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REMARKS ON GLOBAL EXISTENCE FOR A FAMILY OF INEQUALITIES AND APPLICATIONS TO MULLINS AND NAVIER-STOKES EQUATIONS

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Abstract. We give a new method based on the existence of a family of inequalities to prove the global existence of a time evolution equation. We apply this method to various problems.

Introduction

In this paper, we shall first present some abstract global-existence results related to a family of energy inequalities with small data.

As a few applications of such results, we shall consider the Mullins equations established in [7] 1957, related to a surface diffusion. The system of fourth order reads as follows: Find $u : [0,+\infty) \times [0,1] \to \mathbb{R}$ satisfying

$$(M_u) \qquad \begin{cases} u_t = -B\partial_x \Big\{ (1+u_x^2)^{-\frac{1}{2}} \partial_x \big[(1+u_x^2)^{-\frac{3}{2}} u_{xx} \big] \Big\}, \\ u \text{ is periodic}, \quad u(0,x) = u_0(x) \text{ (initial data)}. \end{cases}$$

Here $B > 0$ is a given constant that we choose to be equal to 1 in the sequel, ∂_x (respectively ∂_t) stands for the space (respectively time) derivative, and we denote $\partial_t u = \frac{\partial u}{\partial t}$, $\partial_x u = u' = \frac{\partial u}{\partial x}$, $u_{xx} = u^{(2)} = \frac{\partial^2 u}{\partial x^2}, \dots$. We shall show that

There is a computable constant $c > 0$, scale invariant, such that if the initial data $u_0 \in H^2_{per}(0,1)$ with $c|u_0|_{L^2(0,1)} \cdot \big(1 + |u_0^{(2)}|^2_{L^2(0,1)}\big)^{10} < 1$, then the system (M_u) possesses a unique global solution $u \in L^\infty\big(\mathbb{R}_+; H^2_{per}(0,1)\big) \cap L^2\big(0,T; H^4_{per}(0,1)\big)$ for all finite T. Furthermore, as $t \to +\infty$, $u(t)$ converges

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exponentially to $\overline{u_0} = \int_0^1 u_0(x)dx$ *in* $C_b(\mathbb{R}_+)$. *More precisely, there exist* $c(u_0) > 0$ *and* $\overline{\lambda}(u_0) > 0$ *such that*

$$|u(t) - \overline{u}_0|_\infty \leqslant |\frac{\partial u}{\partial x}(t)|_\infty \leqslant c(u_0)e^{-\overline{\lambda}(u_0)t}.$$

Additional results on (M_u) will be given; namely, we will show that if a blow-up appears at a time $T_{\max} < +\infty$, then there is a point $x^* \in [0,1]$ such that

$$\limsup_{(t,x)\to(T_{\max},x^*)} |\frac{\partial u}{\partial x}(t,x)| = +\infty.$$

The uniqueness of the local solution for (M_u) is included in a general framework [6]. Special approximations of (M_u) are given in [2]. Numerical results confirm the above results on global existence and the nature of blow-up.

For the second application, we shall reconsider the Navier-Stokes equations and make a slight improvement, in the case of three dimensions, on the class of initial data for which we have a global strong solution. Borrowing the notation used in [3], [9], [10], and [12], we shall look at the following abstract equation: Let $(V, \|\cdot\|)$ and $(H, |\cdot|)$ be two Hilbert spaces (see the appendix for the exact choice for Navier-Stokes). The norm $\|\cdot\|$ of V (respectively $|\cdot|$ of H) is associated to the scalar product $((\cdot,\cdot))$ (respectively $(\cdot,\cdot)$).

Let $f \in L^2(0,T;H)$ and $u_0 \in V$. A strong solution u is a function satisfying

$$(N.S) \qquad \begin{cases} i)\ \dfrac{du}{dt} + \nu Au + B(u,u) = f \text{ in } V' \text{ (dual of } V), \\ ii)\ u(0) = u_0, \\ iii)\ u \in L^2\big(0,T;D(A)\big) \cap L^\infty(0,T;V). \end{cases}$$

For the properties of A and B, we shall refer to the above references (A is linear such that $\langle Av, w\rangle_{V',V} = ((v,w))\ \forall v, w \in V \times V$, and B is associated to the bilinear operator).

In order to compare the usual well-known result with our result, we shall recall Leray's theorem (stated in [3], [9], and [10]) in the case $f = 0$ (for simplicity).

Theorem 1 (Leray's theorem for global existence in dimension 3)**.** *Assume that* $f = 0$. *There exists a scale-independent positive constant* c *such that for* $u_0 \in V$ *satisfying*

$$\|u_0\|^2 < \frac{1}{4}\frac{\nu^2}{c^2}\lambda_1^{\frac{1}{2}}, \tag{0.1}$$

there exists a strong solution u *of Navier-Stokes equations (N.S). Here* $\lambda_1 > 0$ *is the first eigenvalue of* A *and satisfies*

$$\|v\| \leqslant \frac{1}{\sqrt{\lambda_1}}|Av|,\ \forall v \in D(A), \quad \textit{and} \quad \lambda_1 = \inf_{v\in V,\ v\neq 0} \frac{\|v\|^2}{|v|^2}.$$

Our result will give the following statement, an improvement of Leray's theorem (when $f = 0$):

There exists a scale-independent positive constant $c > 0$ *such that for* $u_0 \in V$ *satisfying*

$$|u_0|\|u_0\| < \frac{2}{c^2}\cdot \nu^2, \tag{0.2}$$

there exists a strong solution of the Navier-Stokes equations. The constant c *is the same as before.*

Remark 1. Condition (2) is better than condition (1) since

(1) Condition (1) implies condition (2).
(2) The class of u_0 satisfying condition (2) (when $f = 0$) is strictly larger than the class of functions satisfying condition (1), for instance, if u_0 has **a small norm in H** and satisfies with ν and λ_1

$$|u_0|\|u_0\| < \frac{2}{c^2}\nu^2 < 8\frac{\|u_0\|^2}{\lambda_1^{\frac{1}{2}}}.$$

Here, as in relation (0.2), the constant c could be chosen as

$$c = \sup_{\varphi\in D(A)\setminus\{0\}} \frac{|B(\varphi,\varphi)|}{\|\varphi\|^{\frac{3}{2}}|A\varphi|^{\frac{1}{2}}}.$$

(3) The method works for nonzero f.
(4) Global existence for small data has been investigated by various authors [4], [5]. The main differences with respect to those previous works are the method and the conditions on the initial data. For instance in [4], they use the integral representation of the solution u and the Picard fixed point. For the application in dimension 3, their condition is that u_0 is small in $D(A^{\frac{1}{4}})$. By an interpolation argument, our smallness condition implies the smallness in $D(A^{\frac{1}{4}})$. Furthermore, in their case a strong solution is obtained under the condition that $t \to f(t)$ is Hölder continuous and the smallness of the data is not explicit as we did.

