

**INCOMPRESSIBLE LIMIT OF THE NONISENTROPIC
EULER EQUATIONS WITH THE SOLID WALL
BOUNDARY CONDITIONS**

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Abstract. We study the zero Mach number limit of classical solutions to the compressible Euler equations for *nonisentropic* fluids in a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3). We consider the case of *general* initial data. For a domain Ω , bounded or unbounded, we first prove the existence of classical solutions for a time independent of the small parameter. Then, in the exterior case, we prove that the solutions converge to the solution of the incompressible Euler equations.

1. INTRODUCTION

This work is devoted to the study of the so-called incompressible limit for classical solutions of the compressible Euler equations for nonisentropic fluids. We consider the case of a flow in a domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with the solid-wall boundary condition. After the usual rescalings and changes of variables, see [11, 15], we are led to analyze a quasilinear hyperbolic system depending on a small parameter ε , which is linked to the Mach number,

$$\begin{aligned} a^\varepsilon (\partial_t q^\varepsilon + v^\varepsilon \cdot \partial_x q^\varepsilon) + \frac{\operatorname{div} v^\varepsilon}{\varepsilon} &= 0, \\ r^\varepsilon (\partial_t v^\varepsilon + v^\varepsilon \cdot \partial_x v^\varepsilon) + \frac{\nabla q^\varepsilon}{\varepsilon} &= 0, \\ \partial_t S^\varepsilon + v^\varepsilon \cdot \partial_x S^\varepsilon &= 0, \\ v^\varepsilon|_{\partial\Omega} \cdot \nu &= 0, \\ (q^\varepsilon, v^\varepsilon, S^\varepsilon)|_{t=0} &= (q_0^\varepsilon, v_0^\varepsilon, S_0^\varepsilon), \end{aligned} \tag{1.1}$$

where $q^\varepsilon = q^\varepsilon(t, x) \in \mathbb{R}$, $v^\varepsilon = v^\varepsilon(t, x) \in \mathbb{R}^d$ ($d = 2$ or 3), and $S^\varepsilon = S^\varepsilon(t, x) \in \mathbb{R}$. We denote by ν the unit outward normal on the boundary $\partial\Omega$, $v \cdot \nu$ denotes the standard inner product of v and ν in \mathbb{R}^d , and $v \cdot \partial_x = \sum_{j=1}^d v_j \partial_{x_j}$.

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An important feature of (1.1) is that the coefficients a^ε and r^ε depend on S^ε . They read

$$a^\varepsilon = a(S^\varepsilon, \varepsilon q^\varepsilon), \quad r^\varepsilon = r(S^\varepsilon, \varepsilon q^\varepsilon),$$

where a and r are smooth, positive functions given by the state laws of the fluid.

The nature of the incompressible limit depends on several factors: the flow may be *isentropic* or *nonisentropic*. The initial data may be *prepared* (the initial acoustic components, namely $(\operatorname{div} v_0^\varepsilon, \nabla q_0^\varepsilon)$, are $O(\varepsilon)$), or *general*, which means that $\{(q_0^\varepsilon, v_0^\varepsilon, S_0^\varepsilon)\}_{\varepsilon>0}$ is a bounded family in the Sobolev space $H^s(\Omega)$, where $s > d/2 + 1$. Finally, the domain may be the torus, the whole space or a domain $\Omega \subset \mathbb{R}^d$.

First, to study this singular limit one has to prove an existence and uniform boundedness result for a time independent of ε .

Solutions of (1.1) are known to exist for a time interval which is independent of the small parameter ε in the equations whenever the flow is isentropic (see [11, 12]), whenever the initial data is prepared (see [19]), and whenever the domain is the torus or the whole space (see [15]).

These existence results cover all cases of the above-mentioned factors except for the nonisentropic equations with general initial data in the boundary case. The first result in this paper is a uniform existence result for this case. More precisely, we improve the result of G. Métivier and S. Schochet ([15]) by assuming that Ω is a general domain, bounded or unbounded, with compact, smooth boundary. Their proof relies upon the fact that one can establish uniform bounds by applying some good operators to the equations. In the boundary case, the main difficulty is to find other operators which respect the boundary condition $v^\varepsilon|_{\partial\Omega} \cdot \nu = 0$.

The solutions are known to converge to the solution of the corresponding incompressible Euler equations with the limit initial data whenever the initial data are prepared (see [11, 12, 19]). For the isentropic equations with general initial data, the velocity v^ε is the sum of the limit flow v , which is a solution of the incompressible equations whose initial data is the incompressible part of the original initial data, and a highly oscillatory term created by the sound waves (see [5]). In that case the solutions are known to converge, although this convergence is not uniform for time close to zero (see [1, 8, 9, 10, 20]). Finally, we refer the reader to [16, 17] for the case of the nonisentropic Euler equations with spatially periodic boundary conditions.

For the general nonisentropic Euler equations in an exterior domain with general initial data, we prove strong compactness in $L^2([0, T], H_{\text{loc}}^{s'}(\Omega))$ for

all $s' < s$. Therefore, we can prove a convergence theorem. The proof makes use of semiclassical defect measures as in [15].

We use the following background, which includes (1.1):

$$\begin{aligned} E(S^\varepsilon, \varepsilon u^\varepsilon)(\partial_t u^\varepsilon + b(S^\varepsilon, u^\varepsilon) \cdot \partial_x u^\varepsilon) + \frac{1}{\varepsilon} L(\partial_x) u^\varepsilon &= 0, \\ \partial_t S^\varepsilon + b(S^\varepsilon, u^\varepsilon) \cdot \partial_x S^\varepsilon &= 0, \\ v^\varepsilon|_{\partial\Omega} \cdot \nu &= 0. \end{aligned} \tag{1.2}$$

In addition to (1.2), we prescribe the initial data at $t = 0$:

$$u^\varepsilon|_{t=0} = u_0^\varepsilon, \quad S^\varepsilon|_{t=0} = S_0^\varepsilon.$$

The setting is defined by the following hypotheses:

$$\begin{aligned} u^\varepsilon &= \begin{pmatrix} q^\varepsilon(t, x) \\ v^\varepsilon(t, x) \end{pmatrix} \in \mathbb{R} \times \mathbb{R}^d, \quad S^\varepsilon = S^\varepsilon(t, x) \in \mathbb{R}, \quad d \geq 1, \quad x \in \Omega, \\ E(S^\varepsilon, \varepsilon u^\varepsilon) &= \begin{pmatrix} a(S^\varepsilon, \varepsilon u^\varepsilon) & 0 \\ 0 & r(S^\varepsilon, \varepsilon u^\varepsilon) \mathbf{I}_d \end{pmatrix}, \quad L(\partial_x) = \begin{pmatrix} 0 & \text{div} \\ \nabla & 0 \end{pmatrix}, \end{aligned}$$

together with the following assumptions.

Assumptions:

- (A1) We suppose that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is an open, connected set lying on one side of its compact, smooth boundary $\partial\Omega$, it can be bounded or unbounded. We can also assume that Ω is the whole space (we remove the boundary condition) or the torus (with periodic boundary conditions).
- (A2) The coefficients a and r are smooth, positive functions of $(S, u) \in \mathbb{R}^{2+d}$.
- (A3) Also, $b = (b_1, \dots, b_d) \in \mathbb{R}^d$ is a smooth function of $(S, u) \in \mathbb{R}^{2+d}$, and $b \cdot \partial_x$ denotes the operator $\sum_{j=1}^d b_j \partial_{x_j}$.
- (A4) Let ν denote the outward normal on the boundary. We assume that $\nu \cdot v|_{\partial\Omega} = 0$ implies that $\nu \cdot b(S, q, v)|_{\partial\Omega} = 0$ for all functions (S, q) .
- (A5) We require some compatibility conditions. The matrix E is invertible, so $\partial_t^k v(0)$ can be expressed as $A_k(u(0), S(0))$. We say that the solutions satisfy compatibility conditions up to order p if, on the boundary $\partial\Omega$, one has $\nu \cdot A_k(u(0), S(0)) = 0$ for all $k \leq p$.

We verify that the hypotheses (A2)–(A4) are satisfied by the system (1.1). In the context defined by (A1)–(A5) we prove that the solutions of (1.2) exist and are uniformly bounded for a time interval which is independent of ε .

Theorem 1.1 (Ω is bounded or unbounded). *Let $s > 1 + d/2$ be an integer. For all real M_0 , there is a positive $T = T(M_0)$ such that for all $\varepsilon \in (0, 1]$,*

and all initial data $(u_0^\varepsilon, S_0^\varepsilon)$ in the Sobolev space $H^s(\Omega)$ satisfying

$$\|(u_0^\varepsilon, S_0^\varepsilon)\|_{H^s(\Omega)} \leq M_0, \quad (1.3)$$

and compatibility conditions (A5) up to order $s - 1$, the Cauchy problem for (1.2) has a unique solution $(u^\varepsilon, S^\varepsilon) \in C^0([0, T], H^s(\Omega))$. Furthermore, there exists $\Gamma = \Gamma(M_0) < +\infty$ such that for all $t \in [0, T]$ and $\varepsilon \in (0, 1]$,

$$\|(u^\varepsilon(t), S^\varepsilon(t))\|_{H^s(\Omega)} \leq \Gamma, \quad \|\partial_t S^\varepsilon(t)\|_{H^{s-1}(\Omega)} \leq \Gamma, \quad (1.4)$$

$$\|\partial_t \text{curl}(r(S^\varepsilon(t), 0)v^\varepsilon(t))\|_{H^{s-2}(\Omega)} \leq \Gamma. \quad (1.5)$$

Remark 1.

- (1) We prepare the initial data only on the boundary $\partial\Omega$ (which is necessary), and not in the interior, so we consider *general initial data*.
- (2) We verify that the compatibility conditions (A5) are satisfied up to order $s - 1$ for any initial data in $H^s(\Omega)$ vanishing in a neighborhood of $\partial\Omega$.
- (3) The assumptions (A1)–(A5) are sufficient to ensure uniform existence of solutions but do not imply uniform stability of the linearized equations since the matrix, $E(S^\varepsilon, \varepsilon u^\varepsilon)$, multiplying the time-derivative depends on S^ε . Therefore we cannot obtain in standard fashion the nonlinear energy estimates from the L^2 estimate by an elementary argument using differentiation of the equations.

We now turn to consideration of the limit of solutions of (1.1) as $\varepsilon \rightarrow 0$. We still assume that $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) is an open, connected set lying on one side of its compact, smooth boundary $\partial\Omega$. In addition we require for the forthcoming theorem that Ω is the *exterior of a bounded domain*, or the whole space \mathbb{R}^d .

When the initial data are prepared, $(\text{div } v_0^\varepsilon, \nabla q_0^\varepsilon) = O(\varepsilon)$, it is proved in [19] that the limit system reads

$$\begin{aligned} \text{div } v &= 0, \\ r(S, 0)(\partial_t v + v \cdot \partial_x v) + \nabla \pi &= 0, \\ \partial_t S + v \cdot \partial_x S &= 0, \\ v|_{\partial\Omega} \cdot \nu &= 0, \end{aligned} \quad (1.6)$$

for some π such that $\nabla \pi$ belongs to $C^0([0, T], H^{s-1}(\Omega))$.

We consider the same problem when the initial data are not so constrained. We suppose only that S_0^ε decays sufficiently rapidly at infinity.

