Advances in Differential Equations Volume 10, Number 1, Pages 65–88

SMOOTHING OF QUASILINEAR PARABOLIC OPERATORS AND APPLICATIONS TO FORWARD-BACKWARD STOCHASTIC SYSTEMS

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(Submitted by: Reza Aftabizadeh)

Abstract. We solve the Cauchy problem for a quasilinear parabolic equation in $[0,T]\times\mathbb{R}^n$ with quadratic nonlinearity in the gradient and with Hölder-continuous, not necessarily differentiable, initial datum. We get the same smoothing properties of linear parabolic equations, and we use them to improve the results now available in the literature on a class of stochastic forward-backward systems.

1. Introduction

In this paper we study a quasilinear parabolic equation in $[0,T]\times\mathbb{R}^n$,

$$(P)\begin{cases} \dfrac{\partial u}{\partial t}(t,x)+\displaystyle\sum_{i,j}^{n} a_{i,j}(x,u(t,x))\frac{\partial^2}{\partial x_i\partial x_j}u(t,x)=f(x,u(t,x),\nabla_x u(t,x)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t\in[0,T], x\in\mathbb{R}^n, \\ u(T,x)=\psi(x), \qquad\qquad\qquad\qquad\qquad x\in\mathbb{R}^n. \end{cases}$$

Under suitable regularity and growth assumptions on the nonlinear functions $a_{i,j}$ and f, we prove that for every $\psi\in C^\theta(\mathbb{R}^n)$, $\theta\in(0,1)$, there exists a global classical solution u, which is unique in an appropriate (standard) class, and which satisfies

$$(S)\ \sup_{0\le t<T}(T-t)\|u(t,\cdot)\|_{C^{2+\theta}(\mathbb{R}^n)}<\infty,$$

Accepted for publication: August 2004.
AMS Subject Classifications: 35K55, 60H10.
Partially supported by the RTN Project "Evolution Equations for Deterministic and Stochastic Systems" HPRN-CT-2002-00281.

so that we have the same smoothing effect of the linear case.

We are motivated to this investigation by an application to stochastic forward-backward systems. Indeed, problems such as (P) arise in the study of stochastic systems of the type

$$(FB)\begin{cases} dX_s = g(X_s, Y_s, Z_s)\, ds + \sigma(X_s, Y_s)\, dW_s, & s \in [0, T], \\ dY_s = h(X_s, Y_s, Z_s)\, ds + Z_s\, dW_s, \\ X_0 = x, \qquad Y_T = \psi(X_T), \end{cases}$$

where the unknown $(X_s, Y_s, Z_s)_{s\in[0,T]}$ is a triplet of processes with values respectively in $\mathbb{R}^n$, in $\mathbb{R}$, and in $\mathbb{R}^n$. In these systems a forward Itô differential equation is coupled with a backward Itô differential equation; the nonlinearities g, σ, and h are regular functions. Such systems have several applications to mathematical finance and to stochastic optimal control—see e.g. [11, Chapter 8] for a systematic review—and recently they have been widely studied—see for instance [13], [10], [15], [14], [12], and the bibliography therein.

The nonlinearities $a_{i,j}$ and f in (P) are defined by $a_{i,j}(x, u) = (\sigma\sigma^*)_{i,j}(x, u)$ and $f(x, u, p) = -\langle g(x, u, p^*\sigma(x, u)), p\rangle + h(x, u, p^*\sigma(x, u))$, and they are Lipschitz continuous with respect to x and locally Lipschitz continuous with respect to u and p, with at most quadratic growth in the gradient and linear growth in u. Relying on analytic results for system (P), in [10] the so-called *four-step scheme* was introduced—see also [11, Chapter 4]—to solve the stochastic system. Whenever σ does not depend on Z, this scheme reduces to three steps. More precisely it consists in solving (P) as a first step, then in solving

$$\begin{cases} dX_s = \sigma(X_s, u(s, X_s))dW_s + g(X_s, u(s, X_s), \nabla u(s, X_s)^*\sigma(X_s, u(s, X_s)))ds, \\ \qquad\qquad\qquad\qquad\qquad\qquad s \in [r, T], \\ X_r = \xi, \end{cases}$$

for each $r \in [0, T)$ and for each $\mathcal{F}_r$-measurable and square-integrable random variable ξ (where $\mathcal{F}_t$ is the filtration associated to the Brownian motion $\{W_t\}_{t\geq 0}$), and then in showing that

$$(X_s^{r,\xi}, Y_s^{r,\xi}, Z_s^{r,\xi}) := (X_s^{r,\xi}, u(s, X_s^{r,\xi}), \nabla u(s, X_s^{r,\xi})^*\sigma(X_s^{r,\xi}, u(s, X_s^{r,\xi})))$$

is a solution to

$$\begin{cases} dX_s^{r,\xi} = g(X_s^{r,\xi}, Y_s^{r,\xi}, Z_s^{r,\xi})\, ds + \sigma(X_s^{r,\xi}, Y_s^{r,\xi})\, dW_s, & s \in [r, T], \\ dY_s^{r,\xi} = h(X_s^{r,\xi}, Y_s^{r,\xi}, Z_s^{r,\xi})\, ds + Z_s^{r,\xi}\, dW_s, \\ X_r = \xi, \qquad Y_T = \psi(X_T). \end{cases}$$

The deterministic results of [6, Chapter 5, Theorem 6.1] were used to prove existence and uniqueness of a regular solution to problem (FB). Strong regularity conditions on the nonlinearities $a_{i,j}$ and f and on the final data were required. In particular, ψ must belong to $C^{2+\theta}(\mathbb{R}^n)$. This approach has been followed and widely improved in [2], where existence and uniqueness results for the systems (FB) are proved requiring much less regular coefficients and the final datum to be just Lipschitz continuous. In the present paper, still needing the coefficients $a_{i,j}$ and f to be regular enough, we relax the hypothesis on the final datum assuming ψ to be merely Hölder continuous, and we prove existence of a solution to (FB) also in the case where the function h has quadratic growth with respect to Z.

Setting as usual $u(t) = u(t, \cdot)$, we study problem (P) as an evolution equation in the space $C^\alpha(\mathbb{R}^n)$ for some $\alpha \in (0, \theta)$:

$$(P_T) \begin{cases} u'(t) + A(u(t))u(t) = F(u(t)), & t < T, \\ u(T) = \psi, \end{cases}$$

where $A(u(t))$ is the operator $\sum_{i,j}^n a_{i,j}(\cdot, u(t)) \frac{\partial^2}{\partial x_i \partial x_j}$ and

$$F(u(t)) := f(\cdot, u(t), \nabla_x u(t)).$$

As a first step, we look for a local solution in a small time interval $[r, T]$, as a fixed point of the operator $u \mapsto v$, where v is the solution to the linear problem

$$\begin{cases} v'(t) + A(u(t))v(t) = F(u(t)), & t \in [r, T], \\ v(T) = \psi. \end{cases}$$

If $T - r$ is small enough, the fixed-point argument works in a suitably weighted space of functions $v : [r, T] \mapsto C^\alpha(\mathbb{R}^n)$, thanks to optimal smoothing estimates in Hölder spaces for linear equations; such estimates are also used to prove further regularity of the solution and to arrive at (S).

So, the (unique) fixed point u satisfies (S) with $[r, T)$ replacing $[0, T)$; moreover, from the proof of the local existence theorem we see that u may be extended to the whole of $[0, T]$ (still satisfying (S)) provided we can bound $\|u(t)\|_{C^\theta(\mathbb{R}^n)}$ by a constant independent of t. This is done using classical arguments for *a priori* bounds in nonlinear parabolic problems.

The rest of the paper is organized as follows. In Section 2 notation and assumptions are given. The main results will be proved in Section 3, while in Section 4 the application to the stochastic forward-backward system is discussed.

2. Notation and assumptions

Let $\theta \in (0,1)$. We denote by $C^\theta(\mathbb{R}^n)$ the usual space of bounded and uniformly θ-Hölder-continuous functions from $\mathbb{R}^n$ to $\mathbb{R}$, and by $C^{k+\theta}(\mathbb{R}^n)$, $k = 1, 2$, the space of the bounded and differentiable (respectively, twice differentiable) functions with first order (respectively, first and second order) derivatives in $C^\theta(\mathbb{R}^n)$. They are endowed with the norms

$$\|\phi\|_{C^{k+\theta}(\mathbb{R}^n)} = \sum_{|\alpha|\le k} \Big(\|D^\alpha \phi\|_\infty + \sup_{x,y\in\mathbb{R}^n,\, x\ne y} \frac{|\phi(x)-\phi(y)|}{|x-y|^\theta} \Big).$$

If X is any Banach space and $y_0 \in X$, $R > 0$, $B(y_0, R)$ is the closed ball in X centered at y_0 with radius R. If $a, b \in \mathbb{R}$, $B((a,b); X)$ is the space of the bounded functions from (a,b) to X endowed with the sup norm; $C^\theta([a,b]; X)$ is the space of all θ-Hölder-continuous functions from $[a,b]$ to X, endowed with the norm

$$\|f\|_{C^\theta([a,b];X)} = \|f\|_\infty + \sup_{t,s\in[a,b], t\ne s} \frac{\|f(t)-f(s)\|_X}{|t-s|^\theta};$$

$C_\theta((a,b]; X)$ is the space of the continuous functions from $(a,b]$ to X, such that $t \mapsto (t-a)^\theta \|f(t)\|_X$ is bounded, endowed with the norm $\|f\|_{C_\theta((a,b];X)} = \sup_{t\in(a,b]} (t-a)^\theta \|f(t)\|_X$. This is meaningful for $\theta = 0$ too.