A complete survey on the Navier-Stokes topics has been recently given by Temam [11].

1. Some fundamental lemmas

The first simple and fundamental lemma that we need is

Lemma 1. *Let y be nonnegative and absolutely continuous on some interval $[0, T_{\max})$ satisfying the differential inequality*

$$y'(t) \leqslant g(t)y(t)^m, \qquad \textit{for a.e. } t > 0,$$

with $m > 1$ and g a nonnegative function such that there is a constant $M > 0$ independent of $T_{\max}$ such that

$$\int_0^t g(\sigma)d\sigma \leqslant M,\ \forall t \geqslant 0.$$

If $(m-1)y^{m-1}(0)M < 1$, then for all $t \geqslant 0$

$$y(t) \leqslant \frac{y(0)}{[1-(m-1)y^{m-1}(0)M]^{\frac{1}{m-1}}}.$$

Proof. Let $\delta_0 > 0$ be such that $(m-1)\big(y(0)+\delta_0\big)^{m-1}M < 1$. Then for all $\delta \in (0, \delta_0)$, if we set $y_\delta(t) = y(t) + \delta$, it satisfies the differential inequality $y'_\delta(t) \leqslant g(t)y_\delta(t)^m$. Integrating this last differential, we derive

$$y_\delta(t) \leqslant \frac{y_\delta(0)}{\big[1-(m-1)y_\delta(0)^{m-1}M\big]^{\frac{1}{m-1}}}.$$

Letting $\delta \to 0$, we get the result. □

For the applications, we shall present the following corollary.

Corollary 1. *Let H be a vector space and p, q, and N be three (nonlinear) mappings from $D \subset H$ into $\mathbb{R}_+$. Let $[0, T_{max})$ be an interval of $\mathbb{R}_+$ and $u : [0, T_{\max}) \to D$ be such that $p(u)$ and $q(u)$ are absolutely continuous and $N(u)$ is measurable. We assume that u satisfies the following energy inequalities:*

$$\begin{cases} \dfrac{d}{dt}q\big(u(t)\big) + N\big(u(t)\big) \leqslant f(t), \\ \dfrac{d}{dt}p\big(u(t)\big) \leqslant c_1 N\big(u(t)\big)p\big(u(t)\big)^m, \end{cases}$$

with $m > 1$ and $c_1 > 0$ constants and $f \in L^1_+(\mathbb{R}_+)$. We set

$$M = c_1\Big[\int_0^{+\infty} f(\sigma)d\sigma + q(u_0)\Big].$$

If $(m-1)Mp(u_0)^{m-1} < 1$*, then*

$$p\big(u(t)\big) \leqslant \frac{p(u_0)}{[1-(m-1)Mp(u_0)^{m-1}]^{\frac{1}{m-1}}}, \quad \forall t \geqslant 0.$$

Proof. Setting $g(t) = c_1 N\big(u(t)\big)$, we derive from the first inequality that for all $t \geqslant 0$

$$\int_0^t g(\sigma)d\sigma \leqslant c_1\Big[\int_0^{+\infty} f(\sigma)d\sigma + q(u_0)\Big] = M.$$

We may appeal Lemma 1 for the second inequality to get the result. □

2. Applications

2.1. **Mullins equations.** The Mullins equation concerns a surface diffusion (see [7] for more details). If we call $\frac{\partial N}{\partial t}$ the local normal velocity of the surface, κ its curvature, and s the arc length, then the equation can be written as

$$\frac{\partial N}{\partial t} = B\frac{\partial^2 \kappa}{\partial s^2}. \tag{2.1}$$

Using Cartesian coordinates, introducing $u : \frac{dx}{ds} = \big(1+u_x^2\big)^{-\frac{1}{2}}$, and expressing $N_t = -\frac{u_t}{\sqrt{1+u_x^2}}$, which corresponds to the projection of the velocity on the axis directed by the unit normal $\begin{pmatrix} u_x \\ -1 \end{pmatrix} \frac{dx}{ds}$, the equation (2.1) implies (see [2] for details)

$$u_t = -B\partial_x \left\{(1+u_x^2)^{-\frac{1}{2}}\partial_x\left((1+u_x^2)^{-\frac{3}{2}}u_{xx}\right)\right\}.$$

The local existence of such an equation has been investigated in a different framework and in a more general form in [6]. To announce this local existence theorem, we shall recall the following spaces for periodic functions:

For $m \geqslant 1$, we denote by

$$H^m_{per}(0,1) = \Big\{\varphi \in H^m(0,1) : \varphi^{(j)}(0) = \varphi^{(j)}(1), \quad j = 0, \ldots, m-1\Big\}.$$

This space is a Hilbert space with the following scalar product:

$$((\varphi,\psi))_m = ((\varphi,\psi)) = \int_0^1 \varphi(x)\psi(x)dx + \int_0^1 \psi^{(m)}(x)\varphi^{(m)}(x)dx.$$

The associated norm is $|\varphi|_{H^m_{per}} = \sqrt{((\varphi,\varphi))}$, which is equivalent to

$$\Big|\int_0^1 \varphi(x)dx\Big| + \Big(\int_0^1 \Big(\varphi^{(m)}(x)\Big)^2 dx\Big)^{\frac{1}{2}} \cdot \frac{d^j\varphi}{dx^j} = \varphi^{(j)},$$

which denotes the j^{th} derivative of φ. The norm in the usual space $L^p(0,1)$ will be denoted $|\cdot|_p = |\cdot|_{L^p(0,1)}$.