Theorem 1.2 (Exterior case). *Assume that Ω is the exterior of a bounded domain. Assume that $(q^\varepsilon, v^\varepsilon, S^\varepsilon)$ satisfy (1.1) and are uniformly bounded in $C^0([0, T], H^s(\Omega))$, for some fixed $T > 0$ and $s > 1 + d/2$. Suppose that*

the initial data $(v_0^\varepsilon, S_0^\varepsilon)$ satisfy compatibility conditions up to order $s - 1$ and converge in $H^s(\Omega)$ to (v_0, S_0) . Assume further that S_0^ε decays sufficiently rapidly at infinity in the sense that

$$|S_0^\varepsilon(x)| \leq C |x|^{-1-\delta}, \quad |\nabla S_0^\varepsilon(x)| \leq C |x|^{-2-\delta}, \quad (1.7)$$

for all ε and some fixed C and $\delta > 0$. Then $(v^\varepsilon, q^\varepsilon, S^\varepsilon)$ converge weakly $*$ in $L^\infty([0, T], H^s(\Omega))$ and strongly in $L^2([0, T], H_{\text{loc}}^{s'}(\Omega))$ for all $s' < s$ to a limit $(v, 0, S)$. Moreover, (v, S) is the unique solution in $C^0([0, T], H^s(\Omega))$ of (1.6) with initial data (w_0, S_0) , with w_0 being the unique solution in $H^s(\Omega)$ of

$$w_0|_{\partial\Omega} \cdot \nu = 0, \quad \text{div } w_0 = 0, \quad \text{curl}(r_0 w_0) = \text{curl}(r_0 v_0), \quad \text{where } r_0 = r(S_0, 0).$$

We also want to point out that the situation is completely different between the isentropic and nonisentropic case. In the isentropic regime, the pressure can be regarded as a function of the density alone (for instance $P = a\rho^\gamma$). As the Mach number goes to zero the variation in pressure converges to zero, as do the variations in density. In contrast, for nonisentropic fluids, the pressure depends on the density as well as the temperature. Thus, as the Mach number goes to zero, the fact that the variation in pressure converges to zero does not preclude large density fluctuations (provided that they are compensated by temperature fluctuations of opposite sign). And small perturbations of the density might result in nonsmall velocity variations. It means that the linearized equations are not uniformly well-posed.

Theorem 1.1 is proved in Section 2. The main new estimates are embodied in Lemmas 2.3, 2.4, 2.6, 2.9, and 2.10. In Section 3 we prove Theorem 1.2.

2. PROOF OF THEOREM 1.1

Consider the equations (1.2) for fixed $\varepsilon > 0$, and assume that assumptions (A1)–(A4) are satisfied, and that the initial data satisfy compatibility conditions (A5) up to order $s - 1$. The system is symmetric hyperbolic; therefore, if $\Omega = \mathbb{R}^d$ we know from [14] that (1.2) is well-posed for regular initial data. In the boundary case it is shown in [19] that the problem for nonisentropic fluids is well-posed locally in time. Such a theorem is in no way a trivial one. As usual, the essential part lies in establishing the well-posedness of the linearized problem. Yet the linearized equations of the boundary-value problem are not included in classical frameworks. Indeed, the boundary matrix is singular and fails to be of constant rank near the boundary. For a general theory we refer to [18], where J. Rauch has studied the symmetric positive systems with boundary of constant multiplicity. He has proved the well-posedness in L^2 (by means of a “weak=strong” Lemma based on his tangential regularization) and the conormal regularity. Here,

we use the special structure of the equations so as to estimate the normal derivatives.

Given $s > 1 + d/2$, for all fixed $\varepsilon > 0$, we let $T_\varepsilon = T(\varepsilon, M_0) > 0$ denote the lifespan, *i.e.*, the supremum of all the positive times T such that for all initial data which satisfies (1.3) and $s - 1$ compatibility conditions, the Cauchy problem has a unique solution on $C^0([0, T], H^s(\Omega))$. It results from the above-mentioned theorem of Schochet that either $T_\varepsilon = +\infty$ or

$$\limsup_{t \rightarrow T_\varepsilon} \|(u^\varepsilon, S^\varepsilon)(t)\|_{H^s(\Omega)} = +\infty. \quad (2.1)$$

On account of this alternative the problem reduces to establishing the following *a priori* bounds.

Proposition 2.1. *Given $s > 1 + d/2$ and $M_0 \in \mathbb{R}$, there is a constant C_0 and a nonnegative function $C(\cdot)$, such that for all $T \in (0, T_\varepsilon)$, $\varepsilon \in (0, 1]$ and $(u^\varepsilon, S^\varepsilon) \in C^0([0, T], H^s(\Omega))$ a solution of (1.2), with initial data satisfying (1.3), the norm*

$$M_\varepsilon(T) := \sup_{t \in [0, T]} \|(u^\varepsilon, S^\varepsilon)(t)\|_{H^s(\Omega)} \quad (2.2)$$

satisfies the estimate

$$M_\varepsilon(T) \leq C_0 + (T + \varepsilon)C(M_\varepsilon(T)). \quad (2.3)$$

The proof of this proposition is completed only after Lemma 2.11. Let us explain why Proposition 2.1 implies that T_ε is bounded from below by a positive constant for all ε less than 1.

Proof of Theorem 1.1 given Proposition 2.1. Choose first $M_1 > C_0$ and next ε_1 and T_1 such that

$$C_0 + (T_1 + \varepsilon_1)C(M_1) < M_1. \quad (2.4)$$

For $t < \inf\{T_\varepsilon, T_1\}$, and $\varepsilon \leq \varepsilon_1$, the inequalities (2.3) and (2.4) imply that $M_\varepsilon(t) \neq M_1$, where $M_\varepsilon(t)$ is the norm defined by (2.2) on $[0, t]$. Besides we can assume without restriction that $M_\varepsilon(0) < M_1$. Using a continuity argument, we infer $M_\varepsilon(t) < M_1$. Consequently, the continuation principle (2.1) shows that $T_\varepsilon > T_1$ for all $\varepsilon \leq \varepsilon_1$.

On the other hand, we can extract from the proof that the problem is well posed locally in time for fixed ε , and that T_ε is bounded from below by $T'_1 > 0$ for all ε in $[\varepsilon_1, 1]$, a compact subset of $(0, 1]$. \square

So as to prove (2.3) we use in the sequel the following well-known nonlinear estimates. To simplify notation, let $\|\cdot\|_s$ denote the norm in the Sobolev space $H^s(\Omega)$.

For $k \geq 0$, $l \geq 0$, $k+l \leq \sigma$, and $\sigma > d/2$, the product maps continuously $H^{\sigma-k}(\Omega) \times H^{\sigma-l}(\Omega)$ to $H^{\sigma-k-l}(\Omega)$, and

$$\|uv\|_{\sigma-k-l} \leq K \|u\|_{\sigma-k} \|v\|_{\sigma-l}. \quad (\text{R1})$$

Similarly, if F is a smooth function such that $F(0) = 0$, and $u \in H^\sigma(\Omega)$ with $\sigma > d/2$, then $F(u) \in H^\sigma(\Omega)$ and

$$\|F(u)\|_\sigma \leq C(\|u\|_\sigma) \|u\|_\sigma. \quad (\text{R2})$$

To take care of the hypothesis $F(0) = 0$, we use the notation $\tilde{F} = F - F(0)$.

As a consequence, for any smooth function $F: \mathbb{R}^n \rightarrow \mathbb{R}^p$, suppose $u \in H^\sigma(\Omega)$ and $v \in H^m(\Omega)$, with $m \in \{0, \dots, \sigma\}$ and $\sigma > d/2$; then $F(u)v \in H^m(\Omega)$ and

$$\|F(u)v\|_m \leq (|F(0)| + C(\|u\|_\sigma)) \|v\|_m. \quad (\text{R3})$$

It appears easier to estimate M_ε by a stronger norm of $(u^\varepsilon, S^\varepsilon)$, which we introduce here. Given $\mu \in \mathbb{N}$, we set

$$X^\mu([0, T] \times \Omega) := \bigcap_{k=0}^{\mu} C^k([0, T], H^{\mu-k}(\Omega)).$$

We define the weighted norms

$$\begin{aligned} \|w(t)\|_{\mu, \varepsilon} &:= \sum_{k=0}^{\mu} \left\| (\varepsilon \partial_t)^k w(t) \right\|_{H^{\mu-k}(\Omega)}, \\ \|w\|_{\mu, \varepsilon, T} &:= \sup_{t \in [0, T]} \|w(t)\|_{\mu, \varepsilon}. \end{aligned} \quad (2.5)$$

We use the following properties. For all $u \in X^\sigma([0, T] \times \Omega)$, $v \in X^n([0, T] \times \Omega)$, $\sigma > d/2$, $t \in [0, T]$, we have

$$\begin{aligned} \|v(t)\|_n &\leq \|v(t)\|_{n, \varepsilon}, \\ \|u(t)v(t)\|_{n, \varepsilon} &\leq K \|u(t)\|_{\sigma, \varepsilon} \|v(t)\|_{n, \varepsilon}. \end{aligned} \quad (\text{R4})$$

To prove (2.3) we first give an estimate for the entropy S^ε .

Lemma 2.2. *There are a constant C_0 and a function $C(\cdot)$ depending only on M_0 , such that*

$$\forall \varepsilon \in (0, 1], \quad \forall t \in [0, T_\varepsilon), \quad \|S^\varepsilon(t)\|_s \leq C_0 + tC(M_\varepsilon(t)). \quad (2.6)$$

Remark 2. In all the proofs,

- (1) the symbols C_0 and $C(\star)$, where the \star is replaced by relevant quantities, stand for various constants and functions depending only on M_0 and s ; K stands for constants depending only on s . They may vary from relation to relation.

- (2) We omit the indexation by ε , we write S , u for S^ε and u^ε . We denote by $b(t)$, $E(t)$, $\tilde{b}(t)$, and $\tilde{E}(t)$ the functions $b(S^\varepsilon(t), u^\varepsilon(t))$, $E(S^\varepsilon(t), \varepsilon u^\varepsilon(t))$, $(b-b(0))(S^\varepsilon(t), u^\varepsilon(t))$, and $(E-E(0))(S^\varepsilon(t), \varepsilon u^\varepsilon(t))$.

Proof. For $\alpha \in \mathbb{N}^d$, $|\alpha| \leq s$, introduce $f_\alpha = \partial_x^\alpha S$. We commute the equation with ∂_x^α ,

$$\partial_t f_\alpha + b(S, u) \cdot \partial_x f_\alpha = [b(S, u) \cdot \partial_x, \partial_x^\alpha] S. \quad (2.7)$$

The force term $g_\alpha(\tau) = [b(\tau) \cdot \partial_x, \partial_x^\alpha] S(\tau)$, is a sum of terms $\partial_x^\beta \tilde{b}(\tau) \partial_x^\gamma S(\tau)$ with $|\beta| + |\gamma| \leq s + 1$, $\beta > 0$, $\gamma > 0$, so the rules (R1) and (R2) imply that g_α belongs to $C^0([0, T_\varepsilon[, L^2(\Omega))$, together with the bound

$$\forall \tau \leq t < T_\varepsilon, \quad \|g_\alpha(\tau)\|_0 \leq K \|\tilde{b}(\tau)\|_s \|S(\tau)\|_s \leq C(M_\varepsilon(t)). \quad (2.8)$$

Multiply the equation (2.7) by f_α and integrate over $[0, t] \times \Omega$, for $t < T_\varepsilon$. Assumption (A4) ensures $b(S, u) \cdot \nu = 0$, hence the boundary terms vanish when we integrate by parts the term $\int b f_\alpha \partial_x f_\alpha dx$. We obtain

$$\begin{aligned} \|f_\alpha(t)\|_0^2 &\leq \|f_\alpha(0)\|_0^2 + \|\partial_x b\|_{L^\infty([0, t] \times \Omega)} \int_0^t \|f_\alpha(\tau)\|_0^2 d\tau \\ &\quad + 2 \int_0^t \|g_\alpha(\tau)\|_0 \|f_\alpha(\tau)\|_0 d\tau, \end{aligned}$$

Since $s - 1 > d/2$, we get

$$\|\partial_x b\|_{L^\infty([0, t] \times \Omega)} \leq \sup_{\tau \in [0, t]} \|\partial_x b(\tau)\|_{s-1} \leq C(M_\varepsilon(t)),$$

thus

$$\begin{aligned} \|f_\alpha(t)\|_0^2 &\leq \|f_\alpha(0)\|_0^2 + C(M_\varepsilon(t)) \int_0^t M_\varepsilon(t)^2 d\tau + \int_0^t C(M_\varepsilon(t)) M_\varepsilon(t) d\tau, \\ &\leq C_0 + tC(M_\varepsilon(t)). \end{aligned}$$

We can assume that $C_0 \geq 1$, therefore $\|f_\alpha(t)\|_0 \leq C_0 + tC(M_\varepsilon(t))$. The claim then follows by adding up these estimates for $|\alpha| \leq s$. \square

Our goal is to prove a bound analogous to (2.6) for u^ε . We cannot apply the previous works. Indeed, the operators $(E^{-1}L(\partial_x))^m$ which were used in [15] to estimate the acoustic components do not commute with the equations for the boundary condition is violated. And it is the only paper which deals with the non-isentropic equations and general initial data. Nevertheless, the operators $(\varepsilon \partial_t)^m$ are relevant to the boundary case. In this proof they supersede $(E^{-1}L(\partial_x))^m$.

