

**THE INITIAL-VALUE PROBLEM FOR  
THE NAVIER-STOKES EQUATIONS  
WITH A FREE SURFACE IN  $L^q$ -SOBOLEV SPACES**

HELMUT ABELS

Department of Mathematics, Darmstadt University of Technology  
Schloßgartenstraße 7, 64289 Darmstadt, Germany

(Submitted by: Y. Giga)

**Abstract.** We prove small-time existence of strong solutions of a free-boundary-value problem, which describes the motion of an incompressible viscous fluid occupying a semi-infinite domain bounded above by a free surface. This problem was studied by Beale [6] and others in  $L^2$ -Sobolev spaces. In contrast to the latter contribution we study solutions in  $L^q$ -Sobolev spaces for  $q > n$  in space dimension  $n \geq 2$ . This approach has the advantage that the regularity assumptions can be reduced in comparison to [6]. In order to solve the linearized system, we use the nonstationary *reduced* Stokes equations with a mixed boundary condition and the maximal regularity of the associated reduced Stokes operator.

1. INTRODUCTION

In the present contribution we consider the time-dependent flow of an incompressible, viscous fluid bounded above by a free surface and below by a fixed bottom under the influence of gravity, e.g., water in an infinite ocean. The effect of surface tension is neglected. This problem was studied by Beale [6] in the setting of anisotropic  $L^2$ -Sobolev spaces in  $\mathbb{R}^3$ . He proved short-time existence of strong solutions for arbitrary large initial data. The large-time existence was studied by Sylvester [16] and Tani and Tanaka [18]. Moreover, there are several works considering this problem with the effect of surface taken into account in three and two dimensions; cf. Beale [7], Beale and Nishiada [8], Tani [17], [18], and Allain [4]. As far as the author knows, there are no results in anisotropic  $L^q$ -Sobolev spaces for  $q \neq 2$ .

We show short-time existence of strong solutions of this free-boundary-value problem in the setting of anisotropic  $L^q$ -Sobolev spaces for  $q > n$  in space dimension  $n \geq 2$ . The approach is similar to Solonnikov [15], where

---

Accepted for publication: August 2004.

AMS Subject Classifications: 35Q30, 76D07, 35R35.

the motion of an isolated finite volume of viscous incompressible fluid is considered. It has the advantage that lower regularity of the data is needed since  $W_q^1$  is an algebra under pointwise multiplication if  $q > n$  in contrast to  $W_2^m$ , where  $m > \frac{n}{2}$  is needed. In order to work in  $L^q$ -Sobolev spaces we need the unique solvability in these spaces of the linearized system—the nonstationary *generalized* Stokes equations. These results are much harder to obtain than in the case  $q = 2$ , where Hilbert space methods can be used; cf. [6, 16]. Moreover, in contrast to [15] the domain we have to consider is unbounded, which causes additional difficulties.

The motion of the fluid is described by the solution of the Navier-Stokes equations

$$\partial_t v + (v \cdot \nabla)v - \Delta v + \nabla q = -g_0 e_n \quad \text{with } x \in \Omega(t), t \in (0, T), \quad (1.1)$$

$$\operatorname{div} v = 0 \quad \text{with } x \in \Omega(t), t \in (0, T), \quad (1.2)$$

$$T_1^+(v, q) = -P_0 \nu \quad \text{with } x \in \partial\Omega(t)^+, t \in (0, T), \quad (1.3)$$

$$v|_{\partial\Omega^-} = 0 \quad \text{on } \partial\Omega(0)^- \times (0, T), \quad (1.4)$$

$$v|_{t=0} = v_0 \quad \text{in } \Omega(0) \quad (1.5)$$

in a *layer-like domain*  $\Omega(t) = \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x', t) < x_n < \gamma^+(x', t)\}$  with  $\partial\Omega(t)^- = \partial\Omega(0)^-$ , where  $\partial\Omega(t)^\pm = \{(x', \gamma^\pm(x', t)) : x' \in \mathbb{R}^{n-1}\}$  and

$$T_1^+(u, p) = (\nu \cdot Su - p\nu)|_{\partial\Omega^+}, \quad Su = \nabla u + (\nabla u)^T.$$

Here  $v_0$  is a given initial velocity,  $\Omega(0)$  is the initial domain filled by the fluid,  $g_0 > 0$  is the constant due to the acceleration of gravity,  $e_n$  the  $n^{\text{th}}$  canonical unit vector,  $\nu$  the exterior normal, and  $P_0$  is the atmospheric pressure, which is assumed to be constant. Moreover, the velocity field  $v$  and the domain  $\Omega(t)$  have to satisfy a kinematic relation: Let  $X(\xi, t)$ ,  $t > 0$ , be the trajectory of the mass particle; i.e.,  $X(\xi, t)$  solves

$$\partial_t X(\xi, t) = v(X(\xi, t), t) \quad \text{for } t > 0, \quad X(\xi, 0) = \xi.$$

Then  $X(\Omega(0), t) = \Omega(t)$  for  $t > 0$ .

Before we pass to Lagrangian coordinates, we replace the actual pressure  $q$  by the *hydrostatic pressure*  $\tilde{q}(x, t) = q(x, t) - P_0 + g_0 x_n$ . Then the term  $-g_0 e_n$  disappears in (1.1) and (1.3) has to be replaced by

$$T_1^+(v, \tilde{q}) = -g_0 x_n \nu \quad \text{on } \partial\Omega(t)^+, t \in (0, T). \quad (1.6)$$

Now let  $u(\xi, t) = v(X(\xi, t), t)$  and  $p(\xi, t) = \tilde{q}(X(\xi, t), t)$  be the velocity and the pressure of the fluid in Lagrangian coordinates. Then

$$X(\xi, t) = X_u(\xi, t) := \xi + \int_0^t u(\xi, \tau) d\tau$$

and the system (1.1)–(1.5) is transformed to

$$\partial_t u - \Delta_u u + \nabla_u p = 0 \quad \text{in } \Omega_T, \quad (1.7)$$

$$\operatorname{div}_u u = 0 \quad \text{in } \Omega_T, \quad (1.8)$$

$$T_{1,u}^+(u, q) = -g_0 X_{u,n} \nu_u \quad \text{on } \partial\Omega_T^+, \quad (1.9)$$

$$u|_{\partial\Omega^-} = 0 \quad \text{on } \partial\Omega_T^-, \quad (1.10)$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \quad (1.11)$$

where, using  $A(u) = (D_\xi X_u)^{-T}(\xi, t)$ ,

$$\begin{aligned} \nabla_u &= A(u)\nabla, & \operatorname{div}_u v &= \nabla_u \cdot v = \operatorname{Tr}(A(u)\nabla v), \\ \Delta_u &= \operatorname{div}_u \nabla_u, & T_{1,u}^+(u, p) &= (\nu_u \cdot (S_u u - p\nu_u))|_{\partial\Omega^+}, \\ S_u v &= \nabla_u v + (\nabla_u v)^T, & \nu_u(\xi, t) &= \frac{A(u)\nu_\xi}{|(A(u)\nu_\xi|}, \end{aligned}$$

$\nu_\xi$  denotes the exterior norm at  $\xi \in \partial\Omega^+$ , and  $\Omega = \Omega(0)$ .

**Theorem 1.1.** *Let  $n \geq 2$ ,  $n < q < \infty$ , and let  $\Omega(0) \subset \mathbb{R}^n$  be an asymptotically flat layer with  $C^{1,1}$ -boundary (cf. Definition 2.1 below) such that  $\gamma^+ \in W_q^{1-\frac{1}{q}}(\mathbb{R}^{n-1})$ . Then for every  $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n$  with  $\operatorname{div} u_0 = 0$ ,  $(\nu_\xi \cdot S u_0)_\tau|_{\partial\Omega^+} = 0$  if  $q > 3$ , and*

$$\int_0^T \int_{\partial\Omega^+} \int_\Omega \frac{|(\nu_\xi \cdot S u_0)_\tau(y)|^q}{(|x' - y|^2 + t)^{1+n/2}} dy d\sigma(x') dt < \infty$$

if  $q = 3$  (and therefore  $n = 2$ ), there is a  $T > 0$  such that (1.7)–(1.11) have a unique solution

$$(u, p) \in W_q^{2,1}(\Omega_T)^n \times W_q^{1,0}(\Omega_T)$$

with  $p|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)$ .

Here  $\tau$  denotes the tangential components.

The proof is done with the aid of the Banach contraction-mapping principle using the unique solvability of the linearized system—the nonstationary *generalized* Stokes equations. In Section 3, the unique solvability of the latter system is reduced to the nonstationary reduced Stokes equations similarly to the approach in Grubb and Solonnikov [12] and other contributions by these authors. Then the unique solvability is proved using the maximal regularity of the associated reduced Stokes operator in *asymptotically flat layers* studied in [2] and [3]. The maximal regularity itself is a consequence of the result of Dore and Venni [9] and the existence of a bounded  $H_\infty$ -calculus

(cf. McIntosh [13]) of the reduced Stokes operator, which was proved in [3]. The latter proof is long and technical, using pseudodifferential operator techniques for boundary-value problems. Therefore it is not included in the present contribution. Finally, the proof of Theorem 1.1 is given in Section 4.