As usual, we shall denote the time evolution spaces

$$L^p\big(0,T;H^m_{per}(0,1)\big),\ C^k\big([0,T];H^m_{per}(0,1)\big).$$

We have shown in particular

Theorem 2 (see [6])**.** *For each $u_0 \in H^2_{per}(0,1)$, there is a time $T_{\max} > 0$ and a unique solution u belonging to*

$$L^2\big(0,T;H^4_{per}(0,1)\big) \cap C\big([0,T];H^2_{per}(0,1)\big)$$

with $\frac{\partial u}{\partial t} \in L^2(0,T;L^2(\Omega))$for all $T < T_{\max}$ satisfying

$$(M_u)\begin{cases} u_t = -\partial_x\Big\{(1+u_x^2)^{-\frac{1}{2}}\partial_x\Big((1+u_x^2)^{-\frac{3}{2}}u_{xx}\Big)\Big\} & \text{in } (0,1),\ t>0,\\ u(0)=u_0. \end{cases}$$

Furthermore, we have the following energy inequality:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u(t,x)^2 dx + \int_0^1 \frac{u_{xx}^2}{(1+u_x^2)^3}dx \leqslant 0. \tag{2.2}$$

If for all $T_0 > 0$, there exists a finite constant $C(T_0,u_0) > 0$ such that $\sup_{t\leqslant T_0}|u(t)|_{H^2_{per}} \leqslant C(T_0,u_0)$, then $T_{\max} = +\infty$ (that is, u is a global solution on $\mathbb{R}_+$).

To complete this theorem, we shall prove the theorem with more accuracy, that is,

Theorem 3. *There exists a computable constant $c > 0$, scale independent, such that if $u_0 \in H^2_{per}(0,1)$ with $\gamma = c|u_0|_{L^2(0,1)}(1+|u_0^{(2)}|^2_{L^2})^{10} < 1$, then the unique local solution u is a global one in $L^\infty(\mathbb{R}_+;H^2_{per}(0,1)) \cap L^2(0,T;H^4_{per}(0,1))$ for all time $T < +\infty$. Furthermore, for all $t > 0$,*

i) $|u^{(2)}(t)|^2_{L^2} \leqslant \frac{1+|u_0^{(2)}|^2_{L^2}}{(1-\gamma^2)^{\frac{1}{20}}} = \lambda_0(u_0)$,

ii) $|u(t)-\overline{u_0}|_{L^2} \leqslant |u_0-\overline{u_0}|_{L^2}e^{-\overline{\lambda}(u_0)t}$, *with* $\overline{u_0} = \int_0^1 u_0(x)dx$, $\overline{\lambda}(u_0) = \frac{1}{[1+\lambda_0(u_0)^2]^3}$,

iii) $|u'(t)|_{L^\infty(0,1)} \leqslant c(u_0)e^{-\frac{3\overline{\lambda}(u_0)}{4}t}$.

Proof. The proof relies on Corollary 1 of Lemma 1, where we choose $q(v) = \frac{1}{2}\int_0^1 v^2(x)dx,\ N(v) = \int_0^1 \frac{(v^{(2)})^2}{(1+(v')^2)^3}dx$, and $p(v) = 1 + \int_0^1 (v^{(2)})^2(x)dx,\ v \in D = H^2_{per}(0,1)$.

The first differential inequality of Corollary 1 of Lemma 1 is then given by relation (2.2) with $f = 0$.

To find the second differential inequality, we shall use the following interpolation lemmas, which can be found in [8] and [10] (see also [1]).

Lemma 2 (see [8])**.** *Let $v \in W^{1,r}(0,1)$, $\int_0^1 v(x)dx = 0$ and $1 \leqslant r \leqslant +\infty$. Let $1 \leqslant n \leqslant s \leqslant +\infty$, where n and s are real numbers. Then*

$$|v|_{L^s(0,1)} \doteq |v|_s \leqslant \Big(1 + \frac{r-1}{r}n\Big)^a |v'|_r^a |v|_n^{1-a},$$

with $a\left(\frac{1}{n} - \frac{1}{r} + 1\right) = \frac{1}{n} - \frac{1}{s}$.

Lemma 3 (see [10])**.** *Let m_1, m_2, and m_3 be three integers with $0 \leqslant m_1 < m_2 < m_3$. Let $\theta \in (0,1)$ be such that $m_2 = \theta m_1 + (1-\theta)m_3$. Then for all $v \in H^{m_3}_{per}(0,1)$ with $\int_0^1 v(x)dx = 0$ we have*

$$|v|_{H^{m_2}_{per}(0,1)} \leqslant |v|^\theta_{H^{m_1}_{per}} |v|^{1-\theta}_{H^{m_3}_{per}},$$

with $H^0_{per}(0,1) = L^2(0,1)/\mathbb{R}$.

We expand the main equation of the system (M_u) to derive that the unique local solution satisfies

$$\frac{\partial u}{\partial t} + \frac{u^{(4)}}{(1+u'^2)^2} = -\frac{(15u'^2-3)\big(u^{(2)}\big)^3}{(1+u'^2)^4} - 10\frac{u'u^{(2)}u^{(3)}}{(1+u'^2)^3}. \tag{2.3}$$

Multiplying relation (2.3) by $u^{(4)}$, and integrating by parts the first term, we then have

$$\frac{d}{dt}p\big(u(t)\big) + 2\int_0^1 \frac{(u^{(4)})^2}{(1+u'^2)^2}dx \leqslant I_1(t) + I_2(t), \tag{2.4}$$

with $I_1(t) = 36\int_0^1 \frac{|u^{(2)}|^3|u^{(4)}|}{(1+u'^2)^3}dx$ and $I_2(t) = 20\int_0^1 \frac{|u^{(2)}||u^{(3)}||u^{(4)}|}{(1+u'^2)^{\frac{5}{2}}}dx$. (The integration by parts is justified by the regularity of u or using the fact that u is a limit of a smooth sequence $u_{\varepsilon,m}$ (see [6]).) From now on, we shall denote by c various constants which are scale independent and independent of u.