First and foremost we estimate $(u^\varepsilon, S^\varepsilon)$ in the big norm $\|\cdot\|_{s, \varepsilon, T}$.

Lemma 2.3. *Given $s > 1 + d/2$, there is a function $C(\cdot)$ from $[0, \infty)$ to $[0, \infty)$, such that*

$$\forall \varepsilon \in]0, 1], \quad \forall T < T_\varepsilon, \quad \|(u^\varepsilon, S^\varepsilon)\|_{s, \varepsilon, T} \leq C(M_\varepsilon(T)). \quad (2.9)$$

Proof. We prove by induction on $m \in \{0, \dots, s\}$ that $(\varepsilon \partial_t)^m(u, S)(t)$ belongs to $H^{s-m}(\Omega)$ with norm bounded by $C(M)$, where $t \leq T < T_\varepsilon$ and $M := M_\varepsilon(T)$.

For $m \geq 1$, we commute $(\varepsilon \partial_t)^m$ with the equations (1.2). This yields

$$(\varepsilon \partial_t)^{m+1}u = -\varepsilon b \cdot \partial_x (\varepsilon \partial_t)^m u - E^{-1}(L(\partial_x)(\varepsilon \partial_t)^m u + f_m), \quad (2.10)$$

$$f_m := [(\varepsilon \partial_t)^m, E(S, \varepsilon u)(\varepsilon \partial_t + \varepsilon b(S, u) \cdot \partial_x)]u, \quad (2.11)$$

$$\partial_t (\varepsilon \partial_t)^m S = -b(S, u) \cdot \partial_x (\varepsilon \partial_t)^m S - g_m, \quad (2.12)$$

$$g_m := [(\varepsilon \partial_t)^m, b(S, u) \cdot \partial_x]S. \quad (2.13)$$

Assume the result for $m = n$, where $n \in [0, s-1]$, *i.e.* for all $0 \leq p \leq n$, $(\varepsilon \partial_t)^p(u, S)(t)$ belongs to $H^{s-p}(\Omega)$ with norm bounded by $C(M)$. Using the rule (R3), to prove the result at order $m = n+1$, it is sufficient to show that $f_n(t)$ and $g_n(t)$ belong to $H^{s-n-1}(\Omega)$ together with the inequality $\|f_n(t)\|_{s-n-1} + \|g_n(t)\|_{s-n-1} \leq C(M)$.

On account of their forms, (2.11) and (2.13), we study the general commutators $\text{Com}(A) = [(\varepsilon \partial_t)^n, A(\varepsilon \partial_t)]$ and $\text{Com}(B) = [(\varepsilon \partial_t)^n, B \partial_{x_j}]$. We get,

$$\begin{aligned} \|\text{Com}(A)u(t)\|_{s-1-n} &\leq K \sum_{p=0}^{n-1} \|(\varepsilon \partial_t)^{n-p} A\|_{s-1-(n-p)} \|(\varepsilon \partial_t)^{p+1}u(t)\|_{s-1-p} \\ &\leq C(M) \sum_{p=0}^{n-1} \|(\varepsilon \partial_t)^{n-p} A\|_{s-1-(n-p)}. \end{aligned} \quad (2.14)$$

$$\begin{aligned} \|\text{Com}(B)\eta(t)\|_{s-1-n} &\leq K \sum_{p=0}^{n-1} \|(\varepsilon \partial_t)^{n-p} B\|_{s-1-(n-p)} \|(\varepsilon \partial_t)^p \partial_{x_j} \eta(t)\|_{s-1-p} \\ &\leq C(M) \sum_{p=0}^{n-1} \|(\varepsilon \partial_t)^{n-p} B\|_{s-1-(n-p)}, \end{aligned} \quad (2.15)$$

where $\eta \in \{u, S\}$. Here we have expanded the commutators by use of the Leibniz' formula, and estimated the terms by means of the rule (R1) (applied with $\sigma = s-1 > d/2$) and the induction hypothesis.

It remains to estimate the sums which appear in (2.14) and (2.15) when $A = E(S(t), \varepsilon u(t))$, $B = b_j(S(t), u(t))$ or $B = \varepsilon b_j(S(t), u(t))E(S(t), \varepsilon u(t))$.

For that purpose we prove that, for all $k \in [1, n]$, both $(\varepsilon \partial_t)^k b(S(t), u(t))$ and $(\varepsilon \partial_t)^k E(S(t), \varepsilon u(t))$ are bounded in $H^{s-k}(\Omega)$ by $C(M)$.

With $U := (S, u) = (u_0, u_1, \dots, u_n)$, $(\varepsilon \partial_t)^k b(S, u)$ expands as

$$\sum_{\substack{\alpha=(q_0, \dots, q_n) \\ 1 \leq |\alpha| \leq k}} \sum_{\beta_\alpha} C_{\alpha, \beta_\alpha} \left\{ \prod_{\substack{p=0, \dots, n \\ q_p \neq 0}} \prod_{\substack{i_p=1, \dots, q_p \\ \beta_p^{i_p} \neq 0}} (\varepsilon \partial_t)^{\beta_p^{i_p}} u_p \right\} (\partial_U^\alpha b)(U),$$

where $|\alpha| = q_0 + \dots + q_n$, $C_{\alpha, \beta_\alpha} \in \mathbb{N}$ (possibly 0) and \sum_{β_α} is taken over all β such that $\sum_{p=0}^n \sum_{i_p=1}^{q_p} \beta_p^{i_p} = k$. The induction hypothesis and the rule (R1) imply that, for all $k \in [1, n]$, $(\varepsilon \partial_t)^k b(U) \in H^{s-k}(\Omega)$ with norm bounded by $C(M)$. Similarly, $(\varepsilon \partial_t)^k E(S, \varepsilon u)$ expands in the same way and we have the same conclusion. These estimates conclude the proof of Lemma 2.3. \square

Remark 3. We infer from the previous identities that, for all $T < T_\varepsilon$,

$$\partial_t S \in X^{s-1}([0, T] \times \Omega) \quad \text{with } \|\partial_t S^\varepsilon\|_{s-1, \varepsilon, T} \leq C(M_\varepsilon(T)), \quad (2.16)$$

$$\tilde{b}(S^\varepsilon, u^\varepsilon) \in X^s([0, T] \times \Omega) \quad \text{with } \|\tilde{b}(S^\varepsilon, u^\varepsilon)\|_{s, \varepsilon, T} \leq C(M_\varepsilon(T)), \quad (2.17)$$

$$\tilde{E}(S^\varepsilon, \varepsilon u^\varepsilon) \in X^s([0, T] \times \Omega) \quad \text{with } \|\tilde{E}(S^\varepsilon, \varepsilon u^\varepsilon)\|_{s, \varepsilon, T} \leq C(M_\varepsilon(T)). \quad (2.18)$$

Lemma 2.4. *We can improve (2.18). Let $T < T_\varepsilon$, we have*

$$\partial_t E(S^\varepsilon, \varepsilon u^\varepsilon) \in X^{s-1}([0, T] \times \Omega) \quad \text{with } \|\partial_t E\|_{s-1, \varepsilon, T} \leq C(M_\varepsilon(T)). \quad (2.19)$$

Proof. With $U := (S, u) = (u_0, u_1, \dots, u_n)$, $\partial_t (\varepsilon \partial_t)^{k-1} E(S, \varepsilon u)$ expands as

$$\sum_{\substack{\alpha=(q_0, \dots, q_n) \\ 1 \leq |\alpha| \leq k}} \sum_{\beta_\alpha} C_{\alpha, \beta_\alpha} \varepsilon^{\ell_{\alpha, \beta_\alpha}} \left\{ \prod_{\substack{p=0, \dots, n \\ q_p \neq 0}} \prod_{\substack{i_p=1, \dots, q_p \\ \beta_p^{i_p} \neq 0}} (\varepsilon \partial_t)^{\beta_p^{i_p}} u_p \right\} (\partial_U^\alpha E)(S, \varepsilon u),$$

with the previous notations and $\ell_{\alpha, \beta_\alpha} = (\sum_{p=1}^n \sum_{i_p=1}^{q_p} \beta_p^{i_p}) - 1$.

We prove that each term is bounded in $H^{s-k}(\Omega)$ by $C(M_\varepsilon(T))$.

Either $\ell_{\alpha, \beta_\alpha} \geq 0$, then the result follows from the rule (R1), or $\ell_{\alpha, \beta_\alpha} = -1$, which implies that $\alpha = (q_0, 0, \dots, 0)$. Which in turn implies

$$\varepsilon^{-1} \prod_{\substack{p=0, \dots, n \\ q_p \neq 0}} \prod_{\substack{i_p=1, \dots, q_p \\ \beta_p^{i_p} \neq 0}} (\varepsilon \partial_t)^{\beta_p^{i_p}} u_p = \partial_t (\varepsilon \partial_t)^{\gamma_1} S (\varepsilon \partial_t)^{\gamma_2} S \dots (\varepsilon \partial_t)^{\gamma_r} S,$$

where $\gamma_1 + \dots + \gamma_r = k - 1$. The proof is therefore completed by applying (2.16) and the rule (R1) with $\sigma = s$. \square

Now, we are looking for L^2 estimates for the partially linearized equations

$$E(\partial_t \dot{u} + b \cdot \partial_x \dot{u}) + \frac{1}{\varepsilon} L(\partial_x) \dot{u} = F, \quad (2.20)$$

$$\dot{v}|_{\partial\Omega} \cdot \nu = 0, \quad (2.21)$$

where $\dot{u} = (\dot{q}, \dot{v})^t$, $E := E(S^\varepsilon, \varepsilon u^\varepsilon)$ and $b := b(S^\varepsilon, u^\varepsilon)$.

Lemma 2.5. *Given $s > 1 + d/2$ and M_0 , there is a constant C_0 and a non-negative function $C(\cdot)$, such that for all $\varepsilon \in]0, 1]$, and all $\dot{u} \in C^0([0, T], H^1(\Omega))$ and $F \in C^0([0, T], L^2(\Omega))$ satisfying (2.20) and (2.21),*

$$\|\dot{u}(t)\|_0^2 \leq C_0 e^{tC(M_\varepsilon(t))} \left(\|\dot{u}(0)\|_0^2 + \int_0^t \|F(\tau)\|_0^2 d\tau \right). \quad (2.22)$$

Proof. Multiply the equations (2.20) by \dot{u} and integrate over $[0, t] \times \Omega$. Since $L(\partial_x)$ is formally skew-adjoint and $\dot{v}|_{\partial\Omega} \cdot \nu = 0$, the Green's identity applied to $\dot{u}(t) \in H^1(\Omega)$ shows that the term in $1/\varepsilon$ cancel out. Thanks to assumption (A4), $\nu \cdot b(S, u)|_{\partial\Omega} = 0$, so the boundary terms vanish when we integrate by parts the term in $\partial_x \dot{u}$. The derivatives of the coefficients which appear are $\partial_t E$, $\partial_x E$ and $\partial_x b$, which are estimated in $L^\infty(\Omega)$ by $C(M_\varepsilon)$ according to (2.19), the rule (R2) and the embedding of H^{s-1} into L^∞ . Using the symmetry of the matrices $E(t)$, this implies that $(E(t)\dot{u}(t), \dot{u}(t))_0$ is bounded by

$$(E(0)\dot{u}(0), \dot{u}(0))_0 + 2 \int_0^t \|F(\tau)\|_0 \|\dot{u}(\tau)\|_0 d\tau + \int_0^t C(M_\varepsilon(\tau)) \|\dot{u}(\tau)\|_0^2 d\tau,$$

where $(\cdot, \cdot)_0$ denotes the scalar product in $L^2(\Omega)$. Moreover, we have

$$\|\dot{u}(t)\|_0^2 \leq \|E^{-1}(t)\|_{L^\infty} (E(t)\dot{u}(t), \dot{u}(t))_0.$$

The proof of Lemma 2.4, with E replaced by E^{-1} , applies, and we find that

$$\|E^{-1}(t)\|_{L^\infty} \leq \|E^{-1}(0)\|_{s-1} + t \sup_{\tau \in [0, t]} \|\partial_t E^{-1}(\tau)\|_{s-1} \leq C_0 + tC(M_\varepsilon(t)).$$

The estimate (2.22) results from these bounds and Gronwall's Lemma. \square

As a consequence of the previous results, we prove the following Lemma.