According to the assumptions of Section 4, the differential operators

$$A(u) := \sum_{i,j}^{n} a_{i,j}(\cdot, u(t)) \frac{\partial^2}{\partial x_i \partial x_j}$$

and the function $F(u(t)) := f(\cdot, u(t), \nabla_x u(t))$ satisfy next assumptions, for each $\alpha \in [0,1]$.

Hypothesis 2.1. *The maps* $A : C^\alpha(\mathbb{R}^n) \to L(C^{2+\alpha}(\mathbb{R}^n), C^\alpha(\mathbb{R}^n))$ *and* $F : C^{1+\alpha}(\mathbb{R}^n) \to C^\alpha(\mathbb{R}^n)$ *satisfy the following hypotheses:*

(H1) *For every* $y \in C^\alpha(\mathbb{R}^n)$ *the operator* $A(y)$ *is sectorial in* $C^\alpha(\mathbb{R}^n)$, *and if* $0 < \alpha < 1$, $D(A(y)) \simeq C^{2+\alpha}(\mathbb{R}^n)$.

(H2) *For every* $y_0 \in C^\alpha(\mathbb{R}^n)$ *there exist* $K = K(y_0) > 0$ *and* $R = R(y_0) > 0$ *such that*

$$\|A(x) - A(y)\|_{L(C^{2+\alpha}(\mathbb{R}^n), C^\alpha(\mathbb{R}^n))} \le K \|x-y\|_{C^\alpha(\mathbb{R}^n)}$$

for every $x, y \in B(y_0, R) \subset C^\alpha(\mathbb{R}^n)$.

(H3) *There exists* $K_1 > 0$ *such that*

$$\|F(x) - F(y)\|_{C^\alpha(\mathbb{R}^n)} \le K_1 (1 + \|x\|_{C^{1+\alpha}(\mathbb{R}^n)} + \|y\|_{C^{1+\alpha}(\mathbb{R}^n)}) \|x-y\|_{C^{1+\alpha}(\mathbb{R}^n)}$$

for every $x, y \in C^{1+\alpha}(\mathbb{R}^n)$.

Note that (H3) implies that there is $K_2 > 0$ such that

(H4) $\|F(x)\|_{C^{\alpha}(\mathbb{R}^n)} \le K_2(1 + \|x\|^2_{C^{1+\alpha}(\mathbb{R}^n)}), \ \ \forall x \in C^{1+\alpha}(\mathbb{R}^n)$.

For every $T > 0$ and for every $r \in [0, T)$ we shall study the problem

$$\begin{cases} u'(t) + A(u(t))u(t) = F(u(t)), & t \in [r, T], \\ u(T) = \psi. \end{cases} \tag{2.1}$$

The main result of this section is the following local existence and uniqueness theorem.

Theorem 2.2. *Let assumptions* 2.1 *hold, for each* $\alpha \in [0, 1]$. *Then for each* $\psi \in C^{\theta}(\mathbb{R}^n)$, *with* $\theta \in (0, 1)$, *there exists* $\delta = \delta(\psi) > 0$ *such that for every* $r \in [T - \delta, T]$ *problem* (2.1) *has a unique solution* $u \in C^{\theta/2}([r, T]; C_b(\mathbb{R}^n)) \cap B([r, T]; C^{\theta}(\mathbb{R}^n))^{(1)}$, *such that* u *has values in* $C^{2+\theta}(\mathbb{R}^n)$ *and it is differentiable with values in* $C^{\theta}(\mathbb{R}^n)$ *for* $t < T$, *and* $(T - t)\|u(t)\|_{C^{2+\theta}(\mathbb{R}^n)}$, $(T - t)\|u'(t)\|_{C^{\theta}(\mathbb{R}^n)}$ *are bounded in* $[r, T)$. *Consequently,* $(T - t)^{(1-\theta)/2}\|Du(t)\|_{\infty}$ *is bounded in* $[r, T)$.

3. PROOF OF THE MAIN RESULTS

In this section we will reverse time in problem (2.1). Therefore we shall study

$$\begin{cases} u'(t) = A(u(t))u(t) + F(u(t)), & t \in (0, r], \\ u(0) = \psi \end{cases} \tag{3.1}$$

for $r \in [0, T]$. Theorem 2.2 is rephrased as follows.

Theorem 3.1. *Let* (H1)–(H2)–(H3) *hold, for each* $\alpha \in [0, 1]$. *Then for each* $\psi \in C^{\theta}(\mathbb{R}^n)$, *with* $\theta \in (0, 1)$, *there exists* $\delta = \delta(\psi) > 0$ *such that for every* $r \in (0, \delta]$ *problem* (2.1) *has a unique solution* $u \in C^{\theta/2}([0, r]; C_b(\mathbb{R}^n)) \cap B([0, r]; C^{\theta}(\mathbb{R}^n))$, *such that* u *has values in* $C^{2+\theta}(\mathbb{R}^n)$ *and it is differentiable with values in* $C^{\theta}(\mathbb{R}^n)$ *for* $t > 0$, *and* $t\|u(t)\|_{C^{2+\theta}(\mathbb{R}^n)}$ *and* $t\|u'(t)\|_{C^{\theta}(\mathbb{R}^n)}$ *are bounded in* $(0, r]$.

Accordingly to the time change we introduce a family of linear nonautonomous Cauchy problems with $0 \le s \le r \le T$,

$$\begin{cases} v'(t) - \Lambda(t)v(t) = f(t), & t \in [s, r], \\ v(s) = \psi, \end{cases} \tag{3.2}$$

[1]This means that the function $(t, x) \mapsto u(t)(x)$ belongs to the parabolic Hölder space $C^{\theta/2,\theta}([r, T] \times \mathbb{R}^n)$.

where for each $t \in [s,r]$, $\Lambda(t) : D(\Lambda(t)) \subset C^{\alpha}(\mathbb{R}^n) \to C^{\alpha}(\mathbb{R}^n)$ is a sectorial operator in $C^{\alpha}(\mathbb{R}^n)$, with $D(\Lambda(t)) \simeq C^{2+\alpha}(\mathbb{R}^n)$, and $t \mapsto \Lambda(t)$ is Hölder continuous with values in $L(C^{2+\alpha}(\mathbb{R}^n); C^{\alpha}(\mathbb{R}^n))$.

A continuous function $v : [s,r] \to C^{\alpha}(\mathbb{R}^n)$ is said to be a classical solution to (3.2) in $[s,r]$ if $v \in C((s,r]; C^{2+\alpha}(\mathbb{R}^n)) \cap C^1((s,r]; C^{\alpha}(\mathbb{R}^n))$ and $v'(t) - \Lambda(t)v(t) = f(t)$ for $s < t \le r$, $v(s) = \psi$.

We shall use the following results.

Proposition 3.2. *Let $[s,r] \subset [0,T]$. For each $t \in [s,r]$, let $\Lambda(t) : D(\Lambda(t)) \subset C^{\alpha}(\mathbb{R}^n) \to C^{\alpha}(\mathbb{R}^n)$ be a sectorial operator in $C^{\alpha}(\mathbb{R}^n)$, with $D(\Lambda(t)) \simeq C^{2+\alpha}(\mathbb{R}^n)$. If the mapping Λ belongs to $C^{\nu}([s,r]; L(C^{2+\alpha}(\mathbb{R}^n);\ C^{\alpha}(\mathbb{R}^n)))$, there exists an evolution operator $G(t,s)$ in $C^{\alpha}(\mathbb{R}^n)$ associated to Λ.*

If $\psi \in C^{\theta}(\mathbb{R}^n)$ and $f \in C_{\gamma}((s,r], C^{\theta_1}(\mathbb{R}^n))$, with $0 \le \gamma < 1$ and $\theta_1, \theta > \alpha$, problem (3.2) has a unique classical solution v, given by the variation-of-constants formula

$$v(t) = G(t,s)\psi + \int_s^t G(t,\sigma) f(\sigma)\, d\sigma, \quad s \le t \le r. \tag{3.3}$$

For all $0 \le \tau \le \sigma \le r$ and $\alpha \le \theta_1 \le \theta_2 \le 2+\alpha$ there exists $C_1 > 0$ such that

$$\|G(\sigma,\tau)\|_{L(C^{\theta_1}(\mathbb{R}^n), C^{\theta_2}(\mathbb{R}^n))} \le \frac{C_1}{(\sigma-\tau)^{(\theta_2-\theta_1)/2}}, \quad s \le \tau < \sigma \le r. \tag{3.4}$$

Moreover, for $\alpha \le \theta_1 \le 2+\theta_2 \le 2+\alpha+2\nu$ there exists $C_2 > 0$ such that

$$\|\Lambda(\sigma)G(\sigma,\tau)\|_{L(C^{\theta_1}(\mathbb{R}^n), C^{\theta_2}(\mathbb{R}^n))} \le \frac{C_2}{(\sigma-\tau)^{1+(\theta_2-\theta_1)/2}}, \quad s \le \tau < \sigma \le r. \tag{3.5}$$

As a consequence of (3.4), for $0 \le \gamma < 1$, $\alpha \le \theta_1 \le \theta_2 \le 2+\alpha$, and $0 \le \gamma < 1$, $\theta_2 - \theta_1 < 1$ there exists $C_3 > 0$ such that if $f \in C_{\gamma}((s,r], C^{\theta_1}(\mathbb{R}^n))$ we have

$$\Big\| \int_s^t G(t,\sigma) f(\sigma)\, d\sigma \Big\|_{C^{\theta_2}(\mathbb{R}^n)} \le C_3 (t-s)^{1-\gamma-(\theta_2-\theta_1)/2} \|f\|_{C_{\gamma}(]s,r], C^{\theta_1}(\mathbb{R}^n))}, \tag{3.6}$$

for $s < t \le r$.

