-t)$ appears with a power

$$\frac{1}{\omega+1-\nu}+\frac{1}{\mathcal{H}}\Big(-\alpha+\frac{N}{s}\Big)\frac{M_1}{\omega+1-\nu}+\Big(\frac{N}{\mathcal{H}}-\mathcal{B}\nu\Big)\frac{M_2-1}{\omega+1-\nu}=-\mathcal{B}\,,$$

as one can check by means of lengthy but elementary computations. Note that the auxiliary parameters ν, s, and ω do not appear in the estimated blow-up rate. □

4. THE CASE OF NONPOWER NONLINEARITIES. I: f NONINCREASING

We stipulate here the assumption

(H1): $\rho\mapsto f(\rho)>0$, $0<\rho<\rho_0$, is a nonincreasing function, and, writing $f(x)=f(|x|)$, we have $f\in C(B_{\rho_0}\setminus\{0\})$, $f\in L^1(B_{\rho_0})$. Moreover, for all b in some right neighbourhoods of $-1/(p-1)$ and of $-(m+\lambda-1)/(p-m-\lambda+1)$,

$$\gamma^{-1}f(\rho)^b\le \fint_{B_\rho} f(x)^b\,\mathrm{d}x\le\gamma f(\rho)^b\,,\qquad \gamma=\gamma(b)>1\,. \tag{4.1}$$

For example the functions

$$f(\rho)=\rho^{-\alpha}|\ln\rho|^{\beta}\,,\qquad 0<\rho<\rho_0=\min(e^{-1},e^{\beta/\alpha})\,, \tag{4.2}$$

satisfy the assumption above for $0<\alpha<N$ and $\beta\in\boldsymbol{R}$ (and in fact also for $\alpha=0$, $\beta>0$, and $\rho_0=e^{-1}$).

Two auxiliary functions come into play:

$$G(\rho)=\Big[\rho^{\lambda+1}f(\rho)\Big]^{-\frac{1}{p-m-\lambda+1}}\,,\qquad \ell(\rho)=c\Big[\rho^{(\lambda+1)(p-1)}f(\rho)^{m+\lambda-2}\Big]^{\frac{1}{p-m-\lambda+1}}\,,$$

where c is a constant depending on the parameters; it must be fixed so that the function $L(\rho)$ in Lemma 2.1 satisfies $L(\rho)\le\ell(\rho)$; this is possible due to assumption (4.1), for $\theta>0$ small enough. By the same token, the quantity on the right-hand side of (2.1) can be bounded above in terms of G. Therefore we have, under the assumptions of Lemma 2.1, also exploiting (4.1),

$$\Big(\fint_{B_\rho} u(x,t)^{1-\theta}\,\mathrm{d}x\Big)^{\frac{1}{1-\theta}}\le\gamma G(\rho)\,, \tag{4.3}$$

provided $\ell(\rho)\le T-t$, for $0<\theta<\theta_0$ (θ_0 depends on f).

We also need to introduce the function (describing the correct space/time scaling)

$$R(\tau)=\sup\{\rho\mid\ell(\rho)\le\tau\}\,,\qquad \tau>0 \tag{4.4}$$

(in principle ℓ need not be monotone). It may be of interest to remark that, when $f(x) = |x|^{-\alpha}$, we have

$$G(\rho) \approx \rho^{-\mathcal{H}\mathcal{B}}, \quad \ell(\rho) \approx \rho^{\mathcal{H}}, \quad R(\tau) \approx \tau^{\frac{1}{\mathcal{H}}}.$$

Let us remark, before we state our main result, that the restriction on f, needed in order to obtain L^∞ bounds for u, is now expressed as an integrability condition, which, when f is a power function, reduces to the assumption we stipulated in that case. Incidentally, the same conditions on α (as those of Theorem 1.2) guarantee that Theorem 4.1 holds in the case of f given by (4.2).

Theorem 4.1. *Let f fulfill* (H1). *Assume that a number $s > \max(1, N/(\lambda + 1))$ exists such that $f \in L^s(B_{\rho_0})$, and that* (4.1) *holds for $b = s$; moreover, assume*

$$0 < m + \lambda - 1 < p < m + \lambda - 1 + \frac{\lambda + 1}{N} - \frac{1}{s}. \tag{4.5}$$

Then, if u solves (1.1) *in $B_{\rho_0} \times (0, T)$, we have for $T/2 < t < T$ and $2R(T - t) < \rho_0$,*

$$u(x, t) \le \gamma G\big(R(T - t)\big), \qquad |x| \le R(T - t)/2. \tag{4.6}$$

Let us remark that the quantity $G(R(T-t))$ actually blows up as $t \to T$: The summability requirement $f \in L^s(B_{\rho_0})$, for $s > N/(\lambda+1)$, together with (4.1), imply that

$$\rho^{\lambda+1} f(\rho) \to 0, \qquad \text{as } \rho \to 0. \tag{4.7}$$

Indeed for $\rho_1 < \rho_2$, exploiting the nonincreasing character of f,

$$\begin{aligned} \rho_1^{\lambda+1} f(\rho_1) &\le \gamma \rho_1^{\lambda+1} \fint_{B_{\rho_1}} f(x)\,\mathrm{d}x \le \gamma \Big(\fint_{B_{\rho_1}} f(x)^s\,\mathrm{d}x \Big)^{\frac{1}{s}} \rho_1^{\lambda+1} \\ &\le \gamma \Big(\int_{B_{\rho_1}} f(x)^s\,\mathrm{d}x \Big)^{\frac{1}{s}} \rho_1^{\lambda+1-\frac{N}{s}} \\ &\le \gamma \Big(\int_{B_{\rho_2}} f(x)^s\,\mathrm{d}x \Big)^{\frac{1}{s}} \rho_1^{\lambda+1-\frac{N}{s}} \le \gamma \rho_2^{\lambda+1} f(\rho_2) \Big(\frac{\rho_1}{\rho_2} \Big)^{\lambda+1-\frac{N}{s}}. \end{aligned}$$