Lemma 2.6. *There exist C_0 and $C(\cdot)$ depending only on M_0 , such that for all $m \leq s$,*

$$\forall \varepsilon \in]0, 1], \quad \forall t \in [0, T_\varepsilon[, \quad \|(\varepsilon \partial_t)^m u^\varepsilon(t)\|_0 \leq C_0 + tC(M_\varepsilon(t)). \quad (2.23)$$

Proof. Let $\tau \leq t < T_\varepsilon$ and $M := M_\varepsilon(t)$. We begin by assuming that $m \leq s - 1$. According to Lemma 2.3, $(\varepsilon \partial_t)^m u$ belongs to $C^0([0, t], H^1(\Omega))$. What is more $(\varepsilon \partial_t)^m u$ satisfies (2.21) and (2.20) with

$$F = G_1 + G_2 := [E \partial_t, (\varepsilon \partial_t)^m] u + \sum_{j=1}^d [b_j E, (\varepsilon \partial_t)^m] \partial_{x_j} u. \quad (2.24)$$

We estimate the L^2 norm of F according to (2.9), (2.17), (2.18), (2.19), the rules (R1) and (R4). First, we have

$$G_1 = - \sum_{p=0}^{m-1} \binom{m}{p} (\partial_t(\varepsilon\partial_t)^{m-p-1}E)(\varepsilon\partial_t)^{p+1}u.$$

Using the rule (R1) (applied with $\sigma = s-1$, $k = p$, $l = m-p-1$), we obtain

$$\begin{aligned} \|G_1(\tau)\|_{s-m} &\leq K \sum_{p=0}^{m-1} \|\partial_t(\varepsilon\partial_t)^{m-p-1}E(\tau)\|_{s-m+p} \|(\varepsilon\partial_t)^{p+1}u(\tau)\|_{s-1-p} \\ \|G_1(\tau)\|_{s-m} &\leq K \|\partial_t E(\tau)\|_{s-1,\varepsilon} \|u(\tau)\|_{s,\varepsilon} \leq C(M_\varepsilon(t)). \end{aligned} \quad (2.25)$$

The technique for estimating G_2 is similar. We get

$$\|G_2(\tau)\|_{s-m} \leq K \sum_{j=1}^d \sum_{p=0}^{m-1} \|b_j(\tau)E(\tau)\|_{s,\varepsilon} \|(\varepsilon\partial_t)^p \partial_{x_j} u(\tau)\|_{s-1-p} \quad (2.26)$$

$$\|G_2(\tau)\|_{s-m} \leq K \|E(\tau)\|_{s,\varepsilon} \|b(\tau)\|_{s,\varepsilon} \|u(\tau)\|_{s,\varepsilon} \leq C(M_\varepsilon(t)). \quad (2.27)$$

In view of the inequalities (2.25) and (2.27) we see that for all $\tau \leq t < T_\varepsilon$, we have $\|F(\tau)\|_0 \leq C(M_\varepsilon(t))$. Next, we apply the energy inequality (2.22) together with the previous L^2 bound on the force term and the elementary inequality $e^{tC(M)} \leq 1 + t\tilde{C}(M)$ for nonnegative t in a compact set. We find that $\|(\varepsilon\partial_t)^m u^\varepsilon(t)\|_0 \leq C_0 + tC(M_\varepsilon(t))$.

When $m = s$, we use the proof given by S. Schochet of the existence theorem (see [19]). He has proved that one can approximate u in $C^0([0, T], H^s(\Omega))$ by a sequence u^n in $C^0([0, T], H^{s+1}(\Omega))$ such that

- (1) For all $n \in \mathbb{N}$, $u^n(0)$ satisfies the compatibility conditions up to order s .
- (2) For all $n \in \mathbb{N}$, u^n is the unique solution of an approximating system.

Then we deduce, from the previous computations applied to u^n , that

$$\|(\varepsilon\partial_t)^s u^n(t)\|_0 \leq C_0(\|u^n(0)\|_s) + tC\left(\sup_{\tau \in [0,t]} \|u^n(\tau)\|_s\right).$$

Taking the limit when n goes to $+\infty$, we get (2.23) at order $m = s$. \square

The forthcoming computations make use of the special structure of (1.2). We denote $\text{curl } w$ the matrix with coefficients $(\text{curl } w)_{i,j} = \partial_{x_j} w_i - \partial_{x_i} w_j$. The idea of the two following estimates is to apply the curl operator to the equations so as to cancel the large term $\varepsilon^{-1} \nabla q^\varepsilon$, and then to estimate the incompressible components.

Introduce the function $r_0(S) = r(S, 0): \mathbb{R} \rightarrow \mathbb{R}$, which is smooth and positive (recall that $E(S, u) = \begin{pmatrix} a(S, u) & 0 \\ 0 & r(S, u)I_d \end{pmatrix}$).

We define $f_1(S, u) = 1 - r_0(S)/r(S, u)$. Hereafter, to shorten notation, we denote $r_0(t) := r_0(S^\varepsilon(t))$ and $f_1(t) := f_1(S^\varepsilon(t), \varepsilon u^\varepsilon(t))$. Note that $r(S^\varepsilon, \varepsilon u^\varepsilon)$ is bounded away from 0 uniformly with respect to ε . Indeed, we have shown in the proof of Lemma 2.5 that the L^∞ norm of $E^{-1}(S^\varepsilon, \varepsilon u^\varepsilon)$ is well-estimated, and so is the L^∞ norm of $1/r(S^\varepsilon, \varepsilon u^\varepsilon)$.

We can factor out $\varepsilon u^\varepsilon$ in f_1 . There exists a smooth function g such that

$$f_1(t) = \varepsilon g(t) := \varepsilon g(S^\varepsilon(t), \varepsilon u^\varepsilon(t)), \quad \text{with} \quad \|g^\varepsilon(t)\|_s \leq C(M_\varepsilon(t)). \quad (2.28)$$

Since $\partial_t S^\varepsilon + b(S^\varepsilon, u^\varepsilon) \cdot \partial_x S^\varepsilon = 0$, the equation for v^ε (recall that $u^\varepsilon = (q^\varepsilon, v^\varepsilon)^t$) is equivalent to the transport equation

$$(\partial_t + b \cdot \partial_x)(r_0 v^\varepsilon) + \frac{1}{\varepsilon} \nabla q^\varepsilon = g \nabla q^\varepsilon, \quad (2.29)$$

with the previous notation.

Using the fact that $\text{curl} \nabla = 0$, we get

$$(\partial_t + b \cdot \partial_x)(\text{curl}(r_0 v^\varepsilon)) = [b \cdot \partial_x, \text{curl}](r_0 v^\varepsilon) + [\text{curl}, g] \nabla q^\varepsilon. \quad (2.30)$$

Lemma 2.7. *There are C_0 and $C(\cdot)$ depending only on M_0 , such that*

$$\forall \varepsilon \in (0, 1], \quad \forall t \in [0, T_\varepsilon), \quad \|\text{curl}(r_0(t)v^\varepsilon(t))\|_{s-1} \leq C_0 + tC(M_\varepsilon(t)). \quad (2.31)$$

Proof. Let $M := M_\varepsilon(t)$. Using the proof of Lemma 2.2, to prove (2.31) it is sufficient to estimate the force term $[b \cdot \partial_x, \text{curl}](r_0 v) + [\text{curl}, g] \nabla q$ in $H^{s-1}(\Omega)$.

For $w \in H^1(\Omega)$, we have $([\text{curl}, g]w)_{i,j} = w_i \partial_{x_j} g - w_j \partial_{x_i} g$. Therefore, by the rule (R1) and the estimate (2.28), we get for all $\tau \leq t < T_\varepsilon$,

$$\|[\text{curl}, g(\tau)] \nabla q(\tau)\|_{s-1} \leq K \|\nabla g(\tau)\|_{s-1} \|\nabla q(\tau)\|_{s-1} \leq C(M). \quad (2.32)$$

Similarly,

$$\|[b_j(\tau), \text{curl}] \partial_{x_j}(r_0(\tau)v(\tau))\|_{s-1} \leq C(M),$$

which concludes the proof of Lemma 2.7. \square

Another result relies upon similar computations.

Lemma 2.8. *Given $s > 1 + d/2$ and M_0 , there exist a constant C_0 and a function $C(\cdot)$ from $[0, \infty)$ to $[0, \infty)$, such that if $\varepsilon \in (0, 1]$, $n \in \{1, \dots, s-1\}$, and $(\dot{u}, F) \in C^0([0, T], H^{n+1}(\Omega))$ satisfies (2.20) and (2.21), then we have*

$$\|\partial_t \text{curl}(r_0(S^\varepsilon(t))\dot{v}(t))\|_{n-1} \leq C(M_\varepsilon(T))(\|F(t)\|_n + \|\dot{u}(t)\|_{n+1}). \quad (2.33)$$

Proof. Let us introduce the notation $F(t, x) = (F_1(t, x), F_2(t, x)) \in \mathbb{R} \times \mathbb{R}^d$. Similar arguments to those above show that $\partial_t \operatorname{curl}(r_0 \dot{v})$ satisfies

$$\partial_t \operatorname{curl}(r_0 \dot{v}) = \operatorname{curl}((1 - f_1)F_2) + [\operatorname{curl}, g] \nabla \dot{q} - \operatorname{curl}(b \cdot \partial_x(r_0 \dot{v})),$$

The claim then follows from the bounds

$$\begin{aligned} \|\operatorname{curl}((1 - f_1)F_2)\|_{n-1} &\leq K \|f_1\|_s \|F_2\|_n + \|F_2\|_n \leq C(M) \|F\|_n, \\ \|[\operatorname{curl}, g] \nabla \dot{q}\|_{n-1} &\leq K \|g\|_s \|\dot{q}\|_n \leq C(M) \|\dot{u}\|_n, \\ \|\operatorname{curl}(b \cdot \partial_x(r_0 \dot{v}))\|_{n-1} &\leq K \|b\|_s \|r_0\|_s \|\dot{v}\|_{n+1} \leq C(M) \|\dot{u}\|_{n+1}. \end{aligned}$$

The foremost estimates are given in Lemma 2.10. We prepare its proof in the following lemma.

Lemma 2.9. *Given $s > 1 + d/2$ and M_0 , there exist a constant C_0 and a function $C(\cdot)$ from $[0, \infty)$ to $[0, \infty)$, such that if $\varepsilon \in (0, 1]$, $n \in \{0, \dots, s-1\}$, and $(\dot{u}, F) \in X^{n+1}([0, T] \times \Omega)$ satisfy (2.20) and (2.21), then*

$$\|L(\partial_x) \dot{u}(t)\|_n \leq \tilde{C} \|\varepsilon \partial_t \dot{u}(t)\|_n + \varepsilon C(M) \|\dot{u}(t)\|_{n+1} + \varepsilon \|F(t)\|_n \quad (2.34)$$

with $\tilde{C} := C_0 + tC(M)$, and $M := M_\varepsilon(T)$.