Proof. We do not give the complete proof since it is very similar to the proofs of [9, Proposition 3.3] and [8, Chapter 6] and of the bibliography quoted therein.

We just stress the fact that our hypotheses imply that for all $t \in [s,r]$ (see e.g. [1, Appendix A])

(i) there are two positive constants ν_1 and ν_2 such that

$$\begin{aligned} &\nu_1(\|\Lambda(t)y\|_{C^\alpha(\mathbb{R}^n)} + \|y\|_{C^\alpha(\mathbb{R}^n)}) \\ &\le \|y\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \nu_2(\|\Lambda(t)y\|_{C^\alpha(\mathbb{R}^n)} + \|y\|_{C^\alpha(\mathbb{R}^n)}); \end{aligned} \tag{3.7}$$

(ii) there exist $\theta \in (\frac{\pi}{2}, \pi]$, $\omega \in \mathbb{R}$, and $M > 0$ such that, for all $t \in [s, r]$,

$$\|R(\lambda, \Lambda(t))\|_{L(C^\alpha(\mathbb{R}^n))} \le \frac{M}{|\lambda - \omega|}, \qquad \forall \lambda \in \omega + S_\theta, \tag{3.8}$$

where $S_\beta = \{z \in \mathbb{C} : \arg|z| \le \beta\}$, for any $\beta \in [0, 2\pi)$.

Moreover, estimates (ii) on the resolvent operator imply that there exists a positive constant M_1 independent of t and depending on α, θ, ρ, and T, and on the constants introduced in (i)–(ii), such that

$$\text{(iii)} \quad \|(\Lambda(t))^k e^{\sigma\Lambda(t)}\|_{L(C^\alpha(\mathbb{R}^n))} \le \frac{M_1 e^{\omega\sigma}}{\sigma^k}, \quad \sigma, t \in [s, r], \quad k = 0, 1, 2. \tag{3.9}$$

Then estimates (3.4) and (3.5) for the evolution operator $G(t, s)$ are a direct consequence of its construction (see e.g. [8, Chapter 6]), of estimate (3.9), and of the characterization of $C^\theta(\mathbb{R}^n)$ as the real interpolation space $(C^\alpha(\mathbb{R}^n), C^{2+\alpha}(\mathbb{R}^n))_{(\theta-\alpha)/2,\infty}$ for $\alpha < \theta < 2 + \alpha$ (see e.g. [8, Corollary 1.2.18]). In particular, the constants C_1 and C_2 that appear in (3.4) and in (3.5) depend on α, θ_1, θ_2, and T, and on the constants introduced in (i)–(ii)–(iii). Inequality (3.6) is an obvious consequence of (3.4). □

We fix now α, β, θ, ρ, and ν such that

$$\begin{cases} 0 < \alpha < \rho < \theta < 1, \;\; \rho < (\alpha + \theta)/2, \quad 0 < \nu < \frac{\theta - \alpha}{2}, \\ \frac{1}{2} - \frac{\theta - \rho}{2} < \beta < \frac{1}{2} - \frac{\theta - \alpha}{4}, \end{cases} \tag{3.10}$$

and we define the set Y as the intersection of the balls

$$B(\psi, R) \subset C^\nu([0, r]; C^\alpha(\mathbb{R}^n)) \;\; \text{and} \;\; B(0, R) \subset C_\beta((0, r]; C^{1+\rho}(\mathbb{R}^n));$$

i.e.,

$$\begin{aligned} Y = \{u \in C^\nu([0, r]; C^\alpha(\mathbb{R}^n)) \cap C_\beta((0, r]; C^{1+\rho}(\mathbb{R}^n)) : \\ \|u(\cdot) - \psi\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le R, \; \sup_{0<t\le r} t^\beta \|u(t)\|_{C^{1+\rho}(\mathbb{R}^n)} \le R\} \end{aligned} \tag{3.11}$$

where R and r will be chosen later. For any $u \in Y$ we consider the linear nonautonomous problem

$$\begin{cases} v'(t) - \Lambda_u(t)v(t) = f_u(t), \quad s < t \le r, \\ v(s) = x, \end{cases} \tag{3.12}$$

where $\Lambda_u(t) = A(u(t))$ and $f_u(t) = F(u(t))$. We collect the results about this problem in the next lemma.

Lemma 3.3. *Let* (H1)–(H2)–(H3) *hold. Fix the parameters* α, β, θ, ρ, *and* ν *satisfying* (3.10), *and fix* $\psi \in C^\theta(\mathbb{R}^n)$. *Then there is* $R > 0$ *such that for each* $r \in (0, T]$, *defining* Y *by* (3.11), *for every* $u \in Y$ *the operators* $\Lambda_u(t)$ *satisfy the assumptions of Proposition* 3.2. *Denoting by* $G_u(t, s)$ *the associated evolution operator, there exists a unique classical solution to problem* (3.12), *given by*

$$v(t) = G_u(t, s)\psi + \int_s^t G_u(t, \sigma) f_u(\sigma)\, d\sigma, \quad s \le t \le r.$$

Estimates (3.4), (3.5), *and* (3.6) *hold with constants* C_1, C_2, $C_3 > 0$ *independent of* u, s, *and* r.

Proof. Thanks to assumptions (H1) and (H2), for all $x_0 \in C^\alpha(\mathbb{R}^n)$ there exists $R' = R'(x_0) > 0$ such that for every $x \in B(x_0, R')$

(i) there exist two positive constants $\nu_1 = \nu_1(x_0)$ and $\nu_2 = \nu_2(x_0)$ such that

$$\begin{aligned} &\nu_1(\|A(x)y\|_{C^\alpha(\mathbb{R}^n)} + \|y\|_{C^\alpha(\mathbb{R}^n)}) \\ &\le \|y\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \nu_2(\|A(x)y\|_{C^\alpha(\mathbb{R}^n)} + \|y\|_{C^\alpha(\mathbb{R}^n)}); \end{aligned}$$

(ii) there exist $\theta = \theta(x_0) \in (\frac{\pi}{2}, \pi]$, $\omega = \omega(x_0) \in \mathbb{R}$, and $M = M(x_0) > 0$ such that, for all $t \in [s, r]$,

$$\|R(\lambda, A(x))\|_{L(C^\alpha(\mathbb{R}^n))} \le \frac{M}{|\lambda - \omega|} \qquad \forall \lambda \in \omega + S_\theta;$$

(iii) there exists $M_1 = M_1(x_0) > 0$, such that

$$\|A(x)^k e^{\sigma A(x)}\|_{L(C^\alpha(\mathbb{R}^n))} \le \frac{M_1 e^{\omega\sigma}}{\sigma^k}, \qquad \sigma \in [0, r], \quad k = 0, 1, 2.$$

This was proved in [9]. Taking $x_0 = \psi$ and $R = R'(\psi)$ in the definition of Y, it follows that for each $u \in Y$

(i) there exist two positive constants $\nu_1 = \nu_1(\psi)$ and $\nu_2 = \nu_2(\psi)$ such that

$$\begin{aligned} &\nu_1(\|\Lambda_u(t)y\|_{C^\alpha(\mathbb{R}^n)} + \|y\|_{C^\alpha(\mathbb{R}^n)}) \\ &\le \|y\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \nu_2(\|\Lambda_u(t)y\|_{C^\alpha(\mathbb{R}^n)} + \|y\|_{C^\alpha(\mathbb{R}^n)}); \end{aligned}$$

(ii) there exist $\alpha = \alpha(\psi) \in (0, \frac{\pi}{2}]$, $\theta = \theta(\psi) \in (\frac{\pi}{2}, \pi]$, $\omega = \omega(\psi) \in \mathbb{R}$, and $M = M(\psi) > 0$ such that, for all $t \in [s, r]$,

$$\|R(\lambda, \Lambda_u(t))\|_{L(C^\alpha(\mathbb{R}^n))} \leq \frac{M}{|\lambda - \omega|}, \qquad \forall \lambda \in \omega + S_\theta;$$

(iii) there exists $M_1 = M_1(\psi) > 0$ such that for all $t \in [0, r]$

$$\|(\Lambda_u(t))^k e^{\sigma \Lambda_u(t)}\|_{L(C^\alpha(\mathbb{R}^n))} \leq \frac{M_1 e^{\omega\sigma}}{\sigma^k}, \qquad \sigma \in [0, r], \quad k = 0, 1, 2.$$

Therefore, Proposition 3.2 may be applied, and it yields all the estimates for $G_u(t, s)$. Moreover, thanks to (H4) we have

$$\begin{aligned} \sup_{0<s\leq r} s^{2\beta} \|f_u(s)\|_{C^\rho(\mathbb{R}^n)} &\leq K_1(r^{2\beta} + \sup_{0<s\leq r} s^{2\beta} \|u(s)\|^2_{C^{1+\rho}(\mathbb{R}^n)}) \\ &\leq K_1(r^{2\beta} + R^2) < +\infty, \end{aligned} \tag{3.13}$$

so that $f_u \in C_{2\beta}((0, r]; C^\rho(\mathbb{R}^n))$. □

Proof of Theorem 3.1. We define a map $\Gamma : Y \to C([0, r]; C^\alpha(\mathbb{R}^n))$ as $\Gamma(u) = v$, where v is the solution to (3.12). Thus,

$$\Gamma(u)(t) = v(t) = G_u(t, 0)\psi + \int_0^t G_u(t, \sigma) f_u(\sigma)\, d\sigma, \quad 0 \leq t \leq r.$$

As a first step, we shall show that if r is small enough, then $\Gamma(Y) \subset Y$; then we shall show that Γ is a $\frac{1}{2}$-contraction, so that it has a unique fixed point in Y, which is a solution to (3.1). As a third step we shall prove further regularity properties and estimates for the fixed point, and eventually we shall see that the solution is unique.