As a consequence of (4.7),

$$G(\rho) \to \infty, \qquad \text{as } \rho \to 0. \tag{4.8}$$

A similar argument shows that

$$R(\tau) \to 0, \qquad \text{as } \tau \to 0, \tag{4.9}$$

so that $G(R(T - t)) \to +\infty$ as $t \to T$.

We indicate below the few modifications needed in the arguments of Section 3 in order to prove Theorem 4.1. First, we extend the definition of $\mathcal{F}$ in the obvious way:

$$\mathcal{F}(r,s,\theta,\tau,h,k)=\Big\{\frac{(h-k)^{-p+1}}{(\theta-\tau)f(s)} + \frac{(h-k)^{-p+m+\lambda-1}}{(s-r)^{\lambda+1}f(s)}C(h,k)^{(m-1)_+}+C(h,k)^p\Big\}C(h,k)^{(1-m)_+}.$$

The quantities F_i below are then defined in terms of this $\mathcal{F}$, though formally replicating the definitions of Section 3.

4.1. **Analog of Lemma 3.1.** This lemma stays unchanged, excepting for the obvious substitution of $|x|^{-\alpha}$ with $f(x)$. In particular, the assumptions are the same, when we drop the requirements on α, which is now replaced with $f \in L^1(B_{\rho_0})$.

4.2. **Analog of Lemma 3.2.** We give the new statement:

Lemma 4.2. *Let f and $s > \max(1, N/(\lambda+1))$ be as in Theorem 4.1, and also assume*

$$p < m+\lambda-1+\nu\Big(\frac{\lambda+1}{N}-\frac{1}{s}\Big), \tag{4.10}$$

for a given $0<\nu<\omega+p$. Let also

$$a_1 > a_2\,, \quad R_2 > R_1\,, \quad t > \tau_1 > \tau_2 > 0\,.$$

Moreover, ω must satisfy the same requirements as in Lemma 3.2. Then

$$\sup_{\tau_1<\tau<t}\int_{B_{R_1}(\tau)}(u-a_1)_+^{\omega+1}\,\mathrm{d}x \le \gamma(t-\tau_2)\big[f(R_2)R_2^{\frac{N}{s}}\big]^{M_1}F_1^{M_1}\Big(\sup_{\tau_2<\tau<t}\int_{B_{R_2}(\tau)}(u-a_2)_+^{\nu}\,\mathrm{d}x\Big)^{M_2}. \tag{4.11}$$

The constants M_1 and M_2 are defined exactly as in Section 3; note that their denominator is now positive by definition of s.

The proof of the lemma is the same as that of Lemma 3.2.

4.3. **Analog of Lemma 3.3.** In the statement, we need only replace (3.22) with

$$\begin{aligned} &\|u\|_{\infty,B_{d_1}\times(T_1,t)} \\ &\le f(d_2)^{-\frac{1}{p-1}}(T_1-T_2)^{-\frac{1}{p-1}} + f(d_2)^{-\frac{1}{p-m-\lambda+1}}(d_2-d_1)^{-\frac{\lambda+1}{p-m-\lambda+1}} \\ &+ \gamma(t-T_2)^{\frac{1}{\omega+1-\nu}}\left[f(d_2)d_2^{\frac{N}{s}}\right]^{\frac{M_1}{\omega+1-\nu}}\Big(\sup_{T_2<\tau<t}\int_{B_{d_2}(\tau)} u^\nu \,\mathrm{d}x\Big)^{\frac{M_2-1}{\omega+1-\nu}}. \qquad (4.12) \end{aligned}$$

The proof stays unchanged.

4.4. **Proof of Theorem 4.1.** It is essentially similar to the proof of Theorem 1.2. We select $0<\nu<1$ so that (4.10) is fulfilled; this is possible by virtue of (4.5). The only other change we need is in the definition of the geometry of the cylinders: choose

$$d_1 = R((T-t))/2\,, \quad d_2 = R((T-t))\,, \quad T_1 = t-(T-t)/2\,, \quad T_2 = t-(T-t)\,.$$

The proof is completed by checking that, with this selection, each one of the three terms on the right-hand side of (4.12) is bounded above by a constant multiple of $G(R(T-t))$. This requires only elementary calculations, and, of course, the use of estimate (4.3), where $1-\theta=\nu$, and $\rho=d_2$. Indeed, we may clearly assume that θ is small enough for (4.3) to hold.