**Acknowledgment.** The present results are a part of the author's Ph.D. thesis, which was partially supported by the German exchange service (DAAD). The author expresses his gratitude to Professor G. Grubb and Professor R. Farwig for their advice during the preparation of the thesis.

## 2. PRELIMINARIES

**2.1. Notation and layer-like domains.** For  $s \in \mathbb{R}$  we denote by  $[s]$  the largest integer  $\leq s$ , set  $\{s\} := s - [s] \in [0, 1)$ , and define  $[s]_+ = \max\{s, 0\}$ .

If  $M \subseteq \mathbb{R}^n$  is measurable,  $L^q(M)$ ,  $1 \leq q \leq \infty$ , denotes the usual Lebesgue space and  $\|\cdot\|_q$  its norm. Moreover,  $L^q(M; X)$  denotes its vector-valued variant, where  $X$  is a Banach space. If  $f \in L^q(M)$  and  $g \in L^{q'}(M)$ , where  $\frac{1}{q} + \frac{1}{q'} = 1$ , then

$$(f, g)_M := \int_M f(x) \overline{g(x)} dx.$$

The Banach space of all functions  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  that are  $k$ -times differentiable and have Lipschitz-continuous  $k^{\text{th}}$  derivatives is denoted by  $C^{k,1}(\mathbb{R}^n)$ ,  $k \in \mathbb{N}_0$ . Moreover,  $C^\theta([0, T]; X)$ ,  $\theta \in (0, 1)$ ,  $T > 0$ , denotes the space of Hölder-continuous functions  $f: [0, T] \rightarrow X$ .

The dual of a topological vector space  $V$  is denoted by  $V'$ . If  $v \in V$  and  $v' \in V'$ , then  $\langle v, v' \rangle := v'(v)$  denotes the duality product. If  $A: V \rightarrow W$  is a continuous linear operator,  $A': W' \rightarrow V'$  denotes its adjoint.

Finally, if  $x \in \mathbb{R}^n$ ,  $n \geq 2$ , then we use the decomposition  $x = (x', x_n)$ , where  $x'$  denotes the first  $n - 1$  components.

**2.2. Function spaces on layer-like domains.** Let  $\Omega \subset \mathbb{R}^n$  be a domain. In the following  $W_q^s(\Omega)$ ,  $s \geq 0$ ,  $1 \leq q < \infty$ , denotes the usual Sobolev-Slobodeckij space normed by

$$\|u\|_{s,q}^q = \sum_{|\alpha| \leq s} \|D^\alpha u\|_q^q \quad \text{if } s \in \mathbb{N}_0,$$

$$\|u\|_{s,q}^q = \sum_{|\alpha| \leq [s]} \|D^\alpha u\|_q^q + \sum_{|\alpha| = [s]} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^q}{|x - y|^{n+q\{s\}}} dx dy \quad \text{if } s \notin \mathbb{N}_0.$$

Moreover,  $W_q^s(\Omega; X)$  denotes its vector-valued variant, where  $X$  is a Banach space. Finally,  $W_q^s(\partial\Omega)$  is defined in the same way as above with the Lebesgue measure replaced by the surface measure.

**Definition 2.1.** *Let  $n \geq 2$ . If  $\gamma = (\gamma^+, \gamma^-) \in C^{1,1}(\mathbb{R}^{n-1})^2$  with  $\gamma^+(x') - \gamma^-(x') \geq c > 0$  for all  $x' \in \mathbb{R}^{n-1}$ , then*

$$\Omega = \{(x', x_n) \in \mathbb{R}^n : \gamma^-(x') < x_n < \gamma^+(x')\}$$

*is called a layer-like domain with  $C^{1,1}$  boundary.*

*If additionally  $\lim_{|x'| \rightarrow \infty} \gamma^\pm = \gamma_\infty^\pm$  for some constants  $\gamma_\infty^\pm \in \mathbb{R}$  and*

$$\lim_{|x'| \rightarrow \infty} \nabla \gamma^\pm = \lim_{|x'| \rightarrow \infty} \nabla^2 \gamma^\pm = 0,$$

*then  $\Omega_\gamma$  is called an asymptotically flat layer. Finally,*

$$\partial\Omega^\pm = \{(x', \gamma^\pm(x')) : x' \in \mathbb{R}^{n-1}\}$$

*denotes the upper and lower boundary, respectively.*

If  $\Omega$  is a layer-like domain with  $C^{0,1}$  boundary, then

$${}^0W_q^1(\Omega) := \{u \in W_q^1(\Omega) : u|_{\partial\Omega^+} = 0\} \quad \text{and} \quad {}_0W_q^{-1}(\Omega) := ({}^0W_q^1(\Omega))'.$$

Now let  $\Omega \subset \mathbb{R}^n$  be a domain and  $T \in (0, \infty]$ . Then  $\Omega_T = \Omega \times (0, T)$  denotes the space-time cylinder and  $\partial\Omega_T = \partial\Omega \times (0, T)$  its boundary in the spatial coordinates. The anisotropic Sobolev-Slobodeckij space is defined as  $W_q^{2s,s}(\Omega_T) = L^q(0, T; W_q^{2s}(\Omega)) \cap W_q^s(0, T; L^q(\Omega))$ ,  $s \geq 0$ , normed by

$$\|u\|_{2s,s,q}^q = \|u\|_{L^q(0,T;W_q^{2s}(\Omega))}^q + \|u\|_{W_q^s(0,T;L^q(\Omega))}^q.$$

Moreover, we define  $W_q^{m,0}(\Omega_T) = L^q(0, T; W_q^m(\Omega))$ ,  $m \in \mathbb{N}$ , and denote by  $\|\cdot\|_{m,0,q}$  the corresponding norm.

If  $a \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+)$  and  $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)$  with  $q = 3$ , we say that  $a$  and  $u_0$  coincide at  $\partial\Omega^+ \times \{0\}$  if

$$I[a, u_0] := \int_0^T \int_{\partial\Omega} \int_{\Omega} \frac{|a(x, t) - u_0(y)|^3}{(|x - y|^2 + t)^{1+n/2}} dy d\sigma(x) dt < \infty.$$

Note that, if  $u_0 = 0$ , then

$$I[a, 0] \leq C \int_0^T \int_{\partial\Omega} |a(x, t)|^3 d\sigma(x) \frac{dt}{t}. \quad (2.1)$$

The following properties are well-known for bounded domains and are easily proved for layer-like domains.

**Lemma 2.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a layer-like domain with  $C^{1,1}$ -boundary and let  $q > \frac{3}{2}$ . Then*

- (1)  $\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_{W_q^{2-\frac{2}{q}}(\Omega)} \leq C \|u\|_{2,1,q}$ , where  $C = C(T) > 0$ .
- (2)  $\|\nabla u|_{\partial\Omega}\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q} \leq C \|u\|_{2,1,q}$ , where  $C = C(T) > 0$ .
- (3) For every  $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)$  with  $u_0|_{\partial\Omega^-} = 0$  there is a  $u \in W_q^{2,1}(\Omega_T)$  with  $u|_{t=0} = u_0$ ,  $u|_{\partial\Omega^-} = 0$ , and  $\|u\|_{2,1,q} \leq C \|u_0\|_{2-\frac{2}{q}, q}$ , where  $C$  can be chosen independently of  $T \in (0, \infty]$ .
- (4) For every  $a^+ \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n$  with  $a^+|_{t=0} = 0$  if  $q > 3$  and  $I[a^+, 0] < \infty$  if  $q = 3$  there is an  $A \in W_q^{2,1}(\Omega_T)^n$  with  $A|_{t=0} = 0$ ,  $A|_{\partial\Omega^-} = 0$ , and

$$(\nu \cdot SA)_\tau|_{\partial\Omega^+} = a_\tau^+, \quad \operatorname{div} A|_{\partial\Omega^+} = a_\nu^+,$$

where  $SA = \nabla A + (\nabla A)^T$ . Moreover,  $\|A\|_{2,1,q} \leq C \|a^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q}$  if  $q \neq 3$  and  $\|A\|_{2,1,q} \leq C \left( \|a^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q} + I[a^+, 0]^{\frac{1}{q}} \right)$  if  $q = 3$ , where  $C$  can be chosen independently of  $T \in (0, \infty]$ .

**Proof.** With the aid of a coordinate transformation  $F: \Omega \rightarrow \mathbb{R}^{n-1} \times (-1, 1)$  and simple cut-off functions the first three statements are easily reduced to the case of a half-space  $\mathbb{R}_+^n$ , which is well-known; cf. Grubb [11, Appendix].