First step. $\Gamma(Y) \subset Y$. We shall prove that
1) $t \to G_u(t, 0)\psi \in C^\nu([0, r]; C^\alpha(\mathbb{R}^n))$ and $\|G_u(\cdot, 0)\psi - \psi\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \leq R/2$ for sufficiently small $r > 0$.

For $0 < s \leq t \leq r$ we have

$$\begin{aligned} \|G_u(t, 0)\psi - G_u(s, 0)\psi\|_{C^\alpha(\mathbb{R}^n)} &= \left\| \int_s^t \Lambda_u(\sigma) G_u(\sigma, 0)\, d\sigma \psi \right\|_{C^\alpha(\mathbb{R}^n)} \\ \leq C_2 \int_s^t \sigma^{(\theta-\alpha)/2-1}\, d\sigma \|\psi\|_{C^\theta(\mathbb{R}^n)} &= c(\alpha, \theta) C_2 (t - s)^\nu r^{(\theta-\alpha)/2-\nu} \|\psi\|_{C^\theta(\mathbb{R}^n)}. \end{aligned}$$

In particular, for $s = 0$ we have

$$\|G_u(\cdot, 0)\psi - \psi\|_{C^\alpha(\mathbb{R}^n)} \leq c(\alpha, \theta) C_2 r^{(\theta-\alpha)/2} \|\psi\|_{C^\theta(\mathbb{R}^n)}.$$

Thus, $G_u(t,0)\psi \in C^\nu([0,r];C^\alpha(\mathbb{R}^n))$ and there exists $\delta_1 > 0$ such that for $0 < r \le \delta_1$,

$$\|G_u(\cdot,0)\psi - \psi\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le c(\alpha,\theta)C_2(\delta_1^{(\theta-\alpha)/2-\nu} + \delta_1^{(\theta-\alpha)/2})\|\psi\|_{C^\theta(\mathbb{R}^n)} \le \tfrac{R}{2}.$$

2) $t \to \int_0^t G_u(t,\sigma)f_u(\sigma)\,d\sigma \in C^\nu([0,r];C^\alpha(\mathbb{R}^n))$ and $\|\int_0^\cdot G_u(\cdot,\sigma)f_u(\sigma)\,d\sigma\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le R/2$.

Now fix $0 \le s \le t \le r$. Then

$$\begin{aligned}
&\Big\|\int_0^t G_u(t,\sigma)f_u(\sigma)d\sigma - \int_0^s G_u(s,\sigma)f_u(\sigma)d\sigma\Big\|_{C^\alpha(\mathbb{R}^n)}\\
&\le \Big\|\int_0^s [G_u(t,\sigma)-G_u(s,\sigma)]f_u(\sigma)d\sigma\Big\|_{C^\alpha(\mathbb{R}^n)} + \Big\|\int_s^t G_u(t,\sigma)f_u(\sigma)d\sigma\Big\|_{C^\alpha(\mathbb{R}^n)}\\
&= I_1 + I_2.
\end{aligned}$$

I_1 is estimated as follows, taking into account (3.5) and (3.13):

$$\begin{aligned}
I_1 &= \Big\|\int_0^s\int_s^t \Lambda_u(\tau)G_u(\tau,\sigma)\,d\tau f_u(\sigma)\,d\sigma\Big\|_{C^\alpha(\mathbb{R}^n)}\\
&\le \int_0^s\int_s^t \|\Lambda_u(\tau)G_u(\tau,\sigma)\|_{L(C^\rho(\mathbb{R}^n),C^\alpha(\mathbb{R}^n))}\,d\tau\|f_u(\sigma)\|_{C^\rho(\mathbb{R}^n)}\,d\sigma\\
&\le C_2 \sup_{0<\sigma\le r}\|\sigma^{2\beta}f_u(\sigma)\|_{C^\rho(\mathbb{R}^n)}\int_0^s \sigma^{-2\beta}\Big(\int_s^t (\tau-\sigma)^{-1+(\rho-\alpha)/2}\,d\tau\Big)\,d\sigma\\
&\le c(\alpha,\rho,\beta,\nu)C_2K_1(r^{2\beta}+R^2)(t-s)^\nu r^{1-\nu-2\beta+(\rho-\alpha)/2}
\end{aligned}$$

In a similar way, taking also (3.6) into account, I_2 is estimated as follows:

$$I_2 \le C_3(t-\sigma)^{1-2\beta}\|f_u\|_{C_{2\beta}((s,r];C^\alpha(\mathbb{R}^n))} \le C_3c(\alpha,\rho)K_1(r^{2\beta}+R^2)(t-s)^\nu r^{1-2\beta-\nu}.$$

In particular, taking $s=0$ the latter estimate gives

$$\Big\|\int_0^t G_u(t,\sigma)f_u(\sigma)\,d\sigma\Big\|_{C^\alpha(\mathbb{R}^n)} \le C_3c(\alpha,\rho)K_1(r^{2\beta}+R^2)r^{1-2\beta}.$$

This implies that there exists $\delta_2 > 0$ such that for $0 < r \le \delta_2$ we have

$$\Big\|\int_0^t G_u(t,\sigma)f_u(\sigma)\,d\sigma\Big\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le C(\delta_2^{1-2\beta-\nu} + \delta_2^{1-2\beta}) \le \frac{R}{2},$$

where C does not depend on u and r.

3) $t \to G_u(t,0)\psi \in C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))$ and $\|G_u(\cdot,0)\psi\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} \le R/2$.

Thanks to (3.4) we have

$$t^\beta \|G_u(t,0)\psi\|_{C^{1+\rho}(\mathbb{R}^n)} \le C_1 t^{\beta-\frac{1}{2}+\frac{(\theta-\rho)}{2}} \|\psi\|_{C^\theta(\mathbb{R}^n)}.$$

Thus there exists $\delta_3 > 0$ such that for $0 < r \le \delta_3$

$$\|G_u(\cdot,0)\psi\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} \le C_1 \delta_3^{\beta-\frac{1}{2}+\frac{(\theta-\rho)}{2}} \|\psi\|_{C^\theta(\mathbb{R}^n)} \le \frac{R}{2}.$$

4) $t \to \int_0^t G_u(t,\sigma) f_u(\sigma)\, d\sigma \in C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))$ and $\| \int_0^\cdot G_u(\cdot,\sigma) f_u(\sigma)\, d\sigma\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} \le R/2$.
Thanks to (3.6), for $0 < t \le r$ we have

$$\begin{aligned} &t^\beta \Big\| \int_0^t G_u(t,\sigma) f_u(\sigma)\, d\sigma \Big\|_{C^{1+\rho}(\mathbb{R}^n)} \le t^\beta C_3 t^{1/2-2\beta} \|f_u\|_{C_{2\beta}((s,r];C^\rho(\mathbb{R}^n))} \\ &\le C_3 K_1 (r^{2\beta} + R^2) r^{1/2-\beta}. \end{aligned}$$

Thus there exists $\delta_4 > 0$ such that, for $0 < r \le \delta_4$,

$$\Big\| \int_0^\cdot G_u(\cdot,\sigma) f_u(\sigma)\, d\sigma \Big\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} \le C\delta_4^{1/2-\beta} \le \frac{R}{2}.$$

1), 2), 3), and 4) prove that taking $\bar\delta = \min\{\delta_1,\delta_2,\delta_3,\delta_4\}$, we have that for every $0 < r \le \bar\delta$, $\Gamma(Y) \subset Y$.

Second step. Γ is a $\frac{1}{2}$-contraction.

Let $u_1, u_2 \in Y$ and set $v_1 = \Gamma(u_1)$ and $v_2 = \Gamma(u_1)$. For all $t \in [0,r]$ we have

$$\begin{aligned} v_1(t) - v_2(t) = [G_{u_1}(t,0) - G_{u_2}(t,0)]\psi + \int_0^t [G_{u_1}(t,s) - G_{u_2}(t,s)] f_{u_1}(s)\, ds \\ + \int_0^t G_{u_2}(t,s)[f_{u_1}(s) - f_{u_2}(s)]\, ds. \end{aligned}$$

To estimate $v_1 - v_2$ we use the identity

$$G_{u_1}(\tau,\sigma) - G_{u_2}(\tau,\sigma) = \int_\sigma^\tau G_{u_1}(\tau,\rho)[\Lambda_{u_1}(\rho) - \Lambda_{u_2}(\rho)] G_{u_2}(\rho,\sigma)\, d\rho, \quad (3.14)$$

which holds for $0 \le \sigma \le \tau \le r$, and arguments similar to the ones in the first step.