By Young's inequality, we derive that for all $\eta > 0$ there exists a constant c_η (depending only on η) such that

$$I_1(t) = 36\int_0^1 \frac{|u^{(2)}|^3|u^{(4)}|}{(1+u'^2)^3}dx \leqslant \eta\int_0^1 \frac{\big(u^{(4)}\big)^2}{(1+u'^2)^2}dx + c_\eta\int_0^1 |u^{(2)}|^6 dx. \quad (2.5)$$

Using Lemma 2, Lemma 3, and the Sobolev embedding $H^1(0,1) \subset L^\infty(0,1)$, we then have

$$\begin{aligned} c_\eta\int_0^1 |u^{(2)}|^6 dx &\leqslant c_\eta |u^{(4)}|_{L^2}|u^{(2)}|^5_{L^2} \quad (2.6)\\ &\leqslant c_\eta \Big|\frac{u^{(4)}}{1+u'^2}\Big|_{L^2}\left(1+|u^{(2)}|^2_{L^2}\right)^{\frac52}\Big|\frac{u^{(2)}}{(1+u'^2)^{\frac32}}\Big|_{L^2}|u^{(2)}|^4_{L^2}. \end{aligned}$$

Setting $N\big(u(t)\big) = \int_0^1 \frac{(u^{(2)})^2}{(1+u'^2)^3}dx$ and using the Cauchy-Schwarz-Young inequality we get, for all $\eta > 0$,

$$c_\eta\int_0^1 |u^{(2)}|^6 dx \leqslant \eta\int_0^1 \frac{(u^{(4)})^2}{(1+u'^2)^2}dx + c_\eta N\big(u(t)\big)\big(1+|u^{(2)}|^2_{L^2}\big)^9. \quad (2.7)$$

Combining relations (2.5) and (2.7), we derive

$$I_1(t) \leqslant 2\eta\int_0^1 \frac{(u^{(4)})^2}{(1+u'^2)^2}dx + c_\eta N\big(u(t)\big)p\big(u(t)\big)^9, \quad (2.8)$$

while for the second term of the inequality (2.4), i.e.,

$$I_2(t) = 20\int_0^1 \frac{|u^{(2)}||u^{(3)}||u^{(4)}|}{(1+u'^2)^{\frac52}}dx,$$

we have from the Hölder and Young inequalities that, for all $\delta > 0$, there exists a constant $c^1_\delta > 0$ such that

$$I_2(t) \leqslant \delta\int_0^1 \frac{\big(u^{(4)}\big)^2}{(1+u'^2)^2}dx + c^1_\delta\int_0^1 |u^{(2)}|^6 + c^1_\delta\int_0^1 |u^{(3)}|^3. \quad (2.9)$$

Arguing as in relations (2.6) and (2.7), we have for all $\delta > 0$,

$$c^1_\delta\int_0^1 |u^{(2)}|^6 dx \leqslant \delta\int_0^1 \frac{\big(u^{(4)}\big)^2}{(1+u'^2)^2}dx + c_\delta N\big(u(t)\big)p\big(u(t)\big)^9. \quad (2.10)$$

From interpolation inequalities and the Sobolev embeddings, we have

$$c^1_\delta\int_0^1 |u^{(3)}|^3 dx \leqslant c^2_\delta |u^{(4)}|^{\frac74}_{L^2}|u^{(2)}|^{\frac54}_{L^2} \leqslant c^3_\delta\Big|\frac{u^{(4)}}{1+u'^2}\Big|^{\frac74}_{L^2}|u^{(2)}|^{\frac54}_{L^2}\big(1+|u^{(2)}|^2_{L^2}\big)^{\frac74}. \quad (2.11)$$

From Young's inequality, we derive from relation (2.10) that $\forall\delta>0$, $\exists c_\delta^4>0$ such that

$$c_\delta^1\int_0^1 |u^{(3)}|^3dx \leqslant \delta\int_0^1 \frac{|u^{(4)}|^2}{(1+u'^2)^2}dx + c_\delta^4|u^{(2)}|_{L^2}^{10}\Big(1+|u^{(2)}|_{L^2}^2\Big)^{14}. \quad (2.12)$$

As before, introducing $p\big(u(t)\big)$ and $N\big(u(t)\big)$, we get from (2.12) that

$$c_\delta^1\int_0^1 |u^{(3)}|^3 \leqslant \delta\int_0^1 \frac{|u^{(4)}|}{(1+u'^2)^2}dx + c_\delta^4 N\big(u(t)\big)p\big(u(t)\big)^{21}. \quad (2.13)$$

Combining relation (2.8) with (2.13), we derive that

$$I_1(t)+I_2(t) \leqslant 3(\eta+\delta)\int_0^1 \frac{|u^{(4)}|^2}{(1+u'^2)^2}dx + c_{\eta\delta}N\big(u(t)\big)p\big(u(t)\big)^{21}. \quad (2.14)$$

Combining this last relation with relation (2.4), choosing $3(\eta+\delta)=1$, one obtains the following differential inequality:

$$\frac{d}{dt}p\big(u(t)\big) + \int_0^1 \frac{|u^{(4)}|^2}{(1+u'^2)^2}dx \leqslant c_0 N\big(u(t)\big)p\big(u(t)\big)^{21}. \quad (2.15)$$

From relation (2.2), we know that

$$\frac{d}{dt}q\big(u(t)\big) + N\big(u(t)\big) \leqslant 0. \quad (2.16)$$

We may appeal to Corollary 1 of Lemma 1 to derive that

$$1+|u^{(2)}(t)|_{L^2}^2 \leqslant \frac{1+|u_0^{(2)}|_2^2}{\Big[1-c|u_0|_2^2\big(1+|u_0^{(2)}|_2^2\big)^{20}\Big]^{\frac{1}{20}}} = \lambda_0(u_0), \quad (2.17)$$

provided that $c|u_0|_2^2\big(1+|u_0^{(2)}|_2^2\big)^{20}<1$, with $c=10c_0$. Thus,

$$\int_0^t\int_0^1 \frac{|u^{(4)}|^2}{(1+u'^2)^2}dx \leqslant c(u_0) < +\infty$$

for all $t\geqslant 0$, which implies that

$$\int_0^t\int_0^1 \big(u^{(4)}\big)^2 \leqslant c(u_0)\big(1+\lambda_0(u_0)^2\big)^2.$$

Thus, $T_{\max}=+\infty$.

Next we want to show that $u(t)$ tends to $\overline{u_0}=\int_0^1 u_0(x)dx$ as t goes to infinity in $C_b(\mathbb{R}_+)$ (the set of continuous and bounded functions). Since

$$\frac{1}{2}\frac{d}{dt}\int_0^1 u^2(t,x)dx + \int_0^1 \frac{(u^{(2)})^2}{(1+u'^2)^3}dx \leqslant 0 \quad (2.18)$$

and $|u'(t)|_\infty \leqslant |u^{(2)}(t)|_{L^2(0,1)}$, we then derive $|u-\overline{u_0}|^2_{L^2(0,1)} \leqslant |u^{(2)}|^2_{L^2(0,1)}$ and

$$\frac{d}{dt}\int_0^1 u^2(t,x)dx + \frac{2}{[1+\lambda_0(u_0)^2]^3}\int_0^1 |u-\overline{u_0}|^2 dx \leqslant 0. \tag{2.19}$$