5. The case of nonpower nonlinearities. II: f nondecreasing

We look here at the case of nondecreasing f, e.g., $f(x)=|x|^{-\alpha}$, with $\alpha<0$. Assume that

> (H2): $\rho\mapsto f(\rho)>0$, $0<\rho<\rho_0$, is a nondecreasing function, and, writing $f(x)=f(|x|)$, we have $f\in C(B_{\rho_0})$. We also stipulate that (4.1) is in force. In this connection, we must assume that the integrals defining $L(\rho)$ in Lemma 2.1 are finite when $\theta\in(0,\theta_0)$, for some $\theta_0>0$ (this amounts to requirement (1.8) when $f(x)=|x|^{-\alpha}$).

The auxiliary functions G, ℓ, and R are then defined in the same way as in Section 4, so that Lemma 2.1 yields the integral estimate (4.3), for $\ell(\rho)\le T-t$.

However, while (4.8) still trivially holds, the behaviour of ℓ, and therefore of R, may be quite different than in Section 4. Even when f is a power, $R(t)=t^{1/\mathcal{H}}$ may become unbounded as $t\to 0$, if $\mathcal{H}<0$. In order to stay within the framework of blow-up estimates for solutions u which are local in space, we stipulate the following assumption (equivalent to $\mathcal{H} > 0$, i.e., (1.7), when $f(x) = |x|^{-\alpha}$):

(H3): $R(t) \to 0, \quad \text{as } t \to 0.$

For example the functions in (4.2) satisfy both (H2) and (H3) for $\alpha < 0$ fulfilling (1.8) and (1.7), and for all $\beta \in \boldsymbol{R}$ (and in fact also for $\alpha = 0$, $\beta < 0$, and $\rho_0 = e^{-1}$).

See Remark 5.3 below for comments on the case when (H3) does not hold.

Remark 5.1. We stress the fact that the integral estimate (4.3) holds whenever $\ell(\rho) \le T - t$, independently of (H3). The latter (which is in fact an assumption on ℓ—see (4.4)), is stipulated only to to meet the requirement $B_{2R(T-t)} \subset \Omega$ as $t \to T$ (see Lemma 2.1).

Let us also remark that the a priori local supremum estimates, i.e., the arguments sketched in Subsections 5.1–5.3, do not make any use of the functions ℓ and R, and therefore of (H3). □

Our main result here is

Theorem 5.2. *Let f fulfill* (H2) *and* (H3)*. Assume*

$$0 < m + \lambda - 1 < p < m + \lambda - 1 + \frac{\lambda + 1}{N} \,. \tag{5.1}$$

Then, for $T/2 < t < T$ and $2R(T - t) < \operatorname{dist}(0, \partial\Omega)$,

$$u(x,t) \le \gamma G\big(R(T-t)\big) \,, \qquad |x| \le R(T-t)/2 \,. \tag{5.2}$$

Essentially, the only technical difference in the proof of the estimate is that, in the case of nonincreasing f, we estimated the energy in terms of an integral containing f (see (3.1)), while here we extract f from the integral. We need a slightly different definition of the key function $\mathcal{F}$: set

$$\mathcal{F}^*(r,s,\theta,\tau,h,k) = \Big\{ \frac{(h-k)^{-p+1}}{(\theta-\tau)f(s)} + \frac{(h-k)^{-p+m+\lambda-1}}{(s-r)^{\lambda+1}f(s)} C(h,k)^{(m-1)_+} + C(h,k)^p \Big\} C(h,k)^{(1-m)_+},$$

and then

$$\mathcal{F}(r,s,\theta,\tau,h,k) = \mathcal{F}^*(r,s,\theta,\tau,h,k) f(s) \,. \tag{5.3}$$

The quantities F_i below are then defined formally as in Section 3, but in terms of $\mathcal{F}$ as in (5.3). (As a side remark, what we call $\mathcal{F}^*$ here formally coincides with the previous definition of $\mathcal{F}$; $\mathcal{F}^*$ plays a role in Subsection 5.3.) *With this new definition of $\mathcal{F}$*, the proofs are much like the ones given in Section 3 in the case $\alpha = 0$.

5.1. **Analog of Lemma 3.1.** The statement is the same, when we drop the requirements on α and we let $\alpha = 0$ in (3.1). Keep in mind that F_0 is defined by $F_0 = \mathcal{F}(r_1, r_2, t_1, t_2, h_1, h_2)$, for $\mathcal{F}$ as in (5.3).

The proof is very similar to that of Lemma 3.1, where we let $\alpha = 0$ in (3.4) and (3.6). That is, the analogs of I_1 and I_2 are *not* majorized in terms of an integral containing f in our present approach. Instead, f is "extracted" from the integral analogous to I_3, as (3.7) is replaced with

$$\iint_{\mathcal{C}_2} f(x) u^p (u - \widetilde{h})_+^{\omega} \,\mathrm{d}x \,\mathrm{d}\tau \le \gamma C(h_1, h_2)^p f(r_2) \iint_{\mathcal{C}_2} (u - h_2)_+^{\omega+p} \,\mathrm{d}x \,\mathrm{d}\tau \,.$$

5.2. **Analog of Lemma 3.2.** The statement is obtained from the one of Lemma 3.2 by setting $\alpha = 0$ and $s = \infty$ there. Note that f enters the estimate only through F_1 (see (5.3)).

The proof of the lemma is the same as that in Section 3, again in the case $\alpha = 0$ and $s = \infty$. Indeed, the proof of this result relies only on the estimate of Lemma 3.1, and does not make further use of the differential equation.