1) Let us estimate $\|v_1 - v_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}$. For $0 \le s \le t \le r$ we have

$$\begin{aligned} &\|v_1(t) - v_2(t) - v_1(s) + v_2(s)\|_{C^\alpha(\mathbb{R}^n)} \\ &\le \|(G_{u_1}(t,0) - G_{u_2}(t,0) - G_{u_1}(s,0) + G_{u_2}(s,0))\psi\|_{C^\alpha(\mathbb{R}^n)} \end{aligned}$$

$$
\begin{aligned}
&+ \Big\| \int_0^t [G_{u_1}(t,\sigma) - G_{u_2}(t,\sigma)] f_{u_1}(\sigma) d\sigma \\
&- \int_0^s [G_{u_1}(s,\sigma) - G_{u_2}(s,\sigma)] f_{u_1}(\sigma) d\sigma \Big\|_{C^\alpha(\mathbb{R}^n)} \\
&+ \Big\| \int_0^t G_{u_2}(t,\sigma)[f_{u_1}(\sigma) - f_{u_2}(\sigma)] d\sigma - \int_0^s G_{u_2}(s,\sigma)[f_{u_1}(\sigma) - f_{u_2}(\sigma)] d\sigma \Big\|_{C^\alpha(\mathbb{R}^n)} \\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

Recalling (3.14) and (3.4), I_1 is estimated as follows:

$$
\begin{aligned}
I_1 &= \|(G_{u_1}(t,0) - G_{u_2}(t,0) - G_{u_1}(s,0) + G_{u_2}(s,0))\psi\|_{C^\alpha(\mathbb{R}^n)} \\
&\le \Big\| \int_s^t G_{u_1}(t,\sigma)[\Lambda_{u_1}(\sigma) - \Lambda_{u_2}(\sigma)] G_{u_2}(\sigma,0)\psi \, d\sigma \Big\|_{C^\alpha(\mathbb{R}^n)} \\
&+ \Big\| \int_0^s [G_{u_1}(t,\sigma) - G_{u_1}(s,\sigma)][\Lambda_{u_1}(\sigma) - \Lambda_{u_2}(\sigma)] G_{u_2}(\sigma,0)\psi \, d\sigma \Big\|_{C^\alpha(\mathbb{R}^n)} \\
&\le C_1^2 K \|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \|\psi\|_{C^\theta(\mathbb{R}^n)} \Big[\int_s^t \sigma^{-1+(\theta-\alpha)/2} \, d\sigma \\
&\qquad + \int_0^s \sigma^{-1+(\theta-\alpha)/2} \left(\int_s^t (r-\sigma)^{-1} \, dr \right) d\sigma \Big] \\
&\le c(\alpha,\theta) C_1^2 K \|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \|\psi\|_{C^\theta(\mathbb{R}^n)} (t-s)^\nu r^{(\theta-\alpha)/2-\nu}.
\end{aligned}
$$

Thus, there exists $\delta_5 > 0$ such that for $0 < r \le \delta_5$ we have

$$
\|(G_{u_1}(\cdot,0) - G_{u_2}(\cdot,0))\phi\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le \frac{1}{9} \|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}.
$$

I_2 is estimated in a similar way, taking also (3.13) into account:

$$
I_2 \le C \|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} (t-s)^\nu r^{1-2\beta-\nu},
$$

where $C > 0$ is independent of u_1, u_2, and r. Thus there exists $\delta_6 > 0$, such that for $0 < r \le \delta_6$ we have

$$
\Big\| \int_0^{\cdot} [G_{u_1}(\cdot,\rho) - G_{u_2}(\cdot,\rho)] f_{u_1}(\rho) d\rho \Big\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le \frac{1}{9} \|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}.
$$

The third addendum I_3, thanks also to hypothesis $H3$, is estimated in a similar way as follows:

$$
I_3 \le C \|u_1 - u_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} (t-s)^\nu r^{1-2\beta-\nu},
$$

where $C > 0$ is independent of u_1, u_2, and r.

Therefore there exists $\delta_7 > 0$, such that for $0 < r \le \delta_7$ we have

$$\Big\| \int_0^{\cdot} G_{u_2}(\cdot,\sigma)[f_{u_1}(\sigma) - f_{u_2}(\sigma)]\, d\sigma \Big\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le \frac{1}{9}\|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}.$$

Summing up, we have proved that for $0 \le r \le \min\{\bar\delta, \delta_5, \delta_6, \delta_7\}$ we have

$$\|v_1 - v_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le \frac{1}{3}\big(\|u_1 - u_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} + \|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}\big). \tag{3.15}$$

2) Now we estimate $\|v_1 - v_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))}$. We have

$$\begin{aligned}\|v_1 - v_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} &= \sup_{0<t\le r}\|t^\beta[v_1(t) - v_2(t)]\|_{C^{1+\rho}(\mathbb{R}^n)}\\ &\le \sup_{0<t\le r}\|t^\beta[G_{u_1}(t,0) - G_{u_2}(t,0)]\psi\|_{C^{1+\rho}(\mathbb{R}^n)}\\ &\quad + \sup_{0<t\le r}\Big\|t^\beta\int_0^t[G_{u_1}(t,\sigma) - G_{u_2}(t,\sigma)]f_{u_1}(\sigma)\, d\sigma\Big\|_{C^{1+\rho}(\mathbb{R}^n)}\\ &\quad + \sup_{0<t\le r}\Big\|t^\beta\int_0^t G_{u_2}(t,\sigma)[f_{u_1}(\sigma) - f_{u_2}(\sigma)]\, d\sigma\Big\|_{C^{1+\rho}(\mathbb{R}^n)} \le I_1 + I_2 + I_3.\end{aligned}$$

I_1 is estimated as follows:

$$\begin{aligned}I_1 &= \sup_{0<t\le r}\Big\|t^\beta\int_0^t G_{u_1}(t,\sigma)[\Lambda_{u_1}(\sigma) - \Lambda_{u_2}(\sigma)]G_{u_2}(\sigma,0)\psi\Big\|_{C^{1+\rho}(\mathbb{R}^n)}\\ &\le c(\rho,\theta)KC_1^2\|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}r^{\beta-1/2+(\theta-\rho)/2}.\end{aligned}$$

Similarly to I_1, and thanks also to (3.13), I_2 is estimated as follows:

$$\begin{aligned}I_2 &= \sup_{0<t\le r}\Big\|t^\beta\int_0^t\int_\sigma^t G_{u_1}(t,\tau)[\Lambda_{u_1}(\tau) - \Lambda_{u_2}(\tau)]G_{u_2}(\tau,\sigma)\, d\tau f_{u_1}(\sigma)d\sigma\Big\|_{C^{1+\rho}(\mathbb{R}^n)}\\ &\le C\|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))}r^{-\beta+1/2},\end{aligned}$$

where $C > 0$ is independent of u_1, u_2, and r. Finally, using also hypothesis (H3), we get

$$I_3 \le C\|u_1 - u_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))}r^{1-\beta-1/2},$$

where $C > 0$ is independent of u_1, u_2, and r. Therefore there exists $\delta_8 > 0$, such that for $0 < r \le \delta_8$,

$$\|v_1 - v_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} \le \frac{1}{6}\big(\|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} + \|u_1 - u_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))}\big). \tag{3.16}$$

We have thus proved that there exists $\delta := \min\{\overline{\delta}, \delta_5, \delta_6, \delta_7, \delta_8\}$, such that for $0 < r \le \delta$ we have

$$\|v_1 - v_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))} + \|v_1 - v_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} \le \frac{1}{2}\big(\|u_1 - u_2\|_{C^\nu([0,r];C^\alpha(\mathbb{R}^n))} + \|u_1 - u_2\|_{C_\beta((0,r];C^{1+\rho}(\mathbb{R}^n))}\big). \quad (3.17)$$

Consequently, Γ is a $\frac{1}{2}$-contraction that maps Y into itself, and hence there exists a unique $u \in Y$ such that $\Gamma(u) = u$; i.e., there exists a unique solution of (2.1) in Y.

Third step. Further regularity. Let u be the unique fixed point of γ in Y. The same estimates of the first step, points 3)–4), show that $t \mapsto G_u(t,0)\psi \in C_{(1-\theta+\rho)/2}((0,r];C^{1+\rho}(\mathbb{R}^n))$ and that

$$z(t) := \int_0^t G(t,\sigma)f_u(\sigma)d\sigma \in C_{2\beta-1/2}((0,r];C^{1+\rho}(\mathbb{R}^n)).$$

Thanks to the choice (3.10), we have $2\beta < 1 - (\theta - \alpha)/2$ and $\alpha < \rho$, so that $2\beta - 1/2 < (1 - \theta + \rho)/2$ and $z \in C_{(1-\theta+\rho)/2}((0,r];C^{1+\rho}(\mathbb{R}^n))$, and hence $u \in C_{(1-\theta+\rho)/2}((0,r];C^{1+\rho}(\mathbb{R}^n))$ and $f_u \in C_{1-\theta+\rho}((0,r];C^\rho(\mathbb{R}^n))$.

Applying once again estimate (3.6) we get that z is bounded with values in $C^\sigma(\mathbb{R}^n)$ for each $\sigma \le 2 + \rho - 4\beta$; in particular, due to the choice of ρ and β, z is bounded with values in $C^\theta(\mathbb{R}^n)$. Since $G_u(\cdot,0)\psi$ too is bounded with values in $C^\theta(\mathbb{R}^n)$, then

$$u = G_u(\cdot,0)\psi + z \in B([0,r];C^\theta(\mathbb{R}^n)). \quad (3.18)$$

Let us prove that $u(t)$ belongs to $C^{2+\theta}(\mathbb{R}^n)$ for $t > 0$ and that $\|tu(t)\|_{C^{2+\theta}(\mathbb{R}^n)}$ is bounded.

Again, we need more than one step. First we estimate $\|u(t)\|_{C^{2+\alpha}(\mathbb{R}^n)}$. By estimates (3.4) and (3.6) we get

$$\|u(t)\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \frac{C_1}{t^{1-(\theta-\alpha)/2}}\|\psi\|_{C^\theta(\mathbb{R}^n)} + \frac{C_3}{t^{1-\theta+\rho-(\rho-\alpha)/2}}\|f_u\|_{C_{1-\theta+\rho}((0,r],C^\rho(\mathbb{R}^n))} \le \frac{C}{t^{1-(\theta-\alpha)/2}},$$

with a suitable C independent of t.

Second, we improve the regularity of u up to $t = 0$ with values in $C^\alpha(\mathbb{R}^n)$. Indeed, the same estimates as in the first step show that u belongs to $C^{(\theta-\alpha)/2}([0,r];C^\alpha(\mathbb{R}^n))$. More precisely, the estimate of point 1) shows that $t \mapsto G_u(t,0)\psi \in C^{(\theta-\alpha)/2}([0,r];C^\alpha(\mathbb{R}^n))$, while using the fact that

$f_u \in C_{1-\theta+\rho}((0,r];C^\rho(\mathbb{R}^n))$, the estimates of point 2) with $1-\theta+\rho$ replacing 2β shows that $z \in C^{(\theta-\alpha)/2}([0,r];C^\alpha(\mathbb{R}^n))$. Summing up, $u \in C^{(\theta-\alpha)/2}([0,r];C^\alpha(\mathbb{R}^n))$.