Since $\int_0^1 u(t,x)dx = \overline{u_0}$, we then have

$$\int_0^1 u^2(t,x)dx = \int_0^1 u^2(t,x)dx - \overline{u_0}^2. \tag{2.20}$$

Thus, we have the Gronwall inequality

$$\frac{d}{dt}\int_0^1 |u-\overline{u_0}|^2 dx + 2\lambda_1(u_0)\int_0^1 |u-\overline{u_0}|^2 dx \leqslant 0, \tag{2.21}$$

with $\lambda_1(u_0) = \frac{1}{[1+\lambda_0(u_0)^2]^3}$. Thus, we have

$$\Big(\int_0^1 |u(t)-\overline{u_0}|^2 dx\Big)^{\frac{1}{2}} \leqslant e^{-\lambda_1(u_0)t}\Big(\int_0^1 |u_0-\overline{u_0}|^2\Big)^{\frac{1}{2}}. \tag{2.22}$$

Setting $V(t) = u(t) - u_0$, we have from the interpolation Lemmas 2 and 3 that

$$|V'(t)|_{L^\infty(0,1)} \leqslant \sqrt{2}|V^{(2)}|^{\frac{1}{2}}_{L^2}|V'|^{\frac{1}{2}}_{L^2} \leqslant \sqrt{2}|u^{(2)}(t)|^{\frac{3}{4}}_{L^2}|u(t)-u_0|^{\frac{3}{4}}_{L^2}. \tag{2.23}$$

Thus, the combination of relations (2.22) and (2.23) leads to

$$|u'(t)|_{L^\infty(0,1)} \leqslant c(u_0)e^{-\frac{3}{4}\lambda_1(u_0)t}, \quad \text{with} \quad c(u_0) = \sqrt{2}\lambda_0(u_0)^{\frac{3}{8}}|u_0-\overline{u_0}|^{\frac{3}{4}}_{L^2}.$$

Remark 2. Since the constants in Lemmas 2 and 3 are explicit, the above method can lead to an estimate of the constant c by making precise the various constants c_η and c^i_j appearing here. Nevertheless, in a later work we will give an alternative formulation of the system (M_u) from which we can compute more easily the constant c.

2.2. **Description of the blow-up for** (M_u)**.** We shall first prove the following theorem, which will give the description of the (eventual) blow-up for (M_u).

Theorem 4. *Let u be a solution of (M_u), $T_0 > 0$. Assume that $u \in L^\infty\big(0,T;H^2_{per}(0,1)\big) \cap L^2\big(0,T;H^4_{per}(0,1)\big)$, for all $T < T_0$. Then*

$$\sup_{0\leqslant t\leqslant T_0} |\frac{\partial u}{\partial x}(t)|_{L^\infty(0,1)}$$

is finite if and only if

$$\sup_{0\leqslant t\leqslant T_0} |\frac{\partial^2 u}{\partial x^2}(t)|_{L^2(0,1)}$$

is finite. In this case, u is a global solution of (M_u) on $[0,T_0]$.

As a first corollary of such a theorem, we have

Corollary 2. *Let $(u,[0,T_{\max}))$ be the unique (maximal) solution of (M_u); i.e., $u \in L^\infty\big(0,T;H^2_{per}(0,1)\big)\cap L^2\big(0,T;H^4_{per}(0,1)\big)$ for all $T<T_{\max}$. If $T_{\max}<+\infty$, then there exists $x^*\in[0,1]$ such that*

$$\limsup_{(t,x)\to(T_{\max},x^*)} |\frac{\partial u}{\partial x}(t,x)| = +\infty.$$

The main key lemma for the proof of Theorem 4 is the existence of Lyapunov functional. We shall consider u under the assumptions of Theorem 4.

Lemma 4. *Let $G:\mathbb{R}\to\mathbb{R}$ be a C^2 function such that $G''(\sigma)=\frac{1}{(1+\sigma^2)^{\frac{3}{2}}}$. Then, we have*

$$\frac{d}{dt}\int_0^1 G(u_x)dx = -\int_0^1 (1+u_x^2)^{-\frac{1}{2}}\Big(\frac{\partial^2}{\partial x^2}G'(u_x)\Big)^2 dx.$$

Proof of Lemma 4. The periodic function $v=u_x$ satisfies the equation

$$\partial_t v = -\partial_x^2\left\{(1+v^2)^{-\frac{1}{2}}\partial_x^2\Big(G'(v)\Big)\right\}. \tag{2.24}$$

Let us multiply this equation by $G'(v)$ and integrate by parts; we have

$$\frac{d}{dt}\int_0^1 G(v)dx = -\int_0^1 (1+v^2)^{-\frac{1}{2}}\Big(\frac{\partial^2}{\partial x^2}G'(v)\Big)^2 dx. \tag{2.25}$$

As a consequence of the above lemma, one has

Lemma 5. *As before, we set $v=u_x$. Then for all $t\geqslant 0$*

i)
$$\int_0^t\int_0^1 (1+v^2)^{-\frac{11}{2}}\left[v_{xx}(1+v^2)-3v_x^2 v\right]^2 d\tau dx \leqslant \int_0^1 G(u_{0x})dx - \int_0^1 G(v)(t,x)dx;$$

ii) *setting $w=(1+v^2)^{-\frac{11}{4}}[v_{xx}(1+v^2)-3v_x^2v]$, then*

$$\int_0^1 v_{xx}^2(1+v^2)dx + \int_0^1 v_x^4(t,x)dx \leqslant \int_0^1 (1+v^2)^{\frac{11}{4}} w v_{xx}\, dx.$$

Proof of Lemma 5. The computation of $\partial_x^2 G'(v)$ gives us

$$\partial_x^2 G'(v) = (1+v^2)^{-\frac{5}{2}}[v_{xx}(1+v^2) - 3v_x^2 v].$$

Using this expression in Lemma 4 and integrating between 0 and t, we have the first statement, i).