Proof. We rewrite (2.20):

$$L(\partial_x) \dot{u} = -E(S, \varepsilon u) (\varepsilon \partial_t \dot{u} + \varepsilon b(S, u) \cdot \partial_x \dot{u}) + \varepsilon F. \quad (2.35)$$

We estimate $\|E(t)\|_{s-1}$ by the bound of its time derivative, see (2.19), and the identity

$$\|E(t)\|_{s-1} \leq \|E(t=0)\|_{s-1} + t \sup_{\tau \in [0, t]} \|\partial_t E(\tau)\|_{s-1}.$$

It follows that $\|E(t)\|_{s-1} \leq \tilde{C} = C_0 + tC(M)$.

Therefore, the multiplicative property (R1), applied with $\sigma = s-1$, yields

$$\|E(t)(\varepsilon \partial_t) \dot{u}(t)\|_n \leq (C_0 + tC(M)) \|\varepsilon \partial_t \dot{u}\|_n.$$

Again, the rule (R1) implies

$$\|\varepsilon E(t) b(t) \cdot \partial_x \dot{u}(t)\|_n \leq \varepsilon C(M) \|\partial_x \dot{u}(t)\|_n.$$

The identity (2.35) and these bounds imply (2.34). \square

We now come to our main estimates. First, we recall the following useful elliptic estimate, for all $v \in \mathbf{H}^{k+1}(\Omega)$:

$$\|v\|_{k+1} \leq K \left(\|\operatorname{div} v\|_k + \|\operatorname{curl} v\|_k + \|v\|_k + \|v \cdot \nu\|_{k+1/2, \partial\Omega} \right).$$

This estimate is proved in [2] when Ω is a bounded domain; it is clear when Ω is the whole space \mathbb{R}^d using the Fourier transform. As a consequence we

can easily extend such a result to the exterior case. Recall that we assume only that Ω has a compact, smooth boundary, possibly empty if $\Omega = \mathbb{R}^d$.

Lemma 2.10. *Given $s > 1 + d/2$ and M_0 , there is a constant C_0 and a function $C(\cdot)$ from $[0, \infty)$ to $[0, \infty)$, such that if $\varepsilon \in (0, 1)$, $m \in \{1, \dots, s\}$, and $\dot{u}, F \in X^m([0, T] \times \Omega)$ satisfy (2.20) and (2.21), then*

$$\begin{aligned} \|\dot{u}(t)\|_{m,\varepsilon} \leq & \tilde{C} \left(\|(\varepsilon \partial_t)^m \dot{u}(t)\|_0 + \|\operatorname{curl}(r_0(t)\dot{v}(t))\|_{m-1} + \|\dot{u}(t)\|_{m-1,\varepsilon} \right) \\ & + \varepsilon C(M) \left(\|F(t)\|_{m,\varepsilon} + \|\dot{u}(t)\|_{m,\varepsilon} \right), \end{aligned} \quad (2.36)$$

with $M = M_\varepsilon(T)$ and $\tilde{C} \equiv C_0 + tC(M)$.

Remark 4. We will apply this lemma with $(\dot{u}, F) = (u^\varepsilon, 0)$, yet we introduce (\dot{u}, F) so as to prove the desired estimates by induction on m .

Proof. We prove (2.36) by induction on $m \in \{1, \dots, s\}$. Assume the result at order $m = n < s$, and suppose that $(\dot{u}, F) \in X^{n+1}([0, T] \times \Omega)$ satisfy (2.20) and (2.21).

By definition (2.5) we have

$$\|\dot{u}(t)\|_{n+1,\varepsilon} = \|\varepsilon \partial_t \dot{u}(t)\|_{n,\varepsilon} + \|\dot{u}(t)\|_{n+1}. \quad (2.37)$$

We use Lemma 2.8 and the induction hypothesis to estimate $\|\varepsilon \partial_t \dot{u}(t)\|_{n,\varepsilon}$. As regards $\|\dot{u}(t)\|_{n+1}$, we use Lemma 2.9 and the estimate

$$\|v\|_{k+1} \leq K \left(\|\operatorname{div} v\|_k + \|\operatorname{curl} v\|_k + \|v\|_k + \|v \cdot \nu\|_{k+1/2, \partial\Omega} \right). \quad (2.38)$$

In view of the boundary condition (2.21), identities (2.37) and (2.38) result in

$$\|\dot{u}(t)\|_{n+1,\varepsilon} \leq \|\varepsilon \partial_t \dot{u}(t)\|_{n,\varepsilon} + K \left(\|L(\partial_x) \dot{u}(t)\|_n + \|\operatorname{curl} \dot{u}(t)\|_n + \|\dot{u}(t)\|_n \right); \quad (2.39)$$

recall that $L(\partial_x) \dot{u} = (\operatorname{div} \dot{v}, \nabla \dot{q})^t$.

By the multiplicative rule (R1) we infer that

$$\|\operatorname{curl} \dot{v}(t)\|_n \leq K \|r_0^{-1}(t)\|_{s-1} \|r_0(t) \operatorname{curl}(\dot{v}(t))\|_n. \quad (2.40)$$

Furthermore, as in (2.32) we have

$$\|r_0(t) \operatorname{curl} \dot{v}(t) - \operatorname{curl}(r_0(t)\dot{v}(t))\|_n \leq K \|r_0(t)\|_s \|\dot{v}(t)\|_n. \quad (2.41)$$

Since $(\partial_t + b \cdot \partial_x) r_0(S) = 0$, we infer from the proof of Lemma 2.2 that $\|r_0(t)\|_s \leq \tilde{C}$; similarly, $\|r_0^{-1}(t)\|_s \leq \tilde{C}$. Therefore, the inequalities (2.40) and (2.41) imply that

$$\|\operatorname{curl} \dot{v}(t)\|_n \leq \tilde{C} \left(\|\operatorname{curl}(r_0(t)\dot{v}(t))\|_n + \|\dot{v}(t)\|_n \right).$$

Thus, the estimate (2.39) turns into

$$\begin{aligned} \|\dot{u}(t)\|_{n+1,\varepsilon} &\leq \|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} + K \|L(\partial_x)\dot{u}(t)\|_n \\ &\quad + \tilde{C} \|\operatorname{curl}(r_0(t)\dot{v}(t))\|_n + \tilde{C} \|\dot{u}(t)\|_n. \end{aligned} \quad (2.42)$$

From Lemma 2.9 and (2.42), we get

$$\begin{aligned} \|\dot{u}(t)\|_{n+1,\varepsilon} &\leq \tilde{C} \left(\|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} + \|\operatorname{curl}(r_0(t)\dot{v}(t))\|_n + \|\dot{u}(t)\|_n \right) \\ &\quad + \varepsilon C(M) \|\dot{u}(t)\|_{n+1,\varepsilon} + \varepsilon K \|F(t)\|_n. \end{aligned} \quad (2.43)$$

Next, we want to apply the induction hypothesis with $\varepsilon\partial_t\dot{u}$. We commute $\varepsilon\partial_t$ with (2.20):

$$\begin{aligned} E(\partial_t(\varepsilon\partial_t)\dot{u} + b \cdot \partial_x(\varepsilon\partial_t)\dot{u}) + \frac{1}{\varepsilon}L(\partial_x)(\varepsilon\partial_t)\dot{u} &= G(t), \\ G(t) &:= \varepsilon\partial_t F - \varepsilon\partial_t E\partial_t\dot{u} - \sum_{j=1}^d \varepsilon\partial_t(Eb_j)\partial_{x_j}\dot{u}. \end{aligned}$$

We apply the rule (R4) in order to estimate the force term $G(t)$; it yields

$$\begin{aligned} \|G(t)\|_{n,\varepsilon} &\leq \|\varepsilon\partial_t F(t)\|_{n,\varepsilon} + K \|\partial_t E(t)\|_{s-1,\varepsilon} \|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} \\ &\quad + K \|\varepsilon\partial_t \sum_{j=1}^d E(t)b_j(t)\|_{s-1,\varepsilon} \|\partial_x\dot{u}(t)\|_{n,\varepsilon} \\ &\leq \|F(t)\|_{n+1,\varepsilon} + K (\|\partial_t E(t)\|_{s-1,\varepsilon} + \|E(t)\|_{s,\varepsilon} \|b(t)\|_{s,\varepsilon}) \|\dot{u}(t)\|_{n+1,\varepsilon} \\ &\leq \|F(t)\|_{n+1,\varepsilon} + C(M) \|\dot{u}(t)\|_{n+1,\varepsilon}. \end{aligned} \quad (2.44)$$

We have estimated $\|\partial_t E(t)\|_{s-1,\varepsilon}$ and $\|E(t)\|_{s,\varepsilon} \|b(t)\|_{s,\varepsilon}$ by $C(M)$ according to (2.17), (2.18), and (2.19).

Next, we apply the induction hypothesis with (\dot{u}, F) replaced by the $(\varepsilon\partial_t\dot{u}, G)$, which belong to $X^n([0, T] \times \Omega)$. We obtain

$$\begin{aligned} \|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} &\leq \tilde{C} \left(\|(\varepsilon\partial_t)^n \varepsilon\partial_t\dot{u}(t)\|_0 + \|\operatorname{curl}(r_0(t)\varepsilon\partial_t\dot{v}(t))\|_{n-1} \right. \\ &\quad \left. + \|\varepsilon\partial_t\dot{u}(t)\|_{n-1,\varepsilon} \right) + \varepsilon C(M) \left(\|G(t)\|_{n,\varepsilon} + \|\varepsilon\partial_t\dot{u}(t)\|_{n,\varepsilon} \right). \end{aligned} \quad (2.45)$$

We have $\operatorname{curl}(r_0(\varepsilon\partial_t)\dot{v}) = \varepsilon\partial_t\operatorname{curl}(r_0\dot{v}) - \varepsilon\operatorname{curl}((\partial_t r_0)\dot{v})$. Therefore, we infer that

$$\|\operatorname{curl}(r_0(\varepsilon\partial_t)\dot{v})\|_{n-1} \leq \varepsilon\|\partial_t\operatorname{curl}(r_0\dot{v})\|_{n-1} + \varepsilon K \|\partial_t r_0\|_{s-1} \|\dot{v}\|_n.$$

Making use of Lemma 2.8 to estimate $\|\partial_t \operatorname{curl}(r_0 \dot{v})\|_{n-1}$, and the bound $\|\partial_t r_0(t)\|_{s-1} = \|b \cdot \partial_x r_0(t)\|_{s-1} \leq C(M)$, we end up with

$$\|\operatorname{curl}(r_0(t)(\varepsilon \partial_t) \dot{v}(t))\|_{n-1} \leq \varepsilon C(M) (\|F(t)\|_{n+1} + \|\dot{u}(t)\|_{n+1}). \quad (2.46)$$

Combining the inequalities (2.44) and (2.46) with (2.45), we obtain

$$\begin{aligned} \|\varepsilon \partial_t \dot{u}(t)\|_{n,\varepsilon} &\leq \tilde{C} \left(\|(\varepsilon \partial_t)^{n+1} \dot{u}(t)\|_0 + \|\dot{u}(t)\|_{n,\varepsilon} \right) + \\ &\quad \varepsilon C(M) \left(\|F(t)\|_{n+1,\varepsilon} + \|\dot{u}(t)\|_{n+1,\varepsilon} \right). \end{aligned} \quad (2.47)$$

Finally, using the estimates (2.43) and (2.47), we conclude that

$$\begin{aligned} \|\dot{u}(t)\|_{n+1,\varepsilon} &\leq \tilde{C} \left(\|(\varepsilon \partial_t)^{n+1} \dot{u}(t)\|_0 + \|\operatorname{curl}(r_0(t) \dot{v}(t))\|_n + \|\dot{u}(t)\|_{n,\varepsilon} \right) \\ &\quad + \varepsilon C(M) (\|F(t)\|_{n+1,\varepsilon} + \|\dot{u}(t)\|_{n+1,\varepsilon}), \end{aligned}$$

which is the lemma at order $m = n + 1$.