Since u is bounded with values in $C^\theta(\mathbb{R}^n)$ and $t \mapsto tu(t)$ is continuous for $t > 0$ and bounded with values in $C^{2+\alpha}(\mathbb{R}^n)$, from the interpolation inequalities

$$\|\varphi\|_{C^{\theta_2}(\mathbb{R}^n)} \le \text{const.}\|\varphi\|_{C^{\theta_1}(\mathbb{R}^n)}^{1-(\theta_2-\theta_1)/(\theta_3-\theta_1)}\|\varphi\|_{C^{\theta_3}(\mathbb{R}^n)}^{(\theta_2-\theta_1)/(\theta_3-\theta_1)} \tag{3.19}$$

with $\theta_1 = \theta$, $\theta_2 = 2$, and $\theta_3 = 2+\alpha$, we get $u \in C_{1-\theta/2}((0,r];C^2(\mathbb{R}^n))$, so that by hypothesis H3 $f_u \in C_{2-\theta}((0,r];C^1_b(\mathbb{R}^n))$. Therefore there is $C > 0$ such that

$$\|u(t/2)\|_{C^{2+\alpha}(\mathbb{R}^n)} \le \frac{C}{t^{1-(\theta-\alpha)/2}}, \ \ \|f_{u|[t/2,r]}\|_{C([t/2,r];C^1_b(\mathbb{R}^n))} \le \frac{C}{t^{2-\theta}}, \ \ 0 < t \le r. \tag{3.20}$$

Using the variation-of-constants formula we write $u(t)$ as

$$u(t) = G_u(t,t/2)u(t/2) + \int_{t/2}^t G_u(t,s)f_u(s)ds, \ \ 0 < t \le r.$$

Using estimates (3.5) with $\theta_1 = 2+\alpha$ and $\theta_2 = \theta$ in the first addendum, and $\theta_1 = 1$ and $\theta_2 = \theta$ in the second addendum, gives $\Lambda_u(t)u(t) \in C^\theta(\mathbb{R}^n)$ and

$$\begin{aligned}&\|\Lambda_u(t)u(t)\|_{C^\theta(\mathbb{R}^n)}\\ &\le \frac{C}{(t/2)^{(\theta-\alpha)/2}}\|u(t/2)\|_{C^{2+\alpha}(\mathbb{R}^n)} + C(t-t/2)^{(1-\theta)/2}\|f_{u|[t/2,r]}\|_{C([t/2,r];C^1_b(\mathbb{R}^n))}\end{aligned}$$

so that (3.20) implies

$$\|\Lambda_u(t)u(t)\|_{C^\theta(\mathbb{R}^n)} \le \frac{C}{t}, \ \ 0 < t \le r,$$

where $C > 0$ is independent of t. Since u is bounded with values in $C^\theta(\mathbb{R}^n)$, assumption (H1) with θ instead of α gives

$$u(t) \in C^{2+\theta}(\mathbb{R}^n), \ \ \|u(t)\|_{C^{2+\theta}(\mathbb{R}^n)} \le \frac{C}{t}, \ \ 0 < t \le r. \tag{3.21}$$

Let us apply again the interpolation estimate (3.19) with $\theta_1 = \theta$, $\theta_2 = 1+\theta$, and $\theta_3 = 2+\theta$. We get $u \in C_{1/2}((0,r];C^{1+\theta}(\mathbb{R}^n))$, so that $f_u \in C_1((0,r];C^\theta(\mathbb{R}^n))$. It follows that

$$tu'(t) = t[\Lambda_u(t)u(t) + f_u(t)] \in B((0,r];C^\theta(\mathbb{R}^n)). \tag{3.22}$$

It remains to show that $u \in C^{\theta/2}([0,r];C_b(\mathbb{R}^n))$. To this aim, we recall that $u \in C_{1-\theta/2}((0,r];C^2(\mathbb{R}^n))$, so that $\Lambda_u u \in C_{1-\theta/2}((0,r];C_b(\mathbb{R}^n))$, and using once again (3.19) with $\theta_1 = \theta$, $\theta_2 = 1$, and $\theta_3 = 2+\theta$, we get $u \in C_{(1-\theta)/2}((0,r];C^1(\mathbb{R}^n))$, so that $f_u \in C_{1-\theta}((0,r];C_b(\mathbb{R}^n))$. Therefore, $u' \in C_{1-\theta/2}((0,r];C_b(\mathbb{R}^n))$, and this implies that $u \in C^{\theta/2}([0,r];C_b(\mathbb{R}^n))$. Recalling (3.18), (3.21), and (3.22), all the claims about the regularity of the solution are proved.

Fourth step. Uniqueness. Let u_1 and u_2 be two solutions with the specified degree of smoothness, and let $t_0 = \sup\{t \in [0,r] :\ u_{1|[0,t]} \equiv u_{2|[0,t]}\}$. We have to show that $t_0 = r$. Assume for the sake of contradiction that $t_0 < r$ and set $\psi_0 := u_1(t_0) = u_2(t_0)$, $K = \max\{\|u_i\|_{C^{\theta/2}([0,r];C_b(\mathbb{R}^n))} + \sup_{0<t\le r} t\|u_i(t)\|_{C^{2+\theta}(\mathbb{R}^n)} + \sup_{0<t\le r} t\|u_i'(t)\|_{C^\theta(\mathbb{R}^n)},\ i=1,2\}$.

The first part of the proof implies that there exists $\delta_0 \in (0, r-t_0]$ such that for each $\delta \in (0,\delta_0]$ the problem

$$\begin{cases} u'(t) = A(u(t))u(t) + F(u(t)), & t \in (t_0, t_0+\delta], \\ u(t_0) = \psi_0, \end{cases} \tag{3.23}$$

has a unique solution in the space

$$\begin{aligned} Y_0 = \{ & u \in C^\nu([t_0,t_0+\delta];C^\alpha(\mathbb{R}^n)) \cap C_\beta((t_0,t_0+\delta];C^{1+\rho}(\mathbb{R}^n)) : \\ & \|u(\cdot)-\psi_0\|_{C^\nu([t_0,t_0+\delta];C^\alpha(\mathbb{R}^n))} \le R,\ \sup_{t_0<t\le t_0+\delta} (t-t_0)^\beta \|u(t)\|_{C^{1+\rho}(\mathbb{R}^n)} \le R\}. \end{aligned}$$

For $s < t \in [t_0, t_0+\delta]$ we have, thanks to (3.19),

$$\begin{aligned} & \|u_i(t)-u_i(s)\|_{C^\alpha(\mathbb{R}^n)} \le \text{const.} \|u_i(t)-u_i(s)\|_\infty^{1-\alpha/\theta} \|u_i(t)-u_i(s)\|_{C^\theta(\mathbb{R}^n)}^{\alpha/\theta} \\ & \le \text{const.}(t-s)^{(\theta-\alpha)/2}(2K)^{\alpha/\theta} \le \text{const.}(t-s)^\nu (2K)^{\alpha/\theta}\delta^{(\theta-\alpha)/2-\nu}, \end{aligned}$$

and

$$\begin{aligned} & \|t^\beta u_i(t)\|_{C^{1+\rho}(\mathbb{R}^n)} \le \text{const.} t^\beta \|u_i(t)\|_{C^\theta(\mathbb{R}^n)}^{1-(1+\rho-\theta)/2} \|u_i(t)\|_{C^{2+\theta}(\mathbb{R}^n)}^{(1+\rho-\theta)/2} \\ & \le \text{const.} t^\beta t^{-(1+\rho-\theta)/2} K \le \text{const.} \delta^{\beta-(1+\rho-\theta)/2} K, \end{aligned}$$

for $i = 1,2$, so that if δ is small enough, both $u_{1|[t_0,t_0+\delta]}$ and $u_{2|[t_0,t_0+\delta]}$ belong to Y_0. This implies that $u_{1|[t_0,t_0+\delta]} \equiv u_{2|[t_0,t_0+\delta]}$, a contradiction.

This concludes the proof of the theorem. □

4. An application to stochastic forward-backward systems

In this section we discuss an application of Theorem 2.2 to a class of forward-backward systems. We make the following assumptions.

Hypothesis 4.1. *We are given the following:*

(i) *a final time* $T > 0$;
(ii) *an* n*-dimensional Brownian motion* $\{W_t\}_{t\geq 0}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*; we denote by* $\mathcal{F}_t$ *its natural filtration completed with the null sets of* $\mathcal{F}$;
(iii) *the maps* $h(x,u,p) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $g(x,u,p) : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, *and* $\sigma(x,u) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n^2}$ *that are Lipschitz continuous in the first variable uniformly with respect to* u *and* p *and twice continuously differentiable in the other variables with second derivatives bounded uniformly with respect to* x;
(iv) *a constant* $K > 0$ *such that for all* $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, *and* $z \in \mathbb{R}^n$,

$$\|\sigma(x,y)\| \leq K(1+|y|), \qquad \|g(x,y,z)\| \leq K(1+|y|+\|z\|),$$
$$|h(x,y,z)| \leq K(1+|y|+\|z\|^2);$$

(v) *a positive constant* c_0 *such that for every* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}$,

$$a_{i,j}(x,y)\eta_i\eta_j \geq c_0\|\eta\|^2 \qquad \forall \eta \in \mathbb{R}^n,$$

where $a_{i,j}(x,y) = (\sigma\sigma^*)_{i,j}(x,y)$ *for every* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}$;
(vi) *a function* $\psi \in C^\theta(\mathbb{R}^n)$, $\theta \in (0,1)$.