In order to prove (4), let  $A \in W_q^{2,1}(\Omega_T)^n$  with  $A|_{t=0} = 0$ ,  $A|_{\partial\Omega} = 0$ , and  $\partial_\nu A|_{\partial\Omega^+} = a^+$  such that  $\|A\|_{2,1,q} \leq C \|a^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q}$  with an additional term  $CI[a^+, 0]^{\frac{1}{q}}$  if  $q = 3$ . (As before, the existence of  $A$  can be reduced to a corresponding statement in  $\mathbb{R}_+^n$ .) Then

$$\begin{aligned} (\nu \cdot SA)_\tau|_{\partial\Omega^+} &= (\nabla_\tau A_\nu + \partial_\nu A_\tau)|_{\partial\Omega^+} = a_\tau^+, \\ \operatorname{div} A|_{\partial\Omega^+} &= (\operatorname{div}_\tau A_\tau + \partial_\nu A_\nu)|_{\partial\Omega^+} = a_\nu^+. \end{aligned}$$

The constant  $C$  can be chosen independently of  $T$  since we can extend  $a^+$  to  $\tilde{a}^+ \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+ \times (0, \infty))^n$  such that  $\|\tilde{a}^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q} \leq C \|a^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q}$ , where  $C$  does not depend on  $T$ , and restrict the corresponding  $\tilde{A} \in W_q^{2,1}(\Omega \times (0, \infty))^n$  to  $(0, T)$  afterwards. The latter extension to  $(0, \infty)$  can be done by first extending  $a^+$  in an even way around  $t = T$  to a function defined on  $(0, 2T)$  and then extending by zero, which yields an  $\tilde{a}^+ \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega^+ \times (0, \infty))^n$  since  $\tilde{a}^+|_{t=2T} = a^+|_{t=0} = 0$  if  $q > 3$  and  $\tilde{a}^+$  coincides with 0 at  $\partial\Omega^+ \times \{2T\}$  if  $q = 3$ .  $\square$

A simple but useful vector-valued variant of the Sobolev embedding theorem is given in the next lemma.

**Lemma 2.2.** *Let  $1 < q < \infty$ ,  $0 < T < \infty$ , and  $X$  be a Banach space. Then  $W_q^1(0, T; X) \hookrightarrow C^{\frac{1}{q}}([0, T]; X)$  is a continuous embedding.*

**Proof.** The lemma is a simple consequence of the fact that  $u: (0, T) \rightarrow X$  is absolutely continuous (cf. Amann [5, Chapter III, Theorem 1.2.2]) and the Hölder inequality.  $\square$

**2.3. Weak mixed Dirichlet-Neumann problem.** If  $u \in L^q(\Omega)^n$  with  $\operatorname{div} u \in L_{loc}^q(\overline{\Omega}) \cap {}_0W_q^{-1}(\Omega)$ , then we define the trace  $\nu \cdot u|_{\partial\Omega^-} \in W_q^{-\frac{1}{q}}(\partial\Omega^-)$  as

$$\langle \nu \cdot u|_{\partial\Omega^-}, v \rangle = (u, \nabla V) + (\operatorname{div} u, V), \quad (2.2)$$

where  $v \in W_{q'}^{1-\frac{1}{q'}}(\partial\Omega^-)$  and  $V \in {}_0W_{q'}^1(\Omega)$  with  $V|_{\partial\Omega^-} = v$ . The definition does not depend on the choice of  $V$ . Moreover,

$$\|\nu \cdot u|_{\partial\Omega^-}\|_{-\frac{1}{q}, q} \leq C \left( \|u\|_q + \|\operatorname{div} u\|_{{}_0W_q^{-1}(\Omega)} \right).$$

Now let  $f \in L^q(\Omega)^n$ . Then we consider the weak mixed Dirichlet-Neumann problem

$$\Delta u = \operatorname{div} f \quad \text{in } \Omega, \quad (2.3)$$

$$u|_{\partial\Omega^+} = 0 \quad \text{on } \partial\Omega^+, \quad (2.4)$$

$$\partial_\nu u|_{\partial\Omega^-} = \nu \cdot f|_{\partial\Omega^-} \quad \text{on } \partial\Omega^-. \quad (2.5)$$

Here  $\partial_\nu u|_{\partial\Omega^-} = \nu \cdot f|_{\partial\Omega^-}$  is understood as  $\nu \cdot (\nabla u - f)|_{\partial\Omega^-} = 0$ .

**Lemma 2.3.** *Let  $1 < q < \infty$  and  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ . Then for every  $f \in L^q(\Omega)^n$  there is a unique solution  $u \in W_q^1(\Omega)$  of (2.3)–(2.5) which satisfies  $\|u\|_{1, q} \leq C\|f\|_q$ . Moreover,  $u$  is uniquely determined by*

$$(\nabla u, \nabla v) = (f, \nabla v) \quad \text{for all } v \in {}_0W_{q'}^1(\Omega). \quad (2.6)$$

The lemma is a consequence of [2, Theorem 3.6]. Finally, we note that by the Hahn-Banach theorem every  $F \in {}_0W_q^{-1}(\Omega)$  has a representation  $\langle F, v \rangle = (f, \nabla v)$  for all  $v \in {}_0W_{q'}^1(\Omega)$  with  $f \in L^q(\Omega)^n$  and  $\|f\|_q \leq C\|F\|_{{}_0W_q^{-1}}$ .

## 3. NONSTATIONARY STOKES EQUATIONS

In order to solve the free-boundary-value problem, we need the solvability of the linearized problem, i.e., the nonstationary generalized Stokes equations

$$\partial_t u - \Delta u + \nabla p = f \quad \text{in } \Omega_T, \quad (3.1)$$

$$\operatorname{div} u = g \quad \text{in } \Omega_T, \quad (3.2)$$

$$T_1^+(u, p) = a^+ \quad \text{on } \partial\Omega_T^+, \quad (3.3)$$

$$u|_{\partial\Omega^-} = 0 \quad \text{on } \partial\Omega_T^-, \quad (3.4)$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega. \quad (3.5)$$

The solvability of this system is considered for data

$$f \in L^q(\Omega_T)^n, g \in W_q^{1,0}(\Omega_T), a^+ \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n, u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n,$$

$n \geq 2$ . We restrict ourselves to the case  $q > \frac{3}{2}$ , which is sufficient for our application. Moreover, we assume that the data satisfy the following *compatibility conditions*:

- (1) The vector-valued distributional derivative  $\partial_t g$  in  $\mathcal{D}'(0, T; {}_0W_q^{-1}(\Omega))$  belongs to  $L^q(0, T; {}_0W_q^{-1}(\Omega))$ , and  $g|_{t=0} = \operatorname{div} u_0$  in  ${}_0W_q^{-1}(\Omega)$ .
- (2)  $a_\tau^+|_{t=0} = (\nu \cdot Su_0)_\tau|_{\partial\Omega^+}$  if  $q > 3$  and  $I[a_\tau^+, (\nu \cdot Su_0)_\tau] < \infty$  if  $q = 3$ . (In the latter case  $\nu \in C^{0,1}(\partial\Omega^+)$  has to be extended suitably to  $\Omega$ .)
- (3)  $u_0|_{\partial\Omega^-} = 0$ .

Here  $\mathcal{D}'(0, T; {}_0W_q^{-1}(\Omega)) := \mathcal{L}(\mathcal{D}(0, T), {}_0W_q^{-1}(\Omega))$  (cf. [5, Chapter III, Section 1.1]) and  $\mathcal{D}(0, T)$  is the set of all smooth  $f: (0, T) \rightarrow \mathbb{R}$  with  $\operatorname{supp} f \subseteq (0, T)$ .

In our application  $g = \operatorname{div} R$  with  $R \in W_q^{2,1}(\Omega_T)^n$ ,  $\nu \cdot R|_{\partial\Omega^-} = 0$ , and  $\operatorname{div} R|_{t=0} = 0$ . Then

$$\int_0^T (\partial_t R(\cdot, t), \nabla v)_\Omega \varphi(t) dt = \int_0^T (g(\cdot, t), v)_\Omega \partial_t \varphi(t) dt$$

for all  $\varphi \in \mathcal{D}(0, T)$  and  $v \in {}_0W_q^1(\Omega)$ . Hence  $F(t) \in {}_0W_q^{-1}(\Omega)$ ,  $t \in (0, T)$ , defined by  $\langle F(t), v \rangle = -(\partial_t R(\cdot, t), \nabla v)_\Omega$ , is the  ${}_0W_q^{-1}(\Omega)$ -valued distributional derivative  $\partial_t g$ . Thus  $\partial_t g \in L^q(0, T; {}_0W_q^{-1}(\Omega))$  and  $\|\partial_t g\|_{L^q(0, T; {}_0W_q^{-1}(\Omega))} \leq \|\partial_t R\|_{L^q(\Omega_T)}$ . Conversely, the following lemma holds.

**Lemma 3.1.** *Let  $1 < q < \infty$  and  $0 < T \leq \infty$ , and let  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 2$ , be an asymptotically flat layer with  $C^{1,1}$ -boundary. Then every  $g \in W_q^{1,0}(\Omega_T)$  with  $\partial_t g \in L^q(0, T; {}_0W_q^{-1}(\Omega))$  and  $g|_{t=0} = 0$  possesses a representation  $g = \operatorname{div} R$*



where  $R \in W_q^1(\Omega_T)^n$  with  $R|_{t=0} = 0$ ,  $\nu \cdot R|_{\partial\Omega^-} = 0$ , and

$$\|R\|_{W_q^1(\Omega_T)} \leq C \left( \|g\|_{1,0,q} + \|\partial_t g\|_{L^q(0,T; {}^0W_q^{-1}(\Omega))} \right),$$

where  $C$  does not depend on  $T$ .