5.3. **Analog of Lemma 3.3.** In the statement, we need only replace (3.22) with

$$\begin{aligned} &\|u\|_{\infty, B_{d_1} \times (T_1, t)} \\ &\le f(d_2)^{-\frac{1}{p-1}} (T_1 - T_2)^{-\frac{1}{p-1}} + f(d_2)^{-\frac{1}{p-m-\lambda+1}} (d_2 - d_1)^{-\frac{\lambda+1}{p-m-\lambda+1}} \\ &+ \gamma (t - T_2)^{\frac{1}{\omega+1-\nu}} f(d_2)^{\frac{M_1}{\omega+1-\nu}} \Big(\sup_{T_2 < \tau < t} \int_{B_{d_2}(\tau)} u^{\nu} \,\mathrm{d}x \Big)^{\frac{M_2 - 1}{\omega+1-\nu}} . \end{aligned} \tag{5.4}$$

Here M_1 and M_2 are defined as in Lemma 3.3, where we formally set $s = \infty$; i.e., $1/s = 0$.

The only modification in the proof is that the alternative (3.25) is now stated in terms of $\mathcal{F}^*$. More exactly, we let

$$F_2 = \mathcal{F}(d_1, d_2, T_1, T_2, k, 0) = F_2^* f(d_2) \,, \quad F_2^* = \mathcal{F}^*(d_1, d_2, T_1, T_2, k, 0) \,.$$

Then the alternative is given as

$$\text{i)} \quad F_2^* > 3 \,, \qquad \text{ii)} \quad F_2^* \le 3 \,,$$

and the argument is concluded as in Section 3.

5.4. **Proof of Theorem 4.1.** It is the same as the proof of Theorem 1.2, with the changes indicated in Subsection 4.4.

Remark 5.3. THE CASE OF $\mathcal{H} < 0$ AND $\mathcal{B} < 0$. We discuss here the instance when (H3) is not satisfied, confining ourselves to the model case $f(x) = (1 + |x|)^{-\alpha}$ for the sake of brevity. Assume therefore that $\mathcal{H}$ and $\mathcal{B}$ are negative, i.e., that the assumptions of Proposition 1.7 are in force. We note in passing that the behaviour of a local solution u near $x = 0$ can be easily obtained applying Remark 1.12, and it is clearly the same as in the case when f is a constant.

However, we are interested here in the behaviour of a global (in space) solution u. Therefore, we keep to the definition of R given in (4.4): for large ρ,

$$\ell(\rho) \approx \rho^{\mathcal{H}}, \quad \text{so that} \quad R(T-t) \approx (T-t)^{\frac{1}{\mathcal{H}}} \to \infty, \quad \text{as } t \to T. \tag{5.5}$$

We also note that (H2) is still in force, and that requirement (1.8) is not needed now, since it was imposed only to guarantee that the integrals defining $L(\rho)$ in Lemma 2.1 were finite. This is certainly the case now, as f is a continuous, strictly positive function. Moreover, the inclusion $B_{2R(T-t)} \subset \Omega = \boldsymbol{R}^N$ is obviously satisfied even if (H3) is not satisfied (here we need that u is defined in the strip $\boldsymbol{R}^N \times (0,T)$).

Then the integral estimate (4.3) holds true for $\rho \geq \gamma(T-t)^{\frac{1}{\mathcal{H}}}$.

As we already remarked, the arguments providing the other ingredient of the proof, that is, the local supremum estimate, are independent of the form of ℓ and R, so that we conclude that estimate (5.2) actually holds, for any solution u defined in the whole strip $\boldsymbol{R}^N \times (0,T)$. More specifically, for $|x| \leq \gamma_0(T-t)^{\frac{1}{\mathcal{H}}}$

$$u(x,t) \leq \gamma G\big(R(T-t)\big) \leq \gamma(T-t)^{-\mathcal{B}} \to 0, \qquad \text{as } t \to T.$$

This proves that $u \equiv 0$, since T can be chosen arbitrarily. □

6. THE BOUND BELOW: THEOREM 1.8

We apply the change of variables (1.15), and denote again by t the new time variable. Note that the assumption that T is the blow-up time for u implies that

$$\limsup_{t\to\infty} e^{\frac{t}{p-1}} \|w(\cdot,t)\|_{\infty,\boldsymbol{R}^N} = \infty. \tag{6.1}$$

Define, for $\theta > 1$

$$\Phi(t) = \Big(\int_{\boldsymbol{R}^N} w(x,t)^{1+\theta} \, \mathrm{d}x \Big)^{\frac{1}{1+\theta}} .$$

Multiply both sides of the equation (1.16) by $\Phi(t)^{-\theta} w^{\theta}$ and integrate by parts. We obtain

$$\Big(\frac{\mathrm{d}\Phi}{\mathrm{d}t}(t) + \frac{\Phi(t)}{p-1} \Big) e^{\frac{t}{p-1}} + C_1(\theta) e^{-\frac{p-m-\lambda}{p-1} t} \Phi(t)^{-\theta} \int \Big| D w^{\frac{m+\lambda+\theta-1}{\lambda+1}} \Big|^{\lambda+1} \mathrm{d}x \\ \leq e^{\frac{t}{p-1}} \Phi(t)^{-\theta} \int w^{p+\theta} \, \mathrm{d}x \, , \quad (6.2)$$

where

$$C_1(\theta) = \theta \Big(\frac{\lambda+1}{m+\lambda+\theta-1} \Big)^{\lambda+1} = \theta^{-\lambda} [(\lambda+1)^{\lambda+1} + o(1)] \, , \qquad (6.3)$$

as $\theta \to \infty$. Indeed, we need to track the dependence of the constants on θ, as we intend to let $\theta \to \infty$ eventually.