For every $r \in [0,T]$ we introduce the following partial differential equation:

$$\begin{cases} \partial_t u(t,x) + \frac{1}{2}\sum_{i,j} a_{i,j}(x,u(t,x))\partial^2_{i,j}u(t,x) = f(x,u(t,x),\nabla u(t,x)), \\ \qquad\qquad\qquad\qquad\qquad\qquad (t,x) \in [r,T] \times \mathbb{R}^n, \\ u(T,x) = \psi(x), \end{cases} \tag{4.1}$$

where

$$f(x,u(t,x),\nabla u(t,x)) := -\langle g(x,u(t,x),\nabla u(t,x)^*\sigma(x,u(t,x))), \nabla u(t,x)\rangle + h(x,u(t,x),\nabla u(t,x)^*\sigma(x,u(t,x))).$$

We recall the abstract formulation of equation (4.1) in the space $C^\alpha(\mathbb{R}^n)$,

$$\begin{cases} u'(t) + A(u(t))u(t) = F(u(t)), \quad t \in (r,T], \\ u(T) = \psi, \end{cases} \tag{4.2}$$

with the identification of Section 2 for A and F, and $u(t)(x) := u(t,x)$.

It easy to verify that under the above assumptions (iii), (iv), and (v) on the coefficients, the functions A and F satisfy Hypotheses 2.1, so that Theorem 2.2 may be applied, and there exists a unique local solution of (4.2), $u \in C^{\theta/2}([T-\delta,T]; C_b(\mathbb{R}^n)) \cap B([T-\delta,T]; C^\theta(\mathbb{R}^n))$ for some $\delta \in$

$(0,T)$, such that $u(t) \in C^{2+\theta'}(\mathbb{R}^n)$ for each $\theta' \in (0,1)$, $t \in [T-\delta, T)$, and $(T-t)^{1+(\theta'-\theta)/2}\|u(t)\|_{C^{2+\theta'}(\mathbb{R}^n)}$ is bounded in $[T-\delta, T)$.

The usual techniques of *a priori* estimates for parabolic equations may be employed to prove that u is extendable to a solution in the whole interval $[0,T]$. Since we have not found any proper reference in the literature, we give a proof below.

Proposition 4.2. *The local solution given by Theorem* 2.2 *has an extension to a global solution* $u \in C^{\theta/2}([0,T]; C_b(\mathbb{R}^n)) \cap B([0,T]; C^\theta(\mathbb{R}^n))$ *such that* $u(t) \in C^{2+\theta'}(\mathbb{R}^n)$ *for each* $\theta' \in (0,1)$, $t \in [0,T)$,

$$\begin{aligned} &\sup_{0\le t<T}(T-t)^{1+(\theta'-\theta)/2}\|u(t)\|_{C^{2+\theta'}(\mathbb{R}^n)} < \infty, \\ &\sup_{0\le t<T}(T-t)^{(1-\theta)/2}\|u(t)\|_{C^{1}(\mathbb{R}^n)} < \infty. \end{aligned} \tag{4.3}$$

Proof. Let us reverse time once again and consider problem (3.1). It is clear from the proof of Theorem 3.1 that the number δ depends only on $\|\psi\|_{C^\theta(\mathbb{R}^n)}$. Therefore it is enough to bound $\|u(t)\|_{C^{\theta'}(\mathbb{R}^n)}$ for some $\theta' \in (0,1)$ by a constant independent of t. Then standard arguments will yield the statement. Let $C>0$ be such that

$$f(x,u,0)u \le C(1+u^2), \;\; x \in \mathbb{R}^n, \; u \in \mathbb{R}. \tag{4.4}$$

As a first step, we estimate $\|u\|_{L^\infty([0,r];C_b(\mathbb{R}^n))}$, for any r in the maximal interval of existence of u, proving that

$$\sup_{a\le t\le r}\|u(t)\|_\infty \le e^{\lambda r}\max\left\{\|u(a)\|_\infty, \sqrt{\tfrac{C}{\lambda-C}}\right\}, \tag{4.5}$$

for each $\lambda > C$ and $a \in (0,r)$.

Fix any $\lambda > C$, and set $v(t,x) = u(t)(x)e^{-\lambda t}$. If $\|v\|_{L^\infty([a,r]\times\mathbb{R}^n)} = |v(t_0,x_0)|$ for some (t_0,x_0), the usual arguments that lead to the maximum principle in bounded sets (see e.g. [6, Theorem 2.9]) yield (4.5). If $|v|$ does not attain a maximum, the arguments of [6, Theorem 2.9] have to be modified. We assume here that $\|v\|_\infty = \sup v > 0$; if $\|v\|_\infty = -\inf v < 0$ we may argue similarly. For each $k \in \mathbb{N}$ let (t_k,x_k) be such that $v(t_k,x_k) \ge \|v\|_\infty - 1/k$. Let φ be a smooth function such that $0 \le \varphi(x) \le 1$, $\varphi \equiv 1$ in $B(0,1)$, $\varphi \equiv 0$ outside $B(0,2)$, and set

$$v_k(t,x) = v(t,x) + \frac{2}{k}\varphi(x-x_k), \;\; a \le t \le r, \; x \in \mathbb{R}^n, \; k \in \mathbb{N}. \tag{4.6}$$

Then each v_k has a maximum, $\|v_k\|_\infty = \max v_k$, and

$$D_t v_k(t,x) = \frac{1}{2}\sum_{i,j} a_{i,j}(x, v(t,x)e^{\lambda t})\Big(\partial^2_{i,j} v_k(t,x) - \frac{2\partial^2_{i,j}\varphi(x-x_k)}{k}\Big) + f(x, v(t,x)e^{\lambda t}, \nabla v(t,x)e^{\lambda t})e^{-\lambda t} - \lambda v_k(t,x) + \frac{2\lambda}{k}\varphi(x-x_k).$$

If the maximum of v_k is attained at $t = a$, then

$$\|v_k\|_\infty \leq \|v(a)\|_\infty + 2/k. \tag{4.7}$$

Otherwise, at any maximum point (t,x) for v_k, with $t > a$, we have

$$\begin{aligned} 0 &\leq v_k D_t v_k - \frac{v_k}{2}\sum_{i,j} a_{i,j}(x, v(t,x)e^{\lambda t})\partial^2_{i,j} v_k \\ &= -\lambda v_k^2 + f(x, v(t,x)e^{\lambda t}, 0)e^{-\lambda t} v(t,x) + \frac{g_0(t,x)}{k} + \frac{g_1(t,x)v_k}{k}, \end{aligned}$$

where

$$\begin{aligned} g_0(t,x) &= 2f(x, v(t,x)e^{\lambda t}, 0)e^{-\lambda t}\varphi(x-x_k), \\ g_1(t,x) &= -\sum_{i,j} a_{i,j}(x, v(t,x)e^{\lambda t})\partial^2_{i,j}\varphi(x-x_k) + 2\lambda\varphi(x-x_k) \\ &\quad - 2\int_0^1 \langle D_p f(x, ve^{\lambda t}, \sigma D v e^{\lambda t}), D\varphi(x-x_k)\rangle d\sigma, \end{aligned}$$

so that $\|g_0\|_\infty \leq 2\sup|f(x,u,0)| := C_0$,

$$\|g_1\|_\infty \leq \sum_{i,j}\|\partial^2_{i,j}\varphi\|_\infty \sup|a_{i,j}(x,u)| + 2\|D\varphi\|_\infty \sup|D_p f(x,u,p)| + 2\lambda := C_1,$$

where the suprema are taken for $x \in \mathbb{R}^n$, $u \in$ Range $u_{|[a,r]\times\mathbb{R}^n}$, and $p \in$ Range $Du_{|[a,r]\times\mathbb{R}^n}$, and hence they are finite. On the other hand, due to (4.4),

$$f(x, v(t,x)e^{\lambda t}, 0)e^{-\lambda t} v(t,x) \leq C(1+v(t,x)^2) \leq C\Big(1 + v_k(t,x)^2 + \frac{4}{k}\|v\|_\infty + \frac{4}{k^2}\Big).$$

Fix any $\varepsilon > 0$, and let k be so large that $C_0/k + 4C/k^2 \leq \varepsilon$ and $(C_1+4C)/k \leq \varepsilon$. Then

$$(\lambda - C)\|v_k\|^2_\infty - \varepsilon\|v_k\|_\infty - \varepsilon - C \leq 0,$$

so that

$$\|v_k\|_\infty \leq \frac{\varepsilon + \sqrt{\varepsilon^2 + 4(\lambda - C)(\varepsilon + C)}}{2(\lambda - C)}.$$

Recalling (4.7) and (4.6), this leads to (4.5).