**Proof.** We define  $R = \nabla p$ , where  $p$  is the solution of

$$\begin{aligned} \Delta p(\cdot, t) &= g(\cdot, t) && \text{in } \Omega, \\ p|_{\partial\Omega^+} &= 0 && \text{on } \partial\Omega^+, \\ \partial_\nu p|_{\partial\Omega^-} &= 0 && \text{on } \partial\Omega^-. \end{aligned}$$

Hence  $\|p\|_{L^q(0,T; W_q^2(\Omega))} \leq C\|g\|_{1,0,q}$  because of [1, Lemma 3.3], which can be extended to arbitrary asymptotically flat layers with  $C^{1,1}$ -boundary using the same method as in [2, Appendix]. Because of the unique solvability of (2.3)–(2.5),  $\partial_t g \in L^q(0, T, {}^0W_q^{-1}(\Omega))$  has a representation

$$\langle \partial_t g(\cdot, t), v \rangle_\Omega = (\nabla \tilde{p}, \nabla v) \quad \text{for all } v \in {}^0W_q^1(\Omega)$$

for almost all  $t \in (0, T)$ , where  $\tilde{p} \in L^q(0, T; {}^0W_q^1(\Omega))$  with

$$\|\nabla \tilde{p}\|_{q, \Omega_T} \leq C \|\partial_t g\|_{L^q(0,T; {}^0W_q^{-1}(\Omega))}.$$

Therefore, we conclude  $\partial_t \nabla p \in L^q(\Omega_T)$  as follows:  $g(\cdot, t)$ ,  $t \in (0, T)$ , is an absolutely continuous  ${}^0W_q^{-1}(\Omega)$ -valued function (cf. [5, Chapter III, Theorem 1.2.2]) and

$$(g(\cdot, t), v)_\Omega = \int_0^t \langle \partial_t g(\cdot, s), v \rangle_\Omega ds = (\nabla \int_0^t \tilde{p}(\cdot, s) ds, \nabla v)$$

for all  $v \in {}^0W_q^1(\Omega)$ . On the other hand,

$$(g(\cdot, t), v)_\Omega = -(\nabla p, \nabla v) \quad \text{for all } v \in {}^0W_q^1(\Omega)$$

and almost all  $t \in (0, T)$ . Because of Lemma 2.3, we conclude

$$\nabla p(\cdot, t) = -\nabla \int_0^t \tilde{p}(\cdot, s) ds \quad \text{for a.a. } t \in (0, T).$$

Thus,  $R = \nabla p \in W_q^1(\Omega_T)$  with  $\nabla p|_{t=0} = 0$ ,  $\partial_\nu p|_{\partial\Omega^-} = 0$ ,  $\|\nabla p\|_{1,q,\Omega_T} \leq C\|g\|_{1,0,q}$ , and  $\|\partial_t \nabla p\|_{q,\Omega_T} \leq C\|\partial_t g\|_{L^q(0,T; {}^0W_q^{-1}(\Omega))}$ . Finally, it is easy to check that all constants in the proof do not depend on  $T$ .  $\square$

As in the case of the generalized Stokes resolvent equations (cf. [1]), (3.1)–(3.5) can (at least formally) be reduced to the *nonstationary reduced Stokes equations*

$$\partial_t u - \Delta u + G_{10}u = f_r \quad \text{in } \Omega_T, \quad (3.6)$$

$$T_1'^+ u = a_r \quad \text{on } \partial\Omega_T^+, \quad (3.7)$$

$$u|_{\partial\Omega^-} = 0 \quad \text{on } \partial\Omega_T^-, \quad (3.8)$$

$$u|_{t=0} = u_0 \quad \text{in } \Omega, \quad (3.9)$$

where  $(T_1'^+ u)_\tau = (\nu \cdot Su)_\tau$  and  $(T_1'^+ u)_\nu = \operatorname{div} u|_{\partial\Omega^+}$  and

$$G_{10}u = \nabla K_{01}(2\partial_\nu u|_{\partial\Omega^+}, \nu \cdot (\Delta - \nabla \operatorname{div})u|_{\partial\Omega^-}).$$

Here  $K_{01}$  denotes the Poisson operator to (2.3)–(2.5); i.e.,  $\Delta K_{01} = 0$  and  $K_{01}a|_{\partial\Omega^+} = a|_{\partial\Omega^+}$ ,  $\partial_\nu K_{01}a|_{\partial\Omega^-} = a|_{\partial\Omega^-}$ .

It is one of the main results of [3] that  $A_{10} := -\Delta + G_{10}$  with domain

$$\mathcal{D}(A_{10}) = \left\{ u \in W_q^2(\Omega)^n : T_1'^+ u = 0, u|_{\partial\Omega^-} = 0 \right\}$$

admits a bounded  $H_\infty$ -calculus in the sense of McIntosh [13]. In particular,  $A_{10}$  possesses bounded imaginary powers, and the result of Dore and Venni [9, Theorem 3.2], respectively its extension by Giga and Sohr [10, Theorem 2.1], implies that  $A_{10}$  has *maximal regularity*; i.e., for  $f \in L^q(0, T; L^q(\Omega)^n)$ ,  $0 < T \leq \infty$ , there is a unique solution

$$u \in W_q^1(0, T; L^q(\Omega)^n) \cap L^q(0, T; \mathcal{D}(A_{10}))$$

of the abstract ordinary differential equation

$$u'(t) + A_{10}u(t) = f(t), \quad t > 0, \quad u(0) = 0.$$

Combining this result with Lemma 2.1, we obtain

**Theorem 3.1.** *Let  $\frac{3}{2} < q < \infty$  and  $0 < T \leq \infty$ , and let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be an asymptotically flat layer with  $C^{1,1}$ -boundary. Moreover, let  $(f_r, a_r^+, u_0) \in L^q(\Omega_T)^n \times W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\Omega_T)^n \times W_q^{2-\frac{2}{q}}(\Omega)^n$  satisfy the compatibility conditions  $u_0|_{\partial\Omega^-} = 0$ ,  $T_1'^+ u_0 = a^+|_{t=0}$  if  $q > 3$ , and  $I[a_r^+, (\nu \cdot Su_0)_\tau] < \infty$ ,  $I[a_\nu^+, \operatorname{div} u_0] < \infty$  if  $q = 3$ . Then there is a unique solution  $u \in W_q^{2,1}(\Omega_T)^n$  of (3.6)–(3.9) which satisfies*

$$\|u\|_{2,1,q} \leq C \left( \|f_r\|_q + \|a_r^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}), q} + \|u_0\|_{2-\frac{2}{q}, q} \right)$$

*with an additional  $I[a_r^+, (\nu \cdot Su_0)_\tau]^{\frac{1}{q}} + I[a_\nu^+, \operatorname{div} u_0]^{\frac{1}{q}}$  on the right-hand side if  $q = 3$ . The constant  $C$  can be chosen independently of  $T \in (0, \infty]$ .*

Using Theorem 3.1, we are able to prove the main result of this section.

**Theorem 3.2.** *Let  $n \geq 2$ ,  $\frac{3}{2} < q < \infty$ , and  $0 < T < \infty$ . Then for every  $f \in L^q(\Omega_T)^n$  and  $g \in W_q^{1,0}(\Omega_T)$  with  $\partial_t g \in L^q(0, T; {}^0W_q^{-1}(\Omega))$ ,  $g|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n$ ,  $a^+ \in W_q^{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q})}(\partial\Omega_T^+)^n$ , and  $u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n$  satisfying the compatibility condition above there is a unique solution  $(u, p) \in W_q^{2,1}(\Omega_T)^n \times W_q^{1,0}(\Omega_T)$  of (3.1)–(3.5). Moreover,*

$$\begin{aligned} & \|u\|_{2,1,q} + \|p\|_{1,0,q} + \|p|_{\partial\Omega^+}\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}),q} \\ & \leq C \left( \|(f, \nabla g)\|_q + \|\partial_t g\|_{-1,0,q} + \|(g|_{\partial\Omega^+}, a^+)\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}),q} + \|u_0\|_{2-\frac{2}{q},q} \right), \end{aligned} \quad (3.10)$$

with an additional  $I[a_r^+, (\nu \cdot Su_0)_\tau]^{\frac{1}{q}}$  on the right-hand side if  $q = 3$ , where  $\|\cdot\|_{-1,0,q} := \|\cdot\|_{L^q(0,T; {}^0W_q^{-1})}$ . The constant  $C$  can be chosen independently of  $T \in (0, \infty)$ .

**Proof.** For every  $t \in (0, T)$  let  $p_2(\cdot, t) \in W_q^1(\Omega)$  be the solution of

$$(\nabla p_2(\cdot, t), \nabla v) = (f + \nabla g - \partial_t R, \nabla v) \quad \text{for all } v \in {}^0W_q^1(\Omega)$$

with  $p_2|_{\partial\Omega^+} = a_\nu^+$  (cf. Lemma 2.3) where  $\text{div } R$  is a representation of  $g$  due to Lemma 3.1. Now we define  $f_r = f - \nabla p_2$ . Then

$$\|f_r\|_q \leq C \left( \|(f, \nabla g, \partial_t R)\|_q + \|a_\nu^+\|_{1-\frac{1}{q}, \frac{1}{2}(1-\frac{1}{q}),q} \right)$$

with  $C$  independent of  $T$ . Moreover, let  $(a_r^+)_\tau = a_\tau^+$  and  $(a_r^+)_\nu = g|_{\partial\Omega^+}$ .