Setting

$$v = w^{\frac{m+\lambda+\theta-1}{\lambda+1}} \, , \qquad \beta = \frac{(\lambda+1)(p+\theta)}{m+\lambda+\theta-1} \, ,$$

we may write (6.2) as

$$\frac{\mathrm{d}}{\mathrm{d}t} \Big(\Phi(t) e^{\frac{t}{p-1}} \Big) + C_1(\theta) e^{-\frac{p-m-\lambda}{p-1} t} \Phi(t)^{-\theta} \int |Dv|^{\lambda+1} \, \mathrm{d}x \leq e^{\frac{t}{p-1}} \Phi(t)^{-\theta} \int v^{\beta} \, \mathrm{d}x \, . \qquad (6.4)$$

As $\theta \to \infty$, $\beta \to \lambda + 1$; thus, we may assume that θ is so large that $\beta > 1$ and $\beta(N - \lambda - 1)_+ < N(\lambda + 1)$. Then we may use Nirenberg-Gagliardo embedding in order to partially absorb the right-hand side of (6.4). The embedding has the form

$$\int v^{\beta} \, \mathrm{d}x \leq C_2(\theta) \Big(\int |Dv|^{\lambda+1} \, \mathrm{d}x \Big)^{\frac{a\beta}{\lambda+1}} \Big(\int v^{\mu} \, \mathrm{d}x \Big)^{(1-a)\frac{\beta}{\mu}} , \qquad (6.5)$$

where we set $\mathcal{K}_\theta = N(m + \lambda - 2) + (1 + \theta)(\lambda + 1)$, and

$$\mu = \frac{(1+\theta)(\lambda+1)}{m+\lambda+\theta-1} < \beta \, , \qquad a = \frac{N(m+\lambda+\theta-1)}{\mathcal{K}_\theta} \, \frac{p-1}{p+\theta} \in (0,1) \, .$$

Note that $\mu > 1$ for θ large enough. Moreover, according to [21], Theorem 2.2, Chapter II, we have $1 \leq C_2(\theta) \leq \gamma^{\frac{1}{\mathcal{K}_\theta}}$, where γ does not depend on θ, so that $C_2(\theta) \to 1$ as $\theta \to \infty$. In fact, this is hardly surprising, as (6.5) in the limit $\theta \to \infty$ approaches a trivial equality.

Then, from (6.4) and (6.5) we obtain, by means of Young's inequality,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi(t)e^{\frac{t}{p-1}}\Big) + C_1(\theta)e^{-\frac{p-m-\lambda}{p-1}t}\Phi(t)^{-\theta}\int|Dv|^{\lambda+1}\,\mathrm{d}x \\
&\quad\le C_2(\theta)e^{\frac{t}{p-1}}\Phi(t)^{-\theta}\Big(\int|Dv|^{\lambda+1}\,\mathrm{d}x\Big)^{\frac{a\beta}{\lambda+1}}\Big(\int v^{\mu}\,\mathrm{d}x\Big)^{(1-a)\frac{\beta}{\mu}} \\
&\quad\le \frac{C_1(\theta)}{2}e^{-\frac{p-m-\lambda}{p-1}t}\Phi(t)^{-\theta}\int|Dv|^{\lambda+1}\,\mathrm{d}x \\
&\quad + C_3(\theta)e^{\frac{N(p-m-\lambda+1)-(1+\theta)(\lambda+1)}{\mathcal{K}_\theta-N(p-1)}t}\Big(\Phi(t)e^{\frac{t}{p-1}}\Big)^{1+\frac{(1+\theta)(p-1)(\lambda+1)}{\mathcal{K}_\theta-N(p-1)}}. \qquad (6.6)
\end{aligned}$$

Note that

$$\begin{aligned}
1\le C_3(\theta) &= 2^{\frac{N(p-1)}{\mathcal{K}_\theta-N(p-1)}}C_1(\theta)^{-\frac{N(p-1)}{\mathcal{K}_\theta-N(p-1)}}C_2(\theta)^{\frac{\mathcal{K}_\theta}{\mathcal{K}_\theta-N(p-1)}} \\
&\le \gamma^{\frac{1}{\mathcal{K}_\theta}}C_1(\theta)^{-\frac{N(p-1)}{\mathcal{K}_\theta-N(p-1)}}\to 1\,,
\end{aligned}$$

as $\theta\to\infty$, according to (6.3). Then, setting $\psi(t)=\Phi(t)e^{\frac{t}{p-1}}$, we obtain

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t)\le C_3(\theta)e^{-Mt}\psi(t)^{1+S}\,, \qquad (6.7)$$

with

$$M=\frac{(1+\theta)(\lambda+1)-N(p-m-\lambda+1)}{\mathcal{K}_\theta-N(p-1)}\to 1\,,$$
$$S=\frac{(1+\theta)(p-1)(\lambda+1)}{\mathcal{K}_\theta-N(p-1)}\to p-1\,,$$

as $\theta\to\infty$. Integrating (6.7) over (t_0,t_1) we get

$$-\frac{1}{S}[\psi(t_1)^{-S}-\psi(t_0)^{-S}]\le C_3(\theta)\frac{e^{-Mt_0}}{M}\,.$$