For the second statement, we compute $(1+v^2)^{\frac{11}{4}} w v_{xx}$ and integrate over $[0,1]$ to obtain

$$\int_0^1 (1+v^2)^{\frac{11}{4}} w v_{xx}\, dx = \int_0^1 v_{xx}^2 (1+v^2) dx - 3\int_0^1 v_x^2 v_{xx} v\, dx. \tag{2.26}$$

But one has by integration by parts

$$-3\int_0^1 v_x^2 v v_{xx}\, dx = -\int_0^1 (v_x^3)_x v\, dx = \int_0^1 v_x^4(t,x) dx. \tag{2.27}$$

We then deduce the result. □

Proof of Theorem 4. If $\sup_{t\leqslant T_0} |\frac{\partial^2 u}{\partial x^2}(t)|_{L^2(0,1)}$ is finite, then from Sobolev embedding

$$\sup_{t\leqslant T_0} |\frac{\partial u}{\partial x}(t)|_{L^\infty(0,1)} \leqslant \sup_{t\leqslant T_0} |\frac{\partial^2 u}{\partial x^2}(t)|_{L^2(0,1)} < +\infty.$$

Conversely, assume that $\sup_{t\leqslant T_0} |\frac{\partial u}{\partial x}(t)|_{L^\infty(0,1)} \doteq c(T_0,u_0) < +\infty$. Then, for all $t \leqslant T_0$, Lemma 5 implies

$$\begin{aligned}
&\int_0^t \int_0^1 w^2(\tau,x) d\tau dx \leqslant \int_0^1 [G(v_0) - G(v)](t,x) dx \\
&\leqslant \sup_{\sigma\in\mathbb{R}} |G'(\sigma)| \int_0^1 |v_0(x) - v(t,x)| dx \\
&\leqslant \sup_{\sigma} |G'(\sigma)| \Big[\int_0^1 |u_{0x}(x)| dx + c(T_0,u_0) \Big] = c_1(T_0,u_0) < +\infty.
\end{aligned}$$

Thus, from the Cauchy-Schwarz inequality, we then have

$$\begin{aligned}
&\int_0^t \int_0^1 (1+v^2)^{\frac{11}{4}} w v_{xx}\, d\tau\, dx \\
&\leqslant \big(1 + c(T_0,u_0)^2\big)^5 \int_0^t \int_0^1 w^2(\tau,x) d\tau\, dx + \frac{1}{2}\int_0^t d\tau \int_0^1 (1+v^2) v_{xx}^2\, dx.
\end{aligned}$$

Combining this last relation with the second statement, ii), of Lemma 5, we derive

$$\int_0^{T_0} d\tau \int_0^1 (1+v^2)v_{xx}^2 dx + \int_0^{T_0}\int_0^1 v_x^4\, d\tau\, dx$$
$$\leqslant 2\big(1+c(T_0,u_0)^2\big)^5 c_1(T_0,u_0) = c_2(T_0,u_0). \tag{2.28}$$

Next, we recall that from relation (2.4), (2.5) and (2.9), we have, $\forall\eta > 0,\ \exists c_\eta > 0$,

$$\frac{d}{dt}|u^{(2)}(t)|^2_{L^2(0,1)} + 2(1-\eta)\int_0^1 \frac{(u^{(4)})^2}{(1+u'^2)^2} \leqslant c_\eta\Big(\int_0^1 |u^{(3)}|^3 dx + \int_0^1 |u^{(2)}|^6 dx\Big). \tag{2.29}$$

From relation (2.28), we deduce that

$$\int_0^{T_0}\int_0^1 \big(u^{(3)}\big)^2 dx d\tau + \int_0^{T_0}\int_0^1 \big(u^{(2)}\big)^4 dx d\tau \leqslant c_2(T_0,u_0). \tag{2.30}$$

By interpolation Lemma 2, we deduce

$$\int_0^1 |u^{(2)}|^6 dx \leqslant |u^{(2)}|^4_{L^\infty(0,1)}|u^{(2)}|^2_{L^2(0,1)} \leqslant 2|u^{(1)}|^2_{L^2}|u^{(3)}|^2_{L^2}|u^{(2)}|^2_{L^2}$$
$$\leqslant 2c(T_0,u_0)^2|u^{(3)}(t)|^2_{L^2}|u^{(2)}|^2_{L^2}, \tag{2.31}$$

and we have

$$\int_0^1 |u^{(3)}|^3 dx \leqslant |u^{(3)}|_\infty |u^{(3)}|^2_{L^2} \leqslant 4^{\frac13}|u^{(3)}|_\infty |u^{(1)}|^{\frac23}_{L^2}|u^{(4)}|^{\frac43}_{L^2}$$
$$\leqslant 8c(T_0,u_0)|u^{(2)}|^{\frac12}_{L^2}|u^{(4)}|^{\frac{11}{6}}_{L^2} \leqslant \eta\int_0^1 \frac{\big(u^{(4)}\big)^2}{(1+u'^2)^2}dx + c_\eta(T_0)|u^{(2)}|^6_{L^2}$$
$$\leqslant c_\eta(T_0)|u^{(2)}|^4_{L^4}|u^{(2)}|^2_{L^2} + \eta\int_0^1 \Big(\frac{u^{(4)}}{1+u'^2}\Big)^2 dx. \tag{2.32}$$

Thus, setting $g(t) = |u^{(3)}(t)|^2_{L^2} + |u^{(2)}(t)|^4_{L^4}$ and choosing $\eta = \frac14$, we deduce the Gronwall inequality, from relations (2.29), (2.31), and (2.32),

$$\frac{d}{dt}|u^{(2)}|^2_{L^2} \leqslant c(T_0)g(t)|u^{(2)}|^2_{L^2},$$

which implies that $\forall T_0$

$$|u^{(2)}(t)|^2_{L^2} \leqslant |u_0{}^{(2)}|^2_{L^2} e^{c(T_0)\int_0^{T_0} g(\sigma)d\sigma}. \tag{2.33}$$

The right-hand side of (2.33) is finite according to relation (2.30). □

Proof of Corollary 2. If $T_{\max} < +\infty$, then necessarily

$$\sup_{t \leqslant T_{\max}} |\frac{\partial^2 u}{\partial x^2}(t)|_{L^2(0,1)} = +\infty;$$

otherwise, u is a global solution on $[0, T_{\max}]$ and can be extended on $[T_{\max}, T_{\max} + \delta]$, which contradicts the fact that it is maximal. Thus, from Theorem 4, we deduce that

$$\sup_{t \leqslant T_{\max}} |\frac{\partial u}{\partial x}(t)|_{L^\infty(0,1)} = \sup_{t \leqslant T_{\max}} |\frac{\partial^2 u}{\partial x^2}(t)|_{L^2(0,1)} = +\infty.$$