Let  $u \in W_q^{2,1}(\Omega_T)^n$  be the solution of the reduced Stokes equations with right-hand side  $(f_r, a_r^+)$ . Then  $(u, p)$  with  $\nabla p = G_{10}u + \nabla p_2$  is a solution of the generalized Stokes equations, where it only remains to prove that  $\text{div } u = g$ . Testing (3.6) with  $\nabla v$ ,  $v \in L^{q'}(0, T; {}^0W_q^1(\Omega))$ , and using

$$(G_{10}u, \nabla v)_{\Omega_T} = \langle \nu \cdot (\Delta - \nabla \text{div})u|_{\partial\Omega^-}, v|_{\partial\Omega^-} \rangle_{\partial\Omega_T} = ((\Delta - \nabla \text{div})u, \nabla v)_{\Omega_T}$$

because of (2.2), we conclude

$$(\partial_t(u - R), \nabla v)_{\Omega_T} + (\nabla(\text{div } u - \text{div } R), \nabla v)_{\Omega_T} = 0 \quad (3.11)$$

for all  $v \in L^{q'}(0, T; {}^0W_q^1(\Omega))$ . Moreover,  $\text{div}(u - R)|_{t=0} = 0$  because of the compatibility conditions.

Now let  $\tilde{f} \in L^{q'}(\Omega_T)^n$  be arbitrary and let  $\tilde{u} \in W_q^{2,1}(\Omega_T)^n$  be a solution of the reduced Stokes equations with right-hand side  $\tilde{f}$ ,  $a^+ = 0$ , and  $u_0 = 0$ . Then a calculation similar to that above yields

$$(\partial_t \tilde{u}, \nabla w)_{\Omega_T} + (\nabla \text{div } \tilde{u}, \nabla w)_{\Omega_T} = (\tilde{f}, \nabla w)_{\Omega_T}$$

for all  $w \in L^q(0, T; {}^0W_q^1(\Omega))$ . If we set  $v(x, t) = \operatorname{div} \tilde{u}(x, T - t)$  in (3.11), we get

$$\begin{aligned} 0 &= (\partial_t(u - R), \nabla \operatorname{div} \tilde{u}(x, T - \cdot))_{\Omega_T} + (\nabla \operatorname{div}(u - R), \nabla \operatorname{div} \tilde{u}(x, T - \cdot))_{\Omega_T} \\ &= (\nabla \operatorname{div}(u - R), (\partial_t \tilde{u})(\cdot, T - \cdot))_{\Omega_T} + (\nabla \operatorname{div}(u - R), (\nabla \operatorname{div} \tilde{u})(\cdot, T - \cdot))_{\Omega_T} \\ &= (\tilde{f}(\cdot, T - \cdot), \nabla \operatorname{div}(u - R))_{\Omega_T}, \end{aligned} \quad (3.12)$$

where we have used

**Proposition 3.1.** *Let  $1 < q < \infty$  and let  $u \in W_q^{2,1}(\Omega_T)^n$  and  $v \in W_{q'}^{2,1}(\Omega_T)^n$  such that  $\nu \cdot u|_{\partial\Omega_T^-} = \nu \cdot v|_{\partial\Omega_T^-} = 0$ ,  $\operatorname{div} u|_{\partial\Omega_T^+} = \operatorname{div} v|_{\partial\Omega_T^+} = 0$ , and  $u|_{t=0} = v|_{t=T} = 0$ . Then*

$$(\partial_t u, \nabla \operatorname{div} v)_{\Omega_T} = -(\nabla \operatorname{div} v, \partial_t v)_{\Omega_T}.$$

**Proof.** The lemma is simply proved by approximating  $\partial_t u$  by difference quotients and using  $u \in C^{\frac{1}{q'}}([0, T]; L^q(\Omega))^n$ ,  $v \in C^{\frac{1}{q}}([0, T]; L^q(\Omega))^n$  due to Lemma 2.2.  $\square$

Since for every  $F \in L^{q'}(0, T; {}^0W_{q'}^{-1}(\Omega)) = (L^q(0, T; {}^0W_q^1(\Omega)))'$  there is an  $\tilde{f} \in L^{q'}(\Omega_T)^n$  such that

$$\int_0^T \langle F(\cdot, t), v(\cdot, t) \rangle_{{}^0W_{q'}^{-1}(\Omega)} dt = (\tilde{f}(\cdot, T - \cdot), \nabla v)_{\Omega_T}, \quad v \in L^q(0, T; {}^0W_q^1(\Omega)),$$

(3.12) and the Hahn-Banach theorem imply  $\operatorname{div}(u - R) = 0$ .

In order to prove uniqueness, let  $(u, p)$  be a solution with right-hand side 0. Then  $\nabla p = G_{10}u$  since  $\Delta p = 0$ ,  $p|_{\partial\Omega^+} = 2\partial_\nu u_\nu$ , and  $\partial_\nu p|_{\partial\Omega^-} = \nu \cdot (\Delta - \nabla \operatorname{div})u|_{\partial\Omega^-}$  by (3.1) and (3.3). Hence  $u$  solves (3.6)–(3.9) with  $f_r = a_r^+ = u_0 = 0$ , and therefore  $u = \nabla p = 0$  due to Theorem 3.1.  $\square$

#### 4. PROOF OF THEOREM 1.1

For simplicity we multiply (1.9) by  $|A(u)\nu|$  and replace it by

$$\tilde{T}_{1,u}^+(u, p) = ((A(u)\nu) \cdot S_u u - pA(u)\nu)|_{\partial\Omega^+} = -g_0 X_{u,n} A(u)\nu \quad (4.1)$$

with  $S_u v = \nabla_u v + (\nabla_u v)^T$ .

We can formulate the initial-/boundary-value problem as an abstract equation:

$$Lv = G(v) + h \quad \Leftrightarrow v = L^{-1}G(v) + L^{-1}h, \quad (4.2)$$

where  $v = (u, p) \in X_T$ ,  $h = (0, 0, -g_0 \xi_n \nu, u_0) \in Y_T$ , and

$$X_T = \{(u, p) : u \in W_q^{2,1}(\Omega_T)^n, u|_{\partial\Omega^-} = 0, \operatorname{div} u|_{t=0} = 0, p \in W_q^{1,0}(\Omega_T)\},$$

$$\begin{aligned}
& p|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q}, \frac{1}{2}-\frac{1}{2q}}(\partial\Omega_T^+), (\nu \cdot Su|_{t=0})_\tau|_{\partial\Omega^+} = 0\}, \\
Y_T &= \{(f, g, a^+, u_0) : f \in L^q(\Omega_T)^n, g \in W_q^{1,0}(\Omega_T), \\
& \partial_t g \in L^q(0, T; {}_0W_q^{-1}(\Omega)), g|_{\partial\Omega^+} \in W_q^{1-\frac{1}{q}, \frac{1}{2}-\frac{1}{2q}}(\partial\Omega_T^+), g|_{t=0} = 0, \\
& a^+ \in W_q^{1-\frac{1}{q}, \frac{1}{2}-\frac{1}{2q}}(\partial\Omega_T^+)^n, a_\tau^+|_{t=0} = 0, u_0 \in W_q^{2-\frac{2}{q}}(\Omega)^n, \\
& \operatorname{div} u_0 = 0, u_0|_{\partial\Omega^-} = 0, (\nu \cdot Su_0)_\tau|_{\partial\Omega^+} = 0\}, \\
L(u, p) &= ((\partial_t - \Delta)u + \nabla p, \operatorname{div} u, T_1^+(u, p), u|_{t=0}), \\
G(u, p) &= ((\Delta_u - \Delta)u + (\nabla - \nabla_u)p, (\operatorname{div} - \operatorname{div}_u)u, (T_1^+ - \tilde{T}_{1,u}^+)(u, p) - \\
& g_0(X_{u,n}A(u)\nu - \xi_n\nu), 0),
\end{aligned}$$

in the case  $q > 3$ . If  $q = 3$ , we replace the conditions  $(\nu \cdot Su|_{t=0})_\tau|_{\partial\Omega^+} = 0$ ,  $(\nu \cdot Su_0)_\tau|_{\partial\Omega^+}$ , and  $a_\tau^+|_{t=0} = 0$  by  $I[0, (\nu \cdot Su|_{t=0})_\tau] < \infty$ ,  $I[0, (\nu \cdot Su_0)_\tau] < \infty$ , and  $I[a_\tau^+, 0] < \infty$ , respectively. In the case  $q < 3$ , we skip the latter conditions. The associated norms are defined in the obvious way with additional terms  $I[0, (\nu \cdot Su|_{t=0})_\tau]^{\frac{1}{q}}$ ,  $I[0, (\nu \cdot Su_0)_\tau]^{\frac{1}{q}}$ , and  $I[a_\tau^+, 0]^{\frac{1}{q}}$  if  $q = 3$ .

In order to apply the Banach contraction-mapping principle to (4.2), we have to estimate  $G(v) - G(w)$  for all  $v, w \in \overline{B_R(0)}$  for some suitable  $R > 0$ .