It remains to prove the Lemma for $m = 1$. Notice that we derived (2.43) without the use of the induction hypothesis. So we can let $n = 0$ in (2.43), which gives the result at order $m = 1$ and completes the proof of Lemma 2.10. \square

Recall that the upshot of all previous lemmas is to prove a bound analogous to (2.6) for u^ε .

Lemma 2.11. *Given $s > 1 + d/2$ and M_0 , there is a constant C_0 and a function $C(\cdot)$ from $[0, +\infty)$ to $[0, +\infty)$ such that*

$$\forall \varepsilon \in (0, 1], \quad \forall t \in [0, T_\varepsilon), \quad \|u^\varepsilon(t)\|_s \leq C_0 + (t + \varepsilon)C(M_\varepsilon(t)). \quad (2.48)$$

Proof. Let $t < T_\varepsilon$ and $M = M_\varepsilon(t)$. The L^2 norm of $u(t)$ is estimated by Lemma 2.5, taking $F = 0$. Next, we prove by induction on $\sigma \in \{0, \dots, s\}$ that there are C_0 and $C(\cdot)$ such that $\|u(t)\|_{\sigma,\varepsilon} \leq C_0 + (t + \varepsilon)C(M_\varepsilon(t))$. Assume the result at order $\sigma = n < s$. The inequality (2.36), with (m, \dot{u}, F) replaced by $(n + 1, u, 0)$, applies, and we find that

$$\begin{aligned} \|u(t)\|_{n+1,\varepsilon} &\leq \tilde{C} \left(\overbrace{\|(\varepsilon \partial_t)^{n+1} u(t)\|_0}^A + \overbrace{\|\operatorname{curl}(r_0(t) v(t))\|_n}^B \right) \\ &\quad + \overbrace{\|u(t)\|_{n,\varepsilon}}^C + \underbrace{\varepsilon C(M)}_D \|u(t)\|_{n+1,\varepsilon}. \end{aligned} \quad (2.49)$$

We list our bounds, with $M := M_\varepsilon(t)$:

$$\text{by Lemma 2.6} \quad A = \|(\varepsilon \partial_t)^{n+1} u(t)\|_0 \leq C_0 + tC(M),$$

$$\begin{aligned}
&\text{by Lemma 2.7} && B = \|\text{curl}(r_0(S)v(t))\|_n \leq C_0 + tC(M), \\
&\text{by induction hypothesis} && C = \|u(t)\|_{n,\varepsilon} \leq C_0 + (t + \varepsilon)C(M), \\
&\text{by Lemma 2.3} && D = \|u(t)\|_{n+1,\varepsilon} \leq C(M).
\end{aligned}$$

Thus, we deduce from (2.49) that

$$\begin{aligned}
\|u(t)\|_{n+1,\varepsilon} &\leq \tilde{C}(\tilde{C} + \tilde{C} + C_0 + (t + \varepsilon)C(M)) + \varepsilon C(M)C(M) \\
&\leq C_0 + (t + \varepsilon)C(M),
\end{aligned}$$

which is the result at order $\sigma = n + 1$. The last estimate, with $\sigma = s$, is (2.48) stated in sharper form. \square

From the estimates (2.6) and (2.48) we easily prove Proposition 2.1, which in turn implies the existence of classical solutions for a time independent of the small parameter. The uniform bounds (1.4) and (1.5) are immediate consequences of the previous computations. For instance, (1.5) results from Lemma 2.8 applied with $(\dot{u}, F) = (u^\varepsilon, 0)$. Theorem 1.1 is proved.

3. PROOF OF THEOREM 1.2

In this section we assume that Ω is the exterior of a bounded domain lying on one side of its compact, smooth boundary (which includes the case of \mathbb{R}^d). Recall that $s > d/2 + 1$ is an integer; in particular, $s \geq 2$.

3.1. Computation of the limit system. The uniform bounds (1.4) and (1.5) imply that, up to the extraction of subsequences, one has the following convergences:

$$(q^\varepsilon, v^\varepsilon) \rightharpoonup (q, v) \text{ weakly } \star \text{ in } L^\infty([0, T], H^s(\Omega)), \quad (3.1)$$

$$S^\varepsilon \rightarrow S \text{ strongly in } C^0([0, T], H_{\text{loc}}^{s'}(\Omega)), \text{ for all } s' < s, \quad (3.2)$$

$$\text{curl}(r_0^\varepsilon v^\varepsilon) \rightarrow \text{curl}(r_0 v) \text{ strongly in } C^0([0, T], H_{\text{loc}}^{s'}(\Omega)), \text{ for all } s' < s - 1. \quad (3.3)$$

We prove that $q = 0$ and $\text{div } v = 0$. First notice that

$$\varepsilon E(S^\varepsilon, \varepsilon u^\varepsilon) \partial_t u^\varepsilon + L(\partial_x) u^\varepsilon = -\varepsilon E(S^\varepsilon, \varepsilon u^\varepsilon) b(S^\varepsilon, u^\varepsilon) \cdot \partial_x u^\varepsilon;$$

using the uniform bounds (1.4) and $E(S^\varepsilon, \varepsilon u^\varepsilon) - E_0(S^\varepsilon) = O(\varepsilon)$, we obtain

$$\varepsilon E_0(S^\varepsilon) \partial_t u^\varepsilon + L(\partial_x) u^\varepsilon = \varepsilon f^\varepsilon, \quad (3.4)$$

where $E_0(S) = E(S, 0)$ and $(f^\varepsilon)_{\varepsilon > 0}$ is a bounded family in $C^0([0, T], H^{s-1})$. Passing to the weak limit shows that $\nabla q = 0$ and $\text{div } v = 0$. Since $q \in L^\infty(H^s)$, and the Lebesgue measure of the open, connected set Ω is $+\infty$, we infer $q = 0$.

In order to compute the limit system the main point is to prove that the convergence (3.1) holds in the strong topology of $L^2([0, T], H_{\text{loc}}^{s'}(\Omega))$. Indeed, once this is established the proof proceeds as in the whole-space case (see [15]). The only difference is that the Leray's projectors, which are useful to derive the term $\nabla\pi$ in (1.6) and the initial data w_0 , are defined in \mathbb{R}^d as Fourier multipliers. In an exterior domain Ω we use the definitions and properties of these operators given in [9] (in particular, they are bounded operators from $H^m(\Omega)$ to $H^m(\Omega)$). Notice that the uniqueness of the solution of the limit system implies that the whole family converges.

We have strong compactness for the incompressible components by (3.3), so it is sufficient to focus attention on the acoustic components.

Proposition 3.1. *Assume that the hypotheses of Theorem 1.2 are satisfied; then q^ε converges strongly to 0 in $L^2([0, T], H_{\text{loc}}^{s'}(\Omega))$ for all $s' < s$, and $\text{div } v^\varepsilon$ converges strongly to 0 in $L^2([0, T], H_{\text{loc}}^{s'}(\Omega))$ for all $s' < s - 1$.*

The end of the paper is devoted to the proof of Proposition 3.1.

Analogous strong compactness was proved in [9] via the spectral and scattering theory for the linearized equations of acoustics, and in [20] by Strichartz estimates. Both proofs rely upon the fact that they have considered isentropic equations.

In the nonisentropic case we follow the analysis of G. Métivier and S. Schochet ([15, 17]). We introduce some semiclassical measures and prove that they vanish, which imply the strong compactness in time. Together with the strong compactness in space proved in Theorem 1.1, this gives Proposition 3.1.

The semiclassical measures will be defined as defect measures of wave-packets transforms. The definitions rely upon the works of P. Gérard (see [4]) and P.-L. Lions and T. Paul (see [13]). In [4] microlocal defect measures are defined for bounded sequences in $L_{\text{loc}}^2(\mathcal{V}, H)$, where \mathcal{V} is an open set of \mathbb{R}^d and H is a separable Hilbert space. It leads to positive measure on the cosphere bundle of \mathcal{V} by use of Gårding's inequality. And in [13] semiclassical measures are defined by means of Wigner transform. It leads to positive measures via the Husimi's transform. Gårding's inequality has been known to be related to the wave-packets transform since the work of A. Córdoba and C. Fefferman [3]. Moreover, in [13], the authors point out the connection between Husimi's transform and the wave-packets transform.

3.2. The wave-packets transform. Here we introduce the wave-packets transform associated to the scale ε^{-1} , W^ε . It appears as a suitable tool to measure in the phase plane how much of a function oscillates at frequencies $O(\varepsilon^{-1})$.

Proposition 3.2. *Let $C_{t,0}^1(\mathbb{R} \times \mathcal{Q})$ denote the subspace of $C^1(\mathbb{R} \times \mathcal{Q})$ whose elements are compactly supported in the time variable t . Let v be a function in $C_{t,0}^1(\mathbb{R} \times \overline{\Omega}) \cap L^2(\mathbb{R} \times \Omega)$. We define its wave-packets transform associated to the scale ε^{-1} by*

$$W^\varepsilon v(t, \tau, x) = c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{i(t-s)\tau - (t-s)^2/\varepsilon} v(s, x) ds.$$

Then $W^\varepsilon v \in C^1(\mathbb{R} \times \mathbb{R} \times \overline{\Omega}) \cap L^2(\mathbb{R} \times \mathbb{R} \times \Omega)$, and, with $c = (2\pi^3)^{-1/4}$, W^ε extends as an isometry from $L^2(\mathbb{R} \times \Omega)$ to $L^2(\mathbb{R} \times \mathbb{R} \times \Omega)$.

Proof. We start from

$$|W^\varepsilon v(t, \tau, x)|^2 = c^2 \varepsilon^{-6/4} \iint e^{i(u-s)\tau - (t-s)^2 - (t-u)^2/\varepsilon} v(s, x) \overline{v(u, x)} ds du.$$

We want to integrate the last integral with respect to $(t, \tau, x) \in \mathbb{R} \times \mathbb{R} \times \Omega$. In order to deal with absolutely convergent integrals we use the usual Gauss summability method. More precisely, we introduce an additional term $e^{-\delta^2 \tau^2}$, and let δ go to 0. Yet, as usual, it is equivalent to say that

$$\int_{\mathbb{R}} e^{i(u-s)\tau/\varepsilon} d\tau = 2\pi\varepsilon\delta_0(u-s), \quad (3.5)$$

where δ_0 is the Dirac function. The end of the proof is a matter of straightforward computations. \square

We aim to introduce the wave-packets transform of u^ε , which means that we have to carefully extend the functions to $t \in \mathbb{R}$. Hereafter, a subscript zero indicates compact support. Let $\chi_\varepsilon \in C_0^\infty((0, T))$ be a family of functions such that $\chi_\varepsilon(t) = 1$ for $t \in [\varepsilon^{1/2}, T - \varepsilon^{1/2}]$ and $\|\varepsilon\partial_t \chi_\varepsilon\|_{L^\infty} \leq 2\varepsilon^{1/2}$.

We set

$$\tilde{u}^\varepsilon = \begin{pmatrix} \tilde{q}^\varepsilon \\ \tilde{v}^\varepsilon \end{pmatrix} = \chi_\varepsilon u^\varepsilon = \begin{pmatrix} \chi_\varepsilon q^\varepsilon \\ \chi_\varepsilon v^\varepsilon \end{pmatrix}.$$

Next, we choose extensions \tilde{S}^ε of S^ε , supported in $t \in [-1, T+1]$, uniformly bounded in $C_0(\mathbb{R}, H^s(\Omega))$, and converging to \tilde{S} in $C_0(\mathbb{R}, H_{\text{loc}}^{s'}(\Omega))$.