This implies that there is a sequence $t_n \nearrow T_{\max}$ and $x_n \in [0,1]$ such that

$$\sup_{t \leqslant t_n} |\frac{\partial u}{\partial x}(t)|_{L^\infty(0,1)} = |\frac{\partial u}{\partial x}(t_n, x_n)|$$

and

$$\lim_n |\frac{\partial u}{\partial x}(t_n, x_n)| = +\infty.$$

There is $x^* \in [0,1]$ and a subsequence $x_{\sigma(n)}$ such that $x_{\sigma(n)} \to x^*$. We then have

$$\limsup_{(t,x) \to (T_{\max}, x^*)} |\frac{\partial u}{\partial x}(t,x)| \geqslant \lim_n |\frac{\partial u}{\partial x}(t_{\sigma(n)}, x_{\sigma(n)})| = +\infty.$$

2.3. **Navier-Stokes equations.** Let us consider the Navier-Stokes equations, following the notation and results that can be found in [3], [9], and [10]. We recall the physical situations and definitions in the appendix. For any $u_0 \in V$, we have a local (maximal) strong solution $(u, [0, T_{\max}))$ such that

$$\begin{cases} \frac{du}{dt} + \nu A u + B(u,u) = 0, \\ u(0) = u_0, \\ u \in L^2(0,T; D(A)) \cap L^\infty(0,T;V) \text{ for all } T < T_{\max}. \end{cases}$$

We have to show that if $|u_0| \|u_0\| < 2\frac{\nu^2}{c^2}$, then $T_{\max} = +\infty$. To apply Corollary 1 of Lemma 1 we shall consider $D = V \subset H$, $p(v) = \frac{\|v\|^2}{2}$, $q(v) = \frac{|v|^2}{2}$, and $N(v) = \nu \|v\|^2$. The following energy inequalities are well-known (see the above references):

$$\begin{cases} \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu \|u(t)\|^2 = 0, \\ \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu |Au(t))|^2 = -\big(B\big(u(t), u(t)\big), Au(t)\big) \leqslant c\|u(t)\|^{\frac{3}{2}} |Au(t)|^{\frac{3}{2}}. \end{cases}$$

By Young's inequality this implies

$$\begin{cases} \frac{d}{dt}q\big(u(t)\big) + N\big(u(t)\big) = 0, \\ \frac{d}{dt}p\big(u(t)\big) + \nu|Au(t)|^2 \leqslant \frac{3}{4}\nu|Au(t)|^2 + \frac{c^4}{4}\nu^{-3}\|u(t)\|^6, \end{cases}$$

from which we get

$$\begin{cases} \frac{d}{dt}q\big(u(t)\big) + N\big(u(t)\big) = 0, \\ \frac{d}{dt}p\big(u(t)\big) \leqslant c^4\nu^{-4}N\big(u(t)\big)p\big(u(t)\big)^2. \end{cases}$$

We may appeal Corollary 1 of Lemma 1 with $f = 0$, $c_1 = c^4\nu^{-4}$, and $m = 2$ so that the condition reads $c_1 q(u_0)p(u_0) < 1 \Longleftrightarrow |u_0|\|u_0\| < 2\frac{\nu^2}{c^2}$.
Thus, we have for all $t \geqslant 0$

$$\|u(t)\| \leqslant \frac{2\|u_0\|}{[4 - c^4\nu^{-4}|u_0|^2\|u_0\|^2]^{\frac{1}{2}}}, \tag{2.34}$$

from which we deduce the theorem.

Theorem 5. *Let c be the Sobolev constant given by*

$$c = \sup_{\varphi \in D(A)\setminus\{0\}} \frac{|B(\varphi,\varphi)|}{\|\varphi\|^{\frac{3}{2}}|A\varphi|^{\frac{1}{2}}}.$$

Then for any $u_0 \in V$ satisfying $|u_0|\|u_0\| < \frac{2}{c^2}\nu^2$, there is a strong solution u of the Navier-Stokes equations such that $u \in L^\infty(\mathbb{R}_+, V)$ and satisfying estimates (2.34).

In the physical problem that we recall below in the appendix, we need initial data $\overrightarrow{u}_0 \in H_0^1(\Omega)^3$ such that div $(\overrightarrow{u}_0) = 0$. Here is a construction of initial data satisfying condition (2), that is, $|u_0|\|u_0\| < \frac{2}{c^2}\nu^2$.

Theorem 6. *Let Ω be an open, bounded set of $\mathbb{R}^3$, starshaped with respect to a point x_0. For all $a > 0$ and all $\overrightarrow{v} \in \mathcal{V} = \{\overrightarrow{\varphi} \in C_c^\infty(\Omega)^3,\ \operatorname{div}(\overrightarrow{\varphi}) = 0\}$ we have a sequence $\overrightarrow{u}_{0j} \in \mathcal{V}$ such that*

i) $|\overrightarrow{u}_{0j}| \doteq |\overrightarrow{u}_{0j}|_{L^2(\Omega)^3} \xrightarrow[j\to\infty]{} 0$ *and* $\|\overrightarrow{u}_{0j}\| \doteq |\nabla\overrightarrow{u}_{0j}|_2 \xrightarrow[j\to\infty]{} +\infty$,

ii) $|\overrightarrow{u}_{0j}|\|\overrightarrow{u}_{0j}\| = a^2|\overrightarrow{v}|\|\overrightarrow{v}\|$.

In particular, if we choose a such that $a^2|\overrightarrow{v}|\|\overrightarrow{v}\| < 2\frac{\nu^2}{c^2}$, then $\overrightarrow{u}_{0j}$ satisfies condition (2) *and* $\|\overrightarrow{u}_{0j}\| \xrightarrow[j\to\infty]{} +\infty$.

Proof of Theorem 6. For $\overrightarrow{v} \in \mathcal{V}$, $j \geqslant 1$, and $a > 0$ the function $\overrightarrow{u}_{0j} = aj\overrightarrow{v}\big(j(x - x_0) + x_0\big)$ belongs to $\mathcal{V}$. Since Ω is starshaped with respect to x_0, one has $|\overrightarrow{u}_{0j}| = \frac{a}{\sqrt{j}}|\overrightarrow{v}|$ and $\|\overrightarrow{u}_{0j}\| = a\sqrt{j}\|\overrightarrow{v}\|$, from which we derive the result. □

Theorem 7. *Under the construction given in Theorem 6, for all $\nu > 0$, for all $v \doteq \overrightarrow{v} \in \mathcal{V}$, $a > 0$ small, there is a strong global solution u_j of*

$$\frac{du_j}{dt} + \nu A u_j + B(u_j, u_j) = 0, \quad u_j(0) = u_{0j}.$$

The norm $\|u_{0j}\| \to +\infty$ as $j \to +\infty$.