First of all, we note that for  $s \in (0, 1)$

$$\|f\|_{W_q^s(0,T;X)}^q = \|f\|_{L^q(0,T;X)}^q + \int_0^T \int_0^t \frac{\|\Delta_h f(t)\|_X^q}{h^{1+qs}} dh dt,$$

where  $\Delta_h f(t) = f(t) - f(t-h)$ .

**Lemma 4.1.** *Let  $n \geq 2$ ,  $q > n$ , and  $R > 0$ . Moreover, let  $F(u) = X_u$  and  $Z = W_q^2(\Omega)^n$  or  $F(u) = A(u)$  and  $Z = W_q^1(\Omega)^{n \times n}$ . Then there is a  $T_0 = T_0(R) > 0$  and a constant  $C > 0$  such that for all  $0 < T \leq T_0$*

$$\sup_{0 \leq t \leq T} \|F(u) - F(v)\|_Z \leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q}, \quad (4.3)$$

$$\sup_{0 \leq t \leq T} \left( \int_0^t \frac{\|\Delta_h(F(u) - F(v))(\cdot, t)\|_Z^q}{h^{1+\frac{q}{2q'}}} dh \right)^{\frac{1}{q}} \leq CT^{\frac{1}{2q'}} \|u - v\|_{2,1,q}, \quad (4.4)$$

$$\left( \int_0^T \int_0^t \frac{\|\Delta_h(F(u) - F(v))(\cdot, t)\|_Z^q}{h^{1+\frac{q}{2q'}}} dh dt \right)^{\frac{1}{q}} \leq CT^{\frac{1}{q} + \frac{1}{2q'}} \|u - v\|_{2,1,q} \quad (4.5)$$

for all  $u, v \in W_q^{2,1}(\Omega_T)^n$  with  $\|u\|_{2,1,q}, \|v\|_{2,1,q} \leq R$ .

**Proof.** First of all, we observe that

$$\|F(u) - F(v)\|_{W_q^1(0,T;Z)} \leq C\|u - v\|_{2,1,q} \quad (4.6)$$

for all  $0 < T \leq T_0$  if  $T_0 > 0$  is chosen sufficiently small.

If  $F(u) = X_u$ , (4.6) is a consequence of

$$X_u(\xi, t) - X_v(\xi, t) = \int_0^t (u(\xi, \tau) - v(\xi, \tau)) d\tau \in W_q^1(0, T; W_q^2(\Omega)^n).$$

Since

$$\int_0^t \nabla u(\cdot, \tau) d\tau \in W_q^1(0, T; W_q^1(\Omega)^{n \times n}) \hookrightarrow C^{\frac{1}{q'}}(0, T; W_q^1(\Omega)^{n \times n}),$$

we have

$$\sup_{0 \leq t \leq T} \left\| \int_0^t \nabla u(\cdot, \tau) d\tau \right\|_{\infty} \leq C \sup_{0 \leq t \leq T} \left\| \int_0^t \nabla u(\cdot, \tau) d\tau \right\|_{1,q} \leq CT^{\frac{1}{q'}} R$$

for all  $u \in \overline{B_R(0)} \subset W_q^{2,1}(\Omega_T)$ . Thus we can find a  $T_0 > 0$  sufficiently small such that  $A(u) = \left( I + \int_0^t \nabla u(x, \tau) d\tau \right)^{-T}$  exists for all  $u \in \overline{B_R(0)}$ . Since matrix inversion is a smooth mapping on  $\text{GL}(n)$ ,  $A(u)$  depends smoothly on  $M = \int_0^t \nabla u(x, \tau) d\tau$ . Therefore,  $A(u) - A(v) \in W_q^1(0, T; W_q^1(\Omega)^n)$  and (4.6) holds for  $F(u) = A(u)$ .

Now (4.3) is a consequence of Lemma 2.2, (4.6), and  $(F(u) - F(v))|_{t=0} = 0$ . Moreover,

$$\|\Delta_h(F(u) - F(v))(\cdot, t)\|_Z \leq Ch^{\frac{1}{q'}} \|u - v\|_{2,1,q},$$

and therefore

$$\begin{aligned} & \sup_{0 < t \leq T} \left( \int_0^t \frac{\|\Delta_h(F(u) - F(v))(\cdot, t)\|_Z^q}{h^{1 + \frac{q}{2q'}}} dh \right)^{\frac{1}{q}} \\ & \leq C \sup_{0 < t \leq T} \left( \int_0^t h^{\frac{q}{q'} - 1 - \frac{q}{2q'}} dh \right)^{\frac{1}{q}} \|u - v\|_{2,1,q} \leq CT^{\frac{1}{2q'}} \|u - v\|_{2,1,q}. \end{aligned}$$

Because of  $\|f\|_{L^q(0,T)} \leq T^{\frac{1}{q}} \|f\|_{L^\infty(0,T)}$ , (4.5) is a consequence of (4.4).  $\square$

**Lemma 4.2.** *Let  $1 < q < \infty$ ,  $T_0 > 0$ , and  $\Omega \subseteq \mathbb{R}^n$ ,  $n \geq 2$ , be a layer-like domain with  $C^{1,1}$  boundary. Then*

$$\|\nabla u\|_{W_q^{\frac{1}{2}-\varepsilon}(0,T;L^q(\Omega))} \leq C_{\varepsilon, T_0} \|u\|_{2,1,q}$$

for every  $u \in W_q^{2,1}(\Omega_T)$ ,  $\varepsilon \in (0, \frac{1}{2})$ , and  $0 < T \leq T_0$ .

**Proof.** Because of  $\|\nabla u(\cdot, t)\|_{q, \Omega} \leq C\|u(\cdot, t)\|_{\frac{1}{2}, q, \Omega}^{\frac{1}{2}}\|u(\cdot, t)\|_{2, q, \Omega}^{\frac{1}{2}}$ ,

$$\begin{aligned} & \left( \int_0^T \int_0^t \frac{\|(\Delta_h \nabla u)(\cdot, t)\|_{q, \Omega}^q}{h^{1+(\frac{1}{2}-\varepsilon)q}} dh dt \right)^{\frac{1}{q}} \\ & \leq \left( \int_0^T \int_0^t \frac{\|\Delta_h u(\cdot, t)\|_{q, \Omega}^q}{h^{1+(1-\varepsilon)q}} dh dt \right)^{\frac{1}{2q}} \left( \int_0^T \int_0^t \frac{\|\Delta_h u(\cdot, t)\|_{2, q, \Omega}^q}{h^{1-\varepsilon q}} dh dt \right)^{\frac{1}{2q}}. \end{aligned}$$

Since  $\|\Delta_h u(\cdot, t)\|_{2, q, \Omega} \leq 2\|u(\cdot, t)\|_{2, q, \Omega}$ , the second term can be estimated by  $C_{\varepsilon, T_0}\|u\|_{2, 1, q}^{\frac{1}{2}}$ . For the first term we use  $\|\Delta_h u(\cdot, t)\|_q^q \leq h^{\frac{q}{t}} \int_{t-h}^t \|\partial_t u(\cdot, \tau)\|_q^q d\tau$ , and obtain

$$\begin{aligned} \int_0^T \int_0^t \frac{\|\Delta_h u(\cdot, t)\|_q^q}{h^{1+(1-\varepsilon)q}} dh dt & \leq \int_0^T \int_h^T \int_{t-h}^t \|\partial_t u(\cdot, \tau)\|_q^q h^{-2+\varepsilon q} d\tau dt dh \\ & \leq \int_0^T h^{-1+\varepsilon q} dh \|\partial_t u\|_{q, \Omega_T}^q \leq C_{\varepsilon, T_0}\|u\|_{2, 1, q}^q. \end{aligned}$$

□

**Lemma 4.3.** *Let  $n \geq 2$ ,  $q > n$ , and  $\kappa > 0$ , let  $\Omega$  be an asymptotically flat layer with  $C^{1,1}$ -boundary such that  $\gamma^+ \in W_q^{1-\frac{1}{q}}(\mathbb{R}^{n-1})$ , and let  $G: X_T \rightarrow Y_T$ ,  $T > 0$ , be defined as above. Then for every  $R > 0$  there is a  $T_0 > 0$  such that*

$$\|G(u) - G(v)\|_{Y_T} \leq \kappa\|u - v\|_{X_T} \quad (4.7)$$

for all  $u, v \in \overline{B_R(0)} \subset X_T$ .

**Proof.** The operator  $G: X_T \rightarrow Y_T$  can be written as  $G(u, p) = H(u)(u, p) - H(0)(u, p) + F(u) - F(0)$ , where

$$\begin{aligned} H(u)(f, p) &= (\Delta_u f - \nabla_u p, -\operatorname{div}_u f, -\tilde{T}_{1, u}^+(f, p), 0), \\ F(u) &= (0, 0, g_0 X_{u, n} A(u)^T \nu|_{\partial\Omega^+}, 0), \end{aligned}$$

and  $H(u)(f, p)$  is a linear operator in  $(f, p)$  depending on  $u$ . Hence

$$\begin{aligned} & G(u, p_1) - G(v, p_2) \\ &= (H(u) - H(v))(u, p_1) + (H(v) - H(0))(u - v, p_1 - p_2) + F(u) - F(v). \end{aligned}$$

Therefore, it is sufficient to prove

$$\|(H(u) - H(v))(f, p)\|_{Y_T} \leq C\varphi(T)\|u - v\|_{2, 1, q}\|(f, p)\|_{X_T}, \quad (4.8)$$

$$\|F(u) - F(v)\|_{Y_T} \leq C\varphi(T)\|u - v\|_{2, 1, q} \quad (4.9)$$

for a function  $\varphi(T)$  such that  $\lim_{T \rightarrow 0} \varphi(T) = 0$  and  $(u, 0), (v, 0), (f, p) \in X_T$  with  $\|u, v\|_{2,1,q}, \|(f, p)\|_{X_T} \leq R$ .