As a second step, we find a bound for the Hölder norm of $u(t)$. Due to (4.5), the functions

$$\widetilde{a}_{i,j}(t,x) := a_{i,j}(x,u(t)(x)), \;\; a \le t \le r, \; x \in \mathbb{R}^n,$$

satisfy

$$c_0|\eta|^2 \le \sum_{i,j=1}^{n} \widetilde{a}_{i,j}(t,x)\eta_i\eta_j \le c_1|\eta|^2, \;\; a \le t \le r, \; x,\eta \in \mathbb{R}^n,$$

where c_0 is given by hypothesis 4.1(v), c_1 depends only on K, C, and λ, and the function $\widetilde{f}(t,x,p) := f(x,u(t)(x),p)$, $a \le t \le r, \; x,p \in \mathbb{R}^n$, satisfies

$$|\widetilde{f}(t,x,p)| \le c_2(|p|^2 + 1),$$

with c_2 depending only on K, C, and λ. Since

$$D_t u(t,x) = \sum_{i,j=1}^{n} \widetilde{a}_{i,j}(t,x) D_{ij} u(t,x) + \widetilde{f}(t,x,Du(t,x)), \;\; a \le t \le r, \; x \in \mathbb{R}^n,$$

we may apply the nonlinear version of the Krylov-Safonov theorem (see e.g. [7, Lemma 11.4]) to get the existence of $\theta' \in (0,1)$ such that

$$\|u(t)\|_{C^{\theta'}(\mathbb{R}^n)} \le c_3, \;\; a \le t \le r,$$

with c_3 depending only on c_0, c_1, and c_2, and on $\sup_{a \le t \le r} \|u(t)\|_\infty$. The statement follows. □

Let us introduce now a family of stochastic differential equations—SDEs—for every $r \in [0,T]$ and every $\mathcal{F}_r$-measurable and square-integrable random variable ξ,

$$\begin{cases} dX_s = \sigma(X_s, u(s,X_s))\, dW_s \\ \qquad +g(X_s, u(s,X_s), \nabla u(s,X_s)^* \sigma(X_s, u(s,X_s))\, ds, \quad s \in [r,T], \\ X_r = \xi. \end{cases} \tag{4.8}$$

We will need the following lemma.

Lemma 4.3. *Assume that* (i)–(ii)–(iii)–(iv) *hold. Then equation* (4.8) *has a unique strong solution that is an adapted process* $\{X_s^{r,\xi} : s \in [r,T]\}$ *with continuous trajectories such that*

$$\mathbb{E} \sup_{s \in [r,T]} \|X_s^{r,\xi}\|^2 < +\infty. \tag{4.9}$$

Sketch of the proof. The solution is found, as usual, by a fixed-point theorem in the Banach space of processes that are adapted and continuous, endowed with the norm $\|X\| \doteq (\mathbb{E} \sup_{s\in[r,T]} \|X_s\|^2)^{1/2}$. Indeed, thanks to (4.3), the coefficients behave like the Lipschitz-continuous coefficients treated by the classical theory of SDEs; see [5] for instance. The details of the proof of similar results can be found in [5] or in [3, Proposition 3.2], which deals with the infinite-dimensional case. □

We have

Theorem 4.4. *Let Hypothesis* 4.1 *hold, fix $x \in \mathbb{R}^n$ and $0 \le r < T$, and consider the system*

$$\begin{cases} dX_s^{r,x} = g(X_s^{r,x}, Y_s^{r,x}, Z_s^{r,x})\, ds + \sigma(X_s^{r,x}, Y_s^{r,x})\, dW_s, \quad s \in [r,T], \\ dY_s^{r,x} = h(X_s^{r,x}, Y_s^{r,x}, Z_s^{r,x})\, ds + Z_s^{r,x}\, dW_s, \\ X_r^{r,x} = x, \quad Y_T^{r,x} = \psi(X_T^{r,x}). \end{cases} \tag{4.10}$$

There exists a unique triplet of adapted processes $(X_s^{r,x}, Y_s^{r,x}, Z_s^{r,x}) : \Omega \times [r,T] \to \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ that satisfies (4.10)*. Moreover, for every $x \in \mathbb{R}^n$ and $r \in [0,T]$,*

$$\mathbb{E} \sup_{s\in[r,T]} \|X_s^{r,x}\|^2 + \mathbb{E} \sup_{s\in[r,T]} |Y_s^{r,x}|^2 + \mathbb{E}\Big(\int_r^T \|Z_s^{r,x}\|^2\, ds \Big) < +\infty.$$

Proof. We already remarked that equations (4.1) and (4.8) have (unique) solutions. Thus it is sufficient to verify that the triplet

$$(X_s^{r,\xi}, Y_s^{r,\xi}, Z_s^{r,\xi}) = (X_s^{r,\xi}, u(s, X_s^{r,\xi}), \nabla u(s, X_s^{r,\xi})^* \sigma(X_s^{r,\xi}, u(s, X_s^{r,\xi})))$$

is a solution to the forward-backward system (4.10).

Indeed, since u solves (4.1) and for every $\varepsilon > 0$ it belongs to $C^1([0,T-\varepsilon], C^\alpha(\mathbb{R}^n)) \cap C([0,T-\varepsilon], C^{2+\alpha}(\mathbb{R}^n))$, we can apply the Itô formula to get

$$\begin{aligned} u(\tau, X_\tau^{r,\xi}) &= u(T-\varepsilon, X_{T-\varepsilon}^{r,\xi}) - \int_\tau^{T-\varepsilon} \partial_s u(s, X_s^{r,\xi})\, ds \\ &- \int_\tau^{T-\varepsilon} \nabla_x u(s, X_s^{r,\xi})^* \sigma(X_s^{r,x}, u(s, X_s^{r,\xi}))\, dW_s \\ &- \frac{1}{2} \int_\tau^{T-\varepsilon} \sum_{i,j}^n a_{i,j}(X_s^{r,\xi}, u(s, X_s^{r,\xi})) \partial_{i,j}^2 u(s, X_s^{r,\xi})\, ds \\ &- \int_\tau^{T-\varepsilon} \langle g(X_s^{r,\xi}, u(s, X_s^{r,\xi}), \nabla u(s, X_s^{r,\xi})^* \sigma(X_s^{r,\xi}, u(s, X_s^{r,\xi}))), \nabla u(s, X_s^{r,\xi}) \rangle\, ds \end{aligned}$$

$$= u(T-\varepsilon, X^{r,\xi}_{T-\varepsilon}) - \int_\tau^{T-\varepsilon} h(X^{r,\xi}_s, u(s, X^{r,\xi}_s), \nabla u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))ds$$
$$+ \int_\tau^{T-\varepsilon} \nabla_x u(s, X^{r,x}_s)^* \sigma(X^{r,x}_s, u(s, X^{r,\xi}_s))\, dW_s.$$

Now, since $u \in C^{(\theta-\alpha)/2}([0,T]; C^\alpha(\mathbb{R}^n))$ and $X^{r,\xi}_s$ has continuous trajectories, letting ε go to 0 we obtain $u(T-\varepsilon, X^{r,\xi}_{T-\varepsilon}) \to \psi(X^{r,\xi}_T)$, $\mathbb{P}$-a.s. Moreover, for some constant $\tilde{C} > 0$,

$$|h(X^{r,\xi}_s, u(s, X^{r,\xi}_s), \nabla u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))|$$
$$\leq K(1 + |u(s, X^{r,\xi}_s)| + |\nabla u(s, X^{r,\xi}_s)|^2) \leq \tilde{C}(1 + (T-s)^{-1+\theta-\rho}), \quad \mathbb{P}\text{-a.s.},$$

with $\rho \in (0,\theta)$. Letting ε go to 0 we find

$$\int_\tau^{T-\varepsilon} h(X^{r,\xi}_s, u(s, X^{r,\xi}_s), \nabla u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))\, ds \to$$
$$\int_\tau^{T} h(X^{r,\xi}_s, u(s, X^{r,\xi}_s), \nabla u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))\, ds, \qquad \mathbb{P}\text{-a.s.}$$

Hypothesis (iv) and (4.3) imply that there exists a constant $\kappa > 0$ such that

$$\mathbb{E}(|\nabla_x u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))|^2) \tag{4.11}$$
$$\leq 2K^2 \mathbb{E}(\|\nabla_x u(s, X^{r,\xi}_s)\|^2 (1 + |u(s, X^{r,\xi}_s)|^2) \leq \kappa(T-s)^{-1+\theta-\rho}.$$

Then, at least along a subsequence $\varepsilon_h > 0$,

$$\int_\tau^{T-\varepsilon} \nabla_x u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))\, dW_s$$
$$\to \int_\tau^{T} \nabla_x u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))\, dW_s \qquad \mathbb{P}\text{-a.s.}$$

Therefore, for every $\tau \in [0,T]$,

$$u(\tau, X^{r,\xi}_\tau) = \psi(X^{r,\xi}_T) - \int_\tau^T h(X^{r,\xi}_s, u(s, X^{r,\xi}_s), \nabla u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s))\, ds$$
$$+ \int_\tau^T \nabla_x u(s, X^{r,\xi}_s)^* \sigma(X^{r,\xi}_s, u(s, X^{r,\xi}_s)))\, dW_s, \quad \mathbb{P}\text{-a.s.}$$

Thus, we have verified that $(X^{r,\xi}_s, Y^{r,\xi}_s, Z^{r,\xi}_s)$ solves system (4.10).

We still have to prove the claimed regularity properties of the processes $\{Y^{r,\xi}_s : s \in [r,T]\}$ and $\{Z^{r,\xi}_s : s \in [r,T]\}$. Thanks to the regularity of u we have, for every $r \in [0, T]$,

$$\mathbb{E} \sup_{s\in[r,T]} \|Y_s^{r,\xi}\|^2 = \mathbb{E} \sup_{s\in[r,T]} \|u(s, X_s^{r,\xi})\|^2 \leq \|u\|^2_{L^\infty([r,T]\times\mathbb{R}^n)} := \kappa_1 < \infty, \tag{4.12}$$

and recalling estimate (4.11) we get

$$\begin{aligned}\int_r^T \mathbb{E}\|Z_s^{r,\xi}\|^2\, ds &= \int_r^T \mathbb{E}\|\nabla u(s, X_s^{r,\xi})^* \sigma(X_s^{r,\xi}, u(s, X_s^{r,\xi}))\|^2\, ds \\ &\leq \kappa \int_r^T (T-s)^{-1+\theta-\rho}\, ds.\end{aligned}$$

This concludes the proof of the theorem. □

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