Taking the limit $\theta\to\infty$ we obtain therefore

$$-\frac{1}{p-1}[\psi_\infty(t_1)^{-p+1}-\psi_\infty(t_0)^{-p+1}]\le e^{-t_0}\,, \qquad (6.8)$$

where we denote

$$\psi_\infty(t)=\|w(\cdot,t)\|_{\infty,\boldsymbol{R}^N}e^{\frac{t}{p-1}}\,.$$

Finally note that $\psi_\infty(t)$ does not stay bounded as $t\to\infty$, owing to (6.1). Then we let $t_1\to\infty$, along a suitable subsequence, in (6.8), so that $\psi_\infty(t_1)\to\infty$, and we get

$$\psi_\infty(t_0)\ge[(p-1)e^{-t_0}]^{-\frac{1}{p-1}}=(p-1)^{-1/(p-1)}e^{\frac{t_0}{p-1}}\,, \qquad (6.9)$$

which is the sought-after estimate when we switch back to the original time variable.

7. Finite speed of propagation up to the blow-up time: Theorem 1.10

We make use here of the change of variables (1.15). Note that the constant $\eta = (p - m - \lambda + 1)/(p - 1)$ is positive, as a result of our assumptions. This fact is going to play an essential role in the proof.

In the following we revert to the old time-variable name t for simplicity. We still denote by $Z(t)$ the quantity defined in in (1.11) as a function of the new time variable.

According to assumption (1.12) we have

$$0 \le w(x,t) \le c\,, \qquad x \in \boldsymbol{R}^N\,, -\ln T < t < \infty\,. \tag{7.1}$$

Actually, it is easy to prove that the support of w is bounded for any finite time (see, e.g., [5]); our purpose is to find a uniform bound for the support, valid for all times.

The proof is divided into two main parts: 1) a local estimate, bounding $Z(t)$ in terms of, say, $Z(t-1)$; and 2) a simple recursive argument allowing us to extend the local estimate to a global uniform bound.

1) *Local estimate.* We start with a local estimate in time of the support. Fix arbitrarily a time $\bar{t} \ge -\ln T$, and assume that

$$\operatorname{supp} w(x,\bar{t}) \subset B_\rho\,, \qquad 0 < \rho < \infty\,. \tag{7.2}$$

Introduce the sequences $\{r'_n\}$ and $\{r''_n\}$, $r'_n < r''_n$, as

$$r'_n = (1 + 2\sigma - 2^{-n}\sigma)\rho\,, \qquad r''_n = (1 - 2\sigma + 2^{-n}\sigma)4\rho\,, \qquad n \ge 0\,,$$

for any fixed $0 < \sigma \le 1/4$. Let us also introduce the annuli, for $n \ge 0$,

$$A_n = \{r'_n < |x| < r''_n\},\ A_\infty = \{(1+2\sigma)\rho < |x| < (1-2\sigma)4\rho\} \subset A_{n+1} \subset A_n,$$

$$\widetilde{A}_{n+1} = \{(r'_{n+1} + r'_n)/2 < |x| < (r''_{n+1} + r''_n)/2\}\,, \qquad A_{n+1} \subset \widetilde{A}_{n+1} \subset A_n\,,$$

and a standard cut-off function $\zeta_n \in C^1_o(\boldsymbol{R}^N)$, such that

$$\zeta_n(x) = 1\,, \quad x \in \widetilde{A}_{n+1}\,; \qquad \zeta_n(x) = 0\,, \quad x \notin A_n\,; \qquad |D\zeta_n| \le 2^{n+4}/(\sigma\rho)\,.$$

Let us select for (1.16) the testing function $\zeta_n^s w^\theta e^{-kt}$, where $s > \lambda + 1$ and $\theta > 0$ are fixed. The constant $k > 0$ will be chosen as large as needed in the following; k depends on p, c, θ, m, and λ only (see (7.4) and (7.8)). Note that for every $n \ge 0$

$$\zeta_n(x)^s w(x,\bar{t})^\theta e^{-k\bar{t}} = 0\,, \qquad \text{for all } x \in \boldsymbol{R}^N,$$

as $\operatorname{supp} w(\cdot,\bar{t})$ does not intersect A_n, owing to (7.2). Thus, after routine calculations (involving the use of Steklov averaging of w), and dropping the positive integral given by the term $w/(p-1)$ on the left-hand side of (1.16), one finds

$$\begin{aligned} &\frac{1}{1+\theta}\int_{\boldsymbol{R}^N}\zeta_n^s e^{-kt}w(x,t)^{1+\theta}\,\mathrm{d}x+\frac{k}{1+\theta}\int_{\bar t}^t\int_{\boldsymbol{R}^N}\zeta_n^s e^{-k\tau}w^{1+\theta}\,\mathrm{d}x\,\mathrm{d}\tau \\ &\qquad+\frac{\theta}{2}\int_{\bar t}^t\int_{\boldsymbol{R}^N}\zeta_n^s e^{-(\eta+k)\tau}w^{m+\theta-2}|Dw|^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau \qquad (7.3)\\ &\le\frac{\gamma}{\theta^\lambda}\frac{2^{n(\lambda+1)}}{(\sigma\rho)^{\lambda+1}}\int_{\bar t}^t\int_{A_n}w^{m+\theta+\lambda-1}e^{-(\eta+k)\tau}\,\mathrm{d}x\,\mathrm{d}\tau+\int_{\bar t}^t\int_{\boldsymbol{R}^N}\zeta_n^s e^{-kt}w^{p+\theta}\,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned}$$