Using (4.3) and  $W_q^1(\Omega) \hookrightarrow L^\infty(\Omega)$ , it is easy to prove that

$$\begin{aligned} \|(\nabla_u - \nabla_v)p\|_{q,\Omega_T} &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \|p\|_{1,0,q}, \\ \|(\operatorname{div}_u - \operatorname{div}_v)f\|_{1,0,q} &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \|f\|_{2,0,q}, \\ \|(\Delta_u - \Delta_v)f\|_{q,\Omega_T} &\leq CT^{\frac{1}{q'}} (T^{\frac{1}{q'}} R + 1) \|u - v\|_{2,1,q} \|f\|_{2,0,q}, \\ \|(S_u - S_v)f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \|f\|_{2,0,q}. \end{aligned}$$

Since  $\nu^+ \in C^{0,1}(\partial\Omega^+)$  and since  $W_q^{1-\frac{1}{q}}(\partial\Omega^+)$  is an algebra under pointwise multiplication,

$$\begin{aligned} &\|(A(u)^T \nu) \cdot S_u f|_{\partial\Omega^+} - (A(v)^T \nu) \cdot S_v f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} \\ &\leq C \left( \sup_{0 \leq t \leq T} \|(A(u) - A(v))^T \nu|_{\partial\Omega^+}\|_{1-\frac{1}{q},q} \right) \|S_u f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} \\ &\quad + C \left( \sup_{0 \leq t \leq T} \|A(v)^T \nu|_{\partial\Omega^+}\|_{1-\frac{1}{q},q} \right) \|(S_u - S_v)f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} \\ &\leq CT^{\frac{1}{q'}} (\|(u, v)\|_{2,1,q} + 1) \|u - v\|_{2,1,q} \|f\|_{2,0,q}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|(A(u) - A(v))^T \nu p|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \|p\|_{1,0,q}, \\ \|(\operatorname{div}_u - \operatorname{div}_v)f|_{\partial\Omega^+}\|_{1-\frac{1}{q},0,q} &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \|f\|_{2,0,q}. \end{aligned}$$

Hence

$$\|(\tilde{T}_u^+ - \tilde{T}_v^+)(f, p)\|_{1-\frac{1}{q},0,q} \leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} (\|f\|_{2,0,q} + \|p\|_{1,0,q}).$$

In the same way we get

$$\begin{aligned} &\|\gamma_0^+(X_{u,n} A(u)^T - X_{v,n} A(v)^T) \nu\|_{1-\frac{1}{q},0,q} \\ &\leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} + C \|X_{u,n} - X_{v,n}\|_{1,0,q} \leq CT^{\frac{1}{q'}} \|u - v\|_{2,1,q} \end{aligned}$$

because of (4.3). But it remains to estimate the seminorm

$$\|a\|_{\cdot,0,\frac{1}{2q'},q} := \left( \int_0^T \int_0^t \|(\Delta_h a)(\cdot, t)\|_{q,\partial\Omega^+}^q h^{-1-\frac{q}{2q'}} dh dt \right)^{\frac{1}{q}}$$



for the boundary terms. Because of (4.4) and

$$\Delta_h(fg)(t) = (\Delta_h f)(t)g(t-h) + f(t)(\Delta_h g)(t),$$

we conclude

$$\begin{aligned} & \| (A(u) - A(v))|_{\partial\Omega^+} a^+ \|_{.,0,\frac{1}{2q'},q} \\ & \leq C \left( \sup_{0 \leq t \leq T} \|A(u) - A(v)\|_{\infty,\Omega} \|a^+\|_{.,0,\frac{1}{2q'},q} \right. \\ & \quad \left. + \sup_{0 \leq t \leq T} \left( \int_0^t \|\Delta_h A(u-v)\|_{\infty,\Omega}^q h^{-1-\frac{q}{2q'}} dh \right)^{\frac{1}{q}} \|a^+\|_{q,\partial\Omega_T^+} \right) \\ & \leq CT^{\frac{1}{2q'}} \|u-v\|_{2,1,q} \|a^+\|_{W_q^{\frac{1}{2q'}}(0,T;L^q(\partial\Omega^+))}. \end{aligned} \quad (4.10)$$

This implies

$$\begin{aligned} \|(\tilde{T}_u^+ - \tilde{T}_v^+)(f,p)\|_{.,0,\frac{1}{2q'},q} & \leq CT^{\frac{1}{2q'}} \|u-v\|_{2,1,q} \\ & \quad \times (\|f\|_{2,1,q} + \|p|_{\partial\Omega^+}\|_{1-\frac{1}{q},\frac{1}{2}(1-\frac{1}{q}),q}), \\ \|(\operatorname{div}_u - \operatorname{div}_v)f|_{\partial\Omega^+}\|_{.,0,\frac{1}{2q'},q} & \leq CT^{\frac{1}{2q'}} \|u-v\|_{2,1,q} \|f\|_{2,1,q} \end{aligned}$$

due to Lemma 2.1.2. Because of (4.5) and

$$\|X_u - X_v\|_{q,\partial\Omega_T^+} \leq T \|(u-v)|_{\partial\Omega^+}\|_{q,\partial\Omega_T^+} \leq CT \|u-v\|_{2,1,q},$$

we have

$$\|(X_{u,n} - X_{v,n})|_{\partial\Omega^+}\|_{W_q^{\frac{1}{2q'}}(0,T;L^q(\partial\Omega^+))} \leq CT^{\frac{1}{q} + \frac{1}{2q'}} \|u-v\|_{2,1,q}.$$

Thus (4.10) and the latter estimate imply

$$\|B^+(u) - B^+(v)\|_{.,0,\frac{1}{2q'},q} \leq CT^{\frac{1}{2q'}} \|u-v\|_{2,1,q}$$

for  $B^+(u) = (X_{u,n} - \xi_n)A(u)^T \nu|_{\partial\Omega^+}$ . Since  $\xi_n|_{\partial\Omega^+} = \gamma^+ \in W_q^{1-\frac{1}{q}}$ , (4.10) yields

$$\|(A(u)^T - A(v)^T) \nu \xi_n|_{\partial\Omega^+}\|_{.,0,\frac{1}{2q'},q} \leq CT^{\frac{1}{2q'}} \|u-v\|_{2,1,q},$$

which implies

$$\|X_{u,n}A(u)^T \nu|_{\partial\Omega^+} - X_{v,n}A(v)^T \nu|_{\partial\Omega^+}\|_{.,0,\frac{1}{2q'},q} \leq CT^{\frac{1}{2q'}} \|u-v\|_{2,1,q}.$$

In order to estimate  $I[\tilde{T}_u(f, p) - \tilde{T}_v(f, p), 0]^{\frac{1}{q}}$  in the case  $q = 3$  and  $n = 2$ , we use (2.1). Then

$$\begin{aligned} & \int_0^T \int_{\partial\Omega} |\tilde{T}_u(f, p) - \tilde{T}_v(f, p)|^3 t^{-1} d\sigma dt \\ & \leq C (\|f\|_{2,1,3} + \|p\|_{1,0,3})^3 \sup_{0 \leq t \leq T} t^{-1} \|(A(u) - A(v))(\cdot, t)\|_{1,3}^3, \end{aligned}$$

where  $\|(A(u) - A(v))(\cdot, t)\|_{1,3} \leq Ct^{\frac{2}{3}} \|u - v\|_{2,1,q}$  because of (4.3). Hence

$$I[\tilde{T}_u(f, p) - \tilde{T}_v(f, p), 0]^{\frac{1}{q}} \leq CT^{\frac{1}{3}} (\|f\|_{2,1,3} + \|p\|_{1,0,3}) \|u - v\|_{2,1,q}.$$

Similarly, using

$$\begin{aligned} & (X_{u,n}A(u)^T - X_{v,n}A(v)^T)|_{\partial\Omega^+} \in W_q^1(0, T; W_q^{1-\frac{1}{q}}) \hookrightarrow C^{\frac{2}{3}}(0, T; W_q^{1-\frac{1}{q}}) \\ & (X_{u,n}A(u)^T - X_{v,n}A(v)^T)|_{t=0} = 0, \text{ and (2.1), we conclude} \end{aligned}$$

$$I[(X_{u,n}A(u)^T - X_{v,n}A(v)^T)|_{\partial\Omega^+}, 0]^{\frac{1}{q}} \leq CT^{\frac{1}{3}} \|u - v\|_{2,1,q}.$$