According to (3.4), \tilde{u}^ε satisfies

$$\varepsilon E_0(\tilde{S}^\varepsilon) \partial_t \tilde{u}^\varepsilon + L(\partial_x) \tilde{u}^\varepsilon = \varepsilon \tilde{f}^\varepsilon, \quad (3.6)$$

where $(\tilde{f}^\varepsilon)_{\varepsilon>0}$ is a bounded family in $C_0(\mathbb{R}, H^{s-1}(\Omega))$. Using the rule (R1), we infer that $((\varepsilon\partial_t)\tilde{u}^\varepsilon)_{\varepsilon>0}$ is a bounded family in $C_0(\mathbb{R}, H^{s-1}(\Omega))$.

Lemma 3.3. *Let $U^\varepsilon = W^\varepsilon \tilde{u}^\varepsilon$; then*

$$F^\varepsilon := (i\tau E_0(\tilde{S}^\varepsilon(t)) + L(\partial_x))U^\varepsilon \rightarrow 0 \quad \text{in } L^2(\mathbb{R}^2, H^1(\Omega)) \quad \text{as } \varepsilon \rightarrow 0. \quad (3.7)$$

Proof. Since $[W^\varepsilon, L(\partial_x)] = 0$, it results from equation (3.6) that

$$F^\varepsilon - \varepsilon W^\varepsilon \tilde{f}^\varepsilon = [E_0(\tilde{S}^\varepsilon), W^\varepsilon](\varepsilon \partial_t \tilde{u}^\varepsilon + E_0(\tilde{S}^\varepsilon)(i\tau W^\varepsilon \tilde{u}^\varepsilon - W^\varepsilon(\varepsilon \partial_t \tilde{u}^\varepsilon))). \quad (3.8)$$

Since $\varepsilon W^\varepsilon \tilde{f}^\varepsilon$ obviously converges to 0 in $L^2(\mathbb{R}^2, H^1(\Omega))$, we aim to prove that the commutators which appear in the right-hand side of (3.8) converge to 0.

If $\varepsilon \partial_t v \in L^2(\mathbb{R} \times \Omega)$, then

$$W^\varepsilon(\varepsilon \partial_t v) - i\tau W^\varepsilon v = 2c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{(i(t-s)\tau - (t-s)^2)/\varepsilon} (s-t)v(s, x) ds;$$

therefore, using (3.5) and $[W^\varepsilon, \partial_x] = 0$, we easily get

$$\|W^\varepsilon(\varepsilon \partial_t v) - i\tau W^\varepsilon v\|_{L^2(\mathbb{R}^2, H^1(\Omega))} \leq K\sqrt{\varepsilon} \|v\|_{L^2(\mathbb{R}, H^1(\Omega))}. \quad (3.9)$$

Now, let $a \in C_0^1(\mathbb{R}, H^s(\Omega))$; we have

$$aW^\varepsilon v - W^\varepsilon(av) = c\varepsilon^{-3/4} \int_{\mathbb{R}} e^{(i(t-s)\tau - (t-s)^2)/\varepsilon} (a(t, x) - a(s, x))v(s, x) ds.$$

Again, (3.5) implies that

$$\begin{aligned} & \|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2 \times \Omega)}^2 \\ &= c_2\varepsilon^{-1/2} \iint e^{-2(t-s)^2/\varepsilon} |a(t, x) - a(s, x)|^2 |v(s, x)|^2 ds dt dx. \end{aligned} \quad (3.10)$$

Using the embedding of $H^{s-1}(\Omega)$ into $L^\infty(\Omega)$, we deduce that

$$\begin{aligned} & \|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2 \times \Omega)}^2 \\ & \leq c_2\varepsilon^{-1/2} \iint e^{-2(t-s)^2/\varepsilon} \|a(t) - a(s)\|_{H^{s-1}(\Omega)}^2 \|v(s)\|_{L^2(\Omega)}^2 dt ds. \end{aligned}$$

Hence, we easily obtain

$$\|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2 \times \Omega)} \leq K\sqrt{\varepsilon} \|a\|_{C_0^1(\mathbb{R}, H^{s-1}(\Omega))} \|v\|_{L^2(\mathbb{R}, L^2(\Omega))}.$$

From the previous bound and $[W^\varepsilon, \partial_x] = 0$, we get

$$\|aW^\varepsilon v - W^\varepsilon(av)\|_{L^2(\mathbb{R}^2, H^1(\Omega))} \leq K\sqrt{\varepsilon} \|a\|_{C_0^1(\mathbb{R}, H^s(\Omega))} \|v\|_{L^2(\mathbb{R}, H^1(\Omega))}. \quad (3.11)$$

Recall that $(E_0(\tilde{S}^\varepsilon))_\varepsilon$ is a bounded family in $C_0^1(\mathbb{R}, H^s(\Omega))$, $(\tilde{u}^\varepsilon)_\varepsilon$ is a bounded family in $L^2(\mathbb{R}, H^1(\Omega))$, and $((\varepsilon \partial_t \tilde{u}^\varepsilon))_{\varepsilon > 0}$ is a bounded family in $L^2(\mathbb{R}, H^1(\Omega))$. Consequently, from (3.8), (3.9), and (3.11) we infer that $(F^\varepsilon - \varepsilon W^\varepsilon \tilde{f}^\varepsilon)$ converges to 0 in $L^2(\mathbb{R}^2, H^1(\Omega))$. This completes the proof. \square

Let us next introduce a series of notations so as to put into relief the relevant wave equation. First, we set

$$\begin{aligned} U^\varepsilon &= (\Psi^\varepsilon, m^\varepsilon) = (W^\varepsilon \tilde{q}^\varepsilon, W^\varepsilon \tilde{v}^\varepsilon) \in L^2(\mathbb{R}^2, \mathbf{H}^s(\Omega)) \times L^2(\mathbb{R}^2, \mathbf{H}^s(\Omega)^d) \\ F^\varepsilon &= (F_1^\varepsilon, F_2^\varepsilon) \in L^2(\mathbb{R}^2, \mathbf{H}^1(\Omega)) \times L^2(\mathbb{R}^2, \mathbf{H}^1(\Omega)^d). \end{aligned}$$

From the definition $F^\varepsilon := (i\tau E_0(S^\varepsilon(t)) + L(\partial_x))U^\varepsilon$, we obtain

$$i\tau a_0(\tilde{S}^\varepsilon(t))\Psi^\varepsilon + \operatorname{div}(m^\varepsilon) = F_1^\varepsilon, \quad (3.12)$$

$$i\tau r_0(\tilde{S}^\varepsilon(t))m^\varepsilon + \nabla\Psi^\varepsilon = F_2^\varepsilon. \quad (3.13)$$

To shorten notation, we denote $a_0(S) := a(S, 0)$, $r_0(S) := r(S, 0)$, and

$$\begin{aligned} P_\varepsilon(t, \tau, \partial_x) &:= a_0(\tilde{S}^\varepsilon(t))\tau^2 + \operatorname{div}\left(\frac{1}{r_0(\tilde{S}^\varepsilon(t))}\nabla\cdot\right), \\ P_0(t, \tau, \partial_x) &:= a_0(\tilde{S}(t))\tau^2 + \operatorname{div}\left(\frac{1}{r_0(\tilde{S}(t))}\nabla\cdot\right). \end{aligned}$$

From (3.12) and (3.13), we obtain

$$P_\varepsilon(t, \tau, \partial_x)\Psi^\varepsilon = -i\tau F_1^\varepsilon + i \operatorname{div}\left(\frac{1}{r_0(\tilde{S}^\varepsilon(t))}F_2^\varepsilon\right). \quad (3.14)$$

The following equation is simple to derive but very important, because it is the only step in which the boundary conditions are useful. Using the fact that $v_{\mathbb{R} \times \partial\Omega}^\varepsilon \cdot \nu = 0$, we obtain that $m^\varepsilon \cdot \nu = 0$ on the boundary $\mathbb{R} \times \mathbb{R} \times \partial\Omega$. Taking the dot product of (3.13) with ν on the boundary, we infer that

$$\partial_\nu \Psi^\varepsilon := \nabla\Psi^\varepsilon \cdot \nu = F_2^\varepsilon \cdot \nu \quad \text{on } \mathbb{R} \times \mathbb{R} \times \partial\Omega, \quad (3.15)$$

which is meaningful since we have seen that $F^\varepsilon \in L^2(\mathbb{R}^2, \mathbf{H}^1(\Omega))$.

Finally it appears useful to introduce a new family in order to change the characteristic variety of the equation (3.14). We set

$$\Theta^\varepsilon := \Psi^\varepsilon - \Delta\Psi^\varepsilon = \Psi^\varepsilon - \sum_{j=1}^d \partial_{x_j}^2 \Psi^\varepsilon.$$

Since $s \geq 2$, $(\Theta^\varepsilon)_\varepsilon$ is a bounded family in $L^2(\mathbb{R}^2 \times \Omega)$.

Conversely, we can use (3.15) to recover Ψ^ε from Θ^ε and F_2^ε ,

$$\Psi^\varepsilon = (1 - \Delta_N)^{-1}\Theta^\varepsilon + N(F_2^\varepsilon \cdot \nu),$$

where the operators are defined below. Let $g \in L^2(\Omega)$ and $\varphi \in \mathbf{H}^{1/2}(\partial\Omega)$; then

$$f = (1 - \Delta_N)^{-1}g \text{ if and only if } (1 - \Delta)f = g \text{ in } \Omega, \text{ and } \partial_\nu f = 0 \text{ on } \partial\Omega,$$

$f = N(\varphi)$ if and only if $(1 - \Delta)f = 0$ in Ω , and $\partial_\nu f = \varphi$ on $\partial\Omega$.

The operator $(1 - \Delta_N)^{-1}$, respectively N , belongs to $\mathcal{L}(L^2(\Omega), H^2(\Omega))$, respectively $\mathcal{L}(H^{1/2}(\partial\Omega), H^2(\Omega))$.

3.3. Defect measures. We denote by \mathcal{L} , \mathcal{K} , and \mathcal{L}^1 , respectively, the space of bounded, compact, and trace class operators, respectively, in $L^2(\Omega)$, and we denote by \mathcal{K}_+ , respectively \mathcal{L}_+^1 , the subspace of nonnegative self-adjoint operators in \mathcal{K} , respectively \mathcal{L}^1 . (We refer to [7, Chapter Ten, Section 1.3] for details and definitions.)

If $A \in \mathcal{L}^1$, we denote by $\text{tr}(A)$ the trace of A . Recall that the dual space of $(\mathcal{K}, \|\cdot\|_{\mathcal{L}})$ is $(\mathcal{L}^1, \|\cdot\|_{\mathcal{L}^1})$ with the duality bracket $\text{tr}(AB)$.

If $A \in C_0(\mathbb{R}^2, \mathcal{K})$, then A acts on $\Theta \in L^2(\mathbb{R}^2 \times \Omega)$ following $(A\Theta)(t, \tau, x) = (A\Theta(t, \tau, \cdot))(x)$.

Proposition 3.4. *Let $\Theta^\varepsilon = \Psi^\varepsilon - \Delta\Psi^\varepsilon$. There are a subsequence, $(\Theta^{\varepsilon_n})_{n \in \mathbb{N}}$, a finite nonnegative Borel measure μ on \mathbb{R}^2 , and $M \in L^1(\mathbb{R}^2, \mathcal{L}_+^1, \mu)$, such that for all $A \in C_0(\mathbb{R}^2, \mathcal{K})$,*

$$\int_{\mathbb{R}^2} (A(t, \tau)\Theta^{\varepsilon_n}(t, \tau), \Theta^{\varepsilon_n}(t, \tau))_0 dt d\tau \rightarrow \int \text{tr}(A(t, \tau)M(t, \tau))\mu(dt, d\tau), \quad (3.16)$$

$$\left(a_0(\tilde{S}(t))\tau^2 + \text{div}\left(\frac{1}{r_0(\tilde{S}(t))}\nabla\cdot\right) \right) (1 - \Delta_N)^{-1}M(t, \tau) = 0 \quad \mu\text{-a.e.} \quad (3.17)$$

Remark 5. If one considers the defect measure (M', μ') of $(\Psi^\varepsilon)_\varepsilon$, and proves the additional regularity $M'(t, \tau) \in \mathcal{L}(L^2, H^2)$, then arguing that $F_\varepsilon \rightarrow 0$ in $L^2(\mathbb{R}^2 \times \Omega)$, one infers $(a_0(\tilde{S}(t))\tau^2 + \text{div}(1/r_0(\tilde{S}(t))\nabla))M'(t, \tau) = 0$, μ' -almost-everywhere. Yet, it appears easier to introduce Θ , so (M, μ) , and to prove the convergence of F_ε to 0 in $L^2(\mathbb{R}^2, H^1(\Omega))$, so as to infer the same conclusion, namely (3.17).