By virtue of (7.1) the last integral above may be majorized by

$$\int_{\bar t}^t\int_{\boldsymbol{R}^N}\zeta_n^s e^{-k\tau}w^{p+\theta}\,\mathrm{d}x\,\mathrm{d}\tau\le c^{p-1}\int_{\bar t}^t\int_{\boldsymbol{R}^N}\zeta_n^s e^{-k\tau}w^{1+\theta}\,\mathrm{d}x\,\mathrm{d}\tau\,,$$

and then absorbed into the left-hand side of (7.3) by choosing

$$k>2(1+\theta)c^{p-1}\,. \qquad (7.4)$$

Thus we obtain

$$\begin{aligned} \sup_{\bar t<\tau<t}\int_{\widetilde A_{n+1}}e^{-k\tau}w(x,\tau)^{1+\theta}\,\mathrm{d}x+\int_{\bar t}^t\int_{\widetilde A_{n+1}}e^{-(\eta+k)\tau}\left|Dw^{\frac{m+\lambda+\theta-1}{\lambda+1}}\right|^{\lambda+1}\mathrm{d}x\,\mathrm{d}\tau \\ \le\frac{\gamma 2^{n(\lambda+1)}}{(\sigma\rho)^{\lambda+1}}\int_{\bar t}^t\int_{A_n}e^{-(\eta+k)\tau}w^{m+\lambda+\theta-1}\,\mathrm{d}x\,\mathrm{d}\tau\,. \qquad (7.5) \end{aligned}$$

Next introduce a cut-off function $\psi_n\in C_o^1(\boldsymbol{R}^N)$ such that

$$\psi_n(x)=1,\quad x\in A_{n+1};\quad \psi_n(x)=0\,,\quad x\notin\widetilde A_{n+1};\quad |D\psi_n|\le 2^{n+4}/(\sigma\rho)\,.$$

Define $v_{n+1}=w^{\frac{m+\lambda+\theta-1}{\lambda+1}}\psi_n$, so that (7.5) yields for $n\ge 0$,

$$\begin{aligned} Y_{n+1}:=\sup_{\bar t<\tau<t}\int_{\boldsymbol{R}^N}e^{-k\tau}v_{n+1}(x,\tau)^q\,\mathrm{d}x+\int_{\bar t}^t\int_{\boldsymbol{R}^N}e^{-(\eta+k)\tau}|Dv_{n+1}|^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau \\ \le\frac{\gamma 2^{n(\lambda+1)}}{(\sigma\rho)^{\lambda+1}}\int_{\bar t}^t\int_{\boldsymbol{R}^N}e^{-(\eta+k)\tau}v_n^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau\,, \qquad (7.6) \end{aligned}$$

where $q=(1+\theta)(\lambda+1)/(m+\lambda+\theta-1)<\lambda+1$, since $m+\lambda>2$. Next we estimate, by means of Nirenberg-Gagliardo inequality,

$$\int_{\boldsymbol{R}^N}v_n^{\lambda+1}\,\mathrm{d}x\le\gamma\Big(\int_{\boldsymbol{R}^N}|Dv_n|^{\lambda+1}\,\mathrm{d}x\Big)^a\Big(\int_{\boldsymbol{R}^N}v_n^q\,\mathrm{d}x\Big)^{\frac{(1-a)(\lambda+1)}{q}}\,,$$

with

$$a = N(m+\lambda-2)/\mathcal{K}_\theta \in (0,1)\,, \qquad \mathcal{K}_\theta = N(m+\lambda-2)+(1+\theta)(\lambda+1)\,.$$

The integral on the right-hand side of (7.6) is therefore majorized by

$$\begin{aligned}
&\int_{\bar t}^{t}\int_{\boldsymbol{R}^N} e^{-(\eta+k)\tau} v_n^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau \\
&\quad\le \gamma \int_{\bar t}^{t} e^{-(\eta+k)\tau}\Big(\int_{\boldsymbol{R}^N}|Dv_n|^{\lambda+1}\,\mathrm{d}x\Big)^a \Big(\int_{\boldsymbol{R}^N} v_n^q\,\mathrm{d}x\Big)^{\frac{(1-a)(\lambda+1)}{q}}\mathrm{d}\tau \\
&\quad= \gamma \int_{\bar t}^{t} e^{(1-a)\left[-(\eta+k)+k\frac{\lambda+1}{q}\right]\tau}\Big(\int_{\boldsymbol{R}^N} e^{-(\eta+k)\tau}|Dv_n|^{\lambda+1}\,\mathrm{d}x\Big)^a \\
&\qquad\qquad \times \Big(\int_{\boldsymbol{R}^N} e^{-k\tau} v_n^q\,\mathrm{d}x\Big)^{\frac{(1-a)(\lambda+1)}{q}}\mathrm{d}\tau \\
&\quad\le \gamma \Big(\sup_{\bar t<\tau<t}\int_{\boldsymbol{R}^N} e^{-k\tau} v_n(x,\tau)^q\,\mathrm{d}x\Big)^{\frac{(1-a)(\lambda+1)}{q}} \\
&\Big(\int_{\bar t}^{t}\int_{\boldsymbol{R}^N} e^{-(\eta+k)\tau}|Dv_n|^{\lambda+1}\,\mathrm{d}x\,\mathrm{d}\tau\Big)^a \Big(\int_{\bar t}^{t} e^{\left[-(\eta+k)+k\frac{\lambda+1}{q}\right]\tau}\,\mathrm{d}\tau\Big)^{1-a}, \qquad (7.7)
\end{aligned}$$