Proof. It is a direct consequence of Theorem 6 and Theorem 3.

Theorem 8. *Let $f \in L^2(\mathbb{R}_+, H)$, $u_0 \in V$. Assume that*

$$\left(\mu \int_0^{+\infty} |f(t)|^2 dt + |u_0|^2\right)\left(1 + \|u_0\|^2\right) < 2\frac{\nu^4}{c^4}, \qquad (c1),$$

with $\mu = \frac{1}{\lambda_1 \nu} + \frac{4\nu^3}{c^4}$. Then, there is a unique global, strong solution of the Navier-Stokes equations, with u_0 and f as data, say

$$\frac{du}{dt} + \nu A u + B(u, u) = f, \quad u(0) = u_0.$$

Proof. The usual energy inequalities associated to the Navier-Stokes equations (see [3] and [9]) are

$$\begin{cases} \frac{1}{2}\frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 = \big(f(t), u(t)\big), \\ \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \nu|Au(t)|^2 \leqslant c\|u(t)\|^{\frac{3}{2}}|Au(t)|^{\frac{3}{2}} + \big(f(t), Au(t)\big), \end{cases}$$

from which we derive (using Young's inequalities)

$$(\mathcal{A}) \qquad \begin{cases} \frac{d}{dt}|u(t)|^2 + \nu\|u(t)\|^2 \leqslant \frac{1}{\lambda_1 \nu}|f(t)|^2, \\ \frac{d}{dt}\|u(t)\|^2 \leqslant \frac{c^4\nu^{-3}}{2}\|u(t)\|^6 + \frac{2}{\nu}|f(t)|^2. \end{cases}$$

We set $N\big(u(t)\big) = \nu\|u(t)\|^2 + \frac{4\nu^3}{c^4}|f(t)|^2$, $\mu = \frac{1}{\lambda_1\nu} + \frac{4\nu^3}{c^4}$, $q\big(u(t)\big) = |u(t)|^2$, and $p\big(u(t)\big) = \|u(t)\|^2 + 1$. Then the system $(\mathcal{A})$ can be written as

$$(\mathcal{B}) \begin{cases} \frac{d}{dt}q\big(u(t)\big) + N\big(u(t)\big) \leqslant \mu|f(t)|^2, \\ \frac{d}{dt}p\big(u(t)\big) \leqslant \frac{c^4\nu^{-4}}{2}N\big(u(t)\big)p\big(u(t)\big)^2. \end{cases}$$

We may apply Corollary 1 of Lemma 1 to derive the result. □

Sufficient conditions for obtaining (c1) when $\|u_0\|$ is large. According to Theorem 6, for each element $v \in \mathcal{V}$, $a > 0$, we have a sequence $u_{0j} \in \mathcal{V}$ such that

i) $|u_{0j}|\|u_{0j}\| = a^2|v|\|v\|$,
ii) $|u_{0j}| \xrightarrow[j\to+\infty]{} 0$, $\|u_{0j}\| \xrightarrow[j\to+\infty]{} +\infty$.

To fulfill condition (c1), we can choose for instance a and u_{0j} such that

$$a^2|v|\|v\| < \frac{1}{\sqrt{2}}\frac{\nu^2}{c^2} \ \text{ and } \ |u_{0j}| < \frac{1}{\sqrt{2}}\frac{\nu^2}{c^2},\ \|u_{0j}\| \geqslant 1.$$

Then for all $f \in L^2\big(\mathbb{R}_+, H\big)$, with $\mu\|u_{0j}\| \int_0^{+\infty} |f(t)|^2 dt < \frac{1}{2}\frac{\nu^4}{c^4}$, the condition (c1) is fulfilled. □

3. Appendix: Navier-Stokes equations ([3], [9])

Let Ω be an open, bounded, smooth set of $\mathbb{R}^3$, $f : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and $u_0 : \Omega \to \mathbb{R}$ being given. The Navier-Stokes equations consist in finding a vector field $u = (u_1,\ u_2,\ u_3) : \mathbb{R}_+ \times \Omega \to \mathbb{R}^3$ and $p : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ a scalar function such that

$$(N.S) \qquad \begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + \sum_{i=1}^3 u_i \cdot \frac{\partial u}{\partial x_i} + \nabla p = f & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{in } \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{(initial data)},\ x \in \Omega. \end{cases}$$

The spaces H and V are defined by

$$H = \text{closure in } L^2(\Omega) \text{ of } \mathcal{V} = \big\{v \in C_c^\infty(\Omega)^3,\ \operatorname{div} v = 0\big\},$$

$$V = \text{closure in } H_0^1(\Omega)^3 \text{ of } \mathcal{V}.$$

For the variational formulation, we need the following bilinear form on $V \times V \times V$ and scalar products:

- $b(u,v,w) = \sum_{i,j=1}^3 \int_\Omega u_i \frac{\partial v_j}{\partial x_i} w_j dx,\ u = (u_1,u_2,u_3),\ v = (v_1,v_2,v_3)$, and $w = (w_1,w_2,w_3)$;
- $((u,v)) = \int_\Omega \nabla u \cdot \nabla v\, dx,\ (u,v) = \int_\Omega u \cdot v\, dx$;
- $\langle T, w\rangle =$ the duality product between $T \in V'$ and $w \in V$.

V' is the dual space of V. The bilinear form and the scalar product $((\cdot,\cdot))$ define operators by setting, for $u \in V$,

$$\begin{cases} \langle B(u,u), w\rangle = b(u,u,w) & \forall w \in V, \\ \langle Au, w\rangle = ((u,w)) & \forall w \in V. \end{cases}$$

Thus, for $u \in L^2(0,T;V)$, $T < +\infty$, we have $B(u,u) \in L^1(0,T;V')$ and $Au \in L^2(0,T;V')$, where A is linear. Thus, the variational formulation of $(N.S)$ for $f \in L^2(0,T;H)$, $u_0 \in V$ gives

$$\begin{cases} \frac{d}{dt}(u,v) + \nu((u,v)) + b(u,u,v) = (f,v) \quad \forall v \in V, \\ u(0) = u_0, \end{cases}$$

which is equivalent to

$$\begin{cases} \frac{du}{dt} + \nu Au + B(u,u) = f \quad \text{(equality in } V'), \\ u(0) = u_0. \end{cases}$$

The main properties of B are given in [9] and [10].

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