Proof. For the proof of the existence part of the proposition and (3.16), we refer the reader to [4] or [15, Lemma 4.3].

Let $\varphi \in C_0(\mathbb{R}^2)$ and $K \in \mathcal{K}$. Since $(\tilde{S}^\varepsilon)_\varepsilon$ is a bounded family in $C_0(H^s)$, the rule (R3) and the convergence of F_ε to 0 proved in (3.7) imply that

$$\varphi P_\varepsilon(t, \tau, \partial_x)N(F_2^\varepsilon \cdot \nu) \xrightarrow{\varepsilon \rightarrow 0} 0, \text{ in } L^2(\mathbb{R}^2 \times \Omega). \quad (3.18)$$

Moreover, the local strong convergence (3.2) of \tilde{S}^ε implies that

$$\varphi K \left(P_\varepsilon(t, \tau, \partial_x)(1 - \Delta_N)^{-1}\Theta^\varepsilon - P_0(t, \tau, \partial_x)(1 - \Delta_N)^{-1}\Theta^\varepsilon \right) \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.19)$$

Furthermore, the right-hand side in identity (3.14) multiplied by φ converges to 0 in $L^2(\mathbb{R}^2 \times \Omega)$ according to (3.7); thus, we infer

$$\varphi P_\varepsilon(t, \tau, \partial_x) \Psi^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \text{ in } L^2(\mathbb{R}^2 \times \Omega). \quad (3.20)$$

Recall that $\Psi^\varepsilon = (1 - \Delta_N)^{-1} \Theta^\varepsilon + N(F_2^\varepsilon \cdot \nu)$. Combining this equality with (3.18), (3.19), and (3.20) we conclude that

$$\varphi K P_0(t, \tau, \partial_x) (1 - \Delta_N)^{-1} \Theta^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0, \text{ in } L^2(\mathbb{R}^2 \times \Omega). \quad (3.21)$$

Using that $P_0(t, \tau, \partial_x) (1 - \Delta_N)^{-1} \in C^0(\mathbb{R}^2, L^2(\Omega))$, it results from (3.21) and (3.16) that, for all $\varphi \in C_0(\mathbb{R}^2)$ and $K \in \mathcal{K}$,

$$\int_{\mathbb{R}^2} \text{tr} (\varphi(t, \tau) K P_0(t, \tau, \partial_x) (1 - \Delta_N)^{-1} M(t, \tau)) \mu(dt, d\tau) = 0.$$

Since φ and K are arbitrary, we obtain (3.17). \square

Assume that the kernel of $(a_0(\tilde{S}(t))\tau^2 + \text{div}(1/r_0(\tilde{S}(t))\nabla))(1 - \Delta_N)^{-1}$ in $L^2(\Omega)$ is reduced to $\{0\}$. Then the identity (3.17) and (3.16) imply that for all $\varphi \in C_0(\mathbb{R}^2)$ and $K \in \mathcal{K}_+$,

$$\int_{\mathbb{R}^2} \varphi(t, \tau) (K \Theta^{\varepsilon_n}(t, \tau), K \Theta^{\varepsilon_n}(t, \tau))_0 dt d\tau \xrightarrow{n \rightarrow \infty} 0. \quad (3.22)$$

We want to show that this convergence holds for $\varphi(t, \tau) = 1$. The idea is that, on the one hand, \tilde{q}^ε is compactly supported in time; so is Θ^ε in the sense given below by (3.23). And, on the other hand, \tilde{q}^ε oscillates in time at most at frequencies $O(\varepsilon^{-1})$.

First, let $\zeta = \zeta(t)$ be in the class $C_0^\infty((-1, T+1))$ such that $\zeta(t) = 1$ for $t \in [0, T]$. In view of (3.10), one infers that

$$\|\zeta \Theta^\varepsilon - W^\varepsilon((1 - \Delta)\zeta \tilde{q}^\varepsilon)\|_{L^2(\mathbb{R}^2 \times \Omega)} \leq C\sqrt{\varepsilon} \|\partial_t \zeta\|_{L^\infty(\mathbb{R})} \|(1 - \Delta)\tilde{q}^\varepsilon\|_{L^2(\mathbb{R} \times \Omega)}.$$

Furthermore, from the definition of \tilde{q}^ε we have $\zeta \tilde{q}^\varepsilon = \tilde{q}^\varepsilon$. Therefore, the previous inequality implies that

$$\int_{\mathbb{R}^2} (1 - \zeta(t))^2 \|\Theta^\varepsilon(t, \tau)\|_{L^2(\Omega)}^2 dt d\tau \xrightarrow{\varepsilon \rightarrow 0} 0. \quad (3.23)$$

Since $(\varepsilon \partial_t \tilde{q}^\varepsilon)_\varepsilon$ is a bounded family in $C_0(\mathbb{R}, H^{s-1}(\Omega))$, it follows from (3.9) that $(\tau \Theta^\varepsilon)_\varepsilon$ is bounded in $L^2(\mathbb{R}^2 \times \Omega)$. Thus with (3.22) and (3.23) we conclude that for all $K \in \mathcal{K}_+$,

$$\int_{\mathbb{R}^2} \|K \Theta^{\varepsilon_n}(t, \tau)\|_{L^2(\Omega)}^2 dt d\tau \xrightarrow{n \rightarrow \infty} 0. \quad (3.24)$$

Recall that, by the definition of $\Theta^\varepsilon = (1 - \Delta)\Psi^\varepsilon$, W^ε is an isometry from $L^2(\mathbb{R} \times \Omega)$ to $L^2(\mathbb{R}^2 \times \Omega)$, and W^ε commutes with $K(1 - \Delta)$. So (3.24) implies that

$$\forall K \in \mathcal{K}_+, \quad \|K(1 - \Delta)\tilde{q}^{\varepsilon n}\|_{L^2(\mathbb{R} \times \Omega)} \xrightarrow{n \rightarrow \infty} 0. \quad (3.25)$$

Given that \tilde{q}^ε is bounded in $L^2(\mathbb{R}, H^s(\Omega))$, the convergence (3.25) implies the convergence of $\tilde{q}^{\varepsilon n}$ to 0 in $L^2(\mathbb{R}, H_{\text{loc}}^{s'}(\Omega))$ for all $s' < s$. Since the limit is 0 the convergence holds for the given family \tilde{q}^ε . We end up with

$$q^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } L^2([0, T], H_{\text{loc}}^{s'}(\Omega)), \text{ for all } s' < s.$$

Arguments similar to those above show that, from (3.12), the previous convergence of \tilde{q}^ε , and (3.7), we have

$$\operatorname{div} v^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} 0 \text{ in } L^2([0, T], H_{\text{loc}}^{s'}(\Omega)), \text{ for all } s' < s - 1. \quad (3.26)$$

Hence, we have seen that in order to prove Proposition 3.1 it is sufficient to prove that $H_0(t, \tau) = \{0\}$ for all $(t, \tau) \in \mathbb{R}^2$, where

$$H_0(t, \tau) = \left\{ u \in L^2 : \left(a_0(\tilde{S}(t))\tau^2 + \operatorname{div}\left(\frac{1}{r_0(\tilde{S}(t))}\nabla\cdot\right)\right)(1 - \Delta_N)^{-1}u = 0 \right\}.$$

Since the speed of propagation $b(S^\varepsilon, u^\varepsilon)$ is uniformly bounded in $H^s \hookrightarrow C^1$, the decay assumption (1.7) is propagated by the transport equation $\partial_t S^\varepsilon + b(S^\varepsilon, u^\varepsilon) \cdot \partial_x S^\varepsilon = 0$. Thus the operator $\operatorname{div}((1/r_0(\tilde{S}(t)))\nabla)$ appears as a perturbation of the Laplacean at infinity. Following the unique continuation principle for a second-order equation ([6, Theorem 17.2.8]), the point is to prove that if u belongs to the kernel H_0 , then u vanishes in Ω , a connected neighborhood of ∞ , in the sense that $|x|^n u \in L^2(\Omega)$ and $|x|^n \nabla u \in L^2(\Omega)$ for all $n \in \mathbb{N}$.

The proof proceeds as in the whole-space case (see [15, Lemma 5.1]). Indeed, we have introduced the operator $(1 - \Delta_N)^{-1}$ in the characteristic variety of the equation; see (3.17). This operator provides us with the boundary condition $\partial_\nu(1 - \Delta_N)^{-1}u = 0$. Therefore, the boundary terms vanish when we integrate by parts. That is why the computations proceed as in the whole-space case.

Then we conclude $H_0(t, \tau) = \{0\}$ for all $(t, \tau) \in \mathbb{R}^2$. This completes the proof of Proposition 3.1.

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REFERENCES

- [1] K. Asano, *On the incompressible limit of the compressible Euler equations*, Japan J. Appl. Math., 4 (1987), 455–488.
- [2] J.-P. Bourguignon and H. Brezis, *Remarks on the Euler equation*, J. Functional Analysis, 15 (1974), 341–363.
- [3] A. Córdoba and C. Fefferman, *Wave packets and Fourier integral operators*, Comm. Partial Differential Equations, 3 (1978), 979–1005.
- [4] P. Gérard, *Microlocal defect measures*, Comm. Partial Differential Equations, 16 (1991), 1761–1794.
- [5] E. Grenier, *Oscillatory perturbations of the Navier-Stokes equations*, J. Math. Pures Appl., 76 (1997), 477–498.
- [6] L. Hörmander, “The Analysis of Linear Partial Differential Operators, Vol. III,” Springer-Verlag, New York, 1985.
- [7] T. Kato, “Perturbation Theory for Linear Operators,” Springer-Verlag, New York, 1966.
- [8] T. Iguchi, *The incompressible limit and the initial layer of the compressible Euler equation in \mathbf{R}_+^n* , Math. Methods Appl. Sci., 20 (1997), 945–958.
- [9] H. Isozaki, *Singular limits for the compressible Euler equation in an exterior domain*, J. Reine Angew. Math., 381 (1987), 1–36.
- [10] H. Isozaki, *Singular limits for the compressible Euler equation in an exterior domain, I Bodies in an uniform flow*, Osaka J. Math., 26 (1989), 399–410.
- [11] S. Klainerman and A. Majda, *Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible*, Comm. Pure Appl. Math., 34 (1981), 481–524.
- [12] S. Klainerman and A. Majda, *Compressible and incompressible fluids*, Comm. Pure Appl. Math., 35 (1982), 629–651.
- [13] P.-L. Lions, and T. Paul, *Sur les mesures de Wigner*, Rev. Mat. Iberoamericana., 9 (1993), 553–618.
- [14] A. Majda, “Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables,” Springer-Verlag, New York, 1984.
- [15] G. Métivier and S. Schochet, *The incompressible limit of the non-isentropic Euler equations*, Arch. Ration. Mech. Anal., 158 (2001), 61–90.
- [16] G. Métivier and S. Schochet, *Averaging theorems for conservative systems and the weakly compressible Euler equations*, to appear in J. Diff. Equations.
- [17] G. Métivier and S. Schochet, *Limite incompressible des équations d’Euler non isentropiques; Séminaire Équations aux Dérivées Partielles*, 2000-2001, École Polytechnique, Palaiseau.
- [18] J. Rauch, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Math. Soc., 291 (1985), 167–187.
- [19] S. Schochet, *The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit*, Comm. Math. Phys., 104 (1986), 49–75.
- [20] S. Ukai, *The incompressible limit and the initial layer of the compressible Euler equation*, J. Math. Kyoto Univ., 26 (1986), 323–331.