where we used Hölder's inequality. Next, recalling $\lambda+1>q$, we impose our second and last restriction on k, that is,

$$-(\eta+k)+k(\lambda+1)/q > 0\,. \qquad (7.8)$$

Thus the last integral in (7.7) is bounded above by

$$\int_{\bar t}^{t} e^{\left[-(\eta+k)+k\frac{\lambda+1}{q}\right]\tau}\,\mathrm{d}\tau \le \gamma e^{\left[-(\eta+k)+k\frac{\lambda+1}{q}\right]t} =: H(t)\,.$$

Collecting the estimates above, we may rewrite (7.6) as

$$Y_{n+1} \le \gamma 2^{n(\lambda+1)}(\sigma\rho)^{-\lambda-1} Y_n^{1+(1-a)\frac{\lambda+1-q}{q}} H(t)^{1-a}\,. \qquad (7.9)$$

It is well known (see [21], Lemma 5.6, Chapter II) that $Y_n \to 0$ if

$$(\sigma\rho)^{-\lambda-1} Y_0^{(1-a)\frac{\lambda+1-q}{q}} H(t)^{1-a} \le \gamma_0\,. \qquad (7.10)$$

In this case, according to the definition of Y_n and A_n, we infer

$$w(x,\tau)=0\,, \qquad (1+2\sigma)\rho < |x| < (1-2\sigma)4\rho\,, \quad \bar t<\tau<t\,. \qquad (7.11)$$

Let us make (7.10) more explicit; by calculations similar to the ones leading to (7.5), we prove that

$$Y_0 \le \frac{\gamma}{(\sigma\rho)^{\lambda+1}} \int_{\bar t}^{t} \int_G e^{-(\eta+k)\tau} w^{m+\lambda+\theta-1} \,\mathrm{d}x \,\mathrm{d}\tau \le \frac{\gamma}{(\sigma\rho)^{\lambda+1}} \rho^N e^{-(\eta+k)\bar t} , \tag{7.12}$$

where $G = \{(1+\sigma)\rho < |x| < (1-\sigma)4\rho\}$, and we have exploited (7.1) again.

From now on we select for simplicity $t = \bar t + 1$. On substituting estimate (7.12) in (7.10), and taking into account the definitions of $H(t)$, a, and q, after elementary calculations we find that (7.10) is in force for all ρ such that

$$\rho \ge \gamma \sigma^{-\mu} \exp\Big\{ -\frac{\eta}{\lambda+1} \bar t \Big\} , \tag{7.13}$$

where $\mu = [\mathcal{K}_\theta + (\lambda+1)(m+\lambda-2)]/[(\lambda+1)(m+\lambda+\theta-1)] > 0$.

Thus, from (7.11), we conclude that $Z(t)$ is majorized by $(1+2\sigma)\rho$, for any ρ satisfying the two requirements we stipulated, i.e., (7.2) and (7.13). Hence we may write

$$Z(\bar t + 1) \le (1+2\sigma) Z(\bar t) + \gamma_* (1+2\sigma)(2\sigma)^{-\mu} \exp\Big\{ -\frac{\eta}{\lambda+1} \bar t \Big\} . \tag{7.14}$$

This is the local estimate we need to start our iteration procedure. Note that γ_* is fixed for the rest of this proof.

2) *From the local to the global estimate.* Define the sequence of times

$$t_0 = \max(0, -\ln T) , \qquad t_{n+1} = t_n + 1 , \qquad n \ge 0 .$$

Choose now $\bar t = t_n$ in (7.14). This yields the recursive estimate

$$Z(t_{n+1}) \le (1+2\sigma_n) Z(t_n) + \gamma_* (1+2\sigma_n)(2\sigma_n)^{-\mu} e^{-\frac{\eta}{\lambda+1} n} , \quad n \ge 0 ,$$

where we selected $\sigma = \sigma_n = (n+2)^{-2}/2$. One can readily prove by induction that for $i \ge 1$,

$$Z(t_i) \le \prod_{n=0}^{i-1} \big(1 + (n+2)^{-2}\big) \Big\{ Z(t_0) + \gamma_* \sum_{n=0}^{i-1} (n+2)^{2\mu} e^{-\frac{\eta}{\lambda+1} n} \Big\} .$$

Clearly,

$$\prod_{n=0}^{\infty} \big(1 + (n+2)^{-2}\big) < \infty , \qquad \sum_{n=0}^{\infty} (n+2)^{2\mu} e^{-\frac{\eta}{\lambda+1} n} < \infty ,$$

so that we conclude the proof by noting that

$$Z(t) \le \lim_{i\to\infty} Z(t_i) < \infty , \qquad t_0 < t < \infty .$$

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