Finally, we estimate the  $L^q(0, T; {}_0W_q^{-1}(\Omega))$ -norm of  $\partial_t(\operatorname{div}_u - \operatorname{div}_v)f$ . Because of the definition of the  ${}_0W_q^{-1}(\Omega)$ -valued distributional derivate  $\partial_t$ ,

$$\begin{aligned} & -\langle \langle \partial_t(\operatorname{div}_u - \operatorname{div}_v)f, \varphi \rangle_{(0,T)}, w \rangle_{\Omega} \\ & = \int_{\Omega_T} (\operatorname{div}_u - \operatorname{div}_v)f(x, t) \partial_t \varphi(t) w(x) d(x, t) \\ & = - \int_{\Omega_T} f(x, t) \partial_t \varphi(t) \nabla [(A(u) - A(v))^T w(x)] d(x, t) \\ & = \int_{\Omega_T} \partial_t f \varphi \nabla [(A(u) - A(v))^T w] d(x, t) \\ & \quad - \int_{\Omega_T} \operatorname{div} f \varphi \partial_t (A(u)^T - A^T(v)) w d(x, t) \end{aligned}$$

for all  $\varphi \in \mathcal{D}(0, T)$ ,  $w \in {}_0W_q^1(\Omega)$ , and  $u, v, f \in W_q^{2,1}(\Omega_T)$  with  $(u, 0), (v, 0), (f, 0) \in X_T$ . Firstly,

$$\begin{aligned} & \left| \int_{\Omega} \int_0^T \partial_t f \varphi \nabla [(A(u) - A(v))^T w] dt dx \right| \\ & \leq \|\partial_t f\|_{q, \Omega_T} \|\varphi\|_{q', (0,T)} \sup_{0 \leq t \leq T} \|\nabla [(A(u) - A(v))^T(t, \cdot)w]\|_{q', \Omega}. \end{aligned}$$

Using Sobolev inequalities, we have  $\|w\|_r \leq \|w\|_{1, q'}$  for all  $\frac{1}{q'} \geq \frac{1}{r} \geq \frac{1}{q'} - \frac{1}{n}$ . Therefore  $\|vw\|_{q'} \leq \|v\|_q \|w\|_r \leq C \|v\|_q \|w\|_{1, q'}$  with  $\frac{1}{r} = \frac{1}{q'} - \frac{1}{q} > \frac{1}{q'} - \frac{1}{n}$ .

Hence

$$\begin{aligned}
& \sup_{0 \leq t \leq T} \|\nabla [(A(u) - A(v))^T(\cdot, t)w]\|_{q', \Omega} \\
& \leq \sup_{0 \leq t \leq T} [\|\nabla(A(u) - A(v))\|_{q, \Omega} \|w\|_{r, \Omega} + \|A(u) - A(v)\|_{\infty, \Omega} \|\nabla w\|_{q', \Omega}] \\
& \leq C \sup_{0 \leq t \leq T} \|(A(u) - A(v))(\cdot, t)\|_{1, q, \Omega} \|w\|_{1, q', \Omega} \leq CT^{\frac{1}{q'}} \|u - v\|_{2, 1, q} \|w\|_{1, q', \Omega}
\end{aligned}$$

by (4.3). Secondly,

$$\begin{aligned}
& \left| \int_{\Omega} \int_0^T \operatorname{div} f \partial_t (A(u) - A(v))^T \varphi w \, dt \, dx \right| \\
& \leq \sup_{0 \leq t \leq T} \|\operatorname{div} f(\cdot, t)\|_{q, \Omega} \|\varphi\|_{q', (0, T)} \|\partial_t (A(u) - A(v))\|_{L^q(0, T; L^\infty)} \|w\|_{L^{q'}(\Omega)} \\
& \leq C_\varepsilon T^{\frac{1}{2} - \frac{1}{q} - \varepsilon} \|f\|_{2, 1, q} \|\varphi\|_{q', (0, T)} \|w\|_{1, q', \Omega} \|u - v\|_{2, 1, q}.
\end{aligned}$$

Here we have used (4.6) and  $\sup_{0 \leq t \leq T} \|\operatorname{div} f(\cdot, t)\|_{q, \Omega} \leq C_\varepsilon T^{\frac{1}{2} - \frac{1}{q} - \varepsilon} \|f\|_{2, 1, q}$  for  $\varepsilon \in (0, \frac{1}{2} - \frac{1}{q})$  since  $\operatorname{div} f|_{t=0} = 0$  and

$$\operatorname{div} f \in W_q^{\frac{1}{2} - \varepsilon}(0, T; L^q(\Omega)) \hookrightarrow C^{\frac{1}{2} - \frac{1}{q} - \varepsilon}(0, T; L^q(\Omega))$$

because of Lemma 4.2 and Simon [14, Corollary 26].  $\square$

**Proof of Theorem 1.1.** Let  $R > 0$  be chosen so large that  $\|L^{-1}h\|_{X_T} \leq \frac{R}{2}$  for  $h = (0, 0, g_0 \xi_n \nu, u_0)$  for some  $T > 0$ . Then, because of Lemma 4.3, there is a  $0 < T_0 \leq T$  such that  $L^{-1}G: \overline{B_R(0)}|_{X_{T_0}} \rightarrow \overline{B_R(0)}|_{X_{T_0}}$  is a contraction with Lipschitz constant  $\kappa = \frac{1}{2}$ . Now let  $F(v) := L^{-1}G(v) + L^{-1}h$ ,  $v \in X_{T_0}$ . Then  $\|F(v)\|_{X_{T_0}} \leq R$  for  $v \in \overline{B_R(0)}|_{X_{T_0}}$  since  $\|L^{-1}h\|_{X_{T_0}} \leq \|L^{-1}h\|_{X_T} \leq \frac{R}{2}$  and  $\|L^{-1}G(v)\|_{X_{T_0}} = \|L^{-1}G(v) - L^{-1}G(0)\|_{X_{T_0}} \leq \frac{R}{2}$ . Hence the Banach contraction-mapping principle implies the existence of a unique solution in  $\overline{B_R(0)}$ .

Finally, if  $v \in X_T$  is an arbitrary solution, we choose  $R > 0$  in the construction above a priori so large that  $R \geq \|v\|_{X_T}$ . Then  $v$  coincides with the solution  $u$  obtained by the Banach fixed-point theorem for some  $0 < T_0 \leq T$ . Since this argument can be repeated with initial data  $v|_{t=T_0}$ , the solution is unique as long as the solution exists.  $\square$

## REFERENCES

- [1] H. Abels, *Generalized Stokes resolvent equations in an infinite layer with mixed boundary conditions*, to appear in Math. Nachr.

- [2] H. Abels, *Reduced and generalized Stokes resolvent equations in asymptotically flat layers, part I: unique solvability*, to appear in J. Math. Fluid Mech.
- [3] H. Abels, *Reduced and generalized Stokes resolvent equations in asymptotically flat layers, part II:  $H_\infty$ -calculus*, to appear in J. Math. Fluid Mech.
- [4] G. Allain, *Small-time existence for Navier-Stokes equations with a free surface*, Appl. Math. Optim., 16 (1987), 37–50.
- [5] H. Amann, “Linear and Quasilinear Parabolic Problems,” Birkhäuser, Basel-Boston-Berlin, 1995.
- [6] J.T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure Appl. Math., 34 (1981), 359–392.
- [7] J.T. Beale, *Large-time regularity of viscous surface waves*, Arch. Ration. Mech. Anal., 84 (1984), 307–352.
- [8] J.T. Beale and T. Nishida, *Large-time behavior of viscous surface waves*, Lecture Notes in Num. Appl. Anal., 8, 1–14 (1985), “Recent Topics in Nonlinear PDE II,” North-Holland/Kinokuniya, 1985.
- [9] G. Dore and A. Venni, *On the closedness of the sum of two closed operators*, Math. Z., 196 (1987), 189–201.
- [10] Y. Giga and H. Sohr, *Abstract  $L^p$  estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains*, J. Funct. Anal., 102 (1991), 72–94.
- [11] G. Grubb, *Parameter-elliptic and parabolic pseudodifferential boundary problems in global  $L^p$  Sobolev spaces*, Math. Z., 218 (1995), 43–90.
- [12] G. Grubb and V.A. Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scand., 69 (1991), 217–290.
- [13] A. McIntosh, *Operators which have an  $H_\infty$ -calculus*, in “Miniconference on Operator Theory and Partial Differential Equations,” B. Jefferies, A. McIntosh, W. Ricker, editors, Proc. Center Math. Anal. A.N.U., 14 (1986), 210–231.
- [14] J. Simon, *Sobolev, Besov and Nikolskii fractional spaces: imbeddings and comparisons for vector valued spaces on an interval*, Annali Mat. Pura Applicata, (IV), 157 (1990), 117–148.
- [15] V.A. Solonnikov, *On the transient motion of an isolated volume of viscous incompressible fluid*, Math. USSR Izvestiya, 31 (1988), 381–405.
- [16] D.L.G. Sylvester, *Large time existence of small viscous surface waves without surface tension*, Comm. Part. Diff. Eqs., 15 (1990), 823–903.
- [17] A. Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface*, Arch. Ration. Mech. Anal., 133 (1996), 299–331.
- [18] A. Tani and N. Tanaka, *Large-time existence of surface waves in incompressible viscous fluids with or without surface tension*, Arch. Ration. Mech. Anal., 130 (1995), 303–314.